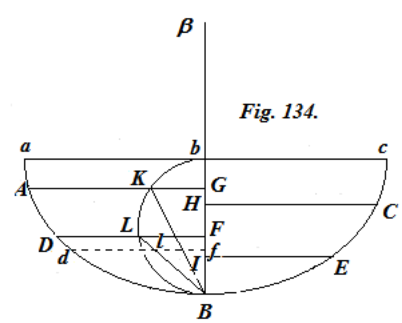
CHAPTER XIX.

Concerning the descent and ascent of weights along the lines of some curves, with the resistance of the medium to be proportional to the square of the velocity.

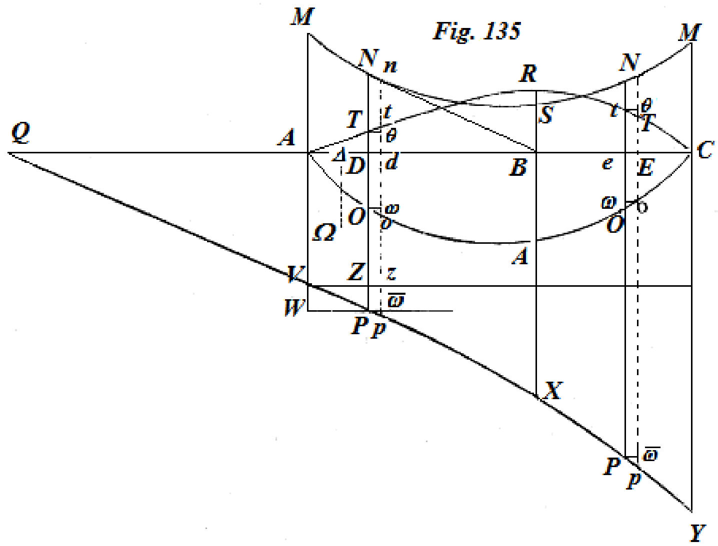
581. Weights, which are carried and fall along some curved lines *in vacuo*, may be agreed at some point of the curve, to have acquired that speed by which they slide past with an accelerated motion, to be the same as that which they would have acquired, as if they may fall vertically along a line to the given point of the curve, drawn from the shortest distance along a horizontal line through the starting point on the curve, that is, as shown generally in the First Book §.142. But the matter is found to be otherwise with bodies, which are moving through air; indeed the speeds of these cannot be shown otherwise than by the quadratures and rectifications of the curves shown, and that only in the case, by which the resistance of the air shall be proportional to the square of the speed of the body moving, as indeed that may be elucidated from the following.

PROPOSITION LXXII.

582. *If, with the whole curve extended into a straight line, which will be described by the weight in its descent and subsequent ascent, at the individual points of this right line perpendiculars may be erected proportional to the resistance of the air, which the moving body endures at the homologous points of the curve. The curvilinear area, which the perpendiculars of this kind contain, will be equal to the rectangle, drawn by the right line which expresses uniform gravity, by the difference of the abscissas of the arcs described of the descent and ascent.*



Let ADB be the curve described by the descent of the moving body and BEC described by the ascent, [Fig. 134], the abscissa of this curve BEC is BH, truly of the other curve BG, and thus GH will be the difference of the abscissas. Thence with the whole curve ABC extended out along the right line ABC, [Fig. 135 ; recall that the cycloid is a rectifiable curve], and at the individual D points of this line, the perpendiculars DT shall be erected expressing the resistance of the medium at the homologous points of the curve D, and forming the curvilinear figure ATRC, which I will show to be equal to the rectangle βB by GH, by assuming the right line βB to express the uniform weight of the body.



On the same DT supposed to be produced, it may be assumed everywhere that DN expresses the tangential action of the weight on the curve at the point D derived from the centre, and $\mathrm{ND}-\mathrm{TD}$ expresses the excess of the tangential gravitational force over the air resistance, which is the acceleration acting of the body falling along the curve ADB. Now the moment of this acceleration acting, that is, $(\mathrm{ND}-\mathrm{TD.})Dd$, or $\mathrm{ND}.\mathrm{D}d-\mathrm{TD}.\mathrm{D}d$, that is, the element of the area AMND – element of the area ATD (§.§.131, 484.) is equal to the moment of the speed of the body acquired at the point D of the curve, or to the element of half the square from the said acquired speed; from which, since at A everything begins from zero, and all the moments of the accelerations acting will be equal to all the moments of the speed, the area AMND – area ATD = half the square of the velocity at D, and the area AMNSB – area ATRB = half the square of the velocity at B.

Again if the body rises along the curve BEC, the whole resistance, at any point E of this (§. 481.) will be $\mathrm{NE}+\mathrm{TE}$, and the moment of this whole resistance or $(\mathrm{NE}+\mathrm{TE}).e\mathrm{E}=\mathrm{NE}.e\mathrm{E}+\mathrm{TE}.e\mathrm{E}=$ elem. of the area MNEC + elem. of the area CTE is equal to the moment of the decreasing speed of the body remaining at the point E, or to the element of half the square from the speed at E; from which, because with NE taken away at MC, everything stops at zero, and the area MNEC + area CTE = half the square from the velocity at E, and thus also MNSBC + CTRB = half the square from the initial velocity at B. And this initial velocity at B, is the same as the velocity acquired at B, after the descent of the body on the curve ADB; therefore AMNSB – ATRBA (= half the square of the speed at B) = CMNSBC + CTRBC, and therefore the whole figure of the resistances ATRCA = AMNSB – CMNSB.

And the area AMNSB is equal always to the rectangle under the right line βB, which expresses uniform gravity, and abscissa BG of the described curve ADB from the descent of the body, taken with some element of the area ND*d* is equal to the rectangle βB.F*f*, or from the moment of the central of gravity βB. By the same argument it follows the area CMNSBC becomes equal to the area βB. BH , therefore AMNSB − CMNSB , which I have shown to be equal to the ATRCA , also will be equal to the rectangle βB.GH. Q.E.D.

COROLLARY I.

583. If the air resistance shall be as the speed acquired by the body, there will be $\text{AMND} = \text{area ATD} + \frac{1}{2}\text{TD}^2$. For TD (following the hypothesis) is as the speed acquired at D, and thus by the preceding paragraph, the moment of the acceleration acting, or $\text{ND.D}d + \text{TD. D}d = \text{elem. AMND} + \text{elem. ATD}$, will be equal to the moment of the speed, or $\text{TD. }t\theta$, which is equal to the moment from $\frac{1}{2}\text{TD}^2$. There on summing, there will become the whole $\text{area AMND} - \text{area ATD} = \frac{1}{2}\text{TD}^2$, and thus :
$\text{area AMND} = \text{area ATD} + \frac{1}{2}\text{TD}^2.$ By the same argument there may be concluded to be $\text{NECM} = \frac{1}{2}\text{TD}^2 - \text{CTE}.$

COROLLARY II.

484. But truly if the air resistance were in the square ratio of the speeds acquired, which is the truer hypothesis, the area

$$\text{AMND} = \text{AQ.TD} + \text{ATD}, \ \ \& \text{ the area CMNE} = \text{AQ.TE} - \text{CTE}.$$

Where AQ is a given or constant magnitude.

For in this case the speed squared acquired at D will be $= 2\text{AQ.TD}$, since the resistance shall be as the square of the speed ; and half the square of the speed AQ.TD, and the element of this [*i.e.* its differential], that is the moment of the speed acquired at D, will be found $\text{AQ.}t\theta$; from which, because the moment of the acceleration acting $\text{ND} - \text{TD}$ is equal to the moment of the speed acquired at D , there becomes $\text{ND.D}d - \text{TD.D}d = \text{AQ.}t\theta$, and summing, all the NO.Dd, or the area AMND − sum of all the TD.Dd, or the area ATD = to the sum of all the AQ.$t\theta$, that is, to the rectangle AQ.TD; and thus $\text{AMND} = \text{ATD} + \text{AQ.TD}.$ By almost the same reasoning it can be shown that $\text{CMNE} = \text{AQ.TE} - \text{CTE}.$

COROLLARY III.

585. If the resistance is uniform, the curve ARC will be changed into a right line parallel to AC itself, and in this case $\text{AMNSB} - \text{CMNSB} = \beta\text{B .GH} = \text{AC.BR},$ as a consequence a uniform resistance of the medium BR will be had to the weight itself, as the difference GH of the abscissas BG and BH, of the arcs of the curves AB & BC of the descent and for the subsequent ascent, to AC the sum of all the arcs ABC.

COROLLARY IV.

586. If the curve ABC is a regular cycloid; of which the axis or diameter of the generating circle is *b*B and the base *ac*; the area ATRTC will be equal to the difference of half the squares described by the arcs of the cyloids ADB and BEC in the descent and ascent.

For, since βB expresses uniform gravity according to these matters, which have been mentioned in §. 179, this βB will be twice the diameter *b*B*;* hence $\beta B.\ BG = 2.bB.BG = 2.BK^2$ (or, because the arc of the cycloid ADB is equal to twice the subtangent BK) $= \frac{1}{2}.AB^2$, and $\beta B.BH = 2.bB.BH = \frac{1}{2}.BC^2$, therefore $\beta B.\ GH = \frac{1}{2}AB^2 - \frac{1}{2}BC^2 = ATRTC.$ that is, the rectangle from half the difference of the arcs ADB and BEC described from the descent and form the subsequent ascent in the cycloid, by the sum ABC of the same arcs, is equal to the area ATRTC.

