CHAPTER XVII.

Concerning the motions of bodies with air resistance, partially in proportion to the speed of the body, and partially to the square of the speed of the same.

540. This hypothesis of the resistances agrees with these fluids, the parts of which appear to have the appearance of sticking together : and indeed, if we may consider such a fluid as the medium through which the body is moving, it will be apparent at once, for the parts of a viscous fluid of this kind required to be separated, that some force is to be used, which is different from that endured by the body due to the continual motion through the particles of the fluid ; and this separating force to be proportional to the extent of fluid to be traversed. Truly the amount of this fluid present being traversed is proportional to the velocity of the body. Therefore the absolute viscosity of the fluid depends on the speed of the body, or what amounts to the same, on the amount of matter requiring to be separated or passed through, which will be one part of the resistance, acting on the body in a fluid of this kind; the other kind of force arising from the fluid, is such as we have considered in the preceding chapter. Therefore the total resistance that a body will experience in a medium of this kind, is as the velocity for the given quantity of fluid passed through, together with the square of the same velocity. Therefore we will consider this said resistive force hence as a dead force and comparable to weight, as well as the resistances, which arise from the slipping past of the fluid for some body moving through it.
[These forces have been considered earlier in Book I; the viscous force is envisaged as one in which bonds between the particle are broken, and thus depend on the distance traversed in a given time through the fluid, or the velocity , while the dynamical forces are taken to depend on the square of the speed, both relative to the fluid, which may be at rest or moving in the lab. frame with a constant speed, as discussed previously. In analytical terms, these forces acting on a body of unit mass or accelerations, can be expressed in the forms : $\frac{dv}{dt} = v\frac{dv}{dx} = -kv - k'v^2 \rightarrow -kv - v^2$; hence, if the body has the initial speed v_0 in a fluid at rest in the ABSOLUTE or lab. frame, then the rate at which the body accelerates becomes

$\frac{dv}{dx} = -k - k'v$; Hence, $\frac{1}{k} \cdot \frac{dv}{1+\frac{k'}{k}v} = -dx$ and $x = \frac{1}{k'}\log\left(\frac{1+\frac{k'}{k}v_0}{1+\frac{k'}{k}v}\right) \to \log\left(\frac{k+v_0}{k+v}\right)$, if $k' = 1$;

or $e^{-k'x} = \frac{1+\frac{k'}{k}v}{1+\frac{k'}{k}v_0}$ to give $\left(1+\frac{k'}{k}v_0\right)e^{-k'x} = 1+\frac{k'}{k}v$ and $v = \frac{k}{k'}\left(\left(1+\frac{k'}{k}v_0\right)e^{-k'x} - 1\right)$.

If we choose $k' = 1$ then $v = k\left(\left(1+\frac{1}{k}v_0\right)e^{-x} - 1\right) = \left(k+v_0\right)e^{-x} - k$. Note that if the velocity v has become very small for large x, then the initial d.e. may become $v\frac{dv}{dx} = -kv$, and the velocity finally decreases to zero in a linear manner.

Again, retaining $k' = 1$, for the time we have : $\frac{dv}{dt} = -kv - v^2 = -v\left(k+v\right)$;

$\therefore \frac{dv}{v(k+v)} = \frac{1}{k}\frac{dv}{v} - \frac{1}{k}\frac{dv}{(k+v)} = -dt$, and $\log\frac{v}{v+k} - \log\frac{v_0}{v_0+k} = \log\frac{v}{v_0}\cdot\frac{v_0+k}{v+k} = -kt$; hence

$\frac{v}{v+k} = \frac{v_0}{v_0+k}e^{-kt}$, where the behaviour can again be investigated as above for near infinite values of the time.

On the other hand, if the body is placed in a fluid moving with the constant speed v_0 in the lab. frame, with the initial speed of the body zero, then the final asymptotic speed approaches close to v_0 in the lab. frame. Using the same initial d.e., with some sign modifications, and now with v the velocity of the body relative to the constant fluid velocity v_0 , we have in this RELATIVE frame:

$v\frac{dv}{dx} = -kv + k'v^2$; here the viscous force still retards the motion, while the dynamical force accelerates the body; Hence, the distance gone x relative to the fluid is found from

$\frac{1}{k} \cdot \frac{dv}{-1+\frac{k'}{k}v} = dx$, giving $\frac{k}{k'}\log\left(\frac{-k+k'v_0}{-k+k'v}\right) = x$; Hence we have :

$\frac{-k+k'v}{-k+k'v_0} = e^{-\frac{k'}{k}x}$ and $v = \left(-\frac{k}{k'}+v_0\right)e^{-\frac{k'}{k}x} + \frac{k}{k'} \to k$ as $x \to \infty$ and $k' = 1$; However, as the viscous force cannot vanish, a point is reached where the accelerating dynamical force just balances the viscous force, and a steady motion independent of distance results $v_f = \frac{k}{k'}$, smaller than the rate of fluid flow.

The time development follows from $\frac{dv}{dt} = -kv + k'v^2 = k'\left(\left(\frac{k}{2k'}\right)^2 - \frac{k}{k'}v + v^2\right) - k'\left(\frac{k}{2k'}\right)^2$

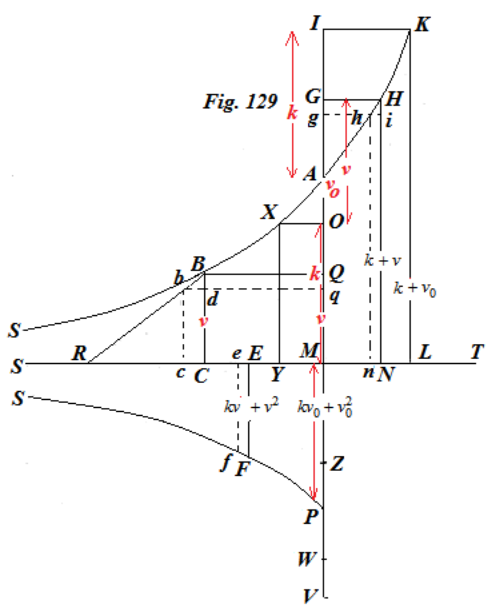
$= k'\left(\left(v - \frac{k}{2k'}\right)^2 - \left(\frac{k}{2k'}\right)^2\right)$; hence $\frac{dv}{\left(\left(v-\frac{k}{2k'}\right)^2 - \left(\frac{k}{2k'}\right)^2\right)} = \frac{1}{k'}dt$, or $\left(\frac{dv}{\left(\left(v-\frac{k}{2k'}\right)-\left(\frac{k}{2k'}\right)\right)} - \frac{dv}{\left(\left(v-\frac{k}{2k'}\right)+\left(\frac{k}{2k'}\right)\right)}\right) = \frac{k}{k'^2}dt$; hence

$\log\left(\frac{v-\frac{k}{k'}}{v}\right) = \frac{1}{k'}t$, and $\frac{v-\frac{k}{k'}}{v} = e^{\frac{t}{k'}} \to v - v_f = ve^t$ or $t = \log\left(\frac{v-v_f}{v}\right)$, where $v > v_f$;

now $\log\left(\frac{v_0-k}{v-k}\right) = \frac{x}{k}$, and $\frac{v-k}{v} = \frac{v_0-k}{v}e^{-\frac{x}{k}}$, then $t = \log\left(\frac{v_0-v_f}{v}\right) - \frac{x}{v_f}$]

PROPOSITION LXVI. THEOREM.

541. *If a body moving in air may be carried forwards with the resistance to the motion to be varied in the said ratio, departing from the case of elementary or equable motion ;* the distance, *which that will describe in some passage of the time, will be expressed by the logarithm of the ratio, which the said initial velocity added to the initial viscous drag, has to that equal viscosity added to the actual speed of the body, and from three proportions for the initial and final speeds and resistances of the body.* The time, *in which this distance is traversed, must be expressed by the excess, by which the logarithm of the ratio, which the initial speed has to the actual speed, surpasses the logarithm of the ratio of the same speeds, but increased by the drag of the air, applied to the initial and final speeds. Evidently with the logarithms put in place to be selected from that logarithmic curve, whose subtangent is found to the initial speed of the body, thus as the rectangle from the sum of the viscous drag of the fluid and of the initial velocity of the body by the initial velocity, to the rectangle formed from the said drag, and from the total resistance of the medium from the start of the motion.*



I. MS shall be the carrying line, which, as we have explained in proposition LVIII, shall be moving towards T with the constant speed AM [$= v_0$], which therefore is the initial speed, truly if the moving body M were to be considered to be carried by its own motion from M towards S, from the continual interaction with the air, and in a certain time it may have acquired the speed AQ on the conveying line, clearly after it has transversed the distance ME on that line, and thus with the speed AQ [$= v_0 - v$] taken from the initial speed AM, by which the carrying line MS proceeds at a constant speed towards T, therefore QM [$= v$] will be the absolute velocity of the body, that it actually has in air; again OM [$= k$] , or equally IA, shall be the force of the fluid arising from its viscosity ; likewise MP shall be put for the air resistance at the start of the motion [$= kv_0 + v_0^2$], and EF for the air resistance, if the actual velocity of the body were QM [$= v$] ; and thus there will be (following the hypothesis) EF to be as $\mathrm{OM.QM} + \mathrm{QM}^2$ [*i.e.* $kv + v^2$], or on making $\mathrm{OG} = \mathrm{MQ}$, as the rectangle GM.QM, and MP as the rectangle IM.AM [*i.e.* $kv_0 + v_0^2 = v_0(k + v_0)$], and thus there may be had

$$\mathrm{MP} : \mathrm{EF} = \mathrm{IM.AM} : \mathrm{GM.QM} = [\tfrac{(k+v_0)\cdot v_0}{(k+v)\cdot v}], \text{ and } \mathrm{MP.GM.QM} = \mathrm{EF.IM.AM}. \text{ Equation I.}$$

[Hermann considers the two situations as before, but which we realize in this case are not equivalent, as the viscous force always opposes motion, and thus there is a sign change;

he supposes on the one hand that the body starts with the speed v_0 at M going to the left and decelerates to rest in the static fluid, and on the other hand, the body starts from rest and accelerates to the right, carried by the fluid travelling to the right with speed v_0 to reach a final speed v_f, in which case the kinetic energy gained by the body is equal to the work done by the fluid. Rather than using energy conservation, unknown at the time, he proceeds from a hypothesis, which essentially allows the original force equations to be integrated; presumably he reasoned backwards from integrable force equations to the point where a geometric condition had to be satisfied, which was then adopted as a hypothesis, which is not very satisfactory to modern eyes : this is my present understanding of Hermann's reasoning. It has to be observed, as was the practice at the time, that methods were indicated in a veiled or incomplete manner, to prevent others from adopting the same and calling them of their own invention ; this is part of the difficulty in analyzing a work such as this. To return to the business at hand : Thus the two alternate methods are not complementary in this case, and we venture to suggest that Herman is incorrect here at least in part of his analysis ; the reader needs to refer to the following chapter 18, in which the analytical methods of producing these results are shown, whether we agree with the physics or not.]

The logarithmic curve HAB shall be drawn through the point A, of which any subtangent RC shall be to AM (following the hypothesis) as

$\text{IM.AM}:\text{OM.MP}$ [*i.e.* $\frac{\text{RC}}{v_0}=\frac{(k+v_0)\cdot v_0}{k\cdot\left(kv_0+v_0^2\right)}=\frac{1}{k}$ and $\text{RC}=\frac{v_0}{k}=Const.$]; and

thus $\text{RC.OM.MP}=\text{IM.AM}^2$. Equation II.

And through the points I, G, O, Q the right lines IK, GH, OX, & QB act as parallels of the asymptote ST of the logarithmic , crossing the logarithmic at the points K, H, X & B, through which the ordinates KL, HN , XY and BC shall be drawn *;* there becomes

$\text{ME}\left[\textit{i.e.}\ \log\left(\frac{kv_0+v_0^2}{kv+v^2}\right)\right]=\text{MC}+\text{LN}$ [$=\log\left(\frac{v_0}{v}\right)+\log\left(\frac{k+v_0}{k+v}\right)$],

II. *Demon.* Since (by construction) there shall be $\text{ME}=\text{MC}-\frac{IM.LN}{AM}$, there will be indeed $\text{E}e=\text{C}c-\frac{IM}{AM}.\text{N}n$, for E*e*, C*c*, & N*n* are elements of the lines ME, MC & LN. Consequently, since (constr.) $\text{OG}=\text{MQ}$ [$=v$], equally there will be $\text{G}g=\text{H}i=\text{Q}q=\text{B}d$ [$=dv$], and thus (§. 491. no. II.) the rectangle $\text{HN}n=$ rectangle $\text{BC}c$, [*i.e.* $(k+v)Nn=vCc$] and therefore

$\text{C}c:\text{N}n=\text{GM:QM}$ [*i.e.* $=\frac{k+v}{v}$], & $\text{Cc}:\frac{IM}{AM}.\text{N}n=\text{GM}:\frac{IM}{AM}.\text{QM}=\text{GM. AM}:\text{IM.QM}$, and [on taking 1 from each ratio and inverting] :

E*e* or $\text{C}c-\frac{IM}{AM}.\text{N}n:\text{C}c=\text{GM.AM}-\text{IM. QM}:\text{GM.AM}$ (or because OM.AQ is equal to $\text{GM.AM}-\text{IM.QM})=\text{OM.AQ}:\text{GM.AM}$, and on account of the similar triangles RCB and *bd*B there is, $\text{C}c:\text{Q}q=\text{RC}:\text{QM}$, therefore from the equation and on account of the composition $Ee:\text{Qq}=\text{RC.OM.AQ}:\text{GM.QM.AM}=\text{RC.OM.MP.AQ}:\text{MP.GM.QM.AM}$,

(or with the respective equal volumes substituted $\mathrm{IM.AM}^2$ & $\mathrm{EF.IM.AM}$ in place of RC.OM.MP and MP.GM.QM with the aid of Equ.II. & I.)
$=\mathrm{IM.AM}^2.\mathrm{AQ}:\mathrm{IM.AM}^2.\mathrm{EF}=\mathrm{AQ}:\mathrm{EF}$, and thus with the first ratio compared with the final, there will be found $\mathrm{E}e:\mathrm{Q}q=\mathrm{AQ}:\mathrm{EF},$ and thus $\mathrm{EF.E}e=\mathrm{AQ.Q}q$. That is, the *moment of the acceleration* EF *acting* on the carrying line MS is equal to the *moment of the speed acquired* AQ on this carrying line.
[This amounts to the differential of the work done by the body being equal to the differential of the loss in its kinetic energy, which would entail a negative sign in the absolute case, while a sign must be reversed in the relative case, as explained above.]

