CONJECTURE CONCERNING THE FORMS OF THE ROOTS OF EQUATIONS OF ANY ORDER

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1. It is seen to be exceedingly paradoxical that, since from the very beginnings of analytical matters, the roots of cubic and biquadratic equations could be found, yet from these times, in which analysis has undertaken the greatest increases, hitherto the way of extracting roots of higher equations seems to lie hidden, especially since this matter may be investigated continually with the greatest studiousness, and with the most outstanding ingenuity. From which study, whatever be sought, little has been satisfied, yet outstanding aids have been found for treating such equations. On which account I do not think there is anyone going to withhold my development here, as I show what the forms the roots of the equations may have, and by what manner these perhaps shall be able to be found, even if more shall not be put in place. Indeed the more to help others, and finally perhaps it will be able to lead towards finding the goal.

2. Since an equation of any power may include all the lesser powers within it, it is easily understood likewise the method to be prepared thus so that any root extracted from each equation may involve the methods of all the lesser equations. On account of which the discovery of a root of the sixth dimension cannot be had, unless the same before is agreed from the equations of the fifth, fourth, and third degree. Thus we see Bombelli's method to lead from the extraction of biquadratic roots to the resolution of cubic equations ; and the root of the cubic equation cannot be defined without the resolution of the quadratic equation.

3. I consider the resolution of cubic equations to depend on quadratics in the follow manner. The cubic equation shall be

$$x^3 = ax + b,$$

in which the second term is absent; I say a root x of this to become

$$= \sqrt[3]{A} + \sqrt[3]{B}\ ,$$

with the two roots A and B proving to be of a certain quadratic equation

$$z^2 = \alpha z - \beta\ .$$

On account of which from the nature of equations, there will be

$$A + B = \alpha \text{ and } AB = \beta.$$

But with α and β requiring to be defined from *a* and *b*, I take the equation

$$x=\sqrt[3]{A}+\sqrt[3]{B}$$

which cubed by multiplication gives

$$x^3=A+B+3\sqrt[3]{AB}(\sqrt[3]{A}+\sqrt[3]{B})=3x\sqrt[3]{AB}+A+B.$$

Which compared with the proposed $x^3=ax+b,$ will give

$$\alpha=3\sqrt[3]{AB}=3\sqrt[3]{\beta} \text{ and } b=A+B=\alpha.$$

Therefore there becomes

$$\alpha=b \text{ and } \beta=\tfrac{a^3}{27};$$

whereby the quadratic equation serving in the resolution of the equation in the said manner $x^3=ax+b$ will be

$$z^2=bz-\tfrac{a^3}{27}.$$

Of which indeed with the known roots *A* and *B* there will become

$$x=\sqrt[3]{A}+\sqrt[3]{B}.$$

4. But since the cube root from some tripled quantity shall have a value, this formula $x=\sqrt[3]{A}+\sqrt[3]{B}$ will contain also all the roots of the proposed equation. For μ and ν, besides unity, shall be the cube roots of unity; also there will be

$$x=\mu\sqrt[3]{A}+\nu\sqrt[3]{B},$$

only if there shall be $\mu\nu=1$. On account of which μ and ν must be

$$\tfrac{-1+\sqrt{-3}}{2} \text{ and } \tfrac{-1-\sqrt{-3}}{2}$$

or in the other order. Therefore besides the root

$$x=\sqrt[3]{A}+\sqrt[3]{B}$$

these two other roots also will satisfy the proposed equation

$$x = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{A} + \frac{-1-\sqrt{-3}}{2}\sqrt[3]{B}$$

and

$$x = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{A} + \frac{-1+\sqrt{-3}}{2}\sqrt[3]{B}.$$

On this account the roots of cubic equations also will be able to be determined, in which the second term is not absent.

5. Biquadratic equations are accustomed to be reduced to cubic equations in various ways, but none of which can be used in my set up. Moreover the method is special and likewise effective for me, by which in the first place cubes are reduced to squares, thus so thence in some way, it may be able to be concluded, any equations of higher order ought to be treated. So that if this equation shall be proposed

$$x^4 = ax^2 + bx + c,$$

in which likewise the second term is absent, I say to become

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C},$$

but *A, B* and *C* to be the three roots from a certain cubic equation

$$z^3 = \alpha z^2 - \beta z + \gamma.$$

On this account there will be

$$\alpha = A + B + C,\ \beta = AB + AC + BC \text{ and } \gamma = ABC.$$

But so that α, β, and γ may be determined, the equation $x = \sqrt{A} + \sqrt{B} + \sqrt{C}$ may be freed from irrationality in this manner. Squares may be taken; there shall become

$$x^2 = A + B + C + 2\sqrt{AB} + 2\sqrt{AC} + 2\sqrt{BC}$$

and hence

$$x^2 - \alpha = 2\sqrt{AB} + 2\sqrt{AC} + 2\sqrt{BC}.$$

With the squares take anew there shall become

$$x^4 - 2\alpha x^2 + \alpha^2 = 4AB + 4AC + 4BC + 8\sqrt{ABC}(\sqrt{A} + \sqrt{B} + \sqrt{C}) = 4\beta + 8x\sqrt{\gamma}$$

or

$$x^4 = 2\alpha x^2 + 8x\sqrt{\gamma} + 4\beta - \alpha^2.$$

This equation compared with the proposed $x^4 = ax^2 + bx + c$, will give

$$2\alpha = a,\ 8\sqrt{\gamma} = b, \text{ and } 4\beta - \alpha^2 = c,$$

from which there arises

$$\alpha = \tfrac{a}{2},\ \gamma = \tfrac{b^2}{64}, \text{ and } \beta = \tfrac{c}{4} + \tfrac{\alpha^2}{16}.$$

Therefore the cubic serving for the resolution of the biquadratic is

$$z^3 = \tfrac{a}{2}z^2 - \tfrac{4c+a^2}{16}z + \tfrac{b^2}{64}.$$

Indeed if the roots of this shall be A, B and C, there will be

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C}$$

But the three remaining roots from the proposed equation will be

$$\sqrt{A} - \sqrt{B} - \sqrt{C},\ \sqrt{B} - \sqrt{A} - \sqrt{C}, \text{ and } \sqrt{C} - \sqrt{A} - \sqrt{B}.$$

6. There may be put $z = \sqrt{t}$ *;* there will be

$$\left(t + \tfrac{4c+a^2}{16}\right)\sqrt{t} = \tfrac{at}{2} + \tfrac{b^2}{64}$$

and from the squares taken, there will be had

$$t^3 + \tfrac{4c+a^2}{8}t^2 + \tfrac{(4c+a^2)^2}{256}t = \tfrac{a^2t^2}{4} + \tfrac{ab^2t}{64} + \tfrac{b^4}{4096}$$

or

$$t^3 = \left(\tfrac{a^2}{8} - \tfrac{c}{2}\right)t^2 + \left(\tfrac{ab^2}{64} - \tfrac{cc}{16} - \tfrac{a^2c}{32} - \tfrac{a^4}{256}\right)t + \tfrac{b^4}{4096}.$$

This equation therefore has this property, that its roots shall be the square roots of the former equation *A, B* and *C.* Whereby if the roots of this equation may be put to be *E, F, G,* there will become

$$x = \sqrt[4]{E} + \sqrt[4]{F} + \sqrt[4]{G}.$$

And thus a cubic equation is given, of which the roots likewise taken may constitute the root of the biquadratic equation proposed. And this method of finding the roots from a biquadratic equation, even if it shall be harder initially, has a greater relationship with the solution of cubic equations, since the root may be extracted of the same power from the roots of an inferior equation, of which that equation has been proposed.