This corollary agrees with proposition XXX. Book II. *Princ. Phil. Nat*. of the illustrious Newton, yet the demonstration of which has been derived from different principles, as will be apparent from any inspection.

[***Newton view of any connected dynamical events between bodies in an isolated system of masses depended on how the velocities of such bodies evolved in time according to his laws of motion, from which the spatial velocities could be deduced via the conservation of linear momentum; an alternative view expressed here by Hermann is in terms of such a system where the velocities evolving spatially, and from which the time evolution can be deduced. Thus, Newton could derive the spatial outcome from the time solution, while likewise Hermann could derive the time evolution from the evolution: the two approaches being complementary. This distinction was and still is, in my opinion, Hermann's great contribution to physics, largely not understood at the time and hence ignored, and even to this day; it was in fact the introduction of energy considerations into dynamical processes, hinted at by Leibniz with his vis viva idea, but not yet set out here properly as a calculating device in terms of work done and the related energy changes; hence the vast amount of time spent in setting up geometrical structures to produce the appropriate differential equation. The actual recognition of the work energy relation and the importance of energy changes had to wait until the turn of the next century, when these were introduced e.g. by Thomas Young; no recognition of Hermann's work was given at that stage***.]

COROLLARY IV.

587. Therefore in the case of this corollary III, the resistance will be in a constant ratio to the weight, that is, RB to βB in fig. 134., just as half the difference of the arcs ADB and BEC to βB, or half the cycloid *aDB,* or, just as the difference of the said arcs to the whole cycloid, *a*B*c*. For there is (§.585.) :

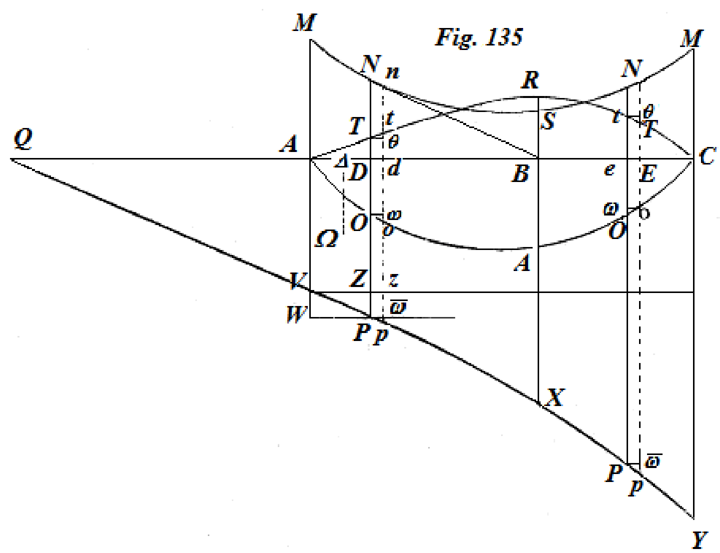
$$\mathrm{BR} : \beta\mathrm{B} = \mathrm{GH} : \mathrm{AC} = \beta\mathrm{B.GH} : \beta\mathrm{B.AC}(\S.586.) = \left(\tfrac{1}{2}\mathrm{AB} - \tfrac{1}{2}\mathrm{BC}\right).\ \mathrm{AC} : \beta\mathrm{B.AC}$$
$$= \tfrac{1}{2}\mathrm{AB} - \tfrac{1}{2}\mathrm{BC} : \beta\mathrm{B}\ \text{vel}\ a\mathrm{DB}, = \mathrm{AB} - \mathrm{BC} : \text{cycloid}.\ a\mathrm{B}c.$$

SCHOLIUM.

588. In the other hypotheses of the resistance, the matter requires to be investigated a little more thoroughly, than in the Corollaries III. and V. of this part ; since in all the other hypotheses the curve ATRTC shall still be required to be found. The illustrious Newton immediately asserts of this the proposal, in corollary IV after the proposition, this curve ARC approximates to an ellipse, if for the present cycloid ABC, the air resistance were proportional to the speed, or to a parabola, if the resistance were in the square proportion to the speeds. But, by considering the matter accurately and mathematically, in the first place the aforementioned ATRTC is not an ellipse in the first case, nor a parabola in the other hypothesis of the resistance; but, however it may be allowed to be classified, it will be transcendental in each case. Because, if the resistances are as the speeds acquired, it cannot be agreed by what reasoning the curve ARC must be constructed; nor indeed with a consensus of the quadratures of the curvilinear figures. Truly for the other hypothesis and of the nature of the nominated curve more agreement can be had for the quadrature and the rectification of the curves. From proposition XXIX of the most lauded Newton in Book II a construction of this kind can be reached in the particular case, in which the descent curve ABC is a cycloid. Indeed in this place I may add a general construction for any curve ABC, and that in two ways.

PROPOSITION LXXIII. THEOREM.

589. *Through any point* V *on the line* AM *produced, with the logarithmic curve* VXY *described having the subtangent equal to the given line* AQ, *and the asymptote* CQ, *and with* ND *extended as far as to meeting the logarithmic* VX *at* P *; if the sum of all the elements* PD.DN.D*d of any volume placed between the two planes* MAV & NDP *be*



erected, as it were at right angles to the plane MABS *above the plane* VAB, *the volume shall be equal to the parallelepiped* $\mathrm{AQ}^2.\mathrm{DO}$, *formed from the square of the subtangent of the logarithmic curve and the ordinate* DO *of a certain curve* AOAC; *the homologous ordinate* DT *of the graph of the resistance* ARC *will be the fourth proportional to the ordinates* PD, DO, *and* AQ *the subtangent of the logarithmic* VX.
[Note that the subtangent is introduced as a scaling factor for the logarithmic or exponential curve, applicable to both increasing and decreasing exponential, and indicating the time taken or the distance gone essentially in modern terms for a quantity to be reduced to e^{-1} of its original value for a decay process, and in a similar manner for exponential increases. The interested reader may wish to compare Hermann's early work with that of E.J.Routh, *A Treatise on the Dynamics of a Particle*, 1898, Sections 129 & 210.]

I. The line *pn* shall be indefinitely close to the other line PN, and through the points T, and P of the curves AT, AO and VP, the infinitesimal lines Tθ, Oω, & Pπ shall be drawn ; and finally, DT shall be the fourth proportional to PD, DO & AQ, or, what amounts to the same, the rectangle $\mathrm{PD.DT} = \mathrm{AQ.DO}$, $\& pd.dt = \mathrm{AQ}.do$, that is,

$$\mathrm{PD}.dt + p\pi.dt = \mathrm{PD.DT} + \mathrm{PD}.t\theta + dt.p\pi = \mathrm{PD.DT}$$
$$+ \mathrm{PD}.t\theta + \mathrm{DT}.p\pi = \mathrm{AQ.DO} + \mathrm{AQ}.\omega o;$$

therefore with PD.DT and AQ.DO taken away there will remain $\mathrm{PD}.t\theta + \mathrm{DT}.p\pi = \mathrm{AQ}.\omega o,$ and with the similar parts multiplied by AQ; there becomes

$$\mathrm{AQ.PD}.t\theta + \mathrm{DT.AQ}.p\pi = \mathrm{AQ}^2.o\omega.$$

II. And $\mathrm{AQ}^2.\omega o - \mathrm{PD.DN.D}d$, since the sum of all $\mathrm{PD.DN.D}d$ shall be (from the construction) $= \mathrm{AQ}^2.\mathrm{DO}$; likewise (§.491. no. II) $\mathrm{AQ}.p\pi$ is equal to PD.D*d,* or $\mathrm{AQ.DT}.p\pi = \mathrm{DT.PD.D}d$; therefore, with these values substituted for the values in the last equality, no.1 above, $\mathrm{PD.AQ}.t\theta + \mathrm{PD.DT}.Dd = \mathrm{PD.DN.D}d$ shall be found and, from the terms adjoining PD : $\mathrm{AQ}.\,t\theta + \mathrm{DT.D}d = \mathrm{DN.D}d$. And therefore the sum of all $\mathrm{AQ}.t\theta$, that is, $\mathrm{AQ.TD} +$ the sum of $\mathrm{DT.D}d$, that is, area ATD = sum of all $\mathrm{DN.D}d$, or equal to the area AMND. Hence there becomes $\mathrm{AQ.TD} + \mathrm{ATD} = \mathrm{AMND}$; and thus (§.584) the curve ATR is the graph of the resistance of the moving body falling along the curve ADB .

III. It may be proven in the same way that $\mathrm{PE}.t\theta - \mathrm{ET.P}\pi$ to be equal to $\mathrm{AQ}.o\omega$ and hence, as before, it may be shown that $\mathrm{AQ.TE} - \mathrm{CTE} = \mathrm{CMNE}$, and as a consequence (§. 584.) the curve CTR is the graph of the opposing air resistance acting on the moving body on the ascending curve BEC [in Fig. 134]. Therefore the curve ATRTC [in Fig. 135] is the graph of the air resistance, when the body falls along some curve ADB, and afterwards rises along the other curve BEC. Q.E.D.

COROLLARY I.

590. The speed of the body, acquired at some point D on the curve, is $\sqrt{(2.AQ.DT)}$, and the speed of the same body remaining at E by ascending along the arc of the curve BE will be $\sqrt{(2.AQ.ET)}$, the ratio of which agrees satisfactorily with the above (§. 484).

