CHAPTER XVIII

A method of finding the properties of the motions of bodies in mediums with any kind of resistance, and with the density taken as it pleases.

In the preceding chapters we have sought out the more celebrated hypotheses of the resistance of the air, and for which motion, and we have defined how they must arise, with the same hypotheses in place, yet supposing the resisting medium to be of the same density everywhere. Truly because the densities are able to be varied one after the other, the laws of the motions also are required to be shown for variable densities of this kind, which we establish in this chapter with regard to rectilinear motions, and to be set out for curvilinear motions in the next two chapters.

PROPOSITION LXIX. LEMMA.

562. *Showing some properties of the elements of logarithms, truly:*
1°. *A* positive *logarithm element, is the element of the logarithmic curve that any magnitude of a variable has of its greater inequality to its minimum magnitude.*
Truly a negative logarithmic element is the element of the logarithm of the same greater ratio of the inequality, which the maximum magnitude of any variable *has to that variable*.
2°. Some multiple or sub multiple of the element of any variable magnitude applied to the magnitude, *of which it is the element, is the element of the logarithm of the ratio of the multiple or submultiple ratio, as it has of the variable magnitude to the variable itself, or as the maximum value of the variable magnitude has to the variable magnitude itself, provided the logarithmic element were positive or negative.*

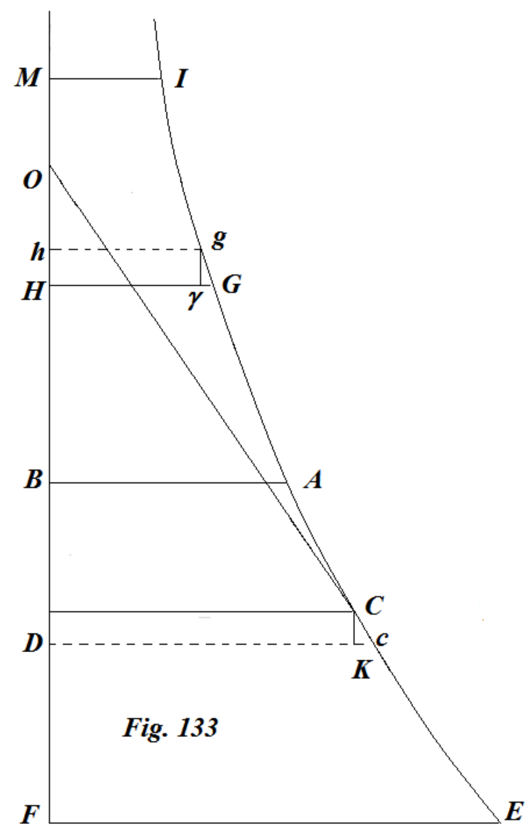


Fig. 133

1. Let u be a variable magnitude, and its *minimum* magnitude shall be *a*, the differential element $+du:u$ of the logarithmic curve will be the element of the logarithmic ratio $(u{:}a)$. But if truly for the magnitude *u*, when it is a *maximum*, that shall be $=a$; then $-du:u=$ the element of the logarithmic ratio $(a:u)$.

2. With the same in place, if there may be had $mdu:u$, it will be the element of the logarithmic ratio *(u:a)* multiplied by *m,* that is, $mdu:u =$ elem. of the ratio $(u^m : a^m)$ in the logarithmic where *m* may indicate some whole number or a fraction, rational or irrational, etc, and $-mdu:u =$ elem. of the ratio $(a^m : u^m)$ on the logarithmic curve, whose tangent is one.

IAE shall be a logarithmic curve about the axis MF, whose subtangent DO is equal to unity 1, and CD shall be a variable magnitude and continually increasing, and its *minimum* magnitude shall be AB, truly the ordinate of the variable GH shall be a magnitude decreasing continually, and its maximum also shall be AB; *cd* and *gh* shall be put as the indefinitely close ordinates of CD & GH, and with $\mathrm{C}k, g\gamma$ drawn parallel to the axis MF, $c\chi$ will be the increment of CD, and thus $c\chi$ will be positive ; truly $G\gamma$ will itself be the decrement of the ordinate GH, and thus negative. And with these in place:

I. Because(§.491.) $+c\chi : \mathrm{CD} = \mathrm{D}d : \mathrm{OD}$ (or because the subtangent OD is equal to one) $= \mathrm{D}d$, and because D*d* is the element of BD or of the log. $(\mathrm{CD}:\mathrm{AB})$, generally there will be $c\chi : \mathrm{CD} =$ element of the log. of the ratio $(\mathrm{CD}:\mathrm{AB})$ that is $+du:u =$ element of the log of the ratio$\left(u:a\right)$, on putting $\mathrm{AB}, a;\ \mathrm{CD}, u\ \ \&\ \ c\chi, du.$

There may be shown by the same argument that $-\mathrm{G}\gamma : \mathrm{GH} =$ element of the log. of the ratio $\mathrm{AB}:\mathrm{GH}$, that is $du:u, =$ elem.log.$\left(a:u\right)$ or of the log. (AB:GH), now we call $\mathrm{GH}, u;$ and $\mathrm{G}\gamma, du$ with AB now being *a*.

II. Now it is required to be shown that $m.c\chi : \mathrm{CD} =$ element of the log $(\mathrm{CD}^m : \mathrm{AB}^m)$. For there shall be $\mathrm{BD}:\mathrm{BF} = 1:m$, and thus $\mathrm{BF} = m.\mathrm{BD}$, thus so that with the ordinate FE raised at F , from the nature of logarithms there shall be $\mathrm{CD}^m : \mathrm{AB}^m = \mathrm{EF}:\mathrm{AB}.$ Indeed $m.c\chi : \mathrm{CD} = m.\mathrm{D}d : \mathrm{DO},$
& the sum of all $m.c\chi.\mathrm{CD} =$ sum of all $m.\mathrm{D}d.\mathrm{DO} = m.\mathrm{BD}:\mathrm{DO}$ (or by constr.) $\mathrm{BF}:\mathrm{DO}$ $\left(\text{or if DO be equal to one}\right) = \mathrm{BF} = \log.\left(\mathrm{EF}:\mathrm{AB}\right) = \log.\left(\mathrm{CD}^m : \mathrm{AB}^m\right).$

And thus $mdu:u =$ element of log.$(u^m : a^m)$.

By similar reasoning, as, making $\mathrm{BM} = m.\mathrm{BH}$ and with the ordinate MI erected , there shall become :

$$-m.\mathrm{G}\gamma : \mathrm{GH} = \text{element of log } (\mathrm{AB}:\mathrm{IM}) = \text{element of } \log.\left(\mathrm{AB}^m : \mathrm{GH}^m\right)$$

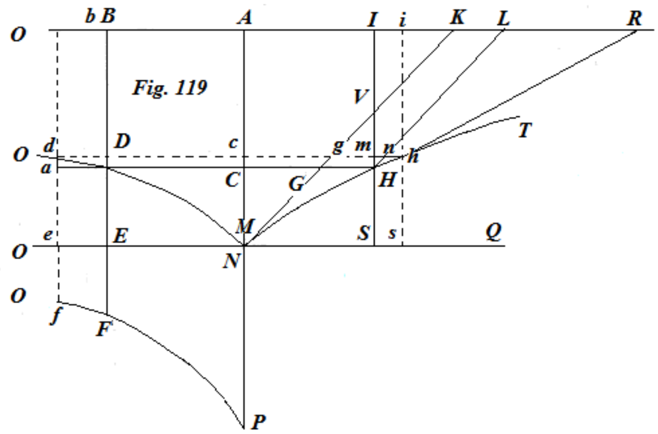
or, which is the same, $-mdu:u =$ element of log ($a^m : u^m$).

Which two propositions were to be shown.

PROPOSITION LXX. PROBLEM.

