HOROLOGII OSCILLATORII

PART III.

Concerning the evolutes and lengths of curved lines.

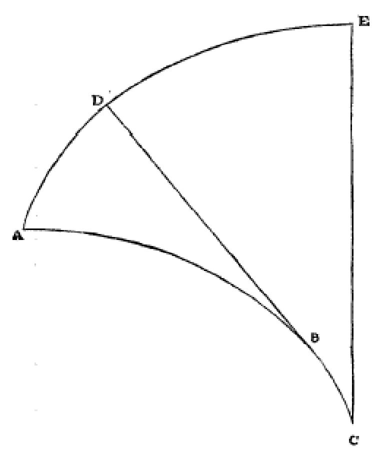
DEFINITIONS.

I.

A line can be considered to be turning to one side when all its nearby tangent straight lines are in contact together on the same side. Moreover if the curve has certain straight parts, these can be produced and considered as tangents.

II.

Moreover, when two curved lines of this kind move apart from the same point, where the convex side of the one is directed towards the concave side of the other, such as with the appointed curves ABC and ADE in the figure, then both can be said to lie in the same concavity.



III.

If in one part of the cavity, a thread or flexible line is understood to be wound around one of the curved lines; and by keeping one end of that thread fixed to the curve, with the other end drawn away thus so that the part which is free is always kept extended ; then another curve is seen to be described by the free end of this thread. Moreover this curve can be said to be described by evolution.

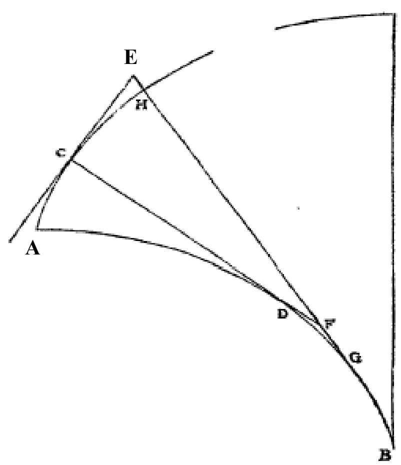
IV.

Truly, the curve on which the thread is initially wound around is called the Evolute. In the above figure, ABC is the evolute, and ADE is described by the evolution from ABC; for indeed when the end of the thread from A reaches D, the extended part of the thread is the right line DB, and the rest of the thread BC is thus still in contact with the curve ABC. Moreover it is clear that DB is a tangent to the evolute at B.

[Modern parlance has the original curve ABC as the evolute as above, while the curve drawn by the end of the taut thread is called the involute as ADE above, and the locus of the centre of curvature of the normals to the involute is the evolute. Thus, such curves form an involute : evolute pair. The evolute is thus the locus of the centre of curvature of the normals to the involute, or the envelope of the normals. The process is not inversive in general. (See Weisstein; *CRC Handbook of Mathematics*; p. 589, for a modern mathematical explanation and numerous references; or just about any calculus book.)]

PROPOSITION I.

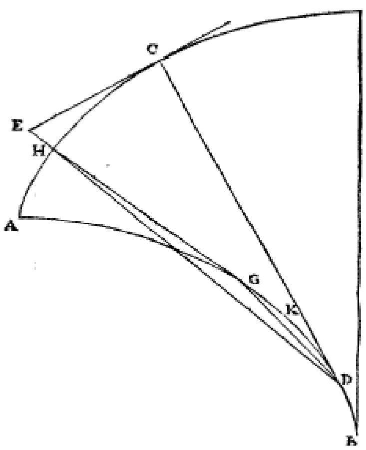
All the right lines drawn from the evolute, and which are tangents to the evolute, cross the curved line* [the involute] *at right angles.



AB is the evolute, and AH is indeed the curve drawn from this evolute. Moreover the right line FDC is drawn touching the curve AD at D and crossing the curve ACH in C. I say that these lines cross at right angles : that is, if CE is drawn perpendicular to the line CD, then CE is a tangent to that curve in C. Since DC is indeed a tangent to the evolute at D, then it is apparent that this line DC refers to the position of the thread itself when the free end of this arrives at C [turning in the anti-clockwise sense]. Therefore, since we can show that the whole length of the thread remaining describes the rest of the curve ACH, then the curve does not touch the line CE anywhere except at the point C, and it will be shown that the right line CE touches the curve ACH in that place.

Some point H other than C is taken on the line AC, and in the first place it is more distant from A than the point C, and by the principle of evolution it is understood that the free part of the thread is HG, when the end of the thread has arrived at H. Hence HG is a tangent to AB at G. And since the part of the curve described between is CH, the [corresponding part of the] evolute is DG, and CD crosses HG in the point F. Moreover, GH is produced to cross the line CE in E. Therefore, since the sum of DF and FG is greater than DG, then the sum of the lines CF and FG is greater than the sum of the right line CD and DG itself. But on account of evolution, it is apparent that the sum of right line CD and the curved line DG is equal to the right line HG. Hence, the sum of the two lines CF and FG is also greater than the right line HG; and with the common line FG taken away, then CF is greater than HF.
[*i. e.* $CF + FG > CD + \text{arc } DG = HG$; or $CF + FG > HG$, giving $CF > HF$].
But FE is greater than CF, since the angle C in triangle FCE is right. Hence FE is everywhere greater than FH. [thus : $FE > CF > HF$] Thus, it is apparent for this placement of the point C, that the end of the thread does not reach to the line CE.

If now the point H is nearer A than the point C [see diagram on this page], then by the principle of evolution, the position of the thread is HG, and then the free end of this shall be at H, and the right lines DG and DH are drawn; DH crosses the line CE in E, and moreover it is apparent that the line cannot be in the direction of HG itself, and thus HGD is a triangle. Now, since the line DG is either less than the curved line DKG, or the same length, if indeed the part DG of the evolution is straight [recall that this is the general situation, so straight parts of the curve are allowed];



[*i. e.* DG $\leq$ arc DKG]
and GH is to be added to both, the sum of the lines DG and GH are either less than or equal to these two, as you wish, DKG and GH, or with these equal to the line DC.
[*i. e.* DG + GH $\leq$ arc DKG + GH = CD]
But the sum of the two lines DG and GH is greater than the line DH. Hence, both these are less than the line DC.
[*i. e.* DH < DG + GH $\leq$ arc DKG + GH = CD]
But DE is greater than DC, since in triangle DCE the angle C is right. Hence DH is much less than DE. Therefore the point H, that is the end of the thread GH, is situated within the angle DCE.
[*i. e.* DH < DG + GH $\leq$ arc DKG + GH = CD < DE.]
Thus it is apparent that the thread is unable to reach the line CE anywhere between A and C. Hence CE is a tangent to the curve AC at C; and hence DC, to which CE has been drawn perpendicular, crosses the curve at right angles. Q.e.d.

Hence it can also be established that the curve AHC is bending in a single direction, and with the same concave nature as AGB, which has been described from the evolution of this line. Indeed, all the tangent lines of the curve AHC fall outside the distance DGAHC : while truly all the tangent lines of the curve AGD lie within the distance discussed, thus to prove the concavity of AHC, the convexity of AGD is to considered.

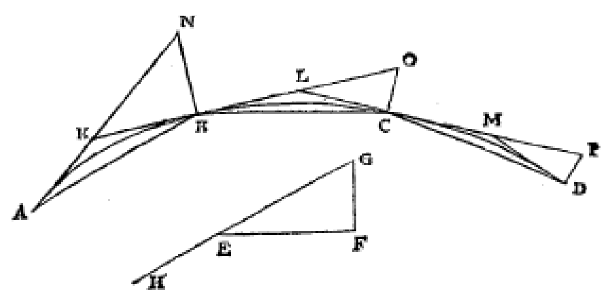
PROPOSITION II.

All curved lines of a certain extend and in a single concave part, such as ABD, can be divided into so many parts, by drawing subtending straight lines for the individual parts, such as AB, BC, CD; and from the same individual points of division, from the ends of the curve itself at the boundary, tangents are drawn to the curve, such as AN, BO, CP, which cross with these lines arising from the nearby following points of division, and where normal lines have been erected, such as BN, CO, DP; in order that, as I say, each subtended chord has to its own adjacent perpendicular to the curve, such as AB to BN, BC to CO, CD to DP, some ratio greater than a proposed ratio.

For let the line GEH be drawn, and the given ratio of the lines adjoined to the right angle at F is EF to FG .

In the first place it is understood that the curve ABD has been cut into many small parts by the points B, C, etc, in order that the tangents which touch any two nearby points of the curve, cross mutually at angles which are greater than the angle FEH; such as the angles AKB, BLC and CMD, which can be made more apparent by the following demonstration. Now with the subtending lines drawn AB, BC, CD, and with the perpendiculars BN, CO, DP erected to the curve, which on being produced cross the lines AK, BL, and CM, in N, O, and P : I say that the individual ratios of the lines: AB to BN, BC to CO, and CD to DP, etc., are greater than the ratio EF to FG.

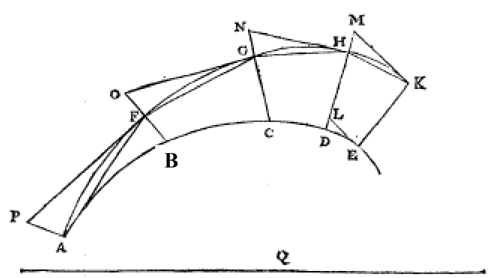
For since the angle AKB is greater than the angle HEF, then for the remainders of these angles taken from two right angles [*i. e.* the supplements], it follows that the angle NKB is less than the angle GEF.



But the angle B of the triangle KBN is right, as thus also is the angle F in triangle EFG. Hence the ratio KB to BN is greater than the ratio EF to FG. But AB is greater than KB, since the angle K in triangle AKB is obtuse, which is greater than the angle HEF which is obtuse from the construction. Hence the ratio AB to BN is greater than the ratio KB to BN, and hence it is in general greater than the ratio EF to FG. In the same manner, the ratio BC to CO, and CD to DP, can be shown to be greater than the ratio EF to FG. Thus the proposition is agreed upon.

PROPOSITION III.

Two curves turning in the same sense, and with a common concavity, are unable to arise from the same point : if they are to be compared in this manner, that a right line which crosses one of the curves at right angles, similarly crosses the other at right angles, and so on for other lines.



Assume that ACE and AGK are curves of this kind, having a common end A, and with some outer point K taken to these lines. Let the line KE be drawn, crossing the curve AGK at right angles, and hence in a similar manner the curve ACE. Now a certain right line Q can be taken greater in length than the curve KGA. Moreover, KGA is itself understood to be divided up, as was said in the previous proposition, into so many parts by the points HGF, so that the individual subtending lines

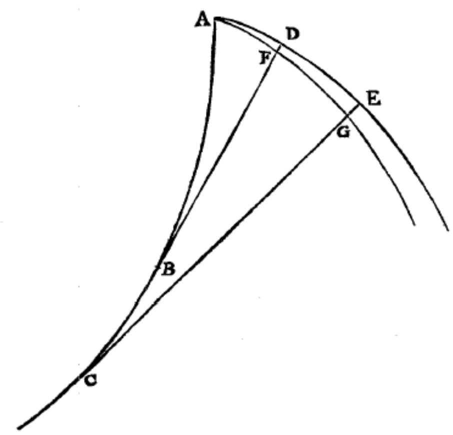
KH, HG, GF, and FA have a greater ratio to the contiguous perpendiculars to the curve, here HM, GN, FO, and AP than the line Q has to the line KE. Thus the sum of all the said subtending chords and the corresponding sum of the perpendiculars have a greater ratio than Q to KE. Moreover, the same perpendiculars can be produced, and cross the curve ACE in D, C, and B, surely at right angles from the hypothesis. Now, KE is less than MD : indeed, by drawing EL perpendicular to KE itself, since KE crosses the curved line ECA at right angles, EL will be a tangent to the curve ACE, and by necessity it crosses the line MD between D and M. Thus, since KE is the shortest of all lines which lies between the paralles lines EL and KM, then this will be less than ML, and hence entirely less too than the whole of MD. In the same way, HD can be shown to be less than NC, and GC less than OB, and FB less than PA.
[*i. e.* KE < MD; HD < NC; GC < OB; FB < PA]
Therefore as PA shall be greater than FB, the sum of PA and OF shall be greater than OB. Likewise since OB is greater than GC, the sum of OB and NG will be greater than NC. But the sum of PA and OF is greater than OB. Thus the sum of PA, OF, and NG is always greater than NC. Again, since NC is greater than HD, the sum of NC and MH is greater than MD. Hence, if in place of NC these three greater lengths are taken PA, OF, and NG, these four PA, OF, NG, and MH will be entirely greater than MD : and hence the same too is entirely greater than the line KE, since MD is itself greater than KE.

[Thus : PA + OF > OB; and OB + NG > NC; hence PA + OF + NG > OB + NG > NC. Again, since NC + MH > MD, then **PA + OF + NG + MH** > MD > **KE**.]

Moreover, we have said that all the subtended chords AF, FG, GH, and HK have a greater ratio to all the perpendiculars PA, OF, NG, MH, than the line Q to KE. Therefore, since KE is less than the sum of the said perpendiculars, all the said subtended chords have a greater ratio than Q to KE. Hence the sum of the subtended chords is greater than the line Q.
[Thus, AF/**PA** > Q/KE; FG/**OF** > Q/KE; GH/**NG** > Q/KE; and HK/**MH** > Q/KE. Hence, AF + FG + GH + HK > (**PA + OF + NG + MH**).Q/KE > Q, as required.]
Moreover, the curve AGK itself was taken to be greater. Hence the sum of the subtended chords AF, FG, GK, HK is greater than the length of the curve AGK subtending the parts; which is absurd, since the individual chords are less than the arcs of the curve. It is therefore not possible to have two curves which have the relation assumed between each other. Q.e.d.

PROPOSITION IV.

If two curves leave the same point turning in the same sense, and are in the same concavity, and thus are compared by the movement that truly all the lines which are tangents to the one curve, cross the other line at right angles; these are described from evolution from the first, after starting from a common point.

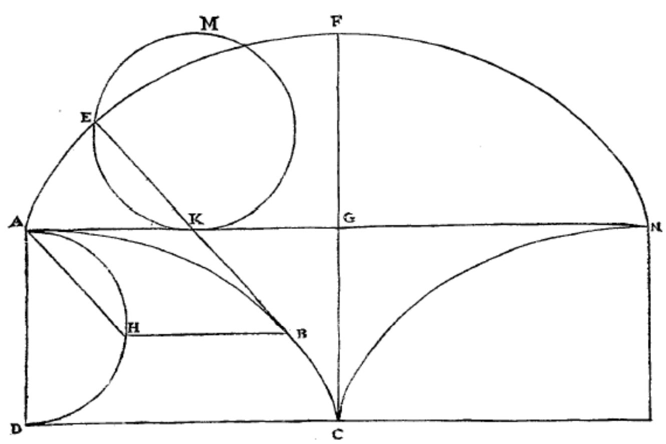


ABC and ADE shall be the curves, turning in one sense, and which are both present in the same cavity, having a common end point A. Moreover all the straight lines touching the curve ABC, such as BD or CE, cross the curve ADE at right angles. I say that from the evolute ABC itself, the curve ADE is to be described from the point of intersection A.

If indeed it were possible, let the evolute describe some other curve AFG. Therefore any straight line, touching the evolute ABC, such as BD, CE, cross the curve AFG at right angles (by Prop. 1, of this section), considered to be in F and G. But the same tangents are also put in place to cross the curve ADE at right angles. Therefore the curves ADF and AFG, with the same terminal point A, and which can be said to be curving in the same sense, and both in the same concavity, obviously each arise from the same curve ABC; for it is agreed from hypothesis that what concerns the curve ADE, truly concerns AFG, which has been asserted from the first part of this proposition. All the lines that cross one of these curves, also similarly cross the other, which indeed has been shown before not to be possible (Prop. 3, of this section). Whereby it is agreed that the curve ADE is described from the evolute ABC. Q.e.d.

PROPOSITION V.

If a line is the tangent to a cycloid at the vertex, upon which as a base, another cycloid similar and equal to the first cycloid is put in position, starting from the said vertex point; any tangent line of the lower cycloid crosses a part of the upper superposed cycloid at right angles.



The line AG is a tangent to the cycloid ABC at the vertex A, upon which as a base, the other similar cycloid AEF can be put in place, with vertex F. Moreover, the line BK is a tangent to the cycloid ABC at B. I say that line produced crosses the cycloid AEF at right angles.

Indeed the generating circle AHD is described about the axis AD of the cycloid ABC, which BH drawn parallel to the base, cross in H, and HA is joined. Therefore, since BK is a tangent to the cycloid at B, it is agreed that it is parallel to the line HA (Prop. 15, part 2). Thus AHBK is a parallelogram, and hence AK is equal to HB, that is, to the arc AH (Prop. 14, parts 2). Now the circle KM is drawn again KM, that is itself equal to the generating circle AHD, which touches the base AG at K, truly cut by BK produced at the point E. Therefore, since AH is parallel to BKE, and hence the angle EKA is equal to the angle KAH, it can be seen that BK product takes an arc from the circle KM equal to that which the line AH takes from the circle AHD. Thus the arc KE is equal to the arc AH, that is to the line HB, and to the line KA. Hence it truly follows, from the property of the cycloid, since the generating circle touches the rule at K, the point describing the cycloid is at E. Thus the line KE crossed the cycloid at E at right angles (Prop. 15, part 2). But KE is BK itself predicted. Thus it is apparent that BK produced crosses the cycloid at right angles. Q.e.d.

PROPOSITION VI.

From the evolute of a semicycloid, taken from the vertex, another semicycloid is described equal and similar to the evolute, the base of which is that line which the cycloid evolute touches at the vertex. .

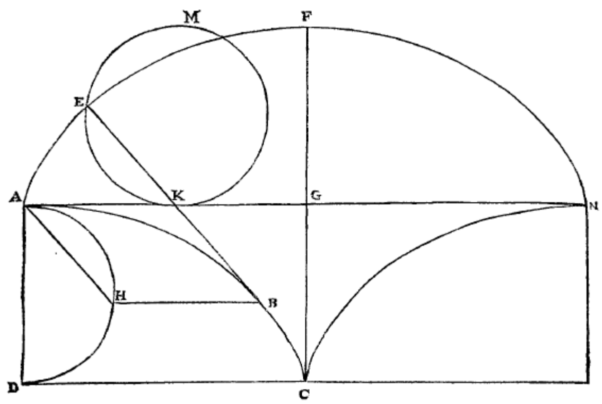
Let ABC be the semicycloid, to which shall be superimposed another similar cycloid AEF, as in the preceding proposition. I say, if the flexible string, applied around the semicycloid ABC, is unwound starting from A, that the semicycloid AEF itself is described by the extremity of the string. Since indeed the semicycloids ABC and AEF emerge from the point A, turning in one direction, and both in the same concavity, and besides thus for comparison, as all the tangents of the semicycloid ABC cross the semicycloid AEF at right angles; this follows from the evolution described of this, starting from the end A. (Prop. 4, of this section). Q.e.d.

And it is apparent, if we place the half cycloid, the twin of ABC itself or CN, on the contrary side of the other line CG, from the evolution of this, as now the string is extended to CF, then the other semicycloid FN is described by the end of the string repeated around that curve, which agrees with the whole arc AEF from before.

