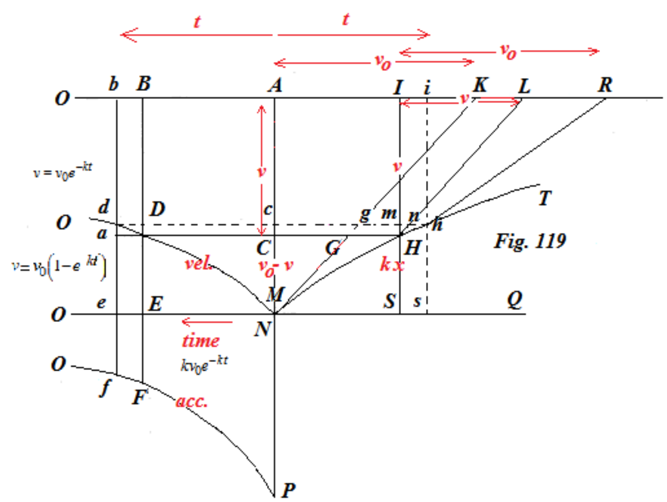
CHAPTER XV.

Concerning the motions of bodies, in which the air resists in the ratio of the speeds of the motion.

PROPOSITION LVIII. THEOREM.

495. *The distance, which a certain body may be sent moving through air in a certain time, with a certain motion varied from the fundamental uniform case, considered to be a proportion of its actual speed, is expressed by the same line, as that on which the initial part of the velocity of the mobile body lost may be represented at this time. And the absolute time, in which a given distance is resolved, must be expressed by the logarithm of the ratio, which the initial velocity of the moving body has to the absolute velocity after this same lapse of time, applied to the subtangent of that logarithmic curve, from which the aforesaid logarithm has been chosen.*

[Note that the subtangent refers to that of the logarithmic or exponential curve to the base *e*. There are two situations to be considered here, which have been handled in §. 487; in the first case, the body initially is placed at rest in a horizontal air stream moving with a constant speed along NO to the left in the diagram, and the body gradually accelerates with a force proportional to the difference between the body's speed and that of the air ; in the second case, the air is at rest and the body is projected with the same initial speed as in the first case to the right, and retarded by a force proportional to the speed of the body in the air. As indicated previously, these two circumstances are deemed to be equivalent. The acceleration or deceleration in each case is taken in proportion to the velocity difference between the body and the air.



In modern terms for the second case , Newton's second law becomes, for unit mass,

$\frac{dv}{dt} = -kv$, giving $\frac{dv}{v} = -kdt$ leading to $\int \frac{dv}{v} = \ln \frac{v}{v_0} = -kt$; or $v = v_0 e^{-kt}$,

and from $v = v_0 e^{-kt}$

the acceleration becomes $-kv_0 e^{-kt}$,

and $t = \frac{1}{k} \log \frac{v_0}{v} = \frac{1}{k} \log v_0 - \frac{1}{k} \log v.$

We may put $v\frac{dv}{dx}=-kv$, leading to $v=v_0-kx$, so that the velocity decreases linearly with the distance *x*, and again, $kx=v_0\left(1-e^{-kt}\right)$; and $x=v_0-v$; in the case to be considered here in which $k=1$. This is examined geometrically in § 487.

For the first case, in an analogous manner, we have $v=v_0\left(1-e^{-kt}\right)$; where the initial velocity is zero rel. to the moving air, the acceleration is kv_0e^{-kt} to the left, and the final velocity is zero relative to the moving air. As above we may put $v\frac{dv}{dx}=kv$, leading to $v=kx=v_0\left(1-e^{-kt}\right)$, so that the velocity increases linearly with the distance *x* to the left, and again, $kx=v_0\left(1-e^{-kt}\right)$; and $x=v_0-v$; in the special case to be considered here in which $k=1$ and *v* is the decreasing velocity of the second case above.

The solution presented here makes use of the logarithmic curve, the name used by Huygens and others used to describe the shape both of the logarithm and of the antilogarithm or logarithmic curves, and which we may call the Alog, being of course the exponential function in some form in modern terms. Huygens has expressed the relation very simply by plotting an arithmetic progression on one axis and a decreasing geometric progression on the other. The geometrical solutions shown in Fig.119 thus can be considered to have both the acceleration or force and the velocity of the moving body expressed by Alog or exponential decay curves placed to the left and right of AN. In all cases, the same or similar exponential decay curves are used, and we have to consider the vertical axis to represent either force or acceleration, velocity, or distance gone; In this case, the origin is A, the graphs are inverted from the usual, and each is proportional to the time on the horizontal axis to the right or left, and the velocity *v* vs *t* and acceleration *a* vs *t* are along the vertically downwards axis on the left hand curves, while *x* vs *t* is on the right. However, when *t* is zero, $v=v_0$, then on choosing the variable to be $\frac{v_0}{v}$ vertically downwards for the right-hand curve, we have :

$\log\frac{v_0}{v}=kt$, while Alog $\log\frac{v_0}{v}=\mathrm{A}\log kt=\frac{v_0}{v}$, or $v=v_0e^{-kt}$, in any case, the initial value of the ratio $\frac{v_0}{v}$ is unity, when the time is zero. Thus, the logarithmic curve on the right serves two purposes : for if we set the vertical independent variable axis to be composed from $\log\frac{v_0}{v}$, then on forming the Alog according to the curve, we have

$\mathrm{A}\log\log\frac{v_0}{v}=\frac{v_0}{v}=\mathrm{A}\log kt=e^{kt}$; while on reversing the process by taking logs, we return to the former. Now, on differention, as it was known at this time, the differential of $y=\log x$ gives $\frac{dy}{dx}=\frac{1}{x}$, as the integral is the area under the rectangular hyperbola for $x>1$, which is the natural log; thus

$\mathrm{A}\log\log x=x$, with $y=\log x$ giving $\frac{d\mathrm{A}\log y}{dx}=\frac{d\mathrm{A}\log y}{dy}\cdot\frac{1}{x}=1$ or $\frac{d\mathrm{A}\log y}{dy}=\mathrm{A}\log y$

i.e. the antilog function is unchanged on differentiation, from which it follows that the subtangent has the constant value 1, as $\frac{dy}{dx} = y/\text{subtangent} = \frac{y}{1} = y$; If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx} = ky = \frac{y}{1/k}$; hence in this case the subtangent is 1/*k*, etc. This is the main result used in this proposition but with *k* assumed to be 1; however, we will retain *k* until the end of the calculations, though in Fig. 119 *k* is taken as 1; these properties were presented without proof by Huygens in his list of the properties of the logarithmic curve, to be found in his dissertation *On the Causes of Weight*, which in turn was inserted originally into the middle of his *Treatise on Light*, originally in French; this topic has been translated here in a nearby file.]

M shall be a mobile body set in motion suddenly on a horizontal plane with a certain speed AN, along the right line NQ, on which henceforth it would move forwards uniformly with the same speed AN, if it were proceeding in a vacuum, but because it is moving in air with a resistance proportional to its speed, at individual moments there will be actual decreases in its speed. Truly these motions are determined readily with the aid of proposition 55 of this second book; as, according to this proposition, they are the very same motions as varied from the fundamental case by the air resistance, which results in the common motion of the moving body being produced along some carrying curve, arising from the impulses of the air acting on the body along this line.

Therefore NO shall be this carrying curve travelling with the speed taken to be AN, the same as that of the object directed towards Q, and after a certain time the mobile object M is understood to have traversed the distance NE on this carrying line, and to have acquired exactly that change in speed DE in the said time, by which it happens that the speed shall have been reduced from its initial speed AN to the actual speed BD (with which clearly the body is moving in the still air) ; therefore the air passes the body with this speed to which the resistance is proportional, which now will be BD, and therefore the impressed force or acceleration acting on the body on the resisting line generally is EF . Therefore the curve PFO will be similar and equal to the curve NDO [as the resisting force is set equal to the speed or *k* is unity], thus so that it remains only to determine (§.488.) what the curve NDO shall be, so that on that BD.B*b* shall be equal to the rectangle *ead* and EF.E*e* to the rectangle BD.*ad*. Where E*e* and B*b* is an element of the distance NE, and *ad* an element of the [velocity] ordinate DE.

[The differential equation indicated here is $vdv = -Fdx$ or $a = -kv$ with $k = 1$.]

The logarithmic curve NHT shall be drawn through the point N, the subtangent of which AK (clearly by placing the right line NK itself to touch at the point N) to equal AN [initially], and on that the ordinate IH equal to BD shall be placed vertical to the asymptote AR, and that is produced to S; and the indefinite right line drawn through the point H of the logarithmic curve parallel to OQ cuts the tangent NK of the logarithmic at G, and the right line NA at C [i.e. the curve can be written as the exponential decay $v = v_0 e^{-kt}$ where $kx = v_0 - v = v_0\left(1 - e^{-kt}\right)$. Thus the geometric progression from the ratio of the velocities is decreasing along the vertical axis as the points on the curve representing the time progress arithmetically along OR.] ; if CD may be taken everywhere on AN equal to GH , the intercept between the tangent NK and the logarithmic curve NHT, the point D will be always on the graph of the proper speed of the mobile on the carrying curve NO. [Recall that we have decided to retain the retardation constant *k* to avoid confusion in equations between the distance *x* and the speed *kx*, which arises by setting $k=1$.]

Further, the tangent HR, and the line HL parallel to the tangent NK , are drawn through the point H of the logarithmic curve [*i.e.* maintaining the 45^0 angle]; HI and IL, and likewise LR and HS or DE will be equal, and in addition with *dh* drawn indefinitely close to the other line DH, the line element *mh* and IR will be cut proportionally by the middle line HL, and thus HI or BD, or indeed EF will be to LR, HS, or indeed DE, [*i.e.* as $\frac{v}{x}$] just as H*m* or *da* to *nh* [*i.e.* as $\frac{dv}{dx}=-k=\left[-1=\frac{v}{x} \text{ when } k=1\right]$]; and thus $\mathrm{EF}.nh=\mathrm{DE}.da$: For, since (by the construction) DC or $\mathrm{EN}=\mathrm{GH}$ & $\mathrm{N}e=ch$; *nh*, the difference between GH and *gh*, will be equal to *Ee* ; for H*n* is parallel to *Gg.* Therefore $\mathrm{EF}.nh=\mathrm{EF}.\mathrm{E}e=\mathrm{DE}.da$. [*i.e.* we have to imagine the incremental triangle *nh*H becoming smaller and smaller towards its limiting ratio.] Q.E.D.

Again $nh:\mathrm{LR}=mh:\mathrm{IR}$ [$\frac{nh}{kx}=\frac{mh}{v_0}$ or $\frac{kdx}{kx}=\frac{dv}{v}$], and $nh:\mathrm{LR}=\mathrm{E}e:\mathrm{DE}$ $(\S.128.)=t\mathrm{E}e$, therefore $t\mathrm{E}e=\mathrm{I}i$ or $mh:\mathrm{IR}$; therefore [the sum of] all the *t*E*e*, that is, the time to pass through NE, divided by IR or AN, $=$ all $\mathrm{I}i:\mathrm{IR}=\mathrm{AI}:\mathrm{IR}=\log.(\mathrm{AN}:\mathrm{HI})$;
[*i.e.* $kx=v_0\left(1-e^{-kt}\right); e^{-kt}=1-\frac{kx}{v_0}; t=\log\frac{v_0}{v_0-kx}\to\log\frac{v_0}{v_0-x}=\log\frac{v_0}{v}$]That is, the absolute time, in which the body travels through the distance NE on the resisting line, is shown by the logarithm of the ratio, which the initial speed AN has to the speed of the body IH remaining in this elapsed time, relative to the subtangent of the logarithm.

