CHAPTER XVI.

Concerning the motion of bodies in air with the resistance in the square ratio of the speed of the moving body.

This hypothesis of the resistance agrees with perfect and rarefied fluids, clearly the parts of which do not therefore adhere together, but are able to depart from each other most freely, when they are struck by solid bodies ; truly in the ratio of the hypothesis given elsewhere (§.427).

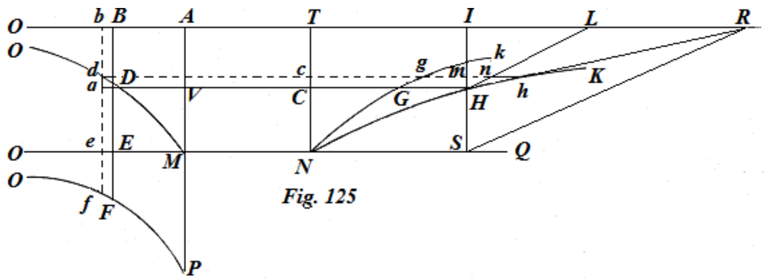


Fig. 125

PROPOSITION LX III. THEOREM.

522 *The motions thus are to be varied from the fundamental uniform motion, with the resistance experienced according to the square ratio of the speeds in air, so that the distance passed through in some time may be expressed by the logarithm of the ratio, which the initial speed of the same moving body has to the remaining speed after a given time. Truly this time shall be in the ratio of the speed of the body lost to the residual speed.*

[In modern terms, we have the differential equations for the motion with the initial speed v_0, assuming unit mass : $\frac{dv}{dt} = -kv^2$ and $v\frac{dv}{dx} = -kv^2$ or $\frac{dv}{dx} = -kv;$ the latter equation can be integrated at once to give : $\frac{dv}{v} = -kdx$; $\ln\frac{v}{v_0} = -kx$ and $v = v_0 e^{-kx}$. In turn, the time dependent equation : $\frac{dv}{dt} = -kv^2$ becomes $\frac{dv}{v^2} = -kdt$, giving $\frac{1}{v} = kt + \frac{1}{v_0}$ or $v = \frac{v_0}{1+kv_0 t}$; in which case, on integrating again : $\frac{dx}{dt} = \frac{v_0}{1+kv_0 t}$ gives $x = \frac{1}{k}\ln\left(1+kv_0 t\right)$.

If we consider the complementary problem, in which the air has the constant speed v_0, and the body is initially at rest at the origin, then
$\frac{dv}{dt} = kv^2$ and $v\frac{dv}{dx} = kv^2$ or $\frac{dv}{dx} = kv;$ hence the same solutions apply with the sign of k changed, and with different starting conditions. Thus we obtain :
$\frac{dv}{v} = kdx$; $\ln\frac{v}{v_0} = kx$ and $v = v_0\left(1-e^{-kx}\right)$, while $\frac{dv}{dt} = kv^2$ gives on integrating,
$x = v_0 t\left(1-e^{-kx}\right)$.]

I. Now MO shall be the carrying line [in the moving air case as in Ch. 15 prelim.

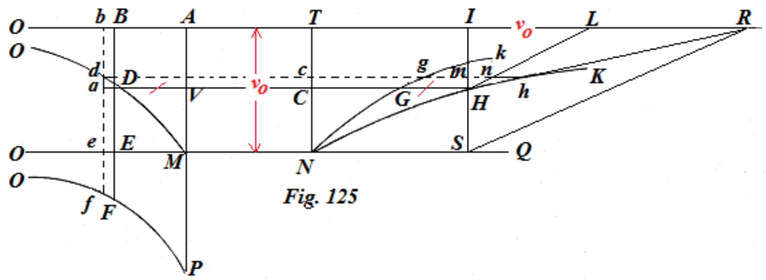


Fig. 125

notes], opposite which NQ is understood equally to be the resting air on which the body to be incident in the same manner, as has been said in proposition LVIII (§ 495) ; indeed the moving body M clearly may be carried towards O in the opposite direction on the carrying line, as which equally it may proceed towards Q [recall that the speeds are complementary, in the sense that they add up to the initial speed of the body to the right, or to the speed of the airflow to the left at any instant]; besides, the graph MDO shall be the speed of the body travelling with its own motion along the carrying line, and PFO the graph of the acceleration acting or of the impulses of the air sliding past the body. From which, if in some time the body M has completed the distance ME on the carrying line, and at the end it has acquired the speed DE, the absolute speed [i.e. relative to the body] of the body will be BD, with which the air will be moving past the body M , from which by the hypothesis there will be MP to EF in the square ratio AM to BD; that is $\mathrm{MP}:\mathrm{EF}=\mathrm{AM}^2:\mathrm{BD}^2$. Indeed AM, MN or NT, may be put to designate the initial velocity. Now with the hyperbola NHK drawn through the point N ; between the asymptotes MA & AR , and NG*k* the logarithmic of the subtangent NT; on the indefinite line DCH parallel to OQ, and drawn somewhere between the parallel lines OR and OQ, VD may be made equal to the intercept GH between the hyperbola NHK and the logarithmic curve NGk, and the point D on the graph of the speed of the body itself MDO. Through the point *d* indefinitely close to the other D, *dh* is acting parallel to OQ, and *bf* parallel to the other BF; and with the right lines IS and HL drawn through the point H of the hyperbola, thus so that $\mathrm{IL}=\mathrm{NT}$ (or equal to the subtangent of the logarithmic NGk at the point G, and thus HL shall be parallel to the tangent of the logarithmic at G, or to the element G*g*) and finally with HR the tangent of the hyperbola, thus so that AI shall be equal to IR, and $\mathrm{LR}=\mathrm{TI}=\mathrm{NS}$; with which done, there will be $nh=\mathrm{E}e=\mathrm{D}a$, and $\mathrm{H}m=da$.

II. The hyperbola gives, $\mathrm{AI}:\mathrm{AT}=\mathrm{TN}:\mathrm{IH}$, [*i.e.* the hyperbola satisfying AI.IH=AT.TN] and on dividing [*i.e.* $\mathrm{AI}:\mathrm{AT}-1=\mathrm{TN}:\mathrm{IH}-1$, i.e. TI:AT = HS:IH]
LR or $\mathrm{TI}:\mathrm{AT}=\mathrm{DE}$ or $\mathrm{HS}:\mathrm{BD}$ or IH, truly from the hypothesis AT or AM or $\mathrm{MP}:\mathrm{BD}=\mathrm{BD}:\mathrm{EF}$, therefore from the equation $\mathrm{LR}:\mathrm{IH}$ or $\mathrm{BD}=\mathrm{DE}:\mathrm{EF}$; truly there is $\mathrm{LR}:\mathrm{IH}=nh$ or $\mathrm{E}e:\mathrm{H}m$ or da, therefore also $\mathrm{DE}:\mathrm{EF}=\mathrm{E}e:da$; and thus $\mathrm{EF}.\mathrm{E}e=\mathrm{DE}.da$.
That is the *moment of the acceleration acting* EF *is equal to the moment of the speed of*

the body M *acquired on the carrying line* DE. Therefore (§. 488.) the curve MDO is the graph of the speed, and PFO the graph of the resistances, or of the impulses of the air, or of the actions of the accelerations of the moving body on the resisting line, indeed these all signify the same.

III. SR may be considered and that shall be parallel to HL ; for $\mathrm{IH : IS}$ or $\mathrm{TN = IL}$ or $\mathrm{AT : IR}$ or Al. And thus [from similar triangles,] $\mathrm{IS : HS = AM : DE = IR : LR} = mh : nh$ or $\mathrm{E}e$, hence $\mathrm{E}e : \mathrm{DE}$ (§.128.) $= t\mathrm{E}e = mh : \mathrm{AM}$; therefore the sum of all the $t\mathrm{E}e$ that is $t\mathrm{ME = CH : AM}$, hence $\mathrm{AM}.t\mathrm{ME = CH}$, & $\mathrm{AM}.t\mathrm{ME}, -\mathrm{ME}$ or (§.489) the absolute distance, which the body has passed through in the air, will be $\mathrm{CH - GH = CG}$, and thus it can be expressed by the logarithm of the ratio, which the initial speed of the body TN has to IH or BD on the logarithmic curve NG*k*, the subtangent of which is NT. Which was the first part.

