CHAPTER III.

Concerning the Figures which flexible Bodies must adopt with forces applied to themselves in whatever manner; and the resultant directions of these forces.

Thus far we have dealt with forces applied to inflexible bodies, and we have assigned the resultant directions of these ; truly in this chapter we will append forces to moveable or flexible bodies, which circumstance requires two kinds of flexibility to be investigated. In the first place, which flexible shape the body must acquire from the applied forces, or rather, which shape must it retain, when the forces themselves have been composed in equilibrium. Secondly with regard to the resultant direction of the forces of this kind acting together in equilibrium, and the force impressed in this resultant direction.

DEFINITIONS .

I.

84. The shape ZBABX of a perfectly flexible body ZAX is said to be remaining with its ends Z, X fixed in some manner, which the body retains, after all the forces $\mathrm{BH}, \beta\mathrm{H}$ &c. applied to the curve, applied to the points through its whole length in some way, put themselves together in equilibrium.

II.

85. The *tenacity* or *firmness* of a string or body at any point or element of the curve, is that force of the string or body it resists, by which the force or strength on that arises from all the applied forces trying to destroy that by pulling in opposite directions ; and thus it is equivalent or equal to that force of destruction, resulting from all the applied forces of the body.

III.

86. *The lateral forces* derived from the tenacity of a string or body at any element of that are two forces along directions occurring at right angles to themselves in turn, from which acting together a force or strength will result equal to the tenacity of the body at the said element , or also of the force trying to tear that apart.

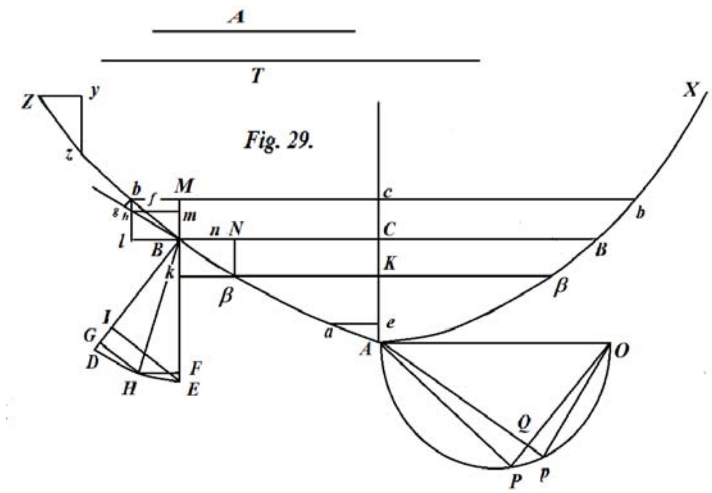


Fig.29. Thus since the string, or rather the element *Bb* of the string, is pulled by a certain resultant force from all the applied forces BH, βH *&c.,* and that by its tenacity it impedes by an equal force lest it be extended or simply broken ; this tenacity or that force striving to equal the tenacity, can be understood, as the force resulting from the side forces, acting at the same time along BM and *Bl*. And the force Bβ by which the element is extended, to which tension it resists by its own tenacity, can be considered as arising from the two lateral forces along the directions BN & *Bk*. Lateral forces of this kind along BM or B*l*, and along BN or *Bk* we will designate from now on by *p*BM, *p*Bl, *p*BN and *p*B*k*. Truly we will denote the tensions of two contiguous elements *Bb* and Bβ by T and *t.*

We suppose the others and indeed the string in these to be flexible in all its parts, but not extensible.

PROPOSITION XI. LEMMA.

87. *If there were some number of decreasing magnitudes* A, B, C, D, E, *the excess of the maximum over the minimum will be the sum of all the differences taken together*.

For besides it is clear, to be $\mathrm{A}-\mathrm{B},\ +\mathrm{B}-\mathrm{C},\ +\mathrm{C}-\mathrm{D},\ +\mathrm{D}-\mathrm{E}=\mathrm{A}-\mathrm{E}$, from which if $\mathrm{E}=0$, the sum of the differences will be equal to the maximum A.

SCHOLIUM.

88. Although this lemma by its simplicity in the first place may seem to be of no use, yet all of the methods of quadratures is based on that, because indeed it can be shown in more detail both in the methods of antiquity as well as in the more recent; but for the sake of brevity only that used in the integral calculus will be shown by one single example. The integral or summitry calculus is the inverse of the differential calculus, and consists there, so that a quantity A may be found from a certain indeterminate quantity and put together with constants, according to its nature, so that, if in place of the indeterminate quantities into that may be substituted successively the same indeterminate quantities, but

with its simplest element [dx] , or twice as great, 3 times as great, 4 times, 5 times, &c. thence the quantities B, C, D, E &c. may arise with a penalty of which the differences $\mathrm{A-B}$, $\mathrm{B-C}$, $\mathrm{C-D}$, $\mathrm{D-E}$ &c. shall be of the same form with the element or with the infinitesimal quantity required to be summed, thus so that the first difference $\mathrm{A-B}$, may show the differential quantity itself, of which the sum or integral itself is sought. So that if a certain curve shall be required to be squared, the axis of which may be called x, and y for the ordinate to this axis, thus so that the element of the area shall become ydx. That area itself may be found, if it may be able to be found, with a certain quantity A being determined from x and composed from certain given quantities or constants, and prepared thus, so that, if into each in place of the indeterminates x there may be substituted the other values $x-dx$, $x-2dx$, $x-3dx$, $x-4dx$ and thus henceforth to infinity, thence the magnitudes may arise B, C, D, E, &c. of which the first difference $\mathrm{A-B}$ may give ydx, second $\mathrm{B-C}$ may give the second element of the area ydx in which the ordinate y now may correspond not to x of the axis, but to $x-dx$, clearly diminished by the element dx, and thus respectively with the rest. Indeed by this method it is evident, all the $\mathrm{A-B}$, $\mathrm{B-C}$, $\mathrm{C-D}$, &c. shows the sum of all the ydx, which are held in the area, or this area itself ; now if in the series of the decreasing quantities A, B, C, D, E &c. the minimum may be called M, the area sought, following Lemma (§.87.) will always be $\mathrm{A-M}$, and if it may come about that occasionally M shall be 0, then the area is equal to A itself. Truly a smaller value M always will be had from the maximum A, clearly if in place of the indeterminate x there may be substituted in that $x-ndx$, where n is the number of elements, into which the axis has been divided, thus so that, ndx sit $=x$, or because it recedes the same, if in place of x there may be substituted 0, since $x-ndx$ is 0 ; and the amount resulting from this substitution will be the minimum M. From which it may arise that, with the substitution made of 0 for the indeterminate x in the quantity A, this vanishes, then M will be $=0$, and thus the area sought, or the sum of all ydx, will be A composed from the indeterminate x diversely affected and with constants involved. Truly in order to find the various values of A itself, from the given element ydx, in place of y there must be substituted is value in terms of x and with constants, which gives rise to the equation of the quadrature of the curve. Because with the quadrature of volumes just as it has been said of areas, that also indeed in the dimension of volumes, equally is to be understood with all the summations.

89. And from these we have substituted the letters A, B, C, D, E &.c. as far as to M to denote a series of decreasing magnitude, but also they are in the case of the infinite, in which the same letters with the same made as in the preceding article with the indeterminates substitutions $x-dx$, $x-2dx$, &c. in place of x in the magnitude A, &c. B, C, D arising greater than A, thus so that the total of these in series may represent a progression of the same increasing thus so that the differences shall be going to become $\mathrm{B-A}$, $\mathrm{C-B}$, $\mathrm{D-C}$, $\mathrm{E-D}$, &c. in which case M, which before was denoting the minimum of the series, now shall be the maximum, and which before it was the maximum now it is necessary A shall be the minimum. An example is required to be added of each case.

90. *Example* 1. The quantity $mdm:\sqrt{(aa+mm)}$ is required to be summed, in which *m* is the variable and *a* given or constant. And in this case there shall be

$\mathrm{A}=\sqrt{(aa+mm)}$, and thus

$\mathrm{B}=\sqrt{\left(aa+mm-2mdm+dm^2\right)}=\sqrt{(aa+mm)},-mdm:\sqrt{(aa+mm)}+\&\text{c}.$

Hence $\mathrm{A}-B=\sqrt{(aa+mm)}-\sqrt{(aa+mm)},+mdm:\sqrt{(aa+mm)}=mdm:\sqrt{(aa+mm)},$

from which the series of magnitudes A, B, C, is descending as far as to M and in place of *m* in the quantity A (§. 89.) 0 may be put, becoming in this case $\mathrm{A}=\mathrm{a}=\mathrm{M}$; therefore with the sum of all $\mathrm{A}-\mathrm{B}$, $\mathrm{B}-\mathrm{C}$, &c. or of all $mdm:\sqrt{(aa+mm)}$ there shall be (87.) $\mathrm{A}-\mathrm{M}$, there will be $\int mdm:\sqrt{(aa+mm)}=\sqrt{(aa+mm)},-a.$

[In modern terms, we may write this as :

$\frac{1}{2}\int du:\sqrt{\left(a^2+u\right)}=\frac{1}{2}\int\frac{du}{\sqrt{\left(a^2+u\right)}}=\sqrt{\left(a^2+u\right)}-a=\sqrt{(aa+uu)}-a.$ We should note that we no longer bother putting successive terms of telescoping difference into the summation, to insure their cancellation, but instead go straight to the final outcome, as the difference of the initial and final values of the differential used. At the time, no doubt, there was still suspicion about the validity of the process!]

91. *Example* 2. But if the element requiring to be summed shall be $aamdm:\overline{aa+mm}\sqrt{aa+mm}$, then the first term will be $\mathrm{A}=aa:\overline{aa+mm}$, and thus $\mathrm{B}=aa:\sqrt{aa+mm}-\overline{2mdm+dm^2}$, from this the root actually extracted from the denominator $=aa:\left(\sqrt{aa+mm}-mdm:\sqrt{aa+mm}\right)$, which fraction on account of the larger denominator, as in the fraction A, here is itself smaller for this fraction, and thus in this case there will be $\mathrm{B}-\mathrm{A}=aamdm:\overline{aa+mm}\sqrt{aa+mm}$. Hence the progression of the magnitudes A, B, C, D &c. is rising or increasing ; and thus the sum of everything (87) $\mathrm{B}-\mathrm{A}$, $\mathrm{C}-\mathrm{B}$, &c. is $\mathrm{M}-\mathrm{A}$, truly in this case M again is *a*, because by making $\mathrm{m}=0$ in the equation $\mathrm{A}=\sqrt{p}$ becomes $\sqrt{(\mu+mm)}=a$, and thus $M-A=a,-aa:\sqrt{aa+mm}$.

92. *Example* 3. By a similar method an infinitely many hyperbolas can be squared within the asymptotes. The general equation of these shall be $x^\alpha y=1$, in which the *x* signify the abscissas from the centre on one of the asymptotes, and *y* the ordinates on the other, the element of the area of the adjacent abscissa will be $ydx=x^{-\alpha}dx.$ Indeed there will be found :

$$A=1:\overline{\alpha-1}.x^{\alpha-1},\ \&\ B=1:\overline{\alpha-1}\left(x-dx\right)^{\alpha-1}=1:\overline{\alpha-1}.x^{\alpha-1}-\overline{\alpha\alpha+2\alpha-1}.x^{\alpha-2}dx+\&\text{c}.$$

hence B is greater than A on account of the smaller denominator than in A, and thus there will be found : $\mathrm{B}-\mathrm{A}=dx:x^{\alpha}=x^{-\alpha}dx=ydx$, from which with the series A, B, C, D shall be ascending, they all become $\mathrm{B}-\mathrm{A},\ \mathrm{C}-\mathrm{B},\ \mathrm{D}-\mathrm{C}$ &c. that is all the $ydx, = \mathrm{M}-\mathrm{A}$. But truly on substituting into the equation $A=1:\overline{\alpha-1}.x^{\alpha-1}$ or $=(1:x)^{\alpha-1}:\alpha-1$, in place of *x*, 0 becomes $\mathrm{M}=(1:0)^{\alpha-1}:\alpha-1$, but truly $1:0$ is infinite in magnitude, that we will indicate by ∞, therefore $\mathrm{M}=\infty^{\alpha-1}:\alpha-1$, and thus $\mathrm{M}-\mathrm{A}$ or the area of the hyperbolas adjacent to the abscissas will be $=\infty^{\alpha-1}:\alpha-1,-1:\overline{\alpha-1}.x^{\alpha-1}$, or because $1:x^{\alpha-1}=xy$, there will be $\mathrm{M}-\mathrm{A}=\left(\infty^{\alpha-1}-xy\right):\alpha-1$.