563. *To show the general canon of the rectilinear motion derived from fundamental uniform motion, by which a body may be carried forwards in air of variable density, and according to some law of the resistance.*

With these things in place, which have been said for Proposition LV. of this book, the distance may be called NE on the carrying line traversed *x*, the time in which it is resolved *t*, the residual speed of the body DB after this time lapse *u*; and thus the speed DE of the body acquired on the carrying line in this time will be $a-u$, if, as we suppose *a* to signify the initial speed of the body AN, by which clearly the carrying line progresses in air in its equable motion (according to the hypothesis) : The absolute distance, which the body is send through in air in the time mentioned *t*, is S. The action of the acceleration, at some point of the carrying line E, or with the right line EF generally indicating the resistance of the air R ; and finally the density of the air Δ. And with these denominations made, the following formulas of paragraphs 489 and 488 are present:



I. $R.dx = udu - adu$. II. $t = f - du : R$. III. & $S = at - x$.

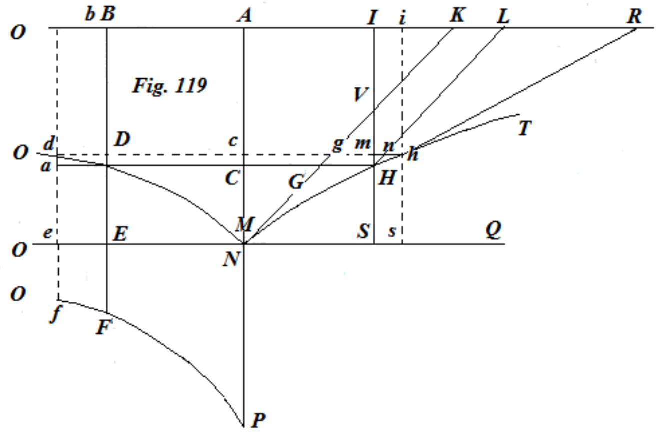
Which were required to be shown.

EXAMPLE.

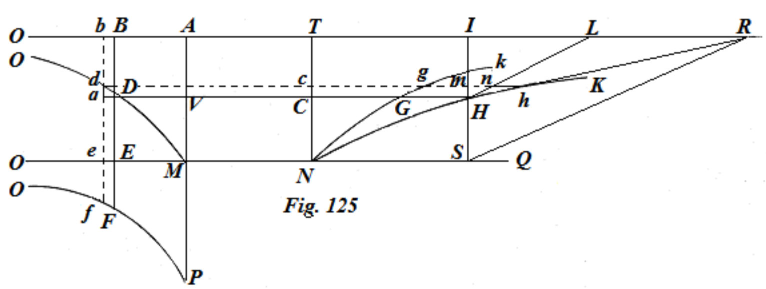
564. Generally there shall be $R = \left(abu^{m} + cu^{m+1}\right).\Delta : a^{m+1}$, where *a, b, c* are constant magnitudes; and *m* is some rational or irrational number. The first general formula, with reductions made, gives $\Delta dx = \left(a^{m+1}udu - a^{m+2}du\right) : abu^{m} + cu^{m+1}$. From which it is apparent *x* to be found by the quadratures of the two curves.

565. If $m = 1$, it becomes $R = (abu + cuu).\Delta : aa$, & $\Delta dx = \left(aaudu - a^{3}du\right) : (abu + cuu)$.

566. Therefore, if in addition $c=0$, there will be $R=bu\Delta:a$, that is, the resistances of the air are in a ratio composed from the density of the air, and velocity of the actual moving body, and the other equation in §. 565, will be changed into $\Delta dx = adu:b, -aadu:bu$. And (§.562) the sum of all $adu:b=(au-aa):b$, and $\int -aadu:bu$ (§.562.) $=\frac{aa}{b}\log.(a:u)$ in the logarithmic curve of which the subtangent is 1, or $=\frac{a}{b}\log.(a:u)$ in the logarithmic curve, of which the subtangent is *a*. In addition, $\int \Delta dx = \frac{a}{b}\log.(a:u)-(aa-au):b.$ Hence, if $\Delta=1$, & $b=a$, there will be $\int \Delta dx = x = \log.(a:u)-a+u$. And thus, if in Fig. 119. $\text{AN}=\text{AK}=a=b$, $\text{BD}=\text{IH}=u$, and $\text{NE}=x$, there will be $\text{CG}:\text{CN}=a-u$, & $\text{CH}=\log.(a:a)=\log.(\text{AN}:\text{IK})$, therefore *x*, or $\text{NE}=\text{CH}-\text{CG}=\text{GH}$, in short, as the construction has in Proposition LVIII. .



567. If in the equations above (§.565.) $b=0$, there will be had $\text{R}=cuu\Delta:aa$, that is, the resistances of the air are in a ratio composed from the density and of the square of the speed of the body, and $\Delta dx = aaudu - a^3du:cuu, = aadu:cu, -a^3du:cuu$. But truly (§.562.) there is $\int -aadu:cu = \frac{a}{c}\cdot\log.(a:u)$ in terms of the logarithmic curve, of which the subtangent $=a, \& \int -a^3du:cuu$ (§.88.) $=(a^3-aau):cu.$ Therefore $\int \Delta dx = a^3-aau:cu\ ;-\frac{a}{c}\log.(a:u)$; or on making $\Delta=1$, & $a=c$, there becomes $x:aa-auu:u,-\log.(a:u)$. Hence, if in fig. 125,



$\mathrm{AM} = \mathrm{AT} = a, \mathrm{BD} = \mathrm{IH} = u, \& \ \mathrm{ME} = x,$ there will be, on account of the hyperbola NHK, $(aa - au) : u = \mathrm{CH};\ \&\ \log.(a : u) = \log.(\mathrm{NT} : \mathrm{IH}) = \mathrm{CG},$ since NGK (constr.) shall be the logarithmic curve whose subtangent $= \mathrm{NT} = a;$ and thus $\mathrm{ME} = \mathrm{CH} - \mathrm{CG} = \mathrm{GH}$, again as the construction has of proposition LXIII.

568. Finally, if the equation of that paragraph 565; $\Delta dx = \left(aaudu - a^3du\right) : (abu + cuu)$ may be constructed, the part of the numerator $-a^3du$ may be divided by the part abu of the denominator, and with the calculations cancelled down, there will be $\Delta dx = \left(\frac{aa}{b} + \frac{aa}{c}\right).du : \ (\frac{ab}{c} + u\ ) - \frac{ab}{c}du : u.$ And the integrals of these can be found (§.562), it is evident that $\int \Delta dx = \log.(a : u) - \frac{b-c}{c}\log.(\frac{ab}{c} + a : \frac{ab}{c} + u)$ in the logarithmic curve, whose subagent is $\frac{ab}{c}$. Now, in figure 129 of proposition LXVI, if there may become $\mathrm{PW} : \mathrm{MP} = \mathrm{OM} : \mathrm{IM}$, & $\mathrm{PV} : \mathrm{MP} = \mathrm{AM} : \mathrm{IM}$, and there may be called $\mathrm{PW} = b,\ \mathrm{PV} = c, \& \ \mathrm{AM} = a, = \mathrm{OI};$ the indeterminates $\mathrm{QM} = \mathrm{OG} = a,$ and $\mathrm{ME} = x$. And there may be found :

$\frac{aa}{b} = \mathrm{IM.AM}^2 : \mathrm{OM.MP}(\text{constr.}) = \mathrm{RC}.$ For (§. 541. no. 1.) $\mathrm{RC} : \mathrm{AM} = \mathrm{IM.AM} : \mathrm{OM.MP}$, and therefore (constr.) $\mathrm{PW} : \mathrm{MP} = \mathrm{OM} : \mathrm{IM}$, and thus $\mathrm{OM.MP} = \mathrm{IM.PW}$, hence $\mathrm{RC} : \mathrm{AM}(= \mathrm{IM.AM} : \mathrm{OM.MP})$ $= \mathrm{IM.AM} : \mathrm{IM.PW} = \mathrm{AM} : \mathrm{PW},$ and therefore $\mathrm{RC} = \mathrm{AM}^2 : \mathrm{PW} = aa : b$, as we have said. Likewise $\frac{b+c}{c} = \frac{\mathrm{IM}}{\mathrm{AM}}$ & $\frac{ab}{c} = \mathrm{OM}$, and thus $\frac{ab}{c} + a = \mathrm{IM}, \&\ \frac{ab}{c} + u = \mathrm{GM}$; hence $\int \Delta dx = \log.(\mathrm{AM} : \mathrm{QM}) - \frac{IM}{AM} \cdot \log.(\mathrm{IM} : \mathrm{GM})$,

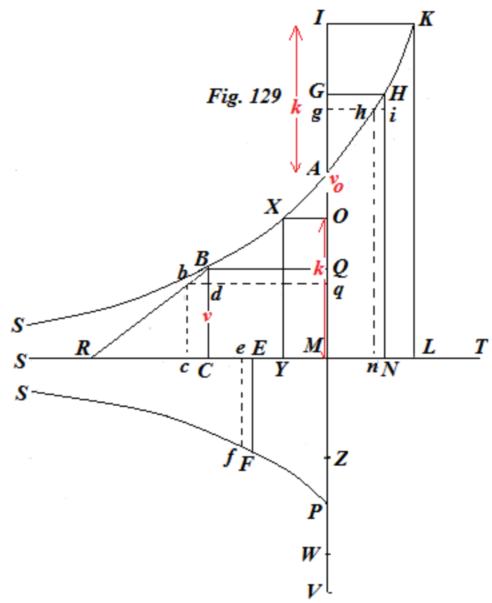


Fig. 129

and thus with $\Delta = 1, x = \mathrm{ME} = \log.(\mathrm{AM} : \mathrm{QM}) - \frac{IM}{AM}\log.(\mathrm{IM} : \mathrm{GM})$ exactly as the construction has of the recent proposition LXVI.