IV. No. III. was $t\mathrm{ME = CH : AM = LR : IL = SH : HI = DE : BD}$; that is, the time in which the body has resolved its absolute distance in air the distance ME, or by its own motion on the carrying line, is expressed by the ratio, which the part AM of the initial speed has been lost DE in that time to the remaining BD. Which was the second part.

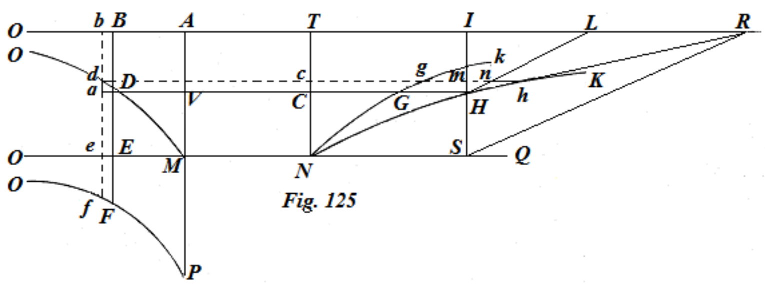


Fig. 125

COROLLARY I.

523. And thus, if the times were in an ascending geometrical progression, and thus the velocities of the body remaining after these times also were in a geometrical progression, but descending, and indeed in the inverse of the progression of the times, the distances past through will be in an arithmetical progression. For, if AT, AI shall be in a geometrical progression, the differences TI, which are as the times, will be in the same ascending progression ; indeed HI themselves, or the speeds AI of the body remaining themselves inversely proportional will be in a descending geometrical progression, truly the distances or CG, will appear in an arithmetical progression. And this corollary is in complete agreement with Prop. V. Book. II. *Princ. Phil. Nat. Math*. by the celebrated Newton.

COROLLARY II.

524. Therefore with regard to this hypothesis of the resistance the body departing to infinity, in an infinite time as CH, where it approaches indefinitely close to the asymptote

TR, also traverses the infinite distance CG ; for, if CG be placed together with the asymptote TR, it shall be infinite. Indeed in the hypothesis of the preceding chapter, the body cannot complete a finite distance in an infinite time, which is set out by the right line indicating the initial speed it can resolve, as shown above (§. 497). Which has been observed by Huygens to be worth mentioning in his tract *De La Cause de la Pesanteur* pag. 175 around the end.

PROPOSITION LXIV. THEOREM.

525. *If a weight may descent by beginning to fall vertically from rest by the force of uniform gravity in air with the resistance in the square ratio of the speed,*

The distance *completed in the descent will be expressed by the logarithm of the ratio of the whole sine to the sine of the complement of that angle, of which the right sine represents the speed acquired;*

The time, *in which that distance is passed through, or of the aforementioned speed of the body acquired, will be expressed by the logarithm of the ratio of the whole sine to the tangent of half the complement of the aforementioned angle, applicable to the logarithm of the subtangent.*

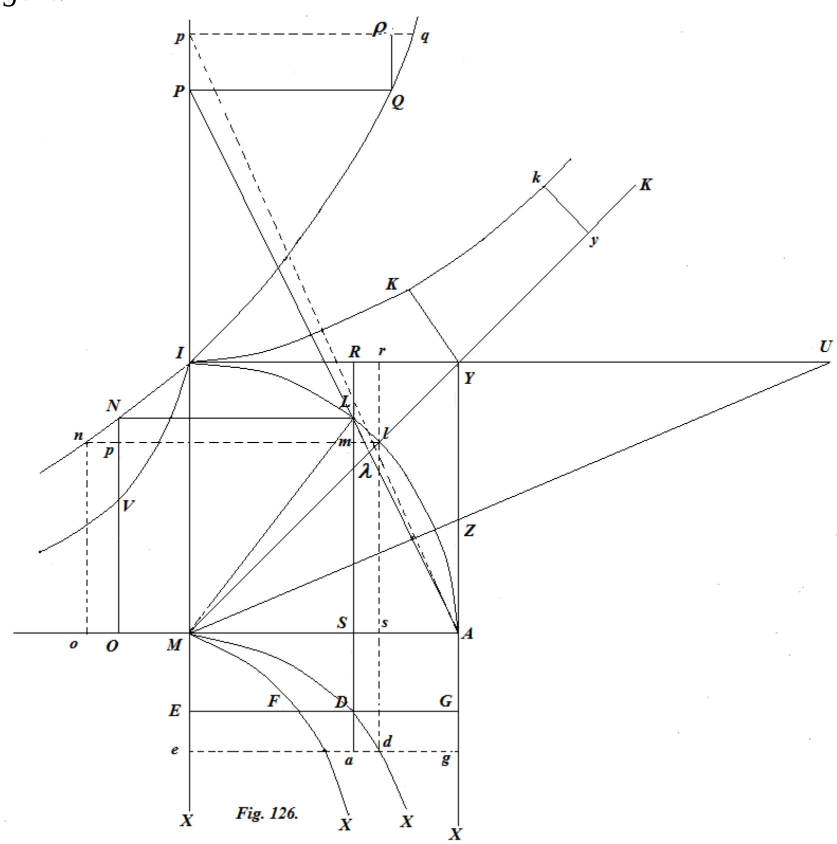


Fig. 126.

[In modern terms, for the downward motion with uniform acc. of grav. g, and with the air resistance proportional to the square of the speed, the body reaches a terminal velocity

v_t, where $kv_t^2 = g$ for unit mass. The acceleration downwards can be written as $\frac{dv}{dt} = k\left(v_t^2 - v^2\right)$ and again, $v\frac{dv}{dx} = k\left(v_t^2 - v^2\right)$; the first equation can be integrated as follows:

$\frac{dv}{v_t^2 - v^2} = \frac{dv}{v_t - v} + \frac{dv}{v_t + v} = 2v_t kdt$, giving $\log\frac{v_t + v}{v_t - v} = 2v_t kt$ or $\frac{v_t - v}{v_t + v} = e^{-2v_t kt}$, with no added constant needed. Hence $v_t - v = \left(v_t + v\right)e^{-2v_t kt}$ or $v = v_t\left(\frac{1-e^{-2v_t kt}}{1+e^{-2v_t kt}}\right) = v_t \tanh v_t kt.$ This equation in turn integrates to give : $x = \frac{v_t}{k}\log\cosh v_t kt$; The other equation $v\frac{dv}{dx} = k\left(v_t^2 - v^2\right)$ can be written as $\frac{d\left(v^2\right)}{v_t^2 - v^2} = 2kdx,$ which integrates to give $v^2 = v_t^2\left(1 - e^{-2kx}\right)$.]

The weight M may begin to fall from rest along the vertical right line MX, to which the other AO shall be attached at right angles ; MA may be taken on AO, which expresses uniform gravity, and with the quadrant of the circle ILA described with centre M and with the radius MA and the equilateral hyperbola IK*k*, and also the logarithmic curve NIQ, of which the subtangent will be equal to the radius of the quadrant, or to the semilatus transversal MI of the hyperbola MI ; the common tangent IU is acting through the point I of the common tangent of the hyperbola and of the quadrant, which will be parallel to MA itself. Then also through some point L of the quadrant LN may be drawn parallel to AO crossing at the point N of the logarithmic curve ; and the right line ALP from the radius MI producted crossing at P, through which point is drawn PQ parallel to MA above and crossing with the logarithmic at the point Q and besides NO shall be the ordinate of the logarithmic curve drawn through the point N. Hence again on the indefinite right line MX making the segment $\mathrm{ME} = \mathrm{MO}$, and thus everywhere respectively; and with the right line EG drawn through the point E, and LS through the point L, respectively parallel to the radii MA and MI, and the common intersection of these D will be on *graph of the speed acquired* MDX, thus indeed, so that the body after it had passed through the distance ME, at the end E of this distance had acquired the speed ED or MS. And everywhere with EF made the third proportional after the right lines EG & ED, the point F will be on the graph of the resistances of the air MFX, which graph also will be the *graph of the acceleration acting* on the falling body on the line MX, but in as much as that refers to the axis AX, towards which the curve MFX is convex ; for, since EF, or $\mathrm{ED}^2 : \mathrm{EG}$, expresses the resistance of the medium at the point E and $\mathrm{EG} = \mathrm{MA}$ expresses uniform gravity, FG expresses everywhere the acceleration acting at the same point E, as has been said now elsewhere (§.481.). Therefore with *eg* drawn parallel to EG and from that with the element of the length E*e* distant; *dlr* produced parallel DL at *a*, and upwards at R, through the point of the quadrant *l*, with the right line A*lp*, and the line *pq* drawn through the point *p* parallel to PQ; and finally with the ordinate distance *no*, $\mathrm{O}o = \mathrm{E}e$ at the distance from the other NO, on which NO shall be VO the third proportional to IM and NO, and with these in place as shown elsewhere nearby (§. 484), it only remains to be proven, FG.E*e*, or the rectangle FG*g*, the *moment of the acceleration acting* FG to be equal to the rectangle ED.*ud*, or to the *moment of the speed* ED *in the case of the body acquired in falling from the height* ME. With which proven the rest will be obtained at once, clearly by expressing the time of descent along the distance ME, and from the nature of the curves MDX, MFX., &c.