7. By similar reasoning also the quadratic equation

$$x^2 = a\,,$$

in which the second term is absent, is resolved with the aid of an equation of one dimension

$$z = a$$

Indeed the root of this is a and the root of the proposed equation

$$x = \sqrt{a} \text{ or } x = -\sqrt{a}\,.$$

But the equation of this kind with the lesser order I will call the *resolving equation*, with the aid of which the superior equation without the second term is resolved. Thus the resolving equation of the quadratic equation

$$x^2 = a$$

will be

$$z = a\,;$$

the resolving equation of the cubic equation

$$x^3 = ax + b$$

will be

$$z^2 = bz - \tfrac{a^3}{27}$$

and the resolving equation of the biquadratic equation

$$x^4 = ax^2 + bx + c$$

is

$$z^3 = (\tfrac{a^2}{8} - \tfrac{c}{2})z^2 - (\tfrac{a^4}{256} + \tfrac{a^2c}{32} + \tfrac{cc}{16} - \tfrac{ab^2}{64})z + \tfrac{b^4}{4096}.$$

Indeed for the quadratic equation, if the root of the resolving equation shall be *A,* there will be

$$x = \sqrt{A}\,;$$

truly for the cubic equation, if the roots of the resolvent shall be *A* and *B,* there will be

$$x = \sqrt[3]{A} + \sqrt[3]{B}\,;$$

and for the biquadratic equation with the resolving roots of the resolving equation *A, B* and *C ,* there will be

$$x = \sqrt[3]{A} + \sqrt[3]{B} + \sqrt[3]{C}\,.$$

8. Even if only from these three cases, it may appear to me to be a sufficient reason to conclude also of higher order equations to give resolving equations of this kind. Thus I assign for the proposed equation

$$x^5 = ax^3 + bx^2 + cx + d$$

to give an equation of the fourth order

$$z^4 = \alpha z^4 - \beta z^2 + \gamma z - \delta,$$

of which the roots shall be *A, B, C* and *D,* to become

$$x = \sqrt[5]{A} + \sqrt[5]{B} + \sqrt[5]{C} + \sqrt[5]{D}.$$

And generally the resolving equation of the equation

$$x^n = ax^{n-2} + bx^{n-3} + cx^{n-4} + \text{etc.},$$

just as I suspect, will be of this form

$$z^{n-1} = \alpha z^{n-2} - \beta z^{n-3} + \gamma z^{n-4} - \text{etc.},$$

of which with the number $n-1$ for all the known roots, which shall be *A, B, C, D* etc., there will be

$$x = \sqrt[n]{A} + \sqrt[n]{B} + \sqrt[n]{C} + \sqrt[n]{D} + \text{etc.}$$

Therefore this conjecture, if it may be agreed to be true, and if the resolving equations may be able to be determined, and of which equations the assigned roots may be able to brought out ; indeed there may arrive always an equation of inferior order and by progressing in this way finally a true root of the proposed equation will become known.

9. But nevertheless, if the proposed equation may have more than four dimensions, at this point I am unable to define the resolving equation, yet at present there are more arguments at hand, by which that same conjecture is confirmed for me. Indeed if the proposed equation has been prepared thus, so that in the resolving equation all the terms may vanish except the first three, then that same resolving equation can be shown always and thus the roots of the proposed equation to be assigned. But the equations, which allow a resolution in this manner, are those themselves, which the celebrated Abr. De Moivre treated in the *Phil. Transact.* no. 809 [*Royal Soc.*]. Indeed the resolving equation shall be

$$z^{n-1}=\alpha z^{n-2}-\beta z^{n-3},$$

or

$$z^{2}=\alpha z-\beta$$

and from this the resolving equation shall be required to be extracted. A and B shall be the roots of this equation ; for the remaining roots all vanish ; the root of the resolving equation will be

$$x=\sqrt[n]{A}+\sqrt[n]{B}.$$

Truly there is

$$\alpha=A+B \text{ and } \beta=AB$$

from the nature of the equations. Hence therefore there will be

$$\sqrt[n]{A^2}+\sqrt[n]{B^2}=x^2-2\sqrt[n]{\beta}$$

and again

$$\sqrt[n]{A^3}+\sqrt[n]{B^3}=x^3-3x\sqrt[n]{\beta},$$
$$\sqrt[n]{A^4}+\sqrt[n]{B^4}=x^4-4x^2\sqrt[n]{\beta}+2\sqrt[n]{\beta^2},$$
$$\sqrt[n]{A^5}+\sqrt[n]{B^5}=x^5-5x^3\sqrt[n]{\beta}+5x\sqrt[n]{\beta^2},$$

and finally

$$\sqrt[n]{A^n}+\sqrt[n]{B^n}=x^n-nx^{n-2}\sqrt[n]{\beta}+\tfrac{n(n-3)}{1\cdot 2}x^{n-4}\sqrt[n]{\beta^2}$$
$$-\tfrac{n(n-4)(n-5)}{1\cdot 2\cdot 3}x^{n-6}\sqrt[n]{\beta^3}+\tfrac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4}x^{n-8}\sqrt[n]{\beta^5}-\text{etc.}=\alpha.$$

Which is the equation requiring to be resolved, of which the resolvent is

$$z^{n-1} = \alpha z^{n-2} - \beta z^{n-3} \text{ or } z^2 = \alpha z - \beta.$$

10. Not only moreover is a single root

$$x = \sqrt[n]{A} + \sqrt[n]{B}$$

of the equation

$$x^n - nx^{n-2}\sqrt[n]{\beta} + \tfrac{n(n-3)}{1\cdot 2}x^{n-4}\sqrt[n]{\beta^2} - \text{etc.} = \alpha$$

found, but also it satisfies some other

$$x = \mu\sqrt[n]{A} + \nu\sqrt[n]{B},$$

provided there shall be $\mu^n = \nu^n = \mu\nu = 1$, which can happen in *n* different ways. So that if there shall be $n = 5$, the fifth roots of the equation

$$x^5 - 5x^3\sqrt[n]{\beta} + 5x\sqrt[n]{\beta^2} = \alpha$$

will be as follows:

I. $x = \sqrt[5]{A} + \sqrt[5]{B}$,

II. $x = \frac{-1-\sqrt{5}+\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1-\sqrt{5}-\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{B}$,

III. $x = \frac{-1-\sqrt{5}-\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1-\sqrt{5}+\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{B}$,

IV. $x = \frac{-1+\sqrt{5}+\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1+\sqrt{5}-\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{B}$,

V. $x = \frac{-1+\sqrt{5}-\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1+\sqrt{5}+\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{B}$.

For all these coefficients are roots of higher order of unity, and the product from two taken together $=1$.
In a similar manner besides unity itself there are the six roots of the seven powers from unity and of these the three pairs produce unity by multiplication, which are the six roots of this equation

$$y^6 + y^5 + y^4 + y^3 + y^2 + y + 1 = 0$$

Moreover for finding these there is a need only for the resolution of a cubic equation ; for the equation of all the sixth powers of this form

$$y^6 + ay^5 + by^4 + cy^3 + by^2 + ay + 1 = 0,$$

which is not changed on putting $\frac{1}{y}$ in place of *y,* can be resolved with the aid of a cubic equation.
Just as which may happen, since it may often be useful for finding roots, I am going to show briefly.

11. Equations of this kind, which do not change form on putting $\frac{1}{y}$ in place of y, I call *reciprocals.* These, if the maximum number of its dimension y is odd, always can be divided by $y+1$ and the reciprocal equation also will be reciprocal, in which the maximum dimension of y will be even. On account of which it will suffice to have considered reciprocal equations of even dimensions only and the manner of resolving these presented.

Therefore this shall be the first equation proposed of the fourth power

$$y^4 + ay^3 + by^2 + ay + 1 = 0;$$

this may be put to be the product from the two quadratics

$$y^2 + \alpha y + 1 = 0$$

and

$$y^2 + \beta y + 1 = 0.$$

From which product there becomes

$$\alpha + \beta = a \text{ and } \alpha\beta + 2 = b \text{ or } \alpha\beta = b - 2.$$

Whereby α and β will be two roots of this equation

$$u^2 - au + b - 2 = 0$$

and from this account the four roots of the proposed equation will become known with the aid of quadratics only.

The reciprocal equation of the sixth power

$$y^6 + ay^5 + by^4 + cy^3 + by^2 + ay + 1 = 0$$

may be put the product of these three equations

$$y^2+\alpha y+1=0,$$
$$y^2+\beta y+1=0,$$

and

$$y^2+\gamma y+1=0.$$

Hence there will become

$$\alpha+\beta+\gamma=a,$$
$$\alpha\beta+\alpha\gamma+\beta\gamma=b-3$$

and

$$\alpha\beta\gamma=c-2\alpha-2\beta-2\gamma=c-2a.$$

Whereby α, β et γ will be the three roots of this cubic equation

$$u^3-au^2+(b-3)u-c+2a=0.$$

Similarly the reciprocal equation of the eighth power

$$y^8+ay^7+by^6+cy^5+dy^4+cy^3+by^2+ay+1=0$$

is the product from these four quadratic equations

$$y^2+\alpha y+1=0,$$
$$y^2+\beta y+1=0,$$
$$y^2+\gamma y+1=0,$$

and

$$y^2+\gamma y+1=0.$$

from which there will be produced

$$\alpha+\beta+\gamma+\delta=a,$$
$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=b-4,$$
$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=c-3a$$

and

$$\alpha\beta\gamma\delta = d - 2b + 2.$$

Therefore the coefficients $\alpha, \beta, \gamma, \delta$ are the four roots of this equation

$$u^4 - au^3 + (b-4)u^2 - (c-3a)u + d - 2b + 2 = 0 .$$

The equation of the tenth order

$$y^{10} + ay^9 + by^8 + cy^7 + dy^6 + ey^5 + dy^4 + cy^3 + by^2 + ay + 1 = 0$$

will be the product of these five

$$y^2 + \alpha y + 1 = 0,$$
$$y^2 + \beta y + 1 = 0,$$
$$y^2 + \gamma y + 1 = 0,$$
$$y^2 + \delta y + 1 = 0,$$
$$y^2 + \varepsilon y + 1 = 0,$$

in which $\alpha, \beta, \gamma, \delta, \varepsilon$ are the five roots of this equation

$$u^5 - au^4 + (b-5)u^3 - (c-4a)u^2 + (d-3b+5)u - e + 2c - 2a = 0.$$

And generally the reciprocal equation

$$y^{2n} + ay^{2n-1} + by^{2n-2} + cy^{2n-3} + dy^{2n-4} + ey^{2n-5} + fy^{2n-6} + \cdots py^n + \cdots$$
$$+ fy^6 + ey^5 + dy^4 + cy^3 + by^2 + ay + 1 = 0$$

will be resolved into these quadratic equations with the number *n*

$$y^2 + \alpha y + 1 = 0,$$
$$y^2 + \beta y + 1 = 0,$$
$$y^2 + \gamma y + 1 = 0,$$
$$y^2 + \delta y + 1 = 0,$$
etc.