Hence, because the absolute distance (§. 489.), which the mobile M passes through in a given time in the moving air, is $\mathrm{AN}.t\mathrm{NE}-\mathrm{NE}$, this distance will be $\mathrm{AI}-\mathrm{NE}=\mathrm{CH}-\mathrm{GH}=\mathrm{CG}=\mathrm{NC}$, because (from the constr.) $\mathrm{AN}=\mathrm{AK}$, likewise $t\mathrm{NE}=\mathrm{AI}:\mathrm{AN}$, and $\mathrm{NE}(\text{by the constr.})=\mathrm{GH}$. Therefore the absolute distance, which the body runs through in the given time, is represented in air by that line NC or DE, which shows the part of the velocity lost. [*c.f.* $kx=v_0\left(1-e^{-kt}\right)=v_0-v$] By which everything has been shown.

COROLLARY I.

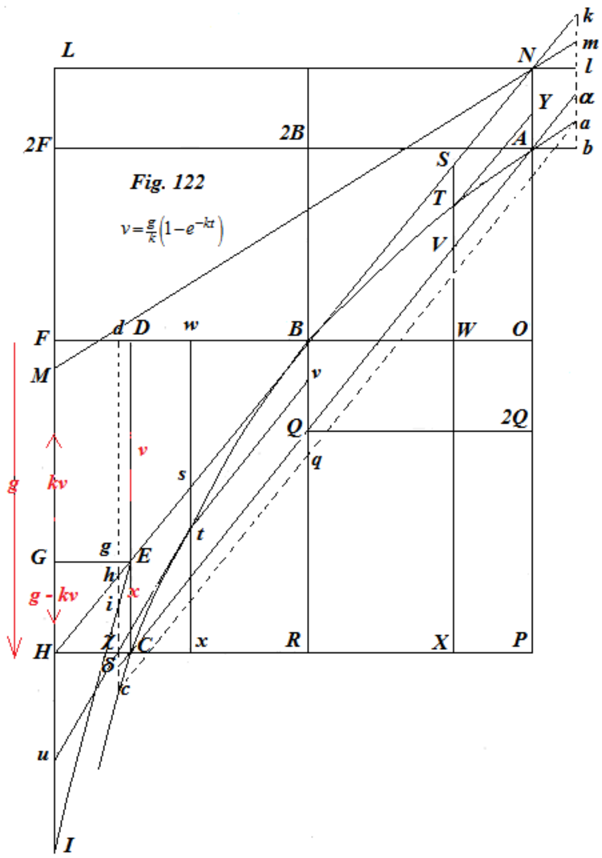
496. Therefore if the actual speeds of the moving body BD or IH were taken in a geometric progression, then the respective times AI:AN will be in an arithmetical progression [Note that the latter time corr. to the speed AN is infinite].

COROLLARY II.

497. And thus the velocity of the mobile body can only be diminished to zero in an infinite time, when the logarithm of the ratio AN to 0, or to some given smaller quantity, shall be infinite. Therefore the body M with a motion of this kind in air derived from the

case of the air moving uniformly can never traverse the distance with the given magnitude of the speed AN.

All these agree precisely with the determinations of Newton : Prop. 2. Book. 2 *Pr. Phil. Nat.* [see this writer's translation of the *Principia* on this website for an extended note by LeSeur and Jacquier on the logistic curve ; the interested reader may also like to look on p. 51, Vol. I, Hutton's *Math. & Phil. Dictionary*, for a contemporary understanding of the logarithmic curve] ; Leibniz *Act. Erud*. 1689, Art.1 ; Wallis *Algebra* Ch.101. And Varignon *Actes de l'Academie Regente des Sciences Paris*. 1707, 13th Aug. Prob. 1. & adjoining corollaries, [available from *Gallica*].



PROPOSITION LIX. THEOREM.

498. *If a weight may fall from rest vertically in air with the resistance following in the ratio of the speeds, the distance, which the mobile may pass through in some time, will be expressed by the excess, by which the product from the time of descent by the terminal velocity of the body exceeds the line, which may represent the speed acquired by the body. Truly the time itself may be expressed by the logarithm of the ratio, which the absolute weight of the mobile body has to the acceleration acting at the end of the said distance, divided by the subtangent of the logarithmic curve.*

But if indeed a weight may be projected vertically upwards to a height in the same air with a given initial velocity, the height of the total ascent of the mobile body will be expressed by the excess, by which the initial velocity exceeds the product from the time of the ascent by the terminal velocity. And the time itself will be expressed by the logarithm of the ratio of the total resistance from the start of the ascent to the absolute weight of the moving body divided by the subtangent of the applied logarithmic curve.

[In modern terms, we have the two differential equations : where + x represents the distance fallen vertically from rest at the origin, and t the time from the start; as above, the same curve appears with the axes chosen to represent either acceleration, velocity, or distance vs. time :

$\frac{dv}{dt} = g - kv$ and $v\frac{dv}{dx} = g - kv$; Hermann's result again sets $k = 1$.

The first equation for the falling body can be written as
$\frac{dv}{g-kv} = dt$, or on integrating, $\log\left(g-kv\right) = -kt + \log g$, or $g - kv = ge^{-kt}$ and $v = \frac{g}{k}\left(1-e^{-kt}\right)$ giving $v_{term.} = \frac{g}{k}$, and $t = \frac{1}{k}\log\left(\frac{g}{g-kv}\right)$; or in terms of distance, $v\frac{dv}{dx} = g - kv$, giving :

$$dx = \frac{vdv}{g-kv}; kdx = \frac{vkdv}{g-kv} = \left(\frac{g}{g-kv}-1\right)dv;$$ on integration:

$$kx + v = -\frac{g}{k}\log\left(1-\frac{kv}{g}\right) = gt;$$ or $x = \frac{gt-v}{k} = v_{term.}t - \frac{v}{k} \to v_{term.}t - v$.]

I. B*t*C shall be the logarithmic curve erected from the axis FI, of which FH shall be the subtangent at some point and BH the tangent of the logarithmic curve at the point B. As it pleases, the line DE may be drawn parallel to FI, crossing the line BF at the point D, and the tangent BH at E, and finally the logarithmic curve at the point C. And the intercept EC between the tangent BH and the logarithmic curve BC will be the distance traversed by the falling body, the [log of the ratio $\mathrm{FH}:\mathrm{GH}$] [DC:FH in the original] expresses the time, in which the distance EC is completed, and last of all DE the intercept between BF and BH expresses the speed acquired by the body in the aforementioned time $\mathrm{DC}:\mathrm{GH}$. Truly FH expresses the terminal or maximum velocity exclusively. Hence with EG drawn parallel BF, the right line GH will represent the acceleration acting at the end of the distance traversed EC. Now because FH represents the absolute weight, and (following the hypothesis) DE or FG the resistance of the medium, the remainder GH expresses completely the driving acceleration of the body, thus so that nothing shall remain to be done, in order that this construction (§. 484) will be verified to provide $\mathrm{GH}.\delta c = \mathrm{DE}.gh$, or *the moment of the acceleration acting* GH *to be equal to the moment of the speed* DE, with *dc* drawn parallel to DE, & δC parallel to BH, thus so that δc shall be the increment of the line EC, and *gh* the increment [element] of the line DE. Where it is to be noted in passing, the point C in the figure by the body falling as far as the line parallel to FO, drawn through the given position of the point H to be found at the common intersection of the line HP and the logarithmic curve BC, since the given point C may be some other point of the logarithmic curve.

Demonst. Let $\mathrm{GI} = \mathrm{FH}$, and there will be $\mathrm{HI} = \mathrm{FG} = \mathrm{DE}$, and with EI acting parallel to the tangent of the curve at C, thus E*i* will be parallel and equal to the element of the logarithmic C*c*, and $\delta c = hi$. Now, because $\mathrm{GH}:\mathrm{HI}$ or $\mathrm{DE}, = gh : hi$ or $c\delta$, there will be $\mathrm{GH}.\delta c = \mathrm{DE}.gh$. Which is the first part to be shown.

Again gi or $\chi c : \mathrm{GI} = gh : \mathrm{GH}$ (§.128.) $= t\delta c$, therefore the sum of all the $\chi c : \mathrm{GI}$, or $\mathrm{DC}:\mathrm{GI} =$ the time of the descent along the distance EC. And DE is the logarithm of the ratio FB to GE or HF to GH, that is, the logarithm of the ratio, which the terminal velocity FH has to the acceleration acting GH. Now, because DE expresses (following the hypothesis) the velocity of the body acquired, and because this shall be FH, where DE were made infinite, it is evident the terminal velocity can be expressed by the subtangent FH, by which also the absolute weight of the body can be represented. Therefore the time of descent through the distance EC is the logarithm of the ratio $\mathrm{FH}:\mathrm{GH}$ divided by the subtangent FH. Therefore DE is the product by the time of the descent through EC by the terminal velocity FH, and the distance EC is the excess of the product in the manner named, more than the right line DE, which expresses the speed acquired. All of which were to be demonstrated.

[It is to be observed that these demonstrations amount to a geometrical version of the integrations produced above analytically.

The second equation for the rising body can be written as $\frac{dv}{g+kv}=-dt$, or on integrating, $\log(g+kv)=-kt+\log(g+kV)$, or $g+kv=(g+kV)e^{-kt}$ and $v=\left(\frac{g}{k}+V\right)e^{-kt}-\frac{g}{k}$ and when $v=0$, $e^{-kt}=\frac{g}{g+kV}$; $t=\frac{1}{k}\log\frac{(g+kV)}{g}$.

In terms of distance, $\frac{dx}{dt}=\left(\frac{g}{k}+V\right)e^{-kt}-\frac{g}{k}$; and $x=\left(\frac{g}{k^2}+\frac{V}{k}\right)\left(1-e^{-kt}\right)-\frac{gt}{k}$, while $x_{\max}$ corresponds to : $\left(\frac{g+kV}{k^2}\right)\left(1-\frac{g}{g+kV}\right)+\frac{g}{k^2}\log\frac{(g+kV)}{g}=\frac{V}{k}-\frac{g}{k^2}\log\frac{(g+kV)}{g}$.]

499. With a second mobile body put in place to rise along the vertical line *ak* or AN with the initial speed NO, it remains to show the height to which the body shall be able to reach, to be the intercept *ak* or AN between the tangent BN and the logarithmic curve BTA , and the time, in which that height may be performed, to be AO : FH . The line *ak* is put indefinitely close to the other AN, whence with Aα drawn parallel to BN, the small line $a\alpha$ will be the decrement of the distance *ak* continually decreasing towards B with the velocity NO. At the start of the motion the resistance of the air (following the hypothesis) is NO, and the weight FH or OP of the body ascending also is put in place, and thus (§.481) NP is the whole resistance. Therefore it is required to show, that always NP.$a\alpha$ *evidently the moment of the whole resistance NP is equal to the moment* NO.*kl of the decreasing speed* NO.