And from these considerations and from proposition 25 concerning the falling of weights, it can be shown now what was said above in the construction of the clock, about the equal motion of the pendulum [*i. e.* the independence of the period on the amplitude]. Indeed it is apparent that the hanging weight placed between the two laminar plates follows the curve of the semicycloid, on the support being disturbed, and by its own motion describes a cycloid, and hence the pendulum can be arranged to have equal oscillations of any desired period. Indeed the motion does not depend on whether the moving particle is attached to the surface and follows the curvature of the cycloid, for the same motion is performed by the thread bound to the curved line itself and the free end traveling through the air, with the same freedom in both places, and at all points on the curve it has the same inclination for motion. [Thus, the frictionless bead on the wire in the shape of an inverted cycloid follows the same motion as the pendulum bob.]

PROPOSITION VII.

The circumference of the cycloid is four times the diameter of the generating circle or of its axis. .



Indeed with the preceding figure repeated : since after the whole semicycloid ABC has been evoluted, the string takes up the line CF, which is double AD, since the axes of the cycloids ABC and AEF are equal; it is apparent that the semicycloid ABC itself, with the string wound round itself is equal to twice the length of the axis AD, and the length of the whole cycloid is four times this amount.

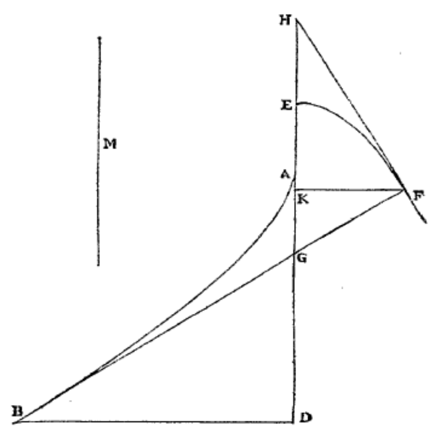
It is also apparent that the tangent BE, which is concerned with the extended part of the string, was applied to the part of the curve BA before, and to have a length equal to this. Moreover, BE is twice BK, or AH, since we have shown in proposition five that KE is equal to AH. Thus the arc of the cycloid AB is twice the length of the line AH or BK: obviously BH is parallel to the base of the cycloid : and wherever that point B may be taken.

That outstanding English geometer, Christopher Wren, first discovered this property of the cycloid by another long way, and then confirmed his result by an elegant demonstration, which has been published in a book on the cycloid by that most outstanding of men, John Wallis. Truly concerning the same line, many other very pretty results have been found by the mathematicians of our time, and for which the occasion is now presented for a certain special problem proposed by the Frenchman Blaise Pascal, who excelled in these studies. Along with his own discoveries, he reviews the discoveries of others, and in the first place everything that Mersenne is said to have turned to concerning the nature of the curve [The cycloid was first studied and named by Galileo, with whom Mersenne had a particular affinity, as he translated some of Galileo's work from Italian into latin, thus making them availaible for a much larger audience.] The tangents first introduced by Roberval are defined for both the plane and solid dimensions. Likewise the centres of gravity both for the whole plane and then for parts of these are to be found. The equality of the cycloidal curve first found by Wren to a line is given. I also have been the first to find the length of a portion of the cycloid, which is taken for a line parallel to the base, through a point on the axis, taken at a quarter part of distance from the vertex, which part obviously is equal to half the length of the equilateral hexagon inscribed in the generating circle. Finally Pascal himself has found the centres of gravity of the solid and semi-solid shapes formed by rotating the curve about the base and axis, and likewise the parts of these. Also, he has found the centre of gravity of the curve (after having received the result for the length of the cycloid from Wren), and the areas of the convex surfaces from which volumes and the parts of these are to be understood, [generated above, as surfaces and volumes of revolution] ; and the centres of gravity of these surfaces. And finally the lengths of any of these cycloidal curves, which are described either more produced or flattened, by taking a point outside or inside the generating circle. And certain demonstrations of these have been published by Pascal. The properties of the same curve have been explained also by Wallis in the most subtle ways, he seems to have discovered the same properties independently, and to have solved and extended to problems proposed by Pascal. Likewise the most learned Lovera claims to have discovered so much himself; truly the contributions of each should be judged by the learned from the writings of these men. So I myself must return so much of the priority of discovery, as these outstanding discoveries are not seen to passed over with silence, for which the matter is, from all the curves, non is now understood better or more fully than the cycloid. Indeed our method that we use to measure the lengths of curved lines, we are free to use for other curves, which we can turn to again now.

PROPOSITION VIII.

From the evolution of this line a parabola is shown to be described.

Let AB be a paraboloid, the axis of which is AD; and the vertex A; moreover the curve has this property: with the axis BD marked off in ordinates [x], the cube of the abscissa [y] measured from the vertex A on the attached vertical axis DA, is equal to a



solid having as base the square of DB, and height equal to that of a certain given line M; indeed this curve was noted by the geometers some time ago; and put in place the line AE joined to the axis DE, which has the length $\frac{8}{27}$ of M. [Thus, for a given value x along BD, $y^3 = \mathrm{M}x^2$ is the relation between x and y; a semi-cubic parabola which is a well-known rectifiable curve, as the next theorem shows.] Now if a continuous thread is connected around EAB, and it starts to unwind from E, I say that the parabola EF is described by evolution [Def. III], the axis of which is EAG, the vertex E, and the [semi-]latus rectum is equal to twice EA. [Thus, what Huygens called the latus rectum, or half the focal chord, we now call the semi-latus rectum, of length 2*a*.]

Indeed take some point B on the curve AB, from which the line BG is drawn tangent to the curve, crossing with the axis EA in G, and from G again GF is drawn, which crosses the parabola EF in F at right angles [The theorem is established by showing that this line BGF is in fact a common straight line]; and GF itself is perpendicular to FH, which touches the parabola in F; and thereafter the ordinate FK is applied to the axis EG [To give the (x, y) coordinates of the point F on the parabola with origin E : $y^2 = 4ax$].

KG is therefore equal to half the semi-latus rectum, that is, to EA itself ; and hence, by adding or taking AK from both sides [depending where the point F lies on the parabola], EK is equal to AG.

[This is a well-known property of the parabola : EA = KG = 2*a* ; also, from the similar right-angled triangles in Δ GFH, $HK \times KG = y^2 = 4ax$, from which HE = EK; See also Lockwood, *A Book of Curves*, page 6; in which some properties of the semi-cubical parabola are also listed; hence EA + AK = **EK** = KG + AK = **AG**.]

Moreover AG is a third of AD, since BG is a tangent to the paraboloid at B : for this can easily be shown from the nature of this curve [by constructing the tangent line and finding its intercept with AD]. Therefore EK is equal to a third of AD : and KH is twice KE from the nature of the parabola, which is equal to two thirds of AD.

[Thus, AG = AD/3 = EK; but KH = 2×KE; hence KH = 2 × AD/3.]

Thus the cube of KH is equal to $\frac{8}{27}$ of the cube of AD, that is, to the volume having the base equal to the square of DB, and with height truly equal to $\frac{8}{27}$M, that is, to AE . On this account, as the square of DB to the square KH, is thus as KH to AE by length, or to KG.

[by definition above; also, $KH^3 = \frac{8}{27}AD^3 = \frac{8}{27}M \times DB^2 = AE \times DB^2$, from which $DB^2/KH^2 = KH/AE = KH/KG$.]

Moreover KH is equal to $\frac{2}{3}$AD, that is to GD itself. Hence as the square BD to the square DG thus HK is to KG.

[$DB^2/DG^2 = KH/KG$.]

Moreover, as HK to KG, thus the square FK is to the square KG.

[$FK^2/KG^2 = HK/KG$ follows from the similar right-angled triangles, for which $FK^2 = HK.KG$.]

Hence as the square BD to the square DG, thus the square FK to the square KG. And hence as BD to DG by length, so FK to KG.

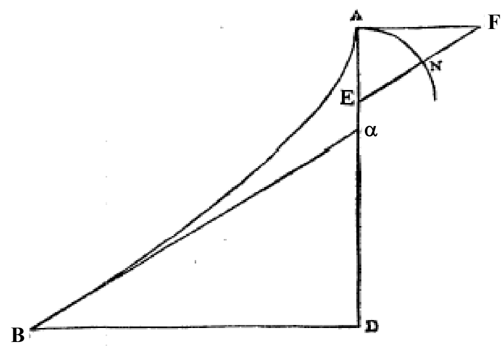
[Thus, $DB^2/DG^2 = KH/KG = FK^2/KG^2$, from which DB/DG = FK/KG.]

Hence it follows that BGF is a straight line. But GF crosses the parabola ED at right angles. Therefore it is apparent that BG produced, a tangent of the paraboloid, cuts the same parabola at right angles. And that the case is similarly for whatever tangent of this curve is demonstrated. Hence it is agreed that by the evolution of the line EAB, beginning from the point E, the parabola EF is described. (Prop. 4, of this section). Q.e.d.

PROPOSITION IX.

The right line is to be found equal to a given part of the curve of the paraboloid, truly of that in which the squares of the ordinates along the axis are as the cubes of the abscissas to the vertical.

How this comes about can be shown from the previous proposition. The parabola EF is not required for the construction, which proceeds thus. Thus, from some part of this paraboloid AB, for which it is necessary to find the right line of equal length, the tangent BG is drawn from the point B, which crosses the axis AG in G. Moreover BG is a tangent if AG is the third part of AD, intercepted between the vertex and the ordinate line BD. Again, take AE equal to $\frac{8}{27}$ of the line M, which is the semi-latus rectum of the paraboloid AB, EF is drawn parallel to BG, and crosses the line AF, which is parallel to BD in F. Now if NF is added to the line BG, the difference of the lines EF and EA, then a straight line is obtained equal in length to the curve AB. The demonstration of which from what has previously been said is easily seen.



Thus the curve AB is always greater than the tangent BG, by the amount that EF is greater than EA.

[In the previous theorem and diagram : BG + GF = arc length BA + EA; hence, arc BA = BG + GF - EA ; these lengths have been transcribed onto the present diagram.]

Moreover, we have inscribed this curve again, of which the lengths of all the previous lines have been measured out. Truly in the year 1659 Johan. van Heuraet of Harlem showed that the curves was equal to this right line, the demonstration of which can be found after van Schooten's commentaries in the Geometry of Descartes, published in that year. And that indeed was the first curved line of all, from a number of these of which the points are defined geometrically, to be reduced to this measure, since at the same time Wren gave the length of the cycloid, an equally ingenious undertaking.

I know equally well, apart from the publication of Heuraet's discovery, that the most learned Wallis wished to attribute the same discovery to a young man of his own country, William Neile, in his book on the Cissoid [1629]. But it seems to me, from that which is referred to, that not much can be taken away from Neile's discovery, nor even that it could be plainly understood. For from the demonstration of Wallis's curve that he produces, it is not apparent that it has been looked into carefully enough : for if that curve of his were to be constructed, the length would seem to be given. And it is plausible, if he had investigated a number of these that had been known by the geometers since ancient times [with his method], either by himself, or by others under his name, then there would have been so much more mature a discovery to communicate to the geometers, like that other, that merited so much praise, that they could exclaim *Eureka!* like Archimedes of olden times. For Fermat, so skilled in geometry, and a Counsellor of Toulouse, has written down proofs of these curves discovered, which were penned in the year 1660, but certainly later than those above.

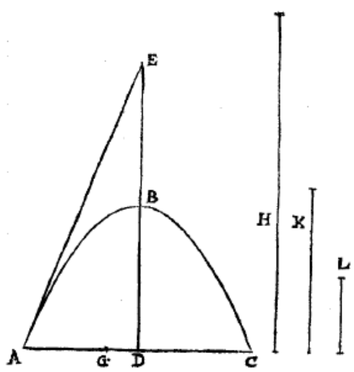
Since truly I have been engaged in these things, it is permitted for me as well to mention an outstanding discovery that I introduce here as an advancement in the subject : indeed I organised a visit so that van Heureat and other prominent individuals could be shown how the length of the parabola follows from the quadrature of a given hyperbola [making use of logarithms], that Heureat had partially found in his work. And indeed, I had stumbled across these two results towards the end of the year 1657 : the length of the parabolic curve, as I have said; and the reduction of the surface of a parabolic conoid to that of a circle. For I had indicated by letter to show these two unusual properties of the parabola that I had found, both to van Schooten [Huygens' mathematics teacher in his youth], and likewise with other friends. One of these results was the reduction of the surface of a conical paraboloid to that of a circle, and these letters were communicated by van Schooten to Heuraet, with whom he was then on friendly terms. Truly the result that the surface of the cone itself is related to the length of the parabolic curve was not too difficult for a man of his intelligence to understand. With each of these found, then on investigating further, he fell upon other parabolic curves, for which the lengths of other right lines could be completely discovered. [*i. e*. the curves could be rectified.]

And indeed for the reduction of the surface of a cone to a plane, stronger evidence could not be desired, than from this extract taken from the letter written to me by that most outstanding of men, and considered today to be among the most distinguished of geometers, Francois de Sluze [1622 - 1685]. This was written in the same year that I made my discovery, and with praise at greater length perhaps than it merits, for which he is given thanks. In which letter dated 24th December, 1657, thus he states : *I add only two results, the first, ... etc. The second is about all these curves in which a line is put in place of the curve, which are to be put as almost nothing before your discovery, by which you have shown the ratio of the surface of a parabolic conoid to that of the base circle. I can present all these pretty results freely for the squaring of the circle, with which in former times I have deduced almost nothing in place of the line, and which I can send to you if you so desire at some time.* [It was thus a natural extension at the time to move from rectifying lines to rectifying areas, such as that of the circle.]

Moreover in the following year I also found the area of the surface of the hyperbolic conoid and of the spheroid, which could be reduced to the area of the circle, and the construction of these problems hardly required anything for the demonstration. Then I was having a friendly exchange of letters with other geometers, in France with Pascal and others, in England I communicated with Wallis, who not long afterwards published his own ideas on these things, together with many more subtle discoveries; and which made the completion of our own demonstrations superfluous. Truly, since our own constructions do not seem to be lacking in elegance, and neither have they been put on public view, it is pleasing to set these out here.

To find the circle equal in area to the surface of a parabolic conoid.

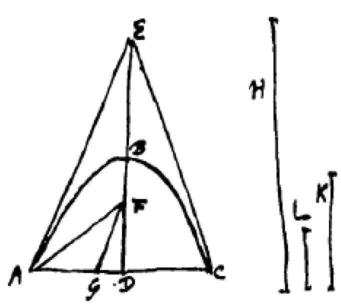
Let there be a conoid given of which a section through the axis is the parabola ABC; the axis of which is BD, the vertex B, and the diameter of the base AC, which shall be at right angles to the axis BD. And it is required to find the circle equal to the portion of the surface.



With the axis produced through the same side as the vertex, BE is taken equal to BD, and EA is joined, which is a tangent to the parabola at A. Again, AD is cut in G, in order that AG to GD is thus as EA to AD. And the sum of AE and DG is taken to stand equal to the right line H. Likewise a third of the base AC is equal to the line L, and the mean proportion K is found between H and L, as by which radius the circle can be described. This circle is equal to the area of the conoid ABC. Hence it follows, if AE were twice AD, the curved surface of the curve would be to the area of the base circle as 14 to 9. If AE is three times AD, as 13 to 6; if AE is four times AD, as 14 to 5. And thus always to be as one number to another number, if AE ad AD have a ratio of this kind.

There now follows an insertion from p. 267, Book XIV, of Huygens' *Oeuvres* relating to the above result:

For a given parabolic conoid, to find a circle equal in area to the surface of the conoid.

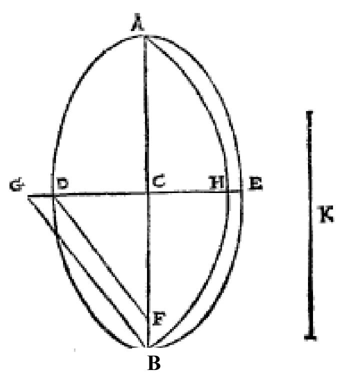


Let ABC be the given parabolic conoid, the axis of which is BD; and it is required to find the circle equal in area to the convex surface of the given conoid surface. The conoid is cut by a plane through the axis, and thus a parabola is present. The axis is produced and the axis is produced and ED is twice BD. With AE and EC joined, AF is drawn which bisects the angle EAD, and crosses the axis in F; then FG is drawn parallel to EA. Then let the line H be equal to the sum of AE and GD. Truly the line L is a third of the diameter of the base. And K is taken as the mean proportional between H L. I say that the circle with radius K is equal to the surface of the convex conoid ABC.

Since indeed the angle EAD is bisected by the right line AF, then EA is to AD thus as EF to FD. and put together such that the sum of EA and AD is thus to AD thus as ED is to DF.
[EA/AD = EF/FD; and thus (EA + AD)/AD = (EF + FD)/FD = ED/FD].
Moreover as ED to DF thus AD to DG. Therefore as the sum of EA and AD to AD thus AD to DG.
[from similar triangles : ED/DF = AD/DG and (EA + AD)/AD = ED/DF = AD/DG.] And with the former ratio taken with twice the latter, thus the perimeter of Δ AEC is to AC thus as AC to twice DG. On account of which, since the sum of twice AE and DG is to $3 \times AC/2$, or with half of each taken, as AE together with DG, that is as H is to $3\times AD/2$, thus the surface of the cone ABC to the base AC. Moreover as the ratio H to $3\times AD/2$, is composed from the ratios H to AD, and from AD to $3\times AD/2$; this is the same as L to AD; for since L is the third part of AC it is the same as $3\times AD/2$. Therefore the ratio that

the surface of the conoid ABC has to the base circle AC, is the same as that composed from the ratios H to AD and L to AD, and thus the same as H to L, to be contained as that which K has to the square of AD. Moreover, the square from K to the square AD, thus is as the circle with radius K described to the base of the circle of which DA is the radius. Therefore the ratio of the surface of the conoid AB to the circle with base AC is the same as the circle from K as radius the same circle AC. And hence as said, the surface of the conoid is equal to the circle of which K is the radius. Q.e.d.
[*It is apparent that this extra information does not actually solve the problem of the integration of the conoidal surface, which appears to come from other work on the paraboloid not presented. Likewise for the following results, which are quoted only, before the next proposition.*]

To find the circle equal in area to the surface of an oblate spheroid.

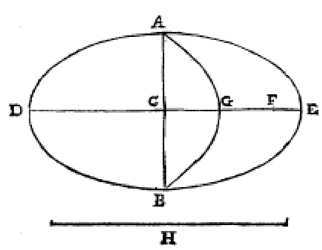


Let a spheroid be given with the long axis AB, centre C, with a section through the axis the ellipse ADBE, the minor diameter of which is DE.

DF is put equal to CB, of F is put at either of the foci of the ellipse ADBE, and the line FD is drawn parallel to BG, cutting BD produced in G. ED in G, and with centre G, with radius GB, the arc of the circumference BHA is described on the axis AB. Between the half diameter CD, and the line equal to the sum of the arc AHB and the diameter DE, the mean proportional shall be the line K. This will be the radius of the circle to which the surface of the spheroid ADBE shall be equal.

To find the circle of area equal to the surface of a wide or compressed spheroid.