From which it is apparent areas of this kind to be infinitely great on a par with diverse orders of infinity, if indeed α shall be some positive number and greater than one. But that is enough from these.

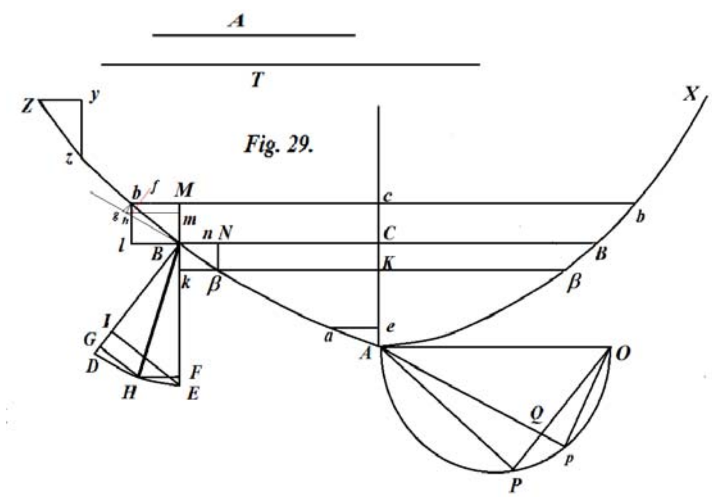
PROPOSITION XII. THEOREM.

93. Fig.29. *The tension* [tenacity] *of the string ZBABX towards maintaining the figure rendered by the forces applied to itself BH,* βH *&c.*,[the latter is force is at an infinitesimal distance from the former; and the forces relate to a normally loaded weightless string under its own tension] *may be equated at any element of the string Bb, to the tension of the string at the vertex A to be increased by all GH, which may be derived from the individual forces BH,* βH *&c. applied to the curve AB, by the resolution of these BH into equivalent perpendicular and parallel lateral forces BG & GH of the individual elements Bb of the curve AB. This is by expressing the tenacity of the string at A; it will become* $\mathrm{T}=\mathrm{A}+\int\mathrm{GH}$.

And with the perpendicular bg sent from the end b of the element of the curve Bb to the tangent Bg of the curve at B, on this tangent it may be assumed, Bb to be equal to the element of the curve $B\beta$ *, and hm acting parallel to the ordinate BC,* bf *truly equidistant to the axis of the curve AC, and through the point B of the curve, also BME to be indefinitely parallel to the same AC, at which the perpendicular HF falls on ME from the end H, with the line BH representing the force applied at the point B to the curve : and then the line increments bh may be joined, and with these done the similar figure bfhg will be had at the angle of contact bBg, and put in place likewise with the larger figure BFHG, if the element Bb were to the element* $\mathrm{B}\beta$ *as the tenacity T of the former to the tenacity t of the latter; and thus there will be some incremental line in the figure bfhg for the element Bb of the curve, just as the homologous line in the greater figure BFHG has to the line T, which expresses the tenacity of the same element of the curve Bb.*

Demonst. I. Because the forces BH (§. 39.) are equivalent to the sides BG and GH, and from these the force GH is taken at the point B in the direction of the tangent of the curve, and since the applied force $\mathrm{B}\beta$ is taken likewise as the force, by which the same $\mathrm{B}\beta$ tries to be extended, to which tension t (§.86.) the tenacity of the element has been said to be equal, these forces acting together, evidently GH and t, are directly equal or opposite to the force T, by which the element B*b* resists being extended, since (following the hypothesis) everything exists in a state of rest or equilibrium, and thus $\mathrm{T} = \mathrm{t} + \mathrm{GH}$, or $\mathrm{T} - \mathrm{t} = \mathrm{GH}$. [Thus the component of the load on the element balances the increase in the tension across the element.] And thus (§. 87.) all the differences $\mathrm{T} - \mathrm{t}$, between the tenacities of the individual contiguous elements of the curve, are equal to the excess of the maximum of T over the minimum, which is the tenacity at the vertex A, therefore $\mathrm{T} - \mathrm{A} = \int \mathrm{GH}$, or $\mathrm{T} = \mathrm{A} + \int \mathrm{GH}$, [on summing over all the increments GH: these integrals are set out in special cases later in the conventional manner].

II. Because (§.39.) $p\mathrm{BM} : p\mathrm{B}b$ that is T, is as $\mathrm{BM} : \mathrm{B}b$, by hypothesis [recall that the prescript *p* (for *potentia*) indicates the force on the increment, which are taken in proportion to the lengths of the increments] truly $\mathrm{T}:t = \mathrm{Bb} : \mathrm{B}\beta$, and $p\mathrm{B}\beta$, that is *t*, is to $p\mathrm{B}k$ as $\mathrm{B}\beta : \mathrm{B}k$; there will be from the equation, $p\mathrm{BM} : p\mathrm{B}k = \mathrm{BM} : \mathrm{B}k$ or B*m* (because



in the similar triangles $\beta\mathrm{BN}$ & $\mathrm{B}hm$ the hypotenuses B*h* and $\mathrm{B}\beta$ are equal ; and thus the triangles themselves are equal), and in the same manner, $(p\mathrm{BM} - p\mathrm{B}k) : p\mathrm{BM} = \mathrm{M}m : \mathrm{BM}$ or $bf : \mathrm{BM}$. And, because all are in equilibrium, the two forces acting together *p*B*k* and *p*BF, which (§. 40.) is derived from the force *p*BH, are placed equally directly opposite to *p*BM, from which $p\mathrm{BM} - p\mathrm{B}k = \mathrm{BF}$, and thus, by putting this value into the preceding proportionality, $(p\mathrm{BM} - p\mathrm{B}k) : p\mathrm{BM} = \mathrm{M}m : \mathrm{BM}$ or $bf : \mathrm{BM}$, we will have

$\mathrm{BF} : p\mathrm{BM} = bf : \mathrm{BM}$, but $p\mathrm{BM} : p\mathrm{B}b$, that is T, $= \mathrm{BM} : \mathrm{B}b$; therefore from these equations [by interchanging] we will have

$\mathrm{BF} : \mathrm{T} = bf : \mathrm{B}b$; [or directly from $\mathrm{BF} : bf = \mathrm{T} : \mathrm{B}b$ or $\mathrm{BG} : (\mathrm{GH} = \mathrm{T}) = bg : gh$]

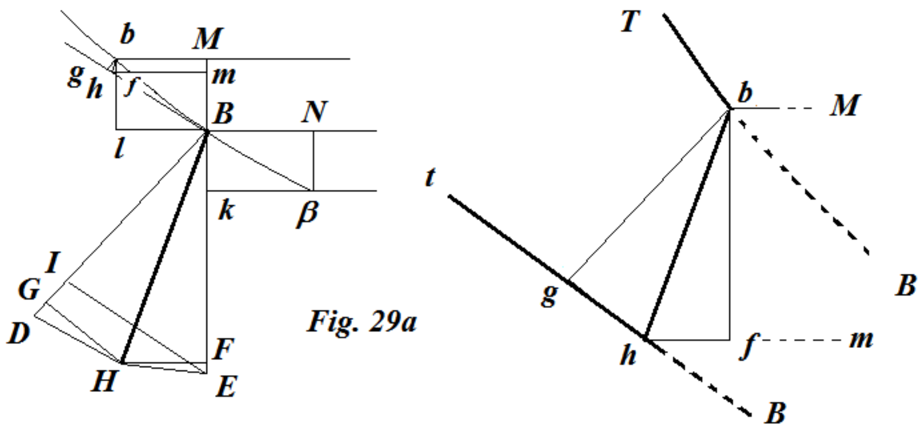


Fig. 29a

By a similar straight forwards argument it will be shown [for the lesser tension *t* to the left], that $p\mathrm{BN} - p\mathrm{B}l = \mathrm{FH}$, and $\left(p\mathrm{BN} - p\mathrm{B}l\right)$ or $\mathrm{FH} : \mathrm{B}l = fh : \mathrm{M}b$; from which because also (§. 39) $p\mathrm{B}l : p\mathrm{B}b$ or $\mathrm{T} = \mathrm{M}b : \mathrm{B}b$; from the equation there will be equally $\mathrm{FH} : \mathrm{T} = fh : \mathrm{Bb}$, or by inverting $\mathrm{T} : \mathrm{FH} = \mathrm{B}b : fh$, and because a little before we had $\mathrm{BF} : \mathrm{T} = bf : \mathrm{B}b$, finally there will be from the equation, $\mathrm{BF} : \mathrm{FH} = bf : fh$, and thus the triangles BFH, *bfh* are similar and therefore the parallels *bf* and BF similarly are put in place. Besides, because as BG and *bg*, themselves perpendicular to the same tangent B*g*, are parallel to each other, and equally the lines BH and *bh*, it is clear also the triangles BGH and *bgh* are similar, and therefore the figures BGHF and *bghf* are composed from similar triangles, are similar and similarly placed. Again since we have found above $\mathrm{FH} : \mathrm{T} = fh : \mathrm{B}b$, on interchanging there will be $\mathrm{FH} : fh = \mathrm{T} : \mathrm{B}b$, and thus any two other homologous lines in the similar figures BFHG & bfhg will be in the same ratio T to Bb, or by interchanging, any small lines in the lesser figure will be to the element of the curve Bb, just as the homologous lines in the greater figure are to the tension T of the same element of the curve. Q.E.D.

COROLLARIUM I.

94. If it pleases, with the right line AO drawn through the vertex A of arbitrary magnitude, to be perpendicular to the axis AC, and with the semicircle APO described upon that, therefore AP will be parallel to the element Bb, and A*p* being equidistant to the other part of the curve [*i.e.* parallel] and to the contiguous element $\mathrm{B}\beta$, and the lines are joined with PO cutting the line A*p* at Q, then also *p*O is drawn; through which lines the forces act, there will be $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP}$. For from (§.93.) there is $\mathrm{BG} : \mathrm{T} = \mathrm{bg} : \mathrm{Bb}$, and on account of the parallel lines AP, B*b* and A*p*, $\mathrm{B}\beta$, the sectors or the triangles B*bg* and APQ are similar, and thus $bg : \mathrm{B}b = \mathrm{PQ} : \mathrm{AP}$; therefore also $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP}$.

COROLLARY II.

95. Any right line in the figure BGHF is to the homologous one in the figure bghf, [see the added diagram Fig. 29a, as the original is far too small and indistinct to be of much use in Fig. 29] as one from the forces *p*B*l* or *p*BM, which are equivalent to the tension T of the element Bb, to the homologous quantities as corresponding to the sides in the characteristic triangle BMb: that is, whatever the line in the greater figure is to the homologous figure in the smaller figure such as *p*B*l* to B*l*, or as *p*BM to BM. For because in each figure any two homologous lines are between themselves, as T to Bb, as *p*B*l* to B*l* or Mb, or as *p*BM to BM, the assertion of this corollary is clear.

COROLLARY III.

96. If all the applied forces BH shall be perpendicular to the curve, such as BD ; the points G and H may be combined as the point D, with GH, *gh* vanishing in each greater or smaller figure ; and thus the tension of the curve everywhere is equal to the tension of the curve at the vertex A, and thus is constant or given. That is everywhere there will be had $\mathrm{T} = \mathrm{A}$.

