CHRISTIAN HUYGENS

Sends greeting to the most illustrious of men,

FRANS VAN SCHOOTEN

*Most Noble Sir,*

*With the publication of the most elegant of your talented works, which you have now in hand, I may understand that with the variety of matters, amongst the other things to be considered by you, the treatment of which you have put in place, you have shown how widely the devine knowledge of analysis may be extended ; also I can understand easily that what we have written about the reasoning of games of chance can serve your purpose very well; for how much less such games, which are fortuitous and uncertain, seem able to be understood in terms of reason, yet by how much more may they be assessed by the admirable art of analysis, by adding these too to that art. Whereby, since I have undertaken this exposition with gratitude to you, which you yourself considered worthy, thus I shall not act contrary to your wishes so that it likewise may be brought to the light with the most subtle of your discoveries, yet I am thinking it to be in my interests especially on this account that these matters also may fall into people's hands. Certainly since otherwise I might be considered to have arranged the work to be light and frivolous, yet not straightforwards, devoid of usefulness, and to be considered of no value, which you might not have wished to include among your own works, yet with much labour you have converted it from our vernacular tongue into Latin. Nevertheless, if anyone has began to examine closely what we have treated, I have no doubt they shall find at once the matter not to be, as it may seem to be, a playful act, but to have established the fundamentals of a most beautiful and subtle line of reasoning. And indeed I am confident the problems of this kind which are proposed are going to be seen to be solved with no less profundity than what are found in the books of Diophantus, [i.e. the answers are always in terms of whole numbers] but with a little more pleasure being had, since they will not terminate so as in that with the consideration of bare numbers. Truly it is to be observed, as these calculations have been a source of discussion amongst all the most outstanding French geometers now for a long time, someone may be granted the glory of the first discovery of these matters, and not to be due to me. In any case, they were accustomed to practice the most difficult questions amongst themselves, each keeping his own method hidden, and thus so that it was necessary for me to develop this general material from first principles. On account of which I still do not know whether they were using that same principle as me ; but in resolving the problems I have found they most often agree beautifully with ours. Some of these problems found by me I have added at the end of the work, yet with the analysis left out, because they were demanding an exceedingly long amount of work, if clearly I had wished to pursue everything, then what would be remaining to be done would be some of our reader exercises, whoever they are.*

*Farewell.*

# CONCERNING REASONING IN GAMES OF CHANCE.

*[Huygens above refers to the correspondence between Pascal and Fermat on the sharing of points in an interrupted game, as raised by the Chevalier de Mere, while Descartes and other members of Mersenne's group also were known to Huygens via his father, who was a diplomat with access to the French intelligensia.]*

Although the outcomes of games which chance alone may control, are accustomed to be uncertain events, nevertheless, by how much someone may be closer to winning rather than losing in these, always has a certain determined value: whether someone may assert to throw a six to win with the first throw of a single die, or whether indeed he will be able to overcome the uncertainty ; but it shall be more plausible for the matter itself to be defined by how much he may lose or win, and to be accounted for by calculation. Thus also, if I may compete [or contend] with someone in this manner, so that victory may be agreed on in three games, and now I may win one game, at this stage there is the uncertainty whether our winner of the first of the three games shall emerge winner of the third. Truly if the game between us may be agreed on as left incomplete, the expectation of the share due to me, and conversely of the amount due to that person, has to be determined, and evidently may follow from the most certain reasoning, by which the greater part of it has been deposited [*i.e.* staked], and hence to define what shall be required to be attributed to me rather than to my opponent ; or also, if someone may wish to succeed in my place and share, at what price by me shall that equity be sold to him. And hence innumerable questions can arise with two, three, or more players. And since a computation of this kind shall be far from common, and often may be consulted usefully, briefly I may expound here by what account or method it may be extricated, and then also I will explain what properly belongs to dice games.

But I may use this as fundamental in both situations : without doubt, in a game of dice of such a size, it is required to estimate the share or expectation of each player, which is required to be found for anyone, just as much as what would be obtained, if he had arrived at the same share or expectation, from an equally sure condition. So that, for example, if someone may conceal unknown to me 3 shillings in the one hand, in the other 7 shillings, and gives me the option to accept the coins from either hand I wish ; I say this is worth just as much to me as if 5 shillings may be given to me. Because again having five shillings, I may be able to arrive at that anew, as I may arrive at in an equal expectation by obtaining 3 or 7 shillings: and competing in that game equally.

## *PROPOSITION I.*

*If I may expect a or b, each of which may occur to me equally easily, my expectation may be said to be* $\frac{a+b}{2}$.

In order not only to demonstrate this rule, but indeed for the first time also eliciting by putting *x* for what shall be equal to my expectation ; it is required by me when I have found *x*, to be able to arrive again at a similar lot which I may contend in the same condition. And thus the game may be considered to be of such a kind, that I may contend with another player according to this condition, that each may deposit *x*, and so that the winner [who in the vernacular receives the jackpot 2*x*] will deliver up an amount *a* to the loser. Moreover, this is to be a fair game, & it is clear to me by this reasoning, to have an equal chance to obtain *a*, clearly if I lose the game, or of obtaining $2x-a$ if I may win : then in this case I obtain 2*x*, surely that which has been deposited, so that *a* is accepted by the other player . Because if moreover $2x-a$ may emerge just as much as *b*, which will occur equally to the value of my expectation for *a* as for *b*. And thus I may consider $2x-a \propto b$ and $x \propto \frac{a+b}{2}$, for the value of my expectation.

The demonstration of which is easy. And indeed now knowing $\frac{a+b}{2}$, I am able to contend with the other player, which also will wish to deposit $\frac{a+b}{2}$, with this condition, that the winner is going to hand over *a* to the loser. By which in a similar account my expectation is reduced to obtaining *a*, if I may lose, or to obtaining *b*, if I may win ; then indeed I keep $a+b$ truly what has been deposited, and to the other player thence I concede *a*.

With numbers. If equal chances may occur to me as 3 or 7, then the expectation 5 prevails for me by this Proposition *;* and it is certain to me 5 can arrive at the same expectation again. For if contending with someone I may deposit a certain 5, and he similary may deposit 5, with this condition, that whoever wins, 3 shall be given to the other: here the game shall be entirely fair, and it is equally evident for me to fall upon the chance of obtaining 3, if I may lose, or 7, if I may win: because then I obtain 10, regarding which I concede 3 to the other.

## *PROPOSITION II.*

*If I may expect a, b, or c, each part of which shall be able to happen to me with equal facility, my expectation is required to be valued* $\frac{a+b+c}{3}$.

Towards which being found again, as before, *x* may be put for the value of my expectation. Therefore it is required by me, when I have *x*, to be able to arrive at the same expectation from a fair game. The game may be put in place to be such, that I may play the game with two others with this condition, that each of us three may deposit $x$, and so with the first player this is agreed upon, if he himself may emerge the winner [*i.e.*, who takes the total deposit $3x$, and returns some of it as follows], to me there shall be given $b$, and if the same may befall to me to be the winner I shall be handing over $b$ to him. But I may enter into this condition with the other player, so that on winning the game the other player shall be handing over $c$ to me, or I shall be handing over $c$ to him, if I may win. [In summary, I receive or give $b$ to the first player as he wins or I win, and I receive or give $c$ to the second player as he wins or I win.] And it is apparent that this game is fair. For according to this reasoning I shall have the chance to obtain $b$, if evidently the first player may win, or $c$, if the second player may win, or also $3x-b-c$ if I may win ; hence indeed $3x$ befalls to me, which is the amount deposited, from which to the one I concede $b$, and to the other $c$. But if $3x-b-c$ were equal to $a$ itself, the same expectation will befall to me for obtaining $a$, which is for $b$, or for $c$. Therefore I put $3x-b-c \propto a$, and there shall be $x \propto \frac{a+b+c}{3}$, for the value of my expectation. In the same manner, the amount is found, if to $a, b, c,$ or $d$ equal chances may befall to me, that is to be the value of the quantity $\frac{a+b+c+d}{4}$ And thus so on.