There is a wide spheroid the axis of which is AB, the centre C, with the section of the ellipse through the axis ADBE.

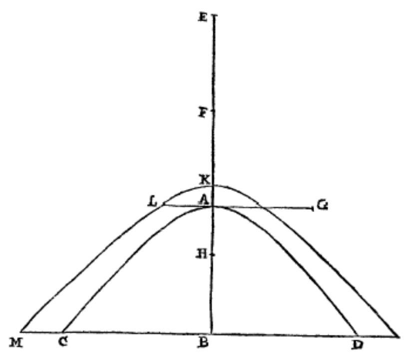


Again F shall be either of the foci, and with FC divided in half at G, the parabola AGB is understood to have as base the axis AB, the vertex truly the point G. And the mean proportional between the diameter and the right line of the curve of the parabola shall be the line H. This will be the radius of the circle which is equal to the surface of the proposed spheroid.

To find the circle of area equal to the surface of a hyperbolic conoid.

Let there be a hyperbolic conoid the axis of which is AB, with the hyperbola CAD the section of the conoid through the axis with the latus transversum [the inter-vertex separation] is EA, the centre F, and the latus rectum AG.

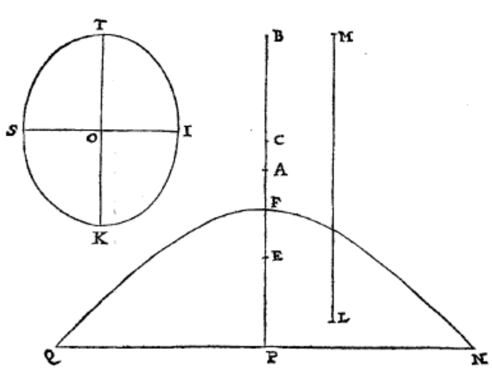
The line AH is taken along the axis, equal to half the latus rectum AG, and as HF is to AF in length, thus AF is to FK in power. And it is understood that from the vertex K another hyperbola KLM is described, with the same axis and centre F as before, and which has its latus rectum and the transverse rectum reciprocally proportional to these of the first curve. Moreover BC produced is cut in M, and AL is drawn parallel BC. Now the area ALMB, taken from three straight lines and the hyperbolic curve, is to half the square from BC, thus as the surface of the conoidal curve to the circle from its own base, the diameter of which is CD. Hence from the construction the rest is easily found, with the quadrature of the hyperbola in place.



Therefore when the surface of conical paraboloid can be reduced to the area of a circle, and equally the surface of a sphere, from the known rules of geometry; as likewise for the surface of an elongated spheroid, it can be accomplished if the length of the arc of the circumference can be put equal to a right line. Truly for the wide [or oblate] spheroid, and likewise the surface of the hyperbolic conoid can be reduced to the same ratio, for which the quadrature of the hyperbola is required. For the length of the parabolic line, as for the case of the spheroid we have referred to, follows from the quadrature of the hyperbola, as we will soon show.

Truly, what is worthwhile is seen from observation, and we can find the circle to be constructed equal to both surfaces taken together, for the wide spheroid and the conoidal hyperbola, without supposing any quadrature of the hyperbola.

Indeed for a given spheroid of any width, it is possible to find the hyperbolic conoid, or on the contrary, the spheroid of this kind can be found, and as each of the surfaces shows the equal circle at the same time, a single example in one case for simplicity is sufficient to be presented.

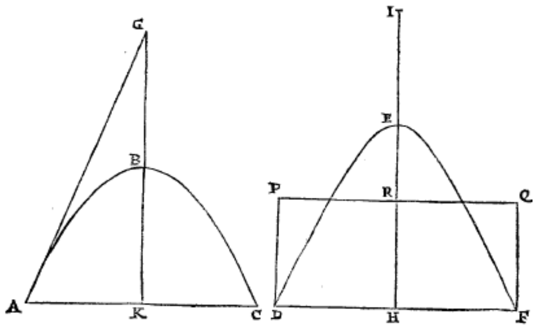


Let there be a spheroid with axis of length SI, with the section through the axis the ellipse STIK, the centre of this ellipse is O, and the major axis TK. Moreover, an ellipse of this kind is put in place, in order that the latus transversum TK has the same ratio to the latus rectum, as the following line is cut in the ratios of the extreme and the mean, from its larger part.

BC is taken as twice the power of SO, likewise BA has twice the power to OK, and these four BC, BA, BF, BE are four continued proportionals, and EP is put equal to EA. It is now understood that the hyperbolic conoid QFN, the axis of which is FP; is to have added to the axis $\frac{1}{2}$ latus transversum FB; while half the latus rectum is equal to BC.

The curved surface of this conoid, together with the surface of the spheroid SI, is equal to the circle of which the radius is given as M, which truly has the square TK with twice the square SI.

To find the right line equal in length to a parabolic curve.

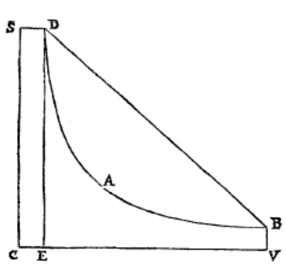


ABC is part of a parabola, the axis of which is BK, with the axis at right angles to the base AC ; and it is required to find the right line equal to the length of the curve ABC.

The line IE is taken equal to half the base AK, which is produced to H, in order that IH is equal to AG, which is a tangent to the base of the parabola at the point A, with the axis produced meets in G. Now DEF is a part of a hyperbola described, with vertex E, centre I, the diameter of which is EH; truly DHF is considered as the ordinate axis to be applied to the diameter. The latus rectum can be taken freely. Concerning which, it is understood that a parallelogram DPQF is set up on the base DG, which is equal to the area of the part DEF of the hyperbola; and the side PQ thus cuts the diameter of the hyperbola in R, in order that RI is equal to the length of the parabola AB, double which is equal to ABC.

It is hence apparent that in some manner the length of the parabola depends on the quadrature of the hyperbola, and vice versa.

Truly whichever problems resolved from these two can be applied to find the solution of the other, however truly an approximate numerical solution has to be taken, from the wonderful invention of logarithms. [This, one presumes, is a witticism on the title of Napier's book *Mirifici Logarithmorum Canonis Descriptio* : A Description of the Wonderful Canon of Logarithms, originally published in Edinburgh in 1614; the type of logarithms Huygens had in mind were base10 logarithms, to which of course Napier's logarithms are related, and we are to understand that Huygens had discovered the connection between these and the area under the hyperbola for himself, or perhaps had perfected the ideas that originally came from Gregorius and his student de Sarasa, for the latter first made the connection between areas under the hyperbola and what came to be natural logarithms. See Eli Maor, *e, The Story of a Number,* for some details on this. The history of mathematics is rather vague on this point.] Since by means of these logarithms, as by us previously found, the quadrature of the hyperbola can be set out numerically in an approximate manner. Moreover, this follows by a rule of this manner.



DAB is a portion of a hyperbola, the asymptotes of which are CS and CV, with DE and BV drawn parallel to the asymptote SC.

The difference of the logarithms which are in agreement with the numbers, have the same ratio between each other as the lines DE and BV; and the logarithm of this difference is required. To this is added the logarithm (which is always the same) 0,36221, 56887. The sum is the logarithm of the area which is designated by DEVBAD, to be taken from three straight lines and the curve DAB, in terms of which the parallelogram DC has an area of 100000, 00000 parts. Thus again the area of the portion DAB is easily found too.

For example, let the proportion of DE to BV be as 36 to 5.

From 1, 55630, 25008, the logarithm of 36
is taken 0, 69897, 00043 the logarithm of 5.

Then 0, 85733, 14965. is the difference of the logarithms.
and 9, 93314, 92866. is the difference from 10 of the log. of this difference.
To which 0, 36221, 56887, the logarithm [of log *e*] is always to be added.

Hence 10, 29536, 49743. is the logarithm of the area DEVBAD.

A number of 11 characters is produced in this logarithm, when the characteristic is 10. Thus the nearest small number is sought from the first, agreeing with the logarithm found, which is the number 19740. Then from the difference of the same logarithms, and following nearby in the table, the remaining places 81026 can be found, to be written after the first, in order that it becomes 197408,10260, with a zero added at the end, so that it effects a number with 11 digits. Hence the area of the surface DEVBAD is nearly 197408,10260 parts, when the parallelogram DC has 100000, 00000 parts.

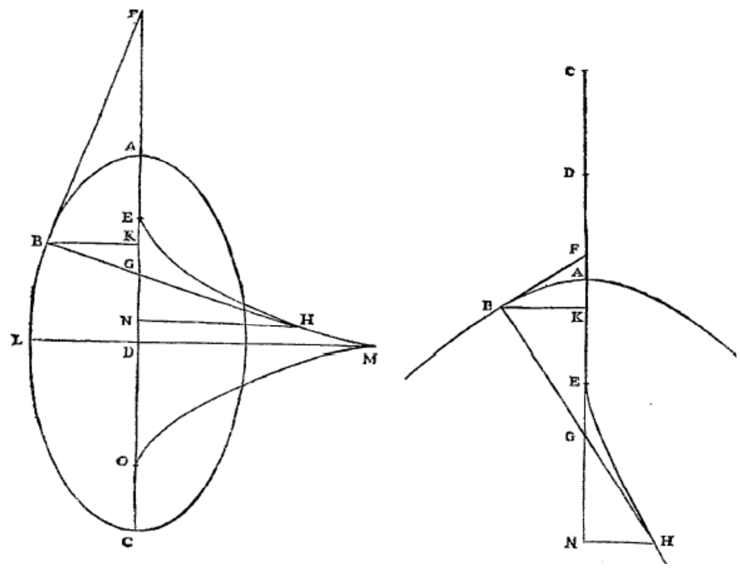
[Initially we note that all the logarithms used are to base 10, and that the modern calculation using natural logs is simply ln(36/5) = 1.974081026 as required; however, Huygens is comparing the area of the rectangle CD of size 10^{10} with the area under the hyperbolic curve between 5 and 36 using logs to base 10, for which he has a set of tables. His initial ratio of the areas is simply that of area par'gram CD : area DEVBAD, or as R = 10^{10} : ln(36/5) in modern terms ; the issue is complicated by taking logs to base 10 of this ratio from the start, thus we have the log R = 10 - log(log e × log (36/5)) = 10 - loglog e - loglog(36/5) = 10 + 0.3622156887 - 0.06685071441 = 9.9331492866 + 0.3622156887 = 10. 2953649743 as required. Note: 'log' means logs to base 10, and 'ln' means natural logs to base e.]

PROPOSITION X.

The curves described by evolution from the ellipse and the hyperbola are described, and also the lengths of the right lines equal to these curves are found.

Let AB be some ellipse or hyperbola, the transverse axis of which is AC; D the centre of the figure; and double the latus rectum AE. And with some point such as B taken in the section, ordinates are applied to the right axis BK, and to the given point B the tangent is drawn which meets the axis at F; and BG is perpendicular to FB, and crosses the axis at G; and BG is produced as far as H, in order that BH to HG has the same ratio as that composed from the ratios GF to FK and AD to DE.

I say that the curve EHM, all the points of which are found in the same manner as the point H, are from the evolution of this, together with the line EA, described by the section AB. Moreover BH itself is a tangent to the curve at H, and is equal to the whole of HEA. On account of which, if EA is taken from HB, the remainder is equal to a portion of the curve HE. Also it is apparent, by taking any points, for a certain ratio is found, that is



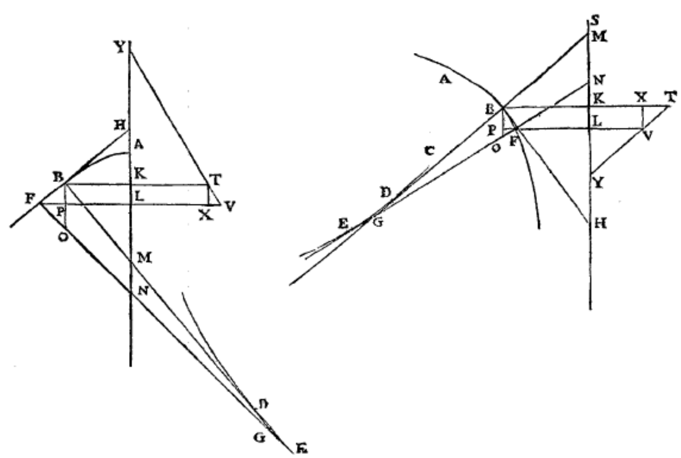
generated from these in both places, which are not considered to be related geometrically. Hence the relation of all these points to the points of the axis AC, can be expressed by some equation, and I have found such an equation which goes as high as the sixth power ; and it has the smallest number of terms when AB is a hyperbola of which the laterus transversus and the latus rectum are equal. Then indeed from some point of the curve such as H, the line HN is drawn perpendicular to the axis CAN, and AC is to be called *a*; CN, *x*; & NH, *y*; then the cube from *xx-yy-aa* is always equal to 27*xxyyaa*.
[i. e. $(x^2 - y^2 - a^2)3 = 27x^2y^2a^2$.]
But also in this case, the points of the curve EHM can be found in a much shorter time than in the previous construction, as is we now proceed to show.

For the remainder, it is noted that for the ellipse, the evolutes are described for the individual quadrants ; thus the quadrant ABL by the evolution of the line AEHM, the quadrant CL by the evolute of that similar to the opposite of this, COM. This is indeed different in each conic section [*i. e.* the ellipse and the hyperbola], since from the beginning of the curve EHM, for the ellipse so for the hyperbola, the point E is taken with AE equal to $\frac{1}{2}$ of the latus rectum ; in the case of the hyperbola the said line thus extends to infinity, at while in the case of the ellipse it is ended in a point M of the minor axis, with LM equal to $\frac{1}{2}$ the latus rectum, following which ordinates can be applied to the said minor axis. And for these end points that there are for this curve, the origin of this curve is easy to be considered, and which is the case of the ellipse is thus AD to DE, thus as LM to MD.

However we will not delay to demonstrate these curves, but move on to discussing that method itself, by which these curves derived from conic sections, and other innumerable curves from any others given, can be found.

PROPOSITION XI.

For a given curved line, to find another curve that describes this given line as its evolute; and to show how from any geometrical curve, another curve with the same geometry can be found, to which it is possible to give an equal right line of the same length.



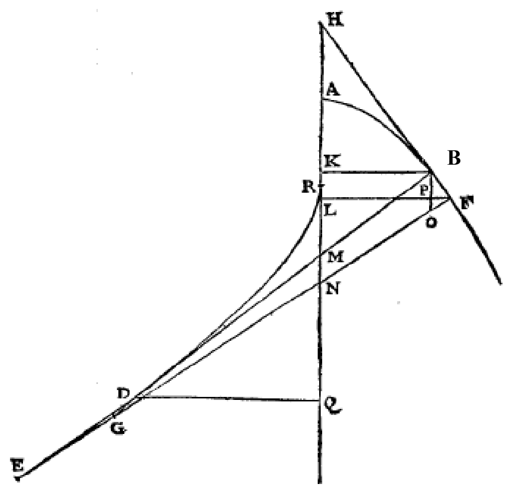
Let ABF be a part of some curve, turning in one direction, and there is a line KL, to which all the points can be referred; it is required to find another curve, such as DE, that is described by evolution from this curve ABF.

[Suppose] the curve to be found is put in place; since all the tangents to the curve DE for the described evolute, by necessity cross the curve ABF at right angles, then it is apparent too that these which follow the curve ABF at right angles, such as BD and FE, touch the evolute CDE in turn.

Moreover the points B and F are understood to be close together; and if a certain line is put in place to start the evolute from the part A, thus a point F further than B, also touches at E more distant than D is separated from A; truly the intersection of the lines BD and FE, which is G, lies beyond the point D on the line BD. For BD and FE to cross each other it is necessary that they remain at right angles to the concave part of the curve.

Moreover when the point F is in the vicinity of B, it is apparent that the points D, G, and E come closer together; and thus, if the distance BF is understood to be indefinitely small, the three said points become one and the same point; and furthermore, the line BH is drawn, which is a tangent to the curve at B, and the same too will be considered as the tangent at F. Let BO be parallel to KL, and the perpendiculars BK and FL fall on this line; and the line FL cuts the line BO in P; and points M and N are to be noted, in which the lines BD and FE themselves cross KL. Therefore, since the ratio BG to GM is the same as BO to MN, with this latter given the former is also known ; and since the line BM is given in size and position, the point G is given in BM produced, or the point D on the curve CDE, since G and D come together in one point as we have said. Moreover, given the ratio BO to MN; indeed in the case of the cycloid simply, where we first investigated all that, we found the ratio to be two to one ; truly for other curves, which we have recently examined, the ratio is composed of two given ratios. For since the ratio BO to MN is composed from the ratios BO to BP, or NH to LH, and from BP (or KL) to MN;

it is apparent that if both these ratios are given, then from the composition of these the ratio of BO to MN is also going to be given.
[For **BO/MN = BO/BP × KL/MN = BG/GM**: This is an important relation.]
Truly these are to be given for all geometrical curves, in what appears here subsequently; and hence these curves can always be assigned, which can be described from the evolutes, and for which likewise the former can be reduced to straight lines [*i. e.* they are rectifiable].



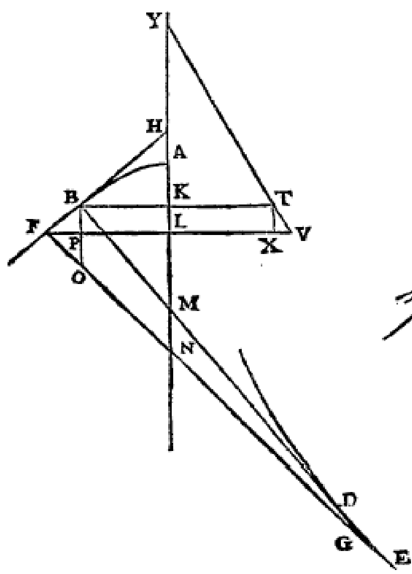
In the first place, let the curve ABF be a parabola, the vertex of which is A, and with the axis AQ. Since the lines BM and FN are at right angles to the parabola; and the perpendiculars BK and FL are drawn to the axis AQ; from the property of the parabola, the individual lengths MK and NL are each equal to half the length of the latus rectum; and by taking away the common length LM, KL and MN are equal to each other. Hence, when the ratio BG to GM is composed from the ratios NH to HL, and KL to MN, as was said, and the second of these ratios is one of equality; it is resolved that the ratio BG to GM is the same as that which NH has to HL; and on division, BM to MG, is in the same ratio as NL to LH, or MK to KH; for LH and KH are put equal, on account of the closeness of the points B and F.
[BG/GM = BO/BP = NH/HL; hence BG/GM - 1 = NH/HL - 1 or **BM/GM** = **LN/HL** = MK/KH as K and L coincide.]
Moreover, the ratio MK to KH is given, for a given point B; since then MK and KH are given with their length : since MK is equal to half the latus rectum, and KH is truly twice KA. The position and length of the line BM is also given. Hence MG is given, and thus the point G, or D, on the curve RDE is given ; which truly is found from BM produced as far as G, in order that BM to MG is thus in the ratio $\frac{1}{2}$ (latus rectum) to 2 × KA.

