E 282 : CONCERNING THE RESOLUTION OF EQUATIONS OF ANY ORDER,

1. Which hitherto have been treated by the algebraic resolution of equations, these, if we may consider them according to general rules, apply only to equations which do not exceed the fourth order; and it is apparent even now no rules have been found, with the aid of which equations of the fifth or higher orders may be able to be resolved, thus so that the whole of Algebra may be restricted to equations of the first four orders. Moreover this restriction is required to be maintained by the general rules, which shall be applied to all equations of the same order: for in any order infinitely many equations are given, which can be resolved by division into two or more equations of a lesser order, whose roots therefore taken together present all the roots of these equations of higher order. Moreover, truly , certain special equations have been observed in any order by the celebrated de Moivre, which are unable to be resolved even by division in factors, yet it may be possible to assign the roots of these.

2. But it is agreed from the known resolution of the general equations of the first, second, third, and fourth orders, indeed equations of the first order can be resolved without any extraction from the roots ; but the resolution of equations of the second order now demands the extraction of the square root. Again, the resolution of equations of the third order implies the extraction both of square roots as well as cube roots, and the resolution of the fourth order demands in addition the extraction of roots of the fourth order. But with care it permitted to conclude the resolution of the equation of the fifth order to demand the general extraction of the root of this fifth order as well as all the inferior roots [The terminology *surdesolidum* used by Euler for the 5th order root, comes from Recorde's *The Whetstone of Witte*]; and the root of any equation of order n in general may be expressed by the form, which will be composed from all the root signs both of order n as will as of lesser orders.

3. I had ventured to advance this conjecture at one time in the *Comm. Acad. Imper. Petrop.* Book VI about the forms of the roots of any equation.
Indeed for the proposed equation of any order

$$x^n + \mathrm{A}x^{n-2} + \mathrm{B}x^{n-3} + \mathrm{C}^{n-4} + \text{etc.} = 0,$$

in which I have assumed the second term to be absent, which indeed is always permitted to be put in place, I have always presumed to give an equation less by one degree, such as

$$y^{n-1} + \mathfrak{A}y^{n-2} + \mathfrak{B}y^{n-3} + \mathfrak{C}^{n-4}y + \text{etc.} = 0,$$

that I have called its resolvent, thus so that, if all the roots of this may be present, which shall be

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \text{etc.},$$

the number of which is $n-1$, from these a root of this other equation may be expressed thus, so that it shall be

$$x=\sqrt[n]{\alpha}+\sqrt[n]{\gamma}+\sqrt[n]{\gamma}+\sqrt[n]{\delta}+\sqrt[n]{\varepsilon}+\text{etc.}$$

Which conjecture I have confirmed showing the resolution of the lower equation actually deduced from this general form ; neither do I doubt even now, why this conjecture may not be in agreement with the truth.

4. But whereas beyond the discovery of the resolving equation, if the proposed equation transcends the fourth order, it shall become most difficult in nature and thus may be seen equally to overcome our strengths, and thus the resolution of the proposed equation itself, so that except for special forms in the de Moivre cases similar to ours, nothing very much will be supplied to us; I have observed other inconveniences in that above form, which have guided me thus, so that I might perhaps choose another form to be given not so very dissimilar to that, but which may not be subjected to the same inconveniences and thus may enable us with greater hope finally to have penetrated further in this arduous algebraic work. Moreover it will be more than a little useful in this matter to have perceived the true form of the roots of each equation more accurately.

5. But in the form elicited by the above conjecture, in the first place I desire this, because not all the roots of the proposed equation may be expressed distinctly enough. For indeed some root sign $\sqrt[n]{\alpha}$ includes as many different values, as the number n contains unities, thus so that, if

$$\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},\ \mathfrak{e},\ \text{etc.}$$

will denote all the values of the formula $\sqrt[n]{1}$, however for $\sqrt[n]{\alpha}$ it may be permitted to write for these formulas

$$\mathfrak{a}\sqrt[n]{\alpha},\ \mathfrak{b}\sqrt[n]{\alpha},\ \mathfrak{c}\sqrt[n]{\alpha},\ \mathfrak{d}\sqrt[n]{\alpha},\ \text{etc.},$$

yet it is evident this variation in the individual terms $\sqrt[n]{\alpha},\ \sqrt[n]{\beta},\ \sqrt[n]{\gamma},\ \sqrt[n]{\delta},$ etc., cannot be put in place as you please. If indeed the combination of these terms with the letters $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},\ \mathfrak{e},$ etc. may be left to our choice, then many more combinations will result, than the equation contains roots, the number of which $=n$.

6. Therefore where likewise the form of the root x shown above may include all the roots of the equation, it is necessary that the combinations of the terms $\sqrt[n]{\alpha},\ \sqrt[n]{\beta},\ \sqrt[n]{\gamma},\ \sqrt[n]{\delta},$ etc., with the letters $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},$ etc. may be written concisely in a certain manner, and the combinations may be excluded which are improper for representing the roots of the equation. Indeed from the resolution of equations of third and fourth orders we see between the roots of unity of the same name $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},$ etc. a

certain reliable order must be put in place, following which also the combinations shall be required to be completed. But finally this similar order will be required to be held by the members of the roots themselves $\sqrt[n]{\alpha}, \sqrt[n]{\beta}, \sqrt[n]{\gamma}, \sqrt[n]{\delta}$, etc., by which the combination may be set in order. Truly since it may not be agreed, how for roots of superior orders such an order may be put in place, it is inconvenient without this conspicuous doubt, by which form of my conjecture it may be depended on to work, which therefore is proposed by me to be removed in this dissertation.

7. But first a certain ordering of any powers in the roots constructed from any power of unity shall be agreed, where generally the sum may be restricted to a variety of combinations. Which I observe to this the end, if besides unity some other value of $\sqrt[n]{1}$ shall be $= \mathfrak{a}$, then also $\mathfrak{a}^2, \mathfrak{a}^3, \mathfrak{a}^4$ etc. show the values of $\sqrt[n]{1}$ to exhibit; for if there shall be $\mathfrak{a}^n = 1$, there will be also $(\mathfrak{a}^2)^n = 1, (\mathfrak{a}^3)^n = 1, (\mathfrak{a}^4)^n = 1$ etc. Hence if the remaining roots may be put to be $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, etc., since from these $\mathfrak{a}^2, \mathfrak{a}^3, \mathfrak{a}^4$ etc. will be found, now indeed a sure order is seen, by which these letters must be placed among themselves.

Thus if after unity, which is agreed to hold the first place always, we may begin with the letter $\mathfrak{a}$, the values of the formula $\sqrt[n]{1}$ will be

$$1, \mathfrak{a}, \mathfrak{a}^2, \mathfrak{a}^3, \mathfrak{a}^4 \ldots\ldots \mathfrak{a}^{n-1},$$

the number of which is *n;* for more cannot occur, since there shall be $\mathfrak{a}^n = 1, \mathfrak{a}^{n+1} = \mathfrak{a}, \mathfrak{a}^{n+2} = \mathfrak{a}^2$, etc.; and the matter will be resolved in a like manner, if after unity we may begin with some other letter $\mathfrak{b}, \mathfrak{c}$, or $\mathfrak{d}$ etc.

8. Hence therefore I suspect deservedly such a root order thus also to be prepared by the terms themselves of the equation present expressing *x*, or the individual root members to be prepared thus, so that with respect of any one root the rest shall be its powers ; but now it will be necessary to attribute indefinite coefficients to the members. Whereby the equation lacking the second term would be

$$x^n + \mathrm{A}x^{n-2} + \mathrm{B}x^{n-3} + \mathrm{C}^{n-4} + \mathrm{D}^{n-5} + \text{etc.} = 0,$$

any root of this equation is seen with the greatest probability to be expressed thus, so that there shall be

$$x = \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{D}\sqrt[n]{v^3} + \ldots\ldots \mathfrak{O}\sqrt[n]{v^{n-1}},$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. shall be either rational or at any rate not involving the root sign $\sqrt[n]{}$, certainly which may affect only the quantity *v* and its powers ; the quantity *v* itself may be involved much less by such a sign.

9. In the first place it appears from this form that no more members are able to be contained, as the number of which shall be $n-1$, ; for even if we may continue that series further according to its own nature, the following terms contained in the preceding now may be taken away. Indeed there will become

$$\sqrt[n]{v^{n+1}} = v\sqrt[n]{v}, \quad \sqrt[n]{v^{n+2}} = v\sqrt[n]{v^2} \quad \text{etc.},$$

thus so that the irrational root sign $\sqrt[n]{\ }$ involving more different kinds may not be allowed, as the number of which is $= n-1$. Therefore even if that series may be continued indefinitely, yet all the terms of the same kind of irrationality in the ratio requiring to be added shall be reduced to the number $n-1$. Therefore since now previously we have seen several terms not present in the expression of the roots, hence this non- trivial argument is found plainly agreeing with the truth; but this truth will be confirmed much more by the following argument.

10. This expression also extends at once to equations, in which the second term is present, while the previous removal of the second term was required, from which this new more natural expression is required to be estimated. For with the continuation of the irrational terms $\sqrt[n]{v}$, $\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$ etc. also will involve the rational terms $\sqrt[n]{v^0}$, $\sqrt[n]{v^n}$, which must be added on account of the second term. Hence more generally we will be able to announce, if the complete equation of the this order *n* were proposed

$$x^n + \Delta x^{n-1} + Ax^{n-2} + Bx^{n-3} + Cx^{n-4} + \text{etc.} = 0,$$

its root expressed by this form

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \mathfrak{D}\sqrt[n]{v^4} + \ldots\ldots \mathfrak{O}\sqrt[n]{v^{n-1}},$$

where ω shows the rational part of the root, as agreed to be $= -\frac{1}{n}\Delta$. But the remaining terms contain irrational root parts involving the powers of *n*, of which, although they are different, the number cannot exceed $n-1$, it is understood with everything as by the previous form.

11. Hence again we see, if *v* were a quantity of this kind, so that from that the root of the power *n* actually to be extracted, or $\sqrt[n]{v}$,may be expressed either rationally or by the root sign of a lesser power, then the irrational order *n* truly departs from the form of the root. But necessarily it must come about by this usage, as often as the proposed equation has been resolved into factors ; for then it will contain no radical root sign $\sqrt[n]{\ }$. Whereby since the nature of the equation may be required, so that in these cases all the root signs $\sqrt[n]{\ }$ vanish and it may be reduced to more simple signs, but it may not be apparent from

the more superior form, how with one sign of this kind vanishing $\sqrt[n]{\alpha}$, the rest $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$ etc. vanish, this expression on account of this reckoning is agreed to be much more applicable to the nature of the equation.

12. Truly besides, this form, which is the main point of the matter, also shows all the roots of the equation without any ambiguity; nor indeed need we hesitate further, just as with all the root signs $\sqrt[n]{\ }$ just as many values of the root $\sqrt[n]{1}$ shall be required to be present. Indeed if all the roots of the powers n from unity shall be $1, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, etc. and we will have combined $\sqrt[n]{v}$ with some one of these $\mathfrak{a}$, therefore so that $\sqrt[n]{v}$ certainly is $\mathfrak{a}\sqrt[n]{v}$ *,* then for $\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$, $\sqrt[n]{v^4}$ etc. it will be required to write $\mathfrak{a}^2\sqrt[n]{v^2}$, $\mathfrak{a}^3\sqrt[n]{v^3}$, $\mathfrak{a}^4\sqrt[n]{v^4}$ etc. But the constant term ω*,* because it represents the form $\omega\sqrt[n]{v^0}$, will be changed into $\mathfrak{a}^0\omega\sqrt[n]{v^0} = 1\cdot\omega$ on account of $\mathfrak{a}^0 = 1$, and thus just as with the remaining members it enters unchanged into all the roots. Because since with the resolution of all the equations it shall itself be evident, hence we have a new and clear criterion of the truth of this new form, which is seen to include the roots of all equations.

13. Hence again moreover it is evident, how one known root of each equation may be able to show all the remaining roots; according to this it is required to know all the roots of the same power from unity, or all the values of $\sqrt[n]{1}$, the number of which $= n$. And if these roots of unity were $1, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, etc. and one root found of the equation shall be

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \ldots\ldots\mathfrak{O}\sqrt[n]{v^{n-1}},$$

the remaining roots will be

$$x = \omega + \mathfrak{A}\mathfrak{a}\sqrt[n]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[n]{v^3} + \ldots\ldots\mathfrak{O}\mathfrak{a}^{n-1}\sqrt[n]{v^{n-1}},$$
$$x = \omega + \mathfrak{A}\mathfrak{b}\sqrt[n]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[n]{v^3} + \ldots\ldots\mathfrak{O}\mathfrak{b}^{n-1}\sqrt[n]{v^{n-1}},$$
$$x = \omega + \mathfrak{A}\mathfrak{c}\sqrt[n]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[n]{v^3} + \ldots\ldots\mathfrak{O}\mathfrak{c}^{n-1}\sqrt[n]{v^{n-1}},$$
etc.

and thus just as many roots always prevail, as the exponent n contains units*,* which designates the order of the equation.

14. By these arguments this new form of the roots most probably has been carried forwards; and most certainly nothing else is required to be shown, except that the rule may be found, with the aid of which for some equation proposed this rule to be defined and the coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. may be able to be assigned with some quantity v ; which if we may be able to put in place, we will have without doubt the general

resolution of all equations with the labour required of all the geometers hitherto in vain. Therefore indeed neither do I attribute so much to myself, that I believe that I can find this rule, but I will be fully content to be showing the roots of all equations certainly to be contained in this form. But this without doubt will bring most light to the resolution of equations, since the known true form of the roots may be provided more easily by means of an outstanding investigation, which would be unable to advance, as long as the form of the roots were unknown.