COROLLARY II.

591. The sum of all the parallelepipeds PD.DN. D*d* are contained in a certain volume MAVN, whose individual sections, at right angles to the plane BAV and parallel to the plane MAV, are the rectangles PDN. Indeed the plane AMSB is to be regarded as erected above the plane VAB.

COROLLARY III.

592. If the descent curve of the moving body ABC is the ordinary cycloid, the curved line MNS will be intersected by the right line through B, and the angle MBA, equal to a semi-right angle contained with the line AB. Therefore the aforementioned volume MAVN will be the logarithmic curve truncated above the base ADPV placed between the plane VAB and some other plane passing through BX, and with the inclination of a semi-right angle to the plane VAB.

For, because the subtended chord [Fig. 134] BL of the circular arc BL is parallel to the tangent of the cycloid at the point D the weight will be to its tangential action on the cycloid at the point D, just as *b*B to $\text{BL} = \beta\text{B} : 2.\text{BL} = \beta\text{B} : \text{arc of the cycloid BD}$, and βB expressed the weight, therefore BD expresses the tangential action of the weight of the body at D, and therefore in Fig.135 BD, which is the same as for the arc of the cycloid BD in Fig.134 on the extended right line ABC, will be equal to the ordinate ND, and $\text{AB} = \text{AM}$; hence the line MNS is right, or the figure AMNS will be a right-angled isosceles triangle, and thus the volume MAVNP is a truncated figure cut from the right prism on the base ADPV by the plane drawn through BX, and inclined at a semi-right angle to the plane base BAV.

COROLLARY VI.

593. From which, if Ω shall be the centre of gravity of the four-sided logarithmic figure ADPV, with the perpendicular ΩΔ dropped to AB, the volume from the figure mentioned ADPV multiplied by BΔ will be equal to the truncated volume MAVNP, and thus (constr.) to the parallelepiped; and with the line VZ drawn parallel to AC, the four-sided figure AVPD will be found to be $= \text{AQ.PZ}$, therefore the volume

$$\text{MAVNP} = \text{AQ. PZ.B}\Delta(\text{constr.}) = \text{AQ}^2.\text{DO}, \;\; \& \;\; \text{PZ.B}\Delta = \text{AQ.DO} \;\; (\text{no. I §.589}) = \text{PD.DT}.$$

Therefore the resistance DT at any point of the cycloid $= \text{PZ.B}\Delta : \text{PD}$.

SCHOLIUM.

594. Since in order to obtain the expression of the ordinates DT also the expression is required of the BΔ themselves, or of the distances of the centre of gravity of the quadrilinear figure ADPV, concerning which I will indicate in a few words, on what account they must be investigated.
On the logarithmic curve there is $\mathrm{DP.D}d = \mathrm{AQ.}p\pi$, and $\mathrm{AD.DP.D}d = \mathrm{AQ.AD.}p\pi$ (or with PW drawn parallel to VZ) $= \mathrm{AQ.PW.}p\pi$, therefore the sum of all $\mathrm{AD.DP.D}d =$ to the sum of all $\mathrm{AQ.PW.}p\pi$ that is $= AQ$ on the trilinear figure VPW. And (§. 44) the quadrilinear figure ADVP at $\mathrm{A\Delta} =$ sum of all the $\mathrm{AD.DP.D}d$; and subsequently $= \mathrm{AQ.VPW}$.

$$\text{Truly ADPV} = \text{AQ. PZ}, \ \ \& \ \text{VPW} = \text{AD.DP} - \text{AQ.PZ}; \ \text{therefore}$$
$$\text{AQ.PZ.A}\Delta = \text{AQ.AD.DP} - \text{AQ}^2\text{.PZ or } PZ.\text{A}\Delta = \text{AD.DP} - \text{AQ. PZ};$$
$$\text{hence PZ.B}\Delta = \text{AB.PZ} - \text{AD. DP} + \text{AQ.PZ} = \text{BO.PZ} - \text{AD. DP}$$
$$\text{(or by putting DP} - \text{AV in place of PZ)} = \text{BQ.DP} - \text{BQ.AV} - \text{AD.DP}.$$

Hence, as I may say finally in passing, the centre of gravity of the logarithmic curve beyond the ordinate DP directed towards Q departs towards infinity with the subtangent to be AQ from its first ordinate DP .

COROLLARY VI.

595. Since DT (§. 593.) shall be $\mathrm{PZ.B\Delta : PD}$, with the value of BΔ found substituted , there will be found

$$\text{DT} = \left(\text{BQ.DP} - \text{AD.DP} - \text{BQ.AV}\right) : \text{PD} = \text{BQ} - \text{AD}, -\text{BQ.AV} : \text{DP}.$$

And thus the acceleration acting on the body at the point D of the cycloid, that is, DN or DB, $-$DT

$$\text{will be BD} - \text{BQ} + \text{AD}, +\text{BQ.AV} : \text{DP} = \text{AV. BQ} : \text{DP}, -\text{AQ}.$$

SCHOLIUM II.

596. Just as I have indicated above (§.588), the resistance on a cycloid was defined some time ago by the cel. Newton in Prop. XXIX. Book. Il. *Princ. Phil. Nat.* [See Book II, section 6 translation plus notes on the website for this proposition, which relates however to the 3rd edition, while Hermann discusses the 1st edition with a few minor changes made.] and that to be determined by hyperbolic distances, by expressions which appear to be very different from ours, but actually agreeing with ours ; but, because the truth of my assertions are not apparent immediately, it pleases a little to show the agreement between

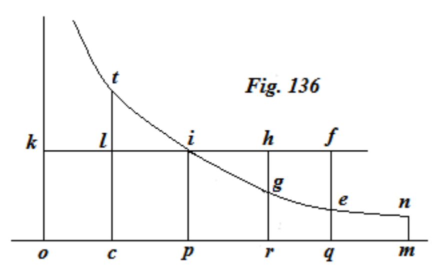


Fig. 136

the Newtonian determination of the resistance and my determination of the resistance. In Fig. 136. I advance Newton's graph with the same letters, but for the sake of avoiding confusion with smaller letters, and with no change made to the actual words, except what in this place, which concern the cycloid, I shall apply to our figure 134, which are concerned with the figure cited in prop. XXV of the book.

597. " ABC shall be the arc described by the whole oscillation, and B shall be the lowest point of the cycloid, and B*a* half the arc of the whole cycloid, equal to the length of the pendulum, and the resistance of the body shall be sought at the location of some point D. The indefinite right line *oq* may be cut at the points *o, c, p, q,* from that rule, so that, (if the perpendiculars *ok, ct, pi, qe,* may be raised, and with centre *o,* and with the asymptotes *ok, oq* the hyperbola *tige* may be described cutting the perpendiculars *ct, pi, qe* at *t, i* & *e,* and *kf* may be acting through the point *i,* parallel to the asymptote *oq* crossing with the asymptote *ok* at *k,* and with the perpendiculars *ct* and *qe* at *l* and *f*) the hyperbolic area *pieq* shall become to the hyperbolic area *pite,* as the arc AB, described in the descent of the body, to the arc BC described in the ascent, and the area *ief* to the area *ilt* as *oq* to *oe.* Thence the hyperbolic area *pinm* may be cut off by the perpendicular *mn,* which shall be to the hyperbolic area *pieq,* as the arc *a*B to the arc AB described in the whole descent. And if the hyperbolic area *pigr* may be cut off by the perpendicular *rg,* which shall be to the area *pieq,* as some arc BD to the arc described in the whole descent AB ; the resistance will be at the position D to the force of gravity, as the area $\frac{or}{oq}ief - igh$ to the area *pienm.* Thus from Newton up to this stage. "

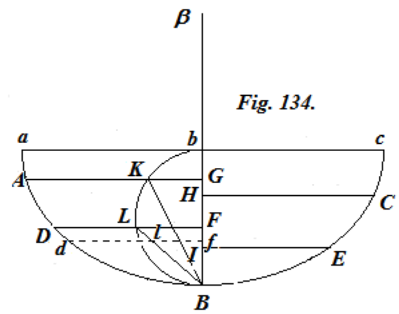


Fig. 134.

598. All three figures 134, 135 and 136 may be consulted and, because (by constr.) the hyperbolic areas *pinm*, *pieq, pigr* and *rgeq* are in proportion to the arcs of the cycloid *a*B, AB, DB & AD, and the ordinates of the logarithmic curve DP, & AV to the abscissas *oq* and *or* of the hyperbola respectively, thus so that there may be had
$\mathrm{AV}:\mathrm{DP} = or : oq$ and $\mathrm{AB}:\mathrm{AQ} = pieq : piko$ and on compounding [*i.e.* adding 1 to each ratio] $\mathrm{BQ}:\mathrm{AQ} = okieq : okip,$ and there will be

$\mathrm{BQ} - \mathrm{AD}$ as $okieq - rgeq$ or $okigr$ and $\mathrm{AV.BQ}:\mathrm{DP}$ will be as $\frac{or}{oq}okieq,$ therefore

$$\mathrm{BQ} - \mathrm{AD} - \frac{AV.BQ}{DP} \text{ as } okigr - \frac{or}{oq}okieq, \text{and thus}$$

$$okigr = orhk - igh, \text{ and } \frac{or}{oq}okieq = (oq.\,pi - ief)\frac{or}{oq} = or.pi - \frac{or}{oq}ief,$$

therefore

$$okigr - \frac{or}{oq}okieq = or.pi - igh - or.\,pi + \frac{or}{oq}ief = \frac{or}{oq}ief - igh,$$

and thus

$$\mathrm{BQ} - \mathrm{AD} - \frac{AV.BQ}{DP} \text{ is, as } \frac{or}{oq}ief - igh, \text{ and } \mathrm{BQ} - \mathrm{AD} - \frac{AV.BQ}{DP} \text{ to } \beta\mathrm{B}$$

or *a*B, that is, the air resistance is to the weight as $\frac{or}{oq}ief - igh$ to the area *pienm.* Q.E.D.