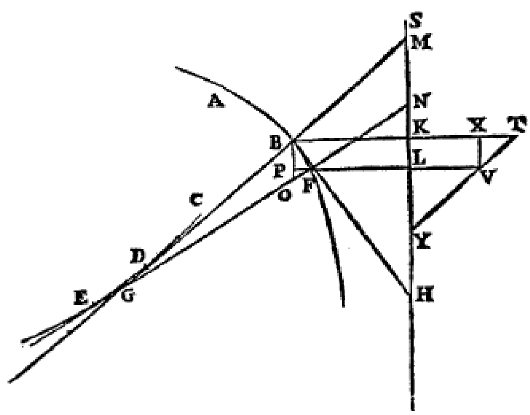
And thus indeed, for any other points taken on the parabola ABF instead of B, all the points on the line RDE are found by the same reasoning; and this line RDE is agreed to be geometrical [one presumes geometrical means rectifiable], and from one known property of this line, all the others can be deduced. Hence if we want to know the equation, as you might ask, that relates the length of the line AQ to a point on the curve GDE for all the points : the line DQ is drawn perpendicular to AQ, and with the latus rectum of the parabola ABF called a [modern usage puts this length as $2a$ for the standard equation $y^2 = 4ax$]; AK, b; AQ, x; QD, y. Since the ratio BM to MD, that is KM to MQ, is in the same ratio as $\frac{1}{2}a$ to $2b$, and KM = $\frac{1}{2}a$, then MQ is equal to 2b. Moreover, MA = $\frac{1}{2}a + b$; hence AQ or x is equal to 3b + $\frac{1}{2}a$. Hence, $b = \frac{1}{3}x - \frac{1}{6}a.$ Again, since the square MK (or $\frac{1}{4}aa$), is to the square KB, (or ab from similar Δ's), thus as the square MQ, (or $4bb$) is to the square QD (again by sim. Δ's); then the square QD or $yy = \frac{16b^3}{a}$. Where, if $\frac{1}{3}x - \frac{1}{6}a$, is substituted in place of b, to which it has been found to be equal, the equation becomes yy = 16 cub. $\frac{1}{3}x - \frac{1}{6}a$, with division by a. And hence $\frac{27}{16}ayy =$ the cube from $x - \frac{1}{2}a$.

[Thus, $y^2 = 16(\frac{1}{3}x - \frac{1}{6}a)^3 : a$; from which $\frac{27}{16}ay^2 = (x - \frac{1}{2}a)^3$]

AR is taken on the axis of the parabola [at the point of intersection] = $\frac{1}{2}a$; and RQ = $x - \frac{1}{2}a$. Therefore the curve sought is of such a nature that the cube of the line RQ is always equal to a parallelepiped, the base of which is the square QD, and with the altitude $\frac{27}{16}a$; and hence this is the paraboloid, the evolute of which is the parabola AB described that we have shown above; of which obviously the latus rectum is equal to $\frac{27}{16}$ of the length of the latus rectum of the parabola AB. Hence the latus rectum of this curve is indeed equal to $\frac{27}{16}$ of the latus rectum of the parabola, as has there been defined.

Again in this way, from the ratio OB to BP, or NH to HL, not only then is it the case that ABF is a parabola, but also it has been shown that some other geometrical curve can be found. Since only with the line FH drawn, which is a tangent to the curve at some point F, and FN itself is perpendicular to FH: then NH and HL are given, and hence the ratio of these lines are given too.





But it is not yet clear how the ratio KL to MN can become know in the general case, thus we show how this can always be found.

The lines KT and LV are perpendicular to the line KL, and KT is equal to KM, and LV is equal to LN, and VX is drawn parallel to LN, which crosses KT at X. Therefore, since there is always the same difference between the two lines LK and NM , and the two line LN and KM, that is, of the two lines LV [= LN] and KT [= KM]; moreover the difference of KT and LV is equal to XT, and XV is equal to LK; hence NM is equal to the sum of VX and XT, or that by which VX is itself greater than XT. And thus, if the

ratio VX to XT is given, the ratio VX to the sum of VX and XT, or to the excess of VX over XT, that is, the given ratio of VX (or LK) to NM.
[KT = KM; LV = LN; hence, MN - LK = KM - LN = KT - LV = XT ;
and MN = VX + XT (convex curve on right) or likewise, MN = VX - XT if LV > KT (concave curve on left). Hence, VX : VX + XT = VX : MN;
or VX : VX - XT = VX : MN]

Moreover, it should be known, since KT is taken equal to KM, and likewise LV to LN, in place of the points T and V, there is to be given some line either a straight line or a curve, as will soon be shown . And if it should be a straight line, as ABF is the part of a conic section, and KL the axis ; it is agreed that the ratio VX to XT is given, with the position of the line VT itself, which is the locus of the points V and T; and thus always having the same stated ratio, for whatever interval KL.

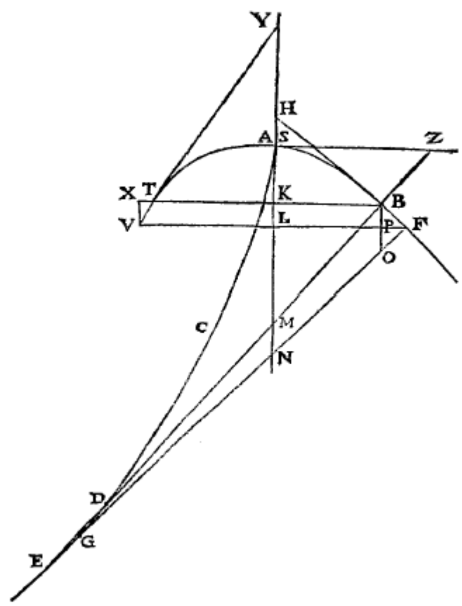
But if the locus were some other curve, the ratio VX to XT would be different, as the interval KL would be larger or smaller. Moreover, it is to be inquired how the ratio will be, when KL is taken to be indefinitely small, since the points B and F are in turn placed close together. Thus similarly the points V and T are understood to intercept the line of the curve with minimum separation; thus the line VT is a tangent that coincides with the curve at the point T. Hence TY is that tangent line ; indeed it is possible to draw some curve on which the points T and V lie, which is geometrical. Therefor the ratio YK to KT is given, and thus also VX to XT, from which we will show also that the ratio LK to NM is given.

Since indeed there is a line on which the points T and V are present, a certain point S is taken on the line KL on the line KL, and calling SK, x; and KT, y. For since the curve ABF is given, and BM is drawn at right angles to this curve, thus the length of the line KM is found by the method of the tangents according to Descartes, to be equal to KT or y, and from this equation, the nature of the curve TV is known, the which the tangent is to be drawn. But everything will be made clear in the following example.

ABF is that paraboloid, for which we found above a line of equal length : in which truly the cubes of the perpendiculars in the line SK are between themselves as the squares of the abscissae SK themselves. And it is required to find the curve CDE from the evolute of which the curve SBF is describes. In the first place here the ratio BO to BP is easily found, since we know how to draw the tangent of the paraboloid at the point B, by taking SH to be equal to $\frac{1}{2}$ SK. To which with BM normal to the tangent put in place, now the lines MH and HK are given, and hence the ratio between these, which is the same as OB to BP.

Moreover, as the ratio BP or KL to MN in known, the perpendiculars LT and LV to KL are put in place, equal to the individual lines KM and LN, and VX is drawn parallel to LK. Now, since from the two lines KL and LN added together by taking the line KM, MN is left; that is, by taking from the lines XV and VL, or XV and XK, KT itself; moreover it is apparent that the remainder is VX and XT : hence these two lines VX and XT are themselves equal to MN, and hence the ratio KL to MN is the same as VX to the sum of VX and XT. Moreover this ratio in known since the interval KL is a minimum; in the second place it is necessary to inquire what the locus shall be, if the line on which the points T and V are present. In order that this shall be done, let the latus rectum of the paraboloid ABF = a; SK = x; KT = y.

Therefore since the proportionals are KH, KB, KM, and HK = $\frac{3}{2}x$: from the nature of the paraboloid KB is equal to R.cub.$aax = y$: KM or KT = $\frac{2}{3}$ R.cub.$aax = y$, and hence $\frac{8}{27}aax = y^3$. Thus it is apparent that the locus of the points T and V is the paraboloid called the cube by geometers. To which the tangent at T is drawn, with SY taken as twice SK, to which YT is added. And now indeed the ratio VX to the sum of VX and XT, is as

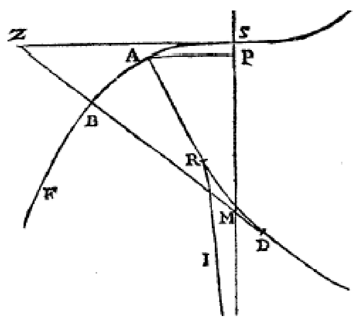


we have said to be the same as KL to MN, is that which YK has to the sum of YK and KT. Moreover this ratio has been given, and hence the ratio KL to MN. But the ratio OB to PB has been shown to be given. Hence, since the ratio BD to DM composed from these two ratios is also known, as is apparent from above; & on division, the ratio BM to MD is known; and thus the point D on the curve DE.

Moreover we give a shorter construction here according to this arrangement, KT or the said KM shall be y. Thus MH is $y + \frac{1}{2}x$. And MH to HK, or OB to BP, as $y + \frac{1}{2}x$ to $\frac{1}{2}x$; or , with everything taken doubled, as $2y + 3x$ to $3x$. Then since YK = $3x$, YK is to YK + KT, or from previously, KL to MN, as $3x$ to $3x + y$. And from the ratios OB to BP, and KL to MN, we have the said composed ratio BD to DM. Hence the ratio BD to DM is composed from the ratios $2y + 3x$ to $3x$, and $3x$ to $3x + y$; and thus it is that ratio $2y + 3x$ to $3x + y$; and by division, the ratio BM to MD, is the same as y to $3x + y$.

Let SZ be perpendicular to SK, and which crosses the line MB produced in Z. Therefore since the ratio BM to MD has been found which is y to $y + 3x$, that is MK to MK + 3KS. Since MK to MK + 3KS, thus MB to MB + 3BZ : and hence MB to MD as MB to MB + 3BZ. **Thus it is clear that MD is equal to the sum of MB + 3BZ**.
[This result, and others like it, enables the points on the evolute to be found for the various curves, as shown finally in tables for paraboloids and hyperboloids.]
And thus it is possible to find any point on the curve CDE. Any portion of the curve such as DS will be equal to the line DB which crosses the curve at right angles. Moreover it is agreed to be geometrical, and if we wish, we can express all of the points by the equation from another relation to the points of the axis SK.

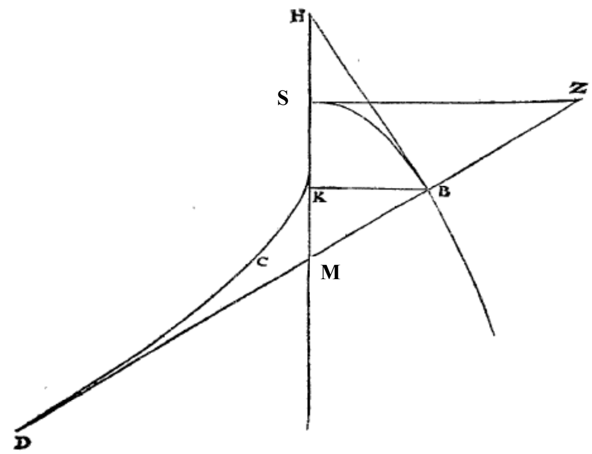
Moreover, in the same manner, if we look in the paraboloid for that cubic or cubic parabola, in which the cubes of the ordinates are as the abscissas along the perpendicular axis of the curve, we will find the curve that is described by the evolute, and hence to which a straight line may be placed equal [for its length from some point], with nothing more difficult in the determination of the points. For if that cubic paraboloid curve is SAB; with axis SM; ([these are Huygens' brackets] it is not correct to talk about the axis of this curve, since its shape is of such as kind, that on drawing SZ, which cuts SM at right angles, these like portions of the curve are on opposite sides;) a line BD is drawn through some point B taken on the paraboloid, cutting the curve SAF at right angles, and crossing the axis at M, and also the line SZ at Z. Then BD is taken equal to half of BM, together with one and a half parts of BZ. D shall be one of the points on the curve RD sought for the evolute of ABF, or after a certain straight line RA has been added, the paraboloid SAB will be described. [Thus, we can imagine a string winding on to the curve DMR until R is reached, upon which it meets the straight section AR and stops; thus one part of the cubic curve is generated.] Moreover, it is noteworthy here, as in certain other cases of paraboloids of this kind, that the two evolutes in opposite parts, both of which start from a certain point A, are each tangents to these paraboloids ; thus, as from the evolute of ARD, again to be continued indefinitely, the infinite part of the ABF is described; moreover by evolution the whole of ARI, similarly in indefinite extension, as described by the small part AS. Moreover the point A is defined, with SP taken, which is in the ratio to the latus rectum of the paraboloid, thus as one to the 8^{th} root of the number 91125, (this number is the cube of 45) with the ordinate PA corresponding to this taken. Thus again, the point R is found, the common end point of the two curves RD and RI, and thus the rest of the points on these curves, that is, in the same manner in which the point D was found.



Finally, for whatever kind of paraboloid is taken for the curve SAB, it is always easy to find another curve, described from the evolute of this same curve, and to which it is therefore possible to equate to a straight line, through points to be found that we can verify. And thus the general construction is shown in the following table, which at some time one is at liberty to extend.

$$\text{If}\left\{\begin{array}{lcl} ax = y^2 & & \text{BM} + 2\text{BZ} \\ a^2x = y^3 & & \tfrac{1}{2}\text{BM} + \tfrac{3}{\text{S}}\text{BZ} \\ ax^2 = y^3 & \text{then} & 2\text{BM} + 3\text{BZ} \\ ax^3 = y^4 & & 3\text{BM} + 4\text{BZ} \\ a^3x = y^4 & & \tfrac{1}{3}\text{BM} + \tfrac{4}{3}\text{BZ} \end{array}\right\} = \text{BD}.$$

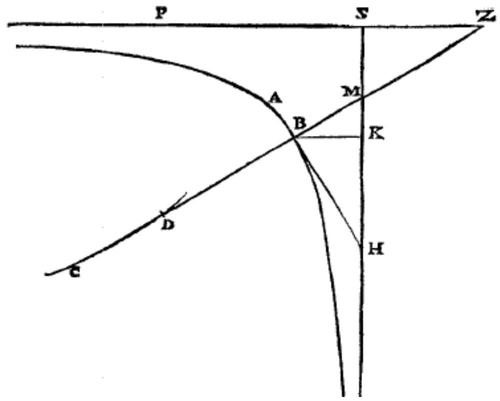
SB is a parabola, or some other paraboloid, the vertex of which is S; the line SK is either the axis, or a perpendicular to the axis, to which the points of the paraboloid are referred by an equation ; and indeed SK is always understood to be drawn in the concave side of the curve; perpendicular to SZ. Now put SK = x;



BK = y, which is the perpendicular to some point on the curve from SK; and the latus rectum of the curve = a; the first part of the table, which is on the left, sets out the nature of the individual parabolas in terms of their equations. To which the corresponding parts on the right hand side relates the length of the line BD, which if it stands upon the curve SAB at right angles, shows the position of the point D on the curve CD sought. For example, if SB is a parabola from a conic section, that we know agrees with the first equation in the table, $ax = y^2$; to which there corresponds from the other part BM + 2BZ = BD. Thus the length of the line BD is known, and thus any points of the curve CD can be found. As indeed in this case, the paraboloid for the above demonstration is that in the third row of the table.

Moreover the table is constructed according to this agreement, in order that BM is taken according to the number which is the exponent of the power of x in equation ; while BZ is taken several times according to the exponent of the power of y; while the numerator is composed from both these, the denominator is taken from the power of the exponent a.

Moreover besides these paraboloid curves, we can likewise find others, from which by a not dissimilar construction, the comparable rectifiable curves can be deduced. Moreover, they can be compared with the hyperbolas, which have their own asymptotes there set at right angles. And the first of these that we set up is the hyperbola itself, derived from the conic section.



Indeed the nature of the rest can be set out as we proceed; let PS and SK be the asymptotes of the curve AB, making a right angle, and from some point B on the curve BK is drawn parallel to PS, and SK = x; KB = y. Therefore if the hyperbola is AB, we know that the rectangle of the lines SK and KB, that is, the rectangle xy is always equal to the square which may be called aa.

Indeed the nearest is the hyperboloid, in which the volume from the square of the line SX, by the height KB is drawn, that is, the volume xxy, to which the cube that can be called a^3 is indeed equal. And thus innumerable other kinds of hyperboloids exist, the following table shows the property of each from their equations, and at the same time the ratio by means of which the curve DC can be constructed, and from which the hyperboloid has been generated from the evolute as well.

$$\text{If}\left\{\begin{array}{l} xy=a^2 \\ x^2y=a^3 \\ xy^2=a^3 \\ x^3y=a^4 \\ xy^3=a^4 \end{array}\right. \quad \text{then} \quad \left.\begin{array}{l} \tfrac{1}{2}\mathrm{BM}+\tfrac{1}{2}\mathrm{BZ} \\ \tfrac{2}{3}\mathrm{BM}+\tfrac{1}{3}\mathrm{BZ} \\ \tfrac{1}{3}\mathrm{BM}+\tfrac{2}{3}\mathrm{BZ} \\ \tfrac{3}{4}\mathrm{BM}+\tfrac{1}{4}\mathrm{BZ} \\ \tfrac{1}{4}\mathrm{BM}+\tfrac{3}{4}\mathrm{BZ} \end{array}\right\}=\mathrm{BD}.$$

The line DBMZ cuts the curve AB at right angles, as before also, and it crosses the asymptotes SK and SP in M & Z. If therefore for example, the hyperbola was AB, and for which the equation is $xy = a^2$, and taking BD = $\tfrac{1}{2}\mathrm{BM}+\tfrac{1}{2}\mathrm{BZ}$, as instructed by the table. And it will be the point D sought in the curve DC, and any other points on this line can thus be found, and the segment of this line is equal to a line of some given length. And this indeed is that same curve, that we have expressed in the equation to the axis above. Moreover the construction of this table is clearly the same as that shown previously.

The remaining constructions of these curves , as well as that from which the paraboloids are generated, require the lines DBZ to be drawn, which cut the curves AB in the given point B, or the tangents themselves BH at right angles; we may say in general how these tangents are to be found. Thus in an equation, in which the nature of the curve is set out, such as the equations presented in the two preceding tables, it is necessary to consider what the powers of the exponents x and y shall be, to be made thus : the ratio SK to KH is as the exponent of the power x to the exponent of the power y. For the line HB is a tangent to the curve at B; just as in the third hyperboloid, the equation of which is $xy^2 = a^3$: since the exponent of the power of x is 1, and the exponent of the power of y is 2; it is required that SK to KH is thus as 1 to 2. However, the demonstration of these are known to those skilled in the analytical arts, who have been considering these lines now for some time; and not only of these paraboloids, but also of certain [higher order] curves of any powers, that are placed between the hyperboloids and their asymptotes, and of plane-volume dimension. And indeed we can establish the tangents also, by this easy and general method, where the demonstration is given in terms of the property of the tangents only. But this is not the place for such demonstrations.

HOROLOGII OSCILLATORII

PARS III.