15. But though from that equation proposed by us at this point may not allow its root or coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. to be assigned with the quantity v, yet the demonstration of the truth will succeed equally, if in turn we may elicit from the assumed root that equation, of which it is the root. But this equation must be free from the root signs $\sqrt[n]{\ }$, since the equations, the roots of which are being investigated, are accustomed to be assumed to agree with the rational terms. Therefore here the question is returned, so that an equation of this kind

$$x=\omega+\mathfrak{A}\sqrt[n]{v}+\mathfrak{B}\sqrt[n]{v^2}+\mathfrak{C}\sqrt[n]{v^3}+\ldots\ldots\mathfrak{O}\sqrt[n]{v^{n-1}},$$

may be freed from irrationality or from the root signs $\sqrt[n]{\ }$ and a rational equation thence may be deduced, from which henceforth we will be able to affirm certainly its root to be that expression assumed itself ; and likewise thence the remaining roots, which agree equally with the same equation, we will be prevailing to assign. Therefore in this manner at least we will be able to show indefinitely many equations, the roots of which will be known to us, and if these equations embrace general equations of all degrees, also the resolution of these will be in our power.

16. Indeed little from us will be going to be seen to be outstanding, if only we may be able to show more equations, the roots of which may be able to be assigned, since from first principles it will be agreed, how an equation of any order must be formed, which may have given roots. For if some number of formulas of this kind $x-a,\ x-b,\ x-c$ etc. may in turn be multiplied together, certainly the equation will be obtained , the roots of which are going to be $x=a,\ x=b,\ x=c$ etc.; but the form from such equations provides little gain for the resolution of equations. But I observe first other equations do not arise in this manner, unless which shall be going to have factors ; but of equations , which can be resolved into factors, the resolution labours under no difficulty. Not at all of greater concern in this matter are equations also, which are produced by the multiplication of two or more lesser equations, the resolution of which clearly in no way is useful for the general resolution requiring to be completed.

17. But if moreover from our form

$$x=\omega+\mathfrak{A}\sqrt[n]{v}+\mathfrak{B}\sqrt[n]{v^2}+\text{etc.}$$

we may come upon a rational equation, that certainly will not have rational factors ; for if it had, its roots, which likewise agree to be roots of an equation of inferior order, not involving the root $\sqrt[n]{\ }$. For the most part it is in outstanding agreement, which for an equation of some higher grade, which may not be resolved into factors, it will assign the roots. On account of which the celebrated de Moivre is owed a huge debt of gratitude, because from the individual orders of equations he showed one in irresolvable factors, of which the roots can be assigned ; and if the formulas of this appear wider, without doubt they are going to have a much greater use, while on the other hand with equations in resolvable factors clearly nothing of benefit can be attributed in this matter.

18. Truly we may return to that form requiring to be freed from the irrationality of the sign $\sqrt[n]{\ }$, and if we may consult a customary method of eliminating the root signs, the resulting equation may be seen generally to rise to more dimensions. Indeed if a single radical sign may be present, for example

$$x=\omega+\mathfrak{A}\sqrt[n]{v},$$

the rational equation will rise to n dimensions of x, from which that is seen to be going to rise to many more dimensions, if more radical signs of this kind may be present; which must without doubt eventuate, if these radical signs are completely independent of each other. But since all are powers of the first, showing perfect rationality is not able to be obtained by rising beyond the power of the exponent n. Thus evidently I will demonstrate the form

$$x=\omega+\mathfrak{A}\sqrt[n]{v}+\mathfrak{B}\sqrt[n]{v^2}+\mathfrak{C}\sqrt[n]{v^3}+\ldots\ldots\mathfrak{O}\sqrt[n]{v^{n-1}},$$

thus able to be freed from irrationality, so that the rational equation thence resulting the power will not exceed x^n . Therefore an equation of this form will be produced

$$x^n+\Delta x^{n-1}+Ax^{n-2}+Bx^{n-3}+\text{etc.}=0,$$

the root of which will be that form assumed; and because the number of roots of this equation is $n,$ from the same form we will be able to designate all the roots of this equation.

19. Now since this will be an excellent criterion for the truth of this form, then it will help also to be noting, because the form of the root contains $n-1$ arbitrary quantities, just as many also arbitrary quantities enter into the rational equation, from which it is evident these quantities thus may be able to be determined, so that the rational equation thus may obtain the given coefficients Δ, A, B, C etc., that is, so that the general equation of this order may be obtained. Which determination if it may actually be able to be put in place, thence we will obtain the general resolution of the equation of any order, from which also the possibility of the resolution requiring to be carried out in this manner may be elucidated. Certain conspicuous difficulties occur in this matter, which we will recognise

there more clearly, if we may adapt our form for any order on beginning with the simplest. Moreover deliberating on the simplicity and conciseness of the calculation, we may set aside the rational part of the root ω, so that in any order we may reach rational equations of this kind, in which the second term may be absent, so that the extent of the resolution is not required to be considered restricted.

I. RESOLUTION OF EQUATIONS OF THE SECOND ORDER

20. Therefore so that we may begin from equations of the second order, there shall be $n = 2$, and putting $\omega = 0$, our form of the root will be

$$x = \mathfrak{A}\sqrt{v},$$

which made rational gives

$$xx = \mathfrak{A}\mathfrak{A}v.$$

This equation may be compared with the general form of the second order equation :

$$xx = A$$

with the second term lacking and there shall be

$$\mathfrak{A}\mathfrak{A}v = A;$$

to which in order that it may be satisfied, there may be put $\mathfrak{A} = 1$ and there will be

$$v = A.$$

From which the proposed equation $xx = A$ if there may be taken $\mathfrak{A} = 1$ and $v = A$, of which the one root will be

$$x = \mathfrak{A}\sqrt{v} = \sqrt{A},$$

and because $\sqrt{1}$ has two values 1 and −1, the other root will be

$$x = -\mathfrak{A}\sqrt{v} = -\sqrt{A},$$

which indeed by itself is evident.

II. RESOLUTION OF EQUATIONS OF THE THIRD ORDER

21. Now on putting $n = 3$ the form of the roots for this case will be

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2};$$

from which so that a rational equation may be elicited, at first the cube may be taken :

$$x^3 = v\mathfrak{A}^3 + 3\mathfrak{A}\mathfrak{A}\mathfrak{B}v\sqrt[3]{v} + 3\mathfrak{A}\mathfrak{B}\mathfrak{B}v\sqrt[3]{v^2} + \mathfrak{B}^3v^2.$$

Now this cubic equation may be formed :

$$x^3 = Ax + B\,,$$

from which by substituting the value assumed for x there will arise also

$$x^3 = A\mathfrak{A}\sqrt[3]{v} + A\mathfrak{B}\sqrt[3]{v^2} + B,$$

which form being rendered equal to that by equating both between the rational parts as well as the irrational parts of each of the kinds $\sqrt[3]{v}$ et $\sqrt[3]{v^2}$.

22. Moreover the comparison of the rational terms provides

$$B = \mathfrak{A}^3v + \mathfrak{B}^3v^2$$

and from the collation of the irrational terms there shall become

$$A\mathfrak{A} = 3\mathfrak{A}\mathfrak{A}\mathfrak{B}v \text{ and } A\mathfrak{B} = 3\mathfrak{A}\mathfrak{B}\mathfrak{B}v,$$

each of which gives

$$A = 3\mathfrak{A}\mathfrak{B}v.$$

Hence, if this cubic equation were proposed

$$x^3 = 3\mathfrak{A}\mathfrak{B}vx + \mathfrak{A}^3v + \mathfrak{B}^3v^2,$$

of which one root will be

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2},$$

and if $1,\ \mathfrak{a},\ \mathfrak{b}$ shall be the three cubic roots of unity, the two remaining roots will be

$$x = \mathfrak{A}\mathfrak{a}\sqrt[3]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[3]{v^2},\quad x = \mathfrak{A}\mathfrak{b}\sqrt[3]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[3]{v^2}.$$

But there is

$$\mathfrak{a} = \mathfrak{b}^2 = \tfrac{-1+\sqrt{-3}}{2} \text{ and } \mathfrak{b} = \mathfrak{a}^2 = \tfrac{-1-\sqrt{-3}}{2}.$$

23. But the quantities $\mathfrak{A}$, $\mathfrak{B}$ and v can be determined in turn, if the cubic equation may be proposed

$$x^3 = Ax + B\,,$$

from the coefficients A and B , so that thence all three roots of the equation may be obtained. But these in the end, since only two of the equations will have to be satisfied, one of the letters $\mathfrak{A}$ and $\mathfrak{B}$ can be assumed as it pleases. Therefore if there shall be $\mathfrak{A}=1$ and the equation

$$A=3\mathfrak{A}\mathfrak{B}v=3\mathfrak{B}v$$

gives

$$\mathfrak{B}=\tfrac{A}{3v},$$

from which there becomes

$$\mathfrak{B}^3=\tfrac{A^3}{27v^3},$$

which value substituted into the first equation

$$B=v+\mathfrak{B}^3v^2$$

gives

$$B=v+\tfrac{A^3}{27v} \quad\text{or}\quad vv=Bv-\tfrac{1}{27}A^3,$$

from which there becomes:

$$v=\tfrac{1}{2}B\pm\sqrt{(\tfrac{1}{4}BB-\tfrac{1}{27}A^3)};$$

but likewise, either of these two values may be assumed.

24. Moreover with the values of $v=\tfrac{1}{2}B\pm\sqrt{(\tfrac{1}{4}BB-\tfrac{1}{27}A^3)}$ found, there will be $\mathfrak{B}=\tfrac{A}{3v}$, and $\mathfrak{B}\sqrt[3]{v^2}=\tfrac{A}{3\sqrt[3]{v}}$. And hence of the equation proposed

$$x^3=Ax+B,$$

the three roots will be

$$\text{I. } x=\sqrt[3]{v}+\tfrac{A}{3\sqrt[3]{v}},\quad \text{II. } x=\mathfrak{a}\sqrt[3]{v}+\tfrac{\mathfrak{b}A}{3\sqrt[3]{v}},\quad \text{III. } x=\mathfrak{b}\sqrt[3]{v^2}+\tfrac{\mathfrak{a}A}{3\sqrt[3]{v}}.$$

But since there shall be

$$\frac{1}{v}=\frac{\tfrac{1}{2}B\mp\sqrt{(\tfrac{1}{4}BB-\tfrac{1}{27}A^3)}}{\tfrac{1}{27}A^3},$$

there will be

$$\sqrt[3]{v}=\sqrt[3]{\left(\tfrac{1}{2}B\pm\sqrt{(\tfrac{1}{4}BB-\tfrac{1}{27}A^3)}\right)}$$

and

$$\frac{A}{3\sqrt[3]{v}}=\sqrt[3]{\left(\tfrac{1}{2}B\mp\sqrt{(\tfrac{1}{4}BB-\tfrac{1}{27}A^3)}\right)},$$

and hence the common formulas arise for the resolution of cubic equations.

III. RESOLUTION OF EQUATIONS OF THE FOURTH ORDER

25. On putting $n=4$ we will consider this form of the root

$$x=\mathfrak{A}\sqrt[4]{v}+\mathfrak{B}\sqrt[4]{v^2}+\mathfrak{C}\sqrt[4]{v^3}$$

and we shall seek the equation of the fourth order, of which this form shall be the root. And indeed in this case the calculation is put in place easily, where the irrationalities are removed ; for on account of $\sqrt[4]{v^2}=\sqrt{v}$ this equation may be taken

$$x-\mathfrak{B}\sqrt{v}=\mathfrak{A}\sqrt[4]{v}+\mathfrak{C}\sqrt[4]{v^3},$$

which squared gives

$$xx-2\mathfrak{B}x\sqrt{v}+\mathfrak{B}\mathfrak{B}v=\mathfrak{A}\mathfrak{A}\sqrt{v}+2\mathfrak{A}\mathfrak{C}v+\mathfrak{C}\mathfrak{C}v\sqrt{v},$$

which with the irrational parts moved to the same side becomes

$$xx+(\mathfrak{B}\mathfrak{B}-2\mathfrak{A}\mathfrak{C})v=2\mathfrak{B}x\sqrt{v}+(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)\sqrt{v}$$

and with the square taken anew this rational equation will be produced :

$$\begin{aligned}&x^4+2(\mathfrak{B}\mathfrak{B}-2\mathfrak{A}\mathfrak{C})vxx+(\mathfrak{B}\mathfrak{B}-2\mathfrak{A}\mathfrak{C})^2vv\\&=4\mathfrak{B}\mathfrak{B}vxx+4(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)\mathfrak{B}vx+(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)^2v,\end{aligned}$$

which ordered will change into this form

$$\begin{aligned}&x^4=2(\mathfrak{B}\mathfrak{B}+2\mathfrak{A}\mathfrak{C})vxx+4(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)\mathfrak{B}vx\\&+\mathfrak{A}^4v-\mathfrak{B}^4vv+\mathfrak{C}^4v^3+4\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv-2\mathfrak{A}\mathfrak{A}\mathfrak{C}\mathfrak{C}vv.\end{aligned}$$

26. Therefore of this biquadratic equation one root is :

$$x=\mathfrak{A}\sqrt[4]{v}+\mathfrak{B}\sqrt[4]{v^2}+\mathfrak{C}\sqrt[4]{v^3}\ ,$$

and if the biquadratic roots of unity may be put $1,\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},$ thus so that there shall be

$$\mathfrak{a}=+\sqrt{-1},\ \mathfrak{b}=-1,\ \text{and}\ \ \mathfrak{c}=-\sqrt{-1},$$

there will be

$$\mathfrak{a}^2 = -1 = \mathfrak{b}, \qquad \mathfrak{a}^3 = -\sqrt{-1} = \mathfrak{c},$$
$$\mathfrak{b}^2 = +1, \qquad \mathfrak{b}^3 = -1 = \mathfrak{b},$$
$$\mathfrak{c}^2 = -1 = \mathfrak{b}, \qquad \mathfrak{c}^3 = +\sqrt{-1} = \mathfrak{a},$$

from which the three remaining roots of the same equation will be :

$$x = \mathfrak{A}\mathfrak{a}\sqrt[4]{v} + \mathfrak{B}\mathfrak{b}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{c}\sqrt[4]{v^3},$$
$$x = \mathfrak{A}\mathfrak{b}\sqrt[4]{v} + \mathfrak{B}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{b}\sqrt[4]{v^3},$$
$$x = \mathfrak{A}\mathfrak{c}\sqrt[4]{v} + \mathfrak{B}\mathfrak{b}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{a}\sqrt[4]{v^3}.$$

27. Hence moreover any biquadratic equation in turn may be reduced to that form and its roots will be able to be assigned. For if this equation shall be proposed :

$$x^4 = Axx + Bx + C,$$

and it is required to find the values of the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with the quantity *v,* with which found likewise the roots of this equation will be known. Indeed there will be

$$A = 2(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v,$$
$$B = 4(\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)\mathfrak{B}v,$$
$$C = \mathfrak{A}^4 v - \mathfrak{B}^4 vv + \mathfrak{C}^4 v^3 + 4\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv - 2\mathfrak{A}\mathfrak{A}\mathfrak{C}\mathfrak{C}vv$$

or

$$C = (\mathfrak{A}\mathfrak{A} + \mathfrak{A}\mathfrak{A}v)^2 v - (\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})^2 vv + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv.$$

But in that place there is

$$(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v = \tfrac{1}{2}A$$

and

$$\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v = \tfrac{B}{4\mathfrak{B}v},$$

which values substituted here give :

$$C = \tfrac{BB}{16\mathfrak{B}\mathfrak{B}v} - \tfrac{1}{4}AA + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv.$$

But the first formula provides $4\mathfrak{A}\mathfrak{C}v = A - 2\mathfrak{B}\mathfrak{B}v$, which value substituted anew gives

$$C = \tfrac{BB}{16\mathfrak{B}\mathfrak{B}v} - \tfrac{1}{4}AA + 2A\mathfrak{B}\mathfrak{B}v - 4\mathfrak{B}^4 vv,$$

thus so that now the two letters $\mathfrak{A}$ and $\mathfrak{C}$ shall be eliminated.