Therefore in this manner the Newtonian determination is reduced to the simpler pure curves shown by us, and in turn our derivations can be reduced to the Newtonian form ; even if most of both are seen to differ amongst themselves at first appearance ; now they appear to agree exceedingly well. Indeed the air resistance at any point on the cycloid at this stage can be determined otherwise, or rather the speed acquired at any point of the curve permits the quadrature of certain areas of the curve with the aid of the following theorem.

PROPOSITION LXXIV. THEOREM.

599. *The squared hyperbola* MO *is described between the asymptotes* AZ, AF *containing right angles, clearly any ordinate of which* MK *shall be as the inverse square of the abscissa* AK, *and the ordinate* FO *expresses uniform gravity, and with the logarithmic curve* FSQ *drawn through* F, *the subtangent of which will be equal to the right line* AF, *and on this the logarithmic ordinate* ID, *of which the abscissa* AD *shall be equal to the arc of the curve* aB *described by the descending body* AD; IM *acts parallel to* AD *and* MZ *parallel to* AF, *and on* MZ *produced beyond the axis* AT, ZN *may be taken equal to* GF *in fig.*134 *and the points* N *will be on a certain curved line* YN. *Again let the rectangle from the ordinate* AF *and from some other* QR *on the logarithmic curve, be equal to twice the area* AYNP, *and at the mid-point* T *of the interval* AR *of the ordinates* AF & RQ, *with the ordinate* TS *drawn, and* TX *equal to the distance* AD, *with the other ordinate* XV; *this* XV *expresses the speed acquired by the moving body at the point* D *on the descending curve* ADB *after sliding through its arc* AD.

With the ordinates drawn for all the curved lines ; evidently with *pn, nm,* parallel and indefinitely near to the known ordinates PN, NM, &c., likewise with the lines *mi, id, st, ux* , and *qr* parallel and indefinitely near to the known lines through the points Q, V & S and with the incremental lines Q*e*, V*v*, & S*c* parallel to the axis AR.

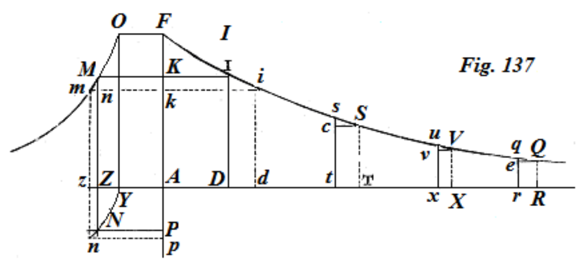


Fig. 137

I: The construction, which requires $2.\text{AYNP} = \text{AF.QR}$ and $2.\text{AY}np = \text{AF.}qr$,

also will effect $2\text{NP.P}p = \text{AF.}qe$

$(\S.\ 491.\ \text{no. II.})$ QR. Rr.

II. Because (from the constr:) $\text{AT} = \text{TR}$ and $\text{A}t = tr$, there will be $\text{R}r = 2.\text{T}t$, and thus $\text{ST.T}t : \text{QR.R}r = \text{ST} : 2.\text{QR}$ (or, on account of the continued proportionals AF, TS & RQ because of the equal intervals AT and TR of the logarithmic ordinates , there will be also) $= \text{AF} : 2.\text{TS} = \text{AF.}s\varepsilon : 2.\text{ST.}s\sigma = \text{ST.T}t : 2S\text{T.}s\sigma$;

therefore $2.\text{ST.}\ \sigma s = \text{QR.R}r$ (no. I. of this) $= 2.\text{NP. P}p$; therefore also

$$\text{ST.}s\sigma = \text{NP.P}p;\ \text{or P}p : s\sigma = \text{ST} : \text{NP}.$$

III. Truly the hyperbola MO gives $AF^2 : AK^2$ (or, on account of the equal intervals AD and TX of the ordinates AF, ID, or AK, ST and VX shall be proportionals by the nature of the logarithmic curve) $= ST^2 : VX^2 (= MK : OF) = AF.\ MK : AF.OF$,there will be, on interchanging, $ST^2 : AF.MK = VX^2 : AF.OF,$ or (no. II of this) there was $Pp : s\sigma = ST : MK,$ or NP ; and the logarithmic curve shows $s\sigma : Tt = ST : AF$ therefore from the equation :

$$Pp : Tt = ST^2 : AF.MK = VX^2 : AF.OF, \text{ and } VX^2.Tt = AF.OF.Pp.$$

IV. Since (from the constr.) there shall be $AD = TX,$ and $Ad = tx = tT + TX - Xx,$ there will be $Dd = Tt - Xx$, or $Xx : Tt - Dd$, that is, $VX^2.Xx = VX^2.Tt - VX^2.Dd$; and, by substituting $VX.Xx$ in place (§. 491. no. I) equally AF.*uv;* and in place of $VX^2.\ Tt$ (no. III of this) equally the volume AF.OF.P*p*, and finally by making $VX^2 = R.AF$, there becomes $AF.VX.vu = AF.OF.Pp - AF.R.Dd$, or, on dividing everything by AF, there will be found $VX.uv = OF.Pp - R.Dd$ (or by putting T.D*d* equal to OF.P*p*) $= VX.uv = T.Dd - R.Dd = (T - R).Dd.$ Now, because D*d* is an element of the distance AD, or $T - R$ shall be some acceleration acting, the moment of its action will be equal to the moment of its velocity acquired at the point D of the curve, and since the same moment of the action shall be found equal to the moment of the ordinate XV, it follows that this ordinate expresses the speed acquired at D. And, because (following the hypothesis) $T.Dd = OF.Pp$, and OF expresses uniform gravity, and OF.P*p* (by constr.) OF.F*f* shall be the moment of the centre of gravity, of which the moment of its tangent, or the right line from this tangent, and the element of the curve D*d* (§. 133) is equal, and likewise T.D*d,* T to be the action of the tangential gravity at the point D of the curve, and, because VX, as now shown, is the speed acquired at some point D, the acceleration acting shall be $T - R$, and $AF.R = VX^2$, that is, R shall be as VX^2; and since the resistance of the medium (following the hypothesis) also shall be as the square of the velocity acquired, the magnitude R expresses the resistance of the air at the point D on the curve D, and thus $T - R$ evidently with the air resistance taken from the tangential action of gravity, expresses the acceleration acting continually of the body descending , with which vigor acting of this kind the speed XV is acquired at D. Q.E.D.

600. If the weight rises on the other side of the curve BEC, Fig. 134, the curve YN always approaches more and more towards AP, as from the same it recedes in the descent case, indeed the construction is the same in each case, only with this difference, that in the case of the descent the part of the hyperbola OM, which is under the ordinate OF, must be used in that construction, and again, which is above the ordinate OF, in the case of the ascent. Truly in the preceding constructions I fell upon the method by using a wire which demonstrations the observant reader would understand readily, therefore, so that I may confine myself to brevity, and I will not dwell further in explaining these matters.

COROLLARY.

601. If the line of descent of the body is again the cycloid *a*DB, the area AYNP in the Figure 137 will be squarable with the aid of logarithms; for by inspecting each Figure 137, and 134 the area will be

$$\mathrm{AYNP} = (\mathrm{AB.YZ} + \tfrac{1}{2}\mathrm{AF.YZ} - \mathrm{AD.AZ}).\mathrm{AF} : 4b\mathrm{B}.\left(\text{constr.}\right) = \tfrac{1}{2}\mathrm{AF.Q.R}$$

(or from the nature of the logarithmic curve) $= \tfrac{1}{2}\mathrm{ST}^2$. Truly, because on account of the equal intervals AD and TX, the ordinates AF, DI or AK, ST and VX are in proportion, there shall be $\mathrm{AF}^2 : \mathrm{AK}^2$ or (on account of the hyperbola OM)

$= \mathrm{AZ} : \mathrm{AY} = \mathrm{ST}^2 : \mathrm{VX}^2 = \left(\mathrm{AB.YZ} + \tfrac{1}{2}\mathrm{AF.YZ} - \mathrm{AD.AZ}\right).\mathrm{AF} : 2b\mathrm{B},$ ad VX^2,

or (constr.) $\mathrm{AF.R} = \left(\mathrm{AB.YZ} + \tfrac{1}{2}\mathrm{AF.YZ} - \mathrm{AD.AZ}\right) : 2b\mathrm{B}$ to R;

or $\mathrm{AB.YZ} - \tfrac{1}{2}\mathrm{AF.YZ} - \mathrm{AD.\ AZ}$ to $\mathrm{R}.2b\mathrm{B}$;

and thus the resistance of the medium, or

$$\mathrm{R} = \left(\mathrm{AB.\ YZ} - \tfrac{1}{2}\mathrm{AF.\ YZ} - \mathrm{AD.AZ}\right).\mathrm{AY} : 2b\mathrm{B.AZ}.$$

Which proof is in accord with the preceding determinations (§§ 591, 596, & 597).