COROLLARY IV.

97. Therefore, since (§. 94.) in Fig. 29, there shall be $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP} = \tan$ of the angle PAQ or of the angle gBb [The original derivation uses the sin of the angle, which is clearly incorrect ; however the angle is very small, and no harm is done; and the ratio is best put as *bg*:t, referring to the infinitesimal triangle *b*B*g*.], that is the sine of the curvature at B to the radius ; in the case of the perpendicular forces to the curve there will be one of these BD applied to some point B of the curve according to the constant tension of the string (§.96.) A, just as the sine of the curvature at the point of the curve, to which the force has been applied to the radius; and by interchanging the force applied BD will be to the sine of the curvature at B just as at A to the radius, that is, in the given ratio.

This elegant property was first considered by the most acute geometer Johann Bernoulli; but that, as much as appears in the Commentaries of the Royal Acad. of Paris, 12th May, 1706, he had deduced from another basis.

COROLLARIUM V.

98. Besides with these in place, which we have deduced in the two immediately preceding Corollaries, because (§.93.) $\mathrm{HF} : \mathrm{hf} = \mathrm{BF} : \mathrm{bf} = \mathrm{T} : \mathrm{Bb}$ (or §. 96.) $= \mathrm{A} : \mathrm{Bb}$, there will be $\int \mathrm{FH} : \int \mathrm{fh} = \int \mathrm{BF} : \int \mathrm{bf} = \mathrm{A} : \mathrm{Bb}$, if the individual elements of the curve were assumed equal. And (§. 87.) $\int \mathrm{hf}$, or all the elements hf or Nn, on putting $\mathrm{Bn} = \mathrm{bM}$, with respect to the portion of the curve $\mathrm{A}\beta\mathrm{B}$ are equal to the excess of the maximum difference of the ordinates over the greatest minimum bM (on putting the little arc of the curve $\mathrm{Aa} = \mathrm{Bb} = \mathrm{B}\beta = \mathrm{Zz}$), that is, $\mathrm{ae} - \mathrm{bM}$ (or also, because aec and aA are equal when the axis CA of the curve is perpendicular at A)

$Aa - bM = Bb - bM$; & $\int bf = \int Mm = BM - Ae$ (or because Ae in this case vanishes before Aa or Bb) $= BM$. Therefore $\int FH : Bb - bM = \int BF : BM = A : BC$. Where all the FH or $\int FH$, and $\int BF$ pertain to the area of the curve $A\beta B$.

But if indeed $\int FH$ and $\int BF$ should pertain to the arc of the curve BZ and the tangent of this arc at Z were parallel to the axis AC, erit $Zy = 0$, on putting as above $Zz = Bb = B\beta$ &c. and thus $zy = Zz = B\beta$; from which $\int hf$ or $\int Nn$ in this case will be (§. 87.) $BN - Zy = BN$, and $\int bf - \int Mm = yz - N\beta = Z\gamma - N\beta = B\beta - N\beta$, and thus $\int FH : BN = \int BF : B\beta - N\beta = A : B\beta$.

COROLLARY VI.

99. If the forces applied to the curve BH, βH &c. are equidistant to the axis AC, then BH and BE coincide, and *bh* and *bf* with HF, *hf* vanishing and thus all the elements of the order bM, BN, &c. are required to be taken equal in this case ; and hence all the forces along the directions of the orders in this hypothesis are to be equal, derived from the tension of the elements of the curve. Therefore, by calling each of these equal forces B, now (§.95.) there arises $BE : bf$ or $Mm = B : bM$, and thus in this case with B & bM constant, there will be $\int BE : \int Mm = B : bM$, or because (§.98.) $\int Mm = BM - Ae = BM$ with Ae vanishing before Aa; there becomes $\int BE : BM = B : bM$ and on interchanging $\int BE : B = BM : bM$. Again, because the ordinate *ae* of the vertex near im itself defines the curve Aa, and B generally a constant force denoting the directions along the orders, that also may be set out, which acts along the direction *ea* or A*a*, but the force along A*a* is the tension of the curve at the vertex A, therefore $A = B$, and thus $\int BE : B = \int BE : A = BM : bM$.

SCHOLIUM.

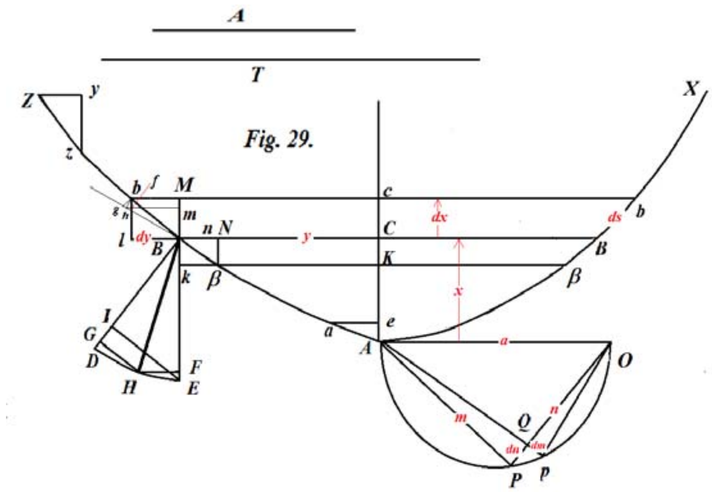
100. It can be agreed well enough from the corollaries added to our theorem that its use may be widely apparent, for actually it contains the solution of an infinite number of problems, of which the problems of the Catenary, Sails, and of the curved figure formed by cloth from liquid at rest, only they are the most specialized cases of our theorem. But before I shall approach the application of this to special cases of this kind, it has to be warned and to be noted, the applied forces BH, βH, even if expressed by simple and finite lines of finite magnitude sometimes signify only infinitely small increments, and

also when they are expressing infinitesimal rectangles ; and the tensions of the string at individual points always emerge indeed as magnitudes of the same kind just as the applied forces, but before these have become infinite ; thus so that it may be clarified more clearly from that application.

101. Therefore there shall be:

$$\mathrm{AC} = x, \mathrm{CB} = y, \mathrm{BM} = dx, \mathrm{bM} = dy, \mathrm{Bb} = ds, \mathrm{AO} = a,$$
$$\mathrm{AP} = m \ \ \& \ \ \mathrm{PO} = n = \sqrt{(aa - mm)}.$$

Hence $\mathrm{PQ} = dn$ & $\mathrm{Qp} = dm$. And with these in place, the similar triangles BbM and OAP are presented:



$ds = adx : n,$ & $dy = mdx : n$. Therefore with these values substituted into the ratios : $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP}$, found in §.94, there will be $\mathrm{BG} : \mathrm{T} = dn : m$, the first rule. And $\mathrm{T} = \mathrm{A} + \int \mathrm{GH}$, from the other general formula.

102. The applied forces BH and perpendicular to the curve will be $= bds$, where b is the magnitude given [of the component BG of the applied force BH at this point], and in this case there will be (by §.96.) $\mathrm{T} = \mathrm{A}$, and A (§100.) shall be a magnitude of the same kind as the force applied, and it may be said to be $= ab$. Hence from BG:T there becomes $bds{:}ab$, or $ds{:}\ a$ and thus $ds : a = dn : m = \mathrm{BG} : \mathrm{T}$, or because (from §.101.) $ds = adx : n$, there will be $(adx : n) : a = dn : m$, and thus $dx : n = dn : m$ therefore $mdx = ndx = -mdm,$ & $dx = -dm$, and by integrating, $x = a - m$ or $m = a - x$, therefore $n = \sqrt{(aa - mm)} = \sqrt{(\ 2ax - xx)}$, hence $dy\,(= mdx : n) = adx - xdx : \sqrt{(2ax - xx)}$, therefore $y = \sqrt{(2ax - xx)}$, which is the equation for a circle.

103. If the forces perpendicular to the curve BD, BG or BH (these three in this case indicate one and the same force) shall be $= dy^2 : ds$, [for $dy \cdot dy : ds$ is the projection of *dy* to *ds*,] which is the case of the sail, so that it may be shown fuller in its place, it becomes with the substitutions made,
$\mathrm{BD} = [dy \cdot dy : ds = \left(m^2 dx^2 / n^2\right) \times n / adx] = mmdx : an$, and on putting $\mathrm{A}\left(=\mathrm{T}\right) = a$, [here the tension is assumed to be greatest at the apex] the ratio (§.101.) $BG : T = dn : m$, now becomes $mmdx : aan = dn : m$, hence for the equation of the sail :
$aandn = m^3 dx$ & $m^3 dx = aamdm$ or $dx = -aadm : mm$, hence
$x = aa : m, \ -a$ [*i.e.* $x = \frac{aa}{m} - a$] and
$m = aa : (a + x)$ & $n = \sqrt{\left(aa - mm\right)} = a\sqrt{(2ax + \ xx)} : a + x$, therefore
$dy\left(= mdx : n\right) = adx : \sqrt{(2ax + xx)}.$

104. If the perpendicular force BD applied to the curve in the same manner shall be *kds*, where *k* denotes some given quantity at *x* and with constants, there may be put as noted nearby above (§. 100.), $\mathrm{T} = \mathrm{A} = \frac{1}{2} aa$; and with this done, from (§.101.) agreeing with the ratio substituted $\mathrm{BG} : \mathrm{T} = \mathrm{d}n : m$, it will be changed into $2kdx : an = dn : m$, hence $2kdx = andn : m = -adm$, or $2kdu = 2kdx = adm$. Putting $x + u =$ to the constant b, therefore $du = -dx,$ & $2kdu = -2kdx = adm$. From which if there becomes $2kdu = adp$, there will be $dp = dm$, and thus $p = m$ & $dy = pdx : \sqrt{\left(aa - pp\right)}$. Which has been found from other principles by the most celebrated Bernoulli's. If $k = u$, there becomes $dy = -uudu : \sqrt{\left(a^4 - n^4\right)}$ for the figure of the cloth, or of the elastic material of the great James Bernoulli.

105. If the applied forces BH are parallel to the axis AC, as BE; EI is drawn perpendicular to BD, from which if $\mathrm{BE} = dq$ on account of the similar triangles BIE, APO the side may be found $\mathrm{EI} = ndq : a,$ and $\mathrm{BI} = mdq : a$. Therefore in place of BG now it is required to take BI, in the ratio $\mathrm{BG{:}T} = dn : m$, and there becomes $\left(mdq : a\right) : t = dn : m,$ or $atdn = mmdq$, but $a + \int \mathrm{IE} = a + \int ndq : a = \mathrm{T} = \mathrm{t}$, and on differentiating, $ndq : a = dt$, or $dq = adt : n$, which substituted into the equation $atdn = mmdq$, gives rise to $atndn = ammdt$ or $dt : t = ndn : mm = -mdm : mm = -dm : m$; hence $t = aa : m$, which substituted into the above ratio gives $\left(mdq : a\right) : \left(aa : m\right) = dn : m$, from which there is found $dq = a^3 dn : m^3$ (or rather $aa = mm + nn)$ $= anmdn + anndn : m^3 = ammdn - amndm : m^3 = amdn - andm : mm$; from which, with the summation made, there will be found $q = an : m$, that is $mq = an$, or with *m* & *n* substituted in place of the proportions *dy*, *dx*; there will be found $dy = adx : q$. Which equation contains all the kinds of catenaries.

But truly even if the equation found now will be contained in corollary six, which gives the ratio $\int \mathrm{BF} : \mathrm{A} = \mathrm{BM} : \mathrm{bM}$, in which if $\int \mathrm{BF} = \int dq = q,\ \mathrm{A} = a,\ \mathrm{BM} = dx,\ \&\ \mathrm{bM} = dy$, there will be had $q : a = dx : dy$, and thus $dy = adx : q$. Yet concerning the matter it is not clear to me, if also in this case I should illustrate the ratio used $\mathrm{BG} : \mathrm{T} = dn : m$.