But the coefficients $\alpha, \beta, \gamma, \delta$ etc. will be the roots of this equation of dimension *n*

$$\begin{array}{llllll}
u^n - au^{n-1} + bu^{n-2} & -cu^{n-3} & +du^{n-4} & -eu^{n-5} \\
\quad -n & +(n-1)a & -(n-2)b & +(n-3)c \\
 & & +\frac{n(n-3)}{1\cdot 2} & -\frac{(n-1)(n-4)}{1\cdot 2}a
\end{array}$$

$$\begin{array}{lll}
+fu^{n-6} & -gu^{n-7} & +hu^{n-8} - \text{etc.} = 0 \\
-(n-4)d & +(n-5)e & -(n-6)f \\
+\frac{(n-2)(n-5)}{1\cdot 2}b & -\frac{(n-3)(n-6)}{1\cdot 2}c & +\frac{(n-4)(n-7)}{1\cdot 2}d \\
-\frac{n(n-4)(n-5)}{1\cdot 2\cdot 3} & +\frac{(n-1)(n-5)(n-6)}{1\cdot 2\cdot 3}a & -\frac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}b \\
 & & +\frac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4}
\end{array}$$

12. Because the final term of any quadratic equation dividing the proposed equation is unity, it is evident the product of the two roots of the proposed equation is unity. Therefore two of this kind always are required to be joined together with the two corresponding members $\sqrt[n]{A}$ and $\sqrt[n]{B}$, by which all the proposed roots of equation § 9 may be obtained.

13. If in the reciprocal equation all the terms may be missing except the first, last, and the middle one, so that in

$$y^{2n} + py^n + 1 = 0,$$

its divisors

$$y^2 + \alpha y + 1 = 0,$$
$$y^2 + \beta y + 1 = 0,$$
$$y^2 + \gamma y + 1 = 0,$$
etc.

will be had with the roots of this equation being substituted for $\alpha, \beta, \gamma, \delta$ etc.

$$u^n - nu^{n-2} + \frac{n(n-3)}{1\cdot 2}u^{n-4} - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}u^{n-6} + \cdots \pm p = 0,$$

where $+p$ must be taken, if n is an even number, and $-p$, if n is odd [The *O.O.* editor of this work, Ferdinand Rudio, indicates that this is not the case for even p, where the final term is $= p \pm 2$]. From which it is apparent this equation to agree with the equation

$$x^n - nx^{n-2}\sqrt[n]{\beta} + \cdots = \alpha$$

§ 9 resolved and that on this account all the divisors are able to be assigned.

14. This same resolution of the formula $y^{2n}+py^n+1=0$ into factors has a great use in the integration of the differential formula

$$\frac{dy}{y^{2n}+py^n+1},$$

now treated more often by the geometers. Indeed with the denominator resolved into its factors $y^2+\alpha y+1=0,\ y^2+\beta y+1=0,$ etc. the whole integration is reduced to the quadrature of the circle or of the hyperbola. Besides this it may help the most, that the equation

$$u^n-nu^{n-2}+\tfrac{n(n-3)}{1\cdot 2}u^{n-4}-\cdots\pm p=0,$$

from which $\alpha,\ \beta,\ \gamma$ etc. may be determined, may consist of the section of a circular arc into *n* parts and thus the coefficients $\alpha,\ \beta,\ \gamma$ etc. may be found easily.

15. But we may revert to the way these equations are required to be resolved by being elicited from resolvable equations. And the resolving equation shall be

$$z^3=\alpha z^2-\beta z+\gamma,$$

of which the three roots shall be *A, B, C;* therefore there will be

$$\alpha=A+B+C,\ \beta=AB+AC+BC \text{ and } \gamma=ABC.$$

And thus a root *x* of the equation requiring to be resolved will be

$$=\sqrt[n]{A}+\sqrt[n]{B}+\sqrt[n]{C}$$

and there may be put

$$p=\sqrt[n]{AB}+\sqrt[n]{AC}+\sqrt[n]{BC}.$$

With these put in place

$$\sqrt[n]{A^2}+\sqrt[n]{B^2}+\sqrt[n]{C^2}=x^2-2p$$

and

$$\sqrt[n]{A^2B^2}+\sqrt[n]{B^2C^2}+\sqrt[n]{B^2C^2}=p^2-2x\sqrt[n]{\gamma};$$

and again, so that there follows:

$$\sqrt[n]{A^3}+\sqrt[n]{B^3}+\sqrt[n]{C^3}=x^3-3px+3\sqrt[n]{\gamma},$$
$$\sqrt[n]{A^3B^3}+\sqrt[n]{B^3C^3}+\sqrt[n]{B^3C^3}=p^3-3px\sqrt[n]{\gamma}+3\sqrt[n]{\gamma^2};$$
$$\sqrt[n]{A^4}+\sqrt[n]{B^4}+\sqrt[n]{C^4}=x^4-4px^2+4x\sqrt[n]{\gamma}+2p^2;$$
$$\sqrt[n]{A^4B^4}+\sqrt[n]{B^4C^4}+\sqrt[n]{B^4C^4}=p^4-4p^2x\sqrt[n]{\gamma}+4p\sqrt[n]{\gamma^2}+2x^2\sqrt[n]{\gamma^2};$$
$$\sqrt[n]{A^5}+\sqrt[n]{B^5}+\sqrt[n]{C^5}=x^5-5px^3+5x^2\sqrt[n]{\gamma}+5p^2x-5p\sqrt[n]{\gamma},$$
$$\sqrt[n]{A^5B^5}+\sqrt[n]{B^5C^5}+\sqrt[n]{B^5C^5}=p^5-5p^3x\sqrt[n]{\gamma}+5p^2\sqrt[n]{\gamma^2}+5px^2\sqrt[n]{\gamma^2}-5x\sqrt[n]{\gamma^3}.$$

Just as it is readily seen, this table may be continued further. And indeed there is

$$\sqrt[n]{A^m}+\sqrt[n]{B^m}+\sqrt[n]{C^m}=x(\sqrt[n]{A^{m-1}}+\sqrt[n]{B^{m-1}}+\sqrt[n]{C^{m-1}})$$
$$-p(\sqrt[n]{A^{m-2}}+\sqrt[n]{B^{m-2}}+\sqrt[n]{C^{m-2}})+\sqrt[n]{\gamma}(\sqrt[n]{A^{m-3}}+\sqrt[n]{B^{m-3}}+\sqrt[n]{C^{m-3}})$$

and

$$\sqrt[n]{A^mB^m}+\sqrt[n]{A^mC^m}+\sqrt[n]{B^mC^m}=p(\sqrt[n]{A^{m-1}B^{m-1}}+\sqrt[n]{A^{m-1}C^{m-1}}+\sqrt[n]{B^{m-1}C^{m-1}}+)$$
$$-x\sqrt[n]{\gamma}(\sqrt[n]{A^{m-2}B^{m-2}}+\sqrt[n]{A^{m-2}C^{m-2}}+\sqrt[n]{B^{m-2}C^{m-2}})$$
$$+\sqrt[n]{\gamma^2}(\sqrt[n]{A^{m-3}B^{m-3}}+\sqrt[n]{C^{m-3}A^{m-3}}+\sqrt[n]{B^{m-3}}C^{m-3}).$$

16. Also we should not disregard other properties of this progression I have observed. For on putting

$$\sqrt[n]{A^m}+\sqrt[n]{B^m}+\sqrt[n]{C^m}=R$$

and

$$\sqrt[n]{A^mB^m}+\sqrt[n]{A^mC^m}+\sqrt[n]{B^mC^m}=S$$

there will become

$$\sqrt[n]{A^{2m}}+\sqrt[n]{B^{2m}}+\sqrt[n]{C^{2m}}=R^2-2S$$

and

$$\sqrt[n]{A^{2m}B^{2m}}+\sqrt[n]{A^{2m}C^{2m}}+\sqrt[n]{B^{2m}C^{2m}}=S^2-2R\sqrt[n]{\gamma^m}.$$

In a similar manner there is also

$$\sqrt[n]{A^{3m}}+\sqrt[n]{B^{3m}}+\sqrt[n]{C^{3m}}=R^3-3RS+3\sqrt[n]{\gamma^m}$$

and

$$\sqrt[n]{A^{3m}B^{3m}}+\sqrt[n]{A^{3m}C^{3m}}+\sqrt[n]{B^{3m}C^{3m}}=S^3-3RS\sqrt[n]{\gamma^m}+3\sqrt[n]{\gamma^{2m}}.$$

And this series proceeds forwards as that itself preceding.