28. Because here hitherto two unknowns $\mathfrak{B}$ and v are present, the value of $\mathfrak{B}$ is left to our choice. Therefore let $\mathfrak{B}=1$ and the quantity v will have to be determined from the following cubic equation

$$v^3-\tfrac{1}{2}Av^2+\tfrac{1}{4}(C+\tfrac{1}{4}AA)v-\tfrac{1}{64}BB=0.$$

But hence the value found for the root v from the previous equations must be sought from the letters $\mathfrak{A}$ and $\mathfrak{C}$
Therefore since there shall be

$$\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v=\tfrac{B}{4v} \text{ and } 2\mathfrak{A}\mathfrak{C}\sqrt{v}=\tfrac{A-2v}{2\sqrt{v}},$$

since both by adding as well as by subtracting , and by extracting the square root, there will be

$$\mathfrak{A}+\mathfrak{C}\sqrt{v}=\sqrt{\left(\tfrac{B}{4v}+\tfrac{A}{2\sqrt{v}}-\sqrt{v}\right)}$$

and

$$\mathfrak{A}-\mathfrak{C}\sqrt{v}=\sqrt{\left(\tfrac{B}{4v}-\tfrac{A}{2\sqrt{v}}+\sqrt{v}\right)},$$

from which there will be found

$$\mathfrak{A}=\tfrac{1}{4\sqrt{v}}\sqrt{\left(B+2A\sqrt{v}-4v\sqrt{v}\right)}+\tfrac{1}{4\sqrt{v}}\sqrt{\left(B-2A\sqrt{v}+4v\sqrt{v}\right)}$$

and

$$\mathfrak{C}=\tfrac{1}{4\sqrt{v}}\sqrt{\left(B+2A\sqrt{v}-4v\sqrt{v}\right)}-\tfrac{1}{4\sqrt{v}}\sqrt{\left(B-2A\sqrt{v}+4v\sqrt{v}\right)}.$$

29. Since there shall be

$$\mathfrak{A}\sqrt[4]{v}\pm\mathfrak{C}\sqrt[4]{v^3}=(\mathfrak{A}\pm\mathfrak{C}\sqrt{v})\sqrt[4]{v},$$

the four roots of the proposed equation

$$x^4=Axx+Bx+C,$$

after the values v were found from the equation

$$v^3-\tfrac{1}{2}Av^2+\tfrac{1}{4}(C+\tfrac{1}{4}AA)v-\tfrac{1}{64}BB=0,$$

will be

$$\text{I.}\quad x=\sqrt{v}+\tfrac{1}{2\sqrt{v}}\sqrt{(B\sqrt{v}+2Av-4vv)},$$

$$\text{II.}\quad x=\sqrt{v}-\tfrac{1}{2\sqrt{v}}\sqrt{(B\sqrt{v}+2Av-4vv)},$$

$$\text{III.}\ x=-\sqrt{v}+\tfrac{1}{2\sqrt{v}}\sqrt{(-B\sqrt{v}+2Av-4vv)},$$

$$\text{IV.}\ x=-\sqrt{v}-\tfrac{1}{2\sqrt{v}}\sqrt{(-B\sqrt{v}+2Av-4vv)}.$$

And in this manner, as agreed, the resolution of biquadratic equations is reduced to the resolution of cubic equations.

IV. RESOLUTION OF EQUATIONS OF THE FIFTH ORDER

30. On putting $n=5$, the form of our root will be

$$x=\mathfrak{A}\sqrt[5]{v}+\mathfrak{B}\sqrt[5]{vv}+\mathfrak{C}\sqrt[5]{v^3}+\mathfrak{D}\sqrt[5]{v^4}$$

and first an equation of the fifth order must be sought, of which this shall become a root, or, which returns the same, the radical signs will be required to be eliminated from this form. But a difficulty is met in this sum, since the operation of elimination by no means may be able to be set up in that way, which I have used in equations of the fourth order. Indeed it is evident, because all the powers of x involve the same radical signs, if the equation sought may be formed

$$x^5=Ax^3+Bx^2+Cx+D,$$

then by substituting the value assumed for x, four equations to be obtained, with the aid of which the four signed radicals may be able to be eliminated ; but then these individual letters assumed *A, B, C* and *D* are to be determined with great difficulty.

31. With these difficulties carefully considered, I have come across another way of setting up this operation, is prepared thus, so that all the roots formed, whatever the orders shall be, may appear equally, and from that likewise a rational equation may be examined never of a higher order, which is indicated to be increasing by the exponent *n*. But here the way depends on the nature of the equation, by which the coefficients of the individual terms are defined from all the roots. Therefore since all five roots of the equation that we look for may agree, from these too the coefficients of the individual terms of this form can be known by rules. Therefore the five roots of unity of the fifth dimension shall be $1,\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d}$, or the roots of this equation $x^5-1=0$, and by putting $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ for the roots of the equation, that we seek, there will be

$$\alpha = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4},$$
$$\beta = \mathfrak{A}\mathfrak{a}\sqrt[5]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4},$$
$$\gamma = \mathfrak{A}\mathfrak{b}\sqrt[5]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4},$$
$$\delta = \mathfrak{A}\mathfrak{c}\sqrt[5]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4},$$
$$\varepsilon = \mathfrak{A}\mathfrak{d}\sqrt[5]{v} + \mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}.$$

32. With these five roots set out if the equation of the fifth degree having these roots may be put in place

$$x^5 - \Delta x^4 + Ax^3 - Bx^2 + Cx - D = 0,$$

these coefficients thus may be defined from the roots $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$, so that there shall be

$\Delta =$ sum of the roots,
$A =$ sum of products in pairs,
$B =$ sum of products in threes,
$C =$ sum of products in fours,
$D =$ product of all five.

But so that we may be able to collect these values more easily, we include these from the sums of the powers of the roots. Therefore there shall be

$$P = \alpha + \beta + \gamma + \delta + \varepsilon,$$
$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2,$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3,$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4,$$
$$T = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5.$$

Indeed from these defined values, we know to use,

$$\Delta = P,$$
$$A = \frac{\Delta P - Q}{2},$$
$$B = \frac{AP - \Delta Q + R}{3},$$
$$C = \frac{BP - AQ + \Delta R - S}{4},$$
$$D = \frac{CP - BQ + AR - \Delta S + T}{5}.$$

33. Now for all the values *P, Q, R, S, T* requiring to be found we must initially render all the powers of the roots of unity $1,\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d}$ into one sum ; which since the roots of the equation shall be $z^5 - 1 = 0$, there will be

$$1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = 0,$$
$$1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2 = 0,$$
$$1 + \mathfrak{a}^3 + \mathfrak{b}^3 + \mathfrak{c}^3 + \mathfrak{d}^3 = 0,$$
$$1 + \mathfrak{a}^4 + \mathfrak{b}^4 + \mathfrak{c}^4 + \mathfrak{d}^4 = 0,$$
$$1 + \mathfrak{a}^5 + \mathfrak{b}^5 + \mathfrak{c}^5 + \mathfrak{d}^5 = 5.$$

The sums of the sixth, seventh, etc. powers as far as up to the tenth again vanish, but the tenth sum again becomes = 5, since there shall be $\mathfrak{a}^5 = 1,\ \mathfrak{b}^5 = 1,\ \mathfrak{c}^5 = 1,$ and $\mathfrak{d}^5 = 1.$ For the sake of brevity in this calculation we will evidently be able to omit the radical signs, provided henceforth we may bear in mind $\sqrt[5]{v},\ \sqrt[5]{v^2},\ \sqrt[5]{v^3},\ \sqrt[5]{v^4}$ are to be connected with the letters $\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D}$.

34. Hence therefore with the roots $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ added we will have

$$P = \mathfrak{A}(1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d}) + \mathfrak{B}(1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2) + \text{etc.} = 0.$$

Moreover with the remaining powers being summed we will elicit in addition,

$$P = 0,$$
$$Q = 10\left(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C}\right),$$
$$R = 15(\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 + \mathfrak{C}^2\mathfrak{D}),$$
$$S = 20(\mathfrak{A}^3\mathfrak{B} + \mathfrak{A}\mathfrak{C}^3 + \mathfrak{B}^3\mathfrak{D} + \mathfrak{C}\mathfrak{D}^3) + 30(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2) + 120\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D},$$
$$T = 5(\mathfrak{A}^5 + \mathfrak{B}^5 + \mathfrak{C}^5 + \mathfrak{D}^5) + 100(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D} + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3)$$
$$+ 150(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}^2\mathfrak{D}).$$

Here other products do not occur, unless which with the root signs attached produce a rational power of v ; or if we may attribute a single dimension to the letter $\mathfrak{A}$, two to the letter $\mathfrak{B}$, three to the letter $\mathfrak{C}$ and four to the letter $\mathfrak{D}$, in all these letters the number of dimensions is divisible by 5 , but the coefficient of any product is a quintuple of this coefficient, which likewise coincides with the law of combinations.

35. Therefore since there shall be $P=0$, also there will be $\Delta=0$ and for the remaining coefficients we will have

$$A=-\tfrac{1}{2}Q,\ B=\tfrac{1}{3}R,\ C=-\tfrac{1}{4}AQ-\tfrac{1}{4}S \text{ and } D=-\tfrac{1}{5}BQ+\tfrac{1}{5}AR+\tfrac{1}{5}T.$$

Hence therefore there will be :

$$\begin{aligned}
&A=-5\left(\mathfrak{A}\mathfrak{D}+\mathfrak{B}\mathfrak{C}\right),\\
&B=5(\mathfrak{A}^2\mathfrak{C}+\mathfrak{A}\mathfrak{B}^2+\mathfrak{B}\mathfrak{D}^2+\mathfrak{C}^2\mathfrak{D}),\\
&C=-5(\mathfrak{A}^3\mathfrak{B}+\mathfrak{B}^3\mathfrak{D}+\mathfrak{A}\mathfrak{C}^3+\mathfrak{C}\mathfrak{D}^3)+5(\mathfrak{A}^2\mathfrak{D}^2+\mathfrak{B}^2\mathfrak{C}^2)-5\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D},\\
&D=\mathfrak{A}^5+\mathfrak{B}^5+\mathfrak{C}^5+\mathfrak{D}^5-5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D}+\mathfrak{A}\mathfrak{B}^3\mathfrak{C}+\mathfrak{B}\mathfrak{C}^3\mathfrak{D}+\mathfrak{A}\mathfrak{B}\mathfrak{D}^3)\\
&\qquad+5(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2+\mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2+\mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2+\mathfrak{A}^2\mathfrak{B}^2\mathfrak{D}).
\end{aligned}$$

with which terms now the powers of v must be adjoined, so that just the values of these may be obtained.

36. But if therefore this equation may be establishes with the signs of the coefficients A and C interchanged :

$$x^5=Ax^3+Bx^2+Cx+D,$$

the coefficients of which may hold these values :

$$\begin{aligned}
&A=5\left(\mathfrak{A}\mathfrak{D}+\mathfrak{B}\mathfrak{C}\right)v,\\
&B=5(\mathfrak{A}^2\mathfrak{C}+\mathfrak{A}\mathfrak{B}^2+\mathfrak{B}\mathfrak{D}^2v+\mathfrak{C}^2\mathfrak{D}v)v,\\
&C=5(\mathfrak{A}^3\mathfrak{B}+\mathfrak{B}^3\mathfrak{D}v+\mathfrak{A}\mathfrak{C}^3v+\mathfrak{C}\mathfrak{D}^3vv)v-5(\mathfrak{A}^2\mathfrak{D}^2+\mathfrak{B}^2\mathfrak{C}^2)v^2+5\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}v^2,\\
&D=\mathfrak{A}^5v+\mathfrak{B}^5v^2+\mathfrak{C}^5v^3+\mathfrak{D}^5v^4-5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D}+\mathfrak{A}\mathfrak{B}^3\mathfrak{C}+\mathfrak{B}\mathfrak{C}^3\mathfrak{D}v+\mathfrak{A}\mathfrak{B}\mathfrak{D}^3v)v^2\\
&\qquad+5(\mathfrak{A}^2\mathfrak{C}^2\mathfrak{D}+\mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2+\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2v+\mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2v)v^2,
\end{aligned}$$

its five roots will be

$$\text{I.}\quad x=\mathfrak{A}\sqrt[5]{v}+\mathfrak{B}\sqrt[5]{v^2}+\mathfrak{C}\sqrt[5]{v^3}+\mathfrak{D}\sqrt[5]{v^4},$$
$$\text{II.}\quad x=\mathfrak{A}\mathfrak{a}\sqrt[5]{v}+\mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4},$$
$$\text{III.}\quad x=\mathfrak{A}\mathfrak{b}\sqrt[5]{v}+\mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4},$$
$$\text{IV.}\quad x=\mathfrak{A}\mathfrak{c}\sqrt[5]{v}+\mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4},$$
$$\text{V.}\quad x=\mathfrak{A}\mathfrak{d}\sqrt[5]{v}+\mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}.$$

with $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ present besides unity for the four remaining fifth roots of unity, for which imaginary values are agreed.