III. As before (no. II of this) there will be $\mathrm{C}c:\mathrm{N}n=\mathrm{GM}:\mathrm{QM}$, and on removing one from each side $\mathrm{C}c-\mathrm{N}n:\mathrm{C}c=\mathrm{OM}(\text{or GQ}):\mathrm{GM},$ & $\mathrm{C}c:\mathrm{Q}q=\mathrm{RC}:\mathrm{QM},$ therefore

$$\mathrm{C}c-\mathrm{N}n:\mathrm{Q}q=\mathrm{RC.OM}:\mathrm{GM.QM}=\mathrm{RC.OM.MP}:$$

$$\mathrm{MP.\ GM.QM}=\mathrm{IM.AM}^2:\mathrm{EF.IM.AM},$$

clearly with the equal volumes $\mathrm{IM.AM}^2$ and EF: IM. AM substituted in place of the volumes RC.OM.MP & MP.GM.QM , which first and second equalities show no. I. ; therefore $\mathrm{C}c-\mathrm{N}n:\mathrm{Q}q=\mathrm{IM.AM}^2:\mathrm{EF.IM.AM}=\mathrm{AM}:\mathrm{EF}$, and by interchanging $Cc-\mathrm{N}n:\mathrm{AM}=\mathrm{Q}q:\mathrm{EF}=\mathrm{E}e:\mathrm{AQ}=t\mathrm{E}e,$ therefore the sum of all the $t\mathrm{E}e$ or the time to traverse $\mathrm{ME}=(\mathrm{MC}-\mathrm{LN}):\mathrm{AM}.$ [The above true result.]

IV. Therefore, with this time designated, the carrying line MS will be described in the air with the velocity AM and by its motion to be equal to the distance $\mathrm{MC}-\mathrm{LN}$, truly by its own motion on this carrying line the body likewise will describe the distance $\mathrm{ME}=\mathrm{MC}-\frac{IM}{AM}.\mathrm{LN}$, and thus the absolute distance, which the body resolves in air, will be the excess of the distance from the common motion and to be equal to the carrying line above the motion of that in the opposite direction of the moving body on its own carrying line,
or $\mathrm{MC}-\mathrm{LN}-\mathrm{MC}+\frac{IM}{AM}.\mathrm{LN}=\mathrm{IM.LN}:\mathrm{AM}.$ Therefore here the absolute distance is expressed by the three proportionals from the initial speed AM, the resistance of the fluid OM or IA, and for the logarithm of the ratio, as KL & HN, that is, of the initial and actual speeds increased by the resistance of the body had in turn. [*i.e.* as $x=\log\left(\frac{k+v_0}{k+v}\right)$.] All of which have been shown.

COROLLARY I.

542. Therefore it is apparent in the presence of the hypothesis of resistances the body not only cannot depart to infinity, but cannot indeed reach the end of the finite distance SM.LY: AM in any magnitude of the time. For with QM vanishing, MC itself shall be infinite, GM becomes OM, and thus LN is changed then into LY, thus so that $\mathrm{MC}-\mathrm{LN}:\mathrm{AM}$ is changed into $\mathrm{MC}-\mathrm{LY}:\mathrm{AM}$ or into MC:AM, because the finite LY vanishes before the infinite MC. Therefore the distance $\mathrm{IM.LN}:\mathrm{AM}$ transversed in the

finites time $MC - LN : AM$, will be changed into IM.LY:AM, as only in an infinite time MC:AM, which cannot happen under any circumstances.

COROLLARY II.

543. If the distances $IM.LN : AM$ may be taken in an arithmetic progression, IM:AM will be of the same magnitude everywhere with LN removed, also will be in an arithmetic progression, and thus the ordinates of the logarithmic curve, HN or equally with these GM, or the actual velocities of the body QM increased by the viscosity of the medium OM or AI, will be in a geometric progression.

COROLLARY III.

544. Because $CM = \log.(AM : QM)$ & $LN = \log.(IM : GM)$ there will be

$$MC - LN = \log.(GM.AM : IM.QM) = \log.(AM.OM + AM.QM : IM.QM).$$

On the right line MP there may be taken MZ, which shall be to OM or $AI = AM : QM$, and there will be $QM.MZ = AM.OM$, and thus $AM.OM + AM.QM : IM.QM = AM + MZ$ or $AZ : IM$, and consequently $MC - LN : \log.(AM + MZ : IM)$, and $(MC - LN) : AM = \log.(AM + MZ : IM)$ applied to AM. And thus, if these $MC - LN : AM$, that is, the times are taken in an arithmetic progression, these $AM + MZ$, or the magnitudes MZ for the velocities QM in reciprocal proportions to the initial speed AM will themselves be in a geometric progression.

These two last corollaries agree closely with the propositions XI. & XII. Book II. *Princ. Phil. Nat. Math*. of the most celebrated Newton, and with these, which the celebrated has shown in the *Actis Acad. Scient*. 1707. 13th August, problem IV. coroll. 11. & 14.

SCHOLIUM.

545. I can see well enough, that it may be objected to me that excessive prolixity has been used in our treatment in this preposition, which would be able to be performed in a much shorter and easier manner, if for the line OM, with the viscosity of the fluid denoted, I could have assumed the right line AM, which expresses the initial velocity. But I respond that with this agreed on, the proposition would have become less general, even if I do not deny, that then it would become much shorter and easier to demonstrate. Now I would wish to introduce such, which also ought to contain the case of our proposition LXIII ; which actually happens, if with OM or AI vanishing, from the construction of the present proposition, all the properties determined from proposition LXIII arise, thus so that only that case or the cases of the corollaries shall be present. But, if for the line representing the viscosity of the fluid I may accept only the line AM, which expresses the initial velocity of the body, deduction from such may not be successful, because AM, since it expresses the initial velocity, never can vanish. But, because the

deduction of the above proposition LXIII. thus cannot be seen from the present, since in this place it is free to be explained more distinctly. If IA or OM vanishes, $AI.LN:AM$ is seen to vanish also ; or the distance travelled by the object in air with the resistance according to the squared proportion of the velocity to become nothing, then also $MC-LM:AM$. For in this case IM will turn into AM, and GM into QM, that is, the ratio IM:GM will become AM:QM, and thus LN the logarithm of this ratio will become MC, thus so that $MC-LN$ shall become $MC-MC=0$. Therefore how from the present construction of the proposition can the construction for proposition LXIII be elicited in the case of $AI=MO=0$?

546. Because in the said proposition (§.522.) the subtangent of the logarithmic curve expresses the initial velocity, the logarithm of the ratio IM:GM is taken on that logarithmic, and there will be (§.492.) $\log.(IM:GM):AM=LN:RC$, and thus

$LN=RC.\log.(IM:GM):AM$, & $IA.LN:AM=RC.IA.\log.(IM:GM):AM^2=RC.OM.$

$MP.\log.(IM:GM):AM^2.MP$. And (§.541. no.1. equat. I. & II.)

there is $RC.OM.MP=IM.AM^2$, therefore

$$IA.LN:AM=IM.AM^2.\log.(IM:GM):AM^2.MP=IM.\log.(IM:GM):MP,$$

and this is a new expression of the distance traversed in the air, whether IA shall be a real quantity or not. Now, if $IA=0$, the ratio will be $lM:GM=AM:QM$, clearly with vanishing $OM=IA$, and with there being $OG=QM$. Therefore the distance traversed in air will be in this case $AM.\log.(AM:QM):MP=\log.(AM:QM)\left[=\log\frac{v_0}{v}\right]$ in the case of proposition LXIII, where MP actually has been made equal to the initial speed AM. Therefore the absolute distance traversed by the body in air is expressed by the logarithm of the ratio, which the initial speed AM has to the actual or residual speed QM, precisely as this proposition has cited.

547. The determination of the time is a little harder to track down. This time is expressed by the present proposition by $(MC-LN):AM$. Now (§.544) there is

$$MC-LN=\log.(AM+MZ:IM)=\log.AM+MZ-\log.IM.\text{ And}$$

$$\log(AM+MZ)=(\tfrac{MZ}{MA}-\tfrac{MZ^2}{2MA^2}+\tfrac{MZ^3}{3MA^3}-\text{etc.})\text{ by }RC,$$

$$=(\frac{MZ.QM}{AM.QM}-\frac{MZ^2.QM^2}{2.AM^2.QM^2}+\frac{MZ^3.QM^3}{3.AM^3.QM^3}-\text{etc.}).RC,$$

because from the construction the rectangle MZ.QM is equal to the given rectangle AM.OM, there will be

$$\log(\mathrm{AM}+\mathrm{MZ})=\left(\frac{RC.OM.AM}{AM.QM}-\frac{RC.OM^2.AM^2}{2MA^2.QM^2}+\frac{RC.OM^3.AM^3}{3.MA^3.QM^3}-\text{etc.}\right)=$$
$$=\frac{RC.OM.MP}{MP.QM}-\frac{RC.OM^2.MP}{2MP.QM^2}+\frac{RC.OM^3.MP}{3.MA.QM^3}-\text{etc.}$$

(or because the volume RC.OM.MP is equal to the volume $\mathrm{IM.AM}^2$, as we have seen a number of times)

$$=\frac{IM.AM^2}{MP.QM}-\frac{IM.AM^2.OM}{2.MP.QM^2}+\frac{IM.AM^2.OM^3}{3.MP.QM^3}-\&c.$$

And in this last infinite series , on account of OM vanishing (following the hypothesis), all the terms vanish after the first term, therefore in this case there will be

$$\log.(\mathrm{AM}+\mathrm{MZ})=\frac{IM.AM^2}{MP.QM}.$$

Because $\mathrm{IM}=\mathrm{AM}+\mathrm{IA}=\mathrm{AM}+\mathrm{OM}$, there will be
$\log.\mathrm{IM}=\left(\frac{OM}{AM}-\frac{OM^2}{2.AM^2}+\frac{OM^3}{3.AM^3}-\&c.\right).\mathrm{RC}$, which is the same series as the first, with OM alone excepted, which in this case is put in place of MZ, in the first series. Therefore also :

$$\log.\mathrm{IM}=\frac{RC.OM.MP}{AM.MP}-\frac{RC.OM^2.MP}{2.AM^2.MP}+\&\text{c.}=\frac{IM.AM^2}{AM.MP}-\frac{IM.AM^2.OM}{2.AM^2.MP}+\&\text{c.}=\frac{IM.AP}{MP}-\frac{IM.OM}{2.MP}+\text{etc.}=\frac{IM.AM}{MP},$$

because also in this infinite series all the terms vanish after the first, with OM or $\mathrm{AI}=0$ (following the hypothesis) . Hence

$$\mathrm{MC}-\mathrm{LN}=\log.(\mathrm{AM}+\mathrm{MZ})-\log.(\mathrm{IM})=\frac{IM.AM^2}{MP.QM}-\frac{IM.AM}{MP}=\frac{IM.AM^2-IM.AM.QM}{MP.QM}$$
$$=\frac{IM.AM.QM}{MP.QM.}=\frac{AM^2.AQ}{MP.QM.},$$

because in the present hypothesis IM becomes AM with IA vanishing, and because in addition, in proposition LXIII, MP was equal to AM; there will be
$\mathrm{MC}-\mathrm{LN}=\frac{AM.AQ}{QM}$, & $\mathrm{MC}-\mathrm{LN}:\mathrm{AM}$, or the time, in which the body has resolved its own distance $\log.(\mathrm{AM}:\mathrm{QM})$ in air, shall be as $\mathrm{AQ}:\mathrm{QM}\left[=\frac{v_0}{v}\right]$, clearly being expressed by the ratio of the fraction of the initial speed sent off to that remaining, or the actual speed of the body in air, just as has been cited in the LXIII. §.522.