17. If there shall be $n=2$, there will be

$$\alpha = x^2 - 2p \text{ and } \beta = p^2 - 2x\sqrt{\gamma}$$

and from these two equations taken together there will be had

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C}$$

and

$$p = \sqrt{AB} + \sqrt{AC} + \sqrt{BC};$$

but *A, B* and *C* are the three roots of this cubic equation

$$z^3 = \alpha z^2 - \beta z + \gamma.$$

Therefore with the letter *p* eliminated from these two equations there will be produced

$$(\tfrac{x^2-\alpha}{2})^2 - 2x\sqrt{\gamma} = \beta$$

or

$$x^4 - 2\alpha x^2 - 8x\sqrt{\gamma} = 4\beta - \alpha^2,$$

and thus the root *x* of which equation is known, evidently $= \sqrt{A} + \sqrt{B} + \sqrt{C}$*;* which equation is agreeing with that, which has been resolved in § 5.

In a similar manner when two equations of this kind occur

$$x^3 - 3px + 3\sqrt[3]{\gamma} = \alpha$$

and

$$p^3 - 3px\sqrt[3]{\gamma} + 3\sqrt[3]{\gamma^2} = \beta,$$

if there will be

$$x = \sqrt[3]{A} + \sqrt[3]{B} + \sqrt[3]{C}$$

and

$$p = \sqrt[3]{AB} + \sqrt[3]{AC} + \sqrt[3]{BC}$$

with *A, B* and *C* being the roots of the equation

$$z^3 = \alpha z^2 - \beta z + \gamma$$

as before. Or by eliminating the letter *p* an equation will be produced between $x, \alpha, \beta, \gamma,$ of which the root *x* will be known.

In the same straight forwards manner, and with the two equations occurring,

$$x^4 - 4px^2 + 4\sqrt[4]{\gamma} + 2p^2 = \alpha$$

and

$$p^4 - 4p^2x\sqrt[4]{\gamma} + 4p\sqrt{\gamma} + 4x^2\sqrt{\gamma} = \beta,$$

there will become

$$x = \sqrt[4]{A} + \sqrt[4]{B} + \sqrt[4]{C}$$

and

$$p = \sqrt[4]{AB} + \sqrt[4]{AC} + \sqrt[4]{BC}$$

and again *A, B* and *C* are the roots of this equation

$$z^3 = \alpha z^2 - \beta z + \gamma.$$

So that *p* may be eliminated more easily, there is put

$$x^2 - 2p = R \text{ and } p^2 - 2x\sqrt[4]{\gamma} = S$$

and there will become

$$R^2 - 2S = \alpha \text{ and } S^2 - 2R\sqrt{\gamma} = \beta.$$

Now with *p* removed from these two equations there will be had

$$x^4 = 2Rx^2 + 8x\sqrt[4]{\gamma} + 4S - R^2.$$

This equation may be compared with that

$$x^4 = ax^2 + bx + c;$$

there will be

$$R = \tfrac{a}{2},\ \sqrt[4]{\gamma} = \tfrac{b}{8} \text{ or } \gamma = \tfrac{b^4}{4096} \text{ and } S = \tfrac{c}{4} + \tfrac{a^2}{16}.$$

Hence there will be had, therefore

$$\alpha = \frac{a^2}{8} - \frac{c}{2} \text{ and } \beta = \frac{c^2}{16} + \frac{a^2c}{32} + \frac{a^4}{256} - \frac{ab^3}{64}.$$

On account of which *A, B* and *C* will be the three roots of this equation

$$z^3 = (\frac{a^2}{8} - \frac{c}{2})z^2 - (\frac{a^4}{256} + \frac{a^2c}{32} + \frac{c^2}{16} - \frac{ab^3}{64})z + \frac{b^4}{4096},$$

that which is in remarkable agreement with that, which was found in § 7.

18. Therefore whenever it happens, that a calculation may lead to two equations containing the two unknowns *x* and *p*, which may be found between the formulas in § 15, and the value of each will be able to be assigned, even if it may be especially arranged with the other equation eliminated. On account of this in these cases it will expedite the calculation not to deduce the unknown for a single equation and for a single unknown, but to retain two equations involving two unknowns and to investigate, whether perhaps a solution may be contained between these two formulas, if the calculation is set up correctly, which I have persuaded myself can happen more often.

19. Moreover just as we have treated solving the equations

$$z^2 = \alpha z - \beta \text{ and } z^3 = \alpha z^2 - \beta z + \gamma,$$

thus also so that it is required to progress further to the equation

$$z^4 = \alpha z^3 - \beta z^2 + \gamma z - \delta$$

on being treated in a similar manner. Evidently if its roots shall be *A, B, C* and *D,* there may be put

$$\sqrt[n]{A} + \sqrt[n]{B} + \sqrt[n]{C} + \sqrt[n]{D} = x,$$

$$\sqrt[n]{AB} + \sqrt[n]{AC} + \sqrt[n]{AD} + \sqrt[n]{BC} + \sqrt[n]{BD} + \sqrt[n]{CD} = p,$$

and

$$\sqrt[n]{ABC} + \sqrt[n]{ABD} + \sqrt[n]{ACD} + \sqrt[n]{BCD} = q$$

and hence the expressions may be sought for

$$\sqrt[n]{A^m} + \sqrt[n]{B^m}\ ,\ \sqrt[n]{A^mB^m} + \sqrt[n]{A^mC^m} + \text{etc. and for } \sqrt[n]{A^mB^mC^m} + \text{etc.}$$

With these completed always three equations will be found containing x, p and q for whatever value of m. And in a similar manner with three equations of this kind occurring, three unknowns will be agreed.

20. But I suspect on putting

$$x = \sqrt[5]{A} + \sqrt[5]{B} + \sqrt[5]{C} + \sqrt[5]{D}$$

a rational equation can be put in place, in which x may not have more than 5 dimensions, even if this may be seen to be almost impossible. For just as in §17 from the equations

$$x^4 - 4px^2 + 4x\sqrt[4]{\gamma} + 2p^2 = \alpha$$

and

$$p^4 - 4p^2x\sqrt[4]{\gamma} + 4p\sqrt{\gamma} + 2x^2\sqrt{\gamma} = \beta$$

by eliminating p we have obtained an equation of not more than 4 dimensions, which equally scarcely may be seen to happen, thus also for the fifth power perhaps a similar artifice can come to be used, so that finally the equation

$$x^5 = ax^3 + bx^2 + cx + d$$

may be able to be resolved. What truly is the greatest help in doing this, so that it responds to my judgement, that α, β, γ and δ not truly in turn, ought to be determined from a, b, c and d ; for in this case the equation may be carried off to a much higher power, than is necessary. But this matter is required to be completed by others, who aid occupations of this kind, or I leave this for myself at another time, only now it pleases me to have shown perhaps a suitable and natural way.

DE FORMIS RADICUM AEQUATIONUM CUIUSQUE ORDINIS CONIECTATIO

Commentatio 30 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 6 (1732/3), 1738, p. 216-231

1. Summopere admirandum videtur, quod, cum ipsis rei analyticae initiis radices aequationum cubicarum et biquadraticarum essent inventae, his tamen temporibus, quibus analysis maxima augmenta accepit, modus adhuc lateat altiorum aequationum radices eruendi, praesertim cum haec res continuo a praestantissimis ingeniis maximo studio sit investigata. Quo studio, quamquam quaesito parum est satisfactum, egregia tamen ad quasque aequationes tractandas subsidia sunt detecta. Quamobrem neminem puto fore, qui hoc meum institutum, quo, quas formas habeant aequationum radices et qua via eae forte inveniri possint, ostendo, etiamsi plus non praestiterim, sit reprehensurus. Alios enim fortasse magis iuvare atque tandem ad inventum scopum perducere poterit.
2. Cum aequatio cuiusque potestatis omnes inferiores in se comprehendat, facile perspicitur methodum quoque radicem ex quaque aequatione extl~hendi ita esse comparatam, ut omnium inferiorum aequationum methodos involvat. Quamobrem inventio radicis ex aequatione sex dimensionum haberi non potest, nisi eadem antea constat de aequationibus quinti, quarti et tertii gradus. lta videmus BOMBELLI methodum ex aequationibus biquadratics radicis extrahendi perducere ad resolutionem aequationis cubicae; atque cubicae aequationis radicem definiri non posse sine quadraticae aequationis resolutione.