CAPUT XIX.

De descenso & ascenso gravium in lineis quibuscunque curvis, posita medii resistentia quadratis celeritatum proportionali.

581. Gravia, quæ in vacuo feruntur atque in lineis curvis incedunt, in quolibet curvæ puncto, quam accelerata motu perlabuntur, eam celeritatem acquisivisse censentur, quam acquisivissent, si via brevissima a linea horizontali per initium descensus in curva ducta, in datum curvæ punctum , id est, juxta lineam rectam horizontali perpendicularem, cecidissent, ut in primo Libro §.142. id generalitcr ostensum. Sed res aliter se habet cum corporibus, quæ in aëre moventur; eorum enim celeritates aliter, quam per quadraturas & rectificationes curvarum exhiberi nequeunt, idque in sola hypothesi, qua resistentiæ aëris quadratis celeritatum mobilis· proportionantur, ut quidem ex sequentibus id elucescet.

PROPOSITIO LXXII.

582. *Si, extensa tota curva, quam grave descensu suo & subsequente ascensu describit in lineam rectam, in singulis hujus rectæ punctis perpendiculares excitentur proportionales resistentiiis aëris, quas mobile in homologis curvæ punctis subit. Area curvilinea, quæ omnes ejusmodi perpendiculares continet, æquabitur rectangulo, a recta, quæ gravitatem uniformem exponit, in differentiam abscissarum arcuum descensu & ascensu descriptorum.*

Esto ADB curva descensu mobilis & BEG ascensu descripta, curvæ hujus BEC abscissa est BH, illius vero BG, adeoque GH erit differentia abscissarum. Extensa deinde tota curva ABC in lineam rectam ABC, atque in singulis hujus rectæ punctis D, erectæ sint perpendiculares DT exponentes resistentiam medii in homologis curvæ punctis B, formantesque figuram curvilineam ATRC, quam æquari ostendam rec-lo βB in GH, supponendo rectam βB exponere gravitatem uniformem.

In eadem DT productum sumatur, ubi DN, quæ solicitationem tangentalem gravitatis in curvæ puncto D ex centrali derivatam exponet ND – TD excessus solicitationis tangentialis gravitatis supra resistentiam aëris , solicitationem acceleratricem mobilis in curva.

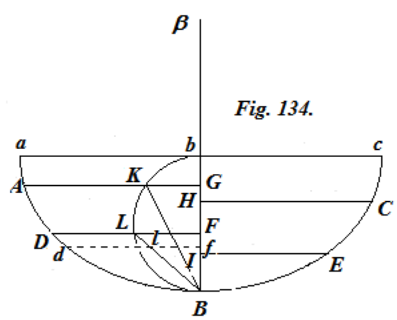


Fig. 134.

ADB descendentis. Jam momentum hujus solicitationis acceleratricis, hoc est, $(\mathrm{ND}-\mathrm{TD.})Dd$, vel $\mathrm{ND.D}d-\mathrm{TD.D}d$, hoc est, elementum areæ AMND – elem. areæ ATD (§.§.13l, 484.) æquatur momento celeritatis mobili in curvæ puncto D acquisitæ , seu elemento dimidii quadrati ex dicta celeritate acquisita; unde, cum in A omnia a nihilo incipiant, & omnia momenta solicitationis acceleratricis æquentur omnibus momentis celeritatis, erit area AMND $-\mathrm{area\ ATD} = \mathrm{dimid.quadr.}$ velocitatis in D, & area $\mathrm{AMNSB} - \mathrm{area\ ATRB} =$ dimid. quadrat. velocitatis acquisitæ in B.

Si porro mobile in curva BEC. ascendit, resistentia totalis, in quolibet ejus puncto E (§. 481.) erit $\mathrm{NE} + \mathrm{TE}$, & momentum hujus resistentiæ totalis seu $\left(\mathrm{NE} + \mathrm{TE}\right).e\mathrm{E} = \mathrm{NE}.e\mathrm{E} + \mathrm{TE}.e\mathrm{E} = \mathrm{elem.\ areæ\ MNEC} - \mathrm{elem.\ areæ\ CTE}$ æquatur momento celeritatis decrescentis mobilique in curvæ puncto E residuæ, seu elemento dimidii quadrati ex celeritate in E; unde, quia delata NE in MC, omnia in nihilum desinunt, erit area $\mathrm{MNEC} + \mathrm{area\ CTE} =$ dimid. quadr. ex veloc. in E, adeoque etiam $\mathrm{MNSBC} + \mathrm{CTRB} =$ dimid. quadrato ex celeritate initiali in B. Atqui hæc celeritas initialis in B, eadem est cum velocitate acquisita in B, post descensum mobilis in curva ADB; ergo $\mathrm{AMNSB} - \mathrm{ATRBA}(=$ dimid. quadrato celeritatis in B$) = \mathrm{CMNSBC} + \mathrm{CTRBC}$, adeoque tota figura resistentiarum $\mathrm{ATRCA} = \mathrm{AMNSB} - \mathrm{CMNSB}.$
Atqui area AMNSB æquatur perpetuo rectangulo sub βB recta, quæ gravitatem uniformem exponit, & abscissa BG curvæ ADB descensu mobilis descriptæ, cum quodlibet elementum areæ ND*d* æquetur rec-lo βB.F*f*, seu momento gravitatis centralis βB. Eodem argumento sequitur fore aream $\mathrm{CMNSBC} = \beta\mathrm{B.\ BH}$, ergo $\mathrm{AMNSB} - \mathrm{CMNSB}$, quod areæ ATRCA æquari ostensum, æquabitur etiam rec-lo βB.GH. Quod erat demonstrandum.

COROLLARIUM I.

583. Si resistentiæ aëris, sint ut celeritates mobilis acquisitæ, erit $\mathrm{AMND} = \mathrm{areæ\ ATD} + \frac{1}{2}\mathrm{TD}^2$. Nam TD (secundum hypothesin) est ut celeritas in D acquisita, adeoque, per præcedentem paragraphum, erit momentum solicitationis acceleratricis, seu $\mathrm{ND.D}d + \mathrm{TD.\ D}d = \mathrm{elem.\ AMND} + \mathrm{elem.\ ATD}$, æquale momento celeritatis, seu $\mathrm{TD.}\ t\theta$, quod æquatur elem. ex $\frac{1}{2}\mathrm{TD}^2$. Ergo summando, erit omnino $\mathrm{area\ AMND} - \mathrm{area\ ATD} = \frac{1}{2}\mathrm{TD}^2$, atqueadeo $\mathrm{area\ AMND} = \mathrm{area\ ATD} + \frac{1}{2}\mathrm{TD}^2$.
Eodem argumento concludetur esse $\mathrm{NECM} = \frac{1}{2}\mathrm{TD}^2 - \mathrm{CTE}.$

COROLLARIUM II.

484. Sin vera resistentiæ aëris fuerint in duplicata ratione celeritatum acquisitarum, quæ verior est hypothesis, erit area
$\mathrm{AMND} = \mathrm{AQ.TD} + \mathrm{ATD},\ \ \&\ \ \mathrm{area\ CMNE} = \mathrm{AQ.TE} - \mathrm{CTE}.$ Ubi AQ est magnitudo data seu constans.
Hoc enim casu celeritatis in D acquisitæ quadratum erit $= 2\mathrm{AQ.TD}$, cum resistentiæ sint ut quadrata celeritatum; & dimidium celeritatis quadratum AQ.TD, hujusque elementum, id est momentum celeritatis acquisitæ in D, reperietur $\mathrm{AQ.}t\theta$*;* unde, quia momentum solicitationis acceleratricis $\mathrm{ND} - \mathrm{TD}$ æquatur momento celeritatis in D acquisitæ, fiet $\mathrm{ND.D}d - \mathrm{TD.D}d = \mathrm{AQ.}t\theta$, & summando, omnia NO.D*d*, seu area

$\text{AMND} - \text{omn. TD.D}d$, seu area $\text{ATD} = \text{omnibus AQ.}t\theta$,id est, rec-lo AQ.TD; adeoque $\text{AMND} = \text{ATD} + \text{AQ.TD.}$ Eadem ferme ratione probatur esse $\text{CMNE} = \text{AQ.TE} - \text{CTE.}$

COROLLARIUM III.

585. Si resistentia est uniformis, curva ARC abit in lineam rectam ipsi AC parallelam, eritque hoc casu $\text{AMNSB} - \text{CMNSB} = \beta\text{B .GH} = \text{AC.BR,}$ ac per consequens = AQ.VPW. consequens resistentia medii BR uniformis ad gravitatem uniformem se habebit, ut GH differentia abscissarum BG & BH arcuum curvæ AB & BC descensu & subsequente ascensu descriptorum ad AC aggregatum ipsorum arcuum ABC.

COROLLARIUM IV.