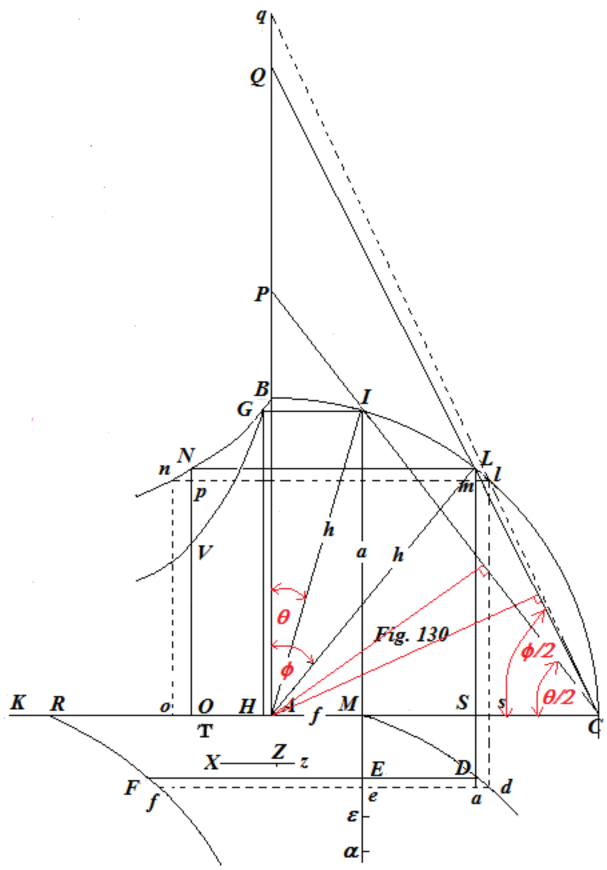
548. Because indeed the more general matter now can be as that, by supposing the resistance from the start of the motion representing MP to be different from the line AM, which expresses the initial speed; and also the absolute distance completed in air may be expressed in that case, as we have found above (§.546.), by

$\log.\mathrm{AM}.(\mathrm{AM}:\mathrm{QM}):\mathrm{MP}\left[=\log\frac{v_0}{v}\cdot\frac{v_0\cdot}{kv_0+v_0^2}=\log\frac{v_0}{v}\cdot\frac{1}{k+v_0}\right]$, and the time, in which that distance has been resolved, by [log.] $\mathrm{AM.AQ}:\mathrm{MP.QM}\left[=\log\frac{v_0\cdot(v_0-v)}{(kv_0+v_0^2)\cdot v}=\log\frac{v_0-v}{(k+v_0)\cdot v}\right]$.

Now indeed it might be better to mention the deeper propositions omitted, just as from that and from these, which follow at once in the same way as in chapter XVI, which equally can be deduced in an easy manner from these, and which will follow at once. But the diverse demonstrations and with the conclusions generally agreed on worthy of note, which may be established by the more observant reader.

PROPOSITION. LXVIL.
THEOREM.

549. *If uniform gravity may be expressed by the sine of the complement of some acute angle given, and a weight falls normally to the horizontal along a right line, together with a resistance, as indicated in the hypothesis in the title of this chapter ; and a certain magnitude may be assumed, which shall be to the logarithm of the ratio, as the tangent of half the complement of the given angle has to the tangent of half the complement of some variable and greater given angle, as the sine of the given angle to the whole sine. Clearly with the logarithms taken from the logarithmic curve, of which the subtangent is equal to the sine of the given complement of the before mentioned angle, with which in place.*



1st. The distance*, which the weight passes through in some time with an accelerated motion, is the excess of the logarithm of the ratio, which the sine of the complement of the given angle has to the sine of the variable angle, above the assumed magnitude.*

2nd. The time*, in which that distance is resolved, will be expressed by the assumed magnitude applied to the sine of the given right angle.*

3rd. The speed*, at the end of any acquired time, is, as the excess of the sine of the variable angle above the sine of the given right angle*.

[As previously, we present here a modern analysis of the problem, assuming the same kind of resistance, but in this case with the body dropped from rest under uniform gravity g :

$\frac{dv}{dt} = g - kv - k'v^2 \rightarrow g - kv - v^2$; Hence, $dt = \frac{dv}{g-kv-v^2} = \frac{dv}{g+\frac{k^2}{4}-\left(\frac{k}{2}+v\right)^2}$; note that

the terminal velocity v_t is found from $g - kv - v^2 = 0$, *i.e.* $v_t = -\frac{k}{2} + \sqrt{\frac{k^2}{4} + g}$;

$= \frac{dv}{K^2-\left(\frac{k}{2}+v\right)^2}$, where $K^2 = g + \frac{k^2}{4}$; hence $dt = \frac{1}{2K}\frac{dv}{K-\left(\frac{k}{2}+v\right)} + \frac{1}{2K}\frac{dv}{K+\left(\frac{k}{2}+v\right)}$, and

$t = -\frac{1}{2K}\log\left(K - v - \frac{k}{2}\right) + \frac{1}{2K}\log\left(K + v + \frac{k}{2}\right) = \frac{1}{2K}\log A \cdot \frac{\left(K+v+\frac{k}{2}\right)}{\left(K-v-\frac{k}{2}\right)}$, provided $K > v + \frac{k}{2}$.

When $t = 0,\ v = 0$, and $\frac{1}{2K}\log A \cdot \frac{\left(K+\frac{k}{2}\right)}{\left(K-\frac{k}{2}\right)} = 0$; hence $A = \frac{K-\frac{k}{2}}{K+\frac{k}{2}}$, and $t = \frac{1}{2K}\log\frac{K-\frac{k}{2}}{K+\frac{k}{2}} \cdot \frac{K+v+\frac{k}{2}}{K-v-\frac{k}{2}}$;

hence $e^{-2Kt} = \frac{K+\frac{k}{2}}{K-\frac{k}{2}} \cdot \frac{K-v-\frac{k}{2}}{K+v+\frac{k}{2}}$ and $\frac{K-v-\frac{k}{2}}{K+v+\frac{k}{2}} = \frac{K-\frac{k}{2}}{K+\frac{k}{2}} e^{-2Kt}$

giving $\left(K - v - \frac{k}{2}\right)\left(K + \frac{k}{2}\right) = \left(K + v + \frac{k}{2}\right)\left(K - \frac{k}{2}\right)e^{-2Kt}$;

$\left(K - v - \frac{k}{2}\right)\left(K + \frac{k}{2}\right) = \left(K + v + \frac{k}{2}\right)\left(K - \frac{k}{2}\right)e^{-2Kt}$; or $\frac{\left(K^2-\frac{k^2}{4}\right)\left(1-e^{-2Kt}\right)}{\left(K+\frac{k}{2}\right)+\left(K-\frac{k}{2}\right)e^{-2Kt}} = v$;

where $v_t = \frac{\left(K^2-\frac{k^2}{4}\right)}{\left(K+\frac{k}{2}\right)} = K - \frac{k}{2} = -\frac{k}{2} + \sqrt{g + \frac{k^2}{4}}$, as above.

Hence, $v = v_t \frac{\left(1-e^{-2Kt}\right)}{1+\left(\frac{K-\frac{k}{2}}{K+\frac{k}{2}}\right)e^{-2Kt}}$.

You may also want to look in Ch.18 at Hermann's analytical derivation.]

The weight falls from rest from M on the right line Mα perpendicular to the horizontal, and with the centre *A*, the quadrant of a circle described BLCA, and in that the right line IM may express the uniform gravity, which is the sine of the complement [$= R\cos\theta$] of a certain given angle BAI [θ] , and again some radius AL may be joined by the line SL drawn parallel to IM ; and there will be this angle BAL [ϕ], which is said to be variable in the proposition, of which the right line LS [$= R\cos\phi$] is the sine of the complement. The logarithmic curve BGN may be drawn about the axis MR, and which passes through the uppermost point of the quadrant B, and to which a line LN parallel to the radius CA shall cross at N, truly with IG parallel to the same MC at the point G, the ordinates GH and NO of the logarithmic pass through these two points ; then a certain XZ may be assumed, which shall be to the logarithm of the ratio QA:AP, as AM the sine of the given angle BAI to the radius AI. For with the right lines CP and CO drawn through the points I and L from the point C, QA: PA will be as the tangent of half the angle IAC to the tangent of the half of LAC, or as the tangent of half the complement of the given angle BAI to the tangent of half the complement of the variable angle BAL; for

$$AQ:AP=[AQ/R:AP/R=]\,\tan.LCA:\tan.ICA\;[=\text{cotan}.SLC:\text{cotan}.MIC]$$

$$=\tan.MIC:\tan.SLC\;[=\text{cotan}.\tfrac{1}{2}LAC:\text{cotan}.\tfrac{1}{2}IAC]=[\tan.\tfrac{\theta}{2}:\tan.\tfrac{\phi}{2}]$$

$$=\tan.\tfrac{1}{2}IAC:\tan.\tfrac{1}{2}LAC.$$

Truly HO itself, or the logarithm of the ratio GH:NO, or IM:LS, is the logarithm of the ratio, which IM, the sine of the complement of the given angle BAI has to the LS, the sine of the complement of the variable angle BAL; therefore there must be shown : 1st the distance described by the body falling ME, to be $=HO-XZ$; then 2nd the time, in which that distance is transversed, or $t\text{ME}=XZ:AM$, and finally MS the difference of the sines AS, AM of the variable and of the given angles, to be the speed acquired at E , thus so that from FE and LS produced, this point D shall be on the graph of the speed MD, clearly with FD drawn through the point E parallel to RS. In addition let RF be the graph of the acceleration acting, and with *dl*, *ln*, *no* and *df* themselves drawn parallel to DL, LN, NO and DF and in the indefinite vicinity and with LD produced to *a*; the moment of the acceleration acting FE must be equal to the moment of the speed acquired ED; or $EF.Ee=ED.ad$, with the above construction put in place. Making $AT=AM$, and VO the third proportional to GH and NO, thus so that hence a new logarithmic GV may emerge. Any subtangent of the logarithmic curve BN or GB will be equal to IM .

Demon. I. Let the resistance at E, which we will indicate by the simple letter R, as gravity by the letter G, to be as I say $G:R=IM^2:2AM.MS+MS^2$, or, because (by constr.) $AM=AT$, there may be put $G:R=IM^2:TS.MS$, and there becomes on re-arranging, $G-R:G$; that is, the acceleration acting at E to gravity [*i.e.* the weight of the body, of unit mass in modern terms], that is,

$EF:IM=IM^2-TS.\,MS:IM^2=IM.EF:IM^2$, and thus

$$IM.EF=IM^2-TS.MS=AI^2-AM^2-TS.MS$$

$$=AI^2-AM^2-2.AM.MS-MS^2=AL^2-AS^2=LS^2=NO^2,$$

or (because NO is the mean proportional between GH or IM & VO) $=IM.VO$, and thus $EF=VO$.

[See the next chapter for Hermann's analytical derivations of these results.]

II. Because (by const.) $ME=HO-XZ$, & $Me=Ho-Xz$ there will be also $Ee=Oo-Zz$; therefore if E*e* may be put $=Oo$, there will be $eo=Zz$, where Z*z* an element of the assumed line XZ.

III. On account of the similar triangles L*lm* & ALS there will be found S*s*:*rl* (or equally L*l*) $=$ LS:AL, & $rl{:}\mathrm{Q}q = rs(\text{or LS}):\mathrm{QA}$; therefore from the equation $\mathrm{S}s:\mathrm{Q}q = \mathrm{LS}^2:\mathrm{AL.AQ} = \mathrm{NO}^2:\mathrm{AL.AQ} = \mathrm{IM.VO}:\mathrm{AI.AQ}$. Truly Q*q*: element of the logarithm $(\mathrm{QA}:\mathrm{PA}) = \mathrm{QA}:$subtangent of the logarithmic IM $= \mathrm{QA.VO}:\mathrm{IM.VO}$, therefore from the equation S*s*: element of the logarithm $(\mathrm{QA}:\mathrm{RA}) = \mathrm{AQ.VO}:\mathrm{AI.AQ} = \mathrm{VO}:\mathrm{AI}$. And (by constr.) the element of the logarithm $(\mathrm{QA}:\mathrm{PA}):\mathrm{Z}z = \mathrm{Al}:\mathrm{AM}$, therefore again from the equation, $\mathrm{S}s:\mathrm{Z}z = \mathrm{VO}:\mathrm{AM}$. And thus $\mathrm{FE}.e\varepsilon$ (nos. I.& II. of this) $= \mathrm{VO}.\mathrm{Z}z:\mathrm{AM}.\ \mathrm{S}s$.

IV. Again $\mathrm{EF.E}\varepsilon$ (or $\mathrm{VO.O}o) = \mathrm{NO.N}p$ by §.491. no. III, $\mathrm{E}t.\mathrm{NO.N}p = \mathrm{LS.\ L}m$ (§.493.) $= \mathrm{AS.S}s$, hence, because (no. III of this) $\mathrm{FE}.e\varepsilon = \mathrm{AM.S}s$, there will be $\mathrm{FE.E}\varepsilon - \mathrm{FE}.e\varepsilon = \mathrm{AS.S}s - \mathrm{AM.S}s$, that is, $\mathrm{FE.E}e = \mathrm{MS.S}s$. That is, the *moment of the force* EF *is equal to the moment of the speed acquired* MS.

V. Because above (no. III of this) we have found $\mathrm{VO.Z}z = \mathrm{AM.S}s$, there becomes $\mathrm{Z}z:\mathrm{AM} = \mathrm{S}s:\mathrm{VO} = da:\mathrm{FE}$ (§.485.) $= t\mathrm{E}e$; and thus the sum of all the *tEe*, that is, the time of the perpendicular fall through the distance ME, which we have designated by the time tME, $=$ sum for all the Z*z*, from which XZ is composed, divided by AM, or $= \mathrm{XZ}:\mathrm{AM}$. Therefore the time through the distance ME is expressed for the assumed magnitude XZ applied to the sine AM of the given angle BAI. Which were all required to be shown.

COROLLARY I.

550. Since the speed acquired at E shall be ED or MS, the terminal speed will be the versed sine MC of the complement IAC of the given angle BAI, and the graph of the speed MD will have the asymptote parallel to Mα but passing through the point C .

COROLLARY II.