3. Resolutionem aequationis cubicae sequenti modo a quadratica pendentem considero. Sit aequatio cubica

$$x^3 = ax + b,$$

in qua secundus terminus deest; huius radicem x dico fore

$$= \sqrt[3]{A} + \sqrt[3]{B}$$

existentibus A et B duabus radicibus aequationis cuiusdam quadraticae

$$z^2 = \alpha z - \beta$$

Quamobrem ex natura aequationum erit

$$A+B=\alpha \text{ et } AB=\beta.$$

Sed ad α et β ex a et b definiendas sumo aequationem

$$x=\sqrt[3]{A}+\sqrt[3]{B}$$

quae cubice multiplicata dat

$$x^3=A+B+3\sqrt[3]{AB}(\sqrt[3]{A}+\sqrt[3]{B})=3x\sqrt[3]{AB}+A+B.$$

Quae cum proposita $x^3=ax+b,$ comparata dabit

$$\alpha=3\sqrt[3]{AB}=3\sqrt[3]{\beta} \text{ et } b=A+B=\alpha.$$

Fiet igitur

$$\alpha=b \text{ et } \beta=\tfrac{a^3}{27};$$

quocirca aequatio quadratica resolutioni aequationis $x^3=ax+b$ dicto modo inserviens erit

$$z^2=bz-\tfrac{a^3}{27}.$$

Huius enim radicibus cognitis A et B erit

$$x=\sqrt[3]{A}+\sqrt[3]{B}.$$

4. Sed cum radix cubica ex quaque quantitate triplicem habeat valorem, haec formula $x=\sqrt[3]{A}+\sqrt[3]{B}$ omnes etiam radices aequationis propositae complectetur. Sint enim μ et ν praeter unitatem radices cubicae ex unitate; erit etiam

$$x=\mu\sqrt[3]{A}+\nu\sqrt[3]{B},$$

si modo sit $\mu\nu=1$. Quamobrem μ et ν esse debebunt

$$\tfrac{-1+\sqrt{-3}}{2} \text{ et } \tfrac{-1-\sqrt{-3}}{2}$$

vel inverse. Praeter radicem igitur

$$x=\sqrt[3]{A}+\sqrt[3]{B}$$

satisfacient quoque aequationi propositae hae duae alterae radices

$$x = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{A} + \frac{-1-\sqrt{-3}}{2}\sqrt[3]{B}$$

et

$$x = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{A} + \frac{-1+\sqrt{-3}}{2}\sqrt[3]{B}.$$

Hacque ratione aequationis cubicae etiam, in qua secundus terminus non deest, radices determinari poterunt.

5. Aequationes biquadraticae variis modis ad cubicas reduci solent, quorum autem nullus instituto meo utilitatem afferre potest. Sed mihi est peculiaris methodus idem efficiendi atque priori, qua cubicae ad quadraticas reducuntur, similis, ita ut exinde quodammodo concludi possit, quomodo aequationes altiorum graduum debeant tractari. Ut si proposita sit haec aequatio

$$x^4 = ax^2 + bx + c,$$

in qua itidem secundus terminus deest, dico fore

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C},$$

at *A, B* et *C* esse tres radices ex aequatione quadam cubica

$$z^3 = \alpha z^2 - \beta z + \gamma.$$

Hanc ob rem erit

$$\alpha = A + B + C,\ \beta = AB + AC + BC \text{ et } \gamma = ABC.$$

Quo autem $\alpha, \beta,$ et γ determinentur, aequatio $x = \sqrt{A} + \sqrt{B} + \sqrt{C}$ ab irrationalitate liberetur hoc modo. Sumantur quadrata; erit

$$x^2 = A + B + C + 2\sqrt{AB} + 2\sqrt{AC} + 2\sqrt{BC}$$

hincque

$$x^2 - \alpha = 2\sqrt{AB} + 2\sqrt{AC} + 2\sqrt{BC}.$$

Sumendis denuo quadratis fit

$$x^4 - 2\alpha x^2 + \alpha^2 = 4AB + 4AC + 4BC + 8\sqrt{ABC}(\sqrt{A} + \sqrt{B} + \sqrt{C}) = 4\beta + 8x\sqrt{\gamma}$$

seu

$$x^4 = 2\alpha x^2 + 8x\sqrt{\gamma} + 4\beta - \alpha^2.$$

Haec aequatio comparata cum proposita $x^4 = ax^2 + bx + c,$ dabit

$$2\alpha = a,\ 8\sqrt{\gamma} = b, \text{ et } 4\beta - \alpha^2 = c,$$

ex quibus prodit

$$\alpha = \tfrac{a}{2},\ \gamma = \tfrac{b^2}{64}, \text{ et } \beta = \tfrac{c}{4} + \tfrac{\alpha^2}{16}.$$

Aequatio ergo cubica resolutioni aequationis biquadraticae inserviens est

$$z^3 = \tfrac{a}{2}z^2 - \tfrac{4c+a^2}{16}z + \tfrac{b^2}{64}.$$

Huius enim radices si sint A, *B* et *C,* erit

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C}$$

At reliquae tres radices ex aequatione proposita erunt

$$\sqrt{A} - \sqrt{B} - \sqrt{C},\ \sqrt{B} - \sqrt{A} - \sqrt{C}, \text{ et } \sqrt{C} - \sqrt{A} - \sqrt{B}.$$

6. Ponatur $z = \sqrt{t}$*;* erit

$$(t + \tfrac{4c+a^2}{16})\sqrt{t} = \tfrac{at}{2} + \tfrac{b^2}{64}$$

et sumendis quadratis habebitur

$$t^3 + \tfrac{4c+a^2}{8}t^2 + \tfrac{(4c+a^2)^2}{256}t = \tfrac{a^2t^2}{4} + \tfrac{ab^2t}{64} + \tfrac{b^4}{4096}$$

seu

$$t^3 = (\tfrac{a^2}{8} - \tfrac{c}{2})t^2 + (\tfrac{ab^2}{64} - \tfrac{cc}{16} - \tfrac{a^2c}{32} - \tfrac{a^4}{256})t + \tfrac{b^4}{4096}.$$

Haec aequatio ergo hanc habet proprietatem, ut eius radices sint quadrata radicum prioris aequationis *A, B* et *C.* Quare si huius aequationis radices ponantur *E, F, G,* erit

$$x = \sqrt[4]{E} + \sqrt[4]{F} + \sqrt[4]{G}.$$

Datur itaque aequatio cubica, cuius radicum radices biquadraticae simul sumtae constituant radicem aequationis biquadraticae propositae. Atque haec methodus inveniendi radices ex aequatione biquadratica, etiamsi sit priori operosior, maiorem habet affinitatem cum resolutione aequationum cubicarum, cum eiusdem potestatis radix extrahatur ex radicibus aequationis inferioris, cuius est ipsa aequatio proposita.

7. Simili ratione etiam aequatio quadratica

$$x^2 = a\,,$$

in qua secundus terminus deest, resolvetur ope aequationis unius dimensionis

$$z = a$$

Huius enim radix est *a* atque radix aequationis propositae

$$x = \sqrt{a} \text{ vel } x = -\sqrt{a}\,.$$

Huiusmodi autem aequationem ordine inferiorem, cuius ope aequatio superior secundo termino carens resolvitur, vocabo *aequationem resolventem.* Ita aequationis quadraticae

$$x^2 = a$$

aequatio resolvens erit

$$z = a\,;$$

aequationis cubicae

$$x^3 = ax + b$$

aequatio resolvens erit

$$z^2 = bz - \tfrac{a^3}{27}$$

atque aequationis biquadraticae

$$x^4 = ax^2 + bx + c$$

aequatio resolvens est

$$z^3 = (\tfrac{a^2}{8} - \tfrac{c}{2})z^2 - (\tfrac{a^4}{256} + \tfrac{a^2c}{32} + \tfrac{cc}{16} - \tfrac{ab^2}{64})z + \tfrac{b^4}{4096}.$$

Pro quadratica enim aequatione, si aequationis resolventis radix sit *A,* erit

$$x = \sqrt{A}$$

pro cubica vero aequatione, si resolventis radices sint *A* et *B,* erit

$$x = \sqrt[3]{A} + \sqrt[3]{B}$$

atque pro biquadratica aequatione existentibus resolventis aequationis radicibus *A, B* et *C* erit

$$x = \sqrt[3]{A} + \sqrt[3]{B} + \sqrt[3]{C} .$$

8. Ex his etiamsi tribus tantum casibus tamen non sine sufficienti ratione mihi concludere videor superiorum quoque aequationum dari huiusmodi aequationes resolventes. Sic proposita aequatione

$$x^5 = ax^3 + bx^2 + cx + d$$

coniicio dari aequationem ordinis quarti

$$z^4 = \alpha z^4 - \beta z^2 + \gamma z - \delta,$$

cuius radices si sint *A, B, C* et *D,* fore

$$x = \sqrt[5]{A} + \sqrt[5]{B} + \sqrt[5]{C} + \sqrt[5]{D}.$$

Et generatim aequationis

$$x^n = ax^{n-2} + bx^{n-3} + cx^{n-4} + \text{etc.}$$

aequatio resolvens, prout suspicor, erit huius formae

$$z^{n-1} = \alpha z^{n-2} - \beta z^{n-3} + \gamma z^{n-4} - \text{etc.},$$

cuius cognitis radicibus omnibus numero $n-1$, quae sint *A, B, C, D* etc., erit

$$x = \sqrt[n]{A} + \sqrt[n]{B} + \sqrt[n]{C} + \sqrt[n]{D} + \text{etc.}$$

Haec igitur coniectura si esset veritati consentanea atque si aequationes resolventes possent determinari, cuiusque aequationis in promtu foret radices assignare; perpetuo enim pervenitur ad aequationem ordine inferiorem hocque modo progrediendo tandem vera aequationis propositae radix innotescet.