106. Let $q = s$, by s it is required to understand the curve $\mathrm{A}\beta\mathrm{B}$, there will be $dy = adx : s$, which is the simplest case of the catenary problem. Indeed there shall become $ms = an\ \&\ s = an : m$, and $ds = \frac{amdn - andm}{mm}$, or because $ds = adx : u$, there will be $dx = mndn - anndm : mm = -mmdm - nndm : mm = -aadm : mm$, therefore as above (§.103.) there will be found $dy = adx : \sqrt{(2ax + xx)}$, the *catenary equation*, which is short is the same as *the sail*. Truly if the direction of gravity is not put to converge at a finite distance , the *catenary* will become an *equilateral hyperbola*, the demonstration of which is left for others for the sake of brevity, though in other respects it shall not be a long calculation, but just a line and can be performed without a calculation.

PROPOSITION XIII. PROBLEM.

107. *With the curve found remaining, in which the sail, the canvas, or any other flexible body may be curved by applied forces, to assign the resultant direction of the forces and the strength of these, along this resultant direction.*

Fig.30. AeD shall be the curve found with its end bound to its small length of unbending wood or right line ACD, the resultant direction of all the forces *eb*, *db* &c. applied to the curve, and the size of the force, or impression, which it must

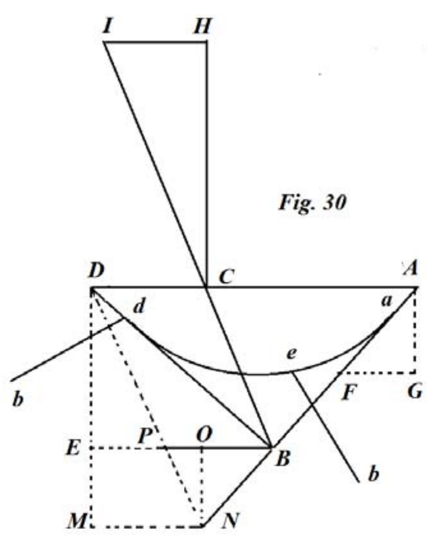


Fig. 30

exert along the resultant direction CB in the beam DCA. [Note that in all these discussions, Hermann uses the term *median* or *average* direction rather than the term *resultant* now used.]

Solution. The tangents AB and DB shall be drawn through the ends of the curve A and D arriving at the point B, in either of which, so that AF may be taken as a segment of AB, which shall be to the remaining whole tangent DB, as the tension of the curve at A to its tension at D, which tensions have been found above (§. 93). The perpendiculars AG and DM to the right line DA are acting through the points A and D, and through F and B equidistant to the same AD, the lines FG and BE are drawn crossing with the perpendiculars at G and E. AD may be divided at C, so that there shall be $AC:DE = DE:AG$, and the part $CH = AG + DE$ may be taken drawn upwards through the point C perpendicular to the line AD, and on the same parallel to the same AD through the point H, $HI = BE - FG$ may be taken to be drawn, and finally IC may be joined, and its continuation CB will be the resultant direction sought, truly the resultant from all the forces applied to the curve, following this mean direction, and it will be to the tension of the string at A or D, thus as IC is to AF or to DB.

Demonst. The line AD cannot undergo any other impression from the forces applied *eb*, *db* &c. to the curve A*e*D, other than what results from the forces, from which it is urged in the directions of the tangents of the curve AB, DB, or from the tensions of the curve at A and D, since the curve AeD has been fastened to the inflexible beam ACD at these points only. Truly by resolving the forces AF and DB into their equivalent lateral forces AG & GF, and DE & EB, because from AG & DE acting together (§.54.) the force arises equal to AG & DE in the direction passing through C clearly through the centre of gravity of AG & DE, and to the perpendicular line AD, and because the force $CH = AG + DE$ acts by construction in the opposite sense, this force CH will be agreed to be in equilibrium with the forces AG & DE. Truly the force, which results from the opposites forces EB & GF (§. 38.) will be equal to the excess of the greater force EB over the lesser GF, and also (by construction) $HI = EB - GF$, therefore from these opposite forces and with the line AD parallel to the forces the force emerges HI, which acts on the line AD from C towards D shall be in equilibrium with the opposite forces EB & GF, therefore the two forces CH & HI will be agreed to be in equilibrium with the forces AG, DF, GF & EB, with which the oblique forces AF & DB are equivalent and thus since from these oblique forces, or because (§.39.) the single force CI arises from the sides CH & HI, this also will remain in equilibrium with the oblique forces AF & DB. Hence (§.37) the continuation IC of the right line CB gives the mean direction of the oblique forces AF, DB, or what is the same, of all the forces *eb*, *db* applied to the curve A*e*D ; indeed IC itself puts in place the force along the resultant direction CB from all the applied forces arising. Q.E.D.

COROLLARY I.

108. The right line IC produced passes through the meeting point B of the tangents AB, & DB. For AB may be drawn forwards to N so that there becomes $BN = AF$, it may be shown that DN is equal and parallel to IC. For if through the point N the lines NM, NO may be acting parallel to the right lines DA or BE, and DE, on account of the parallel lines NO, AG and OB, FG and (constr.) $BN = AF$, the triangles AFG, NBO will be

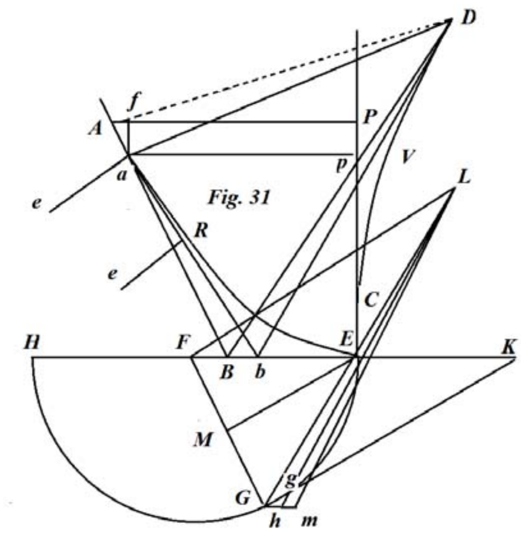
similar and equal, and thus $NO = ME = AG$, and the total $DM = DE + AG$ (constr.) $= HC$, thus also $MN = EO = EB - FG = EB - FG$ (constr.) $= IH$; also therefore $DN = CI$, and therefore DM & CH are parallel and the angles MDN & ICH themselves are equal, and DN & IC will be parallel. Again because (constr.) $AC : DC = DE : AG$ or EM (and on account of the parallel lines EB and MN) AB.BN, and thus $AC : DE = AB : BN$, the lines DN & CB not only are equidistant, but also DN & IC have been shown to be parallel ; therefore IC & CB have been placed in a direction, and hence IC itself produced will pass through the intersection B of the tangents AB and DB.

COROLLARY II.

109. Hence AF is to DB, just as the sine of the angle ABC to the sine of the angle DBC. For in triangle DBN, the sine of the angles NDB, DNB are in proportion with the opposite sides NB, DB, and therefore these angles on account of the parallel lines DN & CB are equal to the angles DBC, ABC. Therefore the preceding, and this other corollary, easily supply the needs for the construction of this problem. With the tangent AB produced to N, so that its continuation BN shall be to the other whole tangent DB, just as the tension of the curve at A, which is expressed by the right line AF, to the tension of the same at D, as DB expresses, the right line BC through the point of intersection B of the tangents AB, DB and parallel to the line DN, joining the points D and N, is the mean [or resultant] direction sought.

COROLLARY III.

110. If the tensions of the string at A & D shall be equal, the mean direction BC of the angle of the tangents ABD shall be divided into two. For since AF equals DB in this case, and AF to DB, to the sine of the angle DBC to the sine of the angle ABC by the preceding corollary, generally the angle ABC will be equal to the angle DBC.



SCHOLION.

111. Fig. 31. The third corollary guides us to that new kind of curves, that came to be called the direction of the mean lines, that appear in the works of the Cel. James Bernoulli in the *Acta Eruditorum* Lips. 1695 page. 548. For if a wire or another flexible body acted on by the forces *ae,* R*e* &c at its individual points applied vertically in the line of an arc ERA, the right line BD dividing the angle ABE formed from the tangents to the curve at each of its ends into two parts (§.110.) is the mean direction of all the forces *ae* &c. applied to the whole curve ERA, and with the tangent drawn from some other point of the curve *a*, with the right line *b*D dividing the angle abE into two parts, is the mean direction of the forces applied to the arc ERa. Now all the mean directions BD, bD &c. are tangents to some curve CVD, which thus may be called the line of the mean directions [*i.e.* the envelope of the tangents].

PROPOSITION XIV. THEOREM.

112. *The right line* AD *perpendicular to the curve* ERA, *and the mean direction* BD *of the forces applied perpendicularly to the curve, will intersect at the point* D *of the line of the mean directions* CVD*.*

Aa shall be the element of the curve, and *ab* the tangent of the curve at *a*, and bD the line bisecting the angle *ab*E, the mean direction of the applied forces to the arc of the curve ER*a* : from which since *b* shall be infinitely close to B itself, it follows one or the other *b*D or BD to cut the curve sought at the point D. Therefore it only remains to be proven, the perpendicular AD to the curve passes through that point of intersection D, of the two infinitely close lines BD, *b*D. With any centre F taken on the tangent BE and with the semicircle EGH described with the radius FE, the radii FG are acting parallel to the tangent AB, and F*g* parallel to the tangent *ab,* and with EG and E*g* joined, G*m* parallel to the radius FE, crossing the right line E*g* produced at *h*, and the radius Fg extended at *m* : finally the tangent GK shall be drawn to the semicircle at the point G, and EM parallel to the same tangent, which therefore will be perpendicular to the radius FG. Now because FG is parallel to the tangent AB, the angle will be equal $\mathrm{ABH} = \mathrm{EFG}$, and with the two FEG & FGE likewise equal to the angle ABE, or to twice the angle DBE, since BD bisects the angle ABE, therefore $\mathrm{FEG} = \mathrm{DBK}$, from which the lines BD & GE are parallel. By a similar argument it is proven, *b*D & *g*E are parallel. Therefore with *m*L drawn parallel to *h*E itself, or *g*E, the figures AB*b*D and FG*m*L will be similar, since they shall be composed from the similar triangles AB*b*, BD*b*, & FG*m*, GL*m*. And with these in place, the parallel lines FK & G*m* will be divided in proportion by the mean line E*h*, and thus $\mathrm{G}m : \mathrm{G}h = \mathrm{FK} : \mathrm{EK}$ (or on account of the parallel lines ME & GK) $= \mathrm{FG} : \mathrm{MG}$. Truly because the triangles GE*h* & GL*m* are similar (for by the construction E*h* & L*m* are parallel), $\mathrm{G}m : \mathrm{G}h = \mathrm{GL} : \mathrm{GE}$. Therefore $\mathrm{FG} : \mathrm{MG} = \mathrm{GL} : \mathrm{GE}$; and thus FL joining the points F, L is parallel to the right line ME, and hence LFG is a right angle, since EM shall be perpendicular to FG. From which on account of the similarity of the figures AB*b*D & FG*m*L, there will be AB:BD= FG:G L, hence, because the angle LFG is right, the angle DAB likewise will be right, and thus AD will be perpendicular to the curve ERA at A. Q.E.D.

COROLLARY.