*PROPOSITION III.*

*If the number of cases, with which a may occur for me, shall be p, but the number of cases with which b may occur for me shall be q, by taking all the cases to have fallen equally : the expectation shall emerge for me*

$$\frac{ap+bq}{p+q}.$$

Towares eliciting this rule, again $x$ may be put for the value of my expectation : therefore it is required by me, when I have found $x$, to be able to arrive at the same expectation as before, with a fair game. Moreover for this I will accept so many fellow players, so that together with me they shall make the number $p+q$ itself, and each of whom deposits $x$, thus so that the deposit shall be $px+qx$, and each himself plays with the equal expectation of winning. Again since from such a number of these fellow players, so many will indicate the number $q$, I will enter into this agreement with each one, so that whoever of these may win shall give $b$ to me, or, on the other hand, likewise I will give $b$ to them, if I may win. Similarly with the remaining players, constituting $p-1$ I may undertake this condition with each one, so that each of these, who may win the game, shall give me $a$, and I just as much to him ($a$ evidently), if I may win. And it is apparent that this game be fair, with this condition, clearly with no one treated unjustly. Then it is apparent to me now to have for $b$, $q$ of the expectations, and $p-1$ expectations for $a$, and 1 expectation (surely for me winning ) for $px+qx-bq-ap+a$, then indeed I obtain $px+qx$, that which has been deposited, from which I must pass up $b$ to each one of the players $q$, and $a$ to each one of the $p-1$ players, who likewise make up $qb + pa - a$. And thus if $qx + bx - bq - ap + a$ shall be equal to $a$, I will have $p$ expectations for $a$, (since now I had $p-1$ expectations for that) and $q$ expectations for $b$, and thus again I may come upon my former expectation. On account of which again $px + qx - bq - ap + a \propto a$, and there shall be $x \propto \dfrac{ap + bq}{p + q}$, for the value of my expectation, as everything was put in place initially.

With numbers. If 3 expectations for me were 13, and 2 expectations were for 8, by this rule I will have as much as 11. And it is easy to show, for me, if I may have 11, to be able to arrive at the same expectation again. For playing the game against 4 others, and each of our five players deposits 11, since with two of these I enter into a pact, so that whoever of these wins shall give me 8, or I to the same give 8, if I may win. Similarly with the two remaining, so that of these five, whoever wins the game, shall be giving 13 to me, or I just as much to them, if I may win. Which game indeed is fair. And it appears to me to have two expectations for 8 in this manner, without doubt, if either of these, who have promised 8 to me, may win, and 3 expectations for 13, without doubt if either of the remaining two, who must hand over 13 to me, or if to them if I may win : for winning the game I obtain the deposit, that is, 55, from which to each of two I must hand over 13 , and to each of the remaining two 8, and thus so that to me there may be left 13.

## *PROPOSITION IV.*

*Therefore so that we may come to the first question proposed [at the end of the work], without doubt, there is a need that we may make a start with [calculating the outcomes for] the most simple game, concerning the distribution [of the prize money] between the different players, when their lots are unequal [and it is wished to end the game at this stage].*

And thus assuming for me to contend with someone, with this agreed on : that whoever has won three times [in succession] first shall gain the deposit ; and now with me having won twice, truly the other once : I would like to know, if we do not wish the game to proceed, but the money about which we compete is to be divided exactly equally, how much of that may due to me.

In the first place it is required to consider the games which are lost on both sides : Indeed it is certain, if it has been agreed between us, for example, that because the deposit has become more valuable he shall win, who first should win twenty times, and I have won 10 times and 9 times, but the other 10 times and 8 times, my odds to win to be so much the better here in this case than in the other, where from [the previous] three games I have won two consecutive games, he truly only one : as without a doubt for me to be deficient only by one game, but the other player by two.

Again towards finding how great a part each may be owed, it is required to consider how it would happen, if we were to proceed with the game. Indeed it is evident, if I should win the first game, I fulfil the required number and with that all of the following deposits, which may be called $a$. But if moreover the other player may win the first game, then our chances shall be equal, (clearly with each one deficient by one game at this stage,) and indeed $\frac{1}{2}a$ might go to each. But it is evident to me to have an equal chance to be winning or losing the first game, thus so that to me now the expectation shall be equal for obtaining $a$ or $\frac{1}{2}a$: which itself by the first Proposition is of such a size and if I may have half of each of the chances, that is $\frac{3}{4}a$; and to the other player by me there is left $\frac{1}{4}a$, which part may be found at once from the beginning in the same way. From which it is apparent, that, whoever may wish to receive my game for himself, must hand over $\frac{3}{4}a$ to me for that; and hence there shall always be able to put three against one, who may contend to win one game, and who may have won the other two before.

[Thus, the players have decided to 'go for broke'; the rules have now been changed, and the winner who would take all is the one who would reach perhaps 20 games first, if they continued to play; the problem is to evaluate the relative probabilities of each player winning, as the game has been terminated.]

### *PROPOSITION V.*

*We may consider myself to be deficient by one game and for my fellow player to be deficient by three games. It is required to make the distribution here [to end the game like the one above].*

And thus we may turn our attention again to a state in which we may be, if either I or himself may win the first game. If I may win, I will obtain the winnings , that is, $a$ ; because if he should win that first game, himself deficient by two games and by one for ; and thus we may be in the same state which was the case in the previous Proposition, and there will fall to me $\frac{3}{4}a$, as it has been shown there. And thus with equal facility to be granted either $a$ or $\frac{3}{4}a$, that which finally is, by the 1st Proposition, $\frac{7}{8}a$. And there remains $\frac{1}{8}a$ to my fellow player; thus so that my share to his share may be has, as 7 to 1.

But just as for this calculation the preceding has been required, thus again and from this to follow can be put in place : without doubt, if we may consider for me to be lacking in one game and my opponent by 4 games. And it is found in the same manner, to me must be owed $\frac{15}{16}$ of the amount deposited, and to him $\frac{1}{16}$.

### *PROPOSITION VI.*

*We may consider for me to be deficient by two games and for my opponent by three games.*

And thus it will happen in the first game ; so that either for me it may fall short by one game and for him by three games (from which I obtain the chance $\frac{7}{8}a$ by the preceding Proposition) ; or so that each of us may fall short by two games, from which I obtain the chance $\frac{1}{2}a$, since thus for each player the odds shall become equal. But there is an equal chance for me to win or lose the first game ; thus so that for me there shall be an equal chance for obtaining $\frac{7}{8}a$ or $\frac{1}{2}a$, so that $\frac{11}{16}a$ prevails for me, by the 1st Proposition. And 11 parts of that deposited is owed to me, and to my fellow-player 5 parts.

## *PROPOSITION VII.*

*We may consider for me to be two games deficient, and for my fellow-player to be four deficient.*

And thus it may come about, so that, if I may win the first game, I must win one game more, and the other player must win four [i.e. if I lose the second game and the other player starts to play]; or, if I may lose the same, I may lose two and the other player wins three. Thus so that an equal chance may befall to me as $\frac{15}{16}a$ or $\frac{11}{16}a$, which prevails to me of such a size $\frac{13}{16}a$, by the 1st Proposition. From which it is apparent, since to have a better chance, whoever wins two games then the other must win four, who wins one while the other wins 2. For in this latter case, without a doubt as of 1 to 2, for my part, by the 4th Proposition, is $\frac{3}{4}a$, which is less than $\frac{13}{16}a$.

## *PROPOSITION VIII.*

*Now truly we may consider three players, of which the first and the second may be deficient by one game, but the third by three games.*

So that therefore the part of the first player may be found, again it is required to consider what may be owed, if either he himself or either of the other two may win the first game. If he himself may win, he will have the deposit, that which shall be *a*. So that if the second player may win, the first shall have nothing, because the second thus has imposed the end to the game. But if the third may win the first game, then each of the three at this stage shall be deficient by one game, and thus both the first player as well as each of the other two may be owed $\frac{1}{3}a$. And the first shall be with one chance for *a*, one for 0, and one for $\frac{1}{3}a$ (since each of the three options can be obtained with equal ease in order that he may win the first game,) which itself may prevail to be as much as $\frac{4}{9}a$, by the 2nd Proposition. And similarly $\frac{4}{9}a$ for the second, and $\frac{1}{9}a$ remains for the third. The part of which also could be found separately, and thence the parts of the remaining determined.

## *PROPOSITION IX.*

*In order that with so many fellow players, from which as many as it may be wished, for one to fall short by several games and the others to fall short by fewer games, it is required to consider the share of each to be found, whether he or whoever it pleases of the remaining players may win the following first game. But these shares, if they may be gathered together into one sum, and the sum may be divided by the number of the players, the quotient will show the share of one sought.*

We may consider three players to be present A, B, & C, and for A himself to fall short by one game, B himself by two games, and C similarly by two games. It is required to find, what may be owed to B himself, of that deposit, which may be called *q*.

In the first place it is required to examine what may be owed to B himself, if either he himself, or A, or C may win the first following game.

If A may win, he would impose an end to the game, and as a consequence B is owed nothing 0. If B may win, at this point he will fall short by one game, and A by one game, but C by two games. On account of which B himself in this case shall be owed $\frac{4}{9}q$, by the 8th proposition.

And then if C may win the first following game, then A and C fall short by a single game , but B himself by two games, and as a consequence B himself may be owed $\frac{1}{9}q$, by the same 8th Proposition. But now there can be gathered together into one sum, that which must be owed to B himself in these three cases : without doubt, $0, \frac{4}{9}q, \frac{1}{9}q$: the sum of which is $\frac{5}{9}q$. So that itself divided by the number 3 of the players, gives $\frac{5}{27}q$. Which is the part of B sought. But the demonstraton of this is apparent from the 2nd Proposition. Because indeed B has equal chances for obtaining $0, \frac{4}{9}q$, or $\frac{1}{9}q$, he has by the 2nd Proposition just as much a chance as $\dfrac{0+\frac{4}{9}q+\frac{1}{9}q}{3}$, that is, $\frac{5}{27}q$. And it is certain that the divisor 3 be the number of players.