569. In order to find the time the following general formula is required to be consulted (§.563.) or $t = \int -du : \mathrm{R},$ or $dt = -du : \mathrm{R}$, or on substituting the value of R from §.565 there will be :

$$\Delta dt = -aadu : abu + cuu = -adu : bu; + \frac{acdu}{b} : ab + cu = -adu : bu, + \frac{a}{b}du : \frac{ab}{c} + u.$$

hence

$\int \Delta dt = \frac{1}{a}.\log.\left(a : u\right) - \frac{1}{a}\log.\frac{ab}{c} + a : \frac{ab}{c} + u\right) = \frac{1}{AM} \cdot \log.\left(\mathrm{AM} : \mathrm{QM}\right) - \frac{1}{AM} \cdot \log.(\mathrm{IM} : \mathrm{GM})$ just $= \left(\mathrm{MC} - \mathrm{LN}\right) : \mathrm{AM};$

as now shown in the aforementioned proposition LXVI, where yet Δ is 1, and thus $\int \Delta dt = t.$

And these few particular examples may be sufficient for the illustration of the general formulas.

PROPOSITION LXXI. PROBLEM.

570. *To show the general formulas for the vertical descent and ascent of a weight with resistance in air with a variable density and with the resistance following some law.*

With the symbols retained above (§.§. 484,485.) now being used : Proposition LIV supplies the formulas for us for each case of the weight descending and ascending, to wit,

$$\text{I. } x = \int udu : g \mp r. \& \text{ II. } t = \int du : g \mp r.$$

Where the upper sign is with respect to the descent and the lower one the descent of the body. In these formulas *g* indicates the weight [if the mass is taken as unity, then this becomes the familiar acceleration of gravity], *r* the resistance of the medium, *u* the velocities of the moving body acquired or remaining, and *x* the distance traversed by descending from rest ; or the complements of the distances of ascent now actually completed for the maximum height A, which the weights with a certain initial velocity prevail to traverse, and finally *t* denoting the times of the descents through the distances *x*, or with which the complements of the aforementioned are able to rise to the maximum heights described. Therefore the distances actually traversed by the moving body ascending will be $\mathrm{A} - x$, and the times, by which these distances are resolved $\mathrm{T} - t$, with T put in place for the times, in which the maximum height A is being completed. Which were required to be shown.

EXAMPLE.

57I. There shall be $r = \left(abu + cuu\right)\Delta : aa$, [where *a* is the initial speed of the body, *b* and *c* are constants of the resistance] and there becomes

$: x = \int aaudu : aag \mp ubu\Delta \mp cuu\Delta, \ \& \ t = \int aadu : aag \mp abu\Delta \mp cuu\Delta$. Just as these formulas are particular cases, yet they include particular special cases, both for the descent as well as for the ascent of the weight; clearly for the variation of the density Δ. We will consider the simplest case first, assuming $\Delta = 1$, and our formulas return to the case, which we have attended to in the immediately preceding chapters.

572. Therefore, as 1 may be substituted in place of Δ, 1, and by retaining the above signs for the descent of the weight, we will have $x=\int aaudu:aag-abu-cuu$, and the other formula $t=\int(aadu:aag-abu-cuu)$. So that these equations may be worked out, there may be put $aag=chh-cff$, likewise $ab=2cf$, and finaly $y=f+u$; where *a, b, c, f, g,* & *h* are constant quantities, and *x, y, u, t*, are variable quantities. With these substitutions put in place, there will be found :

$$aaudu:aag-abu-cuu=aaudu:chh-cff-2cfu-cuu$$
$$=\tfrac{aa}{c}udu:hh-(f+u)^2=\left(\tfrac{aa}{c}ydy-\tfrac{aaf}{c}dy\right):hh-yy.$$

And (§. 562.) all [*i.e.* the integral for the distance gone *x* in terms of the speed *y*] $\frac{aa}{c}ydy:hh-yy=\log.\sqrt{(hh-ff:hh-yy)}$ on the logarithmic curve, of which the subtangent is $\frac{aa}{c}$. For *ydy* is half of the element [*i.e.* derivative] of the element of the denominator $hh-yy$, or with the opposite sign, and $hh-yy$ is the maximum magnitude of all the $hh-yy$, because f is the minimum of all $y=f+u$, which will be obtained when u vanishes.

Again $\int\frac{aaf}{c}dy:hh-yy=$ (§.467.no. III.) $\frac{f}{b}\log.\sqrt{(h+y:h-y)}$, or rather, because in order for *u* itself to vanish also the whole integral must vanish, and with $u=0$, being present, or $y=f$, the integral will be changed into $\frac{f}{b}\log.\sqrt{(h+f:h-f)}$, which quantity is required to be taken form the first, and there will be had :

$$\int\tfrac{aaf}{c}dy:hh-yy=\tfrac{f}{b}\log.\sqrt{(h+y:h-y)}-\tfrac{f}{b}\log.\sqrt{(h+f:h-f)}$$

573. Likewise the other equation of paragraph 572 will be resolved easily, or $t=\int(aadu:aag-abu-cuu)$ (or with the same substitutions made as in the previous article) $=\int\frac{aaf}{c}dy:hh-yy$. Indeed this is equal to the integral found in the final section of the previous article, or by dividing by *f;* therefore there will be found :

$$t\left(=\int\tfrac{aaf}{c}dy:hh-yy\right)=\tfrac{1}{b}\log.\sqrt{(h+y:h-y)}-\tfrac{1}{b}\log.\sqrt{(h+f:h-f)}.$$

574. These equations found resolved most accurately lead us to the construction of Proposition LXVII. For there shall be $a=g=c$, and because (following the hypothesis of §.572.) $aag=chh-cff$, and $ab=2cf$, there will be $aa=hh-ff$, and $b=2f$. And thus if in Fig. 130. there becomes $IM=a$, $AM=\frac{1}{2}b=f$, and thus $TM=b$, and there will

be $IA^2 = aa + ff = hh$, and thus

$AI = AB = AL = H$, and by making $MS = u$, there will be $AS = f + u = y, \& \ LS^2 = hh - yy$, also $IM^2 = hh - ff = aa$, and thus

$$\log.\sqrt{(hh - ff : hh - yy)} = \log.\sqrt{\left(\mathrm{IM}^2 : \mathrm{LS}^2\right)} = \log.\left(\mathrm{lM} : \mathrm{LS}\right)$$
$$= \log.\left(\mathrm{GB} : \mathrm{NO}\right) = \mathrm{HO}.$$

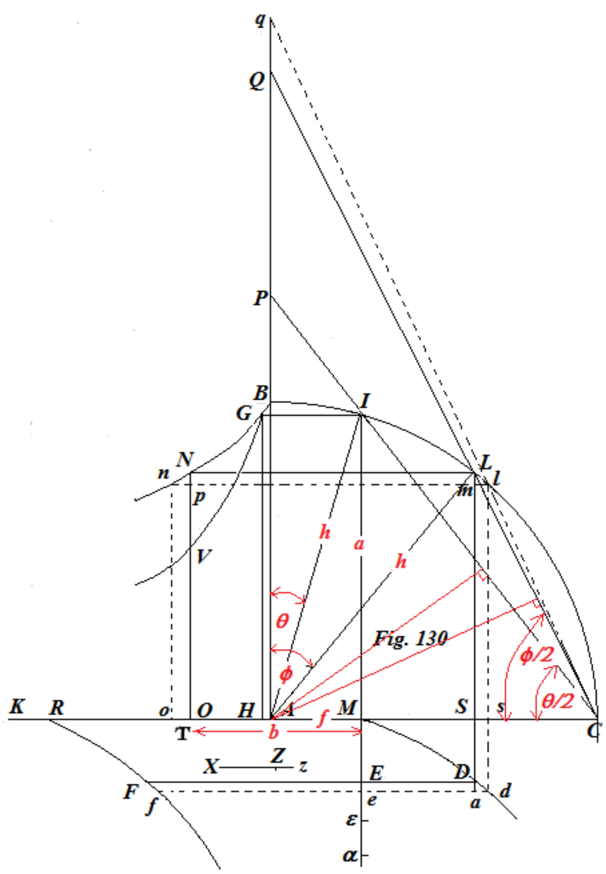
On the logarithmic curve BN, of which the subtangent is equal to $\frac{aa}{c}$ or *a*, because (following the hypothesis) $a = c = \mathrm{IM} = \mathrm{GH}$.