Demonst. I. There is (following the hypothesis) $\mathrm{EG}:\mathrm{EF}=(\mathrm{EG}^2:\mathrm{ED}^2)=\mathrm{ML}^2:\mathrm{MS}^2$, and by interchanging there shall be $\mathrm{EG}:\mathrm{FG}(=\mathrm{ML}^2:\mathrm{LS}^2=\mathrm{IM}^2:\mathrm{NO}^2$ or because IM, NO & VO are in a continued ratio$)=\mathrm{IM}:\mathrm{VO}$, hence $\mathrm{VO}=\mathrm{FG}$, & $\mathrm{VO.O}o=\mathrm{FG.E}e$. And (§.491. no. III.) $\mathrm{VO.O}o:\mathrm{NO.N}p=\mathrm{LS.L}m(\S 493)=\mathrm{MS.S}s:\mathrm{ED.}ad$; therefore FG. E*e*= ED.*ad*. Which was the first part.

II. On account of the similar triangles MLS & Llm, there shall be ad (or ml)$:l\lambda$ (or, §. 463. no. III , L*l*) $=\mathrm{LS}:\mathrm{ML}=\mathrm{NO}:\mathrm{IM}=\mathrm{VO}:\mathrm{NO}$; and $l\lambda:\mathrm{P}p=\mathrm{NO}$ (or L*S*)$:\mathrm{MP}$; and finally from the nature of the logarithmic curve (§. 491. no.1.) $\mathrm{P}p:q\rho=\mathrm{MP}:\mathrm{IM}$, from the equality there will be $ad:pq=\mathrm{VO}:\mathrm{IM}$, or on interchanging $ad:\mathrm{VO}=pq:\mathrm{IM}$. And (§.131. and §.485.) $ad:\mathrm{VO}=t\mathrm{E}e$, that is, the increment from the element of the speed , applied to the acceleration acting VO, expresses the small time interval, in which the element of the distance E*e* is traversed ; therefore also $\rho q:\mathrm{IM}=t\mathrm{E}e$, and thus [the sum of] all the $\rho q:\mathrm{IM}$, that is, PQ.IM = to the sum of all the *tEe*, or to the time of descent through the distance ME.

III. Now MO or ME, that is, the distance completed in the descent, is the logarithm of the ratio IM to NO, or IM to LS, that is the logarithm of the whole sine to the sine of the complement of the angle IML, of which sine the right line MS expresses the speed acquired at E . PQ truly is the logarithm of the ratio PM to IM, or IU to IM, clearly with a line drawn from the centre M perpendicular to the above AP , and with that produced as far as to crossing with the tangent IU ; or finally of the ratio MA to AZ, that is, the whole sine to the tangent of the half angle LMA, that is, of half the complement of the angle IML, the sine of which MS expresses the speed acquired. Therefore the time to pass through ME, which no. II. of this expressed by PQ: IM, must be expressed by the logarithm of the ratio ($\mathrm{MA}:\mathrm{AZ}$) of the whole sine to the tangent of half the complement of the angle IML applied to the radius IM, or to the subtangent of the logarithmic curve . All of which were required to be shown.

COROLLARY I.

526. Therefore with MR drawn as far as to the crossing K with the hyperbola IK [this line has not been drawn on the diagram], the three-lined hyperbolic figure IMK applied to half the quadrant IM, also expresses the time of descent through the distance ME. For (§. 463.) twice the trilinear $\mathrm{IMK}=\mathrm{rect.\ PQ.IM}$, therefore the trilinear form $\mathrm{IMK}=\mathrm{PQ}.\frac{1}{2}\mathrm{IM}$; Now, because (§.525. no. II.) *t*ME = PQ: IM= PQ.IM:IM^2 , there will be $t\mathrm{ME}=$ twice the trilin. $\mathrm{IMK}:\mathrm{IM}^2=\mathrm{trilin.IMK}:\frac{1}{2}\mathrm{IM}^2$.

Equally with the asymptote M*y* drawn to the hyperbola IK*k*, and with the ordinate KY sent from K to the asymptote KY, to which some other *ky* may itself be had, as LS to IM, the four-sided hyperbolic figure K*ky*Y applied to half of IM expresses the distance traversed ME. For, because (constr.) $\mathrm{KY}:ky=\mathrm{IM}:\mathrm{NO}$, there will be (§.368.)
$\mathrm{K}ky\mathrm{Y}:\mathrm{MY.KY}$ or $\frac{1}{2}\mathrm{MI}^2=\mathrm{OM}:\mathrm{IM}$, and thus $\mathrm{OM}=\mathrm{ME}=\mathrm{K}ky\mathrm{Y}:\frac{1}{2}\mathrm{IM}$.

COROLLARY II.

527. The graph of the acceleration acting MFX is the logarithmic, of which the subtangent is half of IM. For, since above (§.525. no.I.) it was shown that $\mathrm{VO}=\mathrm{FG}$ and (by constr.) $\mathrm{OM}=\mathrm{ME}=\mathrm{AG}$, it follows that the curve MFX is similar and equal to the curve IV ; and this curve is the logarithmic, of which the subtangent is the half of IM if the subtangent of the logarithmic IN, since any VO is the third proportional to IM and NO; therefore also MFX is a logarithmic, of which the asymptote is AX. Therefore, if the accelerations acting FG are in a decreasing geometric progression, the distances passed through are in an increasing arithmetical progression.

COROLLARY III.

528. Thus the *terminal or maximum* velocity is MA, which corresponds to the force of uniform gravity, since the line AX shall be parallel to the asymptote MX of each of the curves MFX and MDX.

COROLLARY IV.

529. The time, in which a weight has acquired the speed DE in air, is to the time in which it would acquire the same speed in vacuo, as PQ to DE. For the time (§.151), in which the speed DE is acquired in vacuo, is DE:IM or AM, and the time, in which the same DE is acquired in air, is as $\mathrm{PQ}:\mathrm{IM}$.

COROLLARY V.

530. In turn the speed, which is acquired in the time $\mathrm{PQ}:\mathrm{IM}$ in air, is to the speed, which it can acquire in the same time in vacuo, as DE to PQ. For in the time (§.151.) $\mathrm{PQ}:\mathrm{IM}$ the speed PQ is acquired in vacuo.

SCHOLION.