But so that it may be found, what may be owed to someone in any case, clearly if either he himself or someone remaining may win the first following game : it is necessary to investigate the simpler cases first, and by means of these the following. For here in the same manner the final case cannot be resolved before that calculation of the eight Proposition should be dealt with, in which the games lacking were 1, 1, 2, thus also each part cannot be counted up in such a case, where the games lacking are 1, 2, 3, so that the first calculation undertaken shall be the case of the games lacking 1, 2 , 2 , just as we have made now, and besides that, in which the games lacking are 1, 1, 3; which similarly can be counted up by the 8th Proposition. And indeed with this agreed upon consequently all the cases can be counted up, which are accounted for in the following table, and infinitely many others.

Table for the three players.

| | | | | |
|---|---|---|---|---|
| Games lacking. | 1 ·1·2 | 1 ·2·2 | 1· 1·3 | 1· 2·3 |
| Chances of these. | 4·4·1 | 17·5·5 | 13·13·1 | 19· 6·2 |
| | 9 | 27 | 27 | 27 |

| | | | | |
|---|---|---|---|---|
| Games lacking. | 1 · 1 · 4 | 1 · 1 · 5 | 1 · 2 · 4 | 1 · 2 · 5 |
| Chances of these. | 40·40· 1 | 121·121· 1 | 178·58 · 7 | 542·179· 8 |
| | 81 | 243 | 243 | 729 |

| | | | |
|---|---|---|---|
| Games lacking. | 1 ·3·3 | 1 · 3· 4 | 1 · 3 ·5 |
| Chances of these. | 65·8·8 | 616·82·31 | 629·87·13 |
| | 81 | 729 | 729 |

| | | | | | | |
|---|---|---|---|---|---|---|
| Games lacking. | 2 · 2 · 3 | 2 · 2 · 4 | 2 · 2 · 5 | 2 ·3 · 3 | 2 · 3 · 4 | 2 · 3 · 5 |
| Chances of these. | 34·34· 13 | 338·338· 53 | 353·353· 23 | 133·55· 55 | 451·195·83 | 1433·635·119 |
| | 81 | 729 | 729 | 243 | 729 | 2187 |

Whereas with regard to dice, for those these questions can be proposed : clearly, it shall be required to be tested how many times in turn shall six be thrown by one die, or some other of the remaining points. Likewise, how many times in turn shall two sixes be thrown by two dice, or three sixes by three dice. And several other questions of this kind.

It is required to turn towards solving these questions. In the first place there are six different casts of a single die, of which any number can arise with equal ease. Indeed I assume the die to have the figure of a perfect cube. Again the different casts of two dice shall be 36, any of which similarly can be obtained with equal ease. For on account of each throw of one die able to be one of six, it befalls likewise of the casts of the other dice. And 6 times 6 makes 36 casts. Likewise of three dice there are 216 different casts [or throws]. For on account of each of the 36 throws of the two dice there can be six throws of the one die which come about in the 3$^{rd}$. And 36 times six make 226 throws. It is apparent in the same way, the throws of four dice to be 216 times six, that is 1296; and thus any number of further dice can be totaled up, by assuming always by the increase of a single die, to be of six times the preceding throws.

Again it should be observed, of the two dice thrown only once may make 2 or 12 points, two throws truly which may make 3 or 11 points. For if we may call the dice A and B, it is clear , for 3 points being thrown there will be able to be found either one point in A and two for B, or one point in B and two in A. Similarly for 11 points required to be thrown either five in A and six in B, or six in A and five in B appears possible. There are three throws of four points, clearly of A 1 and B 3 points ; or of A 3 and B 1 point ; or of A 2 and of B 2 points. Similarly there are three throws of ten points.
There are four throws for five or nine points.

There are five throws for six or eight points.
Six throws for seven points.

$$\text{For three dice there will be found } \left\{ \begin{matrix} 3 \text{ or } 18 \\ 4 \text{ or } 17 \\ 5 \text{ or } 16 \\ 6 \text{ or } 15 \\ 7 \text{ or } 14 \\ 8 \text{ or } 13 \\ 9 \text{ or } 12 \\ 10 \text{ or } 11 \end{matrix} \right\} \text{points for } \left\{ \begin{matrix} 1 \\ 3 \\ 6 \\ 10 \\ 15 \\ 21 \\ 25 \\ 27 \end{matrix} \right\} \text{ throws.}$$

## *PROPOSITION X.*

*To find how many throws in turn shall be undertaken by which, so that a 6 may be cast by a single die.*

If someone should contend to cast a six the first time, one would appear to be the case, so that he may win, and that may be had as a pledge in place of a deposit; truly five to be the chances [or cases], by which he may lose, and may have nothing. For there are 5 throws against that itself, and only one for it. But since the deposit may be called *a.* And thus there is a single expectation for obtaining *a*, but five for obtaining 0; and which by the 2nd Proposition is worth as much as $\frac{1}{6}a$. And this chance remains for him which itself offers $\frac{5}{6}a$. Thus so that only as 1 chance against 5 shall it be put in place, who would wish to win in the first turn.

Whereby he may undertake to win in two throws of a single die in turn, the odds of this may be computed with this agreed on: If 6 may be cast in the first turn, he obtains *a*. If it comes out differently, one throw remains for him, which from the preceding may prevail as great as $\frac{1}{6}a$. And in order that the first turn he may throw 6, there is only one case, and 5 cases in which it may turn out differently. And thus from the beginning there is one case, which may give *a* itself ; and five which each may give $\frac{1}{6}a$, by the 2nd Proposition shall be worth $\frac{11}{36}a$ [*i.e.* the expectation value]. From which $\frac{25}{36}a$ goes against the contending player; and thus so that the chance of each, or the estimate of the expectation, provides for him a ratio, that shall be as 11 to 25; that is less than 1 to 2. [*i.e.* $\frac{11}{36}a = \frac{6}{36}a + \frac{5}{6}a \times \frac{}{6}a.$]

Hence in the same way the calculation is accounted for, so that the chance of the player to win in three throws in turn shall be worth $\frac{91}{216}a$; thus so that 91 against 125 shall be put in place [*i.e.* $\frac{91}{216}a = \frac{1}{6}a + \frac{5}{6} \times \frac{1}{6}a + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}a.$] ; that is, a little less than 3 to 4.

Whereby he may undertake to win in four throws in turn, his chance is $\frac{671}{1296}a$; thus so that there shall be put in place 671 to 615 ; that is, more than 1 ad 1.
[*i.e.* $\frac{671}{1296}a = \frac{1}{6}a + \frac{5}{6}\times\frac{1}{6}a + \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}a + \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}a.$]

Whereby he may undertake to win in five throws in turn, his chance is $\frac{4651}{1296}a$, & there may be put in place 4651 against 3125 ; that is, a little less than 3 to 2.

Who undertakes the same in six throws in turn, his chance is $\frac{31031}{46656}a$, and here may be put in place 31031 against 15625; that is, a little less than 2 to 1.

And thus consequently the chance of any number of throws can be found. But clearly progressing towards a greater advantage, as may be shown in the following Proposition ; without which other calculations would become more prolix.

## *PROPOSITION XI.*

*To find how many throws in turn shall be undertaken by which, so that 12 may be cast by two dice.*

If someone may contend to throw two sixes with the first try, it is evident there is a single chance, by which he may win, that is, for obtaining *a*; and to be 35 chances, by which he may lose or have nothing, because there are 36 [possible] throws. And thus he has [an expectation of ] $\frac{1}{36}a$, by the 2nd proposition.

Whereby he may undertake the same by two tries in turn, if with the first turn he may throw two sixed, he will obtain *a;* if truly the first turn may come out differently, one throw [of the two dice] remains for him, as that by itself, by what has now been said, is worth $\frac{1}{36}a$.

And so that in the first turn there is only one chance he may throw two sixes, but 35 chances, in which something different may arise. And thus from the beginning there is one chance, which will give him *a*, and 35 each of which may give $\frac{1}{36}a$; that which by the 5th Proposition is worth $\frac{71}{1296}a$. And the contrary certainty remains, $\frac{1225}{1296}a$.
[*i.e.* $\frac{71}{1296}a = \frac{1}{36}a + \frac{35}{36}\times\frac{1}{36}a$.]

From these it can be found, how great the share or part shall be, which likewise shall be found with four [single] throws, with that case being discarded, when someone undertakes that with three throws.

And indeed, whereby contending with two dice thrown 4 times in turn, if the first or second may be made in turn, he obtains *a*; but if less, two throws remain for him, so that by what has been said above, $\frac{71}{1296}a$ is prevailing. But on that account it has the same ratio also 71 chances, as from the first two thrown, with 1225 opposing chances, by which it may differ. And thus from the beginning he has 71 chances, which may give him *a*, and 1225 chances, which themselves may give $\frac{71}{1296}a$. Which itself by the second Proposition is worth $\frac{178991}{1679616}a$. And contending against that there remains $\frac{1500625}{1679616}a$. So that it can be shown the chances of these between themselves in turn, to be as 178991 to 1500625.