9. Quamquam autem, si aequatio proposita plures quam quatuor habet dimensiones, aequationem resolventem definire adhuc non possum, tamen praesto sunt non nullius momenti argumenta, quibus ista mea coniectura confirmatur. Si enim aequatio proposita ita est comparata, ut in aequatione resolvente omnes termini praeter tres primos evanescant, tum semper ipsa aequatio resolvens poterit exhiberi atque ideo aequationis propositae radices assignari. Aequationes autem, quae hoc modo resolutionem admittunt, sunt eae ipsae, quas Cl. ABR. DE MOIVRE in Transact. n. 809 pertractavit. Sit enim aequatio resolvens

$$z^{n-1} = \alpha z^{n-2} - \beta z^{n-3}$$

seu

$$z^2 = \alpha z - \beta$$

ex hacque aequationem resolvendam erui oporteat. Sint huius aequationis radices A et B; reliquae enim radices omnes evanescunt; erit aequationis resolvendae radix

$$x = \sqrt[n]{A} + \sqrt[n]{B}.$$

Est vero

$$\alpha = A + B \text{ et } \beta = AB$$

ex natura aequationum. Hinc ergo erit

$$\sqrt[n]{A^2} + \sqrt[n]{B^2} = x^2 - 2\sqrt[n]{\beta}$$

atque porro

$$\sqrt[n]{A^3} + \sqrt[n]{B^3} = x^3 - 3x\sqrt[n]{\beta},$$
$$\sqrt[n]{A^4} + \sqrt[n]{B^4} = x^4 - 4x^2\sqrt[n]{\beta} + 2\sqrt[n]{\beta^2},$$
$$\sqrt[n]{A^5} + \sqrt[n]{B^5} = x^5 - 5x^3\sqrt[n]{\beta} + 5x\sqrt[n]{\beta^2},$$

atque tandem

$$\sqrt[n]{A^n} + \sqrt[n]{B^n} = x^n - nx^{n-2}\sqrt[n]{\beta} + \frac{n(n-3)}{1\cdot 2}x^{n-4}\sqrt[n]{\beta^2}$$
$$-\frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}x^{n-6}\sqrt[n]{\beta^3} + \frac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4}x^{n-8}\sqrt[n]{\beta^5} - \text{etc.} = \alpha.$$

Quae est aequatio resolvenda, cuius resolvens est

$$z^{n-1} = \alpha z^{n-2} - \beta z^{n-3} \text{ seu } z^2 = \alpha z - \beta.$$

10. Non solum autem hoc modo aequationis

$$x^n - nx^{n-2}\sqrt[n]{\beta} + \frac{n(n-3)}{1\cdot 2}x^{n-4}\sqrt[n]{\beta^2} - \text{etc.} = \alpha$$

unica radix invenitur

$$x = \sqrt[n]{A} + \sqrt[n]{B},$$

sed satisfacit etiam quaelibet alia

$$x = \mu\sqrt[n]{A} + \nu\sqrt[n]{B},$$

modo sit $\mu^n = \nu^n = \mu\nu = 1$, id quod n diversis modis fieri potest. Ut si sit $n = 5$, aequationis

$$x^5 - 5x^3\sqrt[n]{\beta} + 5x\sqrt[n]{\beta^2} = \alpha$$

radices quinque erunt, ut sequuntur:

$$\text{I. } x = \sqrt[5]{A} + \sqrt[5]{B},$$

$$\text{II. } x = \frac{-1-\sqrt{5}+\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1-\sqrt{5}-\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{B},$$

$$\text{III. } x = \frac{-1-\sqrt{5}-\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1-\sqrt{5}+\sqrt{(-10+2\sqrt{5})}}{4}\sqrt[5]{B},$$

$$\text{IV. } x = \frac{-1+\sqrt{5}+\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1+\sqrt{5}-\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{B},$$

$$\text{V. } x = \frac{-1+\sqrt{5}-\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{A} + \frac{-1+\sqrt{5}+\sqrt{(-10-2\sqrt{5})}}{4}\sqrt[5]{B}.$$

Hi enim coefficientes omnes sunt radices surdesolidae ex unitate et factum ex binis coniunetis est $= 1$.
Simili modo praeter ipsam unitatem sunt sex radices potestatis septimae ex unitate harumque tria paria multiplicatione unitatem producentia, quae sunt sex radices huius aequationis

$$y^6 + y^5 + y^4 + y^3 + y^2 + y + 1 = 0$$

Ad has autem inveniendas tantum opus est resolutione aequationis cubicae; omnis enim aequatio potestatis sextae huius formae

$$y^6 + ay^5 + by^4 + cy^3 + by^2 + ay + 1 = 0,$$

quae non mutatur posito $\frac{1}{y}$ loco y, resolvi potest ope aequationis cubicae.
Quod quemadmodum fiat, cum ad inveniendas radices saepe utilitatem habere possit, brevi sum ostensurus.

11. Aequationes huiusmodi, quae posito $\frac{1}{y}$ loco y formam non mutant, voco *reciprocas*. Hae, si maximus ipsius y dimensionum numerus est impar, semper dividi possunt per $y+1$ et aequatio resultans etiam erit reciproca, in qua maxima ipsius y dimensio erit par. Quamobrem sufficiet parium tantum dimensionum aequationes reciprocas considerasse atque modum earum resolvendarum exposuisse.

Sit igitur primo aequatio proposita quartae potestatis haec

$$y^4 + ay^3 + by^2 + ay + 1 = 0;$$

ponatur haec factum ex duabus quadraticis

$$y^2 + \alpha y + 1 = 0$$

et

$$y^2 + \beta y + 1 = 0.$$

Quo facto fiet

$$\alpha + \beta = a \text{ et } \alpha\beta + 2 = b \text{ seu } \alpha\beta = b - 2.$$

Quare α et β erunt duae radices huius aequationis

$$u^2 - au + b - 2 = 0$$

hacque ratione quatuor aequationis propositae radices ope aequationum tantum quadraticarum innotescent.

Aequatio reciproca sextae potestatis

$$y^6 + ay^5 + by^4 + cy^3 + by^2 + ay + 1 = 0$$

ponatur factum trium harum quadraticarum

$$y^2 + \alpha y + 1 = 0,$$
$$y^2 + \beta y + 1 = 0,$$

et

$$y^2 + \gamma y + 1 = 0.$$

Hinc fiet

$$\alpha + \beta + \gamma = a,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = b - 3$$

et

$$\alpha\beta\gamma = c - 2\alpha - 2\beta - 2\gamma = c - 2a.$$

Quare α, β et γ erunt tres radices huius aequationis cubicae

$$u^3 - au^2 + (b-3)u - c + 2a = 0.$$

Similiter aequatio reciproca octavae potestatis

$$y^8 + ay^7 + by^6 + cy^5 + dy^4 + cy^3 + by^2 + ay + 1 = 0$$

est factum ex quatuor aequationibus quadraticis

$$y^2 + \alpha y + 1 = 0,$$
$$y^2 + \beta y + 1 = 0,$$
$$y^2 + \gamma y + 1 = 0,$$

et

$$y^2 + \gamma y + 1 = 0.$$

et

$$y^2 + \delta y + 1 = 0,$$

ex quo prodibit

$$\alpha + \beta + \gamma + \delta = a,$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = b - 4,$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = c - 3a$$

et

$$\alpha\beta\gamma\delta = d - 2b + 2.$$

Coefficientes ergo $\alpha, \beta, \gamma, \delta$ sunt quatuor radices huius aequationis