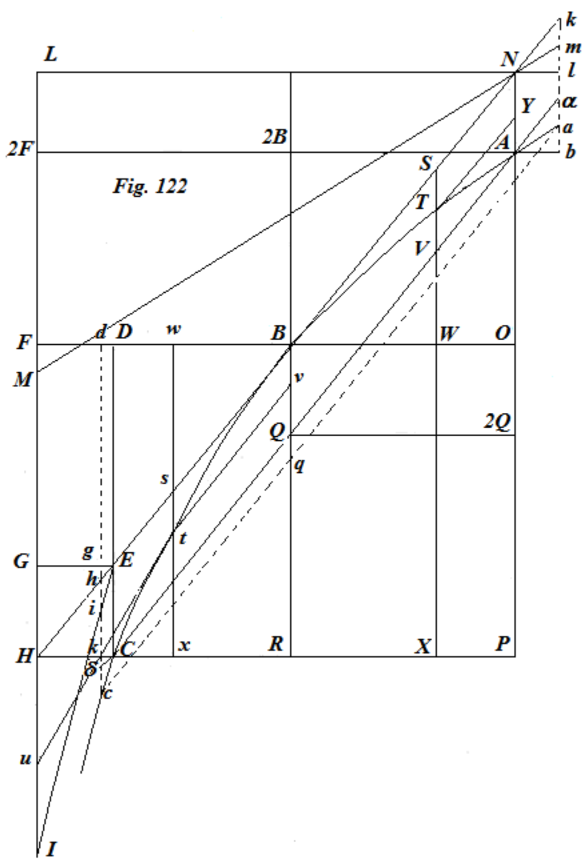


Fig. 122

Indeed again if LM = FH & NM were drawn, there will be MH = NO, $km=a\alpha$, & LH : MH = NP : NO , and thus NP.$a\alpha$ = NO.*kl* . Therefore the sum of all the decrements $a\alpha$, that is *ak*, or AN, is the height, to which the ascending body can rise.

Likewise , $ml:\mathrm{LM} = km$ or $a\alpha:\mathrm{MH}$ or $\mathrm{NO.}(\S.128.)\,ta\alpha$ therefore the sum of all $ml:\mathrm{LM}$, or all $ab:\mathrm{LM}$, that is, $\mathrm{AO}:\mathrm{LM}$ is equal to all the $ta\alpha$, or to the ascent time of the body to the distance *ak* or AN. And AO is the logarithm of the ratio A2F to BF, or of the equal ratio NP to OP, that is, of the ratio, which the whole resistance has initially to the weight [*i.e.* $\log\frac{kV+g}{g}$]. And thus the time to pass through $\mathrm{AN} = \log(\mathrm{NP}:\mathrm{OP})$ divided by LM or FH, and LM in $t\mathrm{AN} = \mathrm{AO} = \log.(\mathrm{NP}:\mathrm{OP})$. Therefore, since AN shall be $\mathrm{NO} - \mathrm{AO}$, the height will be, that the ascending body can traverse , shall be the excess, by which the initial speed NO exceeds the product from the time of the ascent by the terminal speed FH or LM.

COROLLARY I.

500. The time, in which the time of the ascent will be measured out to its height AN or QB, is to the time of descent of the same from the same height BQ, just as AO to OP. This itself is clear enough. For the time of descent through BQ or EC (§.498.) is $\mathrm{DC}:\mathrm{FH}$ or $\mathrm{OP}:\mathrm{FH}$, and the time of ascent to AN (§. 499) is AO: FH. Hence, etc.

COROLLARY II.

501. The time in which the moving body thrown upwards with the initial speed NO ascends through the distance AN, is to the time, in which the same weight and with the same initial speed NO likewise can reach its height in a vacuum, and which can be completed, as AO to NO or A2Q; truly with the right line Q2Q drawn through Q parallel to FO. For the time of the ascent in air through the height AN is, as we have seen, $\mathrm{AO}:\mathrm{TH}$; indeed with the weight ascending in vacuo its altitude will be measured (§.139) by that time, by which that begins to fall from rest can acquire an equal speed NO in that time ; and this is the time (§.151), so that the speed being acquired NO applied to the right line FH, which expresses uniform gravity ; and thus $\mathrm{NO}:\mathrm{FH}$ expresses the said time of ascent of the body in vacuo. Therefore the time of ascent in air to the time of ascent is as $\mathrm{AO}:\mathrm{FH}$ to $\mathrm{NO}:\mathrm{FH}$, that is, just as AO to NO or to A2Q*;* for on account of the equal and parallel lines BN and QA, NO and A2Q also will be equal.

COROLLARY III.

502. The height AN, which a weight projected with a given velocity NO can rise to vertically [in air], will be to the height, which projected upwards in vacuo with the same speed may be able to complete, as the rectangle FBQ or FB.AN to the triangle BNO. For (§.150) the height being resolved in a vacuum is expressed by the square of the initial velocity NO, divided by twice the acceleration of gravity 2FH ; therefore the height in air is to the height in vacuo requiring to be resolved just as AN or BQ to $\mathrm{NO}^2:2\mathrm{FH}$; or as QB to OB.ON : 2FB, since NO shall be to FH as BO to BF*;* therefore the aforesaid heights are as $\mathrm{BF.BQ}$ to $\mathrm{BO.}\frac{1}{2}\mathrm{ON}$, that is, as the rectangle FBQ to the triangle BON. Truly the rectangle FBQ is equal to the trilinear figure BA2B ; for the four sided figure

FBA2F indicated and demonstrated by the celebrated Guido Grandi in his book on Huygens' [treatment of the logistic curve] : *Geometrica Demonstratio Theorematum Hugenianorum circa Logisticam, seu Logarithmicam Lineam,* Chap. 8, no. 14, is equal to the rectangle BO.FH, which can be proven most easily from the above (§.491. no. 11.) ; for because in fig. 119. the rectangle $\mathrm{HI.I}i = \text{rectangle IR.H}m$, they will be [the sum of] all the Hl*i*, which are contained in the area NAIH, that is this area itself is equal to all the IR.H*m* or IR.NC. [Fig. 122.] Therefore it follows from the same argument the four-sided figure FBA2F to be equal to the rectangle FH.BO or to the rectangle formed from the subtangent of the logarithmic curve by the difference of the ordinates A2F and BF. And on account of the similar triangles HFB and NOB, there is $\mathrm{EH.BO = EB.NO} = \text{rectangle LFB}, \quad \text{therefore LFB} = \text{FBA2F}$, and with the common rectangle BF2F taken away, there will remain $\text{L2F2B} = \text{FBQ} = \text{trilinear form BA2B}$.

Hence, because the altitude in air is required to be compared to the altitude being traversed in vacuo, as the rectangle FBQ to the triangle BNO, also the first height will be to the second as the trilinear figure BA2B to the triangle BNO.

COROLLARY IV.

503. The initial speed of the weight ascending in air through the distance AN or QB, is to that speed it will return to the earth, with the same measured distance NA or BQ, as NO to DE, or on account of the similar triangles BON & BDE, as BO or RP to BD or RC.

These four corollaries contain everything, which the most noble Huygens has simply indicated without demonstrating everything, about the motion of bodies ascending and decreasing vertically along right lines in the present hypothesis of the air resistance, as an amendment to his discussion *Discours de la Cause de la Pesanteur* pag.171, and which the celebrated Varignon also has shown in the *Actis Acad. Reg. Paris. Scient*. 1708.

COROLLARY V.

504. With the times of the ascent through the distances AN, and with TS or YN drawn, clearly with TY parallel to BN or QA, they shall be AO: FH, and TW: FH; with the initial speeds NO and SW present ; therefore there will be $t\mathrm{AY} = (\mathrm{AO-TW}):\mathrm{FH} = \log.(\mathrm{OF:WF}):\mathrm{FH},$ in the logarithmic, of which the subtangent is FH, or §.492 simply $= \log.(\mathrm{OF:WF})$ in the logarithmic, of which the subtangent is one.

COROLLARY VI.

505. Thus also the time, in which a weight passes through a distance *st* in air, is $\log.(\mathrm{BF:WF})$ on the logarithmic curve, of which the subtangent one. For the time through $st = wt:\mathrm{FH},$ and $wt = \log.(\mathrm{BF}:w\mathrm{F}),$ therefore (§.492) $wt:\mathrm{FH} = \log.(\mathrm{BF}:w\mathrm{F})$ on the logarithmic, of which the subtangent is one.

COROLLARY VII.

506. Thus, if a certain weight may be projected in the air with a resistance following the present hypothesis, following the direction of the tangent BH of the logarithmic curve B*t*C at the point B, and with that speed, so that the vertical speed FH shall be equal to the *terminal or maximum speed* of the weight exclusively derived from an oblique trajectory BH, the projected body describes the arc B*t*C of the logarithmic curve in air. For the right line *sx* may be considered always to be moving parallel to itself from the position BR towards C, thus so that is extremity *s* always shall be present on the right line BH , and its velocity along the line BH, that may be expressed by BH itself ; therefore, if the point of intersection *s* of the right line *sx* and of the line BH is put to remain in the ratio of the speeds, this point *s* will traverse the distance B*s* in a time, which is expressed by $\log.(\mathrm{BH}:s\mathrm{H}):\mathrm{FH}$ or (§.492.) by $\log.(\mathrm{BH}:\mathrm{SH})$, or on account of the similar triangles BFH & B*ws*, by $\log.(\mathrm{BF}:w\mathrm{F})$ on the logarithmic curve, of which the subtangent is one ; truly with the weight conveyed along in this same time $\log\left(\mathrm{BF}:w\mathrm{F}\right)$ traverses the distance *st*, as said in cor. VI. (§.505) ; therefore in that time, in which the conveying line *sx* arrives at the place *sx* from the situation BR, the weight in that accelerated motion passes through the distance *st*, thus so that always it always shall be advancing on the logarithmic line B*t*C, just as Huygens asserted without demonstration, and Varignon elegantly demonstrated analytically by that calculation in the *Actis Acad. Reg. Paris. Scient*. 1708.

507. Huygens added on page 173 of the tract *De la Cause de la Pesanteur* how to determine a kind of logarithmic curve, because its tangent shall be the double of the height, to which a weight ascending initially with a speed equal to the terminal speed shall be able to reach in vacuo. This can readily be deduced from §.150. For if the square of the subtangent FH, which expresses the terminal speed, will be applied to double of the same subtangent, which also show uniform gravitation, thence the maximum height will result, to which the moving body rising in vacuo with the initial velocity equal to the terminal velocity in air, can reach, shall be equal to half of the subtangent of the logarithmic curve.