[*i.e.* $\frac{178991}{1679616}a = \left(\frac{1}{36}a + \frac{35}{36}\times\frac{1}{36}a\right) + \left(\frac{35}{36}\times\frac{35}{36}\times\frac{1}{36}a\right) + \left(\frac{35}{36}\times\frac{35}{36}\times\frac{35}{36}\times\frac{1}{36}a\right)$.]

From which again the expectation of that can be found by the same reasoning, whoever may contend to throw two sixes in turn once in 8 throws. And thence again the expectation of that, whoever again will undertake to throw 2 sixes in turn in 16 throws. And from the expectation of this, as also from the expectation of that, who undertook these 8 throws in turn, can be found the expectation of that player, who himself undertakes to throw 2 sixed in turn in 24 throws. In which operation, because especially the lot [*i.e.* the amount he can expect to win] is sought from which number of equal throws he may begin, between the player who undertakes it, and he who offers the game, it will be allowed, which in any case may grow to an infinite size, after some the following number characters are removed from the numbers [i.e. rounded]. And thus indeed I find those, who undertakes that 24 in turn, at this point to be a little too small ; and then at last it enters a better condition, when it approaches to 25 throws.

[The process can be speeded up course by summing the G.P. formed, as shown by Jacob Bernoulli, who made this work the first chapter of his own posthumous work on probability, with extensive notes, *Ars Conjectandi*]

## *PROPOSITION XII.*

*To find what number of dice someone must undertake, so that he may throw two sixes at the first turn.*

But this is just the same, as if someone should wish to know, how many times it could be undertaken with any single die, so that he might throw a six twice. So that if someone should undertake that with two throws, he might obtain $\frac{1}{36}a$, through these which have been shown before. Whereby that should be returned by three throws themselves, if the first throw were not of a six, and at this point two throws will be had, each of which must be of a six, that which has been said to be just the same value as $\frac{1}{36}a$. But truly with the first throw of that being a six, there is a need only of one six to be thrown from the two throws. So that by Proposition 10 just the same will prevail as if $\frac{11}{36}a$ were found. And it is certain to have one case itself, in which the first of the sixes may be thrown in turn, and five cases in which it may come out different. And thus from the beginning it has one case for $\frac{11}{36}a$ and 5 cases for $\frac{1}{36}a$, that which prevails by the 2$^{\text{nd}}$ Proposition just the same as $\frac{16}{216}a$ or, $\frac{2}{27}a$. With this agreed on by taking continually one more throw, 10 throws from one die is found, or the first throw from 10 dice can be undertaken, so that two sixes may be thrown with, and thus with profit.

*PROPOSITION XIII.*

*If I may play a game with someone involving one throw of two dice only, with this condition, so that if seven may come out, I may win ; but that player, if ten may be obtained; truly if any other arises, so that then what has been deposited we may divide equally : To find which share of this each of us may be owed.*

Because 36 throws, which arise from two dice, 6 throws are present of 7 points, and 3 throws of 10 points, at this stage 27 throws remain, which are able to level the game ; that which if it happens, each of us is owed $\frac{1}{2}a$. Truly if that may not be obtained, I will have 6 chances, by which I may win, that is, so that I may have *a*; and 3 chances, in which the opposite may happen, and I may have nothing : that which by the 2nd Proposition is just as much as if in such a case I may have $\frac{2}{3}a$. And thus I may have from the beginning 27 chances for $\frac{1}{2}a$ and 9 chances for $\frac{2}{3}a$, that which, by the 2nd Proposition, is as much as $\frac{13}{24}a$. And $\frac{11}{24}a$ remains contended against.

*PROPOSITION XIV.*

*If I and another player may throw with two dice alternately, with this condition, so that I may throw seven and win, as truly thus he wins clearly if he may throw six first with the two dice, as I shall concede the first throw to him : To find the ratio for my share to his share of the winnings.*

Putting my share to be the value *x*, and that amount which has been deposited shall be called *a;* and the share of the other player $\propto a-x$. And it is apparent, whenever his turns to throw come back, then my share again must be $\propto x$. But whenever the turns are mine so that I may throw, my share is to considered more [as I have more agreeing outcomes then he has, i.e. 6 to 5]. And thus for his turn *y* may be put to be the value. Now because from 36 throws there may be found 5 [outcomes adding up to 6] with 2 dice, which can return the game and victory to my competitor ; and 31 throws, for which there may be a different outcome, that is, which promote my turns for throwing : I will have, before he throws first, 5 chances of obtaining 0, and 31 chances for obtaining *y*. Which by the 3rd Proposition is worth $\frac{[0\times a]+31\times y}{36}$. But we have put my chance from the beginning to be $\propto x$. On account of which there will be $\frac{31y}{36}\propto x$, and thus $y\propto\frac{36x}{31}$. Then there was put, with my turns coming, my chance with the value *y*. Therefore I am going to throw, I have 6 chances of obtaining *a,* since 7 points are found in 6, which returns victory to me ; and I have 30 chances, in which the turns revert to my fellow player, that is, I may obtain *x* for me, that which by the 3re Proposition is worth $\frac{6a+30x}{36}$. But since this shall be $\propto y$, there will be as before, on inverting, $\frac{36x}{31}\propto y, \frac{30x+6a}{36}\propto\frac{36x}{31}$. From which there is found $x\propto\frac{31a}{61}$, the value of my share. And as a consequence for my fellow player it will be $\frac{30a}{61}$; thus so that the ratio of my share to his share will be, as 31 to 30.

*In place of a final embellishment the following problems have been added below.*

*Problem 1.* A and B play with two dice, with this one condition, so that A may win, if he may throw a six, but B if he may throw a seven. A at first will set up a single throw; then B consequently two throws ; then again A two throws, and thus henceforth, until this or that one may emerge the winner. What is the ratio of the odds of A to the odds of B ? Resp. as 10355 to l2276.

*Problem* 2. Three players A, B and C taking 12 counters, 4 of which are white and 8 black, play with this condition : so that, who first of these with eyes veiled [or drawn from a bag] selects a white piece may win; and so that A shall have the first selection, B the second, and C the third, and then following again the choice belongs to A, and thus henceforth alternately. It is sought, what shall be the future ratio of their chances ?

*Problem* 3. A competes with B so that from 40 playing cards, that is, 10 or each kind, 4 cards shall be extracted in turn; thus so that the winner may have one card of each kind. And the ratio of the odds of A to the odds of B shall be as 1000 to 8139.

*Problem* 4. On assuming as before, 12 counters, 4 white & 8 black, A contends with B, so that with the eyes veiled [or drawn from a bag] he will remove 7 counters from these, amongst which three will be white. The ratio of the odds of A to the odds of B is sought.

*Problem* 5. A and B individually taking 12 coins, play with three dice under this condition : so that if 11 points may be thrown, A passes up that coin to B; but if 14 points may be thrown, B hands over that coin to A; and thus he shall win the game, who first will have all the coins. And the ratio of the odds of A to the odds of B is found to be as 244140625 to 282429536481.

THE END.

CHRISTIANUS HUGENIUS

Clarissimo Viro,

D. FRANCISCO SCHOTENIO

*S. D.*

*Cum in editione elegantissimorum ingenii tui monumentorum, quam prae manibus nunc habes, vir Clarissime, id inter caetera te spectare sciam, ut varietate rerum, quarum tractationem instituisti, ostendas quam late se protendat divina Analytices scientia, facile intelligo etiam illa plurimum proposito tuo inservire posse, quae de aleae ratiociniis conscripsimus; quanto enim minus rationis terminis comprehendi posse videbantur, quae fortuita sunt atque incerta, tanto admirabilior ars censebitur, cui ista quoque subiacent. Quare cum in tui gratiam primum illa exponenda susceperim, tuque digna existimes, quae simul cum subtilissimis tuis inventis in lucem exeant, adeo tibi non refragabor, ut etiam e re mea esse existimem hac potissimum ratione ipsa in manus hominum pervenire. Quippe cum in re levi ac frivola operam collocasse videri alioqui possim , non tamen prorsus utilitatis expers ac nullius pretii censebitur, quod tu veluti inter tua adoptaveris, nec sine multo labore e vernacula lingua nostra in Latinam converteris. Quanquam, si quis penitius ea quae tradimus examinare coeperit, non dubito quin continuo reperturus sit rem non ut videtur ludicram agi, sed pulchrae subtilissimaeque contemplationis fundamentia explicari. Et Problemata quidem quae in hoc genere proponuntur, nihilo minus profundae indaginis visum iri confido, quam quae Diophanti libris continentur, voluptatis autem aliquanto plus habitura, cum non, sicut illa in nuda numerorum consideratione terminentur. Sciendum vero, quod iam pridem inter praestantissimos tota Gallia Geometras calculus hic agitatus fuerit, ne quis indebitam mihi primae inventionis gloriam hac in re tribuat. Caeterum illi, difficillimis quibusque quaestionibus se invicem exercere soliti, methodum suam quisque occultam retinuere, adeo ut a primis elementis universam hanc materiam evolvere mihi necesse fuerit. Quamobrem ignoro etiamnum an eodem mecum principio illi utantur ; at in resolvendis Problematis pulchre nobis convenire saepenumero expertus sum. Horum Problematum nonnulla in fine operis addidisse me invenies, omissa tamem analysi, cum quod prolixam nimis operam poscebant, si perspicue omnia exequi voluissem, tum quod relinquendum aliquid videbatur exercitationi nostrorum, si qui erunt, Lectorum.*
*Vale.*
Dat. Hagae Com.
27 Apr. 1657·