Moreover, if $\mathrm{AK} = \mathrm{AC} = \mathrm{AI} = h$, there will be $KS = h + y$, and $CS = h - y$; therefore the ratio $h + y : h - y = \mathrm{KS} : \mathrm{CS}$ (or, because KS, LS, & SC are in continued proportion on the circle) $= \mathrm{LS}^2 : \mathrm{SC}^2 = \mathrm{QA}^2 : \mathrm{AC}^2$;

and thus $\sqrt{(h + y : h - y)} = \mathrm{QA} : \mathrm{AC}$.

By a similar argument there is found

$\sqrt{(h + f : h - f)} = \mathrm{IM} : \mathrm{MC} = \mathrm{PA} : \mathrm{AC}$



And thus :

$$\tfrac{f}{b}\log.\sqrt{\left(h + y : h - y\right)}$$
$$-\tfrac{f}{b}\log.\sqrt{\left(h + f : h - f\right)}$$
$$= \tfrac{AM}{AI}\log.\left(\mathrm{QA} : \mathrm{AC}\right)$$
$$-\tfrac{AM}{AI}\log.(\mathrm{PA} : \mathrm{AC})$$
$$= \tfrac{AM}{AI}\log.(\mathrm{QA} : \mathrm{PA}).$$

And (§.549) it is assumed $\mathrm{XZ} : \log.\left(\mathrm{QA} : \mathrm{PA}\right) = \mathrm{AM} : \mathrm{AI}$, therefore $\mathrm{XZ} = AM : AI \cdot \log.\left(\mathrm{QA} : \mathrm{PA}\right)$, and thus

$$x = \log.\sqrt{(hh - ff : hh - yy)} - \tfrac{f}{b}\log.\sqrt{\left(h + y : h - y\right)} + \tfrac{f}{b}\log.\sqrt{\left(h + f : h - f\right)} = \mathrm{HO} - \mathrm{XZ},$$

as cited in Proposition LXVII.

575. Just as for the other equation (§.573.)

$t=\frac{1}{h}\log.\sqrt{(h+y:h-y)}-\frac{1}{h}\log.\sqrt{(h+f:h-f)}$, this time $t=\mathrm{XZ}:\mathrm{AM}$. Whenever

$$\frac{f}{b}\log.\sqrt{\left(h+y:h-y\right)}-\frac{f}{b}\log.\sqrt{\left(h+f:h-f\right)}=\frac{AM}{AI}\log.(\mathrm{QA}:\mathrm{PA})=\mathrm{XZ}$$

and thus

$$t=\frac{1}{b}\log.\sqrt{\left(h+y:h-y\right)}-\frac{1}{b}\log.\sqrt{\left(h+f:h-f\right)}=\mathrm{XZ}:f=\mathrm{XZ}:\mathrm{AM}.$$

As again is cited in the Proposition.

576. For the ascent of the weight the formulas of paragraph 571 are used with the lower signs, from which again Δ shall be $=1$, and there will be $x=\int aaudu:aag+abu+cuu$ (or, if now there may become $aag=chh+cff$, likewise $2cf=ab$, and $f+u$ equals y) $\int(\frac{aa}{c}ydy-\frac{aaf}{c}dy):hh+yy$. And
$\frac{aa}{c}ydy:hh+yy$ (§.562.) $=\log.\sqrt{(hh+yy:hh+ff)}$, because ydy is half the element of the denominator $hh+yy$, and $hh+ff$ is the minimum of all $hh+yy$, since f shall be the minimum of all $y=f+y$. Here the logarithm is taken on the logarithmic curve, of which the subtangent is $\frac{aa}{c}$.

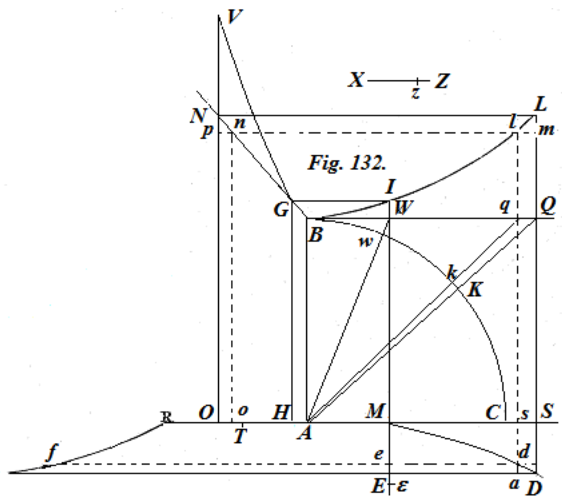
The other part of the equation, or $\int aafdy:hh+yy=\frac{aaf}{cbb}\int hhdy:hh+yy$. And (§. 166.) there is $hhdy:hh+yy=$ element of the circular arc, whose radius is h, and the tangent y, which element is called $d\omega$, and thus the arc itself is ω; from which $\int hhdy:hh+yy=\omega$, therefore $\int aafdy:hh+yy=\frac{aaf}{cbb}\cdot\omega$. Therefore each is had, by being taken together for the integral, as is fitting,

$$x=\int aaudu:aag+bau+cuu=\int(\frac{aa}{c}ydy-\frac{aaf}{c}dy):hh+yy$$
$$=\log.\sqrt{(hh+yy:hh+ff)}-\frac{aaf\omega}{cbb}.$$

577. The other equation regarding the time is :

$$t=\int aadu:aag+abu+cuu=\int\frac{aadu}{c}dy:hh+yy\left(\S.166.\right)=\frac{aa}{cbb}.\omega.$$

With these terms in place, which can be found near the end of the preceding paragraph.



578. These final determinations agree with proposition LXVIII proven ; for if, as above (§.574.) $a = g = c$, and thus $2f = b$, there will be $aa = hh + ff$. From which, Fig. 132, if on calling $\text{AW} = IM = a,\ \text{AM} = f = \frac{1}{2}TM = \frac{1}{2}b,$ there will be $\text{WM} = \text{BA} = h,$ and thus $\sqrt{(hh + ff)} = a = \text{MI},$ and because again $AS = f + u = y$, with $\text{MS} = u;$ there will be $\text{LS} = \sqrt{(hh + yy)}$ and thus :

$$\log.\sqrt{(hh + yy : hh + ff)} : \log.(\text{LS} : \text{IM}) = \log.(\text{NO} : \text{GH}) = \text{HO}.$$

Since the subtangent of the logarithmic $\frac{aa}{c}$ shall be $= a$, with there being $c = a$ (following the hypothesis).

And because $\text{AS} = \text{BQ} = y, \& \ \text{AB} = h$, there will be the arc $\text{BK} = \omega$, and thus $\frac{aaf}{chh}.\omega = \frac{af}{bb}\omega = \frac{WM.AM}{IM^2}$ by BK, but in proposition LXVII I. §.559. there is $\text{XZ} : \text{BK} = \text{WM.AM} : \text{IM}^2$ therefore $\frac{af\omega}{bb} = \text{XZ};$ and hence

$$x(= \log.\sqrt{(hh + yy : hh + ff)} - \frac{aaf\omega}{cbb} = \text{HO} - \text{XZ}.$$

as mentioned in the said proposition. Also

$t = \frac{aac}{bb}\omega,$ will be $= \text{XZ} : \text{AM}$, as shown equally in the said proposition shown.

SCHOLION.

579. I do not advance more examples for the general illustration of the canon, and many more I pass over in silence, which could be elicited by a solved example. Yet it would seem to require the least silence, which could be effected by changing the density of the medium, so that the weights, following some law of the acceleration and retardation, may be able to descent or ascend along vertical lines. Galileo first showed long ago, the speeds of a weight falling in a vacuum to be acquired in the square root ratio of the

distances, which has also been shown in the preceding (§.150) , but from principles different from those of Galileo. This law of the acceleration at this stage was believed only for bodies falling in vacuo, but, as it will be proven soon, since also perversely, weights will be able to fall in air with resistance following the same law of resistance, provided the density of the air may be varied everywhere according to a certain ratio.