37. If now in turn the quantities $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ may be able to be defined from the given coefficients *A, B, C, D* with the letter *v,* the general resolution of all equations of the fifth order may be had. Truly in this itself the greatest difficulty stands, since in no way shall it be apparent the $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, indeed of which one is permitted to be assumed as wished, thus to be required to be eliminated successively, so that an equation involving only the unknown *v* with the given *A, B, C, D* may result, which indeed may include no superfluous roots. But without much risk it may be allowed to suspect, if this elimination may be duly carried out, it may be able to arrive at last at an equation of the fourth order, from which the value of *v* may be defined. If indeed an equation of higher order may be produced, then also the value of *v* with a root sign of the same order may be implied, which may be considered to be absurd. But since the multitude of terms returns this so difficult labour, so that it may not indeed be attempted with any success, yet indeed from this matter certain less general cases may be established, which may be deduced from less complicated formulas.

38. Therefore on descending to particular cases, we may attribute values of this kind to the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, by which the calculation may be reduced into a shortened version; and indeed at first there shall be $\mathfrak{B}=0$, $\mathfrak{C}=0$, and $\mathfrak{D}=0$, from which we will obtain

$$A=0,\ B=0,\ C=0,\ D=\mathfrak{A}^5v.$$

Hence therefore there becomes

$$\mathfrak{A}\sqrt[5]{v}=\sqrt[5]{D}.$$

Whereby if this equation were proposed

$$x^5=D,$$

the five roots of this equation will be

$$\text{I. } x=\sqrt[5]{D},\ \text{II. } x=\mathfrak{a}\sqrt[5]{D},\ \text{III. } x=\mathfrak{b}\sqrt[5]{D},\ \text{IV. } x=\mathfrak{c}\sqrt[5]{D}\ \ \text{V. } x=\mathfrak{d}\sqrt[5]{D};$$

which case since it shall be evident by itself, from that we begin to understand, as therefore may be apparent, how our method may include known cases within it.

39. Now two of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ may vanish, for if indeed three may be considered to vanish, however these may be taken, we may always deduce the preceding case . Therefore $\mathfrak{C}$ and $\mathfrak{D}$ shall be equal to zero, or the equation is sought, of which the root shall become

$$x=\mathfrak{A}\sqrt[5]{v}+\mathfrak{B}\sqrt[5]{v^2},$$

and we will obtain

$$A=0,\ B=5\mathfrak{A}\mathfrak{B}^2v,\ C=5\mathfrak{A}^3\mathfrak{B}v,\ D=\mathfrak{A}^5v+\mathfrak{B}^6v^2,$$

from which the proposed root will agree with this equation

$$x^5=5\mathfrak{A}\mathfrak{B}^2vx^2+5\mathfrak{A}^3\mathfrak{B}vx+\mathfrak{A}^5v+\mathfrak{B}^6v^2.$$

Which equation if it may be compared with this form

$$x^5=5Pxx+5Qx+R,$$

there will be

$$\mathfrak{A}\mathfrak{B}^2v=P,\ \ \mathfrak{A}^3\mathfrak{B}v=Q,$$

from which there is deduced

$$\mathfrak{A}^5v=\tfrac{QQ}{P}\ \text{ and }\ \mathfrak{B}^5v^2=\tfrac{P^3}{Q},$$

thus so that there shall become

$$R=\tfrac{QQ}{P}+\tfrac{P^3}{Q}.$$

40. Hence therefore we deduce for the resolution of this special equation of the fifth order

$$x^5=5Pxx+5Qx+\tfrac{QQ}{P}+\tfrac{P^3}{Q},$$

of which, on account of $\mathfrak{A}\sqrt[5]{v}=\sqrt[5]{\frac{QQ}{P}}$ and $\mathfrak{B}\sqrt[5]{vv}=\sqrt[5]{\frac{P^3}{Q}}$ the five roots will be

$$\text{I. } x=\sqrt[5]{\frac{QQ}{P}}+\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{II. } x=\mathfrak{a}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{a}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{III. } x=\mathfrak{b}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{b}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{IV. } x=\mathfrak{c}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{c}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{V. } x=\mathfrak{d}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{d}^2\sqrt[5]{\frac{P^3}{Q}}.$$

But this equation is not much different from the de Moivre formula, and because it does not appear these can be resolved into factors, its resolution treated here therefore deserves to be noted more.

41. We will be able to free this equation from fractions, if we may put

$$P=MN \text{ and } Q=M^2N;$$

then indeed we will have

$$x^5=5MNxx+5M^2Nx+M^3N+MN^2,$$

of which the root will be

$$x=\sqrt[5]{M^3N}+\sqrt[5]{MN^2},$$

and if $\mathfrak{a}$ may denote some other fifth root of unity, the root of this other equation will be

$$x=\mathfrak{a}\sqrt[5]{M^3N}+\mathfrak{a}^2\sqrt[5]{MN^2}.$$

Thus if for example there may be put $M=1$ and $N=2$, some root of this equation

$$x^5=10xx+10x+6$$

is

$$x=\mathfrak{a}\sqrt[5]{2}+\mathfrak{a}^2\sqrt[5]{4};$$

and this equation has been prepared thus, so that it may not be able to be resolved by any known method.

42. If $\mathfrak{B}$ and $\mathfrak{D}$ shall be equal to zero, we are returned to the same case. For there will become

$$A=0,\ B=5\mathfrak{A}^2\mathfrak{C}v,\ C=5\mathfrak{A}\mathfrak{C}^3vv \text{ and } D=\mathfrak{A}^5v+\mathfrak{C}^5v^3;$$

from which if this equation may be put in place

$$x^5=5Pxx+5Qx+R,$$

so that there shall be

$$P=\mathfrak{A}^2\mathfrak{C}v \text{ and } Q=\mathfrak{A}\mathfrak{C}^3vv,$$

there will be

$$\tfrac{QQ}{P}=\mathfrak{C}^5v^3 \text{ and } \tfrac{P^3}{Q}=\mathfrak{A}^5v$$

And hence there shall become as before

$$R=\tfrac{QQ}{P}+\tfrac{P^3}{Q}$$

and also the same roots are discovered. Also again the same equation is found, whether there may be put $\mathfrak{A}=0$ and $\mathfrak{B}=0,$ or $\mathfrak{A}=0$ and $\mathfrak{C}=0.$ But if either $\mathfrak{A}$ and $\mathfrak{D}$, or $\mathfrak{B}$ and $\mathfrak{C}$ may be assumed to vanish, and each indeed produces the same equation, but different from the preceding cases, which it will be convenient to present thus.

43. Therefore if the shall be both $\mathfrak{B}=0$ and $\mathfrak{C}=0$ and hence we will pursue the following values

$$A=5\mathfrak{A}\mathfrak{D}v,\ B=0,\ C=-5\mathfrak{A}^2\mathfrak{D}^2v^2 \text{ and } D=\mathfrak{A}^5v+\mathfrak{D}^5v^4.$$

From which if we may put $\mathfrak{A}\mathfrak{D}v=P$, there will be

$$A=5P \text{ and } C=-5PP;$$

then truly there will be

$$DD-4P^5=(\mathfrak{A}^5v-\mathfrak{D}^5v^4)^2 \text{ and } \mathfrak{A}^5v-\mathfrak{D}^5v^4=\sqrt{(DD-4P^5)},$$

and thus

$$\mathfrak{A}^5v=\tfrac{1}{2}D+\tfrac{1}{2}\sqrt{(DD-4P^5)} \text{ and } \mathfrak{D}^5v^4=\tfrac{1}{2}D-\tfrac{1}{2}\sqrt{(DD-4P^5)}.$$

Hence if this equation shall be proposed

$$x^5=5Px^3-5PPx+D,$$

any of its roots is

$$x = \mathfrak{a}\sqrt[5]{\left(\tfrac{1}{2}D+\tfrac{1}{2}\sqrt{(DD-4P^5)}\right)}+\mathfrak{a}^4\sqrt[5]{\left(\tfrac{1}{2}D-\tfrac{1}{2}\sqrt{(DD-4P^5)}\right)}$$

and this is that equation itself, the resolution of which the celebrated de Moivre taught.

44. But from the general form of equations of the fifth order innumerable equations may be deduced, the roots of which it is allowed to assign, even if these equations themselves may not be able to be resolved into factors. Indeed for the proposed equation of the fifth order

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

the coefficients of which may have the following values

$$A = \tfrac{5}{gk}(g^3+k^3),$$
$$B = \tfrac{5}{mnr}\Big((m+n)(m^2g^3-n^2k^2)-(m-n)rr\Big),$$

$$C = \tfrac{5}{mngkkrr}\Big(g^3(m^2g^3-n^2k^3)^2-\Big(m(m+n)g^6-(m^2+mn-n^2)g^3k^3+n(m-n)k^6\Big)rr-k^3r^4\Big),$$

$$\begin{aligned}D = {}&\tfrac{gg}{mmnk^4r^3}\Big((m^2g^3-n^2k^3)^3-(m^2g^3-n^2k^3)(m^2g^3+n^2k^3)rr-n^2k^3r^4\Big)\\ &+\tfrac{kk}{mnng^4r}\Big(m^2g^3(m^2g^3-n^2k^3)-(2m^2g^3+n^2k^3)r^2+r^4\Big)\\ &+\tfrac{5(m-n)(g^3-k^3)(m^2g^3-n^2k^3)}{mngkr}-\tfrac{5(m+n)(g^3-k^3)r}{mngk},\end{aligned}$$

the roots of which always can be assigned. [These are the corrected values found by the editor F. Rudio of this commentary in the *O.O.* edition.]

45. Indeed we may put for brevity

$$T = (m^2g^3-n^2k^3)^2-2(m^2g^3+n^2k^3)rr+r^4$$

and there will be

$$\left.\begin{matrix}P\\ Q\end{matrix}\right\} = \tfrac{(m^2g^3-n^2k^3)^3-(m^2g^3-n^2k^3)(m^2g^3+n^2k^3)rr-n^2k^3r^4\pm((m^2g^3-n^2k^3)^2-n^2k^3rr)\sqrt{T}}{2mmnr^3},$$

$$\left.\begin{matrix}R\\ S\end{matrix}\right\} = \tfrac{(m^2g^3-n^2k^3)m^2g^3-(2m^2g^3+n^2k^3)rr+r^4\pm(m^2g^3-rr)\sqrt{T}}{2mmnr},$$

where the upper signs prevail for the values P and $R,$ the lower for Q and S , and any root of the equation will be

$$x = \mathfrak{a}\sqrt[5]{\frac{gg}{k^4}}P + \mathfrak{a}^2\sqrt[5]{\frac{kk}{g^4}}R + \mathfrak{a}^3\sqrt[5]{\frac{kk}{g^4}}S + \mathfrak{a}^4\sqrt[5]{\frac{gg}{k^4}}Q.$$

46. So that we may illustrate the matter with examples, from these forms the following are able to be formed:

I. A root of the equation

$$x^5 = 40x^3 + 70xx - 50x - 98$$

is

$$x = \sqrt[5]{(-31+3\sqrt{-7})} + \sqrt[5]{(-18+10\sqrt{-7})} + \sqrt[5]{(-18-10\sqrt{-7})} + \sqrt[5]{(-31-3\sqrt{-7})}.$$

II. A root of the equation

$$x^5 = 2625x + 61500$$

is

$$x = \sqrt[5]{75(5+4\sqrt{10})} + \sqrt[5]{225(35+11\sqrt{10})} + \sqrt[5]{225(35-11\sqrt{10})} + \sqrt[5]{75(5-4\sqrt{10})},$$

which therefore are more noteworthy, because these equations cannot be solved in any other way.

Moreover in a similar manner investigations can be extended to equations of higher orders, and easily from any order innumerable equations unresolvable by other methods to be shown, of which with the help of this method not only one, but clearly all the roots may be able to be shown.

DE RESOLUTIONE AEQUATIONUM CUIUSVIS GRADUS

Commentatio 282 indices En.
Novi commentarii academiae scientarum Petropolitanae 9 (1762/3), 1764.

1. Quae in Algebra adhuc de resolutione aequationum sunt tradita, ea, si ad regulas generales spectemus, tantum ad aequationes, quae quartum gradum non superant, patent neque etiamnum regulae sunt inventae, quarum ope aequationes quinti altiorisve cuiuspiam gradus resolvi queant, ita ut universa Algebra ad aequationes quatuor primorum ordinum restringatur. Hoc autem de regulis generalibus est tenendum, quae ad omnes aequationes eiusdem gradus sint accommodatae; nam in quovis gradu dantur infinitae aequationes, quae per divisionem in duas pluresve aequationes graduum inferiorum resolvi possunt, quarum idcirco radices iunctim sumtae praebent omnes radices illarum aequationum altiorum graduum. Tum vero a Cel. Moivreano observotae sunt in quovis gradu quaedam aequationes speciales; quae etsi per divisionem in factores resolvi nequeunt, tamen earum radices assignare liceat.

2. Ex cognita autem resolutione generali aequationum primi, secundi, terth et quarti gradus constat quidem aequationes primi gradus sine ulla radicis extractione resolvi posse; at aequationum secundi gradus resolutio iam extractionem radicis quadratae postulat. Resolutio autem aequationum terth gradus tam extractionem radicis quadratae quam cubicae implicat et quarti gradus resolutio insuper extractionem radicis biquadratae exigit. Ex his autem tuto concludere licet resolutionem aequationis quinti gradus generalem extractionem radicis surdesolidae praeter omnes radices inferiores postulare; atque in genere radix aequationis cuiusvis gradus n exprimetur per formam, quae ex omnibus signis radicalibus tam gradus n quam graduum inferiorum erit composita.