DE

# RATIOCINIIS

IN

# LUDO ALEAE.

Et si lusionum, quas solas sors moderatur, incerti solent esse eventus, attamen in his, quanto quis ad vincendum quam perdendum propior sit, certam semper habet determinationem. Ut si quis primo iactu una tessera senarium iacere contendat, incertum quidem an vincet; at quanto verisimilius sit eum perdere quam vincere, reipsa definitum est, calculoque subducitur. Ita quoque, si cum aliquo certem hac ratione, ut ternis lusibus constet victoria, atque ego iam unum lusum vicerim, incertum adhuc uter nostrum prior tertii victor sit evasurus. Verum quanti exspectatio mea, & contra quanti illius, aestimari debeat, certissimo ratiocinio consequi licet, atque hinc definire, si ludum uti est imperfectum linquere inter nos convenerit, quanto major portio eius quod depositum est mihi quam adversario meo tribuenda esset: vel etiam si quis in locum sortemque meam succedere cupiat, quo pretio me eam ipsi vendere aequum sit. Atque hinc innumerae quaestiones exoriri possunt inter duos, tres, pluresque collusores. Cumque minime vulgaris sit hujusmodi supputatio, & saepe utiliter adhibeatur, breviter hic qua ratione aut methodo expedienda sit exponam, ac deinde etiam, quae ad aleam sive tesseras proprie pertinent,explicabo.

Hoc autem utrobique utar fundamento: nimirum, in aleae ludo tanti aestimandam esse cujusque sortem seu expectationem ad aliquid quid obtinendum, quantum si habeat, possit denuo ad similem sortem sive expectationem pervenire, aequa conditione certans. Ut, exempli gratia, si quis me inscio altera manu 3 solidos occultet, altera 7 solidos, mihique optionem det ex utra manu solidi accipere malim; hoc tantundem mihi valere dico, ac si 5 solidi mihi dentur.Quoniam quinque solidos habens, denuo eo pervenire possum, ut aequam expectationem nanciscar ad 3 vel 7 solidos obtinendos: idque aequo lusu contendens.

*PROPOSITIO I.*

*Si a vel b expectem, quorum utrumvis aequi facile mihi obtingere possit, expectatio mea dicenda est valere* $\frac{a+b}{2}$.

Ad hanc regulam non solum demonstrandam, verum etiam primitus eruendam posito $x$ pro eo quod aequivalet expectationi meae, oportet me, quum $x$ habeo, rursus ad similem sortem pervenire posse, aequa conditione certantem. Ponatur itaque lusus esse talis, ut cum altero certem hac conditione, ut quisque deponat $x$, ac ut victor victo traditurus sit $a$. Hic autem lusus justus est, & patet me hac ratione aequam habere sortem ad obtinendum $a$, si lusum perdam scilicet ; aut $2x-a$, si vincam : tum enim obtineo $2x$, id nempe quod depositum est, de quo alteri erogandum est $a$. Quod si autem $2x-a$ tantundem valeret atque $b$, aequa mihi sors obtingeret ad $a$ quam ad $b$. Pono itaque , $2x-a \propto b$ & fit $x \propto \frac{a+b}{2}$, pro valore meae expectationis. Cujus demonstratio facilis est. Etenim habens $\frac{a+b}{2}$ possum cum alio certare, qui etiam $\frac{a+b}{2}$ deponere volet, hac conditione ut vincens victo sit traditurus $a$. Qua ratione similis expectatio mihi obtinget ad obtinendum $a$, si perdam, aut ad obtinendum $b$, si vincam ; tum enim obtineo $a+b$ id nempe quod depositum est, alterique inde concedo $a$.

In numeris. Si ad *3* vel 7 aequa sors mihi obtingat, tum expectatio mea per hanc Propositionem valet *5;* & certum est me 5 habentem rursus ad eandem expectationem pervenire posse. Si enim cum alio certans 5 deponam, atque ille similiter *5* deponat, hac conditione, ut, qui vincit, alteri sit daturus 3: erit hic lusus omnino justus, & patet mihi aequam obtingere sortem ad obtinendum 3, si perdam, aut 7, si vincam: quoniam tunc obtineo 10, de quo alteri concedo 3.

## *PROPOSITIO II.*

*Si a,b, vel c expectem, quorum unumquodque pari facilitate mihi obtingere possit , expectatio mea aestimanda est* $\frac{a+b+c}{3}$.

Ad quod rursus inveniendum, ponatur, ut ante, $x$ pro valore expectationis meae. Oportet ergo me, cum $x$ habeo, ad eandem expectationem pervenire posse justo lusu. Ponatur lusus esse talis, ut cum duobus aliis ludam hac conditione, ut quisque nostrum trium deponat $x$, & ut cum uno hoc pactum aggrediar, si ipse victor evadat, mihi sit daturus $b$, & ego ipsi traditurus sim $b$, si idem mihi obtingat. Cum altero autem hanc ineam conditionem, ut ille ludum vincens mihi traditurus sit $c$, aut ego ipsi sim daturus $c$, si ego vincam. Et patet hunc ludum justum esse. Aequam autem hac ratione sortem habebo ad obtinendum $b$, si nimirum primus vincat, aut $c$, si secundus vincat, aut etiam $3x-b-c$ si ego vincam; tunc enim obtineo $3x$, quod depositum est, dequo uni concede $b$, & alteri $c$. Quodsi $3x-b-c$ aequale fuerit ipsi $a$, eadem mihi obtingeret expectatio ad obtinendum *a*, quae ad *b*, aut ad *c*. Pono itaque $3x - b - c \propto a$ ,& fit $x \propto \frac{a+b+c}{3}$, pro valore meae expectationis. Eodem modo invenitur, si ad *a, b, c*, aut *d* aequa sors mihi obtingat, id tanti valoris esse, quanti $\frac{a+b+c+d}{4}$

Atque ita porro.

*PROPOSITIO III.*

*Si numerus casuum, quibus mihi eveniet a, sit p, numerus autem casuum quibus mihi eveniet b sit q, sumendo omnes casus aeque in proclivi esse: expectatio mea valebit*

$$\frac{ap+bq}{p+q}.$$

Ad hanc regulam eruendam, ponatur rursus *x* pro valore expectationis meae: ergo oportet me, cum *x* habeo, ad eandem expectationem pervenire posse, ut ante, justo lusu. Ad hoc autem tot collusores sumam,ut una mecum numerum ipsius $p+q$ efficiant, quorum deponat quisque *x*, ita ut depositum sit $px+qx$, & quisque sibi ludat aequa expectatione ad vincendum. Porro cum tot ex hisce collusoribus, quot indicat numerus *q*, sigillatim hoc pactum inibo, ut eorum qui vincat mihi sit daturus *b*, aut ego contra ipsi idem *b*, si vincam. Similiter cum reliquis collusoribus, constituentibus $p-1$ sigillatim hanc conditionem aggrediar, ut eorum quisque, qui ludum vincit, mihi sit daturus *a*, & ego tantundem (*a* scilicet) ipsi, si ego vincam. Et patet hunc lusum hac conditione justum esse, nemine videlicet injuriam patiente. Deinde patet me nunc *q* expectationis habere ad *b*, & $p-1$ expectationes ad *a*, & 1 expectationem (me nempe vincente) ad $px+qx-bq-ap+a$, tunc enim obtineo $px+qx$, id quod depositum est, de quo tradere debeo *b* unicuique *q* lusorum, & *a* unicuique $p-1$lusorum, quae simul conficiunt $qb+pa-a$. Si itaque $qx+bx-bq-ap+a$ aequale esset ipsi *a*, haberem *p* expectationes ad *a*, (quandoquidem iam $p-1$ expectationes ad id habebam) & *q* expectationes ad *b*, & sic ad priorem meam expectationem rursus pervenissem. Quocirca porro $px+qx-bq-ap+a \propto a$, & sit $x \propto \frac{ap+bq}{p+q}$, pro valore expectationis meae, omnino ut in initio positum fuit.