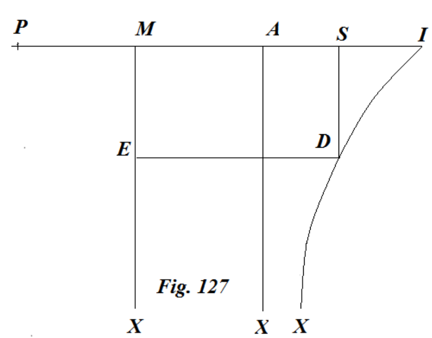


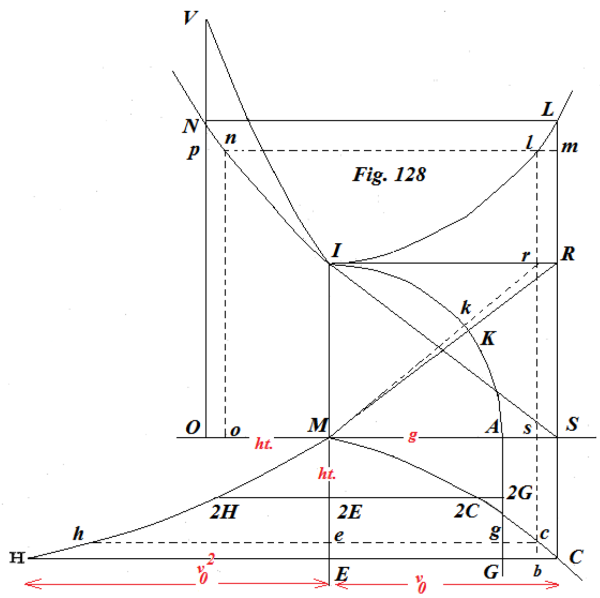
Fig. 127

531. If the weight M may be projected perpendicularly upwards along the right line MX and with the speed MI put in place to be itself greater than the terminal speed MA or MP; the speed of the body is reduced from MI to MS (for, because the velocity of projection MI is greater than AM, the resistance of the medium will be greater than the weight, and therefore the motion

must be continually slowed down) the distance traversed ME being expressed by $\log.\sqrt{(\mathrm{PIA}:\mathrm{PSA})}$ or the logarithm of the square root of the ratio of the rectangle PL.AI to PS.AS; truly the time, in which the body has traversed the distance ME, or in which the initial velocity MI is reduced to ED or MS, is expressed by $\log.\sqrt{(\mathrm{AI}:\mathrm{AS})}-\log.\sqrt{(\mathrm{PI}:\mathrm{PS})}$. Truly on putting $\mathrm{MP}=\mathrm{MA}$, these latter logarithms are taken on the logarithmic, of which the subtangent is unity, truly the log. from $\sqrt{(\mathrm{PIA}:\mathrm{PSA})}$ is taken on the logarithmic, of which the subtangent is AM or MP. If indeed the speed of projection MI were less than MA, now in this case it will be obtained from the present proposition, so that thus it shall not be necessary to tarry longer.

PROPOSITION LXV, THEOREM.

532. *With the right line, which expresses the terminal speed or the force of uniform gravity, taken for the whole sine or radius ; if some body, with the initial speed expressed by the tangent of some angle, may be projected vertically upwards,* the maximum height, *to which the body is able to reach in air, will be expressed by the logarithm of the ratio of the secant of the angle to the radius, and* its time of ascent *will be expressed by that angle, whose tangent expresses the initial speed.*



[In modern terms, for the upwards motion with uniform acc. of grav. g, and with the air resistance proportional to the square of the speed both acting down; the terminal velocity v_t, where $kv_t^2 = g$ for unit mass can still be used. The acceleration downwards can be written as $\frac{dv}{dt}=-k\left(v_t^2+v^2\right)$ and again, $v\frac{dv}{dx}=-k\left(v_t^2+v^2\right)$; the first equation can be integrated as follows:

$\frac{dv}{v_t^2+v^2}=-kdt$; hence $\operatorname{atan}\frac{v}{v_t}=-kv_tt+c$, giving

$v=v_t\tan\left(c-kv_tt\right)$, where $c=\operatorname{atan}\left(\frac{v_i}{v_t}\right)$ and v_i is the initial velocity. This equation in turn integrates to give : $x=\frac{1}{k}\log\left(\cos v_tkt+\frac{v_i}{v_t}\sin v_tkt\right)$; The other equation

$v\frac{dv}{dx} = -k\left(v_t^2 + v^2\right)$ can be written as $\frac{d\left(v^2\right)}{v_t^2+v^2} = -2kdx,$ which integrates to give

$\log\left(v^2 + v_t^2\right) = \left(1 - e^{-2kx}\right) + \log\left(v_i^2 + v_t^2\right)$.]

The body at M may rise with the initial speed EC from E on the vertical line EM, and the quadrant of a circle IKA with centre M, the equilateral hyperbola IL, and the logarithmic IN are described, as above. CE may be produced to H and with EH made the third proportional to EG and EC, and EH will be the resistance, which the ascending body will suffer from the air ; and moreover because EG, or MA, expresses the uniform weight, which likewise acts against the body raised to some height, thus (§. 481) the *total opposing resistance* to the body at E will be equal to GH. And thus a certain curve HM will be the graph of the total resistance, truly the curve CM will be the graph of the decreasing speed EC. And thus (§.484.) matters can be deduced from that, so that these named curves with this property may be constructed, in order that everywhere , there shall be $\text{HE.E}e = \text{EC.}b\text{C}$. To that, with CL and NO drawn parallel to the line MI, and LN drawn parallel to the line MA, respectively, there is made $\text{ME} = \text{MO}$, and thus , just as many points will be given everywhere on the curve C2CM as wished, from which henceforth the other curve HM will not be difficult to construct. *cl*, *ln*, *n*o may be drawn parallel and infinitely near to the former CL, LN and NO, and there is taken on some ON a greater OV, which shall be to the homologous ON, as this NO to MI; which MI likewise also indicates the subtangent of the logarithmic IN, and a new logarithmic IV will result in this ratio, of which the subtangent will be half of the subtangent IM of the logarithmic IN.

Demonst. I. Because the right lines EG, EC & EH (following the hypothesis) are in continued proportion, there will be, there will be $\text{EG} : \text{EH} = \text{EG}^2 : \text{EC}^2$, or by inverting and adding together $\text{HG} : \text{EG} = \text{IS}^2 : \text{lM}^2 = \text{LS}^2 : \text{IM}^2 = \text{NO}^2 : \text{IM}^2$ and (or on account of the continued proportionals VO, NO and IM) $= \text{VO} : \text{IM}$ or EG , therefore $\text{HG} = \text{VO}$; and because $\text{OM} = \text{EM}$, and $o\text{M} = e\text{M}$, and hence $\text{O}o = \text{E}e$ *;* there will be $\text{HG.G}g$, or $\text{HG.E}e = \text{VO.O}o$ (§ 491.no. III.) $= \text{NO. N}p = \text{LS.L}m$ (§ 493.) $\text{MS.S}s = \text{EC.C}b$. That is, the *moment of the decrease of the speed* EC is equal to the moment of the total resistance HG [*i.e*. essentially the work/energy relation : $vdv = \left(g + v^2\right)dh$.]: For ME or MO is the logarithm of the ratio NO to IM, or IS to IM, that is, the logarithm of the ratio of the secant of the angle MIS, of which the tangent $\text{MS} = \text{EC}$, expresses the initial speed, to the radius IM or MA. Which was the first to be shown.

II. Because $\text{VO} : \text{IM} = \text{NO}^2 : \text{lM}^2 = \text{LS}^2 : \text{IM}^2 = \text{MR}^2 : \text{MK}^2$ $(§.165.)$ $= \text{R}r : \text{K}k$, and therefore, because $\text{R}r = \text{C}b$, there will be
$\text{K}k : \text{IM} = \text{C}b : \text{VO} = \text{C}b : \text{HG}$ (§.131, 485.) $= t\text{E}e$, & K*k*:MK or
IM (§.129.) $=$ angle KM*k* , and thus $t\text{E}e =$ angle KM*k* , and therefore the time of the total ascent through the distance EM must be expressed by the angle KMI, of which the tangent is IR or EC. Which was the second part to be shown.

COROLLARY I.

533. Now: the curve of the total resistances M2HH again will be a logarithmic similar and equal to the logarithmic IV. And thus if the total resistance were taken in a descending geometric progression, the distances requiring to be risen will be in a descending arithmetical progression, evidently described by the whole distance risen. The demonstration of which is almost the same as that in Coroll. II. of the preceding prop. §. 527.

COROLLARY II.

534. The initial velocity of the body completing its maximum height EM in air, is to the initial speed, by which the body can complete it maximum height in vacuo in an equal time, as the tangent IK to the arc IK. For, if the body ascends in vacuo with the initial speed IK , that is, it will ascend as far as it can in the time $IK : IM$ (§.§. l41. 151.), but in this time $IK : IM$ with the angle KMI established, in air it will rise the distance EM, with the initial speed EC or IR. Hence, &c. And this agrees with Coroll. 4. Prop. 9. Lib. II. *Princ. Phil. Nat. Math.* of the celebrated Newton.

COROLLARY III.

535. The time resolved, in which the body can arrive at its maximum height in air with the initial velocity IR ; is to the time it would take, in which it can reach its maximum height in vacuo, with the same initial speed present IR, as the arc IK to the tangent IR, or, what is the same, as the sector IMK to the triangle IMR. For the time of the ascent in air is the angle IMK, or $IK : IM$, and the time of the ascent in vacuo (§.151.) is $IR : IM$.