3. Haec perpendens olim in Comment. Acad. Imper. Petrop. Tomo VI coniecturam ausus sum proferre circa formas radicum cuiuscumque aequationis.
Proposita namque aequatione gradus cuiusvis

$$x^n + \mathrm{A}x^{n-2} + \mathrm{B}x^{n-3} + \mathrm{C}^{n-4} + \text{etc.} = 0,$$

in qua secundum terminum deesse assumsi, quod quidem semper ponere licet, suspicatus sum semper dari aequationem uno gradu inferiorem, veluti

$$y^n + \mathfrak{A}y^{n-2} + \mathfrak{B}y^{n-3} + \mathfrak{C}^{n-4}y + \text{etc.} = 0,$$

quam illius resolventem appellavi, ita ut, si huius constent omnes radices, quae sint

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \text{etc.},$$

quarum numerus est $n-1$, ex iis radix illius aequationis ita exprimatur, ut sit

$$x=\sqrt[n]{\alpha}+\sqrt[n]{\gamma}+\sqrt[n]{\gamma}+\sqrt[n]{\delta}+\sqrt[n]{\varepsilon}+\text{etc.}$$

Quam coniecturam confirmavi ostendens resolutionem aequationum inferiorum revera ex hac forma generali deduci; neque etiamnunc dubito, quin haec coniectura veritati sit consentanea.

4. Praeterquam autem quod inventio aequationis resolventis, si proposita quartum gradum transcendit, fit difficillima atque adeo in genere vires nostras aeque superare videtur atque ipsa propositae aequationis resolutio, ita ut praeter formas speciales casibus Moivreanis similes nobis nihil admodum suppeditet, alia insuper incommoda in illa forma observovi, quae me eo induxerunt, ut arbitrarer aliam forte dari formam illi non admodum dissimilem, quae istis incommodis non esset subiecta ideoque maiorem spem nobis faceret in hoc arduo Algebrae opere tandem ulterius penetrandi. Non parum autem in hoc negotio proderit veram formam radicum cuiusque aequationis accuratius perspexisse.

5. In forma autem per superiorem coniecturam eruta hoc imprimis desidero, quod omnes aequationis propositae radices non satis distincte exprimantur. Etsi enim quodvis signum radicale $\sqrt[n]{\alpha}$ tot valores diversos complectitur, quot numerus n continet unitates, ita ut, si

$$\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},\ \mathfrak{e},\ \text{etc.}$$

omnes valores formulae $\sqrt[n]{1}$ denotent, pro $\sqrt[n]{\alpha}$ scribere liceat quamlibet harum formularum

$$\mathfrak{a}\sqrt[n]{\alpha},\ \mathfrak{b}\sqrt[n]{\alpha},\ \mathfrak{c}\sqrt[n]{\alpha},\ \mathfrak{d}\sqrt[n]{\alpha},\ \text{etc.},$$

tamen manifestum [est] hanc variationem in singulis terminis $\sqrt[n]{\alpha},\ \sqrt[n]{\beta},\ \sqrt[n]{\gamma},\ \sqrt[n]{\delta},$ etc., non pro lubitu constitui posse. Si enim combinatio horum terminorum cum litteris $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},\ \mathfrak{e},$ etc. arbitrio nostro relinqueretur, tum multo plures combinationes resultarent, quam aequatio continet radices, quarum numerus est $=n$.

6. Quo igitur forma radicis x supra exhibita omnes aequationis radices simul complectatur, necesse est, ut combinationes terminorum $\sqrt[n]{\alpha},\ \sqrt[n]{\beta},\ \sqrt[n]{\gamma},\ \sqrt[n]{\delta},$ etc., cum litteris $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},$ etc. certo quodam modo circumscribantur atque combinationes, quae ad aequationis radices repraesentandas sunt ineptae, excludantur. Ex resolutione quidem aequationum terth et quarti gradus vidimus inter radices unitatis eiusdem nominis $\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},$ etc. certum quendam ordinem constitui debere, secundum quem etiam combinationes sint perficiendae. Hunc in finem autem similis ordo in ipsis radicis membris $\sqrt[n]{\alpha},\ \sqrt[n]{\beta},\ \sqrt[n]{\gamma},\ \sqrt[n]{\delta},$ etc., erit tenendus, quo combinatio dirigatur. Verum quia non constat, quemadmodum in radicibus superiorum graduum talis ordo sit

constituendus, hoc sine dubio insigne est incommodum, quo forma coniecturae meae innixa laborat, quod igitur removere in hac dissertatione mihi est propositum.

7. Primum autem conveniet ordinem certum in radicibus cuiusvis potestatis ex unitate constituere, quo summa plerumque varietas combinationum restringatur. Quem in finem observo, si praeter unitatem alius quicumque valor ipsius $\sqrt[n]{1}$ sit $=\mathfrak{a}$, tum etiam $\mathfrak{a}^2, \mathfrak{a}^3, \mathfrak{a}^4$ etc. idoneos valores ipsius $\sqrt[n]{1}$ exhibere; nam si sit $a''= 1$, erit quoque $(\mathfrak{a}^2)^n=1, (\mathfrak{a}^3)^n=1, (\mathfrak{a}^4)^n=1$ etc. Hinc si reliquae radices ponantur $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, etc., quoniam in iis reperiuntur $\mathfrak{a}^2, \mathfrak{a}^3, \mathfrak{a}^4$ etc., iam certus quidam ordo perspicitur, quo hae litterae inter se disponi debent.
Ita si post unitatem, quae semper primum locum tenere censenda est, a littera $\mathfrak{a}$ incipiamus, valores formulae $\sqrt[n]{1}$ erunt

$$1, \mathfrak{a}, \mathfrak{a}^2, \mathfrak{a}^3, \mathfrak{a}^4 \ldots\ldots \mathfrak{a}^{n-1},$$

quorum numerus est *n;* plures enim occurrere nequeunt, cum sit $\mathfrak{a}^n=1, \mathfrak{a}^{n+1}=\mathfrak{a}, \mathfrak{a}^{n+2}=\mathfrak{a}^2$, etc.; similique modo res se habebit, si post unitatem a quavis alia littera $\mathfrak{b}$ vel $\mathfrak{c}$ vel $\mathfrak{d}$ vel c vel b etc. incipiamus.

8. Hinc ergo merito suspicor talem quoque ordinem in ipsis terminis radicem aequationis *x* exprimentibus inesse seu singula membra radicalia ita esse comparata, ut respectu uniuscuiusque reliquae sint eius potestates; singulis autem membris nunc necesse erit coefficientes indefinitos tribuere. Quare si aequatio termino secundo destituta fuerit

$$x^n + \mathrm{A}x^{n-2} + \mathrm{B}x^{n-3} + \mathrm{C}^{n-4} + \mathrm{D}^{n-5} + \text{etc.} = 0,$$

maxime probabile videtur radicem quamlibet huius aequationis ita exprimi, ut sit

$$x = \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{D}\sqrt[n]{v^3} + \ldots\ldots \mathfrak{O}\sqrt[n]{v^{n-1}},$$

ubi $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. sint quantitates vel rationales vel saltem non signum radicale $\sqrt[n]{\ }$ involvant, quippe quod tantum quantitatem *v* eiusque potestates afficiat; multo minus ipsa quantitas *v* tale signum involvat.

9. Ex hac forma primum patet eam non plura membra, quam quorum numerus sit $n-1$, continere posse; nam etiamsi seriem illam ex sua indole ulterius continuemus, termini sequentes iam in praecedentibus contenti deprehendentur.
Erit enim

$$\sqrt[n]{v^{n+1}} = v\sqrt[n]{v}, \quad \sqrt[n]{v^{n+2}} = v\sqrt[n]{v^2} \quad \text{etc.},$$

ita ut irrationalitas signum radicale $\sqrt[n]{\ }$ involvens plures diversas species non admittat, quam quarum numerus est $= n-1$. Etiamsi ergo illa series in infinitum continuetur, tamen terminos eiusdem speciei ratione irrationalitatis addendo omnes ad terminos numero $n-1$ redigentur. Cum igitur iam ante viderimus plures terminos in radicis expressionem non ingredi, hinc non leve argumentum habetur hanc novam formam veritati plane esse consentaneam; eius autem veritas per sequentia argumenta multo magis confirmabitur.

10. Haec expressio quoque sponte se extendit ad aequationes, in quibus secundus terminus non deest, dum superior remotionem secundi termini exigebat, ex quo ipso haec nova magis naturalis est aestimanda. Continuatio enim terminorum irrationalium $\sqrt[n]{v}$, $\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$ etc. etiam terminos rationales $\sqrt[n]{v^0}$, $\sqrt[n]{v^n}$ involvit, qui ob aequationis terminum secundum adiici debent. Hinc generalius pronunciare poterimus, si aequatio completa ordinis cuiusque n fuerit proposita

$$x^n + \Delta x^{n-1} + Ax^{n-2} + Bx^{n-3} + Cx^{n-4} + \text{etc.} = 0,$$

eius radicem exprimi huiusmodi forma

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \mathfrak{D}\sqrt[n]{v^4} + \ldots\ldots \mathfrak{O}\sqrt[n]{v^{n-1}},$$

ubi ω partem radicis rationalem exhibet, quam constat esse $= -\frac{1}{n}\Delta$. Reliqui autem termini continent partes irrationales radicem potestatis n involventes, quarum, quatenus sunt diversae, numerus excedera nequit $n-1$, omnino uti per formam superiorem intelligitur.

11. Hinc porro videmus, si v fuerit eiusmodi quantitas, ut ex ea radix potestatis n actu extrahi seu $\sqrt[n]{v}$ vel rationaliter vel per signa radicalia inferiorum potestatum exprimi queat, tum irrationalitatem gradus n prorsus ex forma radicis egredi. Hoc autem necessario usu venire debet, quoties aequatio proposita in factores est resolubilis; tum enim nulla radix signum radicale $\sqrt[n]{}$ continebit. Quare cum natura rei postulet, ut his casibus omnia signa radicalia $\sqrt[n]{}$ evanescant et ad signa simpliciora reducantur, ex forma autem superiori non pateat, quomodo evanescente uno huiusmodi signo $\sqrt[n]{\alpha}$ reliqua $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$ etc. evanescant, ista expressio ob hanc rationem multo magis ad aequationum naturam accommodata est censenda.

12. Praeterea vero haec forma, in quo cardo totius negotii versatur, etiam omnes aequationis radices sine ulla ambiguitate ostendit; neque enim amplius haeremus, quomodo cum omnibus signis radicalibus $\sqrt[n]{}$ totidem valores radicis $\sqrt[n]{1}$ combinandi sint. Si enim omnes radices potestatis n ex unitate sint 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. ac $\sqrt[n]{v}$ cum earum quacumque $\mathfrak{a}$ combinaverimus, propterea quod $\sqrt[n]{v}$ utique est $\mathfrak{a}\sqrt[n]{v}$, tum pro

$\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$, $\sqrt[n]{v^4}$ etc. scribere oportebit $\mathfrak{a}^2\sqrt[n]{v^2}$, $\mathfrak{a}^3\sqrt[n]{v^3}$, $\mathfrak{a}^4\sqrt[n]{v^4}$ etc. Terminus autem constans ω, quia formam $\omega\sqrt[n]{v^0}$ repraesentat, abibit in $\mathfrak{a}^0\omega\sqrt[n]{v^0}=1\cdot\omega$ ob $\mathfrak{a}^0=1$ ideoque in omnibus radicibus nullam mutationem subit quemadmodum reliqua membra. Quod cum ex resolutione omnium aequationum per se sit manifestum, hinc novum ac satis luculentum habemus criterium veritatis huius novae formae, quae omnium aequationum radices in se complecti videtur.

13. Hinc autem porro manifestum est, quomodo una cuiusque aequationis radice cognita reliquae radices omnes exhiberi queant; ad hoc tantum nosse oportet omnes radices eiusdem potestatis ex unitate seu omnes valores ipsius $\sqrt[n]{1}$, quorum numerus $=n$. Ac si istae unitatis radices fueririt $1, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, etc. aequationisque una radix inventa sit

$$x=\omega+\mathfrak{A}\sqrt[n]{v}+\mathfrak{B}\sqrt[n]{v^2}+\mathfrak{C}\sqrt[n]{v^3}+\ldots\ldots\mathfrak{D}\sqrt[n]{v^{n-1}},$$

radices reliquae erunt

$$x=\omega+\mathfrak{A}\mathfrak{a}\sqrt[n]{v}+\mathfrak{B}\mathfrak{a}^2\sqrt[n]{v^2}+\mathfrak{C}\mathfrak{a}^3\sqrt[n]{v^3}+\ldots\ldots\mathfrak{D}\mathfrak{a}^{n-1}\sqrt[n]{v^{n-1}},$$
$$x=\omega+\mathfrak{A}\mathfrak{b}\sqrt[n]{v}+\mathfrak{B}\mathfrak{b}^2\sqrt[n]{v^2}+\mathfrak{C}\mathfrak{b}^3\sqrt[n]{v^3}+\ldots\ldots\mathfrak{D}\mathfrak{b}^{n-1}\sqrt[n]{v^{n-1}},$$
$$x=\omega+\mathfrak{A}\mathfrak{c}\sqrt[n]{v}+\mathfrak{B}\mathfrak{c}^2\sqrt[n]{v^2}+\mathfrak{C}\mathfrak{c}^3\sqrt[n]{v^3}+\ldots\ldots\mathfrak{D}\mathfrak{c}^{n-1}\sqrt[n]{v^{n-1}},$$
etc.

sicque semper tot obtinentur radices, quot exponens *n,* qui aequationis gradum designat, continet unitates.

14. His igitur argumentis nova haec radicum forma iam ad summum probabilitatis est evecta; atque ad plenam certitudinem ostendendam nihil aliud requiritur, nisi ut regula inveniatur, cuius ope pro quavis aequatione proposita ista forma definiri et coefficientes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. cum quantitate v assignari queant; quod si praestare possemus, haberemus sine dubio generalem omnium aequationum resolutionem irrito adhuc omnium Geometrarum labore requisitam. Neque igitur equidem tantum mihi tribuo, ut hanc regulam me invenire posse credam, sed contentus ero plene demonstrasse omnium aequationum radices certo in hac forma esse contentas. Hoc autem sine dubio plurimum luminis foenerabitur ad resolutionem aequationum, cum cognita radicum vera forma via investigationis non mediocriter facilior reddatur, quam ne ingredi quidam licet, quamdiu forma radicum fuerit incognita.