In numeris. Si 3 mihi expectationes forent ad 13, & 2 expectationes ad 8, haberem per hanc regulam tantundem ac 11. Et facile est ostendere, me, si 11 habeam, rursus ad eandem expectationem pervenire posse. Ludens enim contra 4 alios, & quisque nostrum quinque deponens 11, cum duobus ex illis sigillatim pactum inibo, ut horum qui vincat mihi sit daturus 8, aut ego ipsi idem 8, si vincam. Similiter cum duobus reliquis, ut eorum quisque, qui ludum vincit, mihi sit daturus 13, aut ego ipsi tantundem, si ego vincam. Qui quidem lusus justus est. Et patet me hoc modo duas habere expectationes ad 8, nimirum si alteruter eorum, qui mihi 8 promiserunt, vincat, & 3 expectationes ad 13, nimirum si alteruter reliquorum duorum, qui mihi 13 tradere debent, vincat, aut si ipse ludum vincam: ego enim ludum vincens obtineo depositum, id est, 55, de quo unicuique duorum tradere debeo 13 , & unicuique reliquorum duorum 8, ita ut & mihi relinquatur 13.

## *PROPOSITIO IV.*

*Ut igitur ad primo propositam quaestionem veniamus, nimirum, de facienda distributione inter diversos collusores, quando eorum sortes inaequales sunt, opus est ut a facilioribus incipiamus.*

Sumpto itaque me cum aliquo certare, hoc pacto: ut qui prius ter vicerit, quod depositum est, lucretur, & me iam bis vicisse, alterum vero semel. Scire cupio, si lusum prosequi non velimus, sed pecuniam, de qua certamus, prout aequum est, partiri, quantum eius mihi obtingeret.

Primo considerare oportet lusus, qui utrobique deficiunt. Certum enim est, si inter nos convenerit, verbi gratia, ut quod depositum est lucretur is, qui prius vigesies vicerit, & ego decies & novies vicero, at alter decies & octies, tanto meliorem fore eo casu sortem meam quanto hic melior est, ubi a tribus lusibus binos consequutus sum, ille vero unum duntaxat: quia nimirum utrobique mihi unus tantummodo lusus sed ipsi duo deficiunt.

Porro ad inveniendum quanta pars utrique debeatur, advertendum est quid fieret, si in lusu pergeremus. Certum enim est, si primum ludum vincerem, me praescriptum numerum impleturum & omne depositum consecuturum, id quod vocetur $a$. Quod si autem alter primum ludum vinceret, tunc aequata utriusque sors foret, (quippe utrique uno adhuc deficiente ludo, ) adeoque cederet cuique $\frac{1}{2}a$. Manifestum autem est me aequam habere sortem ad primum ludum vincendum aut perdendum, ita ut mihi nunc aequa sit expectatio ad obtinendum $a$ aut $\frac{1}{2}a$: quod ipsum per 1$^{\text{mam}}$ Propositionem tantum est ac si utriusque sortis dimidium, id est $\frac{3}{4}a$, haberem ; & relinquitur alteri meo collusori $\frac{1}{4}a$, quae ipsius portio statim ab initio eodem modo reperiri potuisset. Unde patet, eum, qui ludum meum in se recipere vellet, mihi $\frac{3}{4}a$ pro eo tradere debere; ac proinde semper tria contra unum deponere eum posse, qui unum ludum vincere contendat, priusquam alter duos vincat.

## *PROPOSITIO V.*

*Ponamus unum mihi deficere ludum & collusori meo tres lusus. Oportet hic facere distributionem.*

Advertamus itaque rursus, in quo essemus statu, si ego vel ipse primum vinceret lusum. Si ego vincerem, obtinerem depositum, id est, $a$ ; quod si autem ille primum ludum vinceret, deficerent ipsi duo lusus & mihi unus; ac proinde in eodem statu essemus, qui in praecedenti Propositionc positus fuit, mihique obtingeret $\frac{3}{4}a$, ut ibi ostensum est. Itaque pari facilitate vel $a$ mihi obtinget vel $\frac{3}{4}a$, id quod tantum est, per 1$^{\text{mam}}$ Propositionem, ac $\frac{7}{8}a$. Et relinquitur $\frac{1}{8}a$ collusori meo; ita ut mea sors ad sortem illius se habeat, sicut 7 ad 1.

Quemadmodum autem ad hunc calculum requisitus est praecedens, ita rursus hicce inservit sequenti: nimirum, si ponamus mihi unum ludum deficere & collusori meo 4 or lusus. Et invenitur eodem modo, mihi deberi $\frac{15}{16}$ istius quod depositum est, & ipsi $\frac{1}{16}$.

## *PROPOSITIO VI.*

*Ponamus mihi deficere duos lusus & collusori meo tres lusus.*

Fiet itaque primo lusu; vel ut mihi unus lusus deficiat & ipsi tres ( unde mihi per praecedentem Propositionem obtinget $\frac{7}{8}a$ ) ; velut cuique nostrum ad huc duo lusus deficiant, unde mihi debebitur $\frac{1}{2}a$, quandoquidem sic utrique aequa sors futura est. Est mihi aut est aequalis facilitas ad primum ludum vincendum aut perdendum; ita ut mihi aequa sit expectatio ad obtinendum $\frac{7}{8}a$ aut $\frac{1}{2}a$, id quod mihi valet $\frac{11}{16}a$, per 1mam Propositionem. Et debentur mihi 11 partes eius quod depositum est, & collusori meo 5 partes.

## *PROPOSITIO VII.*

*Ponamus mihi deficere duos lusus & collusori meo quatuor.*

Fiet itaque, ut, si primum ludum vincam, unum ludum vincere debeam & alter quatuor; vel, si eundem perdam, duos & alter tres. Ita ut aequa mihi sors obtingat ad $\frac{15}{16}a$ aut $\frac{11}{16}a$, id quod tantum valet ac $\frac{13}{16}a$, per 1mam Propositionem. Unde patet, cum meliorem habere sortem, qui duos lusus vincere debet dum alter quatuor, quam eum, qui unum dum alter duos. In hoc enim posteriori casu, nimirum ipsius 1 ad 2, portio mea, per 4tam Propositionem, est $\frac{3}{4}a$, quae minor est quam $\frac{13}{16}a$.

## *PROPOSITIO VIII.*

*Nunc vero ponamus tres esse collusores, quorum primo ut & secundo unus lusus deficiat, sed tertio duo lusus.*

Ut igitur inveniatur primi pars, rursus advertendum est, quid ipsi deberetur, si vel ipse vel alter reliquorum duorum primum lusum vinceret. Si ipse vinceret, haberet depositum, id quod sit *a*. Quod si secundus vinceret, primus nihil haberet, quoniam secundus sic lusui finem imposuisset. At si tertius vinceret, tunc cuique trium adhuc unus deficeret lusus, ideoque tam primo quam utrique reliquorum deberetur $\frac{1}{3}a$. Et fit prima una expectatio ad *a*, una ad 0, & una ad $\frac{1}{3}a$ (quandoquidem aeque facile contingere potest cuique trium ut primum ludum vincat,) quod ipsi tantundem valet ac $\frac{4}{9}a$, per 2dam Propositionem. Et fit similiter secundo $\frac{4}{9}a$, & remanet tertio $\frac{1}{9}a$. Cujus pars separatim etiam inveniri potuerat, atque inde reliquorum partes determinari.

## *PROPOSITIO IX.*

*Ut tot collusorum, quot quis voluerit, ex quibus uni plures & alii pauciores lusus deficiunt, cujusque pars inveniatur, considerandum est, quid illi, cuius partem invenire volumus, deberetur, si vel ipse, vel quislibet reliquorum primum sequentem ludum vinceret. Hae autem partes si in unam summam colligantur , & aggregatum per numerum collusorum dividatur, quotiens ostendet unius quaesitam partem.*

Ponamus tres esse collusores A, B, & C, & ipsi A unum ludum deficere, ipsi B duos lusus, & ipsi C similiter duos lusus. Invenire oportet, quid ipsi B, eius quod depositum est, debeatur. Id quod vocetur $q$.

Primo examinandum est, quid ipsi B deberetur, si vel ipse, vel A, vel C primum sequentem ludum vinceret.

Si A vinceret, ludo finem imposuisset, ac per consequens ipsi B deberetur 0. Si ipse B vinceret, deficeret illi adhuc unus lusus , & ipsi A unus lusus, at ipsi C duo lusus. Quocirca ipsi B hoc in casu deberetur $\frac{4}{9}q$, per $8^{\text{vam}}$ Propositionem.
Denique si C primum sequentem ludum vinceret, tunc ipsis A & C singulis unus deficeret lusus , sed ipsi B duo lusus, ac per consequens ipsi B deberetur $\frac{1}{9}q$, per eandem Propositionem $8^{\text{vam}}$, Nunc autem in unam summam colligendum est, id quod in tribus hisce casibus ipsi B deberetur : nimirum, 0, $\frac{4}{9}q$, $\frac{1}{9}q$ : quorum summa est $\frac{5}{9}q$. Quod ipsum divisum per 3 numerum collusorum, dat $\frac{5}{27}q$. Quae ipsius B quaesita pars est.
Demonstratio autem hujus patet ex $2^{\text{da}}$ Propositione. Quoniam enim B aequam habet sortem ad obtinenendum 0, $\frac{4}{9}q$, vel $\frac{1}{9}q$, habet per $2^{\text{dam}}$ Propositionem tantundem ac $\dfrac{0+\frac{4}{9}q+\frac{1}{9}q}{3}$, id est, $\frac{5}{27}q$. Et certum est, hunc divisorem 3 esse numerum collusorum.