586. Si curva ABC est cyclois ordinaria; cujus axis vel diameter circuli generatoris est *b*B & basis *ac;* semissis differentiæ quadraterum ex arcubus cycloidis ADB & BEC descensu & ascensu descriptis æquabitur areæ ATRTC.
Nam, quia βB exponit gravitatem uniformem juxta ea, quæ §. 179 dicta sunt, hæc βB erit dupla diametri *b*B*;* hinc $\beta\text{B. BG} = 2.b\text{B.BG} = 2.\text{BK}^2$ (vel, quia arcus cycloidis ADB æquat duplam subtensam BK) $= \frac{1}{2}.\text{AB}^2$, & $\beta\text{B.BH} = 2.b\text{B.BH} = \frac{1}{2}.\text{BC}^2$, ergo $\beta\text{B. GH} = \frac{1}{2}\text{AB}^2 - \frac{1}{2}\text{BC}^2 = \text{ATRTC.}$ Hoc est, rectangulum ex semisse differentiæ arcuum ADB & BEC descensu & subsequente ascensu in cycloide descriptorum, in aggregatum ABC eorundem arcuum, æquat aream ATRTC.

Hoc corollarium coincidit cum propositione XXX. Lib. II. Princ. Phil. Nat. IIIustr. Newtoni, cujus tamen demonstrationem ex aliis principiis hausit, ut demonstrationem illam inspicienti cuilibet patebit.

COROLLARIUM IV.

587. Ergo, in casu corollarii III. hujus, resistentia uniformis in cycloide erit ad gravitatem, id est, RB ad βB in fig. 134. sicut dimedia differentia arcuum ADB & BEC ad βB, seu hemicycloidem *aDB,* vel, sicut differentia dictorum arcuum ad intcgram cycloidem, *a*B*c*. Est enim (§.585.)

$$\text{BR} : \beta\text{B} = \text{GH} : \text{AC} = \beta\text{B.GH} : \beta\text{B.AC}\,(\S.586.) = \left(\tfrac{1}{2}\text{AB} - \tfrac{1}{2}\text{BC}\right).\ \text{AC} : \beta\text{B.AC}$$

$$= \tfrac{1}{2}\text{AB} - \tfrac{1}{2}\text{BC} : \beta\text{B vel } a\text{DB}, = \text{AB} - \text{BC} : \text{cycloid. } a\text{B}c.$$

SCHOLION.

588. In aliis resistentìæ hypothesibus res evadit paullo altioris indaginis, quam in corollariis III. & V. hujus ; cum in omni alia hypothesi curva ATRTC adhuc invenienda sit. Illustris Newtonus statim post propositionem in corollario IV, hujus citatam asserit, hanc curvam ARC ad ellipsin proxime accedere; si, curva ABC existente cycloide, resistentiæ aëris fuerint celeritatibus proportionales, vel ad parabolam, si resistentiæ fuerint in duplicata proportione celeritatum. Sed, accurate & mathematice rem sumendo,

prænominata ATRTC non est ellipsis in prima, nec parabola in altera resistentiæ hypothesi; sed, quantum conjicere licet, utroque casu erit transcendens. Etenim, si resistentiæ sunt ut celeritates acquisitæ, nondum constat, qua ratione curva ARC construi debeat; ne quidem concessis quadraturis figurarum curvilinearum. Pro altera vero hypothesi ac naturæ magis consentanea nominatæ curvæ constructo haberi potest per quadraturas & rectificationes curvarum. Ex propositione XXIX. laudatissìmi Newtoni Lib. II ejusmodi constructio peti potest in casu particulari, quo curva descensus ABC est cyclois. Hoc loco vero generalem constructionem adducam pro quacunque curva ABC, idque gemino modo.

PROPOSITIO LXXIII. THEOREMA.

589. *Per quodlibet in linea AM producta punctum* V, *descripta logarithmica VXY subtangentem habente æqualem datæ lineæ* AQ, *& asymptotam* CQ, *ac protensa* ND *usque ad occursum* P *cum log-mica* VX ; *si omnia elementa* PD.DN.Dd *alicujus solidi duobus planis* MAV *&* NDP *interjecti erecto scilicet ad angulos rectos plano* MABS *super plano* VAB, *æqualia fuerint parallelepipedo* $\mathrm{AQ}^2.\mathrm{DO}$, *sub quadrato subtangentis log-micæ & ordinatæ* DO *cujusdam curvæ* AOΔG; *homologia ordinata* DT *scalæ resistentiarum* ARC *erit quarta proportionalis ad ordinatas* PD, DO & AQ *subtangentem log-micæ* VX.

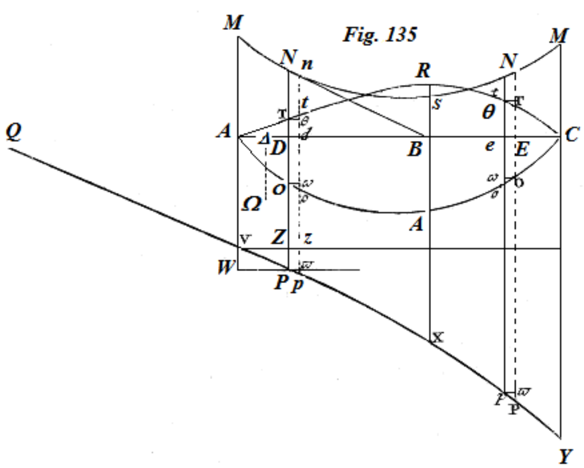


Fig. 135

I. Esto linea *pn* alteri PN indefinite vicina, atque per puncta T, & P curvarum AT, AO & VP ductæ sint lineolæ Tθ, Oω, & Pπ; ac denique, sit DT quarta proportionalis ad PD, DO & AQ, vel, quod eodem recidit, rec-lum $\mathrm{PD.DT} = \mathrm{AQ.DO}$, $\& \, pd.dt = \mathrm{AQ}.do$, id est,

$$\mathrm{PD}.dt + p\pi.dt = \mathrm{PD.DT} + \mathrm{PD}.t\theta + dt.p\pi = \mathrm{PD.DT}$$
$$+ \mathrm{PD}.t\theta + \mathrm{DT}.p\pi = \mathrm{AQ.DO} + \mathrm{AQ}.\omega o;$$

subductis igitur PD.DT & AQ.DO remanebit $\mathrm{PD}.t\theta + \mathrm{DT}.p\pi = \mathrm{AQ}.\omega o$, ductisque sìngulis partibus in AQ; fiet $\mathrm{AQ.PD}.t\theta + \mathrm{DT.AQ}.p\pi = \mathrm{AQ}^2.o\omega.$

II. Atqui $\mathrm{AQ}^2.\omega o - \mathrm{PD.DN.D}d$, cum omnia PD.DN.D*d* sint (constr.) $= \mathrm{AQ}^2.\mathrm{DO}$; item (§.491.num. II) $\mathrm{AQ}.p\pi$ æquatur PD.D*d*, vel $\mathrm{AQ.DT}.p\pi = \mathrm{DT.PD.D}d$; ergo, subrogatis his valoribus in postrema æqualitate num.I hujus, invenietur $\mathrm{PD.AQ}.t\theta + \mathrm{PD.DT}.Dd = PD.\mathrm{DN.D}d$, &, applicando singula membra ad PD, AQ. $t\theta + \mathrm{DT.D}d = \mathrm{DN.D}d$. Adeoque omnia $\mathrm{AQ}.t\theta$, id est, $\mathrm{AQ.TD} + \text{omn. DT.D}d$,id est, area $\mathrm{ATD} =$ omnibus DN.D*d*, seu areæ AMND. Est proinde $\mathrm{AQ.TD} + \mathrm{ATD} = \mathrm{AMND}$; atque adeò (§.584) curva ATR est scala resistentiarum mobilis in curva ADB cadentis.

III. Eodem modo probatur $\mathrm{PE}.t\theta - \mathrm{ET.P}\pi$, æquari $\mathrm{AQ}.o\omega$ atque inde, ut ante, inferetur esse $\mathrm{AQ.TE} - \mathrm{CTE} = \mathrm{CMNE}$, ac per consequens = AQ.VPW. consequens (§. 584.) curvam CTR scalam resistentianun aëris mobili in curva BEC ascendenti oppositarum. Propterea curva ATRTC est scala resistentiarum aëris, cum corpus in curva quacuaque ADB descendit, atque dehinc in altera BEC ascendit. Quod erat demonstrandum.

COROLLARIUM I.

590. Celeritas mobilis, in quolibet curvæ puncto D acquisita, est $\sqrt{(2.\mathrm{AQ.DT})}$, atque velocitas eidem mobili per arcum curvæ BE ascendenti residua in E erit $\sqrt{(2.\mathrm{AQ.ET})}$ quorum ratio ex superioribus (§. 484.) satis constat.

COROLLARIUM II.

591. Omnia parallelepipeda PD.DN. D*d* continentur in solido quodam MAVN, cujus singulæ sectiones plano BAV rectæ planoque MAV parallelæ, sunt rectangula PDN. Planum enim AMSB super plano VAB erectum concipiendum est.

COROLLARIUM III.

592. Si linea descensus mobilis ABC est cyclois ordinaria, linea MNS erit recta transiens per B, angulumque MBA, semirecto æqualem contens cum linea AB. Propterea solidum prædictum MAVN erit truncus log-micus super basi ADPV interjectus plano VAB aliique per BX transeunti, atque angulo semirecto ad planum VAB inclinato.