15. Quamquam autem ex ipsa aequatione proposita nobis adhuc non licet radicem eius seu coefficientes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. cum quantitate v assignare, tamen demonstratio veritatis aeque succedet, si vicissim ex assumta radice illam aequationem, cuius est radix, eliciamus. Haec autem aequatio libera esse debet a signis radicalibus $\sqrt[n]{}$, quoniam

aequationes, quarum radices investigantur, ex terminis rationalibus constare assumi solent. Quaestio ergo huc reducitur, ut huiusmodi aequatio

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \ldots\ldots\mathfrak{O}\sqrt[n]{v^{n-1}},$$

ab irrationalitate seu signis radicalibus $\sqrt[n]{\ }$ liberetur atque aequatio rationalis inde deducatur, de qua deinceps certo affirmare poterimus eius radicem esse ipsam expressionem assumtam; simulque inde reliquas radices, quae eidem aequationi aeque conveniunt, assignare valebimus. Hoc ergo modo saltem infinitas aequationes exhibere poterimus, quarum radices nobis erunt cognitae, atque si hae aequationes in se complectantur omnium graduum aequationes generales, etiam harum resolutio in nostra erit potestate.

16. Parum quidem a nobis praestitum iri videbitur, si tantum plures aequationes, quarum radices assignari queant, exhibuerimus, cum ex primis elementis constet, quomodo cuiusvis gradus aequatio formari debeat, quae datas habeat radices. Si enim quotcumque huiusmodi formulae $x-a,\ x-b,\ x-c$ etc. in se invicem multiplicentur, obtinebitur utique aequatio, cuius radices futurae sunt $x=a,\ x=b,\ x=c$ etc.; sed talis aequationis formatio parum lucri affert ad resolutionem aequationum. Primum autem observo hoc modo alias aequationes non nasci, nisi quae sint habiturae factores; aequationum autem, quae in factores resolvi possunt, resolutio nulla laborat difficultate. Haud maioris quoque momenti sunt in hoc negotio aequationes, quae ex multiplicatione duarum pluriumve inferiorum aequationum producuntur, quarum resolutio nihil plane prodest ad resolutionem generalem perficiendam.

17. Quodsi autem ex nostra forma

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \text{etc.}$$

ad aequationem rationalem perveniamus, ea certo factores rationales non abebit; si enim haberet, eius radices, quae simul assent radices aequationum inferiorum graduum, signum radicale $\sqrt[n]{\ }$ non implicarent. Plurimum is praestare censendus est, qui aequationis cuiuspiam altioris gradus, quae in factores resolvi nequeat, radices assignaverit. Quamobrem etiam Cel. Moivreo ingentes debentur gratiae, quod ex singulis aequationum gradibus unam exhibuerit in factores irresolubilem, cuius radices assignari possunt; atque si eius formulae latius paterent, multo maiorem sine dubio essent habiturae utilitatem, dum contra aequationibus in factores resolubilibus in hoc negotio nihil plane emolumenti attribui potest.

18. Verum revertamur ad illam formam ab irrationalitate signi $\sqrt[n]{\ }$ liberandam, ac si consuetas methodos signa radicalia eliminandi consulamus, aequatio resultans ad plurimas dimensiones plerumque ascendere videatur. Si enim unicum adesset signum radicale, puta

$$x = \omega + \mathfrak{A}\sqrt[n]{v},$$

aequatio rationalis ad n dimensiones ipsius x ascenderet, unde ea ad multo plures dimensiones ascensura videtur, si plura eiusmodi adsint signa radicalia; id quod sine dubio evenire deberet, si illa signa radicalia a se invicem prorsus non penderent. Sed quia omnia sunt potestates primi, ostendam perfectam rationalitatem obtineri posse non ultra potestatem exponentis n ascendendo. Ita scilicet docebo formam

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \ldots\ldots\mathfrak{O}\sqrt[n]{v^{n-1}},$$

ita ab irrationalitate liberari posse, ut aequatio rationalis inde resultans potestatem x^n non superet. Prodibit ergo aequatio huius formae

$$x^n + \Delta x^{n-1} + Ax^{n-2} + Bx^{n-3} + \text{etc.} = 0,$$

cuius radix erit illa forma assumta; et quia radicum huius aequationis numerus est *n,* ex eadem forma omnes huius aequationis radices assignare poterimus.

19. Cum hoc iam sit eximium criterium veritatis huius formae, tum etiam annotasse iuvabit, quoniam forma radicis $n-1$ quantitates arbitrarias continet, totidem quoque quantitates arbitrarias in aequationem rationalem ingredi, unde perspicuum est istas quantitates ita determinari posset, ut aequatio rationalis inde datos coefficientes Δ, A, B, C etc. obtineat, hoc est, ut aequatio generalis huius gradus obtineatur. Quae determinatio si actu institui queat, nanciscemur inde resolutionem generalem aequationum cuiuscumque gradus, ex quo saltem possibilitas resolutionis hoc modo perficiendae elucet. Difficultates quidem insignes in hoc negotio occurrent, quas eo clarius agnoscemus, si nostram formam ad quemvis gradum a simplicissimis incipiendo accommodemus. Simplicitati autem et concinnitati calculi consulentes partem radicis rationalem ω omittamus, ut in quovis gradu ad eiusmodi aequationes rationales pertingamus, in quibus secundus terminus desit, quo ipso amplitudo resolutionis non restringi est censenda.

I. RESOLUTIO AEQUATIONUM SECUNDI GRADUS

20. Ut igitur ab aequationibus secundi gradus incipiamus, sit $n = 2$ et posito $\omega = 0$ forma nostra radicis erit

$$x = \mathfrak{A}\sqrt{v},$$

quae rationalis facta dat

$$xx = \mathfrak{A}\mathfrak{A}v.$$

Comparetur haec aequatio cum forma generali secundi gradus

$$xx = A$$

deficiente secundo termino sitque

$$\mathfrak{A}\mathfrak{A}v = A;$$

cui ut satisfiat, statuatur $\mathfrak{A} = 1$ eritque

$$v = A.$$

Unde proposita aequatione $xx = A$ si sumatur $\mathfrak{A} = 1$ et $v = A$, eius radix una erit

$$x = \mathfrak{A}\sqrt{v} = \sqrt{A},$$

et quia $\sqrt{1}$ duos habet valores 1 et -1, altera radix erit

$$x = -\mathfrak{A}\sqrt{v} = -\sqrt{A},$$

quod quidem per se est perspicuum.

II. RESOLUTIO AEQUATIONUM TERTH GRADUS

21. Posito iam $n = 3$ forma radicis pro hoc casu erit

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2};$$

unde ut aequatio rationalis eruatur, sumatur primo cubus

$$x^3 = v\mathfrak{A}^3 + 3\mathfrak{A}\mathfrak{A}\mathfrak{B}v\sqrt[3]{v} + 3\mathfrak{A}\mathfrak{B}\mathfrak{B}v\sqrt[3]{v^2} + \mathfrak{B}^3v^2.$$

Fingatur iam haec aequatio cubica

$$x^3 = Ax + B,$$

unde pro x valorem assumtum substituendo orietur quoque

$$x^3 = A\mathfrak{A}\sqrt[3]{v} + A\mathfrak{B}\sqrt[3]{v^2} + B,$$

quae forma illi aequalis est reddenda aequandis inter se tam partibus rationalibus quam irrationalibus utriusque speciei $\sqrt[3]{v}$ et $\sqrt[3]{v^2}$.

22. Comparatio autem terminorum rationalium praebet

$$B = \mathfrak{A}^3v + \mathfrak{B}^3v^2$$

et ex collatione irrationalium fit

$$A\mathfrak{A} = 3\mathfrak{A}\mathfrak{A}\mathfrak{B}v \text{ et } A\mathfrak{B} = 3\mathfrak{A}\mathfrak{B}\mathfrak{B}v,$$

quarum utraque dat

$$A = 3\mathfrak{A}\mathfrak{B}v.$$

Hinc, si ista aequatio cubica fuerit proposita

$$x^3 = 3\mathfrak{A}\mathfrak{B}vx + \mathfrak{A}^3 v + \mathfrak{B}^3 v^2,$$

eius radix una erit

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2},$$

et si $1,\ \mathfrak{a},\ \mathfrak{b}$ sint tres radices cubicae unitatis, duae reliquae radices erunt

$$x = \mathfrak{A}\mathfrak{a}\sqrt[3]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[3]{v^2},\ \ x = \mathfrak{A}\mathfrak{b}\sqrt[3]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[3]{v^2}.$$

Est autem

$$\mathfrak{a} = \mathfrak{b}^2 = \tfrac{-1+\sqrt{-3}}{2} \ \text{ et } \ \mathfrak{b} = \mathfrak{a}^2 = \tfrac{-1-\sqrt{-3}}{2}.$$

23. Possunt autem vicissim, si aequatio cubica proponatur

$$x^3 = Ax + B,$$

ex coefficientibus A et B quantitates $\mathfrak{A}$, $\mathfrak{B}$ et v determinari, ut inde omnes tres huius aequationis radices obtineantur. Hunc autem in finem, quia tantum duae aequationes adimplendae habentur, una litterarum $\mathfrak{A}$ et $\mathfrak{B}$ pro lubitu assumi potest. Sit igitur $\mathfrak{A} = 1$ et aequatio

$$A = 3\mathfrak{A}\mathfrak{B}v = 3\mathfrak{B}v$$

praebet

$$\mathfrak{B} = \tfrac{A}{3v},$$

unde fit

$$\mathfrak{B}^3 = \tfrac{A^3}{27v^3},$$

qui valor in prima aequatione

$$B = v + \mathfrak{B}^3 v^2$$

substitutus dat

$$B = v + \tfrac{A^3}{27v} \ \text{ seu } \ vv = Bv - \tfrac{1}{27}A^3,$$

unde fit

$$v = \tfrac{1}{2}B \pm \sqrt{(\tfrac{1}{4}BB - \tfrac{1}{27}A^3)};$$

perinde autem est, uter horum duorum valorum assumatur.

24. Invento autem valore ipsius $v=\frac{1}{2}B\pm\sqrt{(\frac{1}{4}BB-\frac{1}{27}A^3)}$ erit $\mathfrak{B}=\frac{A}{3v}$, et $\mathfrak{B}\sqrt[3]{v^2}=\frac{A}{3\sqrt[3]{v}}$ Hincque tres aequationis propositae

$$x^3=Ax+B$$

erunt radices

$$\text{I. } x=\sqrt[3]{v}+\frac{A}{3\sqrt[3]{v}},\quad \text{II. } x=\mathfrak{a}\sqrt[3]{v}+\frac{\mathfrak{b}A}{3\sqrt[3]{v}},\quad \text{III. } x=\mathfrak{b}\sqrt[3]{v^2}+\frac{\mathfrak{a}A}{3\sqrt[3]{v}}.$$

Cum autem sit

$$\frac{1}{v}=\frac{\frac{1}{2}B\mp\sqrt{(\frac{1}{4}BB-\frac{1}{27}A^3)}}{\frac{1}{27}A^3},$$

erit

$$\sqrt[3]{v}=\sqrt[3]{\left(\frac{1}{2}B\pm\sqrt{(\frac{1}{4}BB-\frac{1}{27}A^3)}\right)}$$

et

$$\frac{A}{3\sqrt[3]{v}}=\sqrt[3]{\left(\frac{1}{2}B\mp\sqrt{(\frac{1}{4}BB-\frac{1}{27}A^3)}\right)},$$

hincque nascuntur formulae vulgares pro resolutione aequationum cubicarum.

III. RESOLUTIO AEQUATIONUM QUARTI GRADUS

25. Posito $n=4$ consideremus hanc radicis formam

$$x=\mathfrak{A}\sqrt[4]{v}+\mathfrak{B}\sqrt[4]{v^2}+\mathfrak{C}\sqrt[4]{v^3}$$

et quaeramus aequationem quarti gradus, cuius haec forma sit radix. Atque hoc quidem casu calculus facile instituitur, quo irrationalitates tolluntur; nam ob $\sqrt[4]{v^2}=\sqrt{v}$ sumatur haec aequatio

$$x-\mathfrak{B}\sqrt{v}=\mathfrak{A}\sqrt[4]{v}+\mathfrak{C}\sqrt[4]{v^3},$$

quae quadrata dat

$$xx-2\mathfrak{B}x\sqrt{v}+\mathfrak{B}\mathfrak{B}v=\mathfrak{A}\mathfrak{A}\sqrt{v}+2\mathfrak{A}\mathfrak{C}v+\mathfrak{C}\mathfrak{C}v\sqrt{v},$$

quae partibus irrationalibus ad eandem partem translatis fit

$$xx+(\mathfrak{B}\mathfrak{B}-2\mathfrak{A}\mathfrak{C})v=2\mathfrak{B}x\sqrt{v}+(\mathfrak{A}\mathfrak{A}+\mathfrak{C}\mathfrak{C}v)\sqrt{v}$$

et sumtis denuo quadratis prodibit haec aequatio rationalis

$$x^4+2(\mathfrak{BB}-2\mathfrak{AC})vxx+(\mathfrak{BB}-2\mathfrak{AC})^2vv$$
$$=4\mathfrak{BB}vxx+4(\mathfrak{AA}+\mathfrak{CC}v)\mathfrak{B}vx+(\mathfrak{AA}+\mathfrak{CC}v)^2v,$$

quae ordinata abit in hanc formam

$$x^4=2(\mathfrak{BB}+2\mathfrak{AC})vxx+4(\mathfrak{AA}+\mathfrak{CC}v)\mathfrak{B}vx$$
$$+\mathfrak{A}^4v-\mathfrak{B}^4vv+\mathfrak{C}^4v^3+4\mathfrak{ABBC}vv-2\mathfrak{AACC}vv.$$