Ut autem inveniatur, quid cuipiam debeatur in quolibet casu, videlicet si vel ipse vel aliquis reliquorum primum sequentem ludum vincat: oportet simpliciores casus primo investigare, & horum medio sequentes. Nam sicut hic ultimus casus solvi non potuit priusquam ille octavae Propositionis calculo subductus esset, in quo deficientes lusus erant 1, 1, 2, ita etiam cujusque pars supputari nequit in tali casu, ubi deficientes lusus sunt 1, 2, 3, quin primum calculo subductus sit casus deficientium lusuum 1, 2 , 2 , quemadmodum iam fecimus, & praeterea ille, in quo lusus dcficientes sunt 1, 1, 3; qui similiter per *8*$^{\text{vam}}$ Propositionem supputari potuisset. Atque hoc quidem pacto consequenter supputare licet casus omnes, qui in sequenti tubula comprehenduntur, & infinitos alios.

Tabula pro 3 collusoribus.

| | | | | |
|---|---|---|---|---|
| Lusus, qui ipsis deficient. | 1 ·1·2 | 1 ·2·2 | 1· 1·3 | 1· 2·3 |
| Eorum partes. | 4·4·1 | 17·5·5 | 13·13·1 | 19· 6·2 |
| | 9 | 27 | 27 | 27 |

| | | | | |
|---|---|---|---|---|
| Lusus, qui ipsis deficient. | 1 · 1 · 4 | 1 · 1 · 5 | 1 · 2 · 4 | 1 · 2 · 5 |

| | | | | |
|---|---|---|---|---|
| Eorum partes. | 40·40·1 | 121·121·1 | 178·58·7 | 542·179·8 |
| | 81 | 243 | 243 | 729 |

| | | | |
|---|---|---|---|
| Lusus, qui ipsis deficient. | 1·3·3 | 1·3·4 | 1·3·5 |
| Eorum partes. | 65·8·8 | 616·82·31 | 629·87·13 |
| | 81 | 729 | 729 |

| | | | | | | |
|---|---|---|---|---|---|---|
| Lusus, qui ipsis deficient. | 2·2·3 | 2·2·4 | 2·2·5 | 2·3·3 | 2·3·4 | 2·3·5 |
| Eorum partes. | 34·34·13 | 338·338·53 | 353·353·23 | 133·55·55 | 451·195·83 | 1433·635·119 |
| | 81 | 729 | 729 | 243 | 729 | 2187 |

Quod ad tesseras attinet, de iis hae quaestiones proponi possunt: videlicet,quota vice una tessera senarium iacere periclitandum sit, aut aliquod reliquorum punctorum. Item quota vice duos senarios duabus tesseris, aut tres senarios tribus tesseris iactus sit tentandum. Et plures aliae hujusmodi quaestiones.

Ad quas solvendas advertendum est. Primo unius tesserae sex esse iactus diversos, quorum quivis aeque facile eveniat. Sumo enim tesseram habere figuram cubi perfectam. Porro duarum tesserarum 36 esse diversos iactus, quorum similiter qui, aeque facile obtingere potest. Nam ratione cuiusque iactus unius tesserae potest unus sex iactuum alterius tesserae simul contingere. Et sexies 6 efficiunt 36 iactus. Item trium tesserarum esse 216 iactus diversos. Nam ratione cujusque 36 iactuum duarum tesserarum potest unus sex iactuum, qui in $3^{tia}$ sunt, evenire. Et sexies 36 efficiunt 226 iactus. Eodem modo patet, quatuor tesserarum iactus esse sexies 216, id est, 1296; atque sic ulterius iactus quotlibet tesserarum supputari posse, sumendo semper pro accessione unius tesserae sexies iactus praecedentis.

Porro notandum, duarum tesserarum unum duntaxat esse iactum, qui 2 aut 12 puncta efficiat, duos vero iactus, qui 3 aut 11 puncta efficiant. Si enim tesseras vocemus A & B , patet , ad 3 puncta iacienda in A unum & in B duo, vel in B unum & in A duo puncta reperiri posse. Similiter ad 11 puncta iacienda in A quinque & in B sex , vel in A sex & in B quinque puncta patere posse. Quatuor punctorum tres sunt iactus, videlicet, ipsius A1 & B3 puncta ; vel ipsius A3 & B1 punctum; vel ipsius A2 & B2 puncta.
Decem punctorum similiter tres sunt iactus.
Quinque vel novem punctorum $4^{or}$ sunt iactus.
Sex vel octo punctorum $5^{que}$ sunt iactus.
Septem punctorum 6 sunt iactus.

$$\text{In tribus tesseris reperiuntur} \left\{ \begin{matrix} 3 \text{ vel } 18 \\ 4 \text{ vel } 17 \\ 5 \text{ vel } 16 \\ 6 \text{ vel } 15 \\ 7 \text{ vel } 14 \\ 8 \text{ vel } 13 \\ 9 \text{ vel } 12 \\ 10 \text{ vel } 11 \end{matrix} \right\} \text{puncturom} \left\{ \begin{matrix} 1 \\ 3 \\ 6 \\ 10 \\ 15 \\ 21 \\ 25 \\ 27 \end{matrix} \right\} \text{iactus.}$$

## *PROPOSITIO X.*

*Invenire, quot vicibus suscipere quis possit, ut una tessera 6 puncta iaciat.*

Si quis prima vice senarium iacere contendat, apparet unum esse casum, quo vincat, habeatque id, quod pignoris loco depositum est; quinque vero esse casus, quibus perdat, & nihil habeat. Sunt enim 5 iactus contra ipsum, & tantum unus pro ipso. Quod autem depositum est vocetur *a*. Est itaque ipsi unica expectatio ad obtinendum *a*, sed quinque ad obtinendum 0; id quod per 2^dam^ Propositionem tanundem valet ac $\frac{1}{6}a$. Et manet pro eo qui ipsi hunc casum offert $\frac{5}{6}a$. Ita ut tantummodo 1 contra 5 deponere possit, qui prima vice suscipere velit.

Qui duabus vicibus semel senarium iacere certet, sors eius hoc pacto computatur. Si prima vice 6 iaciat, obtinet *a*. Si diversum eveniat, unus ipsi restat iactus,qui ex praecedenti tantum valet, quantum $\frac{1}{6}a$. Atqui ut prima vice 6 iaciat, unus tantum casus est, & quinque casus , quibus diversum eveniat. ltaque ab initio unus casus est, qui det ipli.t; & quinque qui dent $\frac{1}{6}a$, id quod per $2^{\text{dam}}$ Propositionem valet $\frac{11}{36}a$. Unde contra certanti lusori cedit reliquum $\frac{25}{36}a$; adeo ut sors utriusque sive aestimatio expectationis cam servet rationem, quam 11 ad 25; id est minus quam 1 ad 2.

Hinc eodem modo calculo subducitur, quod sors eius, qui tribus vicibus semel senarium iacere suscipit, sit futura $\frac{91}{216}a$; ita ut 91 contra 125 deponere possit; id est, paulo minus quam 3 ad 4.

Qui quatuor vicibus idem suscipit, sors eius est $\frac{671}{1296}a$; ita ut 671 contra 615 deponere possit ; id est, plus quam 1 ad 1.

Qui quinque vicibus idem suscipit, sors eius est $\frac{4651}{1296}a$, & potest 4651 contra 3125 deponere; id est, paulo minus quam 3 ad 2.

Qui sex vicibus idem suscipit, sors eius est $\frac{31031}{46656}a$, & potest 31031 contra 15625 deponere; id est, paulo minus quam 2 ad 1.

Atque ita consequenter quilibet iactuum numerus inveniri potest. Sed licet majori compendia progredi, ut in sequenti Propositione ostendetur; sine quo calculus alias multo prolixior foret.

## *PROPOSITIO XI.*

*Invenire, quot vicibus suscipere quis possit, ut duabus tesseris 12 puncta iaciat.*

Si quis prima vice duos senarios iacere contendat, apparet unum esse casum, quo vincat, id est, ad obtinendum *a*; & 35 esse casus, quibus perdat sive nihil habeat, quoniam 36 sunt iactus. Itaque habet, per 2^dam Propostionem, $\frac{1}{36}a$ .

Qui duabus vicibus idem suscipit, si prima vice duos senarios iaciat, obtinebit *a;* si vero prima vice diversum eveniat, unus ipsi restat iactus, id quod ipsi, per illud quod iam dictum est, valet $\frac{1}{36}a$ .