De linearum curvarum evolutione & dimensione.

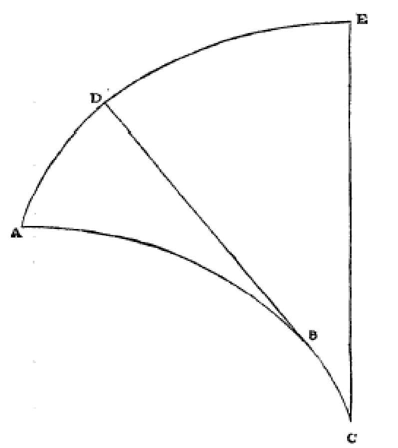
DEFINITIONES.

I.

Linea in unam partem inflexa vocetur quam recta omnes tangentes ab eadem parte contingunt. Si autem portiones quasdam rectas lineas habuerit, hae ipsa producta pro tangentibus habentur.

II.

Cum autem duae hujusmodi lineae ab eodem puncto egrediuntur, quarum



convexitas unius obversa sit ad cavitatem alterius, quales sunt in figura adscripta curvae ABC, ADE. ambae in eandem partem cavae dicantur.

III.

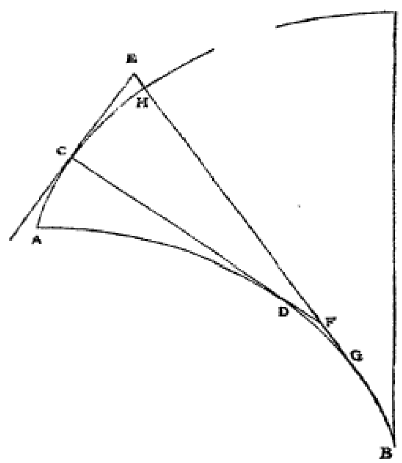
Si lineae, in unam partem cave, filum seu linea flexilis circumplicata intelligatur, & filum seu linea flexilis circumplicata intelligatur, & manente una fili extremitate illi affixa, altera extremitas abducatur, ita ut pars ea quae soluta est semper extensa maneat; manifestum est curvam quandam aliam hac fili extremitate descibi. Vocetur autem ea, Descripta ex evolutione.

IV.

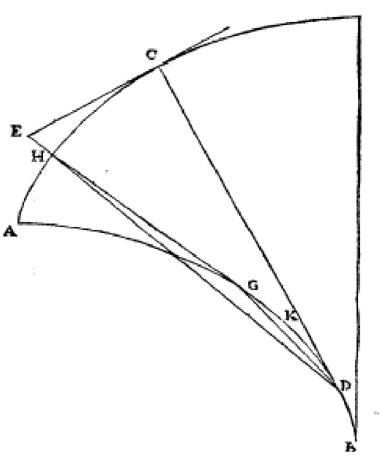
Illa vero cui filum circumplicatum erat, dicatur Evoluta. In figura superiori, ABC est evoluta, ADE descripta ex evolutione ABC, ut nempe cum extremitas fili ex A venit in D, pars fili extensa sit DB recta, reliqua parte BC adhuc applicata curvae ABC. Manifestum est autem DB tangere evolutam in B.

PROPOSITIO I.

Rectae omnis, quae evolutam tangit, occurret linea ex evolutione descripta ad angulos rectos.



Recta autem AB evoluta, AH vero quae ex evolutione illius descripta est. Recta autem FDC, tangens curvam AD in D, occurrat in C curvae ACH. Dico ei occurrere ad angulos rectos : hoc est, si ducatur CE recta perpendicularis CD, dico eam in C tangere curvam ACH.Quia enim DC tangit evolutam in D, apparet ipsam referre positionem fili tunc cum ejus extremitas pervenit in C. Quod si igitur ostenderimus filum in tota reliqua descriptione curvae ACH, nusquam pertingere ad rectam CE praeterquam in C puncto, manifestum erit rectam CE ibidem curvam ACH contingere.



Sumatur punctum aliquod in AC praeter C, quod sit H, sitque primo remotius a principio evolutionis A quam punctum C, & intelligatur pars libera esse HG, cum extremitate sua ad H pervenit. Tangit ergo HG lineam AB in G. Cumque interea dum describitur pars curvae CH, evolutus sit arcus DG, occurret CD a parte D producta ipsi HG, ut in F. Ponatur autem GH occurrere rectae CE in E. Quia igitur duae simul DF, FG, majore sunt quam DG, ut rectae CF, FG simul majores sint recta CD & ipsa DG. Sed propter evolutionem, apparet utrique simul, rectae CD, & lineae DG, aequari rectam HG. Ergo duae simul CF, FG majores quoque erunt rectae HG; & ablata communi FG, erit CF major quam HF. Sed FE omnino major quam FC, quia angulus C triangula FCE est rectus. Ergo FE omnino major quam FH. Unde apparet, ab hac quidem parte puncti C, fili extremitatem non pertingere ad rectam CE.

Sit jam punctum H propinquius principio evolutionis A quam punctum C. sitque fili positio HG, tunc cum ejus extremitas esset in H, & ducantur rectae DG, DH, quarum haec occurrat rectae CE in E, apparet autem DG rectam non posse esse in directum ipsi HG, adeoque HGD fore triangulum. Iam quia recta DG vel minor est quam DKG, vel eadem, si nempe evolutae pars DG recta sit; addita utrique GH, erunt rectae DG, GH simul minores vel aequales duabus istis, scilicet DKG & GH, sive his aequali rectae DC. Duabus autem rectis DG, GH minor est recta DH. Ergo haec minor utique erit recta DC. Sed DE major est quam DC, quia in triangulo DCE angulus C est rectus. Ergo DH multo minor quam DE. Situm est ergo punctum H, hoc est extremitas fili GH, intra angulum DCE. Vnde apparet neque inter A & C usquam illam pertingere ad rectam CE. Ergo CE tangit curvam AC in C; ac proinde DC, cui CE ducta est perpendicularis, occurrit curvae ad angulos rectos. quod erat demonstrandum.

Hinc etiam manifestum est curvam AHC in partem unam inflexam esse, & in eandem partem cavam ac ipsa AGB, cujus evolutione descripta est. Omnes enim tangentes lineae AHC, cadunt extra spatium DGAHC : omnes vero tangentes lineae AGD, intra dictum spatium,unde liquet cavitatem AHC respicere convexitatem AGD.

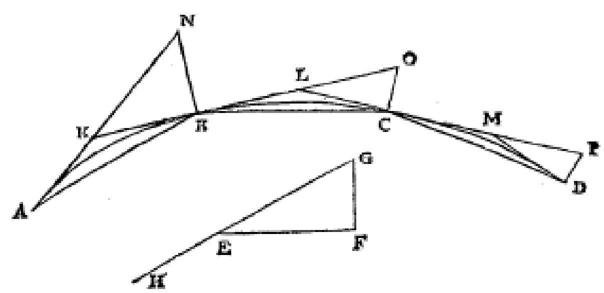
PROPOSITIO II.

Omnis curvae linea terminata, in unam partem cava, ut ABD, potest in tot partes dividi, ut si singulis partibus subtensa recta ducantur, velut AB, BC, CD; & a singulis item divisionis punctis, ipsaque curvae extremitate rectae ducantur curvam tangentes, ut AN, BO, CP, quae occurrant iis quae in proxime sequentibus divisionis punctis curvae ad angulos rectos insistunt, quales sunt lineae BN, CO, DP; ut inquam subtensa quaque habeat ad sibi adjacentem curvae perpendicularem, velut AB ad BN, BC ad CO, CD ad DP, rationem majorem quavis ratione proposita.

Sit enim data ratio lineae EF ad FG, quae recto angulo ad F jungantur, & ducatur recdta GEH.

Intelligatur primo curva ABD in partes tam exiguas secta punctis B, C, ut tangentes quae ad bina quaeque inter se proxima puncta curvam contingunt, occurrant sibi mutuo angulos qui singuli majores sint angulo FEH; quales sunt anguli AKB, BLC, CMD, quod quidem fieri posse evidentius est quam ut demonstratione indigeat. Ductis jam subtensis AB, BC, CD, & erectis curvae perpendicularibus BN, CO, DP, quae occurrant productis AK, BL, CM, in N, O, P : dico rationes singulas rectarum, AB ad BN, BC ad CO, CD ad DP, majores esse ratione EF ad FG.

Quia enim angulos AKB major est angulo HEF, erit residuus illius ad duos rectos, nimirum angulus NKB, minor angulo GEF.

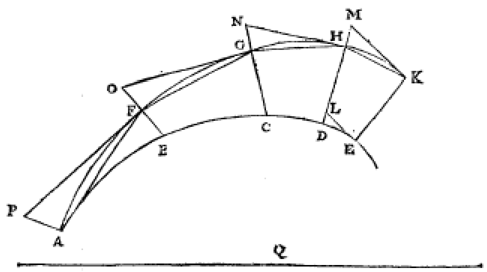


Angulus autem B trianguli KBN est rectus, sicut & angulus F in triangulo EFG. Ergo major erit ratio KB ad BN quam EF ad FG. Sed AB major est quam KB, quoniam angulus K in triangulo AKB est obtusus, est enim major angulo HEF qui est obtusus ex constructione. Ergo ratio AB ad BN major erit ratione KB ad BN, ac proinde omnino major ratione EF ad FG. Eodem modo & ratio BC ad CO, & CD ad DP, major ostendetur ratione EF ad FG. Itaque constat propositam.

PROPOSITIO III.

Duae curvae in unam partem inflexa & in easdem partes cava ex eodem puncto egredi nequeunt, ita ad se invicem comparata, ut recta omnis quae alteri earum ad angulos rectos occurrit, similiter occurrat & reliquae.

Sint enim, si fieri potest, hujusmodi lineae curvae ACE, AGK, communem terminum habentes A, & sumpto in exteriore illarum



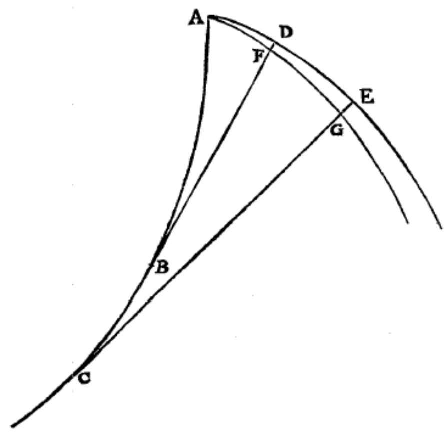
puncto quolibet K, sit inde educta KE recta, curvae AGK occurrens ad angulos rectos, ac proinde etiam curvae ACE. Potest jam recta quaedam sumi major curva KGA, quae sit Q. Divisa autem intelligatur ipsa KGA, ut in propositione antecedenti dictum fuit,in tot partes punctis HGF, ut subtensae singulae KH, HG, GF, FA ad perpendiculares curvae sibi contiguas HM, GN, FO, AP majorem rationem habeant quam linea Q ad rectam KE. Itaque & omnes simul dictae subtensae ad omnes dictas perpendiculares majorem habebunt rationem quam Q ad KE. Producantur autem perpendiculares eaedem & occurrant curvae ACE in D, C, B, nimirum ad angulos rectos ex hypothesi. Erit jam KE minor quam MD. Et enim, ducta EL ipsi KE perpendiculari, quoniam KE occurrit lineae curvae ECA ad angulos rectos, tanget EL curvam ACE, occurretque necessario rectae MD inter D & M. Unde cum KE sit brevissima omnium quae cadunt inter parallelas EL, KM, erit ea minor quam ML, ac proinde minor quoque omnino quam MD. Eodem modo & HD minor ostendetur quam NC, & GC minor quam OB, & FB minor quam PA. Cum sit ergo PA major quam FB, erunt duae simul PA, OF majores quam OB. Item quum OB sit major quam GC, erunt duae simul OB, NG, majores quam NC. Sed duae PA, OF majores erant quam OB. Itaque tres simul PA, OF, NG omnino majores erunt quam NC. Rursus, quia NC major quam HD, erunt duae simul NC, MH majores quam MD. Vnde, si loco NC sumantur tres hae ipsa majores PA, OF, NG, erunt omnino hae quatuor PA, OF, NG, MG majores quam MD : ac proinde eaedem quoque omnino majores recta KE, quia ipsa MD major erat quam KE. Diximus autem subtensas omnes AF, FG, GH, HK, majorem rationem habere ad omnes perpendiculares PA, OF, NG, MH, quam linea Q ad KE. Ergo cum dictis perpendicularibus minor etiam sit KE, habebunt dictae subtensae ad KE omnino majorem rationem quam Q ad KE. Ergo subtensae simul sumptae majores erunt recta Q. Haec autem ipsa curve AGK major sumpta fuit. Ergo subtensae AF, FG, GK, HK simul malores erunt curva AGK cujus partibus subtenduntur; quod est absurdum, cum singulae suis arcibus sint minores. Non igitur poterunt esse duae curvae linea quae quemadmodum dictum sese habeant. quod erat demonstrandum.

PROPOSITIO IV.

Si ab eadem puncto duae lineae exeant in partem unam inflexae, & in eandem partem cava, ita vero mutato comparatae ut rectae omnes, quae alteram earum contingunt,

alteri occurrant ad angulos rectos; posterior haec prioris evolutione, a puncto communi caepta, describetur.

Sunto lineae ABC, ADE, in partem unam inflexae, & quarum utraque in easdem partes cava existat, habeantque communem terminum A punctum. Omnes autem rectae tangentes lineam ABC, velut BD CE, occurrant lineae ADE ad angulos rectos. Dico evolutione ipsius ABC, a termino A incepta, describi ADE.

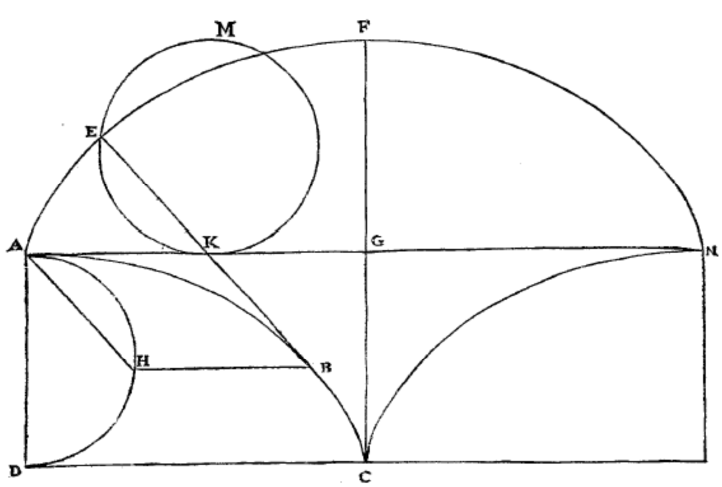


Si enim fieri potest, describatur dicta evolutione alia quaedam curva AFH. Ergo lineae rectae quaelibet, evolutam ABC tangentes, ut BD, CE, occurrent ipsi AFG ad angulos rectos (Prop. 1, hujus), puta in F & G. Sed eaedem tangentes etiam ad rectos angulos occurrere ponantur lineae ADE. Sunt igitur lineae curvae ADF, AFG, eodem puncto A terminatae, inque partem unam flexae, & ambae in eandem partem cavae, quippe utraque in eandem atque ipsa ABC; nam de linea ADE constat ex hypothesi, de AFG vero ex propositione prima huius; & omnes quae uni earum occurrunt ad angulos rectos, etiam alteri similiter occurrunt, quod quidem fieri non posse antea ostensum est (Prop. 3, hujus). Quare constat ipsam ADE descriptum iri evolutione lineae ABC. QED.

PROPOSITIO V.

Si Cycloidem recta linea in vertice contingat, super qua, tanquam basi, alia cyclois priori similis & aequalis constituatur, initium sumens a puncto dicti verticis; recta qualibet inferiorem cycloidem tangens, occurret superioris portioni, sibi superpositae, ad angulos rectos.

Tangat cycloidem ABC in vertice A recta AG, super qua, tanquam basi, similis alia cyclois constituta sit AEF, cujus vertex F. Cycloidem autem ABC tangat recta BK in B. Dico eam productam occurrere cycloidi AEF ad angulos rectos.



Describatur enim circa AD, axem cycloidis ABC, circulus genitor AHD, cui occurrat BH, base parallela, in H, & jungatur HA. Quia ergo BK tangit cycloidem in B, constat eam parallelam esse rectae HA (Prop. 15, partis 2). Itaque AHBK parallelogramum est, ac proinde AK aequalis HB, hoc est, arcui AH (Prop. 14, partis 2). Sit porro jam descriptus circulus KM, genitori circulo, hoc est ipse AHD, aequalis, qui tangat basin AG in K, rectam vero BK productam secet in puncto E. Quia ergo ipsi AH parallela est BKE, ac proinde angulus EKA aequalis KAH, manifestum est BK productam abscindere a circulo KM arcum aequalem ei quem a circulo AHD abscindit recta AH. Itaque arcus KE aequalis est arcui AH, hoc est rectae HB, hoc est rectae KA. Hinc vero sequitur, ex cycloidi proprietate, cum circulus genitor MK tangebat regulam in K, punctum describens fuisse in E. Itaque recta KE occurrit cycloidi in E ad angulos rectos (Prop. 15, partis 2). Est autem KE ipsa BK producta. Ergo patet productam BK occurrere cycloidi ad angulos rectos. QED.

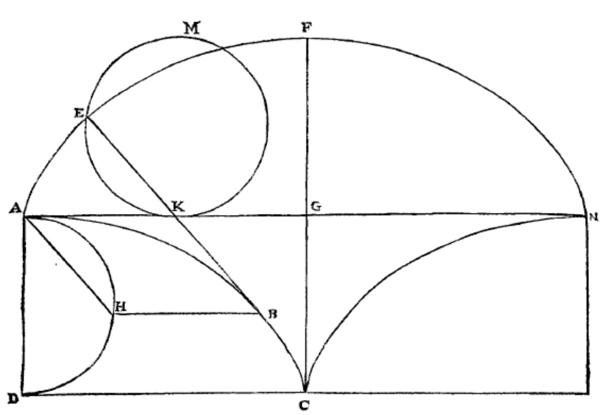
PROPOSITIO VI.

Semicycloidis evolutione, a vertice caepta, alia semicyclois describitur evolutae aequalis & similis, cujus basis est in ea recta quae cycloidem evolutam in vertice contingit.

Sit semicyclois ABC, cui superimposita sit alia similis AEF, quemadmodum in propositione praecedenti. Dico, si linea flexilis, circa semicyloidem ABC applicata, evolvatur, incipiendo ab A, eam describere extremitate sua ipsam semicycloidem AEF. Quia enim ex puncto A egrediuntur semicycloides ABC, AEF, in unam partem inflexae, & ambae in eandem cavae, ac praeterea ita comparatae, ut omnes tangentes semicycloidis ABC occurrant semicycloidi AEF ad angulos rectos, sequitur hanc evolutione illius a termino A incepta, describi (Prop. 4, huius). QED.

Et apparet, si dimidiam cycloidem, ipsi ABC gemellam, contrario situ ab altera parte lineae CG disponamus, velut CN, ejus evolutione, vel etiam dum filum, jam extensum in CF, circa eam replicatur alteram semicycloidem FN fili extremitate descriptum iri, quae simul cum priore AEF integram constituat.