580. So that this law may be treated generally, there shall be now $r = (a^{m+n} + abu^m + cu^{m+1}).\Delta : a^{m+2}$, thus so that the first formula shall be, §.570., $x = \int udu : g \mp r$, or by being differentiated $(g \mp r).\ dx = udu$, now there shall be

$$(a^{m+2}g \mp + a^{m+2}\Delta \ \mp abu^m\Delta \mp cu^{m+1}\Delta).dx = a^{m+2}udu,$$

putting

$$a^{m+2}g \mp + a^{m+2}\Delta \ \mp abu^m\Delta \mp cu^{m+1}\Delta = \tfrac{1}{2}a^{m+2}e,$$

there will become

$\frac{1}{2}a^{m+2}edx = a^{m+2}udu,$ or $edx = 2udu,$ and by integrating $ex = uu,$ or $u = e^{\frac{1}{2}}x^{\frac{1}{2}},$ and $u^m = e^{m:2}x^{m:2}$, and also, $u^{m+1} = e^{m+1:2}x^{m+1:2},$ which values substituted into the equation $a^{m+2}g \mp \&c = \frac{1}{2}a^{m+2}e$ gives rise to this equation,

$$a^{m+2}g \mp a^{m+2}\Delta abe^{\frac{1}{2}m}x^{\frac{1}{2}m}\Delta \mp ce^{\frac{1}{2}m-\frac{1}{2}} = \tfrac{1}{2}a^{m+2}e.$$

Which equation declares the relation between the density of the air and the distances traversed, or between the indeterminates Δ and x, according to that, as the weight may be carried by the same law of acceleration both in this air and in a vacuum ; and indeed this acceleration will be shown by the standard acceleration $uu : ex,$ or $u = \sqrt{ex}$.
Now, since *e* shall be a constant quantity it is apparent the speeds acquired (*u*) to be as $\left(\sqrt{x}\right)$ that is, in the square root ratio of the distances traversed. On this account it is not necessary, that the gravity (g) shall be constant or uniform, even now the same outcome will prevail, provided *g* shall be given by *x* and constants.

We will not tarry with particular examples of these cases, by which the air with weight may resist in a ratio composed from the densities and velocities, or in a ratio composed from the ratio of the densities and the squares of the speeds, or also following the law of the resistance, which arises from each or the mentioned magnitudes, indeed for this is expressed at once by putting $m = 1$ & $c = 0$ into the first , or $b = 0,$ into the second equation, or truly in the third case by retaining each *b, c,* &c., thus just as many different equations will result, as these diverse cases require.

CAPUT XVIII

Methodus inveniendi *symptomata, motuum corporum in mediis utlibet resistentibus, atque densitate pro libitu variantibus.*

In præcedentibus capitibus celebriores resistentiarum aëris hypotheses excussimus, & quinam motus, iisdem hypothesibus positis, nasci debeant, definivimus, supponentes tamen medium resistens ejusdem ubique densitatis esse. Quia vero densitates subinde variare possunt, canones motuum etiam exhibendi sunt pro ejusmodi mediis densitate variantibus, quod in hoc capite respectu motuum rectilineorum præstabimus, de curvilineis in sequentibus duobus capitibus acturi.

PROPOSITIO LXIX. LEMMA.

562. *Exhibens nonnullas elementorum logarithmicorum proprietates, scilicet,*
1^{o}. *Elementum log-micum* positivum, *est elementum log-mi rationis majoris inæqualitatis, quam magnitudo quaecunque variabilis habet ad sum minimam magnitudinem.*
Elementum vero log-micum privativum *est elementum log-mi rationis majoris itidem inæqualitatis, quam* maxima magnitudo-alicujus variabilis *habet ad hanc* variabilem.
2°. Multiplum *quodcunque vel* submultiplam *elementi cujuscunque magnitudinis variabilis applicatum ad ipsam magnitudinem, cujus est elementum, est elementum log-mi rationis* multiplicatæ *vel* submultipli *rationis, quam variabilis magnitudinis habet ad ipsam variabilem, aut quam* maxima *magnitudinis variabilis habet ad ipsam variabilem, prout elementum log-micum positivum aut privativum fuerit.*

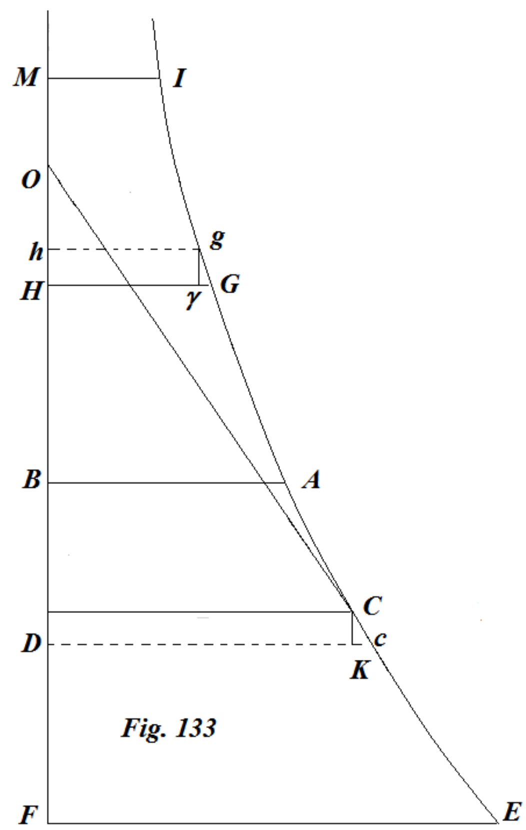


Fig. 133

Esto 1^{o}. quantitas variabilis *u*, ejusque *minima* magnitudo *a,* elementum log-micum $+du:u$ erit elementum log-mi rationis *(u:a).* Sin vero magnitudo *u*, cum *maxima* est, sit $=a$; erit $-du:u=$ elem. rationis $(a:u)$.

2°. Iisdem positis, si habeatur $mdu:u$ erit hoc elementam log-mi rationis *(u:a)* multiplicatæ juxta *m,* id est, $mdu:u=$ elem. rationis $(u^m:a^m)$ ubi *m* significat quemlibet numerum integrum vel fractum, rationalem aut irrationalem, &c. & $-mdu:u=$ elem. rationis $(a^m:u^m)$ in log-mica, cujus subtangens et unitas.

Sit IAE log-mica circa axem MF, cujus subtangens DO æquetur unitati 1, sintque CD magnitudo variabilis ac continue crescens ejusque *minima* magnitudo AB, ordinata vero variabilis GH sit magnitudo continue decresccns, ejusque *maxima* etiam AB; ponantur *cd* & *gh* ordinatæ ipsis CD & GH indefinite propinquæ, ductisque $\text{C}k, g\gamma$, axi MF parallelis, erunt $c\chi$ incrementum ipsius CD, adeoque $c\chi$ erit positivum ; ipsum vero $G\gamma$ erit decrementum ordinatæ GH, atque adeo privativum. Hisce positis,

I. Quia (§.491.) $+c\chi:\text{CD}=\text{D}d:\text{OD}$ (aut quia subtangens OD æquivalet unitati) $=\text{D}d$, & quia D*d* est elementum ipsius BD seu log-mi $(\text{CD}:\text{AB})$, erit omnino $c\chi:\text{CD}=\text{elem. log}-\text{mi rationis }(\text{CD}:\text{AB})$ hoc est $+du:u=\text{elem. log}-\text{mi rat.}(u:a)$, positis AB,a; CD,u & $c\chi,du$.

Eodem argumento probatur esse $-\text{G}\gamma:\text{GH}=$ elem. *log-mi* rationis $\text{AB}:\text{GH}$, hoc est $du:u,=\text{elem.log.}(a:u)$ seu log-mi (AB:GH), vocatus nunc GH,u; ac $\text{G}\gamma,du$ existente AB, etiam nunc *a*.

II. Ostendendum, esse $m.c\chi:\text{CD}=\text{elemento log}-\text{mi}(\text{CD}^m:\text{AB}^m)$. Sit enim $\text{BD}:\text{BF}=1:m$, atque adeo $\text{BF}=m.\text{BD}$, adeo ut excitata in F ordiinata FE, ex natura log-micæ sit $\text{CD}^m:\text{AB}^m=\text{EF}:\text{AB}.$ Verum $m.c\chi:\text{CD}=m.\text{D}d:\text{DO},$
& $\text{omn.}m.c\chi.\text{CD}=\text{omnibus }m.\text{D}d.\text{DO}=m.\text{BD}:\text{DO}$ (seu constr.) $\text{BF}:\text{DO}$
$(\text{vel si DO æquivalet unitati})=\text{BF}=\text{log.}(\text{EF}:\text{AB})=\text{log.}(\text{CD}^m:\text{AB}^m).$

Adeoque $mdu:u,=\text{elem. log.}(u^m:a^m).$

Nec dissimili ratione evincitur, quod, facta $\text{BM}=m.\text{BH}$ erectaque ordinata MI, futurum sit :

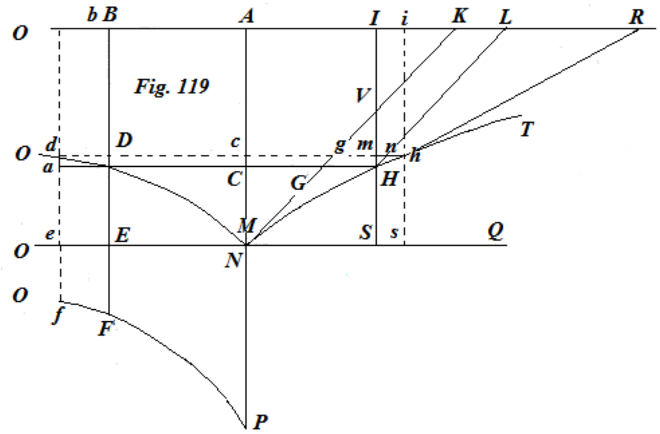
$$-m.\text{G}\gamma:\text{GH}=\text{elem. 1og}-\text{mi }(\text{AB}:\text{IM})=\text{elem.log.}(\text{AB}^m:\text{GH}^m)$$

$$\text{seu, quod idem est,} -mdu:u=\text{elemento log}-\text{mi }(a^m:u^m).$$

Quæ duo erant demonstranda.