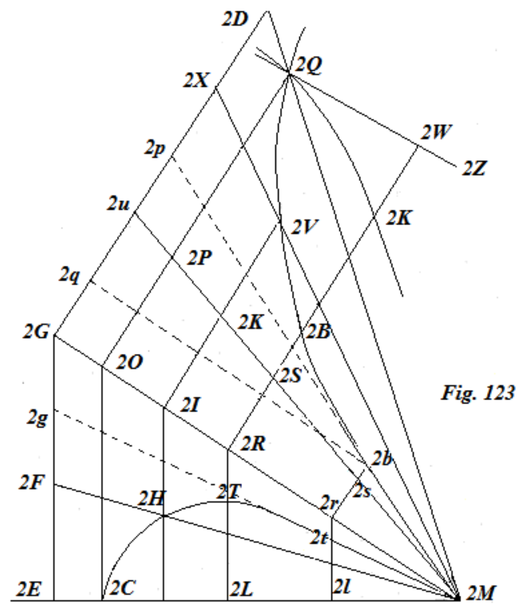
508. And I myself may add with the speed and direction BH of the moving body to be projected in air, and the logarithmic velocity at some point *t* of the describing logarithmic curve B*t*C, must be expressed by the tangent of the logarithmic curve *tu* at this point *t;* thus so that the speed of the moving body always shall be going to be greater at the end of the logarithmic curve, and if it may be less, always to approach closer and closer to that. I have not added the most easy demonstration of this assertion.

COROLLARY VIII.

509. If a weight may be projected downwards with a given speed in air with a resistance nearly in proportion to the speed ; the motions of the body will be determined from the two previous propositions in an easy manner, in the same way as we have shown from the preceding corollary. Indeed let AN be the initial speed or the velocity of projection and because if the moving body were to be moving in a vacuum, its motion would be composed equally from the speed of projection and from the motion due to the acceleration of gravity, thus also its motion in air is from a doubly varies motion, evidently from that, which results from the fundamental uniform motion, and from that which arises from the fundamental acceleration ; both these motions themselves can be considered. In figures 119 and 122 the logarithmic ordinates AN, HI, BF and HC shall be proportionals, and thus there will be (§.491, no. IV.). $AI:DC = AK:FH$, and thus $DC:FH = AI:AK$, and (§.498.) $DC:FH$, or the logarithm of the ratio BF to DF, or FH to GH, that is, the logarithm of the ratio, which the weight has to the acceleration acting in the air divided by the subtangent FH, expresses the time of descent of the weight in air through the distance EC, and that from gravity alone. Truly the other ratio AI:AK (§.495) expresses the time, in which the moving body by its motion varied from the fundamental uniform case is passed through the interval NC, but because it has been shown $DC:FH = AI:AK$, these times will be equal; therefore in the time $DC:FH$ the distance $NC+EC$ will be described by the body in air, by the combined motion from the uniform and from the accelerated motion derived from uniform gravity, (by examining each of the figures 119 & 112): and at the end of the aforementioned time the speed of the body will be $HI+DE$. Indeed there is $NC = AN - HI$, & $EC = DC - DE$, therefore $NC + EC = AN + DC - HI - DE$, and therefore the distance past through in the indicated time will be $DC + AN - HI - DE$, and the speed at the end of this distance, acquired, or remaining, $HI + DE = HI + FH - GH = AN + FH - NC - GH.$ Which quantity with the increase in the time $AI:AK$ or $DC:FH$, and thus with the decrease HI or increase NC, continually decreases, as long as AN is greater than FH; thus still, so that with the minimum excluded, to which always it is approaching more and more, it shall become FH, equal to the terminal velocity. But truly if AN, or the velocity of the throw, were less than the terminal FH, the sum $HI+DE$ continually increases, as long as while HI at infinity becomes vanishingly small, and the other DE will become FH, evidently equal to the terminal velocity.

PROPOSITION LX. PROBLEM.

510. *To find the curve to be described* 2M2T2C, *which a weight projected in air along the direction* 2M2R, *inclined to the horizontal at the given angle* 2R2M2L *with the speed* 2M2G, *with the associated resistance proportional to the speed.*



Solution. At the point 2G, the line 2G2D shall be set up normal to the line 2M2G, the logarithmic curve 2M2B2Q shall be described about the line 2G2D as the axis, passing through the point 2M, having the subtangent 2G2*u*, which just as in the preceding, expresses the constant gravity or also the terminal velocity. 2M2*u* may be joined, which is a tangent to the logarithmic at the point 2M; then through any point 2*r* drawn in the direction projected, with the right lines 2*r*2*l* and *2r*2*b* drawn parallel to the lines 2G2E and 2G2D respectively, of which the latter crosses the tangent of the logarithmic 2M2*u* at 2*s*, and the logarithmic itself at the point *2b* : and finally everywhere for any 2*r*2*l* the part 2*r*2*t* of this is equal to the homologous 2*s*2*b*, of the intercept between the tangent and the logarithmic, and always the point 2*t* will be on the curve sought 2M2T2C.

Demonst. Again a certain line 2M2N is understood, the projected weight itself being carried, itself by its own motion and 2G2E always to be carried parallel, and to be varied from the fundamental motion in air according to the resistance in a ratio of the speed, with the initial speed present the same as with the speed of the projection 2M2G*;* and by being carries on this line 2M2N [there is no point 2N on the original curve of the trajectory] with the weight free to fall with an acceleration considered from the same motion, as what was defined in the preceding proposition ; with which in place, the carrying line 2M2N by this from its own fundamental place may arrive at the place 2*r*2*l*, by being described in the direction of the distance projected 2M2*r*, (§. 495) in the time, which is expressed by the $\log.(2\text{M}2\text{G} : 2r2\text{G}) : 2\text{G}2u = 2r2b : 2\text{G}2u$, and in this same time (§. 505) the weight slips along the resisting line through a distance equal to 2*s*2*b* or (constr.) 2*r*2*t.* And as soon as it has traversed a distance 2M2*r* along the resisting curve 2*r*2*l* , the weight passes along through that distance $2r2t = 2s2b$, and thus it always will be found to proceed partaking of a combined motion of this kind along the curve 2M2T2C. Q.E.D.

COROLLARY I.

511. The tangent 2*g*2*t* drawn through some point 2*t* of the curve, expresses the speed of the moving body at the same point of the curve. For the increments of the right line 2M2*r* and of the curve 2M2*t* with the two right line placed indefinitely close and parallel, of which 2*r*2*t* shall be the other increment, will be described in the same time, and thus the velocities, by which the said elements are traversed, will be as these elements themselves; but these elements are proportional to the lines 2*r*2G and 2*t*2*g* , therefore these lines also must be proportional to the velocities, with which the said elements are traversed. Hence, because (§.495) 2*r*2G expresses the speed, by which the element of the right line 2M2*r* is traversed, thus the tangent 2*t*2*g* will express the speed of the moving body on the curve 2M2T2C at the point 2*t*.

COROLLARY II.

512. The right line 2E2G will be the asymptote of the projection of the curve 2M2T2C. For if 2O2Q falls upon 2G2D, 2P2Q shall become infinite, therefore also2O2C equal to 2P2Q itself will be infinite, when it will fall on the line 2G2E.

COROLLARY III.

513. If 2*u*2D may be equal to 2G2E, and 2M2D may be joined cutting the logarithmic at 2Q, and with the right line 2Q2O drawn from 2Q parallel to 2D2G, and with the right line 2O2C through 2O parallel to 2G2E; the right line 2M2C will be the magnitude of the projection of the curve 2M2T2C.

COROLLARY IV.

514. Thus also the magnitude of the projection of the curve will be found, on some inclined plane to the horizontal 2M2F, by taking $2u2\mathrm{X} = 2\mathrm{F}2\mathrm{G}$, and by drawing the logarithmic curve 2M2X cutting at the point 2V. For, if from this point 2V2I is acting parallel to 2G2D, and 2I2H is parallel to the right line 2E2G, the intercept 2M2H will be the magnitude sought in the plane 2M2F inclined to the horizontal. For there will be $2\mathrm{X}2u : 2\mathrm{V}2\mathrm{K} = 2\mathrm{G}2\mathrm{F} : 2\mathrm{I}2\mathrm{H}$, from which since (from the constr.) $2\mathrm{X}2u = 2\mathrm{G}2\mathrm{F}$, also there will be $2\mathrm{I}2\mathrm{H} = 2\mathrm{V}2\mathrm{K}$.
The preceding corollary is demonstrated by almost the same reasoning.

COROLLARY V.

515. Truly with the magnitude 2M2C given and with the speed of projection 2M2G, the angle 2G2M2E can be found and agreeing with these given. For if from the coordinates 2M2O and 2O2Q, called *x* and *y* respectively, the curve 2Y2Q may be

constructed, of which the equation shall be $y = \frac{cx}{a} + \sqrt{(xx - bb)}$, which hence it is apparent to be some conic section, evidently a hyperbola, in which *a* indicates 2M2G, while *c* and *b* denote the subtangent of the given logarithmic 2G2*u*, and the given magnitude of the projection of the curve 2M2C. 2Q shall be the common intersection of the curve 2Y2Q and of the logarithmic curve 2M2B2Q, through which and through the point 2M the line 2M2D may be drawn crossing the right line 2G2D at the point 2D, the line 2*u*2D will give the sine 2G2E of the angle sought 2G2M2E, with the radius or the total sine 2G2M.

COROLLARY VI.

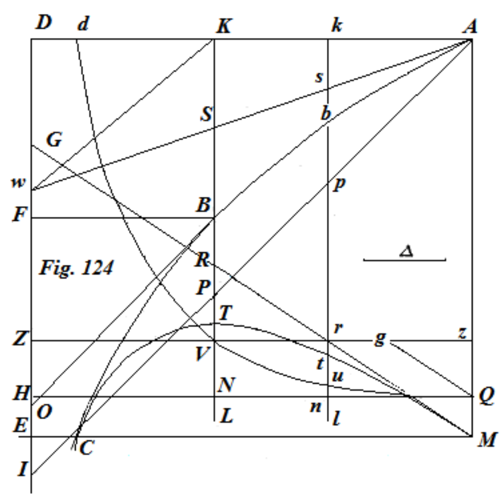
516. It will be no more difficult in determining, which angle of elevation 2G2M2E may be agreed to be the greatest of all the magnitudes 2M2C possible. For, with the same put in place, with the same symbols as in the preceding corollary, if in the case of the curve 2Y2Q the equation were $y = \frac{aax - axx}{cx - 2ac} + \frac{cx}{w}$, as hence it will be evident to be some conic section, the intersection of this new curve and of the common logarithmic 2Q will produce the line 2*u*2D, which always is equal to the sine of the angle sought 2G2M2E. For, if the curve 2Y2Q never intersects the logarithmic curve, then the problem is impossible, which is especially understood from the preceding corollary, in which it can touch often, as at *b*, or with the magnitude of projection assumed to be so great, so that the hyperbola thence resulting at no time may be able to cut the logarithmic curve, and thus the problem will be impossible to solve, since the opposite problem of this sixth corollary may possibly be present always. Sometimes also the hyperbola of corollary V of this can cut the logarithmic curve at two points, from which it arises that the problem allows two different elevations 2G2M2E admitted. The foundations of these two corollaries has been put in place in the corollaries III. and IV. Truly I leave the calculation to the industry of the reader.

COROLLARY VII.