551. The distance, which the weight will resolve in a vacuum by falling in that time, in which it passes through the distance ME in air, is to the half of MI, by which the weight is expressed by the squared ratio taken of XZ to the given AM. For, because the time (§.151.) of descent in vacuo is expressed by the root of twice the distance pertaining to the magnitude, which represents uniform gravity, will be, by calling the distance S traversed in the time XZ:AM , I say will be

$$\sqrt{(2\mathrm{S}:\mathrm{IM})} = \mathrm{XZ}:\mathrm{AM}, \text{or } 2\mathrm{S}:\mathrm{IM} = \mathrm{S}:\tfrac{1}{2}\mathrm{IM} = \mathrm{XZ}^2:\mathrm{AM}^2.$$

COROLLARY III.

552. The velocity, which the moving body may acquire at the end E of the distance ME described in vacuo, will be to the speed acquired in air in traversing the same distance ME, as $\sqrt{(2ME.IM)}$ to MS, as is apparent from §.150.

COROLLARY IV.

553. Truly the speed acquired by the body in air will be to the speed acquired in vacuo in the time $XZ:AM$, as the rectangle AM.MS to the rectangle IM.XZ. For (§.551.) in vacuo the distance traversed $S = IM.XZ^2 : 2.AM^2$ in the time $XZ:AM$; and (§.150.) the velocity acquired at the end of this distance is
$\sqrt{(2.S.IM)} = \sqrt{\left(2.IM^2 : XZ^2 : 2.AM^2\right)} = IM.XZ:AM$; therefore the speed acquired in air is to the speed in vacuo, as MS to IM.XZ:AM, or as the rectangle AMS to the rectangle IM.XZ.

COROLLARY V.

555. Because the right line TA expresses the viscous drag of the medium on the falling body and AM half of this drag, this drag will be to the weight, as the sine of the given angle BAI to half the sine of the complement.

COROLLARY VI.

556. Therefore with the angle BAI vanishing the fluid will be devoid of all viscosity, and it will become a perfect fluid, thus so that it may be return to the case of a perfect fluid of proposition LXIV, which proposition hence can be presented as a corollary only. For in that case AM and XZ vanish, thus so that hence ME simply becomes equal to HO. Therefore the time to traverse ME, which in that proposition LXVII is
$XZ:AM(\text{constr.}) = \log.\ (QA:PA):AI$, in the case of the coincidence of BA and IM, becomes $XZ:AM = \log.\ (QA:BA):AB = t\text{ME}$; precisely as §.525 no. III. has shown.

SCHOLIUM I.

557. The geometrical proposition can also be extricated analytically in the following account. With these things in place, which were present in the preparation for the demonstration of the proposition, no. 1. there will be $FE \text{ or } VO = NO^2 : IM = LS^2 : IM$ (or on putting KC for the diameter of the circle BIC) $= KS.SC:IM$; thence, because generally there must be $EF.Ee = MS.\ Ss = ED.ad$, also there will be
$KS.CS.\ Ee:IM = MS, Ss$, and thus $Ee:IM = MS.Ss:KS.CS$. But

$\mathrm{MS} = \frac{1}{2}\mathrm{KS} - \frac{1}{2}\mathrm{KM} + \frac{1}{2}\mathrm{MC} - \frac{1}{2}\mathrm{SC}$, with which value put in place there will be found :

$$\mathrm{E}e : \mathrm{IM} = \frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}Ss}{KS} - \frac{\frac{1}{2}KM.Ss + \frac{1}{2}CM.Ss}{KS.CS} = \frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}Ss}{KS} - \frac{AM.Ss}{KS.CS};$$

for $\mathrm{KM} - \mathrm{CM} = 2.\mathrm{AM}$. And

$$\frac{AM.Ss}{KS.CS} = \frac{AM}{AC} \cdot \left(\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}St}{CS} \right) \text{ therefore } \mathrm{E}e\text{:IM} = \frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}St}{CS} - \left(\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}St}{CS} \right).$$

Truly $\frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}Ss}{KS}$ = element of the logarithm from $\sqrt{\left(\frac{CM}{CS} : \frac{KS}{KM} \right)}$

$$= \text{elem. log.}\sqrt{\left(\frac{CM.KM}{CS.KS} \right)} = \text{elem.log.}\sqrt{\left(\frac{LM^2}{LS^2} \right)} = \text{elem.log. } (\mathrm{GH} : \mathrm{NO})$$

on the logarithmic curve, whose subtangent is unity.

And $\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}St}{CS}$ = element of the logarithm from $\sqrt{\left(\frac{KS}{KM} : \frac{CM}{CS} \right)}$ = elem.log. $\sqrt{\left(\frac{KS}{CS} : \frac{KM}{CM} \right)}$.

But because SL & MI are the mean proportionals between KS & CS, and between KM & CM, there will be

$$\mathrm{KS} : \mathrm{CS} = \mathrm{LS}^2 : \mathrm{SC}^2 = \mathrm{QA}^2 : \mathrm{AC}^2,$$
$$\& \ \mathrm{KM} : \mathrm{CM} = \mathrm{IM}^2 : \mathrm{MC}^2 = \mathrm{PA}^2 : \mathrm{AC}^2,$$

and thus $\frac{KS}{CS} : \frac{KM}{CM} = \frac{QA^2}{AC^2} : \frac{PA^2}{AC^2} = \frac{QA^2}{PA^2}$; and therefore there will be

$\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}St}{CS}$ = elem. log.$\left(\frac{QA}{PA} \right)$ also on the logarithmic curve, whose subtangent is equal to unity. Hence $\frac{AM}{AC} \cdot \left(\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}St}{CS} \right) = \frac{AM}{AC}$ in the elem. log.(QA:PA) ; and hence also there will be found

$$\mathrm{E}e\text{:IM}\left(= \frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}Ss}{KS} - \frac{AM}{AC} \cdot \left(\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}Ss}{CS} \right) \right) = \text{elem. log.}(\mathrm{GH{:}NO}) - \frac{AM}{AC}.\log(\mathrm{QA{:}PA}),$$

therefore $\int Ee : IM$ = ME:IM = log.(GH : NO) $- \frac{AM}{AC}$.log.(QA:PA), or by multiplying everything by IM, there becomes ME = IM. log.(GH : NO) $- \frac{AM.LM}{AC}$.log.(QA:PA).

But truly IM. log.(GH : NO) following §.492 = HO, since log.(GH : NO) at first was taken on the logarithmic curve, whose subtangent was unity ; thus also IM.log.(QA : PA) is required to be on the logarithmic curve BN, because its subtangent is IM. In the construction if the proposition there was XZ : log.(QA : PA) = AM : AC, and because log.(QA : PA) now is taken on the logarithmic curve, whose subtangent is IM, which before was unity, therefore will be in this case $XZ = \frac{AM.IM}{AC}.\log.(QA:PA)$, and thus $ME = HO - XZ$ as above (§.549.)

Similarly for the expression of the time the fall from the height ME may be found XZ:AM, as now in the place indicated. All of which were required to be found analytically.

SCHOLIUM II.

558 . The fall can be resolved by the same method , where a moving body with a given velocity, yet greater than the terminal, is projected perpendicularly downwards in air according to the present hypothesis of the resistance. For, if the speed of projection were less than the terminal velocity, the solution of the problem will be contained in the last proposition; except for there, as there shall be a need for another solution; if indeed, as now we may suppose, the speed of projection may be greater than the terminal speed, thus another construction will be produced, as now with everything omitted from the previous demonstration, I leave everything from the proceedings to be elicited by the industry of the reader. Let KO be a line of indefinite length,

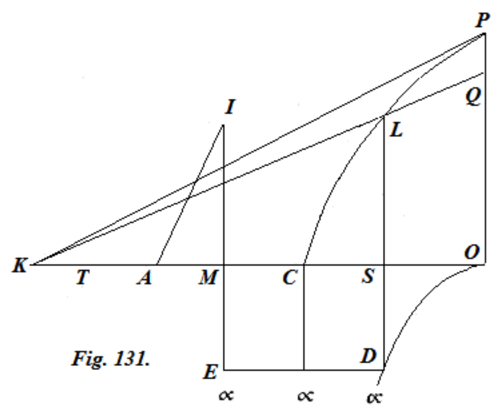


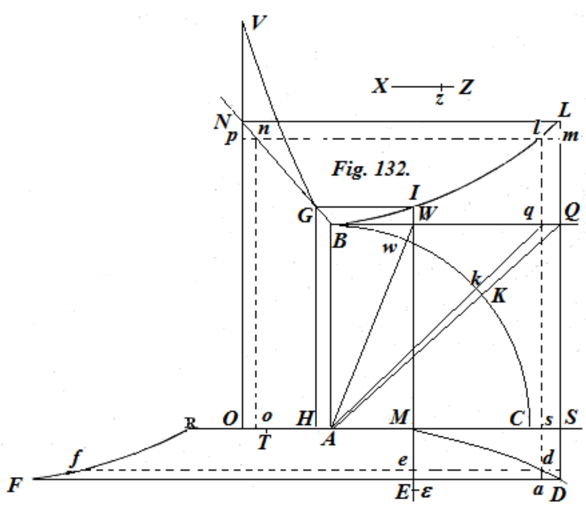
Fig. 131.

whose perpendicular IME of indefinite length α may be put in place, and on that part of the perpendicular IM the uniform gravity may be expressed, and $AM = AT$, half the viscous drag of the medium, and with AI drawn and making $AC = AK = AI$, with the centre A and with the lateral transversal KC the equilateral hyperbola CLP shall be described, on which, if MO may be taken for the initial speed of projection, and MS for the residual speed of the body after some time, and the ordinates OP and SL of the hyperbola may be treated at P and L, through which points the lines KP and KLQ may be drawn finally from K. The distance ME, at whose end E the speed of the body MS or ED may be remaining , will be $\log.(PO:LS) - \frac{AM}{AC}.\log.(PO:QO)$, and the time to pass through the distance $ME = \log.(PO:LS):IM.$ All these logarithms are from the logarithmic curve, of which the subtangent is equal to IM. Now from this construction it is quite apparent, the graph of the speed ODα has the asymptote Cα, and thus the speed of the body certainly decreases continually, but truly it can never be reduced to the speed MC.

PROPOSITION LXVIII. THEOREM.

559. *If the weight may be expressed by the secant of some given arc, and the heavy body may be projected vertically upwards in air according to the hypothesis of the resistance present, and with a speed, which is expressed by the excess of the tangent of the same variable arc over the tangent of the given arc* ; *and with a certain magnitude assumed, which shall be to the difference of the variable arc and of the given arc, as the rectangle from the secant and the tangent of the given arc to the square of the radius.*
1. The distance, *which the body will complete, by ascending as far as it can, shall be as the logarithm of the ratio, which the secant of the variable arc has to the secant of the given arc, with the assumed magnitude taken away*.
2^{o}. The time, *in which the body resolves this height, will be as the assumed magnitude to the tangent of the given arc.*

ABKC shall be the quadrant of a circle, BIL an equilateral hyperbola, and described



from the centre A, with the radius AB. IM or AW expresses the secant of the given arc, Bα the uniform gravity, and the body is understood to be projected up from E with the speed ED or WQ, which is the difference of the tangents BQ, BW of the variable arc BK and of the given arc Bα. And with the logarithmic curve BGN described about the axis AT, of which the subtangent shall be GH or IM; IG and LN act parallel to AS, and pass through the points G, N, &c. of the ordinates of the logarithmic GH and NO; on NO produced , OV shall be the third proportional to GH and NO, and a new logarithmic GV shall pass through all the points V, of which half the subtangent shall be the subtangent of the prior logarithmic GN. Then XZ may be taken, which shall be to the arc αK, as the rectangle IMA to the square of the radius AB, with which in place I say IME or rather $EM = HO - XZ$, and the time to pass through $EM = XZ : AM$.

Demonst. With the lines *dl, ln, no* drawn parallel to the synonymous prior DL, LN, NO for indefinitely small distances, and there becomes as above (§. 549. no.1.) $AT = AM$, and the resistance of the air at E as TS.MS.

I. Therefore there will be $\text{resistance} : \text{weight} = \text{TS.MS} : \text{IM}^2$ and on adding $\text{R} + \text{G}$, or with the whole resistance EF: IM, or $\text{G} = \text{IM}^2 + \text{TS. MS} : \text{IM}^2$,
(or because $\text{AM}^2 + \text{TS. MS} = \text{AS}^2$,
or $\text{TS.MS} = \text{AS}^2 - \text{AM}^2) = \text{AS}^2 + \text{IM}^2 - \text{AM}^2 : \text{IM}^2 = \text{AS}^2 + \text{QS}^2 : \text{IM}^2$
$= \text{LS}^2 : \text{IM}^2 = \text{NO}^2 : \text{GH}^2$ (or because VO, NO and GH are in continued proportion)
$= \text{VO} : \text{IM}$ or GH, therefore we have $\text{EF} : \text{IM} = \text{VO} : \text{IM}$ and thus $\text{EF} = \text{VO}$.

II. From the construction there becomes $\text{EM} = \text{HO} - \text{XZ}, \& \ e\text{M} = \text{H}o - \text{X}z$, and thus $\text{E}e = \text{O}o - \text{Z}z$, from which, if E$t$ were $= \text{O}o$, & $e\text{E} = Zz$, there will be generally $\text{E}e = \text{O}g - Zz$. Now (§.165.)
$\text{Q}q : \text{K}k = \text{AQ}^2 : \text{AK}^2 = \text{LS}^2 : \text{AB}^2 \ \& = \text{NO}^2 : \text{AB}^2 = \text{VO. GH} : \text{BA}^2$, and (by constr.)
$\text{K}k : \text{Z}z = \text{AB}^2 : \text{GH.AM}$, therefore from the equation
$\text{Q}q : \text{Z}z = \text{VO.GH} : \text{GH.AM} = \text{VO} : \text{AM}$ (no. I. of this) $= \text{EF} : \text{AM}$. Therefore
$\text{EF. Z}z = \text{EF.}\varepsilon\text{E} = \text{AM.Q}q = \text{AM.S}s$.