PROPOSITIO LXX. PROBLEMA.

563. *Exhibere canonem generalem motuum rectilineorum ex primitive uniformibus derivatorum, quibus corpora in aëre densitatis variabilis, & juxta quamlibet legem resistente, ferantur.*



Positis iis, quæ ad propositionem LV. hujus Libri Secundi dicta sunt, vocetur spatium NE in linea deferenti percursum x, tempus, quo absolvitur t, celeritas mobili residua DB post effluxum hoc tempus *u*; adeoque celeritas DE mobilis in linea deferenti hoc tempore acquisita erit $a-u$, si, quod supponimus *a* significet velocitatem mobilis initialem AN, qua scilicet linea deferens æquabili motu (secundum hypothesin) in aëre progreditur: Spatium absolutum, quod mobile tempore memorata *t*, in aëre transmittit S. Solicitatio acceleratrix, in quolibet lineæ deferentis puncto E, seu recta EF resistentiam aëris generaliter indicans R; ac denique densitas aëris Δ. Hisce denominationibus factis, paragraphi 489. & 488. sequentes præbent formulas: I. I. $R.dx = udu - adu$. II. $t = f - du : R$. III. & $S = at - x$. Quae erant exhibendæ.

EXEMPLUM.

564. Sit generaliter $R = \left(abu^m + cu^{m+1}\right).\Delta : a^{m+1}$, ubi *a, b, c* sunt magnitudines constantes; & *m* quilibet numerus rationalis vel surdus. Formula prima generalis, reductionibus vel surdus, dat $\Delta dx = \left(a^{m+1}.udu - a^{m+2}du\right) : \ abu^m + cu^{m+1}$. Ex qua liquet abscissas *x* inveniri per quadraturas duarum curvarum.

565. Si $m = 1$, fiet $R = \left(abu + cuu\right).\Delta : aa, \&\ \Delta dx = \left(aaudu - a^3du\right) : \left(abu + cuu\right)$.

566. Igitur, si præterea $c = 0$, erit $R = bu\Delta : a$, hoc est, resistentiæ aëris erunt in composita ratione densitatum & velocitatum mobilis actualium, alteraque æquatio §. 565, mutabitur in $\Delta dx = adu : b, -aadu : bu$. Atqui. (§.562) summa omnium $adu : b = \left(au - aa\right) : b$, & $\int -aadu : \ bu$ (§.562.) $= \frac{aa}{b}\log.(a : u)$ in log-mica; cujus subtangens 1, vel $= \frac{a}{b}\log.(a : u)$ in log-mica, cujus subtangens est *a*. Propterea. $\int \Delta dx = \frac{a}{b}\log.\left(a : u\right) - \left(aa - au\right) : b$. Hinc, si $\Delta = 1$, $\& b = a$, erit

$\int \Delta dx = x = \log.(a:u) - a + u.$ Atque adeo, si in fig.119. $\mathrm{AN} = \mathrm{AK} = a = b$, $\mathrm{BD} = \mathrm{IH} = u$, ac $\mathrm{NE} = x$, erit $\mathrm{CG}:\mathrm{CN} = a - u, \& \ \mathrm{CH} = \log.(a:a) = \log.(\mathrm{AN}:\mathrm{IK})$, ergo *x*, seu $\mathrm{NE} = \mathrm{CH} - \mathrm{CG} = \mathrm{GH},$ prorsus ut constructio propositionis LVIII. habet.

567. Si in æquationibus superioribus (§.565.) $b = 0$, habebitur $\mathrm{R} = cuu\Delta : aa$, id est, resistentiæ aëris sunt in composita ratione densitatum & duplicatæ velocitatum mobilis, $\& \Delta dx = aaudu - a^3du : cuu, = aadu : cu, -a^3du : cuu.$ At vero (§.562.) est $\int -aadu : cu = \frac{a}{c} \cdot \log.(a:u)$ in log-mica, cujus subtangens$= a, \& \int -a^3du : cuu \ (§.88.) = (a^3 - aau) : cu.$ Ergo $\int \Delta dx = a^3aau : cu \ ; -\frac{a}{c} \log.(a:u)$; vel factis $A = 1$, & $a = c$; fiet $x : aa - auu : u, -\log.(a:u)..$ Hinc, si in fig. 125, $\mathrm{AM} = \mathrm{AT} = a, \mathrm{BD} = \mathrm{IH} = u, \& \ \mathrm{ME} = x,$ erit, propter hyperbolam NHK, $(aa - au) : u = \mathrm{CH};$ & $\log.(a:u) = \log.(\mathrm{NT}:\mathrm{IH}) = \mathrm{CG},$ cum NGK (constr.) sit log-mica, cujus subtangens $= \mathrm{NT} = a;$ adeoque $\mathrm{ME} = \mathrm{CH} - \mathrm{CG} = \mathrm{GH}$, iterum ut habet constructio propositionis LXIII.

568. Tandem, si æquatio ipsa paragraphi 565; $\Delta dx = (aaudu - a^3du) : (abu + cuu)$ construenda sit, dividatur numeratoris membrum $-a^3du$ per membrum *abu* denominatoris, reductisque reducendis, erit $\Delta dx = \left(\frac{aa}{b} + \frac{aa}{c}\right).du : \ (\frac{ab}{c} + u \ ) - \frac{ab}{c} du : u.$ Et horum integralia (§.562.) invenientur, scilicet $\int \Delta dx = \log.(a:u) - \frac{b-c}{c} \log.(\frac{ab}{c} + a : \frac{ab}{c} + u)$ in logarithmica, cujus subtangens est $\frac{ab}{c}$.
Jam, in figura 129. propositionis LXVI. si fiat $\mathrm{PW}:\mathrm{MP} = \mathrm{OM}:\mathrm{IM}$, & $\mathrm{PV}:\mathrm{MP} = \mathrm{AM}:\mathrm{IM}$, ac dicantur $\mathrm{PW} = b, \ \mathrm{PV} = c, \& \ \mathrm{AM} = a, = \mathrm{OI};$ indeterminatæ $\mathrm{QM} = \mathrm{OG} = a,$ac $\mathrm{ME} = x$.
Invenieturque $\frac{aa}{b} = \mathrm{IM.AM}^2 : \mathrm{OM.MP}(\mathrm{constr.}) = \mathrm{RC}.$ Nam (§. 541. num. 1.) $\mathrm{RC}:\mathrm{AM} = \mathrm{IM.AM}:\mathrm{OM.MP}$, atqui propter (constr.) $\mathrm{PW}:\mathrm{MP} = \mathrm{OM}:\mathrm{IM}$, atque adeo $\mathrm{OM.MP} = \mathrm{IM.PW}$, ergo $\mathrm{RC}:\mathrm{AM}(= \mathrm{IM.AM}:\mathrm{OM.MP}) = \mathrm{IM.AM}:\mathrm{IM.PW} = \mathrm{AM}:\mathrm{PW}$, ac proinde $\mathrm{RC} = \mathrm{AM}^2 : \mathrm{PW} = aa : b$, ut dicebatur. Item $\frac{b+c}{c} = \frac{\mathrm{IM}}{\mathrm{AM}}$ & $\frac{ab}{c} = \mathrm{OM}$, adeoque $\frac{ab}{c} + a = \mathrm{IM}, \& \ \frac{ab}{c} + u = \mathrm{GM}$; hinc $\int \Delta dx = \log.(\mathrm{AM}:\mathrm{QM}) - \frac{IM}{AM} \cdot \log.(\mathrm{IM}:\mathrm{GM})$, atque adeo existente $\Delta = 1, x = \mathrm{ME} = \log.(\mathrm{AM}:\mathrm{QM}) - \frac{IM}{AM} \log.(\mathrm{IM}:\mathrm{GM})$ prorsus ut habet constructio recensitæ propositionis LXVI.