COROLLARY IV.

536. Thus, if the initial speed, by which bodies ascended in air and in vacuo, were for equal terminal heights, the time will be, for the height of one risen in air, to the time, with its height risen of the other in vacuo, is compared as the circle to the circumscribed square ; for, if $IR = IM$, the sector IMK will be the eighth part of the circle and triangle IMR the eighth part of the circumscribed square to the circle, and thus the one to the other, as the whole circle to the circumscribed square to the circle.

COROLLARY V.

537. At this point, with the same in place, as in coroll. III. the height, that a body in air completes, is to the height being traversed in vacuo, as twice the rectangle IMO to the square of the tangent IR. For the height (§.150.) being described in vacuo will be $IR^2 : 2IM$, and ME is to $IR^2 : 2IM$, as twice the rectangle IMO to IR^2.

COROLLARY VI.

538. Thus, if the initial speed IR were equal to the terminal speed IM , the height being completed in air to the height in vacuo will be, or $2.\mathrm{IMO}$ to IR^2, as 2MO to IM, that is, as $\log.\left(2.\mathrm{IM}^2 : \mathrm{IM}^2\right)$ to IM, that is, as the logarithm of the ratio doubled to the subtangent of the logarithmic, or also (§. 368.) as the hyperbolic four sided figure of each adjacent asymptote, of which the ordinates are in the squared ratio, to the rectangular hyperbola between the asymptotes. As the celebrated Huygens (pag. 174 Diff. *De Causa Gravitatis)* asserted without proof.

COROLLARY VII.

539. The initial speed of the body describing the distance EM in rising, is to the speed, with which it returns again to the ground, as the tangent of the angle IMR to its sine. For, since the distance ME in the figures 126 and 128 are equal to each other (following the hypothesis), therefore the ratio $\mathrm{ON} : \mathrm{IM}$, in fig.126. or $\mathrm{LS} : \mathrm{LM} =$ in fig. 128. shall be $\mathrm{IM} : \mathrm{NO} = \mathrm{RS} : \mathrm{MR}$; and therefore the right-angled triangle LMS in fig. 126. is similar to the triangle RMS, thus so that the angle IML in fig. 126 $= \mathrm{IMK}$, and IR is the tangent and MS in fig. 126 is the sine of one and the same angle IMK or IML, hence etc.

And from these many other corollaries could be added, but on account of brevity I leave such from the preceding to be elicited to the industry of the reader.

CAPUT XVI.
De motu Corporum, in aëre resistente in duplicata ratione celeritatum mobilis.

Hæc resistentiæ hypothesis convenit fluidis perfectis atque raris, quorum scilicet partes prorsus non cohærent, sed liberrime, ubi corporibus solidis impegerunt, recedere queunt; ipsius vero hypotheseos ratio alibi (§. 427) jam data est.

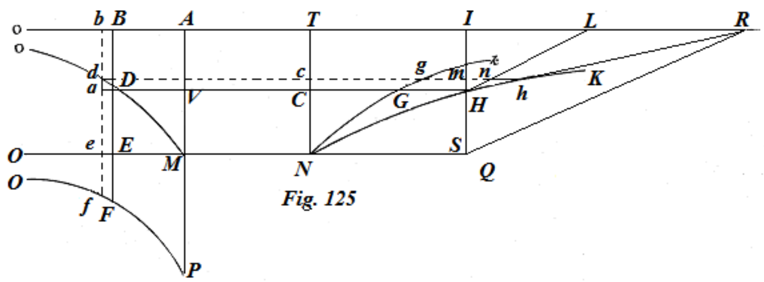


Fig. 125

PROPOSITIO LX III. THEOREMA.

522 *Motus variati ex primitive uniform in aëre juxta duplicatam celeritatum rationem resistente ita se habent, ut spatium transmissum aliquo tempore exponatur log-mo rationis, quam habet celeritas initialis mobilis ad residuam eidem mobili post dictum tempus,ratione celeritatis amissæ mobilis ad residuam.*

I. Sit nunc linea MO deferens, quæ juxta NQ quiescentem æquabiliter incedere intelligitur eodem modo, quo ad propositionem LVIII. dictum est (§.495) *;* mobile vero M feratur ad oppositatem partem scilicet versus O in deferente linea, quæ versus Q æquabiliter procedit; sint præterea MDO scala celeritatum mobilis motu proprio in deferente linea currentis, & PFO scala solicitationum acceleratricium seu impulsuum aëris mobili allidentis. Unde, si aliquo tempore mobile M confecerit in linea deferenti spatium ME, & in ejus termino acquisiverit celeritatum DE, velocitas absoluta mobilis in aëre erit BD, quacum aër mobili M allabetur, unde per hypothesin erit MP ad EF in duplicata ratione AM ad BD; hoc est $\mathrm{MP}:\mathrm{EF}=\mathrm{AM}^2:\mathrm{BD}^2$. Ponitur enim AM, seu MN vel NT, pro designanda velocitate initiali. Jam ductis per punctum N ; intra asymptotas MA & AR hyperbola NHK, & log-mica NGk subtangentis NT; in linea indefinita DCH parallela OQ & quomodocunque inter parallelas OR ac OQ ducta, fiat VD æqualis interceptæ GH inter hyperbolam NHK & log-micam NGk, eritque punctum D in scala celeritatum mobilis propriarum MDO. Per punctum *d* alteri D indefinite vicinum agantur *dh* parallela OQ, & *bf* æquidistans alteri BF; ac ductis per punctum H hyperbolæ rectis IS & HL, ita ut $\mathrm{IL}=\mathrm{NT}$ (seu = subtangenti logmicæ NGk in puncto G,atque adeo HL parallela sit tangenti log-micæ in G, seu elemento G*g*) ac denique tangente hyperbolæ HR, adeo ut AI sit $=\mathrm{IR}$, & $\mathrm{LR}=\mathrm{TI}=\mathrm{NS}$; quibus factis erit $nh=\mathrm{E}e=\mathrm{D}a, \& \ \mathrm{H}m=da$.

II. Hyperbola præbet, $\mathrm{AI}:\mathrm{AT}=\mathrm{TN}:\mathrm{IH}$, ac dividendo LR vel $\mathrm{TI}:\mathrm{AT}=\mathrm{DE}$ vel $\mathrm{HS}:\mathrm{BD}$ seu IH , ex hypothesi vero AT vel AM aut $\mathrm{MP}:\mathrm{BD}=\mathrm{BD}:\mathrm{EF}$, ergo ex æquo $\mathrm{LR}:\mathrm{IH}$ vel $\mathrm{BD}=\mathrm{DE}:\mathrm{EF}$; est vero $\mathrm{LR}:\mathrm{IH}=nh$ vel $\mathrm{E}e:\mathrm{H}m$ seu da, ergo etiam $\mathrm{DE}:\mathrm{EF}=\mathrm{E}e:da$; atque adeo $\mathrm{EF}.\mathrm{E}e=\mathrm{DE}.da$. Hoc est *momentum solititationis acceleratricis* EF *æquatur momento celeratis in deferenti linea mobili* M *acquisitæ* DE. Ergo (§. 488.) curva MDO est scala celeritatum, & PFO scala resistentiarum, seu impulsuum aëris vel solicitatiunum acceleratricium mobilis in linea deferenti, hæc enim omnia idem significant.

III. Agatur SR eritque ea parallela HL ; nam $\mathrm{IH}:\mathrm{IS}$ vel $\mathrm{TN}=\mathrm{IL}$ vel $\mathrm{AT}:\mathrm{IR}$ vel Al. Adeoque $\mathrm{IS}:\mathrm{HS}=\mathrm{AM}:\mathrm{DE}=\mathrm{IR}:\mathrm{LR}=mh:nh$ vel $\mathrm{E}e$, hinc $\mathrm{E}e:\mathrm{DE}$ (§.128.) $=t\mathrm{E}e=mh:\mathrm{AM}$; ergo omnia $t\mathrm{E}e$ id est $t\mathrm{ME}=\mathrm{CH}:\mathrm{AM}$, hinc $\mathrm{AM}.t\mathrm{ME}=\mathrm{CH}$, & $\mathrm{AM}.t\mathrm{ME},-\mathrm{ME}$ seu (§.489) spatium absolutum, quod mobile in aëre transmittit, erit $\mathrm{CH}-\mathrm{GH}=\mathrm{CG}$, adeoque exponi debet logarithmo rationis, quam celeritas initialis mobilis TN habet ad IH vel BD in logarithmicæ NGk, cujus subtangens NT. Quod erat primum.