Atque ex his, & propositione 25 de descensu gravium, manifestum jam est quod supra in Constructione Horologii de aequabili penduli motu dictum fuit. Patet enim perpendiculum, inter laminas binas, secundum semicycloidem inflexas, suspensum agitatumque, motu suo cycloidis arcum describere, ac proinde aequalibus temporibus quaslibet ejus reciprocationes absolvi. Non refert enim utrum in superficie, secundum cycloidem curvata, mobile feratur, an filo alligatum lineam ipsam in aere percurrat, cum utrobique eandem libertatem, eandemque in omnibus curvae punctis inclinationem ad motum habeat.



PROPOSITIO VII.

Cyclois linea sui axis, sive diametri circuli genitoris, quadrupla est.

Repetita enim figura praecedenti : cum post totam semicycloidem ABC evolutam, filum occupet rectam CF, quae dupla est AD, propterea quod axes cycloidum ABC, AEF sunt aequales; apparet semicycloidem ipsam ABC, filo sibi circum applicato aequalem, duplam esse sui axis AD, ac totam cycloidem sui quadruplam.

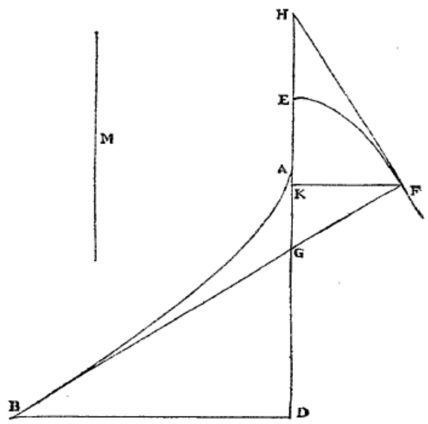
Apparet etiam tangentem BE, quae refert partem fili extensam, antea curvae parti BA applicatam, huic ipsi longitudine aequari. Est autem BE dupla ipsius BK, sive AH, quoniam in propositione quinta ostensum est KE ipsi AH aequalem esse. Itaque pars cycloidis AB rectae AH, sive BK, dupla erit : existente nimirum BH parallela basi cycloidis : idque ubicumque in ea punctum B sumptum fuerit.

Hanc cycloidis dimensionem primus invenit, via tamen longe alia, eximius geometra Christophorus Wren Anglus, eamque deinde eleganti demonstratione confirmavit, quae edita est in libro de cycloide vire clarissimi Ioannis Wallisii. De eadem vero linea, alia quoque multa extant pulcherrima inventa nostri temporis mathematicorum, quibus praecipue occasionem praebuere problemata quaedam a Blasio Paschalio Gallo proposita, qui in his studiis praecellebat. Is cum sua, tum aliorum inventa recensens, primum omnium Mersennum lineam hanc in rerum natura advertisse ait. Primum Robervallium tangentes ejus difinivisse, ac plana & solida dimensum esse. Item centra gravitatis tum plani, tum partium ejus invenisse. Primum Wrennium curvae cycloidis aequalem rectam dedisse. Me quoque primum reperisse dimensionem absolutam portionis cycloidis, quae rectae, basi parallela, abscinditur per punctum axis, quod quarta parte ejus a vertice abest, quae nimirum portio aequatur dimidio hexagono aequilatero, intra circulum genitorem descripto. Seipsum denique solidodrum ac semisolidodrum, tam circa basin quam circa axem, centra gravitatis definivisse, itemque partium eorum. Lineae etiam ipsius (sed haec post acceptam a Wrennio demensionem) centrum gravitatis invenisse, & dimensionem superficierum convexarum, quibus solida ista eorumque partes comprehenduntur; earumque superficierum centra gravitatis. Ac denique dimensionem curvarum cujusvis cycloidis, tam protractae quam contractae : hoc est earum quae describuntur a puncto intra vel extra circumference circuli genitoris sumpto. Et horum quidem demonstrationes a Paschalio sunt editae. A quibus suas quoque, de eadem linea, subtilissimas meditationes exposuit Cl. Wallisius, atque eadem illa omnia suo marte se reperisse, ac problemata a Paschalio proposita solvisse contendit. Quod idem & doctissimus Lovera sibi vindicat Quantum vero unicuique debeatur, ex scriptis eorum eruditi dijudicent. Nos propterea tantum praecedentia retulimus, quod silentio praetereunda non videbantur egregia adeo inventa, quibus factum est ut, ex lineis omnibus, nulla nunc melius aut penitius quam cyclois cognita sit. Methodum vero nostram, qua in hac metienda usi sumus, in aliis quoque experiri libuit, de quibus porro nunc agemus.

PROPOSITIO VIII.

Cujus lineae evolutione parabola describatur ostendere.

Sit paraboloides AB, cujus axis AD; vertex A; proprietas autem ista, ut ordinatim ad axem applicata BD, cubus abscissae ad verticem DA aeqetur solido, basin habenti quadratum DB, altitudinem vero aequalem lineae cuidam datae M; quae quidem curva pridem geometris nota fuit; & ponatur axi DE juncta in directum AE, quae habeat $\frac{8}{27}$ ipsius M. Iam si filum continuum circa EAB applicetur, idque ab E evolvi incipiat, dico descriptam ex evolutione esse parabolam EF, cujus axis EAG, vertex E, latus rectum aequale duplae EA.



Sumpto enim in curva AB puncto quolibet B, ducatur quae in ipso tangat curvam recta BG, occurrens axi EA in G, & ex G ducatur porro GF, quae ad rectos angulos occurrat parabolae EF in F; & sit ipsi GF perpendicularis FH, quae parabolam in F continget; & denique FK ordinatim ad axem EG applicetur.

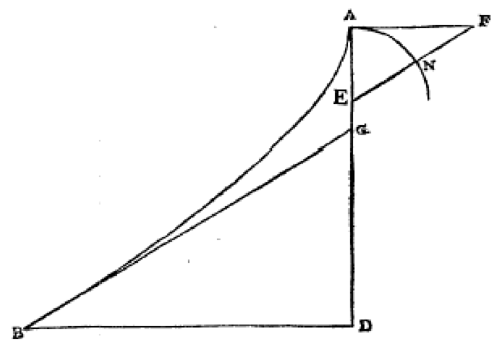
Est igitur KG aequalis dimidio lateri recto, hoc est, ipsi EA; ac proinde, addita vel ablata utrimque AK, erit EK aequalis AG. Est autem AG triens ipsius AD, quoniam BG tangit paraboliodem in B : illud enim ex natura curvae hujus facile demonstrari potest. Ergo & EK aequalis est trienti AD : & KH, quae ex natura parabolae dupla est KE, aequabitur duabus tertius AD. Itaque cubus ex KH aequalis est $\frac{8}{27}$ cubi ex AD, hoc est, solido basin habenti quadratum DB, altitudinem vero aequalem $\frac{8}{27}$ M, hoc est, ipsi AE. Quamobrem ut quadratum DB ad quadratum KH, ita erit KH longitudine ad AE, hoc est ad KG. Erat autem KH aequalis $\frac{2}{3}$ AD, hoc est ipsi GD. Ergo ut quadratum BD ad quadratum DG ita est HK ad KG. Ut autem HK ad KG, ita est quadratum FK ad quadratum KG. Ergo sicut quadratum BD ad quadratum DG, ita quadratum FK ad quadratum KG. Et proinde sicut BD ad DG longitudine, ita FK ad KG. Unde sequitur BGF esse lineam rectam. Sed GF occurrit parabolae ED ad angulos rectos. Ergo apparet BG, tangentem paraboloidis, productam occurrere eidem parabolae ad angulos rectos.

Idque similiter de quavis illius tangente demonstrabitur. Ergo constat ex evolutione lineae EAB, a termino E incepta, describi parabolam EF (Prop. 4, hujus). Q.E.D.

PROPOSITIO IX.

Rectam lineam invenire aequalem data portioni curvae paraboliodis, ejus nempe in qua quadrata ordinatim applicatarum ad axem, sunt inter se sicut cubi abscissarum ad verticem.

Quomodo hoc fiat ex prop. praecedenti manifestum est. Parabola vero EF ad constructionem non requiritur, quae sic peragetur. Data quavis parte paraboloidis hujus AB, cui rectam aequalem invenire oporteat, ducatur BG tangens in puncto B, quae occurrat axi AG in G. Tanget autem si AG fuerit tertia pars AD, inter



verticem & ordinatim applicatam BD interceptae. Porro sumpta AE aequali $\frac{8}{27}$ linea M, quae latus rectum est paraboloidis AB, ducatur EF parallela BG, occurratque linea AF, quae parallela est BD, in F. Iam si ad rectam BG addatur NF, excessus rectae EF supra EA, habebitur recta aequalis curvae AB. Cujus demonstratio ex ante dictis facile perspicitur.

Semper ergo curva AB tantum superat tangentem BG, quantum recta EF rectam EA.

Rursus autem hic lineam incidimus, cujus longitudinem alii jam ante dimensi sunt. Ilam nempe quam anno 1659 Ioh. Heuratius Harlemensis recta aequalem ostendit, cujus demonstratio post commentarios Ioh. Schotenii in Cartesii Geometriam, eodem anno editam, adjecta est. Et ille quidem omnium primus curvam lineam, ex earum numero quarum puncta quaelibet geometrice definiuntur, ad hanc mensuram reduxit, cum sub idem tempus Cycloidis longitudinem dedisset Wrennius, non minus ingenioso epicheremate.

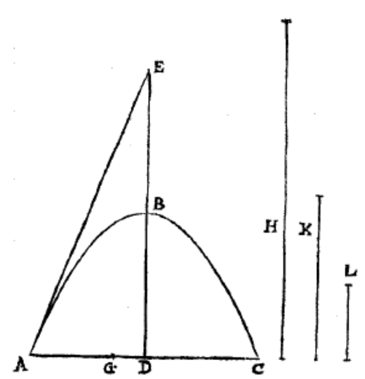
Scio equidem, ab edito Heuratii invento, Doctissimum Wallisium Wilhelmo Nelio, apud suos juveni, idem attribuere voluisse, in libra de Cissoide. Sed mihi, quae illic adfert perpendenti, videtur non multum quidem ab invento illo Nelium abfuisse, neque tamen plane id adsecutum esse. Nam neque ex demonstratione ejus, quam Wallisius affert, apparet satis perspexisse quaenam foret curva illa, cujus, si construeretur, mensuram datam fore videbat, Et credibile est, si scivisset ex earum numero esse quae jampridem Geometris cognitae fuerant, vel ipsum, vel alios ejus nomine, tam nobile inventum Geometris maturius impertituros fuisse, quod, si quod aliud, merebatur ut Archimedeum illud ευρηκα exclamarent. Sane ejusdem inventi, tanquam a se profecti, etiam Fermatius, Tholosanus senator ac Geometra peritissimus, demonstrationes conscripsit, quae anno 1660 excusae sunt, sed illae sero utique.

Cum vero in his simus, etiam de nobis dicere liceat, quid ad promovendum tam eximium inventum contulerimus: sequidem & Heuratio ut eo perveniret occasionem praebuimus, & dimensionem curvae parabolicae ex hyperbolae data quadratura, quae Heuratiani inventi pars est, ante ipsum atque omnium primi reperimus. Etenim sub finem anni 1657 in haec duo simul incidimus, curvae parabolicae quam dixi dimensionem, & superficiei coniodis parabolici in circulum reductionem. Cumque Schotenio, aliisque item amicorum, per literas indicassemus, duo quaedam non vulgaria circa parabolam inventa nobis sese obtulisse, eorumque alterum esse conoidicae superficiei extensionem in circulum, ille litteras eas cum Heuratio, quo tum familiariter utebatur, communicavit. Huic vero, acutissimi ingenii viro, non difficle fuit intelligere, conoidis istius superficiei affinem esse dimensionem ipsius curvae parabolicae. Quae utraque inventa, ulterius inde investigans, in alias istas curvas paraboliodes incidit, quibus rectae aequales absolute inveniuntur.

Ac de Coniodis quidem superficie in planum redacta, ne quis forte testimonium desideret, pauca haec adscribere visum est ex literis viri clarissimi, atque inter praecipuos hodie Geometras censendi, Franc. Slusii, quibus eo ipso anno mihi inventum illud, ac prolixius forte quam pro merito, gratulatus est. In quibus literas 24. Decemb. anni 1657 datis, ista habentur. *Duo tantum addo, unum &c. Alterum est, me has omnes curvas, ipsumque adeo locum linearem integrum, nihili pene facere prae invento hoc tuo, quo superficiei in conoide paralolico rationem ad circulum suae baseos demonstrasti. Hanc pro circuli quadratura pulcherrimam απαγωγην praefero libens iis omnibus, quas ex loco lineari nec paucas olim deduxi, & quas tecum, si ita jusseris, data occasione communicabo.*

Anno autem insequenti etiam superficies coniodum hyperbolicorum & sphaeroidem repperi, quomodo ad circulos reduci possent, constructionesque eorum problematum, non addita tamen demonstratione, Geometris quibuscum tunc litterarum commercium habebam, in Gallia Paschalio aliisque, in Anglia Wallisio impertii, qui non multo post sua quoque super his, una cum aliis multis subtilibus inventis in lucem edidit, fecitque ut nostris demonstrationibus perficiendis supersederem. Quoniam vero non inelegantes visae sunt constructiones nostrae, neque adhuc publice extant, placet hoc loco illas adscribere.

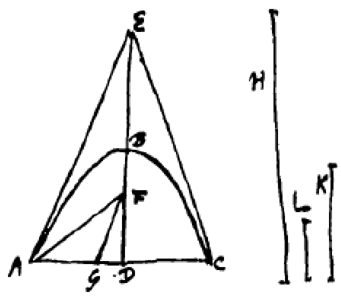
Conoides parabolici superficiei curvae circulum aequalem invenire.



Sit datum conoides cujus sectio per axem parabola ABC; axis ejus BD, vertex B, diameter basis AC, qui sit axi BD ad angulos rectos. Et oporteat superficiei portionis curvae invenire circulum aqualem.

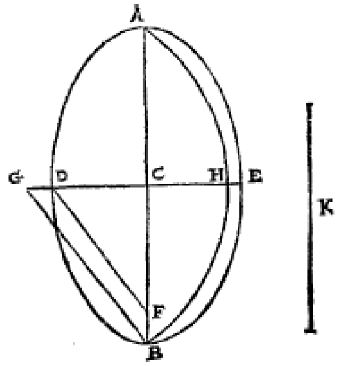
Producto axe a parte vertis, sumatur BE aequalis BD, & jungatur EA, quae parabolam ABC in A continget. Porro secetur AD in G, ut sit AG ad GD sicut EA ad AD. Et utrique simul AE, DG aequalis statuatur recta H. Item trienti basis AC aequalis sit recta L, & inter H & I media proportionalis inveniatur K, qua tanquam radio circulus describatur. Is aequalis erit superficiei curvae conoidis ABC. Hinc sequitur, si fuerit AE dupla AD, superficiem coniodis curvam ad circulum basem fore ut 14 ad 9. Si AE tripla AD, ut 13 ad 6; si AE quadrupla AD, ut 14 ad 5. Atque ita semper fore ut numerus ad numerum, si AE ad AD ejusmodi rationem habuerit.

[***Dato conoide parabolico, invenire circulum conoidis superficiei aequalem.***



Esto datam conoides parabolicum ABC, cujus axis BD; Et oporteat superficiei conoidis convexae ABC aequalem circulum invenire. Secetur conoides plano per axem, unde existat parabola, Et producatur axis et sit ED dupla BD. Junctisque AE, EC, ducatur AF quae bifariam dividat angulum EAD, occuratque axi in F, unde ducatur FG parall. EA. Deinde esto H linea aequalis utrique simul AE et GD. Trienti vero diametri baseos AC sit aequalis recta L. Et inter utramque H et L media proport. sumatur K. Dico circulum a semidiametro K aequalem esse superficiei conoidis convexae ABC.

Quia enim bifariam dividitur angulus EAD a recta AF, Erit ut EA ad AD ita EF ad FD. Et compon. ut utraque simul EA, AD ad AD ita ED ad DF. Ut autem ED ad DF ita est AD ad DG. Ergo ut utraque simul EA, AD ad AD ita erit AD ad DG. Et sumptis tam antecedentium quam consequentium duplis, sicut ambitus Δ^{i} AEC ad AC ita AC ad duplam DG. Quamobrem sicut dupla AE una cum dupla DG ad sesquialteram ipsius AC, vel sumptis utriusque dimidiis ut AE una cum DG, hoc est ut H, ad sesquialteram ipsius AD, ita erit superficies conoidis ABC ad circulum baseos AC. Ratio autem quam habet H ad sesqualteram AD, componitur ex rationibus H ad AD, et AD ad sesquialteram AD; quarum haec eadem est quae rectae L ad AD; nam cum L sit subtripla AC erit eadem subsesquialtera AD. Igitur et ratio quam habet superficies conoidis ABC ad circulum baseos AC, eadem erit compositae ex rationibus H ad AD et L ad AD, ac proinde eadem quae rect. ab H et L contenti hoc est qui K ad quad. AD. Sicut autem quadr. ex K ad qu. AD, ita est circulus radio K descriptus ad circulum baseos cujus semid. DA. Ergo eadem erit ratio superficiei conoidis ABC ad circulum baseos AC, quae circuli a K semid. ad eundem circulum AC. Ac proinde dicta conoidis superficies aequalis erit circulo cujus est semidiameter. Quod erat dem.



Spheroidis oblongi superficiei circulum aequalem invenire.

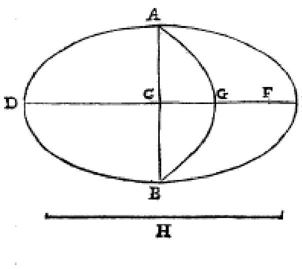
Esto sphaeroides oblongum cujus axis AB, centrum C, sectio per axem ellipsis ADBE, cujus minor diameter DE.

Ponatur DF aequalis CB, seu ponatur F alter focorum ellipseos ADBE, rectaeque FD parallela ducatur BG, occurrens productae

ED in G, centrioque G, radio GB, describatur super axe AB arcus circumferentiae BHA. Interque semidiametrum CD & rectam utrisque aequalem, arcui AHB & diametro DE, media proportionalis set recta K. Erit haec radius circuli que superficiei sphaeroidis ADBE aequalis sit.

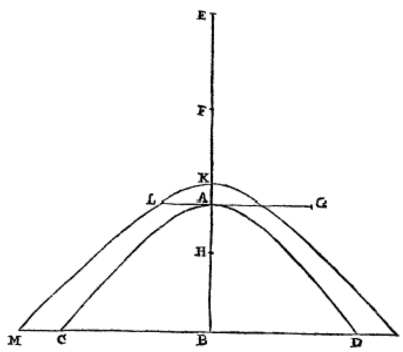
Sphaeroidis lati sive compressi superficiei circulum aequalem invenire.

Sit sphaeroides latum cujus axis AB, centrum C, sectio per axem ellipsi ADBE.



Sit rursus focorum alteruter F, divisaque bifariam FC in G, intelligatur parabola AGB quae basin habeat axem AB, verticem vero punctum G. Sitque inter diametrum DE, & rectam curvae parabolicae AGB aequalem, media proportionalis linea H. Erit haec radius circuli qui superficiei sphaeroidis propositi aequalis sit.