$$u^4 - au^3 + (b-4)u^2 - (c-3a)u + d - 2b + 2 = 0 .$$

Aequatio ordinis decimi

$$y^{10} + ay^9 + by^8 + cy^7 + dy^6 + ey^5 + dy^4 + cy^3 + by^2 + ay + 1 = 0$$

factum erit harum quinque

$$y^2 + \alpha y + 1 = 0,$$
$$y^2 + \beta y + 1 = 0,$$
$$y^2 + \gamma y + 1 = 0,$$
$$y^2 + \delta y + 1 = 0,$$
$$y^2 + \varepsilon y + 1 = 0,$$

in quibus $\alpha, \beta, \gamma, \delta, \varepsilon$ sunt quinque radices huius aequationis

$$u^5 - au^4 + (b-5)u^3 - (c-4a)u^2 + (d-3b+5)u - e + 2c - 2a = 0.$$

Atque generatim aequatio reciproca

$$\begin{aligned} y^{2n} + ay^{2n-1} + by^{2n-2} + cy^{2n-3} + dy^{2n-4} + ey^{2n-5} + fy^{2n-6} + \cdots py^n + \cdots \\ + fy^6 + ey^5 + dy^4 + cy^3 + by^2 + ay + 1 = 0 \end{aligned}$$

resolvetur in has numero n aequationes quadraticas

$$\begin{gathered} y^2 + \alpha y + 1 = 0, \\ y^2 + \beta y + 1 = 0, \\ y^2 + \gamma y + 1 = 0, \\ y^2 + \delta y + 1 = 0, \\ \text{etc.} \end{gathered}$$

At coefficientes $\alpha, \beta, \gamma, \delta$ etc. erunt radices huius aequationis n dimensionum

$$\begin{array}{llllll} u^n - au^{n-1} + bu^{n-2} & -cu^{n-3} & +du^{n-4} & -eu^{n-5} \\ \qquad\quad -n & +(n-1)a & -(n-2)b & +(n-3)c \\ & & +\frac{n(n-3)}{1\cdot 2} & -\frac{(n-1)(n-4)}{1\cdot 2}a \end{array}$$

$$\begin{array}{lll} +fu^{n-6} & -gu^{n-7} & +hu^{n-8} - \text{etc.} = 0 \\ -(n-4)d & +(n-5)e & -(n-6)f \\ +\frac{(n-2)(n-5)}{1\cdot 2}b & -\frac{(n-3)(n-6)}{1\cdot 2}c & +\frac{(n-4)(n-7)}{1\cdot 2}d \\ -\frac{n(n-4)(n-5)}{1\cdot 2\cdot 3} & +\frac{(n-1)(n-5)(n-6)}{1\cdot 2\cdot 3}a & -\frac{(n-2)(n-6)(n-7)}{1\cdot 2\cdot 3}b \\ & & +\frac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4} \end{array}$$

12. Quia cuiuslibet aequationis quadraticae dividentis aequationem propositam terminus extremus est unitas, perspicuum est binarum radicum aequationis propositae factum esse unitatem. Huiusmodi igitur duae semper cum duobus membris $\sqrt[n]{A}$ et $\sqrt[n]{B}$ sunt coniungendae, quo aequationis § 9 propositae omnes obtineantur radices.

13. Si in aequatione reciproca omnes termini praeter extremos et medium deficiant ut in

$$y^{2n} + py^{n} + 1 = 0,$$

divisores eius

$$y^2 + \alpha y + 1 = 0,$$
$$y^2 + \beta y + 1 = 0,$$
$$y^2 + \gamma y + 1 = 0,$$
etc.

habebuntur substituendis pro $\alpha, \beta, \gamma, \delta$ etc. radicibus huius aequationis

$$u^n - nu^{n-2} + \tfrac{n(n-3)}{1\cdot 2}u^{n-4} - \tfrac{n(n-4)(n-5)}{1\cdot 2\cdot 3}u^{n-6} + \cdots \pm p = 0,$$

ubi $+p$ accipi debet, si n est numerus par, et $-p$, si n est impar. Ex quo apparet hanc aequationem convenire cum aequatione

$$x^n - nx^{n-2}\sqrt[n]{\beta} + \cdots = \alpha$$

§ 9 resoluta et hanc ob rem omnes divisores posse assignari.

14. Magnam isthaec in factores resolutio formulae $y^{2n} + py^{n} + 1 = 0$ habet utilitatem in integranda formula differentiali

$$\frac{dy}{y^{2n} + py^{n} + 1}$$

iam saepius a Gcometris pertractata. Denominatore enim in suos factores $y^2 + \alpha y + 1 = 0$, $y^2 + \beta y + 1 = 0$, etc. resoluto tota integratio ad quadraturam circuli vel hyperbolae reducitur. Praeterea hoc plurimum iuvat, quod aequatio

$$u^n - nu^{n-2} + \tfrac{n(n-3)}{1\cdot 2}u^{n-4} - \cdots \pm p = 0,$$

ex qua α, β, γ etc. determinantur, sectionem arcus circularis in n partes complectatur atque ita coefficientes α, β, γ etc. facillime inveniantur.

15. Revertamur autem ad modum ex aequationibus resolventibus ipsas aequationes resolvendas eliciendi. Sitque aequatio resolvens

$$z^3 = \alpha z^2 - \beta z + \gamma,$$

cuius tres radices sint *A, B, C;* erit ergo

$$\alpha = A + B + C,\ \beta = AB + AC + BC \text{ et } \gamma = ABC.$$

Radix itaque aequationis resolvendae x erit

$$= \sqrt[n]{A} + \sqrt[n]{B} + \sqrt[n]{C}$$

atque ponatur

$$p = \sqrt[n]{AB} + \sqrt[n]{AC} + \sqrt[n]{BC}.$$

His factis erit

$$\sqrt[n]{A^2} + \sqrt[n]{B^2} + \sqrt[n]{C^2} = x^2 - 2p$$

et

$$\sqrt[n]{A^2B^2} + \sqrt[n]{B^2C^2} + \sqrt[n]{B^2C^2} = p^2 - 2x\sqrt[n]{\gamma};$$

atque porro, ut sequitur:

$$\sqrt[n]{A^3} + \sqrt[n]{B^3} + \sqrt[n]{C^3} = x^3 - 3px + 3\sqrt[n]{\gamma},$$
$$\sqrt[n]{A^3B^3} + \sqrt[n]{B^3C^3} + \sqrt[n]{B^3C^3} = p^3 - 3px\sqrt[n]{\gamma} + 3\sqrt[n]{\gamma^2};$$
$$\sqrt[n]{A^4} + \sqrt[n]{B^4} + \sqrt[n]{C^4} = x^4 - 4px^2 + 4x\sqrt[n]{\gamma} + 2p^2;$$
$$\sqrt[n]{A^4B^4} + \sqrt[n]{B^4C^4} + \sqrt[n]{B^4C^4} = p^4 - 4p^2x\sqrt[n]{\gamma} + 4p\sqrt[n]{\gamma^2} + 2x^2\sqrt[n]{\gamma^2};$$
$$\sqrt[n]{A^5} + \sqrt[n]{B^5} + \sqrt[n]{C^5} = x^5 - 5px^3 + 5x^2\sqrt[n]{\gamma} + 5p^2x - 5p\sqrt[n]{\gamma},$$
$$\sqrt[n]{A^5B^5} + \sqrt[n]{B^5C^5} + \sqrt[n]{B^5C^5} = p^5 - 5p^3x\sqrt[n]{\gamma} + 5p^2\sqrt[n]{\gamma^2} + 5px^2\sqrt[n]{\gamma^2} - 5x\sqrt[n]{\gamma^3}.$$

Quemadmodum haec tabula sit ulterius continuanda, facile perspicitur. Namque est

$$\sqrt[n]{A^m} + \sqrt[n]{B^m} + \sqrt[n]{C^m} = x(\sqrt[n]{A^{m-1}} + \sqrt[n]{B^{m-1}} + \sqrt[n]{C^{m-1}})$$
$$-p(\sqrt[n]{A^{m-2}} + \sqrt[n]{B^{m-2}} + \sqrt[n]{C^{m-2}}) + \sqrt[n]{\gamma}(\sqrt[n]{A^{m-3}} + \sqrt[n]{B^{m-3}} + \sqrt[n]{C^{m-3}})$$

atque

$$\sqrt[n]{A^mB^m} + \sqrt[n]{A^mC^m} + \sqrt[n]{B^mC^m} = p(\sqrt[n]{A^{m-1}B^{m-1}} + \sqrt[n]{A^{m-1}C^{m-1}} + \sqrt[n]{B^{m-1}C^{m-1}} +)$$
$$-x\sqrt[n]{\gamma}(\sqrt[n]{A^{m-2}B^{m-2}} + \sqrt[n]{A^{m-2}C^{m-2}} + \sqrt[n]{B^{m-2}C^{m-2}})$$
$$+\sqrt[n]{\gamma^2}(\sqrt[n]{A^{m-3}B^{m-3}} + \sqrt[n]{C^{m-3}A^{m-3}} + \sqrt[n]{B^{m-3}}C^{m-3}).$$