III. Indeed there becomes (§.491. no. III.)
$\text{VO.O}o \ (= \text{EF.}\varepsilon e) = \text{NO.N}p = \text{LS.L}m$ (§. 493.) $= \text{AS.S}s$. And thus
$\text{EF.E}e \ (= \text{EF.}\varepsilon e - \text{EF.}\varepsilon\text{E}) = \text{AS.S}s - \text{AM.S}s = \text{MS.S}s$. That is, the moment of the whole resistance EF is equal to the moment of the speed MS.

IV. Part two of this demonstration presents, Ss:EF (no. III of this)
$= \text{E}e : \text{MS} = \text{Z}z : \text{AM}$, and, as said now many times $\text{S}s : \text{EF} = \text{E}e : \text{MS} = t\text{E}e$; therefore $t\text{E}e = Zz : \text{AM}$, and the sum of all the $t\text{E}e =$ all the $\text{Z}z : \text{AM}$ or $= \text{XZ} : \text{AM}$. That is, the time of the ascent through the distance EM, is $\text{XZ} : \text{AM}$, is expressed therefore by the magnitude assumed to be applied to the tangent of the angle BAW. Which were all required to be shown.

COROLLARY I.

560. The height to which a weight projected up can reach, in the same time as it may resolve the height EM in air, again will be to half of IM, which expresses gravity in the square ratio of the assumed XZ to the given AM. Truly the speed of projection in vacuo is to the speed of projection in air, as IM.XZ to AM.MS. The demonstration is shown in corollaries II & IV of the preceding proposition.

COROLLARY II.

561. With AM or AT vanishing, and thus coinciding with the right lines IM, BA and GH, the case will be returned of proposition LXV thus so that everything, which were shown in that proposition, just as much shall be in the present corollaries.

For the sake of brevity I have omitted several corollaries, which can be elicited from this proposition, as well as the arithmetical calculation of tables of sines, of tangents, and of the logarithms also, by means of which the speeds, distances traversed, and the times of rendering these results can be determined. The intelligent reader will supply everything according to his wishes. But, before I bring myself to the consideration of curvilinear motion in resisting mediums, perhaps it would not detract from the matter, if I were to treat algebraically the general rules for rectilinear motion in mediums with variable densities.

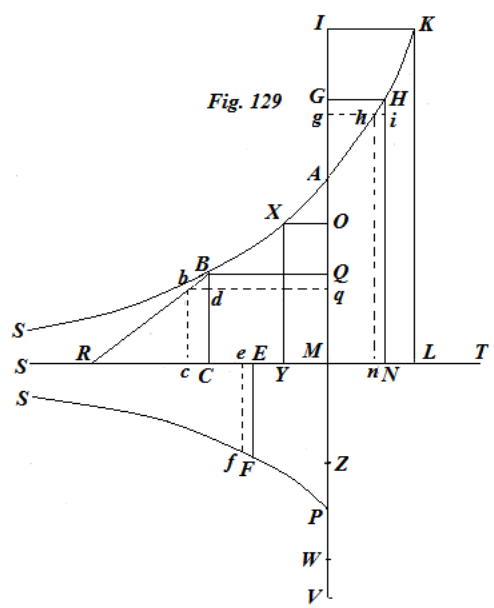
CAPUT XVII.

De motibus Corporum in aëre resistente, partim juxta proportionem celeritatum mobilis, partim etiam juxta duplicatam proportionem earundem .

540. Hæc resistentiarum hypothesis illis fluidis convenit, quorum partes non nihil instar visci cohærent: etenim, si tale fluidum tanquam medium concipiamus mobili trajiciendum, ilico apparebit, ad separanda ejusmodi viscidi fluidi partes, aliquam vim adhibendam esse, diversam ab ea, quæ in mobili tollitur ab allapsu continua particularum fluidi; & hanc vim separatricem partium fluidi proportionari quantitatibus fluidi trajiciendi velocitatibus vero ipsæ fluidi trajiciendæ velocitatibus mobilis proportionales existunt. Igitur tenacitas absoluta fluidi ducta in celeritatem mobilis, seu, quod idem est, in quantitatem materiæ separandæ vel trajiciendæ, erit una pars resistentiæ, quam mobile in ejusmodi fluido latum subibit, altera provenit a fluiditate, qualem in præcedenti capite contemplati sumus. Idcirco resistentia totalis , quam mobile in hujusmodi medio patietur, est ut velocitas in datam quantitatem ducta, cum quadrato ejusdem velocitatis. Idcirco tenacitatem dictam considerabimus deinceps tanquam vim mortuam gravitatique comparabilem, perinde ac resistentias, quæ ab allapsu fluidi ad corpus mobile proveniunt.

PROPOSITIO LXVI. THEOREMA.

541. *Si mobile in aëre dicta ratione resistente motu variato ex primitive uniformi seu æquabili feratur;* spatium, *quod id aliquo effluxo tempore discribet, exponetur tertia proportionali ad celeritatem mobilis initialem tenacitatem fluidi & log-um rationis, quam habet dicta velocitas initialis tenacitate aucta, ad hanc* tenacitatem *auctam pariter, sed celeritate actualis mobilis.* Tempus, *quo hoc spatium percurritur, exponi debet excessu, quo log-us rationis, quam celeritas initialis habet ad actualem, superat log-um rationis earundem celeritatum, sed tenacitate aëris auctarum, applicato ad celeritatem initialem. Posito scilicet log-mos desumtos esse ex log-mica, cujus subtangens se habet ad celeritatem mobilis initialem, sicut rectangulum ex aggregato tenacitatis fluidi & velocitatis initialis corporis in hanc celeritatem initialem ad rec-lum sub dicta tenacitate & resistentia medii totali initio motus.*



I. Esto MS linea deferens, quæ, ut in propositione LVIII explicuimus, uniformiter moveatur versus T celeritate AM, quæ proinde est celeritas initialis, mobile vera M proprio suo motu ab allisione continua aëris feratur ex M versus S, ponaturque certo tempore acquisivisse in linea deferenti celeritatem AQ, postquam scilicet spatium ME in ea transmisit, adeoque detracta celeritate AQ ab initiali AM, qua deferens linea MS versus T æquabiliter incedit, erit residua QM velocitas absoluta mobilis, quam revera in aëre habet; sit porro OM, vel æqualis IA, tenacitas fluidi proveniens ex viscositate ejus ; item ponatur MP pro resistentia aëris initio motus, & EF pro resistentia aëris, ubi actualis mobilis velocitas fuerit QM; eritque adeo (secundiun hypothesin) EF ut $\mathrm{OM.QM}+\mathrm{QM}^2$, seu facta $\mathrm{OG}=\mathrm{MQ}$, ut rec-lum GM.QM, &MP ut IM.AM, atque adeo habetur $\mathrm{MP}:\mathrm{EF}=\mathrm{IM.AM}:\mathrm{GM.QM}$, atque $\mathrm{MP.GM.QM}=\mathrm{EF.IM.AM}$. Æqu. I.

Per punctum A ducta sit log-mica HAB, cujus subtangens quælibet RC sit ad AM (secundum hypothesin) $=\mathrm{IM.AM}:\mathrm{OM.MP}$; atque adeo $\mathrm{RC.OM.MP}:\mathrm{IM.AM}^2$. Æqu.II. Et per puncta I, G, O, Q agantur rectæ IK, GH, OX, & QB asymptotæ log-micæ ST parallelæ, log-micæ occurrentes in punctis K, H, X & B, per quæ ordinatæ KL, HN , XY & BC ductæ sint *;* fiat $\mathrm{ME}=\mathrm{MC}-\frac{IM.LN}{AM}$, atque PFS scala resistentiarum, & tempus, quo spatium ME in linea deferenti MS percurritur, seu $t\mathrm{ME}=(\mathrm{MC}-\mathrm{LN}):\mathrm{AM}$.

II. *Demonstr.* Cum (constr.) sit $\mathrm{ME}=\mathrm{MC}-\frac{IM.LN}{AM}$, erit etiam $\mathrm{E}e=\mathrm{C}c-\frac{IM}{AM}.\mathrm{N}n$, sunt enim E*e*, C*c*, & N*n* elementa linearum ME, MC & LN. Præterea, quia etiam (constr.) $\mathrm{OG}=\mathrm{MQ}$, erit pariter *Gg·=Hi=Qq=Bd*, adeoque (§. 491. num. II.) rec-lum $\mathrm{HN}n=\mathrm{rec-lo\ BC}c$, ac proinde $\mathrm{C}c:\mathrm{N}n=\mathrm{GM:QM}$,
$\&\ \mathrm{Cc}:\frac{IM}{AM}.\mathrm{N}n=\mathrm{GM}:\frac{IM}{AM}.\mathrm{QM}=\mathrm{GM.\ AM}:\mathrm{IM.QM}$, *&* convertendo
E*e* seu $\mathrm{C}c-\frac{IM}{AM}.\mathrm{N}n:\mathrm{C}c=\mathrm{GM.AM}-\mathrm{IM.\ QM}:\mathrm{GM.AM}$ (vel quia OM.AQ æquatur $\mathrm{GM.AM}-\mathrm{IM.QM})=\mathrm{OM.AQ}:\mathrm{GM.AM}$, & propter triangula similia RCB ac *bd*B est, $\mathrm{C}c:\mathrm{Q}q=\mathrm{RC}:\mathrm{QM}$, ergo ex æquo & per rationum compositionem
$Ee:\mathrm{Qq}=\mathrm{RC.OM.AQ}:\mathrm{GM.QM.AM}=\mathrm{RC.OM.MP.AQ}:\mathrm{MP.GM.QM.AM}$,
(vel substitutis loco RC.OM.MP & MP.GM.QM ope.Æqu.II. & I. solidis respective æqualibus $\mathrm{IM.AM}^2$ & $\mathrm{EF.IM.AM})=\mathrm{IM.AM}^2.\mathrm{AQ.IM.AM}^2.\mathrm{EF}=\mathrm{AQ}:\mathrm{EF}$, adeoque comparando primam cum ultima ratione, habetur $\mathrm{E}e:\mathrm{Q}q=\mathrm{AQ}:\mathrm{EF}$, atque adeo $\mathrm{EF.E}e=\mathrm{AQ.QI}$. Id est *momentum solicitationis acceleratricis* EF in linea deferenti MS æquatur *momento celeritatis acquisitæ* AQ in hac linea deferente.

III. Ut antea (num. II hujus) erit $\mathrm{C}c:\mathrm{N}n=\mathrm{GM}:\mathrm{QM}$, & convertendo
$\mathrm{C}c-\mathrm{N}n:\mathrm{C}c=\mathrm{OM}(\mathrm{seu\ GQ}):\mathrm{GM}$, & $\mathrm{C}c:\mathrm{Q}q=\mathrm{RC}:\mathrm{QM}$, ergo

$$\mathrm{C}c-\mathrm{N}n:\mathrm{Q}q=\mathrm{RC.OM}:\mathrm{GM.QM}=\mathrm{RC.OM.MP}:$$

$$\mathrm{MP.\ GM.QM}=\mathrm{IM.AM}^2:\mathrm{EF.lM.AM},$$

substitutis scilicet loco solidorum RC.OM.MP & MP.GM.QM æqualibus $\mathrm{IM.AM}^2$ & EF: IM. AM, quæ æqualitates secunda & prima num. I. exhibent est ergo
$\mathrm{C}c-\mathrm{N}n:\mathrm{Q}q=\mathrm{IM.AM}^2:\mathrm{EF.IM.AM}=\mathrm{AM}:\mathrm{EF}$, & permutando

$Cc - \mathrm{N}n : \mathrm{AM} = \mathrm{Q}q : \mathrm{EF} = \mathrm{E}e : \mathrm{AQ} = t\mathrm{E}e$, ergo omnia $t\mathrm{E}e$ seu tempus per $\mathrm{ME} = (\mathrm{MC} - \mathrm{LN}) : \mathrm{AM}$.

IV. Ergo, assignato hoc tempore, linea defens MS describet in aëre cum velocitite AM motu suo æquabili spatium $\mathrm{MC} - \mathrm{LN}$, motu vero proprio in hac deferente corpus simul describit spatium $\mathrm{ME} = \mathrm{MC} - \frac{IM}{AM}.\mathrm{LN}$, adeoque spatium absolutum, quod mobile in aëre absolvet, erit excessus spatii motu communi & æquabili lineæ deferentis supra motum isti contrarium mobilisque in deferenti proprium, seu $\mathrm{MC} - \mathrm{LN} - \mathrm{MC} + \frac{IM}{AM}.\mathrm{LN} = \mathrm{IM.LN} : \mathrm{AM}$. Hoc ergo spatium absolutum exponitur tertia proportionali ad celeritatem initialem AM, tenacitatem fluidi OM vel IA, atque ad log-mum rationis, quam KL & HN, id est, celeritates initialis & actualis, mobilis tenacitate auctæ ad invicem habent. Quæ omnia erant demonstranda.

COROLLARIUM I.