Nam, quia subtensa BL arcus circularis BL parallela est tangenti cycloidis in puncto D erit gravitas ad solicitationem ejus tangentialem in cycloidis puncto D, sicut *b*B ad $\mathrm{BL} = \beta\mathrm{B} : 2.\mathrm{BL} = \beta\mathrm{B} : \text{arcum cycl. BD}$, atqui βB exponit gravitatem, ergo BD exponet solicitationem gravitatis tangentialem in D, atque adeo in fig. 135 BD quæ eadem est cum arcu cycloidis BD in fig.134 in rectam extenso æquabitur ordinatæ ND, & $\mathrm{AB} = \mathrm{AM}$; hinc linea MNS est recta, seu figura AMNS erit triangulum rectangulum isosceles, atque adeo solidum MAVNP truncus resectus ex prismate recto super basi ADPV a plano secante per BX ducto, atque angulo semirecto ad planum baseos BAV inclinato.

COROLLARIUM VI.

593. Unde, si Ω sit centrum gravitatis quadrilinei log-mici ADPV, demissa perpendiculari ΩΔ ad AB, solidum ex quadrilineo prædicto ADPV in BΔ æquabitur trunco MAVNP, atque adeo (constr.) parallelepipedo ; atqui ducta per VZ parallela AC, quadrilineum AVPD reperietur $= \text{AQ.PZ}$, ergo solidum

$$\text{MAVNP} = \text{AQ. PZ.B}\Delta\left(\text{constr.}\right) = \text{AQ}^2.\text{DO}, \ \& \ \text{PZ.B}\Delta = \text{AQ.DO} \ (\text{num. I §.589})$$
$$= \text{PD.DT}. \ \text{Est proinde resistentia DT in quolibet cycloidis puncto} = \text{PZ.B}\Delta : \text{PD}.$$

SCHOLION.

594. Quia ad obtinendam expressionem ordinatarum DT etiam expressio requiritur ipsarum BΔ, seu distantiarum centri gravitatis quadrilineorum ADPV, idcirco duobus verbis indicabo, qua ratione investigari debeant.
In log-mica est $\text{DP.D}d = \text{AQ.}p\pi, \& \ \text{AD.DP.D}d = \text{AQ.AD.}p\pi$ (vel ducta PW parallela VZ) $= \text{AQ.PW.}p\pi$,ergo omnia $\text{AD.DP.D}d =$ omnibus $\text{AQ.PW.}p\pi$ id est $= AQ$ in trilineum *VPW*. Atqui (§. 44) est quadrilineum ADVP in $\text{A}\Delta =$ omnibus AD.DP.Dd ; ac per consequens = AQ.VPW.

$$\text{Verum ADPV} = \text{AQ. PZ} , \ \& \ \text{VPW} = \text{AD.DP} - \text{AQ.PZ}; \ \text{ergo}$$
$$\text{AQ.PZ.A}\Delta = \text{AQ.AD.DP} - \text{AQ}^2.\text{PZ seu } PZ.\text{A}\Delta = \text{AD.DP} - \text{AQ. PZ};$$
$$\text{hinc PZ.B}\Delta = \text{AB.PZ} - \text{AD. DP} + \text{AQ.PZ} = \text{BO.PZ} - \text{AD. DP}$$
$$(\text{vel ponendo DP} - \text{AV loco ipsius PZ}) = \text{BQ.DP} - \text{BQ.AV} - \text{AD.DP}.$$

Hinc, ut saltem in transitu dicam, centrum gravitatis areæ log-micæ ultra ordinatam D P versus *Q* in infinitum excurrentis intervallo subtangentis AQ distat a sua prima ordinata DP.

COROLLARIUM VI.

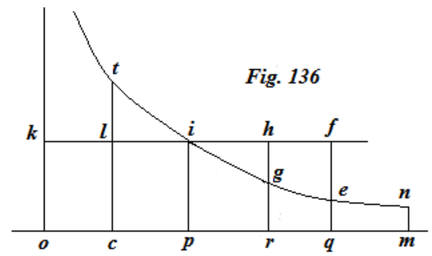
595. Cum DT (§. 593.) sit $\text{PZ.B}\Delta : \text{PD}$, substituto invento valore ipsius BΔ, reperietur
$$\text{DT} = \left(\text{BQ.DP} - \text{AD.DP} - \text{BQ.AV}\right) : \text{PD} = \text{BQ} - \text{AD}, -\text{BQ.AV} : \text{DP}.$$
Adeoque solicitatio acceleratrix mobilis in cycloidis puncto D, hoc est, DN seu DB,
$$-\text{DT erit BD} - \text{BQ} + \text{AD}, +\text{BQ.AV} : \text{DP} = \text{AV. BQ} : \text{DP}, -\text{AQ}.$$

SCHOLION II.

596. Supra jam indicavi (§.588.) resistentias in cycloide jam pridem definitas esse Cel. Newtono Prop. XXIX. Lib. Il. Princ. Phil. Nat. idque præstitit per spatia hyperbolica, expressionibus a nostris diversissimis quo ad apparentiam, sed revera cum nostris conspirantibus; sed, quia assertionis meæ veritas non statim in oculos incurrit, consensum inter Newtonianam atque meam determinationem resistentiarum, paucis ostendere libet. In figura 136. afferam Newtoni schema ipsiusmet literis, ast confusionis vitandæ gratia minusculis insignitum, atque propria ejus verba nulla mutatione facta, nisi

quod hoc loco, quæ cycloidem respiciunt, nostræ figuræ 134 applicaturus sim, quæ ille indicat circa figuram propos. XXV. citati libri.

597. " Sit ABC arcus oscillatione integra descriptus, sitque B infimium cycloidis punctum, & B*a* semissis arcus cycloidis totius, longitudini penduli æqualis, & quæratur resistentia corporis in loco quovis D. Secetur recta infinita *oq*, in punctis *o, c, p, q,* ea lege, ut (si erigantur perpendicula *ok, ct, pi, qe,* centroque *o*, & asymptotis *ok, oq* describatur hyperbola *tige* secans perpendicula *ct, pi, qe* in *t, i* & *e*, & per punctum *i* agatur *kf*, parallela asymptoto *oq* occurrens asymptoto *ok* in *k*, & perpendiculis *ct* & *qe* in *l* & *f*) fuerit area hyperbolica *pieq* ad aream hyperbolicam *pite*, ut arcus AB, descensu corporis descriptus, ad arcum BC ascensu descriptum, & area *ief* ad aream *ilt* ut *oq* ad *oe*. Dein perpendiculo *mn* abscindatur area hyperbolica *pinm*, quæ sit ad aream hyperbolicam *pieq*, ut arcus *a*B ad arcum AB descensu descriptum. Et si perpendiculo *rg* abscindatur area hyperbolica *pigr*, quæ sit ad aream *pieq*, ut arcus quilibet BD ad arcum AB desccnsu toto descriptum ; erit resistentia in loco D ad vim gravitatis, ut area $\frac{or}{oq} ief - igh$ ad aream *pienm*. Hactenus Newtonus. "



598. Consulantur omnino tres figuræ 134, 135 & 136 &, quia (constr.) spatia hyperbolica *pinm*, *pieq*, *pigr* & *rgeq* arcubus cycloidis *a*B, AB, DB & AD proportionalia sunt, & logarithmiæ ordinatæ DP, & AV abscissis hyperbolæ *oq* & *or* respective, adeo ut habeatur $\mathrm{AV} : \mathrm{DP} = or : oq . \& \ \mathrm{AB} : \mathrm{AQ} = pieq : piko$ & componendo

$\mathrm{BQ} : \mathrm{AQ} = okieq : okip,$ erit

$\mathrm{BQ} - \mathrm{AD}$ ut $okieq - rgeq$ seu $okigr$ & $\mathrm{AV.BQ} : \mathrm{DP}$ ut $\frac{or}{oq} okieq,$ ergo

$$\mathrm{BQ} - \mathrm{AD} - \frac{AV.BQ}{DP} \text{ ut } okigr - \frac{or}{oq} okieq, \text{atqui}$$

$$okigr = orhk - igh, \ \ \& \frac{or}{oq} okieq = (oq.\, pi - ie\, f\,)\frac{or}{oq} = or.pi - \frac{or}{oq} ief,$$

ergo

$$okigr - \frac{or}{oq} okieq = or.pi - igh - or.\, pi + \frac{or}{oq} ief = \frac{or}{oq} ief - igh,$$

atque adeo

$$\mathrm{BQ} - \mathrm{AD} - \frac{AV.BQ}{DP} \text{ est, ut } \frac{or}{oq} ief - igh, \ \ \& \ \mathrm{BQ} - \mathrm{AD} - \frac{AV.BQ}{DP} \text{ ad } \beta\mathrm{B}$$

seu *a*B, id est, resistentia aëris ad gravitatem ut $\frac{or}{oq} ief - igh$ ad aream *pienm*. Quod erat demonstrandum.

Hoc ergo modo Newtoniana determinatio reducitur ad simpliciorem meris lineis a nobis exhibitam, & vice versa nostra facile reducitur sub formam Newtonianæ; propterea etsi ambæ plurimum inter se differre prima fronte videntur; inter se jam egregie conspirare apparent. Resistentiæ aëris in cycloide imo in qualibet data curva, aliter adhuc determinari possunt, vel potius celeritates acquisitæ in quolibet curvæ puncto concessa quadratura cujusdam spatii curvilinei ope theorematis sequentis.