569. Pro inventione temporis consulenda est secunda formula generalis (§.563.) seu $t = \int -du : \mathrm{R},$ vel $dt = -du : \mathrm{R}$, vel substituendo ex §. 565. valorem ipsius R erit

$$\Delta dt = -aadu : abu + cuu = -adu : bu; + \tfrac{acdu}{b} : ab + cu = -adu : bu, + \tfrac{a}{b} du : \tfrac{ab}{c} + u.$$

hinc

$$\int \Delta dt = \tfrac{1}{a}.\log.\left(a : u\right) - \tfrac{1}{a}\log.\tfrac{ab}{c} + a : \tfrac{ab}{c} + u) = \tfrac{1}{AM} \cdot \log.\left(\mathrm{AM} : \mathrm{QM}\right) - \tfrac{1}{AM} \cdot \log.(\mathrm{IM} : \mathrm{GM});$$
$$= \left(\mathrm{MC} - \mathrm{LN}\right) : \mathrm{AM}$$

prorsus ut in præmemorata propositione LXVI. jam ostensum, ubi tamen Δ est 1, atque adeo $\int \Delta dt = t.$

Atque hæc pauca cxempla particularia ad illustrationem formularum generalium sufficiant.

PROPOSITIO LXX I. PROBLEMA.

570. *Exhibere generales formulas pro descensu & ascensu gravium resistento & horizonti perpediculari in aëre densitatis variabilis & secundum quamcunque legem resistente.*

Retentis symbolis supra (§.§. 484,485.) jam adhibitis : propositio LIV nobis suppeditat formulas pro utroque casu descensus ascensusque gravium; scilicet,

$$\text{I. } x = \int udu : g \mp r. \& \text{ II. } t = \int du : g \mp r.$$

Ubi signum superius respicit descensum & inferius descensum corporum. In hisce formulis *g* significat gravitatem, *r* resistentiam medii, *u* velocitates mobili acquisitas vel residuas, & *x* spatia a quiete descendendo percursa; vel complementa spatiorum ascensu jam actu confectorum ad maximam altitudinem *A*, quam gravia certa quodam velocitate initiali ascendentia percurrere valent, ac denique *t* denotant tempora descensus in spatiis *x*, vel quibus complementa præmemorata ad maximam altitudinem describi possunt. Idcirco spatia actu mobili ascendendo percursa erunt $\mathrm{A} - x$, & tempora, quo hæc spatia absolvuntur $\mathrm{T} - t$, posita T pro tempore, quo maxima altitudio A conficitur. Quæ erant cxhibenda.

EXEMPLUM.

57I. Sit $r = \left(abu + cuu\right)\Delta : aa$, eritque

$x = \int aaudu : aag \mp ubu\Delta \mp cuu\Delta,$ & $t = \int aadu : aag \mp abu\Delta \mp cuu\Delta$. Utut hæ formulæ particulares sint, infinitos tamen casus particulares complectuntur, tam pro descensu quam pro ascensu gravium; scilicet pro varietate densitatum Δ. Nos primum simplicissimum casum considerabimus, supposentes $\Delta = 1$, & formulæ nostræ redigentur ad casus, quos in capitibtis proxime antecedentibus jam excussimus.

572. Propterca, substituatur tantum loco Δ, 1, & retinendo signa superiora pro descensu gravium , & habebimus $x = \int aaudu : aag - abu - cuu$, alteramque $-t = \int (aadu : aag - abu - cuu).$ Ut hæ æquationes construi queant, ponantur

$aag = chh - cff$, item $ab = 2cf$, denique $y = f + u$; ubi *a, b, c, f, g,* & *h* sunt quantites constantes, ac *x, y, u,* t, variabiles. Factis hisce substitutionibus reperietur,

$$aaudu : aag - abu - cuu = aaudu : chh - cff - 2cfu - cuu$$
$$= \tfrac{aa}{c} udu : hh - (f+u)^2 = \left(\tfrac{aa}{c} ydy - \tfrac{aaf}{c} dy\right) : hh - yy.$$

Atqui (§. 562.) omnia $\frac{aa}{c} ydy : hh - yy = \log.\sqrt{(hh - ff : hh - yy)}$ in logarithmica, cujus subtangens est $\frac{aa}{c}$. Nam *ydy* est dimidium elementi denominatoris $hh - yy$, sed sub signo contrario,& $hh - yy$ est maxima magnitudo ommum $hh - yy$, quia *f* est minima omnium $y = f + u$, quæ habetur ubi *u* evanuit.
Porro $\int \frac{aaf}{c} dy : hh - yy =$ (§.467.num. III.) $\frac{f}{b} \log.\sqrt{(h + y : h - y)}$, vel potius, quia evanescere *u* ipsum etiam integrale evanescere debet, & existente $u = 0$, seu $y = f$, invectum integrale abit in $\frac{f}{b} \log.\sqrt{(h + f : h - f)}$, quæ quantitas detrahenda est a priore, & habebitur

$$\int \tfrac{aaf}{c} dy : hh - yy = \tfrac{f}{b} \log.\sqrt{(h + y : h - y)} - \tfrac{f}{b} \log.\sqrt{(h + f : h - f)}$$

573. Eadem facilitate resolvetur altera æquatio paragraphi 572, seu.
$t = \int (aadu : aag - abu - cuu)$ (seu factis iisdem substitutionibus, quæ in præcedenti articulo factæ sunt) $= \int \frac{aaf}{c} dy : hh - yy$. Hæc enim quantitas æquatur integrali invento in ultima periodo articuli præcedentis, sed diviso per *f;* propterea invenietur
$t\left(= \int \frac{aaf}{c} dy : hh - yy\right) = \frac{1}{b} \log.(h + y : h - y) - \frac{1}{b} \log.(h + f : h - f).$

574. Hæ repertæ æquationes reductæ accuratissime nos ducunt ad constructionem propositionis LXVII. Sint enim $a = g = c$, & quia (secundum hypothesin §.572.) $aag = chh - cff$, ac $ab = 2cf$, erunt $aa = hh - ff$, & $b = 2f$. Si itaque in fig. 130. fiant $IM = a$, $AM = \frac{1}{2} b = f$, adeoque $TM = b$, erit $IA^2 = aa + ff = hh$, atque adeo $AI = AB = AL = H$, factaque $MS = u$, erit $AS = f + u = y$, & $LS^2 = hh - yy$, nec non $IM^2 = hh - ff = aa$, adeoque

$$\log.\sqrt{(hh - ff : hh - yy)} = \log.\sqrt{\left(\mathrm{IM}^2 : \mathrm{LS}^2\right)} = \log.(\mathrm{IM} : \mathrm{LS})$$
$$= \log.(\mathrm{GB} : \mathrm{NO}) = \mathrm{HO}.$$

In log-mica BN, cujus fubtangens æquatur $\frac{aa}{c}$ seu *a,* quia (secundum hypothesin) $a = c = \mathrm{IM} = \mathrm{GH}$.

Præterea, si $\mathrm{AK}=\mathrm{AC}=\mathrm{AI}=h$, erit $KS=h+y$, & $CS=h-y$;
ergo ratio $h+y:h-y=\mathrm{KS}:\mathrm{CS}$ (vel, quia KS, LS, & SC in circulo sunt in continua ratione) $=\mathrm{LS}^2:\mathrm{SC}^2=\mathrm{QA}^2:\mathrm{AC}^2$; atque adeo $\sqrt{(h+y:h-y)}=\mathrm{QA:AC}$. Simillimo argumento invenitur $\sqrt{(h+f:h-f)}=\mathrm{IM}:\mathrm{MC}=\mathrm{PA}:\mathrm{AC}$.
Adeoque

$$\tfrac{f}{b}\log.\sqrt{(h+y:h-y)}-\tfrac{f}{b}\log.\sqrt{(h+f:h-f)}$$
$$=\tfrac{AM}{AI}\log.(\mathrm{QA}:\mathrm{AC})-\tfrac{AM}{AI}\log.(\mathrm{PA}:\mathrm{AC})=\tfrac{AM}{AI}\log.(\mathrm{QA}:\mathrm{PA}).$$

Atqui (§.549) assumta est $\mathrm{XZ}:\log.(\mathrm{QA}:\mathrm{PA})=\mathrm{AM}:\mathrm{AI}$, ergo
$\mathrm{XZ}=AM:AI\cdot\log.(\mathrm{QA}:\mathrm{PA})$, atque adeo

$$x=\log.\sqrt{(hh-ff:hh-yy)}-\tfrac{f}{b}\log.\sqrt{(h+y:h-y)}+\tfrac{f}{b}\log.\sqrt{(h+f:h-f)}=\mathrm{HO}-\mathrm{XZ},$$

ut in citata propositione LXVII.