542. Apparet igitur in præsenti resistentiarum hypothesi mobile non solum non posse in infinitum excurrere, sed ne quidem posse tcrminum finiti spatii SM.LY: AM tempore quantumlibet magno attingere. Nam evanescente QM, ipsa MC sit infinita, GM sit OM, adeoque LN mutatur tunc in LY, adeo ut $\mathrm{MC} - \mathrm{LN} : \mathrm{AM}$ abeat in $\mathrm{MC} - \mathrm{LY} : \mathrm{AM}$ seu in MC:AM, quia finita LY præ infinita MC evanescit. Idcirco spatium $\mathrm{IM.LN} : \mathrm{AM}$. tempore finito $\mathrm{MC} - \mathrm{LN} : \mathrm{AM}$ percursum, mutabitur in IM.LY:AM, quod tempore infinito tantum MC:AM, id est nunquam, absolvi potest.

COROLLARIUM II.

543. Si spatia $\mathrm{IM.LN} : \mathrm{AM}$ sumantur in progressione arithmetica, erunt ipsæ LN abjectis IM:AM ejusdem ubique magnitudinis, etiam in progressione arithmetca, atque adeo ordinatæ log-mici, HN vel æquales hisce GM, seu velocitates actuales mobilis QM tenacitate medii OM vel AI auctæ, in progressione geometrica.

COROLLARIUM III.

544. Quia $\mathrm{CM} = \log.(\mathrm{AM} : \mathrm{QM})$ & $\mathrm{LN} = \log.(\mathrm{IM} : \mathrm{GM})$ erit

$$\mathrm{MC} - \mathrm{LN} = \log.(\mathrm{GM.AM} : \mathrm{IM.QM}) = \log.(\mathrm{AM.OM} + \mathrm{AM.QM} : \mathrm{IM.QM}).$$

In recta MP sumatur MZ, quæ sit ad OM vel $\mathrm{AI} = \mathrm{AM} : \mathrm{QM}$, eritque $\mathrm{QM.MZ} = \mathrm{AM.OM}$, atque adeo $\mathrm{AM.OM} + \mathrm{AM.QM} : \mathrm{IM.QM} = \mathrm{AM} + \mathrm{MZ}$ seu $\mathrm{AZ} : \mathrm{IM}$, & consequenter $\mathrm{MC} - \mathrm{LN} : \log.(\mathrm{AM} + \mathrm{MZ} : \mathrm{IM})$, atque $(\mathrm{MC} - \mathrm{LN}) : \mathrm{AM} = \log.(\mathrm{AM} + \mathrm{MZ} : \mathrm{IM})$ applicatus ad AM. Adeoque, si hæ $\mathrm{MC} - \mathrm{LN} : \mathrm{AM}$, hoc est, tempora sumantur in progressione arithmetica, erunt ipsæ $\mathrm{AM} + \mathrm{MZ}$, seu magnitudines MZ velocitatibus QM reciproce proportionales celeritate initiali AM auctæ in progressione geometrica.

Hæc duo postrema corollaria probe conspirant cum propositionibus XI. & XII. Lib. II. Princ. Phil. Nat. Math. Cl. Newtoni, & cum iis, quæ Cl. Virignon in Actis Acad. Scient. 1707. d.13. Augusti probl. IV. coroll. 11. & 14. exhibuit.

SCHOLION.

545 . Satis prævideo fore, ut mihi objiciatur nimia prolixitate in hac propositione nostra ea tradita esse, quæ multo brevius atque facilius potuissent perfici, si pro linea OM, tenacitatem fluidi denotante, assumsissem rectam AM, quæ velocitatem initialem exponit. At respondeo hoc pacto propositionem minus generalem futurant fuisse, etsi non negem, multo brevius atqae simplicius tunc demonstrari potuisse. Nam talem adducere, volui, quæ etiam casum propositionis nostræ LXIII contineret; quod revera præstat, nam evanescente OM vel AI, ex constructione præscentis propositionis, nascuntur omnes determinationes prepositionis LXIII, adeo ut illa casus tantum sit seu corollarium præsentis. Sed, si pro linea tenacitatem fluidi repræsentatrice accepissem tantum rectam AM, quæ velocitatem initialem mobilis exponit, talis deductio non successisset, quia AM, quatenus velocitatem initialem exprimit, nunquam potest evanescere. Sed, quia deducto propositionis superioris LXIII. ex præsenti non adeo obvia esse videtur, cum hoc loco distinctius explicare libet. Si IA vel OM evanescit, etiam $AI.\,LN:AM$ evanescere videtur; seu spatium a mobili in aëre juxta duplicatam proportionem celeritatum resistente in nihilum abire , tum etiam $MC-LM:AM$. Hoc enim casu IM abiret in AM, & GM in QM, id est, ratio IM:GM fieret AM:QM, adeoque LN log-us illius rationis fieret MC, adeo ut $MC-LN$ futura sit $MC-MC=0$. Quomodo ergo ex constructione præsentis propositionis elici potest constructio propositione LXIII in casu ipsius $AI=MO=0$?

546. Quia in dicta propositione (§.522.) log-micæ subtangens exponit celeritatem initialem, sumatur log-us rationis IM:GM in ea log-mica, eritque (§.492.)
$\log.\left(IM:GM\right):AM=LN:RC$, adeoque

$LN=RC.\log.(IM:GM):AM,$ & $IA.LN:AM=RC.IA.\log.(IM:GM):AM^2=RC.OM,$

$MP.\log.\ \left(IM:\ GM\right):\ AM^2.MP.$ Atqui (§.541. num.1. æqu; I. II.)

est $RC.OM.MP=IM.AM^2$, ergo

$$IA.LN:AM=IM.AM^2.\log.\left(IM:GM\right):AM^2.MP=IM.\log.\left(IM:GM\right):MP,$$

& hæc est nova expressio spatii in aëre decursi, sive IA sit quantitas realis sive non. Jam, si $IA=0$, erit ratio $lM:GM=AM:QM$, evanescente scilicet $OM=IA$, & existente $OG=QM.$ Ergo spatium in aëre percursum erit hoc casu

$AM.\log.\left(AM:QM\right):MP=\log.\left(AM:QM\right)$ in casu propositionis LXIII.

ubi MP revera æqualis facta erat AM celeritati initiali. Est ergo spatium absolutum a mobili in aëre percursum exponendum log-mo rationis, quam celeritas initialis AM habet ad residuam mobilis seu actualem QM, prorsus ut habet propositio citata.

547. Determinatio temporis est paulo altioris indaginis. Hoc tempus exponitur juxta præsætem propositionem per $(\mathrm{MC}-\mathrm{LN}):\mathrm{AM}$. Jam (§.544) est
$\mathrm{MC}-\mathrm{LN}=\log.\left(\mathrm{AM}+\mathrm{MZ}:\mathrm{IM}\right)=\log.\mathrm{AM}+\mathrm{MZ},-\log.\mathrm{IM}$. Et

$$\log(\mathrm{AM}+\mathrm{MZ})=\left(\tfrac{MZ}{MA}-\tfrac{MZ^2}{2MA^2}+\tfrac{MZ^3}{3MA^3}-\text{etc.}\right)\text{ in RC},$$
$$=\left(\tfrac{MZ.QM}{AM.QM}-\tfrac{MZ^2.QM^2}{2.AM^2}+\tfrac{MZ^3.QM^3}{3.AM^3.QM^3}-\text{etc.}\right).\mathrm{RC},$$

quia ex constructione rec-lum MZ.QM æquatur dato rect-angulo AM.OM, erit

$$\log(\mathrm{AM}+\mathrm{MZ})=\left(\tfrac{RC.OM.AM}{AM.QM}-\tfrac{RC.OM^2.AM^2}{2MA^2.QM^2}+\tfrac{RC.OM^3.AM^3}{3.MA^3.QM^3}-\text{etc.}\right)=$$
$$=\tfrac{RC.OM.MP}{MP.QM}-\tfrac{RC.OM^2.MP}{2MP.QM^2}+\tfrac{RC.OM^3.MP}{3.MA.QM^3}-\text{etc.}$$

(aut quia solidum RC.OM.MP æquatur solido $\mathrm{IM.AM}^2$, ut aliquoties jam vidimus)

$$\tfrac{IM.AM^2}{MP.QM}-\tfrac{IM.AM^2.OM}{2.MP.QM^2}+\tfrac{IM.AM^2.OM^3}{3.MP.QM^3}-\&c.$$

Atqui in hac ultima serie infinita, ob evanescentem (secundun hypothesin) OM, evanescent omnia post primum terminum membra, propterea erit hoc casu

$$\log.\ \left(\mathrm{AM}+\mathrm{MZ}\right)=\tfrac{IM.AM^2}{MP.QM}.$$

Quia $\mathrm{IM}=\mathrm{AM}+\mathrm{IA}=\mathrm{AM}+\mathrm{OM}$, erit $=\log.\mathrm{IM}=\left(\tfrac{OM}{AM}-\tfrac{OM^2}{2.AM^2}+\tfrac{OM^3}{3.AM^3}-\&c.\right).\mathrm{RC}$, quae est eadem series cum prima,excepta sola OM, quæ hoc loco est posita loco MZ, in prima. Ergo etiam
$\log.\mathrm{IM}=\tfrac{RC.OM.MP}{AM.MP}-\tfrac{RC.OM^2.MP}{2.AM^2.MP}+\&c.=\tfrac{IM.AM^2}{AM.MP}-\tfrac{IM.AM^2.OM}{2.AM^2.MP}+\&c.=\tfrac{IM.AP}{MP}-\tfrac{IM.OM}{2.MP}+\text{etc.}=\tfrac{IM.AM}{MP}$,
quia etiam in hac serie infinita omnes, post primum, termini evanescunt, existente (secundum hypothesin) OM vel $\mathrm{AI}=0$. Hinc

$$\mathrm{MC}-\mathrm{LN}=\log.\left(\mathrm{AM}+\mathrm{MZ}\right)-\log.\left(\mathrm{IM}\right)=\tfrac{IM.AM^2}{MP.QM}-\tfrac{IM.AM}{MP}=\tfrac{IM.AM^2-IM.AM.QM}{MP.QM}$$
$$=\tfrac{IM.AM.QM}{MP.QM.}=\tfrac{AM^2.AQ}{MP.QM.},$$

quia in præsenti hypothesi IM fit AM evanescente IA, & quia insuper, in propositione LXIII, MP æqualis erat AM; erit
$\mathrm{MC}-\mathrm{LN}=\tfrac{AM.AQ}{QM}$, & MC – LN:AM, seu tempus, quo mobile suum spatium
$\log.\left(\mathrm{AM}:\mathrm{QM}\right)$ in aëre absolvit sit $=\mathrm{AQ}:\mathrm{QM}$, exponendum scilicet ratione celeritatis

initialis partis amissæ ad residuam, seu velocitatem mobilis actualem in aëre, prorsus ut habet citatu propositio LXIII.§.522.

548. Quin imo res universalius adhuc quam illic tradi potest, supponendo resistentiæ ab initio motus repræsentatricem MP diversam esse a recta AM, quæ celeritatem initialem exponit; etenim spatium absolutum in aëre confectum eo casu exponetur, ut supra (§.546.) invenimus, per $AM.\log.(AM:QM):MP$, & tempus, quo hoc spatium absolvitur per $AM.AQ:MP.QM$. Idcirco potuisset sæpius jam memorata propositio penitus omitti, quemadmodum & eæ, quæ post eam in eodem capite XVI. immediate sequuntur, quod pariter ex iis deduci possint facillimo negotio illæ, quæ mox sequentur. Sed demonstrationum diversitas atque in conclusionibus concentus omnino digna sunt, quæ distinctius curiosi Lectoris oculis exponantur.

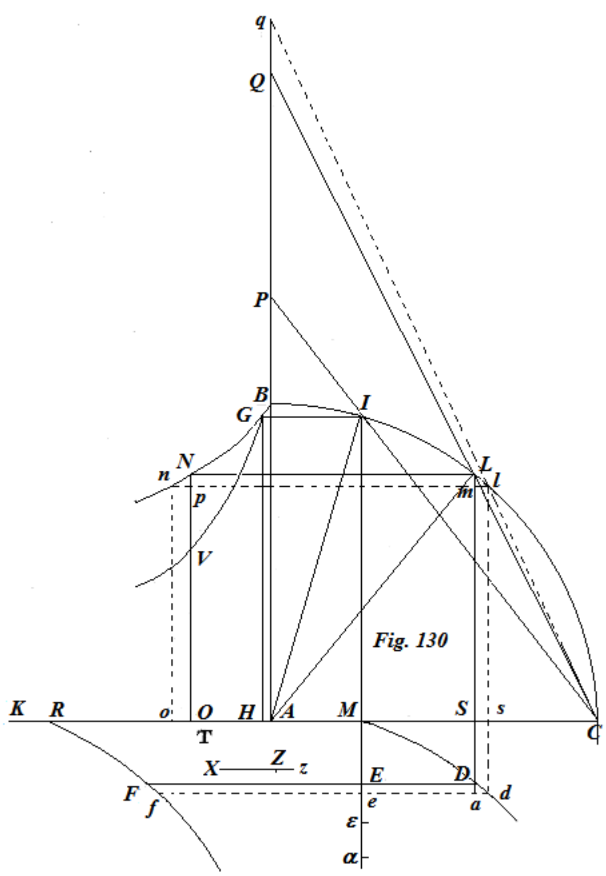
PROPOSITIO. LXVIL. THEOREMA.