517. The tangent 2*b*2*p* may be drawn through the point 2*b* of the logarithmic curve, and initially there will be $2u2p = 2\text{G}2g$. For the element of the line 2*s*2*b* is to the element of the line 2M2*s* as $2u2p - 2s2b$ to $2u2s$, and the element of the line 2M2*s* to the element 2M2*r* as 2*u*2*s* to 2G2*r*, therefore from the equation of the element of the line 2*s*2*b* to the element of the line 2M2*r* is itself just as $2u2p - 2s2b$ to 2G2*r*; and the element of the line 2*s*2*b* or (constr.) of the equal 2*r*2*t* is to the element 2M2*r* just as $2\text{G}2g - 2r2t$ or $2s2b$ to $2\text{G}2r$, therefore $2u2p - 2s2b : 2\text{G}2r$ is obtained as $2\text{G}2g - 2s2b : 2\text{G}2r$, and thus $2\text{G}2g - 2s2b = 2u2p - 2s2b$, that is, $2\text{G}2g = 2u2p$, and thus everywhere. In the second place 2*u*2*p* is equal everywhere to the respective 2*r*2*b* . For with the right line 2*b*2*q* drawn through 2*b*, parallel to the right line 2M2G, 2*p*2*q* will be the subtangent, and thus itself will be equal to 2G2*u*, with the common 2*q*2*u* removed (or subsequently added) , the equal amounts will remain, 2*u*2*p* and 2*G*2*q* or 2*r*2*b*, therefore also, $2\text{G}2g = 2r2b$, thence $2\text{G}2g - 2r2s = 2r2b - 2s2b = 2r2s$, therefore the element 2*t*2*r* is to the element of the line 2M2*r*, as 2*r*2*s* to 2*r*2*G*. Now

$2s2r : 2r2\mathrm{M} = 2\mathrm{G}2u : 2\mathrm{G}2\mathrm{M}$, and $2r2\mathrm{M} : 2r2\mathrm{G} = 2r2\mathrm{M} : 2r2\mathrm{G}$, therefore by arranging the ratios, from the equality, $2s2r : 2r2G = 2r2\mathrm{M}.\ 2\mathrm{G}2u : 2\mathrm{G}2\mathrm{M}.\ 2r2\mathrm{G}$. Also there will be the element of the right line $2r2t$ to the element of the right line $2\mathrm{M}2r = 2r2\mathrm{M}.\ 2\mathrm{G}2u : 2\mathrm{G}2\mathrm{M}.2r2\mathrm{G}$. Hence, if the lines will be called, as follows, evidently 2M2G, *b;* th subtangent $2G2u$, *a;* the unknowns 2M2*r*, *y* ; 2*s*2*t*, *x*, the elements of these *dy* and *dx*, and in a similar manner it will be found that the element. $2r2t : \text{elem.}2\mathrm{M}2r = 2r2\mathrm{M}.2\mathrm{G}2u{:}2\mathrm{G}2\mathrm{M}.2r2\mathrm{G}$ will adopt this other form, in analytical terms, $dx : dy = ay : bb - by$; and thus the differential equation of the curve will be $dx = aydy : bb - by$. As the celebrated Varigon first found in the *Actes de l'Academie Royale des Sciences. Paris*. 1708. for the 18th of July, Coroll. III ; this most distinguished author also indicated in this work the construction of the problem he used in the present proposition to be shown to be most similar to what we have adopted above (§.510), but shown otherwise than done by us. And finally in the work of 22nd August of the same year 1708, in the presence of that prelect society, the identity of the construction of the curves of the trajectories by the most outstanding geometers Newton and Huygens with the effect added by themselves (as from that, which we have taken, now we have said not to differ except in the demonstration) from the more elegant calculation made known. Truly, because the same identity of the curves, which both the constructions of Newton and Huygens between themselves, as well also with the construction of Varignon at first sight bearing not a few discrepancies, except by a more elegant calculation also can be demonstrated, the same identity can be shown geometrically that I will not be troubled to confirm in this place. Towards facilitating a demonstration I will begin from the Huygens construction.

PROPOSITION LXI. THEOREM.



518. *The curve of the trajectory, the construction of which has been shown in the preceding proposition, is similar and equal to the curve, which results from the Huygens construction* (*see* Discourse on the cause of weight pag.171.) *according to the hypothesis of the resistances of the media proportional to the actual velocities of the moving body.*

Huygens' construction is obtained thus. MG shall be the direction of the throw, ME the horizontal, DE perpendicular to the horizontal, just as about the axis the logarithmic ABC having the subtangent D*w*, or FO, and with AD divided in some manner at K parallel to the other EM, so that AK shall be to KD, just as the vertical speed of the throw derived from the oblique MG, is to the terminal speed, or to the exclusive maximum, as evidently the weight cannot attain, even if by falling it may approach that more and more . Through K, the right line KBL acts parallel to DE, cutting the logarithmic curve at B, and the horizontal at L; and again with the tangents A*w*, BO drawn through the points A , B of the logarithmic curve, while also AC is drawn parallel to the tangent BO, which KL will cut at P, and the logarithmic curve at C; if it may happen on some line KL, with the lines A*w*, AC, MG and ME cut at the points S, P, R and L, and truly the logarithmic at B, so that as RL to TL thus SP to BP, and $rl : tl = sp : bp$, the points T, *t*, &c. will be on the chosen curve MTC, that I say to be the same as the curve 2M2T2C constructed in proposition LX. §.510, if the subtangents of the angles GME, 2G2M2E, of the logarithms D*u* and 2G2*u*, and finally the speeds of the projections MG and 2M2G, were equal.

Demonst. Therefore M*r* and 2M2*r* shall be equal, and *rl* and 2*r*2*l* will be equal as well. Now, because MG expresses the speed of the throw, EG will denote the vertical velocity derived from the oblique MG , and D*u* or FO (§.498) will express the terminal speed, and there will be (by construction) $\mathrm{AK} : \mathrm{DK} = \mathrm{EG} : \mathrm{D}w$; and, because if to the equal D*w* and FO the common *w*F is added, thence there comes about DF or KB and *w*O to be equal, and thus the right lines *w*K, HB and AI are parallel, also there will be $\mathrm{AK} : \mathrm{DK} = w\mathrm{I} : \mathrm{D}w = \mathrm{EG} : \mathrm{D}w$; and consequently $w\mathrm{I} = \mathrm{EG}$, and thus everywhere $sp = rl = 2r2l$, since M*r* and 2M2*r* will be equal to the angles GME and 2G2M2E (following the hypothesis) ; again $\mathrm{AD} : k\mathrm{D}\ (= \mathrm{MG} : rG) = 2\mathrm{M}2\mathrm{G} : 2r2\mathrm{G}$, therefore (§.492) $kb : \mathrm{D}w = 2r2b : 2\mathrm{G}2u$, for *kb* is the logarithm of the ratio AD to *k*D in the logarithmic ABC, and 2*r*2*b* the logarithm of the other equal ratio, 2M2G to 2*r*2G in the logarithmic 2M2B2Q. Or, because (following the hypothesis) $\mathrm{D}w = 2\mathrm{G}2u$, there becomes $kb = 2r2b$, and because $2\mathrm{G}2u : 2r2s (= 2\mathrm{G}2\mathrm{M} : 2r2\mathrm{M} = \mathrm{GM} : r\mathrm{M} = \mathrm{DA} : k$

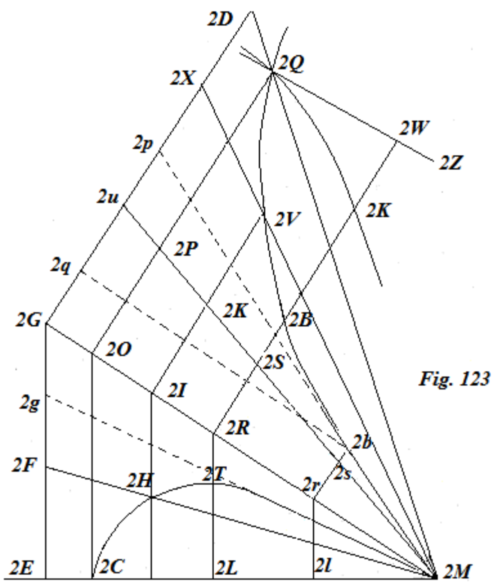


Fig. 123

and $2\mathrm{G}2u = \mathrm{D}w$, there will be $2r2s = ks \,\&\, sb = 2s2b$*;* hence also from the equalities *sp* and 2*r*2*l* with the equal amounts removed *sb* and 2*s*, and 2*b* or 2*r*2*t*, there remains $bp = 2t2l$ *;* but the construction gives $rl : tl = sp : bp$ *;* or because $rl = sp$, there shall be also $tl = bp$, therefore there will be had equally $tl = 2t2l$, and thus the Huygenian curve MTCis the same everywhere as the other curve 2M2T2C. Q.E.D.

COROLLARY I.

519. Hence it is apparent its Huygens construction would be treated somewhat more simpler, if in place of the ratio $\mathrm{RL} : \mathrm{TL} = \mathrm{SP} : \mathrm{BP}$ it would be justified to assume everywhere the divisor TL to be equal to the homologous intercept BP between the logarithmic curve ABC and its subtangent AC.

COROLLARY II.

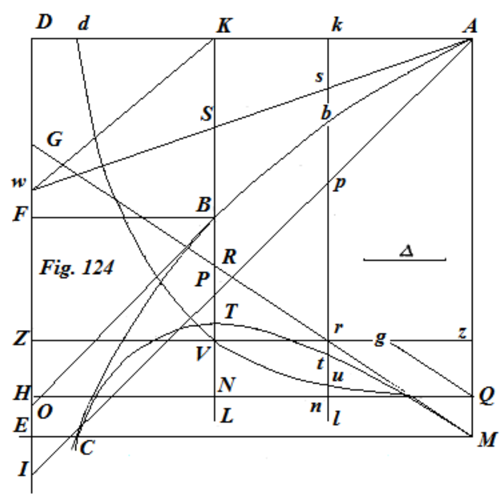
520. The parameter of the parabola, which the missile will describe along the direction MG and with that speed expressed correctly in a vacuum, will be $2.\mathrm{EM}^2 : \mathrm{D}w$. For, because the vertical speed derived from the oblique projection is EG, and uniform gravity is expressed by the subtangent of the logarithmic D*w*, the maximum altitude, to which the weight can reach with the initial speed EG, will be (§.150) $\mathrm{EG}^2 : 2\cdot \mathrm{D}w$. But GE is to EM, as twice the height of the parabola to half the width, and therefore this half width is $\mathrm{EM}.\,\mathrm{EG} : \mathrm{D}w$. And the square of half the width of the parabola applied to its height produces the parameter, therefore here the parameter is $2\cdot \mathrm{EM}^2 : \mathrm{D}w$. Which is in agreement with the determinations of Huygens and Varignon, as expressed in other terms.

PROPOSITION LXII. THEOREM.