26. Huius igitur aequationis biquadraticae radix una est

$$x=\mathfrak{A}\sqrt[4]{v}+\mathfrak{B}\sqrt[4]{v^2}+\mathfrak{C}\sqrt[4]{v^3}$$

ac si radices biquadratae unitatis ponantur $1,\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},$ ita ut sit

$$\mathfrak{a}=+\sqrt{-1},\ \mathfrak{b}=-1,\ \text{et}\ \mathfrak{c}=-\sqrt{-1},$$

erit

$$\mathfrak{a}^2=-1=\mathfrak{b},\qquad \mathfrak{a}^3=-\sqrt{-1}=\mathfrak{c},$$
$$\mathfrak{b}^2=+1,\qquad \mathfrak{b}^3=-1=\mathfrak{b},$$
$$\mathfrak{c}^2=-1=\mathfrak{b},\qquad \mathfrak{c}^3=+\sqrt{-1}=\mathfrak{a},$$

unde tres reliquae radices eiusdem aequationis erunt

$$x=\mathfrak{Aa}\sqrt[4]{v}+\mathfrak{Bb}\sqrt[4]{v^2}+\mathfrak{Cc}\sqrt[4]{v^3},$$
$$x=\mathfrak{Ab}\sqrt[4]{v}+\mathfrak{B}\sqrt[4]{v^2}+\mathfrak{Cb}\sqrt[4]{v^3},$$
$$x=\mathfrak{Ac}\sqrt[4]{v}+\mathfrak{Bb}\sqrt[4]{v^2}+\mathfrak{Ca}\sqrt[4]{v^3}.$$

27. Hinc autem vicissim aequatio biquadrata quaecumque ad illam formam reduci eiusque radices assignari poterunt. Sit enim proposita haec aequatio

$$x^4=Axx+Bx+C$$

et quaeri oportet valores coefficientium $\mathfrak{A},\mathfrak{B},\mathfrak{C}$ cum quantitate *v,* quibus inventis simul huius aequationis radices innotescent. Erit autem

$$A = 2(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v,$$
$$B = 4(\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)\mathfrak{B}v,$$
$$C = \mathfrak{A}^4 v - \mathfrak{B}^4 vv + \mathfrak{C}^4 v^3 + 4\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv - 2\mathfrak{A}\mathfrak{A}\mathfrak{C}\mathfrak{C}vv$$

seu

$$C = (\mathfrak{A}\mathfrak{A} + \mathfrak{A}\mathfrak{A}v)^2 v - (\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})^2 vv + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv.$$

Illinc autem est

$$(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v = \tfrac{1}{2}A$$

et

$$\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v = \tfrac{B}{4\mathfrak{B}v},$$

qui valores hic substituti dant

$$C = \tfrac{BB}{16\mathfrak{B}\mathfrak{B}v} - \tfrac{1}{4}AA + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv.$$

Prima autem formula praebet $4\mathfrak{A}\mathfrak{C}v = A - 2\mathfrak{B}\mathfrak{B}v$, qui valor denuo substitutus dat

$$C = \tfrac{BB}{16\mathfrak{B}\mathfrak{B}v} - \tfrac{1}{4}AA + 2A\mathfrak{B}\mathfrak{B}v - 4\mathfrak{B}^4 vv,$$

ita ut iam duae litterae $\mathfrak{A}$ et $\mathfrak{C}$ sint eliminatae.

28. Quia hic adhuc duae incognitae $\mathfrak{B}$ et v supersunt, valor ipsius $\mathfrak{B}$ arbitrio nostro relinquitur. Sit igitur $\mathfrak{B} = 1$ et quantitas v ex sequenti aequatione cubica determinari debebit

$$v^3 - \tfrac{1}{2}Av^2 + \tfrac{1}{4}(C + \tfrac{1}{4}AA)v - \tfrac{1}{64}BB = 0.$$

Inventa autem hinc radice v ex prioribus aequationibus quaeri debent litterae $\mathfrak{A}$ et $\mathfrak{C}$ Cum igitur sit

$$\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v = \tfrac{B}{4v} \text{ et } 2\mathfrak{A}\mathfrak{C}\sqrt{v} = \tfrac{A-2v}{2\sqrt{v}},$$

erit tam addendo quam subtrahendo et radicem quadratam extrahendo

$$\mathfrak{A} + \mathfrak{C}\sqrt{v} = \sqrt{\left(\tfrac{B}{4v} + \tfrac{A}{2\sqrt{v}} - \sqrt{v}\right)}$$

et

$$\mathfrak{A} - \mathfrak{C}\sqrt{v} = \sqrt{\left(\tfrac{B}{4v} - \tfrac{A}{2\sqrt{v}} + \sqrt{v}\right)},$$

unde reperietur

$$\mathfrak{A} = \frac{1}{4\sqrt{v}}\sqrt{\left(B+2A\sqrt{v}-4v\sqrt{v}\right)} + \frac{1}{4\sqrt{v}}\sqrt{\left(B-2A\sqrt{v}+4v\sqrt{v}\right)}$$

et

$$\mathfrak{C} = \frac{1}{4\sqrt{v}}\sqrt{\left(B+2A\sqrt{v}-4v\sqrt{v}\right)} - \frac{1}{4\sqrt{v}}\sqrt{\left(B-2A\sqrt{v}+4v\sqrt{v}\right)}.$$

29. Cum sit

$$\mathfrak{A}\sqrt[4]{v} \pm \mathfrak{C}\sqrt[4]{v^3} = (\mathfrak{A} \pm \mathfrak{C}\sqrt{v})\sqrt[4]{v},$$

erunt aequationis propositae

$$x^4 = Axx + Bx + C,$$

postquam valor v inventus fuerit ex aequatione

$$v^3 - \tfrac{1}{2}Av^2 + \tfrac{1}{4}(C + \tfrac{1}{4}AA)v - \tfrac{1}{64}BB = 0,$$

quatuor radices

$$\text{I.} \quad x = \sqrt{v} + \frac{1}{2\sqrt{v}}\sqrt{(B\sqrt{v} + 2Av - 4vv)},$$

$$\text{II.} \quad x = \sqrt{v} - \frac{1}{2\sqrt{v}}\sqrt{(B\sqrt{v} + 2Av - 4vv)},$$

$$\text{III.} \quad x = -\sqrt{v} + \frac{1}{2\sqrt{v}}\sqrt{(-B\sqrt{v} + 2Av - 4vv)},$$

$$\text{IV.} \quad x = -\sqrt{v} - \frac{1}{2\sqrt{v}}\sqrt{(-B\sqrt{v} + 2Av - 4vv)}.$$

Hocque modo, ut constat, resolutio aequationis biquadraticae ad resolutionem aequationis cubicae reducitur.

IV. RESOLUTIO AEQUATIONUM QUINTI GRADUS

30. Posito $n = 5$ erit forma nostra radicis

$$x = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{vv} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4}$$

ac primo quaeri debet aequatio quinti gradus, cuius haec futura sit radix, seu, quod eodem redit, ex hac forma signa radicalia eliminari oportet. In hoc autem ipso summa occurrit difficultas, cum operatio haec eliminationis neutiquam eo modo, quo in aequationibus quarti gradus sum usus, institui queat. Manifestum quidem est, quia omnes potestates ipsius x eadem signa radicalia involvunt, si aequatio quaesita fingatur

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

tum substituendo pro x valorem assumtum quatuor obtineri aequationes, quarum ope quaterna signa radicalia eliminari liceat; sed tum litterae hae assumtae *A, B, C* et *D* singulae difficillime determinabuntur.

31. His difficultatibus perpensis in alium incidi modum hanc operationem instituendi, qui ita est comparatus, ut ad omnes radicum formas, cuiuscumque sint gradus, aeque pateat, et ex quo simul perspicietur aequationem rationalem numquam ultra gradum, qui exponente n indicatur, esse ascensuram. Hic autem modus innititur ipsi naturae aequationum, qua singulorum terminorum coefficientes ex omnibus radicibus definiuntur. Cum igitur omnes quinque radices aequationis, quam quaerimus, constent, ex iis quoque coefficientes singulorum terminorum eius formari possunt per regulas cognitas. Sint igitur $1,\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d}$ quinque radices surdesolidae unitatis seu radices huius aequationis $x^5-1=0$ ac ponendo $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ pro radicibus aequationis, quam quaerimus, erit

$$\begin{aligned}
\alpha &= \mathfrak{A}\sqrt[5]{v}+\mathfrak{B}\sqrt[5]{v^2}+\mathfrak{C}\sqrt[5]{v^3}+\mathfrak{D}\sqrt[5]{v^4},\\
\beta &= \mathfrak{A}\mathfrak{a}\sqrt[5]{v}+\mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4},\\
\gamma &= \mathfrak{A}\mathfrak{b}\sqrt[5]{v}+\mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4},\\
\delta &= \mathfrak{A}\mathfrak{c}\sqrt[5]{v}+\mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4},\\
\varepsilon &= \mathfrak{A}\mathfrak{d}\sqrt[5]{v}+\mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2}+\mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3}+\mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}.
\end{aligned}$$

32. His quinque radicibus expositis si aequatio quinti gradus has radices habens statuatur

$$x^5-\Delta x^4+Ax^3-Bx^2+Cx-D=0,$$

hi coefficientes ex radicibus $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ ita definiuntur, ut sit

$\Delta =$ summae radicum,
$A =$ summae productorum ex binis,
$B =$ summae productorum ex ternis,
$C =$ summae productorum ex quaternis,
$D =$ producto ex omnibus quinis.

Quo autem hos valores facilius colligere queamus, eos ex summis potestatum radicum concludamus. Sit igitur

$$P = \alpha + \beta + \gamma + \delta + \varepsilon,$$
$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2,$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3,$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4,$$
$$T = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5.$$

His enim valoribus definitis erit, uti novimus,

$$\Delta = P,$$
$$A = \frac{\Delta P - Q}{2},$$
$$B = \frac{AP - \Delta Q + R}{3},$$
$$C = \frac{BP - AQ + \Delta R - S}{4},$$
$$D = \frac{CP - BQ + AR - \Delta S + T}{5}.$$

33. Iam ad valores *P, Q, R, S, T* investigandos debemus prius radicum unitatis $1,\ \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d}$ omnes potestates in unam summam redigere; quae cum sint radices aequationis $z^5 - 1 = 0$, erit

$$1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} = 0,$$
$$1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2 = 0,$$
$$1 + \mathfrak{a}^3 + \mathfrak{b}^3 + \mathfrak{c}^3 + \mathfrak{d}^3 = 0,$$
$$1 + \mathfrak{a}^4 + \mathfrak{b}^4 + \mathfrak{c}^4 + \mathfrak{d}^4 = 0,$$
$$1 + \mathfrak{a}^5 + \mathfrak{b}^5 + \mathfrak{c}^5 + \mathfrak{d}^5 = 5.$$

Summae potestatum sextarum, septimarum etc. usque ad decimas iterum evanescunt, at decimarum summa iterum fit = 5, cum sit $\mathfrak{a}^5 = 1,\ \mathfrak{b}^5 = 1,\ \mathfrak{c}^5 = 1,$ et $\mathfrak{d}^5 = 1.$ Brevitatis gratia in hoc calculo poterimus signa radicalia plane omittere, dummodo deinceps recordemur cum litteris coniungenda $\sqrt[5]{v},\ \sqrt[5]{v^2},\ \sqrt[5]{v^3},\ \sqrt[5]{v^4}.$

34. Nunc igitur addendis radicibus $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ habebimus

$$P = \mathfrak{A}(1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d}) + \mathfrak{B}(1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2) + \text{etc.} = 0.$$

Reliquis autem potestatibus sumendis eliciemus insuper

$$P = 0,$$
$$Q = 10(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C}),$$
$$R = 15(\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 + \mathfrak{C}^2\mathfrak{D}),$$
$$S = 20(\mathfrak{A}^3\mathfrak{B} + \mathfrak{A}\mathfrak{C}^3 + \mathfrak{B}^3\mathfrak{D} + \mathfrak{C}\mathfrak{D}^3) + 30(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2) + 120\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D},$$
$$T = 5(\mathfrak{A}^5 + \mathfrak{B}^5 + \mathfrak{C}^5 + \mathfrak{D}^5) + 100(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D} + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3)$$
$$+150(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}^2\mathfrak{D}).$$

Hic alia producta non occurrunt, nisi quae adiungendis signis radicalibus potestatem ipsius v rationalem producunt; seu si litterae $\mathfrak{A}$ unam dimensionem tribuamus, litterae $\mathfrak{B}$ duas, litterae $\mathfrak{C}$ tres et litterae $\mathfrak{D}$ quatuor, in omnibus his productis numerus dimensionum est per 5 divisibilis, coefficiens autem cuiusvis producti est quintuplum eius coefficientis, qui eidem producto ex lege combinationum competit.

35. Cum igitur sit $P = 0$, erit quoque $D = 0$ et pro reliquis coefficientibus habebimus

$$A = -\tfrac{1}{2}Q,\ B = \tfrac{1}{3}R,\ C = -\tfrac{1}{4}AQ - \tfrac{1}{4}S \text{ et } D = -\tfrac{1}{5}BQ + \tfrac{1}{5}AR + \tfrac{1}{5}T.$$

Hinc ergo erit

$$A = -5(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C}),$$
$$B = 5(\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 + \mathfrak{C}^2\mathfrak{D}),$$
$$C = -5(\mathfrak{A}^3\mathfrak{B} + \mathfrak{B}^3\mathfrak{D} + \mathfrak{A}\mathfrak{C}^3 + \mathfrak{C}\mathfrak{D}^3) + 5(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2) - 5\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D},$$
$$D = \mathfrak{A}^5 + \mathfrak{B}^5 + \mathfrak{C}^5 + \mathfrak{D}^5 - 5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D} + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3)$$
$$+5(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}^2\mathfrak{D}).$$

cum quibus terminis iam debitae potestates ipsius v coniungi debent, ut obtineantur eorum iusti valores.