Conoidis hyperbolici superficiei curvae circulum aequalem invenire.



Esto conoides hyperbolicum cujus axis AB, sectio per axem hyperbola CAD, cujus latus transversum EA, centrum F, latus rectum AG.

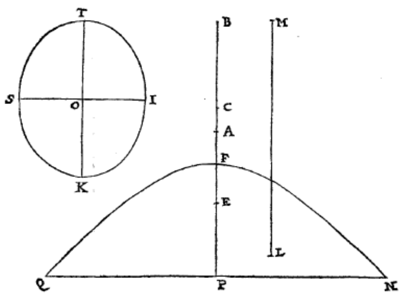
Sumatur in axe recta AH, aequalis dimidio lateri recto AG, & ut HF ad AF longitudine ita, sit AF ad FK potentia. Et intelligatur vertice K alia hyperbola descripta KLM, eodem axe & centro F cum priore, quaeque latere rectum & transversum illi reciproce proportionalia habeat. Occurrat autem ipsi producta BC in M, sitque AL parallela BC. Erit jam spatium ALMB, tribus rectis lineis & curva hyperbolica comprehensum, ad dimidium quadratum ex BC, ita superficies conoidis curva ad circulum baseos suae, cujus diameter CD. Unde constructio reliqua facile absolvetur, posita hyperbolae quadratura.

Quum igitur conoidis parabolici superficies ad circulum redigatur, aeque ac superficies sphaerae, ex notis geometriae regulis, in superficie sphaeroidis oblongi, ut idem fiat, ponendum est arcus circumferentiae longitudinem aequari posse lineae rectae. Ad sphaeroidis vero lati, itemque ad conoidis hyperbolici superficiem eadem ratione complanandam, hyperbolae quadratura requiritur. Nam parabolicae lineae longitudino, quam in sphaeroide hoc adhibuimus, pendet a quadratura hyperbolae, ut mox ostendemus.

Verum, quod non indignum animadversione videtur, invenimus absque ulla hyperbolicae quadraturae suppositione, circulum aequalem construi superficiei utrique simul, sphaeroidis lati & conoidis hyperbolici.

Dato enim sphaeroide quovis lato, posse inveniri conoides hyperbolicum, vel contra, dato conoide hyperbolico, posse inveniri sphaeroides latum ejusmodi, ut utriusque simul superficiei exhibeatur circulos aequalis, cujus exemplum in casu uno caeteris simpliciore sufficiet attulisse.

Sit sphaeroides latum cujus axis SI, sectio per axem ellipsis STIK, cujus ellipsis centrum O, axis major TK. Ponatur autem ellipsis haec ejusmodi, ut latus transversum TK habeat ad latus rectum eam rationem, quam linea secundum extremam & mediam rationem secta, ad partem sui majorem. S



Sumatur BC potentia dupla ad SO, item BA potentia dupla ad OK, & sint hae quatuor continue proportonales BC, BA, BF, BE, & ponatur EP aequalis EA. Intelligatur jam conoides hyperbolicum QFN, cujus axis FP; axi adjecta, sive $\frac{1}{2}$ latus transversum FB; dimidium latus rectum aequale BC.

Hujus conoidis superficies curva, una cum superficie sphaeroidis SI, aequabitur circulo cujus datus erit radius M, qui nempe possit quadratum TK cum duplo quadrato SI.

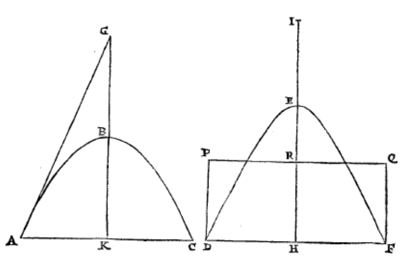
Curvae parabolicae aequalem rectam lineam invenire.

Sit parabolae portio ABC, cujus axis BK, basis AC axe ad angulos rectos; & oporteat curvae ABC rectam aequalem invenire.

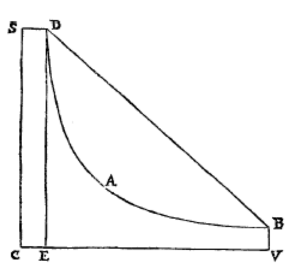
Accipiatur basi dimidiae AK aequalis recta IE, quae producatur ad H, ut sit IH aequalis AG, quae parabolam in puncto basis A contingens, cum axe producto convenit in G. Sit jam portio hyperbolae DEF, vertice E, centro I descriptae, cujusque diameter sit EH; basis vero DHF ordinatim ad diametrum applicata. Latus rectum pro lubitu sumi potest. Quod si jam super base DG intelligatur parallelogrammum constitutum DPQF, quod portioni DEF aequale sit; ejus latus PQ ita secabit diametrum hyperbolae in R, ut RI sit aequalis curvae parabolicae AB, cujus dupla est ABC.

Apparet igitur hinc quomode a quadratura hyperbolae pendeat curvae parabolicae mensura, & illa ab hac vicissim.

Quaecunque vero problemata ad alterum e duobus hisce reducuntur, quamlibet verae proximam solutionem per numeros accipiunt, logarithmorum admirabili invento. Cum per hos hyperbolae quadratura, ut olim invenimus, numeris quam proxime explicetur. Est autem regula hujusmodi.



Sit DAB portio hyperbolae, cujus asymptoti CS, CV, ductis DE, BV parallelis asymptoto SC.



Accipiatur differentia logarithmorum qui conveniunt numeris, eandem inter se rationem habentibus quam rectae DE, BV; ejusque differentiae quaeratur logarithmus. Cui addatur logarithmus (qui semper est idem) 0,36221, 56887. Summa erit logarithmus numeri qui spatium DEVBAD designabit, tribus rectis & curva DAB comprehensi, in partibus qualium parallelogrammum DC est 100000, 00000. Unde porro facile quoque habebitur area portionis DAB.

Sit ex. gr. proportio DE ad BV ea quae 36 ad 5.

Ab	1, 55630, 25008, logar[a]. 36
auferatur	0, 69897, 00043 logar[us]. 5.
Erit	0, 85733, 14965. differ. logar[orum].
Et	9, 93314, 92856. logar[us]. differentiae.
Cui addatur	0, 362214, 56887. logar[us]. semper addendus.
Fit	10, 29536, 49743. logar[us]. spatii DEVBAD.

Habebit hujus logarithmi numerus 11 characteres, quum characteristica sit 10. Quaeratur itaque primo numerus proxime minor, conveniens invento logarithmo, qui numerus est 19740. Deinde ex differentia logarithmi ejusdem, & proxime eum in tabula sequentis, reliqui characteres eliciantur 81026, scribendi post priores, ut fiat 197408,10260, addito ad finem zero, ut efficiatur numerus characterum 11. Est ergo area spatia DEVBAD proxime partium 197408,10260, qualium partium parallelogrammum DC est
100000, 00000.

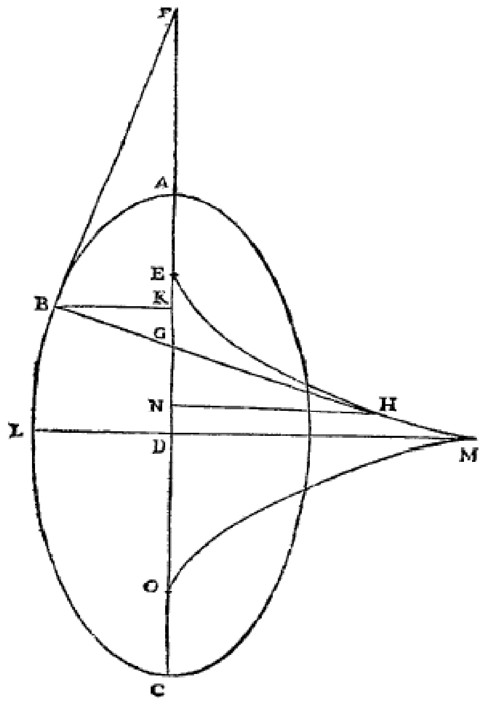
PROPOSITIO X.

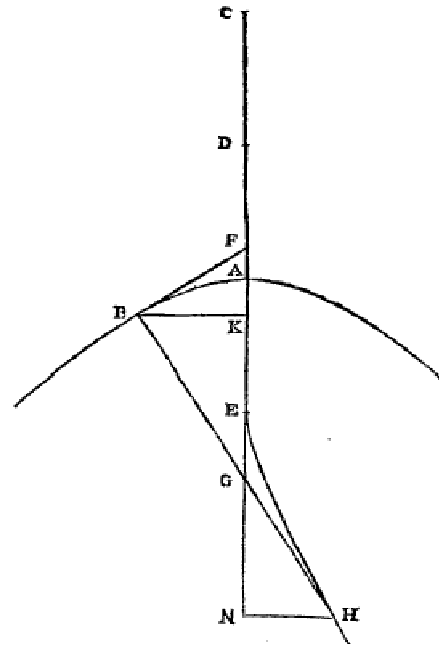
Lineas curvas exhibere quarum evolutione ellipses & hyperbola describantur, rectasque invenire iisdem curvis aequales.

Sit ellipsis vel hyperbole quaelibet AB, cujus axis transversus AC; centrum figurae D; latus rectum duplum ipsius AE. Et sumpto in sectione quovis puncto, ut B, applicetur ordinatim ad axem recta BK, & ad dictum punctum B tangens ducatur quae conveniat cum axe in F; sitque BG ipsi FB perpendicularis, axeque occurrat in G; & producatur BG usque ad H, ut BH ad HG habeat rationem eam quae componitur ex rationibus GF ad FK, & AD ad DE.

Dico curvam EHM, cujus puncta omnia inveniuntur eodem modo quo punctum H, esse eam cujus evolutione , una cum recta EA, describetur sectio AB. Ipsam autem BH tangere curvam in H, & esse toti HEA aequalem. Quamobrem, si ab HB auferatur EA, reliqua recta portioni curvae HE aequabitur. Apparet autem, cum curvae puncta quaevis indifferenter , certaque ratione inveniantur, esse eam utrobique ex earum genere, quae mere geometricae censentur. Unde & relatio horum omnium punctorum ad puncta axis AC, aequatione aliqua exprimi poterit, quam aequationem ad sextam dimensionem ascendere invenio; minimumque habere terminorum, si fuerit AB hyperbola cujus latera transversum rectumque aequalia. Tunc enim ducta ex quovis curvae puncto, ut H, ad axem CAN perpendiculari HN; vocataque AC, a; CN, x; & NH, y; erit semper cubus ab $xx\text{-}yy\text{-}aa$ aequalis $27xxyyaa$. Sed hoc casu brevius quoque multo, quam praedicta constructione, curvae EHM puncta reperiri possunt, ut in sequentibus ostendetur.

Caeterum notandum est, in ellipsi singulos quadrantes singularum linearum evolutione describi; sicut quadrans ABL evolutione lineae AEHM, quadrans CL evolutione similis





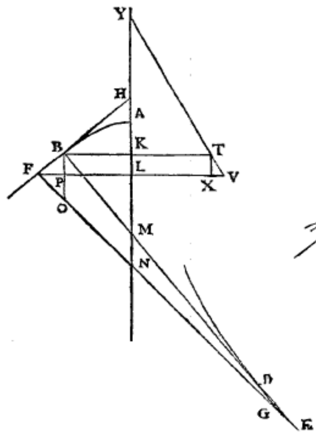
huic oppositae COM. Est enim haec in sectione utraque diversitas, quod cum principium quidem curvae EHM, tam in ellipsi quam in hyperbola, sit punctum E, sumpta AE aequali $\frac{1}{2}$ lateris recti; in hyperbola in infinitum inde dicta linea extenditur, at in ellipsi finitur in puncto axis minoris M, sumpta LM aequali $\frac{1}{2}$ lateris recti, secundum quod possunt ordinatim applicatae ad dictum minorem axem. Namque hos terminos esse hujus curvae, facile apparebit ortum ejus consideranti, quodque in ellipse est sicut AD ad DE, ita LM ad MD.

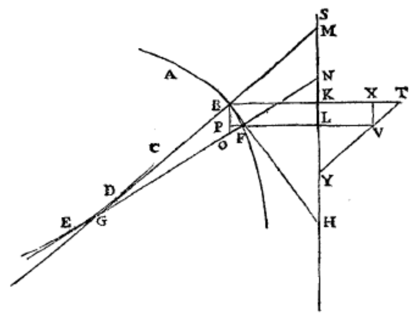
Horum autem demonstrationi non immorabimur,sed ad ipsam methodum tradendam pergemus, qua & hae curvae ex sectionibus conicis, & alia innumerae ex aliis quibuscunque datis inveniuntur.

PROPOSITIO XI.

Data linea curva, invenire aliam cujus evolutione illa describatur; & ostendere quod ex unaquaque curva geometrica, alia curva itidem geometricae existat, cui recta linea aequalis dari possit.

Sit curva quaepiam, vel pars ejus, in partem unam inflexa ABF, & recta KL, ad quam puncta omnia referantur; & oporteat invenire curvam aliam, ut DE, cujus evolutione ipsa ABF descdribatur.

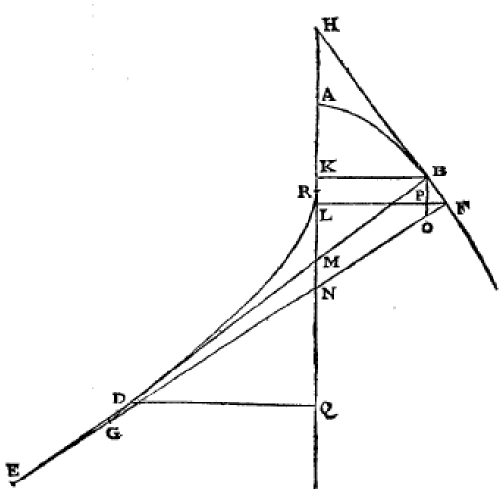




Ponatur jam inventa; & quoniam tangentes omnes curvae DE, necesse est occurrere lineae ABF, ex evolutione descriptae, ad angulos rectos ; patet quoque vicissim eas quae ipsi ABF ad rectos angulos insistunt, ut BD, FE, tacturas evolutam CDE.

Intelligantur autem puncta B, F, inter se proxima; & si quidem a parte A evolutio incipere ponatur, ulteriusque inde distet F quam B, etiam contactus E ulterius quam D distabit ab A; intersectio vero rectarum BD, FE, quae est G, cadet ultra punctum D in recta BD. Nam concurrere ipsas BD, FE necesse est, cum curvae BF ad partem cavam insistant rectis angulis.

Quanto autem punctum F ipsi B propinquius fuerit, tanto propius quoque puncto D, G, & E cenvenire apparet; ideoque, si interstitium BF infinite parvum intelligatur, tria dicta puncta pro uno eodemque erunt habenda; ac praeterea, ducta recta BH, quae curvam in B tangat, eadem quoque pro tangente in F consebitur. Sit BO parallela KL, & in hanc perpendiculares cadant BK, FL; secetque FL rectam BO in P, & sint puncta notata M, N, in quibus rectae BD, FE, occurrant ipsi KL. Quia igitur ratio BG ad GM est eadem quae BO ad MN, data hac dabitur & illa; & quia recta BM datur magnitudine ac positione, dabitur & punctum G in producta BM, sive D in curva CDE, quia G & D in unum convenire diximus. Datur autem ratio BO ad MN; simpliciter quidem in Cycloide, ubi primum omnium illam investigavimus, invenimusque duplam; in aliis vero curvis, quas hactenus examinavimus, per duarum datarum rationum compositionem. Nam quia ratio BO ad MN componitur ex rationibus BO ad BP, sive NH ad LH, & ex BP sive KL ad MN; patet si rationes hae utraeque dentur, etiam ex iis compositam rationem BO ad MN datum iri. Illas vero dari in omnibus curvis geometricis, in sequentibus patebit; ac proinde iis semper



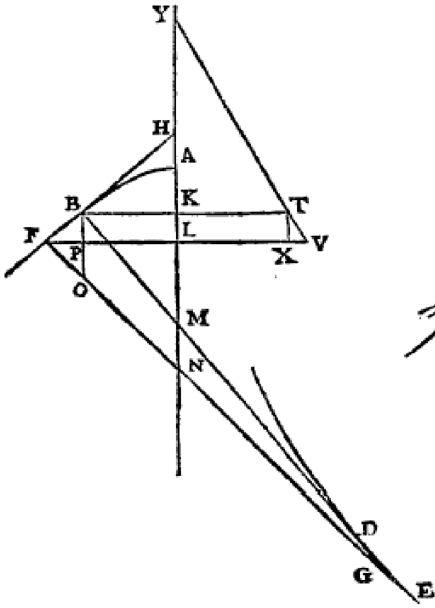
curvas adsignari posse, quarum evolutione describantur, quaeque ideo ad rectas lineas sint reducibiles.

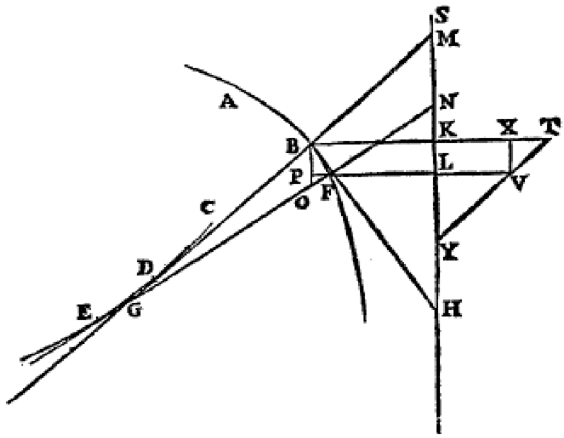
Ponatur primo parabola esse ABF, cujus vertex A, axis AQ. Cum igitur lineae BM, FN, sint parabolae ad angulos rectos; ductaeque sint ad axem AQ perpendiculares BK, FL, erunt, ex proprietate parabolae, singulae MK, NL dimidio lateri recto aequales; & ablata communi LM, aequales inter se KL, MN. Hinc, quum ratio BG ad GM componatur ex rationibus NH ad HL, & KL ad MN, uti dictum fuit, sitque earum posterior ratio aequalitatis; liquet rationem BG ad GM fore eandem quae NH ad HL; & dividendo, BM ad MG, eandem quae NL ad LH, sive MK ad KH; nam LH, KH pro eadem habentur, propter propinquitatem punctorum B, F, Data autem est ratio MK ad KH, dato puncto B; quoniam tam MK, quam KH dantur magnitudine; nam MK aequatur dimidio lateri recto, KH vero duplae KA. Dataque etiam est positione & magnitudine recta BM. Ergo & MG data erit, adeoque & punctum G, sive D, in curva RDE; quod nempe invenitur producta BM usque in G, ut sit BM ad MG sicut $\frac{1}{2}$ lateris recti ad duplam KA.