113. And thus also the triangles ADB & MEG are similar, from which the value of the line AD is elicited easily. For if $\mathrm{FE} = a, \mathrm{AP} = y, \mathrm{Af} = dy, \mathrm{EP} = x, \mathrm{Pp} = dx$, & $\mathrm{Aa} = ds$: there will be AB =*xds:dx*; and because the angle $\mathrm{MFE} = a\mathrm{Af}$, and thus the triangles FME & Aaf are similar, there is found $\mathrm{FM} = ady : ds$; & $\mathrm{MG} = (ads - ndy) : ds$, then also $\mathrm{ME} = adx : ds$; hence because $\mathrm{MG} : \mathrm{ME} = \mathrm{AB} : \mathrm{AD}$, there will be found $\mathrm{AD} = xds : ds - dy$.

The most enlightened James Bernoulli (Acta Erud. Lips. cited above here) used an analytical calculation, which still was not apposite for that, he found $\mathrm{AD} = xds^2 + xdyds : dx^2$, from which formula ours is a little simpler and to some extend by a different account elicited from the preceding from the Bernoulli account, may be derived with little trouble, only be substituting the quantity $ds^2 - dy^2$ in place of the equal quantity dx^2, and by dividing the numerator and denominator by $ds + dy$.

CAPUT III.

De Figuris, quas Corpora flexibilla induere debent a potentiis ipsis quomodocunque applicatis; & de Mediis directionibus harum potentiarum.

Egimus hactenus de potentiis corporibus inflexibilibus applicatis, earum medias directones assignavimus ; in hoc vero capite potentias corporibus cedentibus seu flexibilibus appendemus, quae circumstantia flexibilitatis duo indaganda jubet. Primum quamnam corpus flexile figuram debeat acquirere ab applicatis potentiis, vel potius, quamnam retinere debeat formam, ubi potentiae ad aequilibrium sese composuerunt. Secundum respicit mediam directionem cjusmodi potentiarum in aequilibrio consistentium, & vim impulsus secundum hanc mediam directionem.

DEFINITIONES .

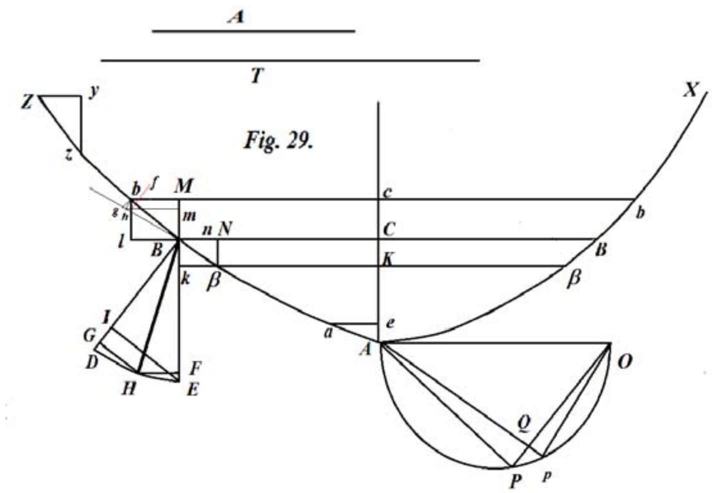
I.

84. Corporis perfecte flexilis ZAX terminis suis Z, X alicubi affixi *Figur*a ZBABX *manens* dicitur, quam corpus retinet, posteaquam omnes potentiae BH, βH & c. curvae, punctis per totam ejus longitudinem quomodocunque applicatae, ad aequilibrium se composuerunt.

85. *Tenacitas* vel *Firmitas* fili aut corporis in quolibet ejus puncto seu elemento curvae, est vis illa fili aut corporis, qua potentiae illi aut vi ex omnibus applicatis potentiis nascenti, atque filum in contrarias partes trahendo id dilacerare conanti, resistit ; atque adeo aequipollet, vel aequatur, ipsi vi dilaceranti, ex omnibus corpori applicatis potentiis resultanti.

III.

86. *Potentiae laterales* ex fili aut corporis tenacitate in quolibet ejus elemento, *derivatae* sunt duae potentiae secundum directiones sibi invicem ad angulum rectum occurrences,



ex quibus conjunctim agentibus resultaret vis aut potentia aequalis corporis tenacitati in dicto ejus elemento, aut etiam vi ejus dilaceranti.

Fig.29. Sic cum filum, vel potius fili elementum *Bb*, vi quadam ex omnibus applicatis BH, βH *&c.* resultante tenditur, idque sua tenacitate aequali vi tendenti impedit ne extendatur aut prorsus rumpatur, haec tenacitas, aut vis illa tendens tenacitati aequalis, intelligi potest, tanquam potentia resultans ex potentiis lateralibus, secundum BM & *Bl* simul agentibus. Et vis qua tenditur elementum Bβ, cui tensioni id sua tenacitate resistit, spectari potest tanquam nascens ex duabus lateralibus potentiis juxta directiones BN & *Bk*. Ejusmodi potentias laterales juxta BM vel B*l*, & juxta BN aut *Bk* deinceps designabimus per *p*BM, *p*B*l*, *p*BN & *p*B*k*. Tenacitates vero duorum elementorum contiguorum *Bb* & Bβ indicabimus per T& *t*.

Caeterum in hisce supponimus filum quidem flexile esse in omnibus suis partibus, non vero extensibile.

PROPOSITIO XI. LEMMA.

87. *Si fuerit quotcunque decrescentes magnitudines* A, B, C, D, E, *erunt omnium differentiae simul sumtae aequales excessui maximae supra minimam.* Nam ultro liquet, esse $\mathrm{A-B},\ +\mathrm{B-C},\ +\mathrm{C-D},\ +\mathrm{D-E}=\mathrm{A-E}$, unde si $\mathrm{E}=0$, erit summa differentiarum maximae A aequalis.

SCHOLION.

88. Quanquam hoc lemma simplicitate sua nullius usus prima fronte videatur, in eo tamen omnes quadraturarum methodi fundantur, quod prolixius quidem ostendi posset tam in antiquorum methodis quam recentiorum; sed brevitatis gratia id tantum in usu Calculi integralis uno alteroque exemplo illustrabitur. Calculus integralis vel summatorius est inversus calculi differentalis, & consistit in eo, ut inveniatur quantitas A ex indeterminata quadam & constantibus composita, ejus indolis, ut, si loco indeterminatae in ea substituantur successive eadem indeterminata, sed elemento suo simplici, ejusve 2plo, 3plo, 4lo, 5plo &c. mulctata inde exsurgant quantitates B, C, D, E &c. quarum differentiae $\mathrm{A}-\mathrm{B}$, $\mathrm{B}-\mathrm{C}$, $\mathrm{C}-\mathrm{D}$, $\mathrm{D}-\mathrm{E}$ &c. sint ejusdem formae cum elemento seu quantitate infinitesima summanda, ita ut prima differentialem $\mathrm{A}-\mathrm{B}$, exhibeat ipsam quantitatem differentialem, cujus summa seu integralis quaeritur. Ut si curva quaedam sit quadranda, cujus axis dicatur x, & ordinatae ad hunc axem y, ita ut areae elementum futurum sit ydx. Area ipsa inventetur, si haberi possit, quaedam quantitas A ex indeterminata x & quantitatibus datis seu constantibus composita, itaque comparara, ut, si in ea substituantur loco indeterminatae x aliae $x-dx$, $x-2dx$, $x-3dx$, $x-4dx$ & sic deinceps in infinitum, inde proveniant magnitudines B, C, D, E, &c. quarum prima differentia $\mathrm{A}-\mathrm{B}$ det ydx, secunda $\mathrm{B}-\mathrm{C}$ det secundum elementum areae ydx in quo ordinata y jam respondeat non axi x sed axis $x-dx$, suo scilicet elemento dx imminuto, atque sic de reliquis respective. Hac enim ratione perspicuum est, omnes $\mathrm{A}-\mathrm{B}$, $\mathrm{B}-\mathrm{C}$, $\mathrm{C}-\mathrm{D}$, &c. exhibere aggregatum omnium ydx, quae in area continentur, seu hanc aream ipsam; jam si in serie quantitatum decrescentium A, B, C, D, E &c. minima dicatur M, area quaesita, juxta Lemma (§.87.) perpetuo erit $\mathrm{A}-\mathrm{M}$, & si contingat, ut aliquando M sit O, tunc area aequatur ipsi A. Minima vero quantitas M semper habebitur ex Maxima A, si scilicet loco indeterminatae x substituatur in ea $x-ndx$, ubi n est numerus elementorum, in quae axis divisus est, adeo ut, ndx sit $=x$, vel quod eodem recidit, si loco x ibi substituatur 0, quandoquidem $x-ndx$ est 0 ; quantitasque ex hac substitutione resultans erit minima M. Unde si contigat ut, facta substitutione ipiius 0 loco indeterminatae x in quantitate A, haec evanescat, tunc M erit $=0$, atque adeo area quaesita, seu summa omnium ydx, erit A composita ex indeterminata x varie affecta & cum constantibus implicata. Pro inveniendo vero valore ipsius A, ex dato elemento ydx, loco ipsius y substitui debet ejus valor in x & constantibus, quem aequatio curvae quadrandae praebet. Quod de quadratura spatiorum seu arearum dictum est, id etiam dimensione solidorum imo, de omnibus summationibus, pariter est intelligendum.

89. In hisce supposuimus literas A, B, C, D, E &.c. usque ad M denotare seriem magnitudinum decrescentium, sed infiniti etiam sunt casus, in quibus eaedem literae iisdem ac praecedenti articulo factis substitutionibus indeterminutarum $x-dx$, $x-2dx$, &c. loco ipsius x in magnitudine A, proveniant B, C, D &c. majores quam A, adeo ut tota earum series repraesentet progressionem earundem crescentem ita ut differentiae futurae sint $\mathrm{B}-\mathrm{A}$, $\mathrm{C}-\mathrm{B}$, $\mathrm{D}-\mathrm{C}$, $\mathrm{E}-\mathrm{D}$, &c. quo casu M, quae antea minimam seriei magnitudinem denotabat, nunc sit maxima, & quae antea erat maxima A nunc sit minima, necesse est. Utriusque casus exemplum est adducendum.

90 *Exempl.* 1. Sit summanda quantitas $mdm:\sqrt{(aa+mm)}$, in qua *m* est variabilis & *a* data seu constans. Eritque hoc casu
$\mathrm{A}=\sqrt{(aa+mm)}$, adeoque
$\mathrm{B}=\sqrt{\left(aa+mm-2mdm+dm^2\right)}=\sqrt{(aa+mm)},-mdm:\sqrt{(aa+mm)}+\&\mathrm{c.}$
Hinc $\mathrm{A}-B=\sqrt{(aa+mm)}-\sqrt{(aa+mm)},+mdm:\sqrt{(aa+mm)}=mdm:\sqrt{(aa+mm)},$
unde series magnitudinum A, B, C, usque ad M est descendens & si loco ipsius *m* in quantitate A (§. 89.) ponatur 0, fiet eo casu $\mathrm{A}=\mathrm{a}=\mathrm{M}$; cum igitur summa omnium $\mathrm{A}-\mathrm{B},\ \mathrm{B}-\mathrm{C}$, &c. vel omnium $mdm:\sqrt{(aa+mm)}$ sit (87.) $\mathrm{A}-\mathrm{M}$, erit
$\int mdm:\sqrt{(aa+mm)}=\sqrt{(aa+mm)},-a.$

91. *Exempl*. 2. Sin detur elementum summandum $aamdm:\overline{aa+mm}\sqrt{aa+mm}$, tunc erit $\mathrm{A}=aa:\overline{aa+mm}$, atque adeo $\mathrm{B}=aa:\sqrt{aa+mm}-\overline{2mdm+dm^2}$, hoc est extrahendo revera radicem ex denominatore $=aa:\left(\sqrt{aa+mm}-mdm:\sqrt{aa+mm}\right)$
quae fractio ob denominatorem majorem, quam in fractione A, hac ipsa fractione minor est, atque adeo hoc casu erit $\mathrm{B}-\mathrm{A}=aamdm:\overline{aa+mm}\sqrt{aa+mm}$. Hinc progressio quantitatum A, B, C, D &c. est ascendens seu crescens;
atque adeo summa omnium (87) $\mathrm{B}-\mathrm{A},\ \mathrm{C}-\mathrm{B}$, &c. est $\mathrm{M}-\mathrm{A}$, verum hoc casu M iterum est *a*, quoniam facta $\mathrm{m}=0$ in aequatione $\mathrm{A}=\sqrt{p}$ fiet $\sqrt{(\mu+mm)}=a$, atque adeo $M-A=a,-aa:\sqrt{aa+mm}$.