Atqui ut prima vice duos senarios iaciat, unus tantum est casus, sed 35 casus, quibus diversum eveniat. Itaque ab initio unus casus est, qui det ipsi *a*, & 35 qui dent $\frac{1}{36}a$ ; id quod per 5^dam Propositionem valet $\frac{71}{1296}a$ . Et remanet contra certanti $\frac{1225}{1296}a$ .

Ex his invenire licet, qualis sit ei sors aut pars, qui idem suscipit quaternis iactibus, praetereundo casum eum, cum quis illud ternis iactibus suscipit.

Etenim, qui 4^or vicibus duos senarios iacere contendit, si illud 1^ma aut 2^da vice faciat, obtinet *a*; sin minus, restant ipsi duo iactus, qui per illud quod superius dictum est, valent $\frac{71}{1296}a$ . Sed propter eandem rationem habet etiam 71 casus, ut ex duobus primis iactibus semel duos senarios iaciat, contra 1225 casus, quibus diversum eveniat. Habet itaque ab initio 71 casus, qui ipsi dent *a*, & 1225 casus, qui dent ipsi $\frac{71}{1296}a$ . Quod ipsi per 2^dam Propositionem valet , $\frac{178991}{1679616}a$ . Et remanet contra certanti $\frac{1500625}{1679616}a$ . Id quod ostendit eorum sortes esse ad se invicem, ut 178991 ad 1500625.

E quibus porro eadem ratione invenitur expectatio eius, qui 8 vicibus semel duos senarios iacere certat. Ac inde rursus expectatio eius, qui idem suscipit 16 vicibus. Atque ex hujus expectatione, ut etiam ex expectatione illius, qui istud 8 vicibus suscipit, invenitur expectatio eius, qui illud 24 vicibus in se recipit. In qua operatione, quoniam praecipue quaeritur in quo numero iactuum aequalis sors incipiat, inter eum qui id suscipit & eum qui offert, licebit a numeris, qui alioquin in immensum excrescerent, posteriores aliquot characteres auferre. Atque ita quidem reperio ei, qui illud 24 vicibus suscipit, adhuc aliquid deficere; tumque demum eum potiorem conditionem inire, cum 25 iactibus aggreditur.

## *PROPOSITIO XII.*

*Invenire quot tesseris suscipere quis possit, ut prima vice duos senarios iaciat.*

Hoc autem tantundem est, ac si quis scire velit, quoto iactu quispiam una tessara suscipere possit, ut bis senarium iaciat, Quod si quis duobus iactibus susciperet, obtingeret ei, per ea quae ante ostensa sunt, $\frac{1}{36}a$ . Qui illud tribus iactibus in se reciperet, si primus eius iactus senarius non foret, haberet adhuc duos iactus, quorum uterque senarius esse deberet, id quod tantundem valere dictum est ac $\frac{1}{36}a$. At vero primo eius iactu existente senario, opus est ut ex duobus iactibus non nisi semel senarium iaciat. Quod per 10 Propositionem tantundem valet ac si $\frac{11}{36}a$ haberet. Atqui certum est ipsum unum habere casum, quo prima vice senarium iaciat, & quinque casus quibus diversum eveniat. Habet itaque ab initio unum casum ad $\frac{11}{36}a$ & 5 casus ad $\frac{1}{36}a$, id quod per 2$^{\text{dam}}$ Propositionem tantundem valet ac $\frac{16}{216}a$ seu, $\frac{2}{27}a$. Hoc pacto assumendo continue unum iactum amplius, invenitur 10 iactibus una tessera, aut 10 tesseris prima iactu suscipi posse, ut duo senarii iaciantur, idque cum lucro.

## *PROPOSITIO XIII.*

*Si cum alio ludam duabus tesseris unum solummodo iactum , hac conditione, ut, si septenarius eveniat, ego vincam; at ille , si denarius obtingar; si vero quidquam aliud accidat, ut tum id quod depositum est aequaliter dividamus: Invenire qualis istius pars cuique nostrum debeatur.*

Quoniam 36 iactuum, qui duabus tesseris proveniunt, 6 jactus existent septem punctorum, & 3 iactus decem punctorum, restant adhuc 27 iactus, qui ludum aequare possunt; id quod si fiat, cuique nostrum debebitur $\frac{1}{2}a$. Verum si id non obtingat, habebo 6 casus, quibus vincam, id est, ut *a* habeam; & 3 casus, quibus diversum eveniat, nihilque habeam : id quod per 2$^{\text{dam}}$ Propositionem tantundem est ac si tali casu $\frac{2}{3}a$ haberem. Habeo itaque ab initio 27 casus ad $\frac{1}{2}a$ & 9 casus ad $\frac{2}{3}a$, id quod, per 2$^{\text{dam}}$ Propositionem,tantundem est ac $\frac{13}{24}a$. Et remanet contra certanti $\frac{11}{24}a$.

## *PROPOSITIO XIV.*

*Si ego & alius duabus tesseris alternatim iaciamus, hac conditione, ut ego vincam simul atque septenarium iaciam, ille vero quam primum senarium iaciat ; ita videlicet, ut ipsi primum iactum concedam : Invenire rationem mea ad ipsius sortem.*

Ponatur, sortem meam valere *x,* & id quod depositum est vocari *a;* eritque sors alterius $\propto a - x$. Et patet, quandocunque ipsius vices iacicndi revertuntur, sortem meam tum rursus dcbere esse $\propto x$. At quandocunque meae vices sunt ut iaciam, sors mea pluris aestimanda est. Ponatur itaque pro eius valore *y.* Iam quoniam ex 36 iactibus reperiuntur 5 in 2 tesseris, qui collusori meo senarium dare lususque victorem reddere possunt; & 31 iactus, quibus deversum eveniat, id est, qui meas iaciendi vices promovent : habebo, priusquam iacit, 5 casus ad obtinendum 0, & 31 casus ad obtinendum *y*. id quod per 3$^{\text{tiam}}$ Propositionem valet $\frac{31y}{36}$. Posuimus autem casum meum a principio esse $\propto x$. Quocirca erit $\frac{31y}{36} \propto x$, adeoque $y \propto \frac{36x}{31}$. Deinde positum fuit, vicibus meis venientibus, sortem meam valere *y*. Ego vero iacturus, habeo 6 casus ad obtinendum *a*, quandoquidem 6 iactus reperiuntur 7 punctorum, qui me victorcm reddunt; habeoque 30 casus, quibus vices collusoris mei revertuntur, id est, ut mihi obtineam *x*, id quod per 3tiam Propositionem valet $\frac{6a+30x}{36}$. Hoc autem cum sit $\propto y$ , erit, invento, ut ante, $\frac{36x}{31} \propto y, \frac{30x+6a}{36} \propto \frac{36x}{31}$. Unde invenitur $x \propto \frac{31a}{61}$, valor meae sortis. Et per consequens collusoris mei erit $\frac{30a}{61}$; ita ut ratio sortis mea ad illius sortem fit, ut 31 ad 30.

*Coronidis loco subjungantur sequentia Problemata.*

*Probl.1.* A & B una ludunt duabus tesseris, hac conditione, ut A vincat, si senarium iaciat, at B si septenarium iaciat. A primo unum iactum instituet; deinde B duos iactus consequenter; tum rursus A duos iactus, atque sic deinceps, donec hic vel ille victor evadat. Quaeritur ratio sortis ipsius A ad sortem ipsius B ? Resp. ut 10355 ad l2276.

*Probl.*2. Trcs collusores A, B & C assumentes 12 calculos, quorum 4 albi & 8 nigi existunt, ludunt hac conditione: ut, qui primus ipsorum velatis oculis album calculum elegerit, vincat; & ut prima electio sit penes A, secunda penes B, & tertia penes C, & tum sequens rursus penes A, atque sic deinceps alternatim. Quaeritur, quaenam futura sit ratio illorum sortium?

*Probl.*3. A certat cum B quod ipse ex 40 chartis lusoriis, id est, 10 cujusque speciei, 4 chartas extracturus sit; ita ut ex unaquaque specie habeat unam. Et invenitur ratio sortis A ad sortem B ut 1000 ad 8139.

*Probl.*4. Assumptis, ut ante, 12 calculis, 4 albis & 8 nigris, certat A cum B, quod velatis oculis 7 calculos ex iis exempturus sit, inter quos 3 albi crunt. Quaeritur ratio sortis ipsius A ad sortem ipsius B.

*Probl.*5. A & B assumentes singuli 12 nummos ludunt tribus tesseris hac conditione: ut 11 puncta iaciantur, A tradat nummum ipsi B; at si 14 puncta iaciantur, B tradat nummum ipsi A; & ut ille ludum victurus sit, qui primum omnes habuerit nummos. Et invenitur ratio sortis ipsius A ad sortem ipsius B, ut 244140625 ad 282429536481.

F I N I S.