521. *With the same hypotenuse of the resistance put in place, the curve of the projection which has arisen from the construction of the most celebrated Newton* (Princ. Phil. Nat. Math. Lib. II. Prop. IV.) *has been shown, to be the same as that of Huygens, about which have acted in the nearest preceding proposition, or with our sixty-first proposition.*

Lest there be some need for the reader to question the construction of Newton in some regard, but to be advanced by others rather than from the writings designated in the works of the author, that is a pleasure to address here .
The same truly thus is found :

With MG assumed for the speed and direction of the throw, and with ME drawn through its end to the horizontal, and to the vertical ED, on which is put its segment the vertical speed EG derived from the oblique MG ; between the asymptotes DEM some hyperbola QV*d* is understood to be described cutting the right line AM at Q , through which point in addition QH is acting parallel to the horizontal ME, and here with ME divided at L, so that ML shall be to LE just as the initial resistance of the medium to the motion of the projectile struggling to a greater height, or the vertical velocity EG to gravity, or its terminal velocity, before set out by the subtangent FO of the logarithmic ABC, LK may be drawn parallel AM, cutting the hyperbola in V, which crosses the logarithmic at B and AD at K, as indeed in the Huygens construction (§. 518) AK also is to KD as EG to FO or D*w*. The line Z*z* is acting parallel to EM through the point V of the hyperbola, and *gz* is taken in that portion, as Newton designates by the letter N , which shall be to *z*Q or VN*r*, as ME to EG, or more simply, Q*g* may be drawn



parallel to MG. With which in place, if on some *rl* parallel to KL there may be taken *rt* equal to the three-lined figure from the hyperbola Q*un* applied to the given *gz*, the point *t* will be on the curve, which evidently is the same as that, which the constructions of the preceding proposition LX and LXI produced.

Demonst. I. The hyperbola QV*d* gives, $\mathrm{EM}:\mathrm{EL}=\mathrm{VL}:\mathrm{QM}$ on dividing by $\mathrm{LM}:\mathrm{EL}=z\mathrm{Q};\mathrm{QM}$, and (by constr.) $\mathrm{LM}:\mathrm{EL}=\mathrm{EG}:\mathrm{D}u=rl:ks,$ therefore also $z\mathrm{Q}:\mathrm{QM}=rl:ks$ then also (by constr.) $gz:z\mathrm{Q}\left(=\mathrm{ME}:\mathrm{EG}\right)=\mathrm{M}l:rl$, therefore from the equation $gz:\mathrm{QM}=\mathrm{M}l:ks$; and therefore $\mathrm{M}l.\mathrm{QM}=gz.ks$.

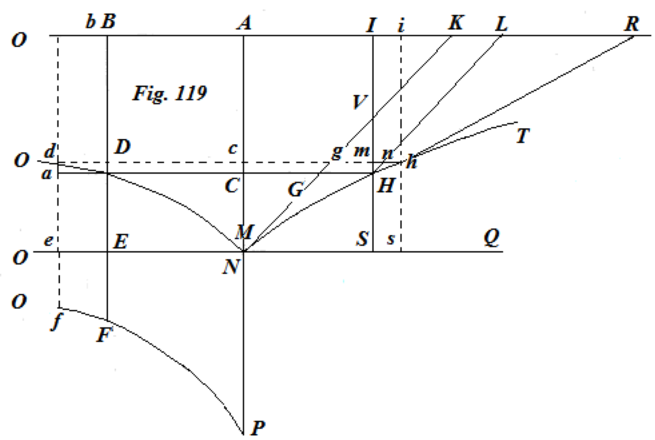
II. With the ratios resumed $z\mathrm{Q}:\mathrm{QM}=\mathrm{GE}:\mathrm{D}w,$ and $gz:z\mathrm{Q}=\mathrm{EM}:\mathrm{GE}$, therefore from the equation $gz:\mathrm{QM}=\mathrm{EM}:\mathrm{D}w$. And thus $\mathrm{EM.QM}=\mathrm{D}w.gz$.

III. From the construction the trilinear part of the hyperbola $Qnu=gz.rt$, and with $\mathrm{M}l.\mathrm{QM}$ added (no.1. of this) $=gh.ks$, making $\mathrm{Q}ul\mathrm{M}=rt.gz+ks.gz.$ And (§.368) $\mathrm{QM}lu:\mathrm{EM.MQ}=kb:\mathrm{D}u$ & $\left(\text{no. II. of this}\right)\mathrm{EM.QM}=\mathrm{D}w.gz$; therefore $rt.gz+ks.gz:\mathrm{D}w.gz=rt+ks:\mathrm{D}w=kb:\mathrm{D}w$; and thus $rt+ks=kb=sb+ks:$ hence finally $rt=sb$, as the constructions of the propositions LX and LXI provide ; therefore the three different constructions of the three last propositions supply one and the same curve. Q.E.D.

CAPUT XV.
De motibus Corporum, quibus aër resistit in ratione celeritatum mobilis.

PROPOSITIO LVIII. THEOREMA.

495. *Spatium, quod mobile quodam in aëre, juxta proportionem celeritatum ejus actualium, variato motu ex primitive uniformi certo quodam tempore transmittit, exponitur eadem linea, qua velocitatis mobilis initialis pars hoc tempore amissa repræsentatur. Et tempus absolutum, quo dictum spatium absolvitur, exponi debet per logarithmum rationis, quam habet velocitas initalis mobilis ad velocitatem absolutam ejusdem elapso hoc tempore, applicatum ad subtangentem illius logarithmicæ, ex qua prædictus logarthmus desumtus est.*



Sit M mobile in plano quodam horizontali celeritate AN impulsum, juxta rectam NQ, in qua deinceps æquabiliter incederet celeritate eadem AN, si in vacuo ferretur, sed quia in aëre juxta proportionem celeritatum suarum resistente movetur; singulis momentis aliquid de sua velocitate actuali decedet. Ipsi vero motus ope propositionis 55 hujus secundi libri facile determinantur ; quandoquidem, juxta hanc propositionem, iidem sunt motus variati ex primitive uniformi in aëre resistente, qui resultarent ex motu communi alicujus lineæ deferentis, & ex motu proprio mobilis in hac deferenti linea orto ab appulsu aëris corpori in linea deferente delato.

Sit ergo NO hæc linea deferens celeritate mobilis initiali AN, æquabili motu tendens versus Q, & post aliquod tempus mobile M in hac linea deferente spatiuan NE percurrisse intelligatur, atque in ea celeritatem DE exacto dicto tempore acquisivisse, quo fiet, ut velocitas ejus initialis AN reducta sit ad velocitatem actualem (qua scilicet in aëre movetur) BD ; hac ergo velocitate aër mobili ad abitur & resistentia est ut hæc celeritas , ergo impressio seu solicitatio acceleratrix mobilis in linea deferenti, quæ generaliter est EF, jam erit BD. Propterea curva PFO erit similis & æqualis curvæ NDO, adeo ut tantum (§.488.) determinandum restet, quænam sit curvæ NDO, ut in ea BD.B*b* seu EF.E*e* æquetur rec-lo *ead* seu ED.*ad.* Ubi E*t* vel B*b* est elementum spatii NE, & *ad* elementum ordinatæ DE.

Per punctum N ducta sit log-mica NHT, cujus subtangens AK (posito scilicet rectam NK ipsam contingere in puncto N) æqualis AN, atque in ea ad asymptotam AR aptetur ordinata IH æqualis BD, eaque producamr in S; ductaque per logarithmicæ punctum

H recta indefinita HCD parallela OQ, tangentem log-micæ NK in G, rectamque NA interceptæ inter tangentem NK & log-micam NHT, erit semper punctum D in scala celeritatum propriarum mobilis in linea deferente NO.

Ducantur enim per punctum log-micæ H tangens HR & linea HL parallela tangenti NK, eruntque HI & IL, item LR & HS vel DE æquales, ductaque insuper *Jh* alteri DH indefinite vicina lineola *mh* & IR a media HL proportionaliter secentur, eritque adeo HI vel BD aut EF ad LR vel HS seu DE, sicut Hm seu *da* ad *nh*; adeoque $\mathrm{EF}.nh=\mathrm{DE}.da:$Jam, quia (constr.) DE vel $\mathrm{EN}=\mathrm{GH}$ & $\mathrm{N}e=ch$, erit *nh*, differentia inter GH & *gh*, æqualis *Ee* ; nam H*n* est parallela ipsi *Gg*. Ergo $\mathrm{EF}.nh=\mathrm{EF.E}e=\mathrm{DE}.da.$ Quod erat demonstrandum.

Porro est $nh:\mathrm{LR}=mh:\mathrm{IR}$, atqui $nh:\mathrm{LR}=\mathrm{E}e:\mathrm{DE}\ (\S.128.):t\mathrm{E}e$, ergo $t\mathrm{E}e=\mathrm{I}i$ vel $mh:\mathrm{IR}$; ergo omnia *t*E*e*, id est, tempus per $\mathrm{NE}=$ omnibus $\mathrm{I}i:\mathrm{IR}=\mathrm{AI}:\mathrm{IR}:\log.(\mathrm{AN}:\mathrm{HI})$ applicatus ad IR vel AN. Hoc est, tempus absolutum, quo mobile percurrit spatium NE in linea deferente, exponitur log-mo rationis, quam habet celeritas initialis AN ad celeritatem mobilis residuam IH elapso hoc tempore, applicato ad log-micæ subtangentem.

Hinc, quia (§. 489.) spatium absolutum, quod mobile M dato illo tempore in aëre transmittit, est $\mathrm{AN}.t\mathrm{NE}-\mathrm{NE}$, erit hoc spatium $\mathrm{AI}-\mathrm{NE}=\mathrm{CH}-\mathrm{GH}=\mathrm{CG}=\mathrm{NC}$, quoniam (constr.) $\mathrm{AN}:\mathrm{AK}$, item $t\mathrm{NE}=\mathrm{AI}:\mathrm{AN},$ & $\mathrm{NE}(\text{constr.})=\mathrm{GH}$. Ergo spatium absolutum, quod mobile nominato tempore percurrit, in aëre repræsentatur per eam lineam NC vel DE, quæ velocitatis amissam partem exhibet. Quæ omnia erant demonstranda.

COROLLARIUM I.

496. Ergo si mobilis celeritates actuales BD seu IH sumantur in progressione geometrica, tempora respectiva AI:AN erunt in progressione arithmetica.

COROLLARIUM II.

497. Adeoque velocitas mobilis non nisi tempore infinito extingui potest, cum log-us rationis AN ad o, seu ad quantitatem qualibet data minorem, sit infinitus. Propterea mobile M ejusmodi motu ex primitive uniformi derivato in aëre nunquam percurrere potest spatium AN datæ magnititudinis.

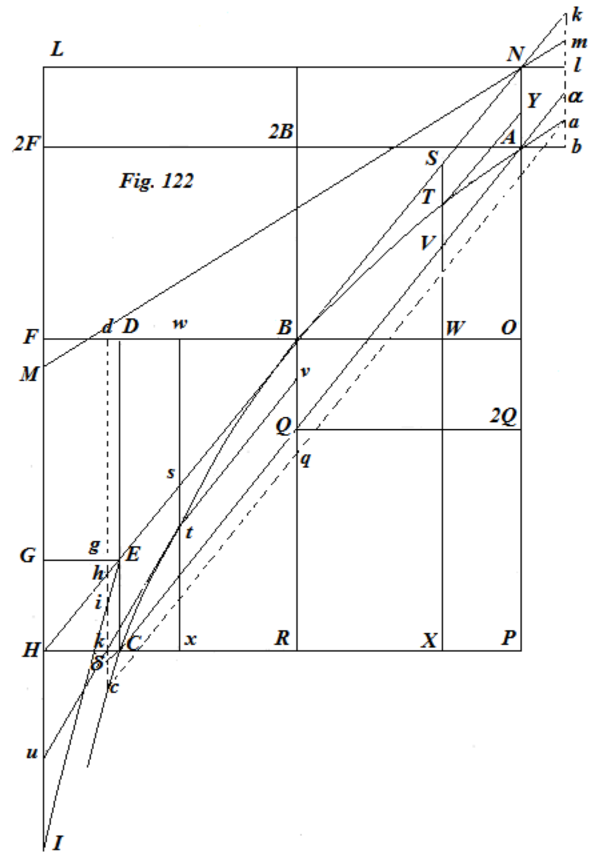
Hæc omnia ad amussim conveniunt cum determinationibus Newtoni Prop. 2. Lib.2 Pr. Phil. Nat. ; Leibnitii Act. Erud. 1689, art.1 ;Wad isii Algebr.cap.101. Et Varignonii Act.Acad. Reg. Paris. Scient. 1707, die 13 Aug. probl. 1. & coroll. annexis.