549. *Si gravitas uniformis exponatur sinu complementi alicujus anguli acuti & dati, & grave in linea recta horizontali normali cadat in aëre, juxta indicatam in titulo hujus capitis hypothesin, resistente ; assumaturque quædam magnitudo, quæ sit ad log-mum rationis, quam tangens semissis complementi dati anguli habet ad tangentem semiss complementi alicujus anguli variabilis & majoris dato, ut sinus dati anguli ad sinum totum. Sumptis scilicet log-mis in logarithmica, cujus subtangens est æqualis sinui complementi dati anguli præmemorati, quibus positis.*

1°. Spatium, *quod grave aliquo tempore perlabitur accelero motu, est excessus log-mi rationis, quam habet sinus complementi dati anguli ad sinum complementi variabilis, supra assumptam magnitudinem:*

2°.Tempus, *quo spatium istud absolvitur, exponetur assumpta magnitudine applicata ad sinum rectum anguli dati.*

3°. Celeritas, *in fine cujusque temporis acquisita, est, ut excessus sinus anguli variabilis supra sinum rectum anguli dati.*



Cadat grave a quiete ex M in linea recta Mα horizonti perpenendiculari, ac centro *A* descripto circuli quadrante BLCA, exponat in eo recta IM, quæ est sinus complemcnti cujusdam dati anguli BAI, gravitatem uniformem, & ducta porro qualibet SL parallela IM, jungatur AL*;* eritque angulus BAL is, qui in propositione variabilis dicitur, cujus recta LS est sinus complementi. Circa axem MR & per supremum quadrantis. punctum transeat log-micaBGN, cui linea LN parallela: radio CA occurrat in N, recta vero IG eidem MC æquidistans in puncto G, per hæc duo log-micæ, puncta transeant ordinatæ GH & NO ; dein assumatur quædam XZ, quæ sit ad log-um rationis QA:AP sicut AM sinus dati anguli BAI ad radium AI. Ductis enim per puncta I & L ex puncto C rectis CP & CO, erit QA: PA, ut tangens dimidii anguli IAC ad tangentem dimidii LAC, seu ut tangens semissis complementi anguli dati BAI ad. tangentem semissis complementi anguli variabilis BAL; nam

$$\mathrm{AQ:AP = tang.LCA : tang.ICA}$$

$$\mathrm{= tang.MIC : tang.SLC = tang.\tfrac{1}{2}IAC : tang.\tfrac{1}{2}LAC.}$$

Ipsa vero HO, seu log-us rationis GH:NO, seu IM:LS, est log-us rationis, quam habet sinus complementi IM anguli dati BAI ad sinum complementi LS variabilis BAL; propterea ostendi debet 1°. spatium cadendo descriptum ME, esse $=\mathrm{HO}-\mathrm{XZ}$; deinde 2°. tempus, quo spatium istud percurritur, seu $t\mathrm{ME}=\mathrm{XZ:AM}$, ac denique MS differentiam sinuum AS, AM angulorum variabilis & dati, celeritatem in E acquisitam, adeo ut productis FE & LS in D hoc punctum futurum sit in scala celeritatum MD, ducta scilicet FD per punctum E parallela RS. Esto insuper RF scala solicitationum acceleratricium, ductisque *dl*, *ln*, *no* ac *df* ipsis. DL, LN, NO ac DF parallelis ac indefinite vicinis productaque LD in *a*; probari debet momentum solicitationis acceleratricis FE æquari momento celeritatis æquisitæ ED; seu $\mathrm{EF.E}e=\mathrm{ED.}ad$, posita constructione supra exposita. Fiat $\mathrm{AT}=\mathrm{AM}$, & VO tertia proportionalis ad GH & NO, adeo ut inde nova log-mica GV exurgat. Log-micæ BN subtangens quælibet æquetur IM vel GB.

Demonstr. I. Esto resistentia in E, quam simplici litera, R indicabimus, ut gravitatem per G, esto inquam $\mathrm{G:R}=\mathrm{IM}^2:2\mathrm{AM.MS}+\mathrm{MS}^2$, seu, quia (constr.) $\mathrm{AM}=\mathrm{AT}$, ponatur $\mathrm{G:R}=\mathrm{IM}^2:\mathrm{TS.MS}$, eritque convertendo $\mathrm{G-R:G}$; id est, solicitatio acceleratrix in E ad gravitatem, hoc est, $\mathrm{EF:IM}=\mathrm{IM}^2-\mathrm{TS.\,MS}:\mathrm{IM}^2=\mathrm{IM.EF}:\mathrm{IM}^2$, atque adeo

$$\mathrm{IM.EF}=\mathrm{IM}^2-\mathrm{TS.MS}=\mathrm{AI}^2-\mathrm{AM}^2-\mathrm{TS.MS}$$

$$=\mathrm{AI}^2-\mathrm{AM}^2-2.\mathrm{AM.MS}-\mathrm{MS}^2=\mathrm{AL}^2-\mathrm{AS}^2=\mathrm{LS}^2=\mathrm{NO}^2,$$

aut (quia. NO est media proportionalis inter GH vel IM & VO)$=\mathrm{IM.VO}$, atque adeo $\mathrm{EF}=\mathrm{VO}$.

II. Quia (constr.) $\mathrm{ME}=\mathrm{HO}-\mathrm{XZ}$, & $\mathrm{M}e=\mathrm{H}o-\mathrm{X}z$ erit etiam $\mathrm{E}e=\mathrm{O}o-\mathrm{Z}z$; propterea si E*e* ponatur$=\mathrm{O}o$, erit $eo=\mathrm{Z}z$, ubi Z*z* est elementum lineæ assumtæ XZ.

III. Propter triangula similia L*lm* & ALS habetur S*s*:*rl* (vel æqualem L*l*) $=\mathrm{LS:AL}$, & $rl:\mathrm{Q}q=rs(\text{vel LS}):\mathrm{QA}$; ergo ex æquo

$\mathrm{S}s:\mathrm{Q}q=\mathrm{LS}^2:\mathrm{AL.AQ}=\mathrm{NO}^2:\mathrm{AL.AQ}=\mathrm{IM.VO}:\mathrm{AI.AQ}.$ Verum Q*q*: elementum log-mi $(\mathrm{QA}:\mathrm{PA})=\mathrm{QA}:\text{subtang. log-micæ IM}=\mathrm{QA.VO}:\mathrm{IM.VO}$, ergo ex æquo S*s*: elem.log-mi $(\mathrm{QA}:\mathrm{RA})=\mathrm{AQ.VO}:\mathrm{AI.AQ}=\mathrm{VO}:\mathrm{AI}.$ Et (constr.) element log-mi $(\mathrm{QA}:\mathrm{PA}):\mathrm{Z}z=\mathrm{Al}:\mathrm{AM},$ ergo denuo ex æquo $\mathrm{S}s:\mathrm{Z}z=\mathrm{VO}:\mathrm{AM}.$
Atque adeo $\mathrm{FE}.e\varepsilon$ (num. I.& II.hujus) $=\mathrm{VO}.Zz:\mathrm{AM}.\ \mathrm{S}s.$

IV. Rursus $\mathrm{EF.E}\varepsilon$ (seu $\mathrm{VO.O}o)=\mathrm{NO.N}p$ pcr §.491. num.III,
$\mathrm{E}t.\mathrm{NO.N}p=\mathrm{LS.\ L}m\ (§.493.)=\mathrm{AS.S}s,$ hinc, quia (num.III hujus) $\mathrm{FE}.e\varepsilon=\mathrm{AM.S}s$, erit $\mathrm{FE.E}\varepsilon-\mathrm{FE}.e\varepsilon=\mathrm{AS.S}s-\mathrm{AM.S}s,$ hoc est, $\mathrm{FE.E}e=\mathrm{MS.S}s.$ Id est, *momentum solicitationis* EF *est æquale momento celeritatis acquisitæ* MS.

V. Quoniam supra (num.III hujus) invenimus $\mathrm{VO.Z}z=\mathrm{AM.S}s$, erit $\mathrm{Z}z:\mathrm{AM}=\mathrm{S}s:\mathrm{VO}=da:\mathrm{FE}(§.485.)=t\mathrm{E}e$; adeoque omnia *t*E*e*, hoc est, tempus casus perpendicularis per spatium ME, quod tempus designavimus per *t*ME, = omnibus Z*z*, quibus tota XZ componitur, divisis per AM, seu $=\mathrm{XZ}:\mathrm{AM}$. Ergo tempus per spatium ME exponitur per assumtam magnitudinem XZ applicatam ad sinum AM anguli dati BAI. Quæ omnia erant demonstranda.

COROLLARIUM I.

550. Cum celeritas in E acquisita sit ED vel MS, celeritas terminalis erit MC sinus versus complementi IAC anguli dari BAI, & scala celeritatum MD asymptotam habebit parallelam ipsi Mα sed per punctum C transeuntem.

COROLLARIUM II.

551. Spatium, quod grave in vacuo absolveret cadendo eo tempore, quo in aëre spatium ME perlabitur, est ad semissem MI, quæ gravitatem exponit in duplicati ratione assumtæ XZ ad datam AM. Nam, quia tempus (§.151.) descensus in vacuo exponitur radice ex duplo spatio applicato ad magnitudinem, quæ gravitatem uniformem repræsentat, erit, vocando spatium tempore XZ:AM in vacuo percurrendem S, erit inquam
$\sqrt{(2\mathrm{S}:\mathrm{IM})}=\mathrm{XZ}:\mathrm{AM},\text{seu } 2\mathrm{S}:\mathrm{IM}=\mathrm{S}:\tfrac{1}{2}\mathrm{IM}=\mathrm{XZ}^2:\mathrm{AM}^2.$

COROLLARIUM III.

552. Velocitas, quam mobile in termino E spatii ME in vacuo descripti acquireret, erit ad celeritatem in aere acquisitam percurso eodem spatio ME, sicut $\sqrt{(2\mathrm{ME.IM})}$ ad MS, id liquet ex §.150.

COROLLARIUM IV.

553. Celeritas vero in aëre erit ad celeritatem in vacuo mobili acquisitam tempore $XZ:AM$, sicut rec-lum AM.MS ad rec-lum IM.XZ. Nam (§.551.) in vacuo percurritur spatium $S = IM.XZ^2 : 2.AM^2$ tempore $XZ:AM;$ atqui (§.150.) est velocitas in termino huius spatii acquisita $\sqrt{(2.S.IM)} = \sqrt{\left(2.IM^2 : XZ^2 : 2.AM^2\right)} = IM.XZ:AM;$ ergo celeritas in aëre est ad celertatem in vacuo acquisitam, sicut MS ad IM.XZ:AM, vel sicut rec-lum AMS ad rec-lum IM.XZ.

COROLLARIUM V.

555. Quia recta TA tenacitatem medii & AM dimidium hujus tenacitatis exponit, erit hæc tenacitas ad gravitatem, sicut sinus anguli dati BAI ad semissem sinus complementi.

COROLLARIUM VI.

556. Idcirco evanescente angulo BAI fluidum omni tenacitate carebit, eritque perfecte fluidum, adeo ut redeat casus propositionis LXIV, quæ propositio proinde tantum corollarium est præsentis. Evanescent enim eo casu AM & XZ, adeo ut tunc ME simpliciter æqualis fiat ipsi HO. Tempus vero per ME, quod in hac propositione LXVII. est $XZ:AM(\text{constr.}) = \log.\ (QA:PA):AI,$ in casu coincidentiæ ipsarum BA & IM, fiet $XZ:AM = \log.\ (QA:BA):AB = t\text{ME}$; prorsus ut §.525 num.III. ostensum.

SCHOLION I.

557. Propositio etiam analysi geometrica sequenti ratione expediri potest. Positis iis, quæ in præparatione ad demonstrationem propositionis, numero 1. erit $FE \text{ vel } VO = NO^2 : IM = LS^2 : IM$ (vel posita KC diametro circuli BIC) $= KS.SC:IM;$ unde, quia generaliter esse debet $EF.Ee = MS.\ Ss = ED.ad,$ erit etiam $KS.CS.\ Ee:IM = MS, Ss$, atque adeo $Ee:IM = MS.Ss:KS.CS.$ Sed $MS = \frac{1}{2}KS - \frac{1}{2}KM + \frac{1}{2}MC - \frac{1}{2}SC$, quo valore subrogato reperietur

$$Ee:IM = \frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}Ss}{KS} - \frac{\frac{1}{2}KM.Ss + \frac{1}{2}CM.Ss}{KS.CS} = \frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}Ss}{KS} - \frac{AM.Ss}{KS.CS};$$

nam $KM - CM = 2.AM.$ Atqui

$$\frac{AM.Ss}{KS.CS} = \frac{AM}{AC}\cdot\left(\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}St}{CS}\right) \text{ ergo } Ee:IM = \frac{\frac{1}{2}Ss}{CS} - \frac{\frac{1}{2}St}{CS} - \left(\frac{\frac{1}{2}Ss}{KS} + \frac{\frac{1}{2}St}{CS}\right).$$

Verum $\frac{\frac{1}{2}Ss}{CS}-\frac{\frac{1}{2}Ss}{KS}=$ elem. log – mi ex $\sqrt{\left(\frac{CM}{CS}:\frac{KS}{KM}\right)}$

$$=\text{elem. log.}\sqrt{\left(\frac{CM.KM}{CS.KS}\right)}=\text{elem.log.}\sqrt{\left(\frac{LM^2}{LS^2}\right)}=\text{elem.log. (GH : NO)}$$

in log-mica, cujus subtangcns est unitas.

Et $\frac{\frac{1}{2}Ss}{KS}+\frac{\frac{1}{2}St}{CS}=$ elem. log – mi ex $\sqrt{\left(\frac{KS}{KM}:\frac{CM}{CS}\right)}=$ elem.log. $\sqrt{\left(\frac{KS}{CS}:\frac{KM}{CM}\right)}$..