575.Quantum ad alteram (§.573.) $t=\tfrac{1}{h}\log.\sqrt{(h+y:h-y)}-\tfrac{1}{h}\log.\sqrt{(h+f:h-f)}$, hæc $t=\mathrm{XZ}:\mathrm{AM}$. Quandoquidem

$$\tfrac{f}{b}\log.\sqrt{(h+y:h-y)}-\tfrac{f}{b}\log.\sqrt{(h+f:h-f)}=\tfrac{AM}{AI}\log.(\mathrm{QA}:\mathrm{PA})=\mathrm{XZ}$$

adeoque

$$t=\tfrac{1}{b}\log.\sqrt{(h+y:h-y)}-\tfrac{1}{b}\log.\sqrt{(h+f:h-f)}=\mathrm{XZ}:f=\mathrm{XZ}:\mathrm{AM}.$$

Iterum ut in citata propositione asseritur.

576. Pro ascensu gravium faciunt formulæ paragraphi 571. cum signis inferioribus, unde si iterum Δ sit $=1$, erit $x=\int aaudu:aag+abu+cuu$ (vel, si nunc fiat $aag=chh+cff$, item $2cf=ab$, & $f+u$ æquale y) $\int(\tfrac{aa}{c}ydy-\tfrac{aaf}{c}dy):hh+yy.$ Atqui $\tfrac{aa}{c}ydy:hh+yy$ (§.562.) $=\mathrm{Log}.\sqrt{(hh+yy:hh+ff)}$, quia ydy est dimidium elementum denominatoris $hh+yy,$ & $hh+ff$ est minima omnium $hh+yy$, cum *f* sit minima omnium $y=f+y$. Hic log-us sumitur in log-mica, cujus subtangens est $\tfrac{aa}{c}$.

Altera pars æquationis, seu $\int aafdy:hh+yy=\tfrac{aaf}{cbb}\int hhdy:hh+yy.$ Atqui (§. 166.) est $hhdy:hh+yy=$ elemento arcus circularis, cujus radius *h*, & tangens *y*, quod elementum

dicatur $d\omega$, atque adeo arcus ipse ω; unde $\int hhdy : hh + yy = \omega$, ergo
$\int aafdy : hh + yy = \frac{aaf}{cbb} \cdot \omega$. Ergo
utrumque integrale, ut decet, conjungendo, habetur

$$x = \int aaudu : aag + bau + cuu = \int (\tfrac{aa}{c} ydy - \tfrac{aaf}{c} dy) : hh + yy$$
$$= \log.\sqrt{(hh + yy : hh + ff)} - \frac{aaf\omega}{cbb}.$$

577. Altera æquatio tempus respiciens est

$$t = \int aadu : aag + abu + cuu = \int \frac{aadu}{c} dy : hh + yy \left(\S.166.\right) = \frac{aa}{cbb}.\omega.$$

Positis iis, quæ in periodo paragraphi proxime antecedentis ultimo.

578. Etiam hæ postremæ determinationes cum propositione LXVIII. probe conspirant; nam si, ut supra (§.574.) $a = g = c$, atque adeo $2f = b$, erit $aa = hh + ff$. Unde, si dicantur $\mathrm{AW} = IM = a,\ AM = f = \frac{1}{2}TM = \frac{1}{2}b,$ erit $WM = BA = h,$ adeoque $\sqrt{\left(hh + ff\right)} = a = \mathrm{MI},$ & quia iterum $AS = f + u = y$, existente $\mathrm{MS} = u;$ erit $\mathrm{LS} = \sqrt{\left(hh + yy\right)}$ adeoque

$$\log.\sqrt{\left(hh + yy : hh + ff\right)} : \log.\left(\mathrm{LS} : \mathrm{IM}\right) = \log.\left(\mathrm{NO} : \mathrm{GH}\right) = \mathrm{HO}.$$

Cum log-micæ subtangens $\frac{aa}{c}$ sit $= a$, existente (secundum hypothesin) $c = a$.
Et quia $\mathrm{AS} = BQ = y,$& $\mathrm{AB} = h$, erit arcus $\mathrm{B} = \omega$, atque adeo

$\frac{aaf}{chh}.\omega = \frac{af}{bb}\omega = \frac{WM.AM}{IM^2}$ in BK, sed in propositione LXVII I. §.559. est
$\mathrm{XZ} : \mathrm{BK} = \mathrm{WM.AM} : \mathrm{IM}^2$ ergo $\frac{af\omega}{bb} = \mathrm{XZ};$ ac proinde

$$x (= \log.\sqrt{\left(hh + yy : hh + ff\right)} - \frac{aaf\omega}{cbb} = \mathrm{HO} - \mathrm{XZ}.$$

ut in memorata propositione dictum.

$t = \frac{aac}{bb}\omega,$ erit $= \mathrm{XZ} : \mathrm{AM}$, pariter ut in dicta propositione ostensum.

SCHOLION.

579. Plura alia exempla ad illustrationem canonum generalium non affero, ac multa alia silentio prætereo, quæ ex soluto exemplo potuissent elici. Id tamen minime videtur subticendum, quod densitates medii varie modificando effici possit, ut gravia, juxta quamlibet accelerationis retardationisque legem, possint descendere vel ascendere in lineis horizonti perpendicularibus. Galilæus olim primus demonstravit, celeritates gravi in

vacuo cadenti acquisitas esse in subduplicata ratione spatiorum, quod etiam in antecedentibus (§. 150.*)* ostensum est, sed ex principiis a Galilæanis diversis. Hæc accelerationis lex solis corporibus in vacuo cadentibus convenire hactenus credebatur, sed, ut mox probabitur, perperam, cum etiam in aere resistente juxta hanc eandem accelerationis legem descendere possint gravia, modo aeris densitas certa quodam ratione ubique varietur.

580. Ut res generaliter tradatur, sit nunc $r = (a^{m+n} + abu^m + cu^{m+1}).\Delta : a^{m+2},$ adeo ut formula prima §.570. $x = \int udu : g \pm r$, vel differentiando $(g \mp r).\, dx = udu$, nunc futura sit

$$(a^{m+2}g \mp +a^{m+2}\Delta \ \mp abu^m \Delta \mp cu^{m+1}\Delta).dx = a^{m+2}udu,$$

fiat

$$a^{m+2}g \mp +a^{m+2}\Delta \ \mp abu^m \Delta \mp cu^{m+1}\Delta = \tfrac{1}{2}a^{m+2}e,$$

eritque

$\frac{1}{2}a^{m+2}edx = a^{m+2}udu,$ vel $edx = 2udu,$ ac integrando $ex = uu,$ seu $u = e^{\frac{1}{2}}x^{\frac{1}{2}},$ ac $u^m = e^{m:2}x^{m:2}$, nec non, $u^{m+1} = e^{m+1:2}x^{m+1:2},$ qui valores, in æquatione $a^{m+2}g \mp \&c = \frac{1}{2}a^{m+2}e$ substituti, præbent æquationem,

$$a^{m+2}g \mp a^{m+2}\Delta abe^{\frac{1}{2}m}x^{\frac{1}{2}m}\Delta \mp ce^{\frac{1}{2}m-\frac{1}{2}} = \tfrac{1}{2}a^{m+2}e.$$

Quæ æquatio relationem inter densitates aëris & spatia percursa, seu inter indeterminatas Δ & x, declarat, ad id, ut grave in hoc aëre eadem accelerationis lege feratur ac in vacuo ; etenim hæc accelerationis norma exhibetur æquatione $uu : ex,$ vel $u = \sqrt{ex}$.
Jam, cum *e* sit quantitas constans liquet celeritates acquisitas (*u*) esse ut $\left(\sqrt{x}\right)$ id est, in subduplicata ratione spatiorum percursorum. Ad hoc ne quidem necesse est, ut gravitas (g) sit constans seu uniformis, res etiamnum obtinebit, dummodo *g* datæ sint per *x* & constantes.

Non immorabimur particularibus examplis illorum casuum, quibus aër gravibus resistit in ratione composita densitatum & velocitatum, vel in composita ratione ex densitatum ratione & duplicata celeritatum, vel etiam juxta resistentiæ legem, quæ ex utraque memoratarum participat, ad hæc enim duntaxat in æquatione ponendus est exponens $m = 1$ & $c = 0$ primo , vel $b = 0,$ secundo, aut vero utramque *b, c,* &c. retinendo tertio casu, inde tot resultabunt diversæ æquationes, quot hi diversi casus exigunt.