IV. Num.III. erat $t\mathrm{ME}=\mathrm{CH}:\mathrm{AM}=\mathrm{LR}:\mathrm{IL}=\mathrm{SH}:\mathrm{HI}=\mathrm{DE}:\mathrm{BD}$; id est, tempus quo mobile spatium suam absolutum in aëre absolvit, seu motu proprio in linea deferenti spatium ME, exponitur ratione, quam habet celeritatis initialis AM pars hoc tempore extincta DE ad residuam BD. Quod erat secundum.

COROLLARIUM I.

523. Adeoque, si tempora fuerint in progressione geometrica ascendente, atque adeo velocitates mobili post hæc tempora residuæ etiam in progressione geometrica, sed descendente, & quidem reciproca progressionis temporum, spatia transmissa erunt in progressione arithmetica. Nam, si AT , AI sint in progressione geometrica, differentiæ TI, quæ sunt ut tempora, erunt in eadem progressione ascendente ; ipsæ vero HI, seu celeritates mobili residuæ ipsos AI reciproce proportionales erunt in progressione geometrica descendente, spatia vero, seu CG, existent in progressione arithmetica. Atque cum hoc corollario penitus consentit Prop. V.Lib. II. Princ. Phil. Nat. Math. Celeberr. Newtoni.

COROLLARIUM II.

524. ldcirco in ista resistentiæ hypothesi mobile in infinitum excurrens, tempore infinito ut CH, ubi infinite accesserit ad asymptotam TR, percurret etiam spatium infinitum CG ; nam, si CG confunditur cum asymptota TR, sit infinita. In hypothesi vero capitis præcedentis, mobile tempore infinito ne quidem spatium finitæ magnitudinis, quod per rectam celeritatem initialem indicantem exponitur, absolvere potest, ut supra (§. 497) ostensum. Quod Hugenio notatu dignum merito visum est in Tractatu *De La Cause de la Pesanteur* pag. 175 circa finem.

PROPOSITIO LXIV. THEOREMA.

525. *Si grave vi gravitatis uniformis in aëre juxta duplicatum rationem celeritatum mobilis resistente verticaliter descendat a quiete suum incipiendo,*

Spatium *descensu confectum exponetur log-mo rationis sinus totius ad sinum complementi illius anguli, cujus sinus rectus celeritatem mobile acquisitam repræsentar;*

Tempus *vero, quo spatium illud pertransitur, aut prædicta velocitas mobili acquiritur, exponetur log-mo rationis sinus totius ad tangentem semissis complementi præmemorati anguli, applicato ad logarithmicæ subtangentem.*

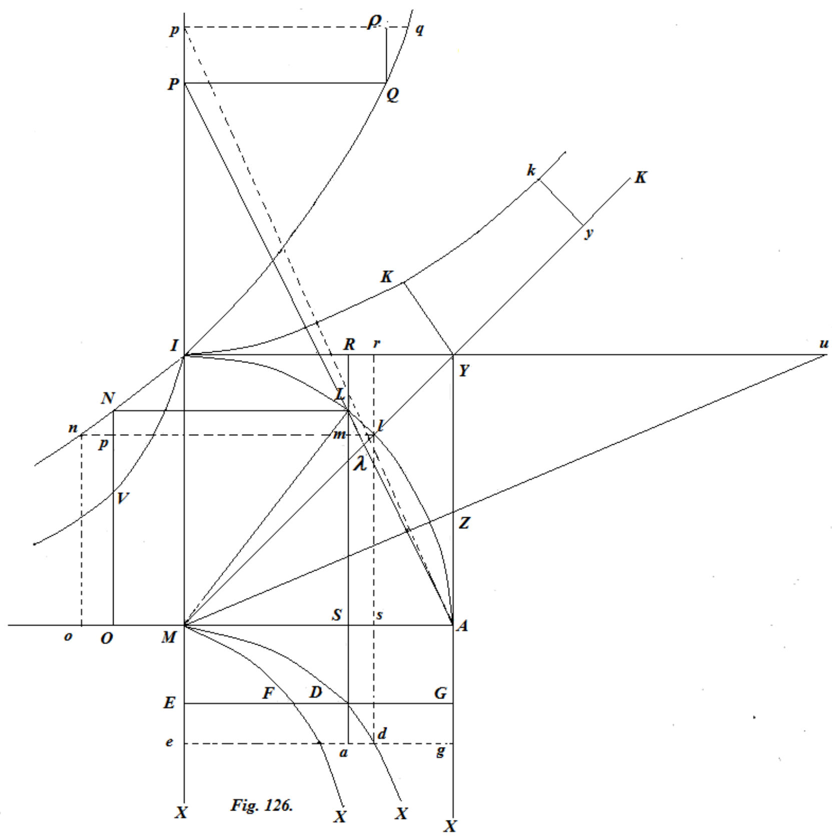


Fig. 126.

Grave M descendere incipiat a quiete in recta verticali MX, cui alia AO ad angulos rectos aptata sit; in hac AO capiatur MA, quæ gravitatem uniformem exponat, ac descriptis centro M intervallo MA quadrante circuli ILA & hyperbola æquilatera IK*k* , necnon log-mica NIQ, cujus subtangens æquet radium quadrantis vel semilatus transversum MI hyperbolæ; per hyperbolæ & quadrantis punctum I agatur tangens communis IU, quæ æquidistans erit ipsim MA. Tum etiam per quodvis quadrantis punctum L ducatur LN parallela AO log-micæ occurrens in puncto N; & recta ALP radio MI producto occurrens in P, per quod punctum ducatur insuper PQ æquidistans MA & log-micæ occurrens in puncto Q sitque adhuc NO ordinata log-micæ per punctum N ducta. Dein fiat porro in recta indefinita MX segmentum $ME = MO$, & sic ubique respective; eritque ductis per puncta E & L rectis EG, LS radiis MA & MI respective. parallelis, communis earum intersectio D in *scala celeritatum acquisitarum* MDX, adeo quidem, ut mobile postquam spatium ME perlapsum fuerit, ist termino E hujus spatii æquisiverit celeritatem ED vel MS. Factaque ubique EF tertia proportionali post rectas EG & ED, punctum F erit in *scala resistentiarum* aëris MFX, quæ scala etiam erit *scala solicitationum acceleratricium* mobilis cadentis in recta MX, sed quatenus ea refertur ad axem AX, versus quem curva MFX convexa est; nam, quia EF, seu $ED^2 : EG$, exponit resistentiam medii in puncto E & $EG : MA$ gravitatem uniformem, exponet FG omnino solicitationem acceleratricem in puncto eodem E, ut alibi (§.481.) jam dictum est. Igitur ductis *eg* parallela EG & ab ea elemento spatii E*e* distante; *dlr* parallela DL productæ in *a*, & sursum in R, per punctum quadrantis *l*, recta A*lp*, & per *p* linea *pq* æquidistante PQ; ac denique ordinata *no* distantia $\mathrm{O}o = \mathrm{E}e$ distante ab altera NO, in qua NO sit VO tertia proportionalis ad IM & NO, & his positis juxta alibi (§. 484) ostensa, tantum probandum superest, FG.E*e*, seu rec-lum FG*g*, *momentum solicitationis* acceleritatus FG æquari rec-lo ED.*ud*, seu *momento celeratatis* ED *casu mobilis ex altitudine* ME *acquisitæ*. Quo probato reliqua sponte sua obtinebuntur, scilicet expressio temporis descensus in spatio ME, & natura curvarum MDX, MFX., &c.

Demonstr. I. Est (secundum hypothesin) $EG : EF = (EG^2 : ED^2) = ML^2 : MS^2$, & convertendo sit $EG : FG (= ML^2 : LS^2 = IM^2 : NO^2$ seu quia IM, NO & VO sunt in continua ratione$) = IM : VO$, hinc $VO = FG$, & $VO.\mathrm{O}o = FG.\mathrm{E}e$. Atqui (§.491. num. III.) $VO.\mathrm{O}o : NO.\mathrm{N}p = LS.\mathrm{L}m\,(\S 493) = MS.\mathrm{S}s : ED.ad$; ergo FG. E*e*= ED.*ad*. Quod erat primum.