Sed quia. SL & MI sunt media proportionales inter KS & CS, atque inter KM & CM, erit

$$\text{KS : CS}=\text{LS}^2:\text{SC}^2=\text{QA}^2:\text{AC}^2,$$
$$\text{\& KM : CM}=\text{IM}^2:\text{MC}^2=\text{PA}^2:\text{AC}^2,$$

atque adeo $\frac{KS}{CS}:\frac{KM}{CM}=\frac{QA^2}{AC^2}:\frac{PA^2}{AC^2}=\frac{QA^2}{PA^2}$; ac propterea erit $\frac{\frac{1}{2}Ss}{KS}+\frac{\frac{1}{2}St}{CS}=$ elem. log.$\left(\frac{QA}{PA}\right)$ etiam in lGg-mica, cujus subtangens est æqualis unitati. Hinc $\frac{AM}{AC}\cdot\left(\frac{\frac{1}{2}Ss}{KS}+\frac{\frac{1}{2}St}{CS}\right)=\frac{AM}{AC}$ in elem. log.(QA:PA). ; ac proinde habebitur etiam

$$\text{E}e\text{:IM}\left(=\frac{\frac{1}{2}Ss}{CS}-\frac{\frac{1}{2}Ss}{KS}-\frac{AM}{AC}\cdot\left(\frac{\frac{1}{2}Ss}{KS}+\frac{\frac{1}{2}Ss}{CS}\right)\right)=\text{elem. log.}(\text{GH:NO})-\frac{AM}{AC}.\log(\text{QA:PA}),\text{ ergo}$$

$$\int Ee:IM=\text{ME:IM}=\log.(\text{GH : NO})-\frac{AM}{AC}.\log.(\text{QA:PA})$$

vel ducendo omnia in IM, erit ME : IM. log.$(\text{GH : NO})-\frac{AM.LM}{AC}.\log.(\text{QA:PA})$.

At vero IM. log.(GH : NO) juxta §.492 = HO , quandoquidem log.(GH : NO) primum sumebatur in log-mica, cujus subtangens erat unitas ; sic etiam IM.log.(QA : PA) sumenda est in log-mica BN, quia ejus subtangens est IM. In constructione propositionis erat XZ : log.(QA : PA) = AM : AC, & quia log.(QA : PA) jam sumitur in log-mica, cujus subtangens est IM, quæ antæ erat unitas, erit proinde hoc casu

$\text{XZ}=\frac{AM.IM}{AC}.\log.(\text{QA : PA})$, atque adeo ME = HO – XZ ut supra (§.549.)

Similiter pro expressione temporis casus ex altitudine ME invenietur XZ:AM, ut in jam citato loco. Quæ omnia analytice erant invenienda.

SCHOLION II.

558 . Eadem methodo solvi potest casus, quo mobile data velocitate, majore tamen terminali, perpendiculariter deorsum projicitur in aëre resistente juxta præsentem resistentiarum hypothesin. Nam, si celeritas projectonis terminali minor sit, problema jam solutum

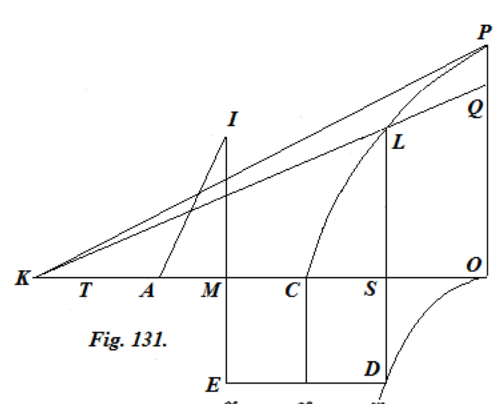


Fig. 131.

continetur in propositione postrema ; absque eo, ut alia solutione opus sit; sin vero, ut nunc supponemus, celeritas projectionis major sit quam terminalis, nonnihil alia inde prodibit constructio, quam nunc adducam omissa demonstratione, quam industriæ Lectoris ex præcedentibus eliciendam relinquo. Esto KO linea indefinitæ longitudinis, cui perendicularis insistat IME ad partes α indefinita, in hac perpendiculari portio IM exponat gravitatem uniformem, & $AM = AT$ semissem tenacitatis medii, ductaque AI factisque $AC = AK = AI$, centro A & latere transverso KC descripta sit hyperbola æquilatera CLP, in qua, si MO accipiatur pro celeritate initiali projectionis, & MS pro velocitate mobili post aliquod tempus residua, agantur ordinatæ OP & SL hyperbolæ occurrentes in P & L, per quæ puncta ducantur denique ex K rectæ KP, KLQ. Spatium ME, in cujus termino E mobili relinquitur celeritas MS vel ED, erit $\log.(PO:LS) - \frac{AM}{AC}.\log.(PO:QO)$, & tempus per spatium $ME = \log.(PO:LS):IM.$ Omnes hi logarithmi sunt ex log-mica, cujus subtangens est æqualis IM. Ex hac constructione jam satis apparet, scalam celeritatum ODα asymptotam habere Cα, atque adeo mobilis celeritatem quidem continuo decrescere, nunquam vero ad celeritatem MC reduci posse.

PROPOSITIO LXVIII. THEOREMA.

559. *Si gravitas exponatur per secantem alicujus arcus dati, & corpus grave verticaliter in altum projiciatur in aëre præsentis resistentiæ hypotheseos, ac celeritate, quæ exponitur excessu tangentis arctus cujusdam variabilis supra tangentem dati arcus* ; *assumaturque magnitudo quædam, quæ sit ad differentiam arcuum variabilis & dati, ut rec-lum ex secante* & *tangente dati arcus ad quadratum radii.*
Erit 1^o. Spatium, *quod mobile, ascendendo quousque potest, conficiet, ut log-us rationis, quam secans arcus variabilis habet ad secantem dati arcus, dempta magnitudine assumta*.
2^o. Tempus, *quo mobile hanc altitudinem absolvet, erit ut magnitudino assumta applicata ad tangentem dati arcus.*

Sint ABKC quadrans circuli, BIL hyperbola æquilatera ex centro A & semilatere AB

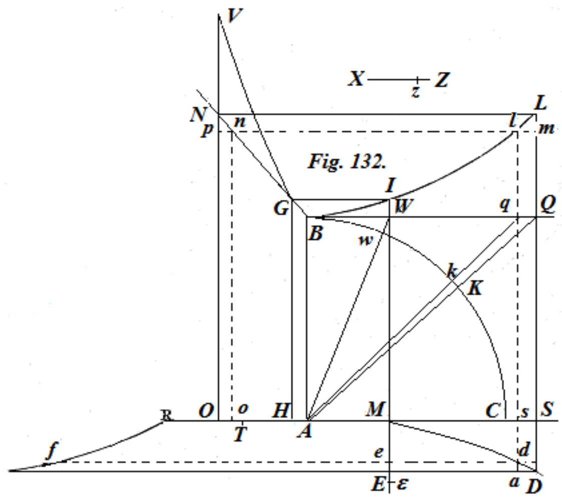


Fig. 132.

descripta. Exponat IM seu AW secans dato arcus Bα gravitatem uniformem, ac mobile ex E in altum projici intelligatur celeritate ED vel WQ, quæ est differentia tangentium BQ, BW arcuum variabilis BK & dati Bα. Descriptaque circa axem AT log-mica BGN, cujus subtangens sit GH vel IM; agantur IG, LN parallelæ AS, ac per puncta G, N, &c. transeant ordinatæ log-micæ GH & NO, in NO productaa ut OV tertia proportionalis ad GH & NO, & per omnia puncta V transibit nova log-mica GV, cujus subtangens dimidia erit subtangentis priora log-micæ GN. Accipiatur dcinde XZ, quæ, sit ad arcum αK, ut rec-lum IMA ad quadratum radii AB, quibus positis dico fore I.ME vel potius $\mathrm{EM} = \mathrm{HO} - \mathrm{XZ}$, & temp. per $\mathrm{EM} = \mathrm{XZ} : \mathrm{AM}$.

Demonstr. Ductis lineis *dl, ln, no* prioribus cognonimis DL, LN, NO ad distantias indefinite parvas parallelis, fiatque ut supra (§. 549. num.1.) $\mathrm{AT} = \mathrm{AM}$, & resistentia aëris in E ut TS.MS.

I. Eritque adeo $\text{resist.} : \text{grav.} = \mathrm{TS.MS} : \mathrm{IM}^2$ & componendo $\mathrm{R} + \mathrm{G}$, seu resistentia totalis EF: IM, seu $\mathrm{G} = \mathrm{IM}^2 + \mathrm{TS.\,MS} : \mathrm{IM}^2$, (seu quia $\mathrm{AM}^2 + \mathrm{TS.\,MS} = \mathrm{AS}^2$, seu $\mathrm{TS.MS} = \mathrm{AS}^2 - \mathrm{AM}^2) = \mathrm{AS}^2 + \mathrm{IM}^2 - \mathrm{AM}^2 : \mathrm{IM}^2 = \mathrm{AS}^2 + \mathrm{QS}^2 : \mathrm{IM}^2$ $= \mathrm{LS}^2 : \mathrm{IM}^2 = \mathrm{NO}^2 : \mathrm{GH}^2$ (aut quia VO, NO &GH sunt in continua ratione) $= \mathrm{VO} : \mathrm{IM}$ vel GH, habemus ergo $\mathrm{EF} : \mathrm{IM} = \mathrm{VO} : \mathrm{IM}$ atque adeo $\mathrm{EF} = \mathrm{VO}$.

II. Ex constructione est $\mathrm{EM} = \mathrm{HO} - \mathrm{XZ}$, & $e\mathrm{M} = \mathrm{H}o - \mathrm{X}z$, adeoque $\mathrm{E}e = \mathrm{O}o - \mathrm{Z}z$, unde, si E*t* fuerit $= \mathrm{O}o$, & $e\mathrm{E} = Zz$,erit omnino $\mathrm{E}e = \mathrm{O}g - Zz$. Jam (§.165.) $\mathrm{Q}q :\ \mathrm{K}k = \mathrm{AQ}^2 : \mathrm{AK}^2 = \mathrm{LS}^2 : \mathrm{AB}^2 \,\&= \mathrm{NO}^2 : \mathrm{AB}^2 = \mathrm{VO.\,GH} : \mathrm{BA}^2$, & (constr.) $\mathrm{K}k : \mathrm{Z}z = \mathrm{AB}^2 : \mathrm{GH.AM}$, ergo ex æquo $\mathrm{Q}q : \mathrm{Z}z = \mathrm{VO.GH} : \mathrm{GH.AM} = \mathrm{VO} : \mathrm{AM}$ (num. I. hujus) $= \mathrm{EF} : \mathrm{AM}$. Adeoque $\mathrm{EF.\,Z}z = \mathrm{EF}.\varepsilon\mathrm{E} = \mathrm{AM.Q}q = \mathrm{AM.S}s$.

III. Est vero etiam (§.491. num. III.)
$\mathrm{VO.O}o\ (= \mathrm{EF}.\varepsilon e) = \mathrm{NO.N}p = \mathrm{LS.L}m\ (§.\ 493.) = \mathrm{AS.S}s.$ Adeoque

$\mathrm{EF.E}e\ (=\mathrm{EF.}\varepsilon e-\mathrm{EF.}\varepsilon\mathrm{E})=\mathrm{AS.S}s-\mathrm{AM.S}s=\mathrm{MS.S}s$. Hoc est, momentum resistentiæ totalis EF æquatur momento celeritatis MS.

IV. Numerus secundus hujus demonstrationis præbet, Ss:EF (num. III hujus) $=\mathrm{E}e:\mathrm{MS}=\mathrm{Z}z:\mathrm{AM},$ atqui, ut toties jam dictum $\mathrm{S}s:\mathrm{EF}=\mathrm{E}e:\mathrm{MS}=t\mathrm{E}e;$ ergo $t\mathrm{E}e=Zz:\mathrm{AM}$, & omnia $t\mathrm{E}e=$ omnibus $\mathrm{Z}z:\mathrm{AM}$ seu $=\mathrm{XZ}:\mathrm{AM}.$ Hoc est, tempus ascensionis in spatio EM, est $\mathrm{XZ}:\mathrm{AM}$, exponitur ergo per magnitudinem assumtam applicatam ad tangentem anguli BAW. Quæ omnia erant demonstranda.

COROLLARIUM I.

560. Altitudo, ad quam grave in altum projectum pertingere potest, eodem tempore, quo in aëre altitudinem EM absolvet, erit iterum ad semissem IM, quæ gravitatem exponit in duplicata ratione assumtæ XZ ad datam AM. Celeritas vero projectionis in vacuo est ad celeritatem projectionis in aëre, sicut IM.XZ ad AM.MS. Demonstratio continetur in corollariis II. & IV. propositionis præcedentis.

COROLLARIUM II.

561. Evanescente AM vel AT, atque adeo coincidentibus rectis IM, BA & GH, redibit iterum casus propositionis LXV. adeo ut omnia, quæ in propositione illa exhibita sunt, tantum corollaria fint præsentis.

Ut brevitati consulerem omisi perplura corollaria, quæ ex hac propositione adhuc elici potuissent, tum etiam calculum arithmeticum, quoque tabularum sinuum tangentium & log-morum, celeritares, spatia percursa, & tempora lationis determinari potuissent. Intelligens Lector omnia sua sponte supplebit. Sed, priusquam ad contemplationem motuum curvilineorum in mediis resistentibus me conferam, forte haud abs re fuerit, si calculo algebraico regulas generales tradidero pro motibus rectilineis in mediis densitate variantibus.