PROPOSITIO LIX. THEOREMA.

498. *Si grave in aëre juxta rationem celeritatum resistente verticaliter a quiete descendat, spatium, quod mobile aliquo tempore per labitur, exponetur excessu, quo factum ex tempore descensus in velocitatemcorporis terminalem excedit lineam, quæ celeritatem mobili acquisitam repræsentat. Tempus vero ipsum exponetur log-mo*

rationis,quam gravitas absoluta mobilis habet ad solicitationem acceleratricem in termino dicti spatii, applicato ad log-micæ subtangentem.

Sin vero grave in aëre eodem verticaliter in altum projiciatur data cum celeritate initiali, altitudo totius ascensus mobilis exponetur excessu, quo velocitas initialis excedit factum ex tempore ascensus in velocitatem terminalem. Et tempus ipsum exponetur log-mo rationis resistentiæ totalis initio ascensus ad gravitatem absolutam mobilis applicato subtangentem logarithmicæ.



I. Sit B*t*C log-mica ad axem FI extructa, cujus subtangens FH sit quæcunque & BH tangens log-micæ in puncto B. Ducatur pro lubitu linea DE æquidistans ipsi FI, rectæ BF in puncto D, tangenti BH in E, & denique log-micæ in puncto C occurrens. Eritque EC intercepta inter tangentem BH & log-micam BC spatium percursum descensu mobilis, ratio $\mathrm{DC}:\mathrm{FH}$ exponet tempus, quo spatium EC conficitur, & denique DE intercepta inter BF & BH exponet celeritatem mobili acquisitam tempore prædicto $\mathrm{DC}:\mathrm{FH}$. FH vero exponet celeritatem tcrminalem, seu maximam, exclusive. Hinc ducta EG parallela BF, recta GH repræsentabit solicitationem acceleratricem in fine spatii percursi EC. Nam quoniam FH repræsentat gravitatem absolutam, & (secundum hypothesin) DK vel FG resistentiam medii, residua GH exponet omnino solicitationem acceleratricem mobilis, ut adeo aliud non restet faciendum, quam (§. 484) ut probetur hanc constructionem, præbere $\mathrm{GH}.\delta c = \mathrm{DE}.gh$, seu *momentum solicitationis acceleratricis* GH *æquari momento celeritatis* DE, ductis *dc* parallela DE, & $\delta\mathrm{C}$ parallela BH, adeo ut δc sit elementum linea EC, & *gh* elementum linæ DE. Ubi obiter notandum, punctum C in figura per accidens tantum reperiri in communi intersectione rectæ HP, æquidistantis FO per datum positione punctum H ductæ & log-micæ BC, cum dictum punctum C quodlibet aliud log-micæ punctum esse possit.

Demonstr. Fiat $\mathrm{GI} = \mathrm{FH}$, eritque $\mathrm{HI} = \mathrm{FG} = \mathrm{DE}$, & acta EI erit parallela tangenti log-micæ in C, atque adeo E*i* erit parallela & æqualis elemento logarithmicæ C*c*, & $\delta c = hi$. Jam, quia $\mathrm{GH}:\mathrm{HI}$ vel DE, $= gh:hi$ seu $c\delta$ erit $\mathrm{GH}.\delta c = \mathrm{DE}.gh$. Quod erat unum.

Porro gi vel $xc:\mathrm{GI}=gh:\mathrm{GH}$ (§.128.) $=t\delta c$, ergo omnia $\chi c:\mathrm{GI}$, seu $\mathrm{DC}:\mathrm{GI}=$ tempori descensus in spatio EC. Atqui DE est log-us rationis FB ad GE seu HF ad GH, hoc est, rationis, quam habet velocitas terminalis FH ad solicitationem acceleratricem GH. Nam, quia DE exponit (secundum hypothesin) velocitatem mobili acquisitam, & quia hæc sit FH, ubi DE facta fuerit infinita, manifestum est celeritatem terminalem per subtangentem FH exponendam esse, qua etiam gravitas absoluta mobilis repræsentatur. Idcirco tempus descensus per spatium EC est log-us rationis $\mathrm{FH}:\mathrm{GH}$ divisus per subtangent FH. Adeoque DE est factum ex tempore descensus per EC in velocitatem terminalem FH, & spatium EC est excessus facti modo nominati, supra rectam DE, quæ celeritatem acquititam exponit. Quæ omnia erant demonstranda.

499. Ponatur secundo mobile ascendere in linea verticali *ak* vel AN celeritate initiali NO, ostendendum restat, altitudinem, ad quam mobile pertingere possit, esse *ak* vel AN interceptam inter tangentem BN & log-micam BTA, tempusque, quo altitudo ista conficiatur, esse $\mathrm{AO}:\mathrm{FH}$. Linea *ak* alteri AN indefinite vicina ponitur, unde ducta Aα parallela BN, lineola $a\alpha$ erit decrementum spatii *ak* versus B continue decrescentis cum velocitate NO. Initio motus resistentia aëris (secundum hypothesin) est NO, & gravitas FH vel OP corpori ascendenti etiam opponitur, adeoque (§.481) resistentia totalis est NP. Propterea ostendendum, constanter esse $\mathrm{NP}.a\alpha$ *momentum scilicet resistantiæ totalis NP æquale NO.kl momento celeratis decrescentis* NO.
Si enim iterum $\mathrm{LM}=\mathrm{FH}$ & NM ducta fuerit, erunt
$\mathrm{MH}=\mathrm{NO}$, & $\mathrm{km}=a\alpha$, & $\mathrm{LH}:\mathrm{MH}=\mathrm{NP}:NO=kl:km$ vel $a\alpha$, adeoque $\mathrm{NP}.a\alpha=\mathrm{NO}.kl$.
Idcirco omnia decrementa $a\alpha$, id est *ak*, seu AN, sunt altitudo, quam ascendens mobile absolvere potest.

Item $ml:\mathrm{LM}=km$ seu $a\alpha:\mathrm{MH}$ vel NO.$(§.128.)\,ta\alpha$, ergo omnia $ml:\mathrm{LM}$, seu omnia $ab:\mathrm{LM}$, hoc est, $\mathrm{AO}:\mathrm{LM}$ æquantur omnibus $ta\alpha$, seu tempori ascensionis mobilis in spatio *ak* vel AN. Atqui AO est log-us rationis A2F ad BF, seu rationis æqualis NP ad OP, id est, rationis, quam habet resistentia totalis initio motus ad gravitatem.
Adeoquc temp. per $\mathrm{AN}=\log.(\mathrm{NP}:\mathrm{OP})$ divis. per LM vel FH, & LM in
$t\mathrm{AN}=\mathrm{AO}=\log.(\mathrm{NP}:\mathrm{OP})$. Ergo,cum AN sit $\mathrm{NO}-\mathrm{AO}$, erit altitudo, quam mobile ascendens percurrere potest, excessus, quo celeritas initialis NO excedit factum ex tempore ascensionis in celeritatem terminalem FH vel LM.

COROLLARIUM I.

500. Tempus, quo ascendens grave suam altitudinem AN vel QB emetietur, est ad tempos descensus ejusdem ex eadem altitudine BQ, sicut AO ad OP. Hoc per se satis clarum. Nam tempus descensus per BQ vel EC (§.498.) est $\mathrm{DC}:\mathrm{FH}$ seu $\mathrm{OP}:\mathrm{FH}$, &c tempus ascensus in AN (§. 499) est AO: FH. Ergo &c.

COROLLARIUM II.

501. Tempus quo mobile spatium AN ascendendo trajicit celeritate initiali NO, est ad tempus, quo idem grave in vacuo & celeritate eadem initiali NO suam altitudinem, quousque pertingere potest, absolveret, ut AO ad NO vel A2Q; ducta scilicet per Q recta Q2Q parallela FO. Nam tempus ascensionis in aëre per altitudinem AN est, ut vidimus, $AO:TH$; grave vero in vacuo ascendens emetietur (§.139) suam altitudinem eo tempore, quo id a quiete cadere incipiens accelerato motu celeritatem NO initiali æqualem acquirere potest ; atqui (§.151) hoc tempus est, ut celeritas acquirenda NO applicata ad rectam FH, quæ gravitatem uniformem exponit ; atque adeo $NO:FH$ exponit dictum tempus ascensionis mobilis in vacuo. Est ergo tempus ascensionis in aëre ad tempus ascensionis in vacuo ut $AO:FH$ ad $NO:FH$, id est, sicut AO ad NO seu A2Q*;* nam ob æquales & parallelas BN & QA, etiam æquales erunt NO A2Q.

COROLLARIUM III.

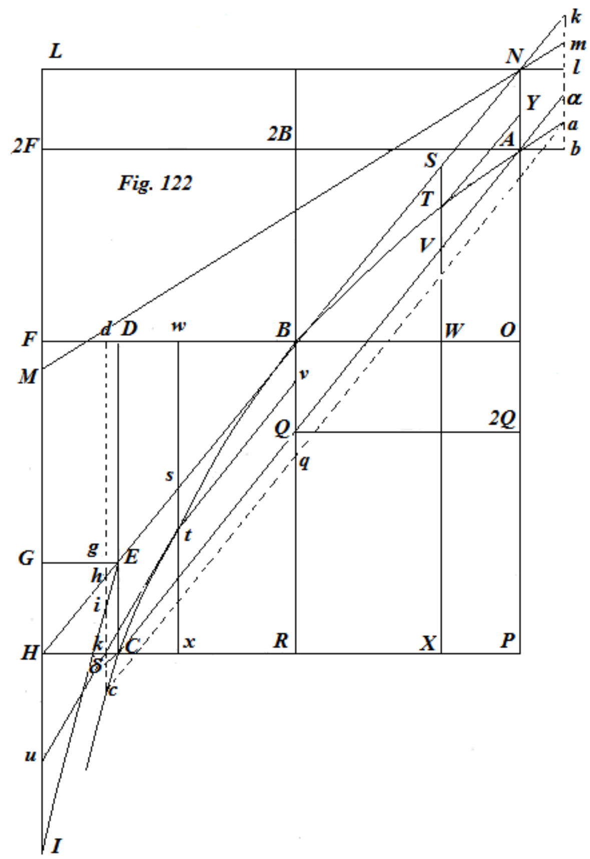


Fig. 122

502. Altitudo AN, quam grave celeritate NO verticaliter in altum projectum percurrere potest, erit ad altitudinem, quam eadem celeritate in vacuo sursum projectam absolvere posset, ut rec-lum FBQ vel FB.AN ad triangulum BNO. Nam (§.150) altitudo in vacuo absolvenda exponitur quadrato velocitatis initialis NO, applicato ad duplum gravitatis, seu 2FH ; ergo altitudo in aëre est ad altitudinem in vacuo absolvendam sicut AN vel BQ ad $NO^2 : 2FH$; vel ficut QB ad OB.ON : 2FB, cum NO sit ad FH sicut BO ad BF*;* ergo prædictæ altitudines sunt ut $BF.BQ$ ad $BO.\frac{1}{2}ON$ id est, ut rec-lum FBQ ad triangulum BON. Verum rec-lum FBQ æquatur trilineo BA2B *;* nam quadrilineum FBA2F, indicante Hugenio & demonstrante Cel. Viro Guidone Grando in suis Hugenianis cap. 8, num. 14, æquatur rec-lo BO.FH, quod quidem ex superioribus (§. 491. num. 11.) facillime probari potest ; nam quia fig. 119. rec-lum $HI.Ii = rec-lo\ IR.Hm$, erunt omnia Hl*i*, quæ in area NAIH, continentur, id est hæc area ipsa = omnibus IR.H*m* seu IR.NC. Eodem ergo argumemo sequitur quadrilineum FBA2F æquari rec-lo FH.BO seu rec-lo ex subtangente log-micæ in differentiam ordinatarum A2F & BF. Atqui propter similitudinem triangulorum HFB & NOB, est