II. Propter similitudinem triangulorum MLS & Llm, sit $ad\ (\text{vel } ml) : l\lambda$ (vel, §. 463. num. III , L*l*) est $= LS : ML = NO : IM = VO : NO$; & $l\lambda : \mathrm{P}p = NO\ (\text{vel } \mathrm{L}S) : MP$; ac denique ex natura log-micæ (§. 491. num.1.) $\mathrm{P}p : q\rho = MP : IM$, erit ex æquo $ad : pq = VO : IM$, seu permutando $ad : VO = pq : IM$. Atqui (§. 131. & 485.) $ad : VO = t\mathrm{E}e$, hoc est, incrementum celeritatis elementare, applicatum ad solicitationem acceleratricem VO, exponit tempusculum, quo elementum spatii E*e* percurritur ; ergo etiam $\rho q : IM = t\mathrm{E}e$, atque adeo omnia $\rho q : IM$, id est, PQ.IM = omnibus *tEe*, seu tempori descensus per spatium ME.

III. Jam MO seu ME, id est, spatium descensu confectum, est log-us rationis IM ad NO, seu IM ad LS, hoc est log-us sinus totius ad sinum complementi anguli IML, cujus sinus rectus MS celeritatem in E acquisitam exponit. PQ vero est log-us rationis PM ad IM, seu IU ad IM ; ducta scilicet ex centro M super AP perpendiculari, eaque producta usque ad occursum cum tangente IU ; aut denique rationis MA ad AZ, id est, sinus totius ad tangentem semissis anguli LMA, id est, semissis complementi anguli IML, cujus sinus MS celeritatem acquisitam exponit. Idcirco tempus per ME, quod num. II. hujus exponitur per PQ: IM, exponi debet per log-mum rationis $(MA : AZ)$ sinus totius ad tangentem semissis complement anguli IML applicatum ad radium IM, seu ad log-micæ subtangentem. Quæ omnia. erant demonstranda.

COROLLARIUM I.

526. Igitur ducta MR usque ad occursum K cum hyperbola IK, trilineum hyperbolicum IMK applicatum ad semissem quadrati IM; exponet etiam tempus descensus per spatium ME. Nam (§. 463.) est duplum trilinei $\mathrm{IMK} = \mathrm{rec}-\mathrm{lo}\ \mathrm{PQ.IM}$, ergo trilineum $\mathrm{IMK} = \mathrm{PQ}.\frac{1}{2}\mathrm{IM}$; Jam, quia (§.525. num. II.) tME = PQ: IM= PQ.IM:IM^2 , erit

$t\mathrm{ME} = \mathrm{dupl.trilin.IMK} : \mathrm{IM}^2 = \mathrm{trilin.IMK} : \frac{1}{2}\mathrm{IM}^2.$

Pariter ducta asymptota M*y* hyperbolæ IK*k*, atque ex K demissa ad asymptotam ordinata KY, ad quam alia quædam *ky* se habeat, ut LS ad IM, quadrilineum hyperbolicum K*ky*Y applicatum ad semissem IM exponet spatium percursum ME. Nam, quia (constr.) $\mathrm{KY} : ky = \mathrm{IM} : \mathrm{NO}$,erit (§.368.) $\mathrm{K}ky\mathrm{Y} : \mathrm{MY.KY}$ seu

$$\tfrac{1}{2}\mathrm{MI}^2 = \mathrm{OM} : \mathrm{IM},\ \text{adeoque}\ \mathrm{OM} = \mathrm{ME} = \mathrm{K}ky\mathrm{Y} : \tfrac{1}{2}\mathrm{IM}.$$

COROLLARIUM II.

527. Scala solicitationum acceleratricium MFX est log-mica, cujus subtangens est semissis ipsius IM. Nam, quia supra(§.525. num.I.) ostensum $\mathrm{VO} = \mathrm{FG}$ & (constr.) $\mathrm{OM} = \mathrm{ME} = \mathrm{AG}$, sequitur curvam MFX similem & æqualem esse curvæ IV ; atqui hæc curva est log-mica, cujus subtangens est dimidia ipsius IM subtangentis logarithmicæ IN, quandoquidem quælibet VO est tertia proportionalis ad IM & NO; ergo etiam MFX est log-mica, cujus asymptota est AX. Propterea, si solicitationes acceleratrices FG sunt in progressione geometrica descendente, spatia transmissa sunt in progressione arithmetica ascendente.

COROLLARIUM III.

528. Adeoque velocitas *terminalis* seu *maxima*, est MA, quæ gravitatem uniformem repræsentat, cum linea AX parallela MX asymptota sit utriusque curvæ MFX & MDX.

COROLLARIUM IV.

529. Tempus, quo grave in aëre celeritatem DE acquirit, est ad tempus, quo eandem celeritatem in vacuo acquireret, ut PQ ad D. Nam (§.151) tempus, quo celeritas DE in vacuo acquiritur, est DE:IM vel AM, & tempus, quo in aëre eadem DE acquiritur, $\mathrm{PQ} : \mathrm{IM}$.

COROLLARIUM V.

530. Vicissim celeritas, quæ tempore $\mathrm{PQ} : \mathrm{IM}$ in aëre acquiritur, est ad celeritatem, quæ eadem tempore in vacuo acquiri potest, ut DE ad PQ. Nam (§. 151.) tempore $\mathrm{PQ} : \mathrm{IM}$ acquiritur in vacuo celeritas PQ.

SCHOLION.

531. Si grave M perpendiculariter deorsum projiciatur in recta MX celeritate MI, majore ipsi terminali MA vel MP; positoque, mobilis velocitatem reductam esse ex MI ad MS (nam, quia velocitas projectionis MI major est quam AM, resistentia medii major erit gavitate, ac propterea motus continuo debet retardiri) erit spatium percursum ME exponendum per $\log.\sqrt{(\mathrm{PIA}:\mathrm{PSA})}$ seu log-mum subduplicatæ

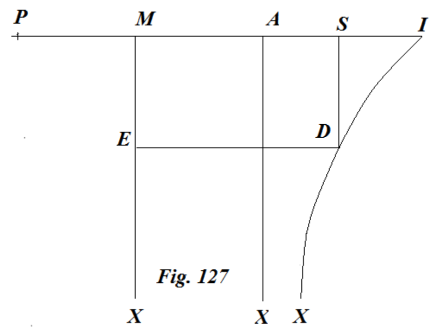


Fig. 127

rationis rectanguli PL.AI ad PS.AS; tempus vero, quo mobile spatium ME percurrit, seu quo velocitas initialis MI reducitur ad ED vel MS, exponitur per $\log.\sqrt{(\mathrm{AI}:\mathrm{AS})}-\log.\sqrt{(\mathrm{PI}:\mathrm{PS})}$. Ponitur vero $\mathrm{MP}=\mathrm{MA}$, hi posteriores log-mi sumuntur in log-mica, cujus subtangens est unitas, log. vero ex $\sqrt{(\mathrm{PIA}:\mathrm{PSA})}$ sumitur in log-mica, cujus subtangens est AM vel MP. Sin vero celeritas projectionis MI minor fuerit quam MA, hic casus jam continetur in præsenti propositione, ut ideo eidem diutius immorari plane necessum non sit.

PROPOSlTIO LXV, THEOREMA.