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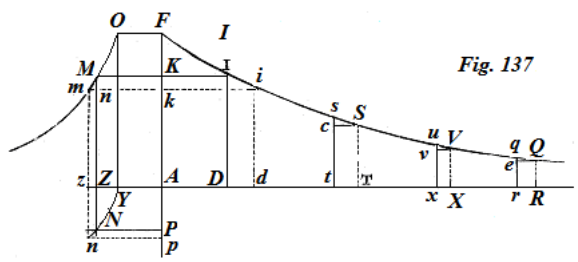
PROPOSITIO LXXIV. THEOREMA.

599. *Inter asymptotes* AZ, AF *angularum rectum continentes describatur hyperbola quadratica* MO, *cujus scilicet quælibet ordinata* MK *sit ut quadratum abscissæ* AK *inverse, ordinataque* FO *gravitatem uniformem exponat, ducta per* F *log-mica* FSQ, *cujus subtangents æquet rectam* AF, *atque in hac log-mica ordinata* ID, *cujus abscissa* AD *æqualis sit arcui curvæ* aB *descensu mobilis discripto* AD; *agantur* IM *parallel* AD *&* MZ *parallela* AF, *atque in* MZ *producta infra axem* AT, *capiatur* ZN *æqualis* GF *in fig.*134 *eruntque puncta* N *in linea quadam curva* YN. *Esto porro rec-lum ex ordinata* AF *aliaque* QR *in log-mica, æquale duplo areæ* AYNP, *& in medio* T *intervalli* AR *ordinatarum* AF *&* RQ, *ducta ordinata* TS, *atque ad distantiam* TX *æqualem* AD, *alia ordinata* XV; *hæc* XV *exponet celeritatem acquisitam mobili in curvæ descensus* ADB *puncto* D *post lapsum per ejus arcum* AD.

Ductis ordinatis aliisve lineis, videlicet *pn, nm, mi, id, st, ux* , & *qr* lineis cognominis PN, *NM,* &c. parallelis & indefinite vicinis, item per puncta Q, V & S lineolis Q*e*, V*v*, & S*c* axi AR parallelis.



I: Constructio, quæ præbet $2.\mathrm{AYNP} = \mathrm{AF.QR}$ & $2.\mathrm{AY}np = \mathrm{AF}.qr$, etiam efficiet $2\mathrm{NP.P}p = \mathrm{AF}.qe$ (§. 491. num.II.) QR. R*r*.

II. Quia (constr:) $\mathrm{AT} = \mathrm{TR}$ & $\mathrm{A}t = tr$, erit $\mathrm{R}r = 2.\mathrm{T}t$, atque adeo $\mathrm{ST.T}t : \mathrm{QR.R}r = \mathrm{ST} : 2.\mathrm{QR}$ (seu, propter continue proportionales AF, TS & RQ ob æqualia harum ordinatarum log-micæ intervalla AT & TR, erit etiam) $= \mathrm{AF} : 2.\mathrm{TS} = \mathrm{AF}.s\varepsilon : 2.\mathrm{ST}.s\sigma = \mathrm{ST.T}t : 2S\mathrm{T}.s\sigma$;

ergo $2.\mathrm{ST}.\ \sigma s = \mathrm{QR.R}r$ (num. I. hujus) $= 2.\mathrm{NP.\ P}p$; ergo etiam

$$\mathrm{ST}.s\sigma = \mathrm{NP.P}p;\ \text{vel } \mathrm{P}p : s\sigma = \mathrm{ST} : \mathrm{NP}.$$

III. Hyperbola vero MO præbet $\mathrm{AF}^2 : \mathrm{AK}^2$ (aut, quia ob æqualia intervalla AD & TX ordinatæ AF, ID, seu AK, ST ac VX per naturam log-micæ proportionales sunt) $= \mathrm{ST}^2 : \mathrm{VX}^2 (= \mathrm{MK} : \mathrm{OF}) = \mathrm{AF.\ MK} : \mathrm{AF.OF}$,erit, permutando, $\mathrm{ST}^2 : \mathrm{AF.MK} = VX^2 : \mathrm{AF.OF}$, sed (num.II. hujus) erat $\mathrm{P}p : s\sigma = \mathrm{ST} : \mathrm{MK}$, vel NP ; & log-mica exhibet $\mathrm{s}\sigma : \mathrm{T}t = \mathrm{ST} : \mathrm{AF}$ ergo ex æquo

$$\mathrm{P}p : \mathrm{T}t = \mathrm{ST}^2 : \mathrm{AF.MK} = \mathrm{VX}^2 : \mathrm{AF.OF}, \& \mathrm{VX}^2.\mathrm{T}t = \mathrm{AF.OF.P}p.$$

IV. Cum (constr.) sit $\mathrm{AD} = \mathrm{TX}$, & $Ad = tx = t\mathrm{T} + \mathrm{TX} - \mathrm{X}x$, erit $\mathrm{D}d = \mathrm{T}t - \mathrm{X}x$, vel $\mathrm{X}x : \mathrm{T}t - \mathrm{D}d$, id est, $\mathrm{VX}^2.\mathrm{X}x = \mathrm{VX}^2.\mathrm{T}t - \mathrm{VX}^2.\mathrm{D}d$; atqui, loco $\mathrm{VX.X}x$ substituendo (§. 491. num. I) æquale AF.*uv;* locoque $\mathrm{VX}^2.\ \mathrm{T}t$ (num.III. hujus) æquale solidum

AF.OF.P*p*, ac denique faciendo $VX^2 = R.AF$, fiet $AF.VX.vu = AF.OF.Pp - AF.R.Dd$, vel, applicando omnia ad AF, reperietur $VX.uv = OF.Pp - R.Dd$ (seu ponendo T.D*d* æquale OF.P*p*) $= VX.uv = T.Dd - R.Dd = (T - R).Dd$. Jam, quoniam D*d* est elementum spatii AD, si $T - R$ sit solicitatio quæcunque acceleratrix, erit momentum hujus solicitationis æquale momento velocitatis acquisitæ in curvæ puncto D, & cum idem solicitationis momentum repertum sit æquale momento ordinatæ XV, sequitur hanc ordinatam exponere celeritatem in D acquisitam. Atqui, quoniam (secundum hypothesin) $T.Dd = OF.Pp$, atque OF gravitatem uniformem exponit, & OF.P*p* (constr.) OF. F*f* sit momentum gravitatis centralis, cui momentum ejusdem tangentialis, seu rectum ex hac tangentiali & elemento curvæ D*d* (§. 133) æquatur, perinde ac T.D*d*, liquet T esse solicitationem gravitatis tangentialem in curvæ puncto D, &, quia VX, ut jam ostensum, est celeritas acquisita in D quæcunque sit solicitatio acceleratrix $T - R$, atque $AF.R = VX^2$, id est, R ut VX^2; & cum resistentia medii (secundum hypothesin) etiam sit ut quadratum celeritatis acquisitæ, magnitudo R exponet resistentiam aëris in curvæ puncto D, atque adeo $T - R$ detracta scilicet resistentia aëris a solicitatione gravitatis tangentiali, exponit solititationem acceleratricem mobiii descendenti continue applicatam, cui vigore ejusmodi solicitationum acquiritur in D celeritas XV. Quod erat demonstrandum.

600. Si grave in altera curvæ parte BEC, fig. 134 ascendit, curva YN semper magis magisque ad AP accedit, ut ab eadem recedit in casu descensus, constructio vero eadem est utroque casu, hac sola cum differentia, quod in casu descensus hyperbolæ portio OM, quæ est subter ordinatam OF, in constructione illa adhiberi debet, & porro, quæ supra ordinatam OF, in casu ascensus. Quonam vero artificio in præcedentes constructiones inciderim, ex demonstrationis filo facile intelliget perspicax Lector, propterea, ut brevitati consulam, eidem explicando non diutius immorabor.

COROLLARIUM.

601. Si linea descensus mobilis *a*DB est iterum cyclois, area AYNP in figura 137 erit quadrabilis ope logarithmorum; nam inspiciendo utramque figuram 137, & 134 erit area

$$AYNP = (AB.YZ + \tfrac{1}{2}AF.YZ - AD.AZ).AF : 4bB.(\text{constr.}) = \tfrac{1}{2}AF.Q.R$$

(seu ex natura log – micae) $= \frac{1}{2}ST^2$. Verum, quia, ob æqualia intervalla AD & TX, ordinatæ AF, DI vel AK, ST & VX proportiones sunt, sit $AF^2 : AK^2$ vel (propter hyperbolam OM)

$= AZ : AY = ST^2 : VX^2 = \left(AB.YZ + \frac{1}{2}AF.YZ - AD.AZ\right).AF : 2bB$, ad VX^2,

seu (constr.) $AF.R = \left(AB.YZ + \frac{1}{2}AF.YZ - AD.AZ\right) : 2bB$ ad R;

seu $AB.YZ - \frac{1}{2}AF.YZ - AD.\ AZ$ ad $R.2bB$;

atque adeo erit resistentia medii, seu $\mathrm{R} = \left(\mathrm{AB.\ YZ} - \frac{1}{2}\mathrm{AF.\ YZ} - \mathrm{AD.AZ}\right)\mathrm{.AY} : 2b\mathrm{B.AZ}.$
Quod præcedentibus determinationibus (§.§.591, 596, & 597) probe consonum est.