16. Alias etiam harum progressionum non contemnendas observavi proprietates. Posito enim

$$\sqrt[n]{A^m}+\sqrt[n]{B^m}+\sqrt[n]{C^m}=R$$

et

$$\sqrt[n]{A^mB^m}+\sqrt[n]{A^mC^m}+\sqrt[n]{B^mC^m}=S$$

erit

$$\sqrt[n]{A^{2m}}+\sqrt[n]{B^{2m}}+\sqrt[n]{C^{2m}}=R^2-2S$$

et

$$\sqrt[n]{A^{2m}B^{2m}}+\sqrt[n]{A^{2m}C^{2m}}+\sqrt[n]{B^{2m}C^{2m}}=S^2-2R\sqrt[n]{\gamma^m}.$$

Simili modo est quoque

$$\sqrt[n]{A^{3m}}+\sqrt[n]{B^{3m}}+\sqrt[n]{C^{3m}}=R^3-3RS+3\sqrt[n]{\gamma^m}$$

et

$$\sqrt[n]{A^{3m}B^{3m}}+\sqrt[n]{A^{3m}C^{3m}}+\sqrt[n]{B^{3m}C^{3m}}=S^3-3RS\sqrt[n]{\gamma^m}+3\sqrt[n]{\gamma^{2m}}.$$

Atque hoc modo haec series procedit prorsus ut ipsa praecedens.

17. Si sit $n=2$, erit

$$\alpha=x^2-2p \text{ et } \beta=p^2-2x\sqrt{\gamma}$$

hisque duabus aequationibus coniunctis habebitur

$$x=\sqrt{A}+\sqrt{B}+\sqrt{C}$$

et

$$p=\sqrt{AB}+\sqrt{AC}+\sqrt{BC};$$

sunt autem *A, B* et *C* tres radices huius aequationis cubicae

$$z^3=\alpha z^2-\beta z+\gamma.$$

Eliminata ergo ex illis duabus aequationibus littera *p* prodibit

$$(\tfrac{x^2-\alpha}{2})^2-2x\sqrt{\gamma}=\beta$$

seu

$$x^4-2\alpha x^2-8x\sqrt{\gamma}=4\beta-\alpha^2,$$

cuius aequationis itaque radix *x* est cognita, quippe $=\sqrt{A}+\sqrt{B}+\sqrt{C}$ *;* quae aequatio illi est consentanea, quae § 5 est resoluta.

Simili modo si quando duae huiusmodi aequationes occurrent

$$x^3 - 3px + 3\sqrt[3]{\gamma} = \alpha$$

et

$$p^3 - 3px\sqrt[3]{\gamma} + 3\sqrt[3]{\gamma^2} = \beta,$$

erit

$$x = \sqrt[3]{A} + \sqrt[3]{B} + \sqrt[3]{C}$$

et

$$p = \sqrt[3]{AB} + \sqrt[3]{AC} + \sqrt[3]{BC}$$

existentibus *A, B* et *C* radicibus aequationis

$$z^3 = \alpha z^2 - \beta z + \gamma$$

ut ante. Vel eliminata littera p prodibit aequatio inter $x,\ \alpha,\ \beta,\ \gamma,$ cuius radix x innotescet.

Eodem prorsus modo occurrentibus duabus hisce aequationibus

$$x^4 - 4px^2 + 4\sqrt[4]{\gamma} + 2p^2 = \alpha$$

et

$$p^4 - 4p^2x\sqrt[4]{\gamma} + 4p\sqrt{\gamma} + 4x^2\sqrt{\gamma} = \beta$$

erit

$$x = \sqrt[4]{A} + \sqrt[4]{B} + \sqrt[4]{C}$$

et

$$p = \sqrt[4]{AB} + \sqrt[4]{AC} + \sqrt[4]{BC}$$

et iterum sunt *A, B* et *C* radices huius aequationis

$$z^3 = \alpha z^2 - \beta z + \gamma\ .$$

Quo p facilius eliminetur, ponatur

$$x^2 - 2p = R \ \text{ et } \ p^2 - 2x\sqrt[4]{\gamma} = S$$

eritque

$$R^2 - 2S = \alpha \text{ et } S^2 - 2R\sqrt{\gamma} = \beta.$$

Iam ex illis duabus aequationibus exterminata *p* habebitur

$$x^4 = 2Rx^2 + 8x\sqrt[4]{\gamma} + 4S - R^2.$$

Comparetur haec aequatio cum ista

$$x^4 = ax^2 + bx + c\,;$$

erit

$$R = \tfrac{a}{2},\ \sqrt[4]{\gamma} = \tfrac{b}{8} \text{ seu } \gamma = \tfrac{b^4}{4096} \text{ et } S = \tfrac{c}{4} + \tfrac{a^2}{16}.$$

Hinc igitur habebitur

$$\alpha = \tfrac{a^2}{8} - \tfrac{c}{2} \text{ et } \beta = \tfrac{c^2}{16} + \tfrac{a^2c}{32} + \tfrac{a^4}{256} - \tfrac{ab^3}{64}.$$

Quamobrem erunt *A, B* et *C* tres radices huius aequationis

$$z^3 = (\tfrac{a^2}{8} - \tfrac{c}{2})z^2 - (\tfrac{a^4}{256} + \tfrac{a^2c}{32} + \tfrac{c^2}{16} - \tfrac{ab^3}{64})z + \tfrac{b^4}{4096},$$

id quod mire consentit cum eo, quod § 7 est inventum.

18. Quoties igitur accidit, ut calculus perducatur ad duas aequationes duas incognitas *x* et *p* continentes, quae reperiantur inter formulas § 15, utriusque valor poterit assignari, etiamsi eliminata altera aequatio prodeat maxime composita. Hanc ob rem in his casibus expediet calculum non ad unicam aequationem unicamque incognitam deducere, sed duas aequationes duas incognitas involventes retinere atque investigare, num forte inter illas formulas contineantur, id quod saepius, si calculus recte instituatur, evenire posse mihi persuasum est.

19. Quemadmodum autem aequationes resolventes

$$z^2 = \alpha z - \beta \text{ et } z^3 = \alpha z^2 - \beta z + \gamma$$

tractavimus, ita etiam ulterius est progrediendum ad aequationem

$$z^4 = \alpha z^3 - \beta z^2 + \gamma z - \delta$$

simili modo pertractandam. Scilicet si eius radices sint *A, B, C* et *D,* ponatur

$$\sqrt[n]{A} + \sqrt[n]{B} + \sqrt[n]{C} + \sqrt[n]{D} = x$$

et

$$\sqrt[n]{AB}+\sqrt[n]{AC}+\sqrt[n]{AD}+\sqrt[n]{BC}+\sqrt[n]{BD}+\sqrt[n]{CD}=p$$

atque

$$\sqrt[n]{ABC}+\sqrt[n]{ABD}+\sqrt[n]{ACD}+\sqrt[n]{BCD}=q$$

et quaerantur hinc expressiones pro

$$\sqrt[n]{A^m}+\sqrt[n]{B^m} \text{ et } \sqrt[n]{A^m B^m}+\sqrt[n]{A^m C^m}+\text{etc. atque pro } \sqrt[n]{A^m B^m C^m}+\text{etc.}$$

His perficiendis semper trinae invenientur aequationes *x, p* et *q* continentes pro quovis ipsius *m* valore. Atque simili modo occurrentibus tribus huiusmodi aequationibus tres incognitae constabunt.

20. Suspicor autem posito

$$x=\sqrt[5]{A}+\sqrt[5]{B}+\sqrt[5]{C}+\sqrt[5]{D}$$

aequationem rationalem posse concinnari, in qua *x* plures quam 5 non habeat dimensiones, etiamsi hoc fere impossibile videatur. Nam quemadmodum §17 ex aequationibus

$$x^4-4px^2+4x\sqrt[4]{\gamma}+2p^2=\alpha$$

et

$$p^4-4p^2x\sqrt[4]{\gamma}+4p\sqrt{\gamma}+2x^2\sqrt{\gamma}=\beta$$

eliminanda *p* aequationem non plurium quam 4 dimensionum obtinuimus, quod pariter vix fieri posse videatur, ita etiam pro quinta potestate forte simile artificium usu venire potest, ut aequatio

$$x^5=ax^3+bx^2+cx+d$$

tandem resolvi queat. Quod vero maximum in hoc perficiendo est subsidium, eo redit meo iudicio, ut α, β, γ et δ ex *a, b, c* et *d* debeant determinari, non vero vicissim; hoc enim casu aequatio ad multo altiorem eveheretur potestatem, quam opus est. Aliis autem, quos huiusmodi occupationes iuvant, hanc rem perficiendam vel mihi ad aliud tempus relinquo hoc solo nunc contentus me fortasse idoneam atque genuinam viam ostendisse.