36. Quodsi ergo mutatis signis coefficientium A et C proponatur haec aequatio

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

cuius coefficientes hos teneant valores

$$A = 5(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C})v,$$
$$B = 5(\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 v + \mathfrak{C}^2\mathfrak{D}v)v,$$
$$C = 5(\mathfrak{A}^3\mathfrak{B} + \mathfrak{B}^3\mathfrak{D}v + \mathfrak{A}\mathfrak{C}^3 v + \mathfrak{C}\mathfrak{D}^3 vv)v - 5(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2)v^2 + 5\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}v^2,$$
$$D = \mathfrak{A}^5 v + \mathfrak{B}^5 v^2 + \mathfrak{C}^5 v^3 + \mathfrak{D}^5 v^4 - 5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D}v + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3 v)v^2$$
$$+5(\mathfrak{A}^2\mathfrak{C}^2\mathfrak{D} + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 v + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 v)v^2,$$

erunt eius quinque radices

$$\text{I.}\quad x = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4},$$
$$\text{II.}\quad x = \mathfrak{A}\mathfrak{a}\sqrt[5]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4},$$
$$\text{III.}\quad x = \mathfrak{A}\mathfrak{b}\sqrt[5]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4},$$
$$\text{IV.}\quad x = \mathfrak{A}\mathfrak{c}\sqrt[5]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4},$$
$$\text{V.}\quad x = \mathfrak{A}\mathfrak{d}\sqrt[5]{v} + \mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}.$$

existentibus $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ praeter unitatem reliquis quatuor radicibus surdesolidis unitatis, quarum valores imaginarii constant.

37. Si nunc vicissim ex datis coefficientibus *A, B, C, D* definiri possent quantitates $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D},$ cum littera *v,* haberetur resolutio generalis omnium aequationum quinti gradus. Verum in hoc ipso summa difficultas consistit, cum nulla via pateat litteras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D},$ quarum quidem unam pro lubitu assumere licet, successive ita eliminandi, ut aequatio solum incognitam *v* cum datis *A, B, C, D* involvens resultet, quae quidem nullas radices superfluas complectatur. Satis tuto autem suspicari licet, si haec eliminatio rite administretur, tandem ad aequationem quarti gradus perveniri posse, qua valor ipsius *v* definiatur. Si enim aequatio altioris gradus prodiret, tum quoque valor ipsius *v* signa radicalia eiusdem gradus implicaret, quod absurdum videtur. Quoniam autem multitudo terminorum hunc laborem tam difficilem reddit, ut ne tentari quidem cum aliquo successu queat, haud abs re erit casus quosdam minus generales evolvere, qui non ad formulas tantopere complicatas deducant.

38. Ad casus ergo particulares descensuri tribuamus litteris $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D},$ eiusmodi valores, quibus calculus in compendium reducatur; ac primo quidem sint $\mathfrak{B} = 0, \mathfrak{C} = 0, \text{ et } \mathfrak{D} = 0,$, unde nanciscemur

$$A = 0,\ B = 0,\ C = 0,\ D = \mathfrak{A}^5 v.$$

Hinc igitur fit

$$\mathfrak{A}\sqrt[5]{v} = \sqrt[5]{D}.$$

Quare si haec proposita fuerit aequatio

$$x^5 = D,$$

erunt huius aequationis quinque radices

$$\text{I. } x = \sqrt[5]{D},\ \text{II. } x = \mathfrak{a}\sqrt[5]{D},\ \text{III. } x = \mathfrak{b}\sqrt[5]{D},\ \text{IV. } x = \mathfrak{c}\sqrt[5]{D}\ \ \text{V. } x = \mathfrak{d}\sqrt[5]{D};$$

qui casus cum per se sit manifestus, ab eo exordium capere visum est, ut pateat, quomodo nostra methodus casus cognitos in se complectatur.

39. Evanescant iam duae litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, et $\mathfrak{D}$; si enim tres evanescentes ponantur, quaecumque eae sumantur, semper ad casum praecedentem deducimur. Sint igitur $\mathfrak{C}$ et $\mathfrak{D}$ nihilo aequales seu aequatio quaeratur, cuius radix sit futura

$$x = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2},$$

atque obtinebimus

$$A = 0,\ B = 5\mathfrak{A}\mathfrak{B}^2 v,\ C = 5\mathfrak{A}^3\mathfrak{B}v,\ D = \mathfrak{A}^5 v + \mathfrak{B}^6 v^2,$$

unde proposita radix conveniet huic aequationi

$$x^5 = 5\mathfrak{A}\mathfrak{B}^2 v x^2 + 5\mathfrak{A}^3\mathfrak{B}vx + \mathfrak{A}^5 v + \mathfrak{B}^6 v^2.$$

Quae aequatio si comparetur cum hac forma

$$x^5 = 5Pxx + 5Qx + R,$$

erit

$$\mathfrak{A}\mathfrak{B}^2 v = P,\ \ \mathfrak{A}^3\mathfrak{B}v = Q,$$

unde deducitur

$$\mathfrak{A}^5 v = \tfrac{QQ}{P}\ \text{ et }\ \mathfrak{B}^5 v^2 = \tfrac{P^3}{Q},$$

ita ut sit

$$R = \tfrac{QQ}{P} + \tfrac{P^3}{Q}.$$

40. Hinc ergo deducimur ad resolutionem huius aequationis specialis quinti gradus

$$x^5 = 5Pxx + 5Qx + \tfrac{QQ}{P} + \tfrac{P^3}{Q},$$

cuius ob $\mathfrak{A}\sqrt[5]{v} = \sqrt[5]{\frac{QQ}{P}}$ et $\mathfrak{B}\sqrt[5]{vv} = \sqrt[5]{\frac{P^3}{Q}}$ quinque radices erunt

$$\text{I. } x=\sqrt[5]{\frac{QQ}{P}}+\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{II. } x=\mathfrak{a}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{a}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{III. } x=\mathfrak{b}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{b}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{IV. } x=\mathfrak{c}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{c}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{V. } x=\mathfrak{d}\sqrt[5]{\frac{QQ}{P}}+\mathfrak{d}^2\sqrt[5]{\frac{P^3}{Q}}.$$

Aequatio autem haec non multum absimilis est formulae Moivreanae, et quia se in factores resolvi non patitur, eius resolutio hic tradita eo magis notari meretur.

41. Hanc aequationem a fractionibus liberare poterimus, si ponamus

$$P=MN \text{ et } Q=M^2N;$$

tum enim habebitur

$$x^5=5MNxx+5M^2Nx+M^3N+MN^2,$$

cuius radix erit

$$x=\sqrt[5]{M^3N}+\sqrt[5]{MN^2},$$

et si $\mathfrak{a}$ quamlibet aliam radicem surdesolidam unitatis denotet, erit huius aequationis quaelibet alia radix

$$x=\mathfrak{a}\sqrt[5]{M^3N}+\mathfrak{a}^2\sqrt[5]{MN^2}.$$

lta si exempli gratia statuatur $M=1$ et $N=2$, huius aequationis

$$x^5=10xx+10x+6$$

radix quaecumque est

$$x=\mathfrak{a}\sqrt[5]{2}+\mathfrak{a}^2\sqrt[5]{4};$$

haecque aequatio ita est comparata, ut per nullam methodum cognitam resolvi posse videatur.

42. Si $\mathfrak{B}$ et $\mathfrak{D}$ sint nihilo aequales, ad eundem casum revolvimur. Fiet enim

$$A=0,\ B=5\mathfrak{A}^2\mathfrak{C}v,\ C=5\mathfrak{A}\mathfrak{C}^3vv \text{ et } D=\mathfrak{A}^5v+\mathfrak{C}^5v^3;$$

unde si statuatur haec aequatio

$$x^5=5Pxx+5Qx+R,$$

ut sit

$$P = \mathfrak{A}^2\mathfrak{C}v \text{ et } Q = \mathfrak{A}\mathfrak{C}^3vv,$$

erit

$$\tfrac{QQ}{P} = \mathfrak{C}^5v^3 \text{ et } \tfrac{P^3}{Q} = \mathfrak{A}^5v$$

Hincque fit ut ante

$$R = \tfrac{QQ}{P} + \tfrac{P^3}{Q}$$

atque etiam eaedem reperiuntur radices. Eadem porro etiam aequatio reperitur, sive ponatur $\mathfrak{A} = 0$ et $\mathfrak{B} = 0$ sive $\mathfrak{A} = 0$ et $\mathfrak{C} = 0$. Sin autem vel $\mathfrak{A}$ et $\mathfrak{D}$ vel $\mathfrak{B}$ et $\mathfrak{C}$ evanescere assumantur, utrimque quidem eadem prodit aequatio, sed diversa a praecedentibus casibus, quam ideo evolvere conveniet.

43. Sit igitur et $\mathfrak{B} = 0$ et $\mathfrak{C} = 0$ atque hinc consequemur sequentes valores

$$A = 5\mathfrak{A}\mathfrak{D}v,\ B = 0,\ C = -5\mathfrak{A}^2\mathfrak{D}^2v^2 \text{ et } D = \mathfrak{A}^5v + \mathfrak{D}^5v^4.$$

Unde si statuamus $\mathfrak{A}\mathfrak{D}v = P$, erit

$$A = 5P \text{ et } C = -5PP;$$

tum vero erit

$$DD - 4P^5 = (\mathfrak{A}^5v - \mathfrak{D}^5v^4)^2 \text{ et } \mathfrak{A}^5v - \mathfrak{D}^5v^4 = \sqrt{(DD - 4P^5)},$$

ideoque

$$\mathfrak{A}^5v = \tfrac{1}{2}D + \tfrac{1}{2}\sqrt{(DD - 4P^5)} \text{ et } \mathfrak{D}^5v^4 = \tfrac{1}{2}D - \tfrac{1}{2}\sqrt{(DD - 4P^5)}.$$

Hinc si proposita sit haec aequatio

$$x^5 = 5Px^3 - 5PPx + D,$$

quaelibet eius radicum est

$$x = \mathfrak{a}\sqrt[5]{\left(\tfrac{1}{2}D + \tfrac{1}{2}\sqrt{(DD - 4P^5)}\right)} + \mathfrak{a}^4\sqrt[5]{\left(\tfrac{1}{2}D - \tfrac{1}{2}\sqrt{(DD - 4P^5)}\right)}$$

atque haec est ipsa illa aequatio, cuius resolutionem Cel. Moivrus docuit.

44. Possunt autem ex forma generali innumerabiles deduci aequationes quinti ordinis, quarum radices assignare licet, etiamsi ipsae illae aequationes in factores resolvi nequeant. Proposita enim aequatioue quinti gradus

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

cuius coefficientes habeant sequentes valores

$$A = \tfrac{5}{gk}(g^3 + k^3),$$

$$B = \tfrac{5}{mnr}\Big((m+n)(m^2g^3 - n^2k^2) - (m-n)rr\Big),$$

$$C = \tfrac{5}{mngkkrr}\Big(g^3(m^2g^3 - n^2k^3)^2 - \Big(m(m+n)g^6 - (m^2 + mn - n^2)g^3k^3 + n(m-n)k^6\Big)rr - k^3r^4\Big),$$

$$\begin{aligned} D = {} & \tfrac{gg}{mmnk^4r^3}\Big((m^2g^3 - n^2k^3)^3 - (m^2g^3 - n^2k^3)(m^2g^3 + n^2k^3)rr - n^2k^3r^4\Big) \\ & + \tfrac{kk}{mnng^4r}\Big(m^2g^3(m^2g^3 - n^2k^3) - (2m^2g^3 + n^2k^3)r^2 + r^4\Big) \\ & + \tfrac{5(m-n)(g^3-k^3)(m^2g^3-n^2k^3)}{mngkr} - \tfrac{5(m+n)(g^3-k^3)r}{mngk}, \end{aligned}$$

eius radices semper assignari possunt.

45. Ponatur enim brevitatis gratia

$$T = (m^2g^3 - n^2k^3)^2 - 2(m^2g^3 + n^2k^3)rr + r^4$$

sitque

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \tfrac{(m^2g^3-n^2k^3)^3-(m^2g^3-n^2k^3)(m^2g^3+n^2k^3)rr-n^2k^3r^4 \pm((m^2g^3-n^2k^3)^2-n^2k^3rr)\sqrt{T}}{2mmnr^3},$$

$$\left.\begin{matrix} R \\ S \end{matrix}\right\} = \tfrac{(m^2g^3-n^2k^3)m^2g^3-(2m^2g^3+n^2k^3)rr+r^4 \pm(m^2g^3-rr)\sqrt{T}}{2mmnr},$$

ubi signa superiora pro valoribus *P* et *R,* inferiora pro *Q* et *S* valent, ac quaelibet radix aequationis erit

$$x = \mathfrak{a}\sqrt[5]{\tfrac{gg}{k^4}}P + \mathfrak{a}^2\sqrt[5]{\tfrac{kk}{g^4}}R + \mathfrak{a}^3\sqrt[5]{\tfrac{kk}{g^4}}S + \mathfrak{a}^4\sqrt[5]{\tfrac{gg}{k^4}}Q.$$

46. Ut rem exemplis illustremus, ex his formis sequentia formari possunt:

I. Aequationis

$$x^5 = 40x^3 + 70xx - 50x - 98$$

radix est

$$\begin{aligned} x = {} & \sqrt[5]{(-31+3\sqrt{-7})} + \sqrt[5]{(-18+10\sqrt{-7})} + \sqrt[5]{(-18-10\sqrt{-7})} \\ & + \sqrt[5]{(-31-3\sqrt{-7})}. \end{aligned}$$

II. Aequationis

$$x^5 = 2625x + 61500$$

radix est

$$x = \sqrt[5]{75(5+4\sqrt{10})} + \sqrt[5]{225(35+11\sqrt{10})} + \sqrt[5]{225(35-11\sqrt{10})} + \sqrt[5]{75(5-4\sqrt{10})},$$

quae eo magis sunt notatu digna, quod hae aequationes nullo alio modo resolvi possunt.

Simili autem modo huiusmodi investigationes ad aequationes altiorum graduum extendi possunt facileque erit ex quovis gradu innumerabiles aequationes per alias methodos irresolubiles exhibere, quarum huius methodi ope non solum una, sed omnes plane radices exhiberi queant.