92. *Exempl*. 3. Pari ratione infinitae hyperbolae intra asymptotas possunt quadrari. Sit earum aequatio generalis $x^{\alpha}y=1$, in qua *x* abscissas a centro in alterutra asymptotarum, & *y* ordinatas significant, elementum areae abscissae adjacentis erit $ydx=x^{-\alpha}dx$. Invenietur vero,

$$A=1:\overline{\alpha-1}.x^{\alpha-1},$$
$$\& \ B=1:\overline{\alpha-1}\left(x-dx\right)^{\alpha-1}=1:\overline{\alpha-1}.x^{\alpha-1}-\overline{\alpha\alpha+2\alpha-1}.x^{\alpha-2}dx+\&\mathrm{c.}$$

hinc B major est quam A ob denominatorem minorem quam in A, atque adeo invenietur $\mathrm{B}-\mathrm{A}=dx:x^{\alpha}=x^{-\alpha}dx=ydx$, unde cum A, B, C, D sit series ascendens, fient omnes $\mathrm{B}-\mathrm{A},\ \mathrm{C}-\mathrm{B},\ \mathrm{D}-\mathrm{C}$ &c. id est omnia $ydx,=\mathrm{M}-\mathrm{A}$, At vero substituendo in aequatione $A=1:\overline{\alpha-1}.x^{\alpha-1}$ vel $=(1:x)^{\alpha-1}:\alpha-1,$ loco *x*, 0 fiet $\mathrm{M}=(1:0)^{\alpha-1}:\alpha-1,$ at vero $1:0$ est magnitudo infinita, quam designabimus per ∞, ergo $\mathrm{M}=\infty^{\alpha-1}:\alpha-1,$ atque adeo $\mathrm{M}-\mathrm{A}$ seu area hyperbolarum abscissae adjacens, erit $=\infty^{\alpha-1}:\alpha-1,-1:\overline{\alpha-1}.x^{\alpha-1}$, vel quia $1:x^{\alpha-1}=xy$, , erit $\mathrm{M}-\mathrm{A}=\left(\infty^{\alpha-1}-xy\right):\alpha-1.$

Ex quo apparet ejusmodi areas juxta diversos gradus infinite magnas esse, siquidem α sit quilibet numerus positivus & unitate major. Sed de his sufficit.

PROPOSITIO XII. THEOREMA.

93. Fig.29. *Tenacitas fili ZBABX ad figuram manentem redacti a potentiis BH,* βH *&c. ipsi applicatis, aequatur in quolibet fili elmemento Bb, tenacitati fili in vertice A auctae omnibus GH, quae ex singulis potentiis BH,* βH *&c. curvae AB applicatis derivantur, resolutione illarum BH in aequipollentes potentias laterales BG & GH singulis curvae AB elementis Bb perpendiculares & parallelas. Hoc est exponendo tenacitatem si fili in A; erit* $\mathrm{T} = \mathrm{A} + \int \mathrm{GH}$.

Et demissa ex elementi curvae Bb termino b perpendicula bg ad tangentem Bg curvae in B, sumatur in hac tangente, Bb aequalis elemento curvae βH*, & agantur hm parallela ordinatae BC, bf vero aequidistans axi curvae AC, & per curvae punctum B, indefinita BME eidem AC etiam parallela, in quam ME cadat perpendicularis HF, ex termino H lineae BH potentiae curvae puncto B applicatae repraesentatricis : ac denique jungantur lineola bh, atque hisce factis habebitur in angulo contactus bBg figuras bfhg similis, & similiter posita cum figura majore BFHG, si elementum* Bβ *ut illius tenacitas T ad hujus tenacitatem t ; atque adeo erit quaelibet lineola in figura bfhg ad elementum curvae Bb, sicut homologa linea in figura majore BFHG ad lineam T, quae tenacitatem ejusdem curvae elementi Bb exponit.*

Demonstr. I. Quoniam potentiae BH (§. 39.) laterales BG & GH aequipollent, & ex hisce potentia GH curvae puncto B in directione tangentis vel dementi Bβ applicata potentia conspirans est cum potentea, qua idem elementum Bβ tenditur, cui tensioni (§.86.) tenacitas elementi *t* aequalis dicta est, hae potentiae conspirantes scilicet GH & *t* aequales erunt directe contratriae vel oppositae potentiae T, qua elementum B*b* tensioni resistit, cum (secundum hypothesin) omnia in statu manenti seu aequilibrii existant, atque adeo $\mathrm{T} = \mathrm{t} + \mathrm{GH}$, vel $\mathrm{T} - \mathrm{t} = \mathrm{GH}$. Atqui (§. 87.) omnes differentiae $\mathrm{T} - \mathrm{t}$, inter singulorum curvae contiguorum elementorum tenacitates ; aequantur excessui maximae T supra minimam, quae est tenacitas in vertice, A, ergo

$$\mathrm{T} - \mathrm{A} = \int \mathrm{GH}, \text{vel } \mathrm{T} = \mathrm{A} + \int \mathrm{GH}.$$

II. Quia (§. 39.) $p\mathrm{BM} : p\mathrm{Bb}$ quae est $\mathrm{T}, = \mathrm{BM} : \mathrm{Bb}$, per hypothesin vero $\mathrm{T:t} = \mathrm{Bb} : \mathrm{B}\beta$, & $p\mathrm{B}\beta$ id est t ad $p\mathrm{Bk} = \mathrm{B}\beta : \mathrm{Bk}$, erit ex aequo $p\mathrm{BM} : p\mathrm{Bk} = \mathrm{BM} : \mathrm{Bk}$ vel B*m* (quoniam in triangulis similibus βBN & Bhm hypothenusae Bh, Bβ, aequales sunt; atque adeo ipsa triangula aequalia), & convertendo $p\mathrm{BM} - p\mathit{Bk} : p\mathrm{BM} = \mathrm{M}m$ vel $bf : \mathrm{BM}$. Atqui, quoniam omnia in aequilibrio sunt, duae potentiae conspirantes *p*Bk & BF, quae (§. 40.) ex potentia BH derivantur, aequales erunt directe oppositae *P*bm, unde $p\mathrm{BM} - p\mathrm{Bk} = \mathrm{BF}$, atque adeo, ponendo hunc valorem in anticedenti analogia, habebimus $\mathrm{BF} : p\mathrm{BM} = bf : \mathrm{BM}$, sed $p\mathrm{BM} : p\mathrm{Bb}$, id est $\mathrm{T} = \mathrm{BM} : \mathrm{Bb}$ ergo ex aequo $\mathrm{BF} : \mathrm{T} = \mathrm{b}f : \mathrm{B}b$. Simili prorsus argumento probabitur, quod $p\mathrm{BN} - p\mathrm{Bl} = \mathrm{FH}$, & $\left(p\mathrm{BN} - p\mathrm{B1}\right)$ vel $\mathrm{FH} : p\mathrm{B1} = fh : \mathrm{Mb}$; unde quia etiam (§. 39) *p*B1: *p*Bb vel $\mathrm{T} = \mathrm{Mb} : \mathrm{Bb}$; erit pariter ex aequo $\mathrm{FH} : \mathrm{T} = fh : \mathrm{Bb}$, vel invertendo $\mathrm{T} : \mathrm{FH} = \mathrm{Bb} : fh$, & quia paullo ante habuimus $\mathrm{BF} : \mathrm{T} = \mathrm{bf} : \mathrm{Bb}$, erit denique ex aequo $\mathrm{BF} : \mathrm{FH} = \mathrm{bf} : \mathrm{fh}$, atque adeo triangula BFH, *bfh* sunt similia & propter parallelas *bf* ac BF similiter posita. Praeterea, quia ipsae BG, & *bg* utpote eidem tangenti B*g* perpendiculares inter se parallelae sunt, aeque ac lineae BH & bh, liquet etiam triangula BGH & bgh similia esse, ac propterea figurae BGHF & bghf ex triangulis similibus compositae, similes erunt & similiter positae. Porro cum invenerimus supra $\mathrm{FH} : \mathrm{T} = fh : \mathrm{Bb}$, erit permutando $\mathrm{FH} : \mathrm{fh} = \mathrm{T} : \mathrm{Bb}$, atque adeo duae quaelibet aliae lineae homologae in figuris similibus BFHG & bfhg erunt in hac eadem ratione T ad Bb, vel permutando, quaelibet lineola in figura minore erit ad elementum curvae Bb, sicut homologa linea in figura majore ad tenacitatem T ejusdem curvae elementi. Quae erant demonstranda.

COROLLARIUM I.

94. Ducta, si placet, per verticem A recta AO arbitraria magnitudinis, axi AC perpendiculari, ac super ea descripto semicirculo APO, agantur AP parallela elemento Bb, & Ap aequidistans alteri curvae & elemento contiguo $\mathrm{B}\beta$, & jungantur cum Po rectam Ap secans in Q, tum etiam Po ; quibus peractis, erit $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP}$. Nam (§.93.) est $\mathrm{BG} : \mathrm{T} = \mathrm{bg} : \mathrm{Bb}$, atque ob parallelas AP, B*b* & A*p*, $\mathrm{B}\beta$, sectores seu triangula B*bg* & APQ similia sunt, atque adeo $bg : \mathrm{B}b = \mathrm{PQ} : \mathrm{AP}$; ergo etiam $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP}$.

COROLLARIUM II.

95. Quaelibet recta in figura BGHF est ad homologam in figurae bghf, ut alterutra ex potentiis pBI vel pBM, quae elementi Bb tenacitati T aequipollent, ad homologum, id est respondens latus in triangulo characteristico BMb: hoc est, quaelibet linea in figura majori est ad homologam in minori sicut *p*BI ad BI, vel sicut *p*BM ad BM. Nam quia in utraque figura duae quaelibet lineae homologae sunt inter se, sicut T ad Bb, sicut *p*BI ad BI vel Mb, aut sicut *p*BM ad BM, liquet assertio hujus corollarii.

COROLLARIUM III.

96. Si omnes applicatae potentiae BH curvae perpendiculares sint, ut BD; puncta G & H confundentur cum puncto D, evanescentibus GH, *gh* in utraque figura majore & minore ; atque adeo tenacitas curvae ubique aequalis et tenacitas curvae in vertice A, atque adeo constans seu data. Id est habebitur ubique $\mathrm{T} = \mathrm{A}$.

COROLLARIUM IV.

97. Idcirco, cum (§. 94.) sit $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP} = \text{sin. anguli PAQ}$ seu anguli gBb, id est sinus curvitatis in B ad radium ; erit in casu potentiarum curvae perpendicularium una earum BD cuilibet curvae puncto B applicata ad tenacitatem fili (§.96.) constantem A,

sicut sinus curvitatis in curvae puncto, cui potentia applicata est ad radium; & permutando erit potentia applicata BD ad sinum curvititis in B sicut A ad radium, id est , in ratione data.

Elegans haec proprietas ab Acutissimo Geometra Joh. Bernoulli primum animadversa est; sed eam, quantum ex Commentariis Acad. Reg. Scient. Paris. 1706. ad diem 12. Maji apparet, ex alio fundamento deduxit.

COROLLARIUM V.