532. *Linea recta, quæ celeritatem terminalem seu gravitatem uniformem exponit, sumta pro sinu tota seu radio; si mobile quoddam, celeritate initiali, expressa tangente alicujus anguli,verticaliter in altum projiciatur*, maxima altitudo, *ad quam mobile in aëre pervenire potest, exponetur log-mo rationis secantis anguli ad radium, &.* tempus *hujus ascensionis exponetur per ipsum angulum, cujus tangens celeritatem initialem exponit.*

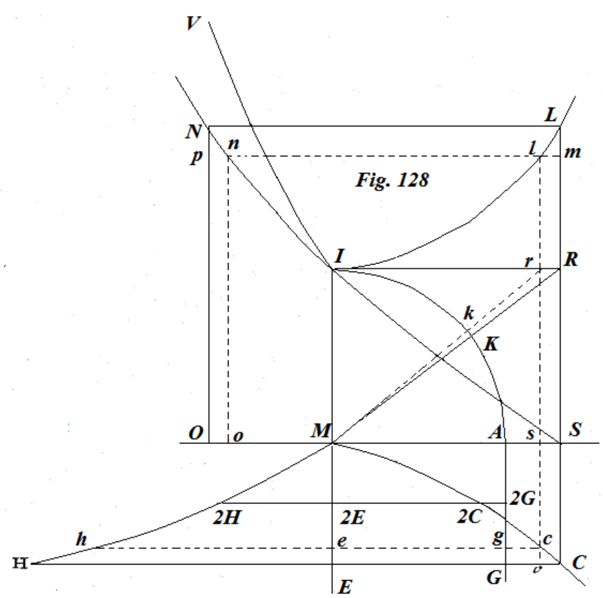


Fig. 128

Ascendat mobile celeritate initiali EC ex E in M in linea verticali EM, centroque M describantur, ut in præcedenti , circuli quadrans IKA, hyperbola æquilatera IL, & logarithmica IN. Producatur CE in H factaque EH tertia proportionali ad EG & EC, eritque EH resistentia, quam ascendens mobile ab aëre patietur in E; & quia præteræ EG, seu MA, gravitarem uniformem exponit, quæ mobili in altum lato quoque resistit, erit ideo (§. 481) *resistentia totalis* mobili opposita in E æqualis GH. Adeoque curva quædam HM erit scala resistentiarum totalium, curva vero CM scala celeritatis decrescentis EC. Adeoque (§.484.) res eo deducitur, ut construantur hæ nominatæ curvæ ejus proprietatis, ut sit ubique $HE.Ee = EC.bC$. Ad id, ductis CL, LN atque NO parallelis rectis MI & MA respective, fiat $ME = MO$,& sic ubique, dabunturque tot puncta in curva C2CM, quot libuerit, ex qua deinceps alteram HM construere non erit difficile. Ducantur *cl*, *ln*, *n*o prioribus CL, LN & NO æquidistantes & indefinite vicinæ, sumaturque in qualibet ON major OV, quæ sit ad homologam ON, ut hæc NO ad MI; quæ MI simul etiam log-micæ IN subtangentem significat, & hac ratione resultabit nova log-micæ IV, cujus subtangens erit semissis IM subtangentis log-micæ IN.

Demonstr. I. Quia rectæ EG, EC & EH (secundum hypothesin) sunt in continua ratione,erit $EG : EH = EG^2 : EC^2$, vel invertendo & componendo $HG : EG = IS^2 : lM^2 = LS^2 : IM^2 = NO^2 : IM^2$ & (vel ob continue proportionales VO, NO & IM) $= VO : IM$ vel EG , ergo $HG = VO$; & quia $OM = EM$, ac $oM = eM$, & proinde $Oo = Ee$; erit $HG.Gg$, vel $HG.Ee = VO.Oo$ (§ 491.num. III.) $= NO.\ Np = LS.Lm$ (§ 493.) $MS.Ss = EC.Cb$. Id est *momentum celeritatis decrescentis* EC æquatur momento *resistentiæ totalis* HG: Jam ME vel MO est log-us rationis NO ad IM, seu IS ad IM, id est, log-us rationis secantis anguli MIS, cujus tangens $MS = EC$ celeritatem initialem exponit, ad radium IM vel MA. Quod erat primum.

II. Quia $VO : IM = NO^2 : lM^2 = LS^2 : IM^2 = MR^2 : MK^2\ (§.165.) = Rr : Kk$, atque adeo, quia $Rr : Cb$, erit $Kk : IM = Cb : VO = Cb : HG$ (§.131, 485.) $= tEe$, & Kk:MK vel IM $(§.129.) =$ angulo KMk, adeoque $tEe =$ angul KMk, ac proinde tempus totius ascensus per spatium EM exponi debet angulo KMI, cujus tangens est IR vel EC. Quod erat alterum.

COROLLARIUM I.

533. Nunc: iterum curva resistentiarum totalium M2HH erit log-mica similis & æqualis log-micæ IV. Atque adeo si resistentiæ totales sumantur in progressione geometrica descendente, spatia ascendendo confecta erunt in progressione arithmetica descendente, scilicet spatia toto ascensu descripta.Horum demonstratio eadem ferme est cum ea coroll. II. prop. præced. §. 527.

COROLLARIUM II.

534. Velocitas initialis mobilis suam maximam in aëre altititudinem EM absolventis, est ad celeritatem initialem, qua mobile pari tempore suam in vacuo maximam altitudinem conficeret, ut tangens IK ad arcum IK. Nam, si corpus in vacuo celeritate initiali IK ascendit, tempore $IK : IM$ id (§.§. l41. 151.) ascendet quousque potest, sed hoc ipso tempore $IK : IM$ expositum angulo KMI, ascendet in aëre spatio EM, celeritate initiali EC vel IR. Ergo, &c. Atque hoc consentit cum coroll. 4. prop. 9. Lib. II. Princ. Phil. Nat. Math. Cel. Newtoni.

COROLLARIUM III.

535.Tempus, quo mobile suam in aëre altitudinem, ad quam celeritate initiali IR pervenire potest, absolvit; est ad tempus, quo in vacuo suam altitudinem, celeritate initiali existente eadem IR, absolveret, sicut arcus IK ad tangentem IR, seu, quod idem est, ut sector IMK ad triangulum IMR. Nam tempus ascensionis in aëre est angulus IMK, seu $\mathrm{IK}:\mathrm{IM}$, & tempus ascensionis in vacuo (§.151.) $\mathrm{IR}:\mathrm{IM}$.

COROLLARIUM IV.

536. Adeoque, si celeritas initialis, qua corpora in aëre & in vacuo ascendunt, terminalem æquaverit, erit tempus, quo altitudo illius quod in aëre, ad tempus, quo altitudo ejus, quod in vacuo ascendit, conficitur, ut circulus ad quadratum circumscriptum; nam, si $\mathrm{IR}=\mathrm{IM}$, erit sector IMK octava circuli & triangulum IMR octava quadrati circulo circumscripti pars, adeoque ille ad hoc, ut circulus totus ad quadratum circulo circumscriptum.

COROLLARIUM V.

537. Iisdem adhuc positis, quæ in coroll. III. altitudo, quam mobile in aëre absolvit, est ad altitudinem in vacuo percurrendam, ut duplum rec-lum IMO ad quadratum tangentis IR. Nam (§.150.) altitudo in vacuo describenda erit $\mathrm{IR}^2:2\mathrm{IM}$, & ME est ad $\mathrm{IR}^2:2\mathrm{IM}$, ut duplum rec:lum IMO ad IR^2.

COROLLARIUM VI.

538. Adeoque, si celeritas initialis IR terminali IM par fuerit, erit altitudo in aëre ad altitudinem in vacuo absolvendam, vel $2.\mathrm{IMO}$ ad IR^2, sicut 2MO ad IM, id est, sicut $\log.\left(2.\mathrm{IM}^2:\mathrm{IM}^2\right)$ ad IM, hoc est, sicut log-us rationis duplæ ad subtangentem log-micæ, vel etiam (§. 368.) ut quadrilineum hyperbolicum alterutri asymptotæ adjacens, cujus ordinatæ sunt in ratione dupla, ad rec-lum hyperbolæ inter asymptotas. Ut Celeb. Hugenius (pag. 174 Diff. *De Causa Gravitatis)* sine demonstratione asseruit.

COROLLARIUM VI1.

539. Celeritas initialis mobilis spatium EM ascendendo describentis, est ad celeritatem, quacum denuo in terram reddit, sicut tangens anguli IMR ad ejusdem sinum. Nam, quia in figuris 126 & 128 spatia ME utrinque sunt (secundum hypothesin) æqualia, erit ratio $\mathrm{ON}:\mathrm{IM}$, in fig.126. seu $\mathrm{LS}:\mathrm{LM}=$ in fig. 128. $\mathrm{IM}:\mathrm{NO}=\mathrm{RS}:\mathrm{MR}$; adeoque triangulum rec-lum LMS in fig. 126. est simile triangulo RMS, adeo ut angulus IML in fig. 126 $=\mathrm{IMK}$, atque IR est tangens & MS in fig. 126 sinus unius ejusdemque anguli IMK vel IML, ergo &c.

Hisce multa alia potuissent addi corollaria, sed brevitati consulens talia Lectoris industriæ ex præcedentibus elicienda relinquo.