Et sic quidem, asumptis in parabola ABF aliis quotlibet punctis praeter B, totidem quoque puncta lineae RDE simili ratione, invenientur; atque hoc ipso lineam RDE geometricam esse constat, unaque proprietas ejus innotescit, ex quae caeterae deduci possunt. Ut si inquirere diende velimus, quanam aequatione exprimatur relatio punctorum omnium curvae CDE ad rectam AQ: ducta in hanc perpendiculari DQ, vocatoque latere recto parabolae ABF, a; AK, b; AQ, x; QD, y. Quoniam ratio BM ad MD, hoc est KM ad MQ, est ea quae $\frac{1}{2}a$ ad $2b$, estque ipsa KM = $\frac{1}{2}a$, erit aequalis 2b. Est autem MA = $\frac{1}{2}a$ + b; ergo AQ sive x aequalis 3b + $\frac{1}{2}a$. Unde $b=\frac{1}{3}x-\frac{1}{6}a.$ Porro quoniam, sicur quadratum MK, hoc est, $\frac{1}{4}aa$ ad quadratum KB, hoc est , ab, ita qu. MQ, hoc est, $4bb$ ad qu. QD; erit qu. QD, sive $yy=\frac{16b^3}{a}$. Ubi, si in locum b substituatur $\frac{1}{3}x-\frac{1}{6}a,$ quod illi aequale inventum est, fiet yy = 16 cub. $\frac{1}{3}x-\frac{1}{6}a$ divisis per a. Ac proinde $\frac{27}{16}ayy$ = cubo ab $x-\frac{1}{2}a$. Accipiatur AR in axe parabolae = $\frac{1}{2}a$; eritque RQ = $x-\frac{1}{2}a.$ Curvam igitur CD ejus naturae esse liquet, ut semper cubus lineae RQ aequetur parallelepipedo, cujus basis qu. QD, altitudo $\frac{27}{16}a$; ac proinde ipsam paraboliodem esse, cujus evolutione describi parabolam AB supre ostendimus ; cujus nimirum paraboloidis latus rectum aequetur $\frac{27}{16}$ lateris recti parabolae AB. tunc enim hujus latus rectum aequale sit $\frac{27}{16}$ lateris recti paraboloidis, quemadmodum ibi fuit definitum.

Quomodo porro ratio OB ad BP, sive NH ad HL, non tantum cum ABF parabola est, sed etiam alia quaelibet curva geometrica, semper inveniri possit manifestum est.

Quoniam tantum recta FH ducenda est, quae curvam in adsumpto puncto F tangat, & FN ipse FH perpendicularis : unde NH & HL datae erunt, ac proinde ratio quoque earum data.





At non aeque liquet quo pacto ratio KL ad MN innotescat quam tamen semper quoque reperiri posse sic ostendemus.

Sint rectae KT, LV, perpendiculares super KL, sitque KT aequalis KM, & LV aequalis LN, & ducatur VX parallela LN, quae occurrat ipsi KT in X. Quoniam ergo semper eadem est differentia duarum LK, NM, quae duarum LN, KM, hoc est, quae duarum LV, KT; est autem differentiae ipsarum LV, KT aequalis XT, & XV ipsi LK; erit proinde NM aequalis duabus simul VX, XT, vel ei quo VX ipsam XT superat. Atque adeo, si data fuerit vatio VX ad XT, data quoque erit ratio VX ad utramque simul VX, XT, vel ad excessum VX supra XT, hoc est, data erit ratio VX sive LK ad NM.

Sciendum est autem, quoniam KT ipsi KM, & LV ipse LN, aequales sumptae sunt, locum punctorum T, V, fore lineam quandam vel rectam vel curvam datam, ut mox ostendetur. Et siquidem sit linea recta; ut contingit ABF coni sectio fuerit, & KL axis ejus; constat rationem VX ad XT datam fore, data positione ipsius lineae VT, quae locus est punctorum V, T; semperque eandem tunc haberi dictam rationem, qualecunque fuerit intervallum KL.

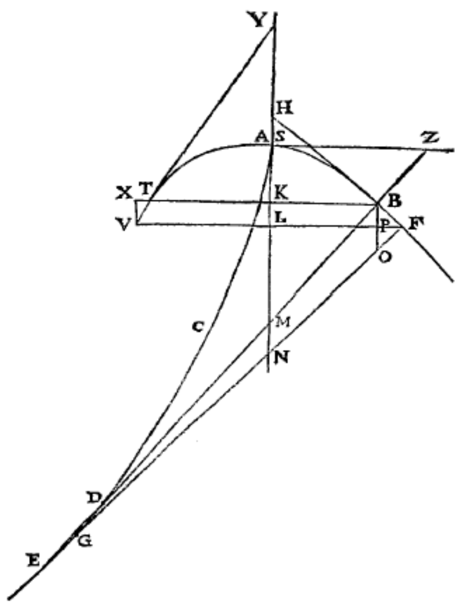
At si locus alia curva fuerit, diversa erit ratio VX ad XT, prout majus minusve fuerit intervallum KL. Inquirendum est autem quaenam futura sit ista ratio, cum KL infinite parvum imaginamur, quoniam & puncta B, F, proxima invicem posuimus. Similiter itaque & puncta V, T, lineae curvae minimam particulam intercipere intelligendum est; unde recta VT, cum ea quae in T curvam contingit coincidet. Sit ergo tangens illa TY; potest enim duci quoniam curva, ad quam sunt puncta T, V, geometrica est. Ratio igitur YK ad KT data erit, adeoque & VX ad XT, ex aue etiam rationem LK ad NM dari ostendimus.

Quaenam vero sit linea ad quam sunt puncta T, V, invenitur ponendo certum punctum s in recta KL, & vocando SK, *x*; KT, *y*. Nam quia data est curva ABF, eique BM ad angulos rectos ducta, invenietur inde quantitas linea KM. per methodum tangentium a

Cartesio traditam, quae ipsi KT, sive y aequabitur, & ex ea aequatione, natura curvae TV innotescet, ad quam deinde tangens ducenda est. Sed clariora omnia fient sequenti exemplo.

Sit ABF paraboloides illa, cui superius rectam aequalem invenimus : in qua nempe cubi perpendicularium in rectam sK, sint inter se sicut quadrata ex ipsa SK abscissarum. Et oporteat invenire curvam CDE cujus evolutione paraboliodes SBF describatur. Hic primum ratio BO ad BP facile invenitur, quia tangentem paraboloidis in puncto B duci scimus, sumpta SH aequali $\frac{1}{2}$ SK. Cui tangenti cum BM ad angulos rectos insistat, dantur jam lineae MH, HK, ac proinde earum inter se ratio, quae est eadem quae OB ad BP.

Ut autem ratio BP, sive KL ad MN innotescat, ponantur ad KL perpendiculares rectae LT, LV, aequales singulis KM, LN, sitque VX parallela LK. Iam quia ex duabus simul KL, LN, auferendo KM, relinquitur MN; hoc est, auferendo ex duabus XV, VL, sive XV, XK, ipsam KT; hinc autem relinqui apparet VX & XT : erunt igitur hae duae VX, XT ipsi MN aequales, ac proinde ratio KL ad MN eadem quae VX ad duas simul VX, XT. Ut autem haec ratio innotescat cum intervallum KL est minimum; oportet secundum praedicta inquirere quis sit locus, sive linea ad quam sunt puncta T, V. Quod ut fiat sit latus rectum paraboloidis ABF = a; SK = x; KT = y.



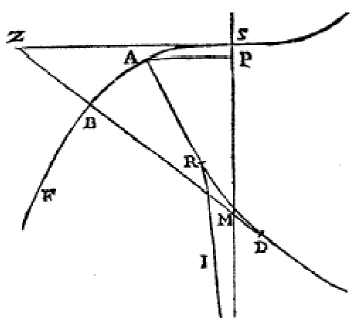
Quia igitur proportionales sunt KH, KB, KM, estque HK = $\frac{3}{2}x$: KB ex natura paraboloidis aequalis R.cub.aax = y: fiet KM, hoc est KT = $\frac{2}{3}$ R.cub.aax = y, ac proinde $\frac{8}{27}aax = y^3$. Unde patet locum punctorum T, V, esse paraboloidem illam, quam cubicam vocant geometricae. Cui proinde ad T tangens ducetur, sumpta SY dupla ipsius sK, junctaque YT. Et jam quidem ratio VX ad duas simul VX, XT, quam diximus eadem esse ac KL ad MN, erit ea quae YK ad utramque simul YK, KT. Haec autem ratio data est, ergo & ratio KL ad MN. Sed & rationem OB ad PB datam esse ostensum est. Ergo, cum ex duabus hisce componatur ratio BD ad DM, ut supra patuit, dabitur & haec; & dividendo, ratio BM ad MD; adeoque & punctum D in curva DE.

Ad constructionem autem brevissimam hoc pacto hic perveniemus, KT sive KM dicta fuit y. Itaque MH erit $y + \frac{1}{2}x$. Et MH ad HK, sive OB ad BP, ut $y + \frac{1}{2}x$ as $\frac{1}{2}x$; sive, sumptis omnium duplis, ut $2y + 3x$ ad $3x$. Deinde quia YK = $3x$, erit YK ad YK + KT, sive per praedicta, KL ad MN, ut $3x$ ad $3x + y$. Atqui ex rationibus OB ad BP, & KL ad MN, componi diximus rationem BD ad DM. Ergo ratio BD ad DM erit composita ex rationibus $2y + 3x$ ad $3x$, & $3x$ ad $3x + y$; ideoque erit ea quae $2y + 3x$ ad $3x + y$; & dividendo, ratio BM ad MD, eadem quae y ad $3x + y$.

Sit SZ perpendicularis ad SK, eique occurrat MB producta in Z.
Quia ergo ratio BM ad MD inventa est ea quae y ad y + 3x, hoc est quae MK ad MK + 3KS. Sicut autem MK ad MK + 3KS, ita MB ad MB + 3BZ : erit proinde MB ad MD ut

MB + 3BZ. Unde liquet MD aequalem sumendam ipsi MB + 3BZ. Atque ita quotlibet puncta curvae CDE invenire licebit. Cujus curvae portio qualibet ut DS, rectae DB, quae paraboloidi SAB ad angulos rectos occurrit, aequalis erit. Constat a esse, & si velimus, possumus aequatione aliqua relationem exprimere punctorum omnium ipsius ad puncta axis SK.

Simili modo autem, si inquiramus in paraboloide illa sive parabola cubica, in qua cubi ordinatim applicatarum ad axem, sunt inter se sicut portiones axis abscissae, inveniemus curvam cujus evolutione describitur, quaeque proinde rectae lineae aequari poterit, nihilo difficiliori constructione per puncta determinari. Nam si fuerit illa SAB; axis SM; (dicitur autem improprie axis in hac curva, cum forma ejus sit ejusmodi, ut ducta SZ, quae secet SM ad angulos rectos, ea portiones similes curvae habeat ad partes oppositas;)

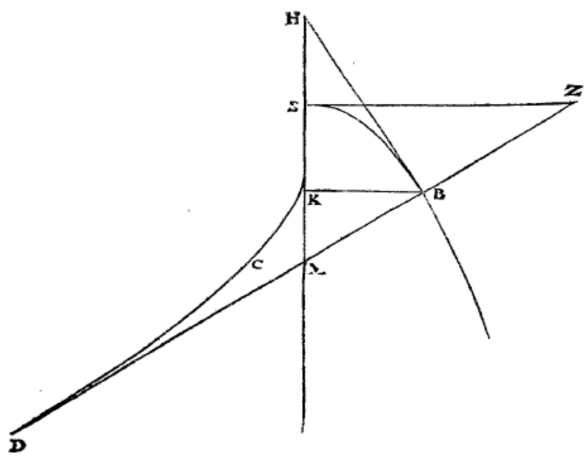


agatur per punctum quodlibet B, in paraboloide sumptum, recta BD, quae curvam ad angulos rectos secet, axique ejus occurrat in M, rectae veros SZ in Z. Deinde sumatur BD aequalis dimidiae BM, una cum sesquialtera BZ. Eritque D unum e punctis curvae quasitae RD vel RI, cujus evolutione, juncta tamen recta quadam RA, describetur paraboloides SAB. Sunt autem hic, quod notatu dignum est, quodque in aliis etiam nonnullis harum paraboloidum contingit, duae evolutiones in partes contrarias, quarum utraque a puncto certo A initium capit; ita ut evolutione ipsius ARD, in infinitum porro continuata, describatur paraboloidis pars infinita ABF; evolutione autem totius ARI, similiter in infinitum extensae, tantum particula AS. Punctum autem A definitur, sumpta SP quae sit ad latus rectum paraboloidis, sicut unitas ad radicem quadrato-quadraticam numeri 91125, (is cubus est ex 45) applicataque ordinatim PA. Unde porro punctum R, confinium duarum curvarum RD, RI, invenitur sicut caetera omnia harum curvarum, hoc est, sicut punctum D modo inventum fuit.

Denique, quaecunque fuerit ex paraboloidum genere curva SAB, semper aeque facile curvam aliam, cujus evolutione ipsa describatur, quaeque propterea rectae adaequari possit, per puncta inveniri comperimus. Atque adeo constructionem universalem sequenti tabella exhibemus, quae quousque libuerit extendi poterit.

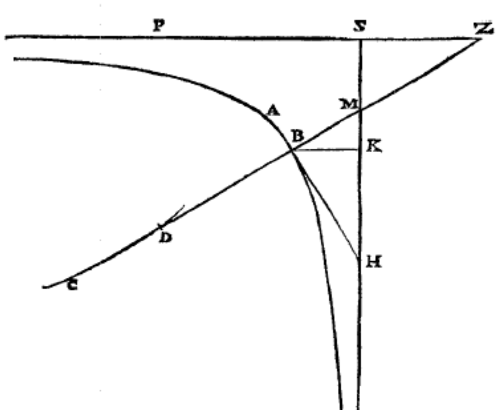
$$\text{Si}\left\{\begin{array}{lcl} ax = y^2 & & \mathrm{BM} + 2\mathrm{BZ} \\ a^2x = y^3 & & \tfrac{1}{2}\mathrm{BM} + \tfrac{3}{S}\mathrm{BZ} \\ ax^2 = y^3 & \text{Erit} & 2\mathrm{BM} + 3\mathrm{BZ} \\ ax^3 = y^4 & & 3\mathrm{BM} + 4\mathrm{BZ} \\ a^3x = y^4 & & \tfrac{1}{3}\mathrm{BM} + \tfrac{4}{3}\mathrm{BZ} \end{array}\right\} = \mathrm{BD}.$$

Sit SB parabola , vel paraboloidum aliqua, cujus vertex S; recta SK vel axis, vel axi perpendicularis, ad quam referuntur aequatione puncta paraboloidis; & ipsa quidem semper ad partem cavam ducta intelligitur; perpendicular SZ. Ponendo jam SK = x;



BK = y, quae a puncto quovis curvae perpendicularis est ipsi SK; & latere recto curvae = a; prior pars tabellae, quae ad sinistram est, naturam singularum paraboloidum singulis aequationibus explicat. Quibus respondent in parte dextra quantitates lineae quantitates lineae BD, quae si curvae SAB insistat ad angulos rectos, exhibitura sit punctum D in curva quaesita CD. Exempli gratia, si SB est parabola quae ex coni sectione sit, ei scimus convenire aequationem tabella primam, $ax = y^2$; cui respondet ab altera parte BM + 2BZ = BD. Unde longitudino lineae BD cognoscitur, adeoque inventio quotlibet punctorum curvae CD. Quam quidem, hoc casu, paraboloidem esse supra demonstratum fuit, eam nempe, cujus aequatio tertia est hujus tabellae.

Construitur autem tabella hoc pacto, ut BM sumatur multiplex secundum numerum qui est exponens potestatis x in aequatione; BZ vero, multiplex secundum exponentem potestatis y; ex his autem utrisque compositae accipiatur pars denominata ab exponente potestatis a.



Praeter hasce autem paraboloides lineas, alias item invenimus, a quibus, non absimili constructione, deducuntur curvae rectis comparabiles. Assimilantur autem hyperbolis, eo quod asymptotes suas habent, sed tantum angulum rectum constituentes. Et harum primam quidem statuimus hyperbolam ipsam, quae est e coni sectione.

Reliquarum vero naturam ut explicemus; sunto PS, SK, asymptoti curvae AB, rectum angulum comprehendentes, & a curvae puncto quolibet B ducatur BK parallela PS, sitque SK = x; KB = y. Si igitur hyperbola sit AB, scimus rectangulum linearum SK, KB, hoc est, rectangulum xy semper eidem quadrato aequale esse, quod vocetur aa.

Proxima vero hyperboloidum erit, in qua solidum ex quadrato lineae SX, in altitudinem KB ductum, hoc est, solidum xxy, cubo certo aequabitur, qui vocetur a^3. Atque ita innumerae aliae hujus generis hyperboloides existunt, quarum proprietatem sequens tabella singulis aequationibus exhibet, simulque rationem construendi curvam DC, cujus evolutione quaeque generetur.

$$\text{Si}\left\{\begin{array}{lcl} xy = a^2 & & \tfrac{1}{2}\text{BM} + \tfrac{1}{2}\text{BZ} \\ x^2y = a^3 & & \tfrac{2}{3}\text{BM} + \tfrac{1}{3}\text{BZ} \\ xy^2 = a^3 & \text{Erit} & \tfrac{1}{3}\text{BM} + \tfrac{2}{3}\text{BZ} \\ x^3y = a^4 & & \tfrac{3}{4}\text{BM} + \tfrac{1}{4}\text{BZ} \\ xy^3 = a^4 & & \tfrac{1}{4}\text{BM} + \tfrac{3}{4}\text{BZ} \end{array}\right\} = \text{BD}.$$

Recta DBMZ curvam AB, ut antea quoque, secat ad angulos rectos, occurritque asymptotis SK, SP, in M & Z. Si igitur exempli gratia hyperbola fuerit AB, cujus aequatio est $xy = a^2$, sumetur BD = $\tfrac{1}{2}\text{BM} + \tfrac{1}{2}\text{BZ}$, quemadmodem tabella praecipit. Eritque punctum D in curva DC quaesita, cujus alia quotlibet puncta sic inveniri poterunt, & portio ejus quaelibet rectae lineae adaequari. Et haec quidem eadem illa est curva, cujus relationem ad axem hyperbolae superius aequatione expdressimus. Constructio autem tabellae hujus plane eadem est quae superioris.

Caeterum, quoniam tum ad harum curvarum, tum ad earum quae ex paraboloidibus nascuntur constructionem, ducendae sunt lineae DBZ, quae ad datum punctum B secent curvas AB, sive ipsarum tangentes BH, ad angulos rectos; dicemus in universum quomodo hae tangentes inveniantur. In aequatione itaque quae cujusque curvae naturam explicat, quales aequationes duabus tabellis praecedentibus exponuntur, considerare oportet quae sint exponentes potestatum *x* & *y*, facere ut, sicut exponens potestatis *x* ad exponentem potestatis *y*, ita sit SK ad KH. Iuncta enim HB curvam in B contindet. Velut in tertia hyperboloide, cujus aequatio est $xy^2 = a^3$: quia exponens potestatis *x* est 1, potestatis autem y exponens 2; oportet esse ut 1 ad 2 ita SK ad KH. Horum autem demonstrationem noverunt analyticae artis periti, qui jam pridem omnes has lineas contemplari coeperunt; & non solum paraboloidum istarum, sed & spatiorum quorundam infinitorum, inter hyperboloides & asymptotos interjectorum, plana solidaque dimensi sunt. Quod quidem & nos, facili atque universali methodo, expedire possemus, ex sola tangentium proprietate sumpta demonstratione. Sed illa non sunt huius loci.