98. Iisdem adhuc positis, quae in duobus proxime antecedentibus Corollariis, quia (§.93.) $\mathrm{HF:hf = BF:bf = T : Bb}$ (vel§. 96.) $= \mathrm{A : Bb}$, erit $\int \mathrm{FH} : \int \mathrm{fh} = \int \mathrm{BF} : \int \mathrm{bf} = \mathrm{A : Bb}$, si singula curvae elementa aequalia assumta fuerint. Atqui (§. 87.) $\int \mathrm{hf}$, seu omnes hf vel Nn, posita $\mathrm{Bn = bM}$, respectu curvae portionis $\mathrm{A\beta B}$ aequantur excessui maximae ordinatarum differentiae ac (posito arculo curvae $\mathrm{Aa = Bb = B\beta = Zz}$) supra minimam bM, id est, $\mathrm{ae - bM}$ (vel etiam, quia aec ac aA aequales sunt cum axis CA curvae perpendicularis est in A)
$\mathrm{Aa - bM = Bb - bM}$; & $\int \mathrm{bf} = \int \mathrm{Mm = BM - Ae}$ (vel quia Ae hoc casu evanescit prae Aa vel Bb) $= \mathrm{BM}$. Ergo $\int \mathrm{FH : Bb - bM} = \int \mathrm{BF : BM = A : BC}$. Ubi omnes FH seu $\int \mathrm{FH}$, & $\int \mathrm{BF}$ pertinent ad areum curvae $\mathrm{A\beta B}$.

Sin vero $\int \mathrm{FH}$ & $\int \mathrm{BF}$ pertinuerint ad arcum Curvae BZ & tangens hujus arcus in Z axi AC parallela fuerit, erit $\mathrm{Zy} = 0$, posita ut supra $\mathrm{Zz = Bb = B\beta}$ &c. atque adeo $\mathrm{zy = Zz = B\beta}$; unde $\int \mathrm{hf}$ vel $\int \mathrm{Nn}$ erit hoc casu (§. 87.) $\mathrm{BN - Zy = BN}$, &
$\int \mathrm{bf} - \int \mathrm{Mm = yz - N\beta = Z\gamma - N\beta = B\beta - N\beta}$, atque adeo
$\int \mathrm{FH : BN} = \int \mathrm{BF : B\beta - N\beta = A : B\beta}$.

COROLLARIUM VI.

99. Si potentiae curvae applicatae BH, βH &c. axi AC aequidistantes sunt, coincident BH, BE, & *bh* ac *bf* evanescentibus HF, *hf* atque adeo omnia ordinarum elementa bM, BN, &c. hoc casu aequalia sumenda sunt; ac proinde omnes potentiae secundum directiones ordinatarum ex tenacitate elementorum curvae derivatae in ista hypothesi aequales existent. Propterea, vocando unamquamque harum aequalium potentiarum B, fiet nunc (§.95.) $\mathrm{BE : bf}$ vel $\mathrm{Mm = B : bM}$, adeoque cum B & bM constantes sint hoc casu, erit $\int \mathrm{BE} : \int \mathrm{Mm = B : bM}$, vel quia (§.98.) $\int \mathrm{Mm = BM - Ae = BM}$ evanescente

Ae prae Aa; fiet $\int \mathrm{BE} : \mathrm{BM} = \mathrm{B} : \mathrm{bM}$ & permutando $\int \mathrm{BE} : \mathrm{B} = \mathrm{BM} : \mathrm{bM}$. Porro, quia ordinata *ae* vertici A proxima in ipsam curvam Aa definit, & B generaliter constantem potentiam juxta ordinatarum directiones denotans, etiam eam, quae juxta directionem *ea* vel A*a* agit, exponet, sed potentia juxta A*a* est tenacitas curvae in vertice A, ergo $\mathrm{A} = \mathrm{B}$, adque adeo $\int \mathrm{BE} : \mathrm{B} = \int \mathrm{BE} : \mathrm{A} = \mathrm{BM} : b\mathrm{M}$.

SCHOLION.

100. Ex Corollariis theorematі nostro adjectis satis constare potest quam late pateat ejus usus, revera enim infinitorum id problematum solutionem continet, quorum Problema Catenariae, Velariae, & figurae lintei ab incumbente liquore inflexi nonnisi casus sunt specialissimi nostri theorematis. Sed priusquam ad applicationem ejus nonnullis specialibus, ejusmodi casibus accedam, monendum est atque notandum, potentias applicatas BH, βH, etsi simplicibus & finitae magnitudinis lineis expressas, nonnunquam lineolas tantum infinite parvas, quandoque etiam rectangula infinitesima significare ; &. tenacitates fili in singulis punctis ejusdem semper generis magnitudines existere quidem cum potentiis applicatis, sed prae hisce infinitas; ut id ex ipsa applicatione clarius elucescet.

201. Sint ergo

$$\mathrm{AC} = x, \mathrm{CB} = y, \mathrm{BM} = dx, \mathrm{bM} = dy, \mathrm{Bb} = ds, \mathrm{AO} = a,$$
$$\mathrm{AP} = m \ \& \ \mathrm{PO} = n = \sqrt{(aa - mm)}.$$

Hinc $\mathrm{PQ} = dn$ & $\mathrm{Qp} = dm$. Hisce positis, triangula similia BbM, & OAP praebent $ds = adx : n$, & $dy = mdx : n$. Idcirco substitutis his valoribus in analogia $\mathrm{BG} : \mathrm{T} = \mathrm{PQ} : \mathrm{AP}$, §.94. reperta, erit $\mathrm{BG} : \mathrm{T} = dn : m$, primus Canon. Et $\mathrm{T} = \mathrm{A} + \int \mathrm{GH}$, altera formula generalis.

102. Sint potentiae applicatae BH curvaeque perpendiculares $= bds$, ubi b est magnitudo data, eritque hoc casu (§. 96.) $\mathrm{T} = \mathrm{A}$, & A (§100.) ejusdem generis magnitudo cum potentia applicata, atque adeo dicatur $= ab$. Hinc ex BG:T fiet *bds*:*ab*, vel *ds: a* atque adeo $ds : a = dn : m = \mathrm{BG} : \mathrm{T}$, vel quia (§. 101.) $ds = adx : n$, erit $(adx : n) : a = dn : m$, atque adeo $dx : n = dn : m$ ergo $mdx = ndx = -mdm$, & $dx = -dm$, & integrando $x = a - m$ vel $m = a - x$, ergo $n = \sqrt{(aa - mm)} = \sqrt{(2ax - xx)}$, hinc $dy (= mdx : n) = adx - xdx : \sqrt{(2ax - xx)}$, ergo $y = \sqrt{(2ax - xx)}$ quae est aequatio ad circulum.

103. Si potentia curvae perpendicularis BD vel BG aut BH (hae tres in hoc casu unum indemque significant) sint $= dy^2 : ds$, qui est casus Velariae, ut suo loco plenius

ostendetur, erit, factis substitutionibus, $\mathrm{BD} = mmdx : an$, atque posita $\mathrm{A}(=\mathrm{T}) = a$, analogia (§.101.) $BG : T = dn : m$, nunc fiet $mmdx : aan = dn : m$, hinc $aandn = m^3dx$ & $m^3dx = aamdm$ vel $= -aadm : mm$, hinc $x = aa : m, \ -a$ atque $m = aa : a + x$ & $n = \sqrt{(aa - mm)} = a\sqrt{(2ax + xx)} : a + x$, ergo $dy(= mdx : n) = adx : \sqrt{(2ax + xx)}$. Pro aequatione *Velariae*.

104. Si potentia applicata curvae itidem perpendicularis BD sit *kds*, ubi *k* denotat quantitatem utlibet datam in *x* & constantibus, ponatur juxta superiorem (§. 100.) notam $\mathrm{T} = \mathrm{A} = \frac{1}{2}aa$, factisque; ex (§.101.) convenientibus substitutionibus analogia $\mathrm{BG} : \mathrm{T} = \mathrm{d}n : m$, mutabitur in $2kdx : an = dn : m$, hinc $2kdx = andn : m = -adm$, vel $2kdu = 2kdx = adm$. Ponatur $x + u =$ constanti b, ergo $du = -dx$, & $2kdu = -2kdx = adm$. Unde si fiat $2kdu = adp$, erit $dp = dm$, atque adeo $p = m$ & $dy = pdx : \sqrt{(aa - pp)}$. Quae ex allis principiis reperta est a Celeberrimis Bernoulliis. Si $k = u$, fiet $dy = -uudu : \sqrt{(a^4 - n^4)}$ pro aequatione figurae lintei vel Elasticae Acutissimi Jac. Bernoullii.

105. Si applicatae potentiae BH sunt axi AC parallelae, ut BE; ducatur EI perpendicularis ad BD, unde si $\mathrm{BE} = dq$ propter triangulorum BIE, APO similitudinem invenietur latus $\mathrm{EI} = ndq : a$, & $\mathrm{BI} = mdq : a$. Idcirco loco BG nunc est sumenda BI, in analogia $\mathrm{BG{:}T} = dn : m$, fietque $(mdq : a) : t = dn : m,$ vel $atdn = mmdq$, sed $a + \int \mathrm{IE} = a + \int ndq : a = \mathrm{T} = \mathrm{t}$, & differentiando $ndq : a = dt$, vel $dq = adt : n$, quod in aequatione $atdn = mmdq$ suffectum, praebet $atndn = ammdt$ vel $dt : t = ndn : mm = -mdm : mm = -dm : m$; hinc $t = aa : m$, quod in superiori analogia substitutum, dat $(mdq : a) : (aa : m) = dn : m$, ex qua elicitur $dq = a^3dn : m^3$ (vel propter $aa = mm + nn$) $= anmdn + anndn : m^3 = ammdn - amndm : m^3 = amdn - andm : mm$; unde, facta summatione, invenietur $q = an : m$, id est $mq = an$, vel substitutis loco *m* & *n* ipsarum proportionibus *dy*, *dx*; reperietur $dy = adx : q$. Quae aequatio omnis generis Caterarias contenit.

Etsi vero modo inventa aequatio jam continetur in Corollario sexto, quod dat analogiam $\int \mathrm{BF} : \mathrm{A} = \mathrm{BM} : \mathrm{bM}$, in qua si $\int \mathrm{BF} = \int dq = q, \ \mathrm{A} = a, \ \mathrm{BM} = dx,$ & $\mathrm{bM} = dy$, habebitur $q : a = dx : dy$, atque adeo $dy = adx : q$. Non tamen abs re mihi visum est, si usum analogiae $\mathrm{BG} : \mathrm{T} = dn : m$ etiam hoc casu illustrarem.

106. Sit $q = s$, per *s* intelligendo curvam $\mathrm{A}\beta\mathrm{B}$, erit $dy = adx : s$, qui casus est simplicissimus problematis Catenaria. Foret enim $ms = an$ & $s = an : m,$ ac $ds = \frac{amdn - andm}{mm}$, vel quia $ds = adx : u$, erit

$dx = mndn - anndm : mm = -mmdm - nndm : mm = -aadm : mm,$ ergo ut supra (§.103.) invenietur $dy = adx : \sqrt{(2ax + xx)}$, aequato *Catenariae*, quae eadem prorsus est cum *Velaria*. Sin vero gravium directiones in puncto ad distantiam finitam ponantur convergere, *Catenariae* erit *hyperbola aequilatera*, cujus demonstratio brevitatis gratia aliis relinquenda est, quanquam caeteroquin non longa sit, sed ferme una linea & absque calculo potest perfici.

PROPOSITIO XIII. PROBLEMA.

107. *Inventa curva manente, in quam velum, linteum, aut quodvis aliud corpus flexile a potentiis applicatis arcauatur, assignare mediam directionem potentiarum & impressionem earum, juxta hanc mediam directionem.*

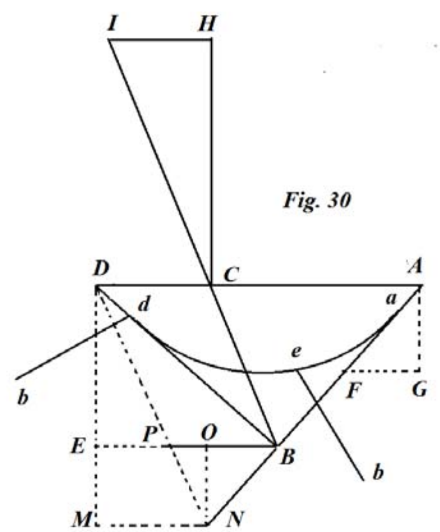


Fig. 30. Sit AeD inventa curva terminis suis tigillo seu lineae inflexili ACD alligata, quaeritur media directio omnium potentiarum *eb*, *db* &c. curvae applicatum, & quantitas impulsus, seu impressionis, qui secundum mediam. directionem CB in tigillum DCA exere debet.

Solutio. Per terminos curvae A, D ductae sint tangentes curvae AB, DB convenientes in puncto B, in quarum alterutra, ut AB, capiatur segmentum AF, quod sit ad reliquam tangentem totam DB, ut tenacitas curvae in A ad tenacitatem ejus in D, quae tenacitates ex superius (§. 93) demonstratis habentur. Per puncta A & D agantur perpendiculares AG, DM, rectae DA, &. per F ac B eidem AD aequidistantes FG, BE perpendicularibus occurrentes in G & E. Dividatur AD in C, ut sit $AC : DE = DE : AG$, & in perpendiculari lineae AD per punctum C sursum ducta sumatur portio $CH = AG + DE$, & in parallela eidem AD per punctum H ducta sumatur $HI = BE - FG$, & jungatur denique IC, ejusque continuatio CB erit media directio quaesita, impulsus vero ex omnibus curvae applicatis potentiis resultans, secundum hanc mediam directionem, erit ad tenacitacem fili in A vel D, sicut IC est ad AF vel ad DB.

Demonstr. Linea AD non aliam a potentiis *eb*, *db* &c;. curvae A*e*D applicatis impressionem subire potest, quam quae resultat ex potentiis, quibus in directionibus tangentium curvae AB, DB urgetur, seu ex tenacitatibus curvae in A & D, quandoquidem in his tantum punctis curva AeD lineae inflexili ACD alligata est. Verum resolvendo potentias AF & DB in suas aequipollentes laterales AG & GF, atque DE & EB, quia ex conspirantibus AG & DE (§.54.) nascitur potentia ipsis aequalis AG & DE in directione per C transeunte scilicet per ipsarum AG & DE centrum gravitatis, & rectae AD perpendiculari, & quia potentia $CH = AG + DE$ per constructionem contario sensu agit, haec CH in aequilibrio consistet cum potentiis AG & DE. Potentia vero, quae contariis EB, & GF resultat (§. 38.) aequabitur excessui majoris EB supra minorem GF, atqui etiam (constr.) $Hl = EB - GF$, ergo ex contrariis & lineae AD parallelis potentiis resultat potentia HI, quae in recta AD agens ex C versus D in aequilibrio foret cum contrariis EB & GF, ergo duae potentia CH & HI in aequilibrio consisterent cum potentiis AG, DF, GF & EB, quibus obliquae AF & DB aequipollent atque adeo cum hisce obilquis, vel quia (§.39.) ex lateralibus CH & HI unica potentia CI nascitur, haec etiam in aequilibrio maneret cum potentiis obliguis AF & DB. Hinc (§.37) rectae IC continuatio CB dar mediam directionem obliquarum AF, DB, vel, quod idem est , omnium potentiarum *eb*, *db* curvae A*e*D applicatarum; ipsa vero IC exponit impressionem secundum mediam directionem CB ex omnibus applicatis potentiis provenientem. Quod erat demonstradum.

COROLLARIUM I.

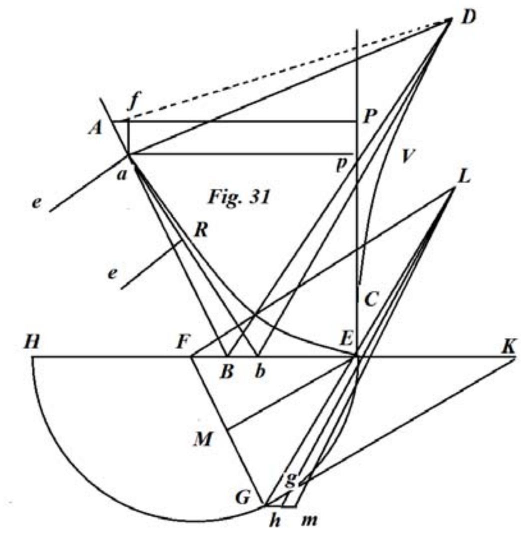
108. Recta IC producta transit per concursum B tangentium AB, & DB. Protrahatur enim AB in N ut fiat $BN = AF$, ostendetur DN aequalis ; & parallela ipsi IC. Nam si per punctum N agantur NM, NO rectis DA vel BE & DE aequidistantes, propter parallelas NO, AG & OB, FG ac (constr.) $BN = AF$, erunt triangula AFG, NBO similia & aequalia, adeoque $NO = ME = AG$, totaque $DM = DE + AG$ (constr.) $= HC$, sic etiam $MN = EO = EB - FG = EB - FG$ (constr.) $= IH$; idcirco etiam $DN = CI$, atque propter parallela DM & CH atque angulos aequales MDN & ICH ipsae DN & IC parallelae erunt. Porro quia (constr.) $AC : DC = DE : AG$ vel EM (& propter parallelas EB ac MN) AB.BN, atque adeo $AC : DE = AB : BN$, lineae DN, & CB non tantum sunt aequidistantes, sed etiam DN & IC parallelae ostensae sunt; ergo IC & CB in directum positae sunt, ac proinde ipsa IC producta per concursum B tangentium AB & DB transibit.

COROLLARIUM II.

109. Hinc AF est ad DB, sicut sinus anguli ABC ad sinum anguli DBC. Nam in triangulo DBN, sinus angulorum NDB, DNB lateribus oppositis NB, DB sunt proportionales, & anguli illi propter parallelas DN & CB angulis DBC, ABC aequantur. Propterea praecedens, & hoc alterum corollarium, hanc facillimam problematis suppeditant constructionem. Producta tangente AB in N, ut eius continuatio BN sit ad alterum tangentem totam DB, sicut tenacitas curvae in A, quae exponitur per rectam AF, ad tenacitatem ejusdem in D, quam exponit DB, recta BC per punctum concursus B tangentium AB, DB transiens & parallela lineae DN, jungenti puncta D & N, ent media directio quaesita.

COROLLARIUM III.

110. Si tenacitates fili in A & D aequales sint, media directio BC angulum tangentium ABD bifariam dividit. Nam quia AF aequalis DB hoc casu, & AF ad DB, ut sinus anguli DBC ad sinum anguli ABC per praecedens Corollarium, erit omnino angulus ABC aequalis angulo DBC.



SCHOLION.

Fig. 31. 111. Corollorium tertium nos manuducit ad novum illud curvarum genus, illud apud Cel. Jac. Bernoulli in Act. Lipf. 1695 pag. 548. *Linearum mediarum directionum* nomine venit. Nam si filum aliudve corpus flexile a potentiis *ae*, R*e* &c in singulis punctis ipsi perpendiculariter applicatis in lineam ERA arcuatur, recta BD angulum ABE a tangentibus curvae in utroque sui termino formatum bifariam dividens (§.110.) est media directio omnium potentiarum *ae* &c. toti curvae ERA applicatarum, & ducta tangente ab per punctum aliud curvae *a*, recta *b*D, angulum abE bifariam dividens, est media directio potentiarum arcui ERa applicatarum. Jam omnes mediae directiones BD, bD &c. contingent aliquam curvam CVD, quae ideo *Linea mediarum directionum* vocatur.

PROPOSITIO XIV. THEOREMA.

112. *Recta* AD *curvae* ERA *perpendicularis, mediae directioni* BD *potentiarum curvae perpendiculariter applicatarum, occurret in puncto* D *lineae mediarum directionum* CVD.

Sit Aa curvae elementum, & *ab* tangens curvae in *a*, & bD linea bisecans angulum, *ab*E, erit media directio potentiarum arcui curvae ER*a* applicatarum: unde cum *b* ipsi B infinite vicinum sit, sequitur unam *b*D alteram BD secare in puncto D curvae quaesitae. Igitur probandum tantum restat, perpendicularem curvae AD transire per punctum illud intersectionis D, duarum infinite vicinarum BD, *b*D. Centro quolibet F in tangente BE assumto, & radio FE descripto semicirculo EGH, agantur radii FG tangenti AB, &. Fg tangenti *ab* paralleli, junctisque EG, E*g*, agatur G*m* radio FE parallela, rectae E*g* productae occurrens in *h*, & radio Fg prolongato in *m* : denique ductae sint semicirculi tangens GK in puncto G, & EM eidem tangenti parallela, quae proinde radio FG perpendicularis erit. Jam quia FG tangenti AB parallela est, erit angulus $\mathrm{ABH} = \mathrm{EFG}$, & duo FEG & FGE simul aequales angulo ABE, seu duplo anguli DBE, quandoquidem BD angulum ABE bifariam dividit, igitur $\mathrm{FEG} = \mathrm{DBK}$, unde rectae BD & GE sunt parallelae. Pari argumento probatur, parallelas esse *b*D & *g*E. Igitur ducta *m*L parallela ipsi *h*E, vel *g*E, figura AB*b*D & FG*m*L erunt similes, cum compositae sint ex triangulis similibus AB*b*, BD*b*, & FG*m*, GL*m*. Hisce positis, parallelae FK & G*m* a media E*h* proportionaliter dividentur, atque adeo $\mathrm{G}m : \mathrm{G}h = \mathrm{FK} : \mathrm{EK}$ (vel propter parallelas ME & GK) $= \mathrm{FG} : \mathrm{MG}$. Triangula vero similia GE*h*, & GL*m* (nam per constr. E*h* & L*m* parallelae sunt) praebent, $\mathrm{G}m : \mathrm{G}h = \mathrm{GL} : \mathrm{GE}$. lgitur $\mathrm{FG} : \mathrm{MG} = \mathrm{GL} : \mathrm{GE}$; atque adeo FL jungens puncta F, L parallela est rectae ME, ac proinde LFG est rectus, cum EM perpendicularis sit ipsi FG. Unde propter similitudinem figurarum AB*b*D, & FG*m*L, erit AB:BD= FG:G L, hinc, quia angulus LFG rectus est, angulus DAB itidem rectus erit, atque adeo AD curvae ERA perpendicularis in A. Quod erat demonstrandum.

COROLLARIUM.

113. Adeoque etiam triangula ADB & MEG similia sunt, ex quo facile elicitur valor lineae AD. Nam si $\mathrm{FE} = a, \mathrm{AP} = y, \mathrm{Af} = dy, \mathrm{EP} = x, \mathrm{Pp} = dx$, & $\mathrm{Aa} = ds$: erit AB $= xds:dx$; & quia angulus $\mathrm{MFE} = a\mathrm{Af}$, atque adeo triangula rectangula FME, & Aaf similia sunt, invenietur $\mathrm{FM} = ady : ds$; & $\mathrm{MG} = (ads - ndy) : ds$, tum etiam $\mathrm{ME} = adx : ds$; hinc quea $\mathrm{MG} : \mathrm{ME} = \mathrm{AB} : \mathrm{AD}$, invenietur $\mathrm{AD} = xds : ds - dy$.

Cl. Jac. Bernoullius (Act. Lipf. loco supra citato) calculo analytico usus, quem tamen illic non apposuit, reperit $\mathrm{AD} = xds^2 + xdyds : dx^2$, ex qua formula nostra paulo simplicior & nonnihil diversa ratione a Bernoulliana ex praecedentibus elicita, nullo negotio derivatur, substituendo tantum loco dx^2 aequalem quantitatem $ds^2 - dy^2$, & numeratorem ac dominatorem dividendo per $ds + dy$.