$\mathrm{EH.BO = EB.NO = rec-lum\ LFB}$, ergo $\mathrm{LFB = FBA2F}$, & ablato communi rec-lo BF2F, restabit $\mathrm{L2F2B = FBQ = trilineo\ BA2B}$.

Hinc, quia altitudo in aëre conficienda est ad altitudinem in vacuo percurrendam, sicut rec-lum FBQ ad triangulum BNO, erit etiam prior altitudo ad alteram ut. trilineum BA2B ad triangulum BNO.

COROLLARIUM IV.

503. Celeritas initialis gravis spatio AN vel QB in aëre ascendentis, est ad celeritatem quacum in terram recideret, emenso eodem spatio NA vel BQ, sicut NO ad DE, vel propter triangula similia BON & BDE, sicut BO seu RP ad BD vel RC.

Hæc quatuor corollaria continent omniæ, quæ Nobil. Hugenius circa motus corporum in lineis rectis verticaliter ascendentium & descendentium in præsenti resistentiæ aëris hypothesi absque omni demonstratione simpliciter indicavit, ad calcem suæ Diatribæ *De Causa gravitatis* pag.171, & quæ Cl. Varignonius postea eleganter etiam demonstravit in Actis Acad. Reg. Paris. Scient. 1708.

COROLLARIUM V.

504. Cum tempora ascensus per spatia AN, & TS vel YN ducta, scilicet parallela TY ad BN vel QA, sint AO: FH, & TW: FH; existentibus celeritatibus initialibus NO & SW*;* erit ideo $t\mathrm{AY} = (\mathrm{AO - TW}):\mathrm{FH} = \log.(\mathrm{OF:WF}):\mathrm{FH}$, in log-mica, cujus subtangens est FH, vel §. 492 simpliciter $= \log.(\mathrm{OF:WF})$ in log-mica, cujus subtangens est unitas.

COROLLARIUM VI.

505. Sic etiam tempus, quo grave spatium *st* in aëre perlabitur, est $\log.(\mathrm{BF:WF})$ in log-mica, cujus subtangens est unitas. Nam tempus per $st = wt:\mathrm{FH}$, & $wt = \log.(\mathrm{BF}:w\mathrm{F})$, ergo (§.492) $wt:\mathrm{FH} = \log.(\mathrm{BF}:w\mathrm{F})$ in log-mica, cujus subtangens est unitas.

COROLLARIUM VII.

506. Unde, si grave quoddam projiciatur in aëre resistente juxta præsentem hypothesin, secundum directionem tangentis BH log-micæ B*t*C in puncto B, & celeritate ea, ut celeritas verticalis FH ex obliqua jactus BH derivata par sit celeritati *terminali gravis* seu *maximœ* exclusive, corpus projectum arcum log-micum B*t*C in aëre describet. Fingatur enim rectam *sx* ex situ BR versus C sibi semper parallelam moveri, ita ut ejus extremitas *s* semper in recta BH existat, ejusque velocitas, juxta lineam BH, hac ipsa BH exponatur; idcirco, si ponatur puncto intersectionis *s* rectæ *sx* & BH resisti in ratione celeritatum, hoc punctum *s* tempore, quod exponitur per $\log.(\mathrm{BH}:s\mathrm{H}):\mathrm{FH}$ vel (§.492.) per $\log.(\mathrm{BH:SH})$ vel propter triangula similia BFH & B*ws*, per $\log.(\mathrm{BF}:w\mathrm{F})$ in log-mica,

cujus subtangens est unitas, percurret spatium B*s* ; grave vero hoc eodem tempore $\log\left(\mathrm{BF}:w\mathrm{F}\right)$ in linea deferente perlabetur spatium *st*, ut in coroll. VI. (§.505) dictum ; eo ergo tempore, quo deferens linea *sx* ex situ BR venit in situm *sx*, grave in ea motu accelerato perlabitur spatium *st*, adeo ut id semper incessurum sit in linea logarithmica B*t*C, prorsus ut Hugenius absque demonstatione asseruit, & Varignonius analytico id calculo in Actis Acad. Reg. Paris.Scient. 1708 eleganter demonstravit.

507. Hugenius addit pag. 173 Dissertationis *De la Cause de la Pesanteur* hujus log-micæ speciem eo determinari, quod ejus subtangens dupla sit altitudinis, ad quam grave celeritate initiali terminalem æquante ascendens, in vacuo pervenire possit. Hoc facile deducitur ex §.150. Nam si quadratum subtangentis FH, quæ exponit celeritatem terminalem, applicetur ad duplum subtangentis ejusdem, quæ etiam gravitatem uniformem exponit, resultabit inde altitudo maxima, ad quam mobile in vacuo ascendens velocitate initiali terminalem in aëre æquante, pervenire potest, æqualis semissi subtangentis log-micæ.

508. Addam & ego mobilis celeritate & directione BH *in* aëre projecti, & log-micam B*t*C describentis velocitatem in quolibet log-micæ puncto *t*, exponi debere tangente log-micæ *tu* in hoc puncto *t;* ut adeo mobilis celeritas in logarithmica terminali semper major futura sit, etsi decrescat, eique semper magis magisque accedat. Assertionis demonstrationem utpote facillimam non adduco.

COROLLARIUM VIII.

509. Si grave, data cum celeritate, verticaliter deorsum projiciatur in aëre resistente juxta proportionem celeritatum ; motus corporis ex duabus præcedentibus propositionibus facili negotio determinabuntur, perinde ac præcedens corollarium ex iisdem elicuimus. Esto enim AN celeritas initialis seu velocitas projectionis, & quia sit mobile in vacuo ferretur, ejus motus mixtus foret ex æquibili projectionis & ex motu accelerato gravitats, ita etiam ejus motus in aëre mixtus est duplici motu variato, scilicet ex eo, qui resultat a motu primitive uniformi, & ex eo qui nascitur a primitive accelerato ; ambo hi motus seorsim considerari possunt. In figuris 119 & 122 sint ordinatæ logarithmicæ AN, HI, BF atque HC proportionales, atqui adeo erit (§.491. nom. IV.). $\mathrm{AI}:\mathrm{DC}=\mathrm{AK}:\mathrm{FH}$, atque adeo $\mathrm{DC}:\mathrm{FH}=\mathrm{AI}:\mathrm{AK}$, atqui (§.498.) DC:FH, seu logarithmus rationis BF ad DF, seu FH ad GH, id est, log-us rationis, quam gravitas habet ad solicitationem acceleratricem in aëre divisus per subtangentem FH, exponit tempus descensus gravis in aëre per spatium EC, idque a sola gravitate. Altera vero ratio AI:AK (§.495) exponit tempus, quo mobile motu ex primitivo æquabili variato in aëre transmittit spatium NC, sed quia ostensum est $\mathrm{DE}:\mathrm{FH}=\mathrm{AI}:\mathrm{AK}$, hæc tempora erunt æqualia; propterea tempore DE : FH mobile describet in aëre, motu mixto motibus ex æquabili & accelerato a gravitate uniformi derivatis , spatium $\mathrm{NC}+\mathrm{EC}$ (inspiciendo utramque figuram 119 & 112): atque in fine prædicti temporis celeritas mobilis erit $\mathrm{HI}+\mathrm{DE}$. Est vero $\mathrm{NC}=\mathrm{AN}-\mathrm{HI}$, & $\mathrm{EC}=\mathrm{DC}-\mathrm{DE}$, ergo $\mathrm{NC}+\mathrm{EC}=\mathrm{AN}+\mathrm{DC}-\mathrm{HI}-\mathrm{DE}$, atque adeo

spatium indicato tempore transmissum erit $\mathrm{DC}+\mathrm{AN}-\mathrm{HI}-\mathrm{DE}$, & celeritas, in termino hujus spatii acquisita vel residua, $\mathrm{HI}+\mathrm{DE}=\mathrm{HI}+\mathrm{FH}-\mathrm{GH}=\mathrm{AN}+\mathrm{FH}-\mathrm{NC}-\mathrm{GH}.$ Quæ quantitas crescente tempore $\mathrm{AI}:\mathrm{AK}$ vel $\mathrm{DC}:\mathrm{FH}$, atque adeo decrescente HI aut crescente NC, continue decrescit, quandiu AN major est quam FH; ita tamen, ut minima exclusive, cui semper magis magisque appropinquatur, futura sit FH, æqualis velocitati terminali. Sin vero AN, seu velocitas jactus, minor fuerit terminali FH, summa $\mathrm{HI}+\mathrm{DE}$ continue crescit, usque dum HI in infinitum imminuta evanuerit, alteraque DE facta fuerit FH, æqualis scilicet velocitati terminali.

PROPOSITIO LX. PROBLEMA.

510. *Invenire curvam* 2M2T2C, *quam grave secundum directionem* 2M2R, *dato angulo* 2R2M2L, *ad horizontem inclinatem celeritate* 2M2G *projectum, in aëre, juncta proportionem resistente, describet.*

Solutio. In puncto 2G excitata normali 2G2D ad 2M2G, circa 2G2D tanquam axem descripta sit per punctum 2M log-mica 2M2B2Q, subtangentem habens 2G2*u*, quæ juxta præcedentia, gravitatem uniformem seu etiam velocitatem terminalem exponit. Jungatur 2M2*u*, quæ log-micam in puncto 2M continget; tunc per quodlibet punctum 2*r* directionis jactus, ductis 2*r*2*l* & 2*r*2*b* rectis 2G2E, & 2G2D æquidistantibus, quaram altera tangenti log-micæ in 2*s* & log-micæ ipsi in 2*b* occurrat: denique fiat ubique qualibet 2*r*2*l* ejus pars 2*r*2*t* æqualis homologæ 2*s*2*b*, interceptæ inter tangentem & log-micam, eritque semper punctum 2*t* in curva quæsita 2M2T2C.

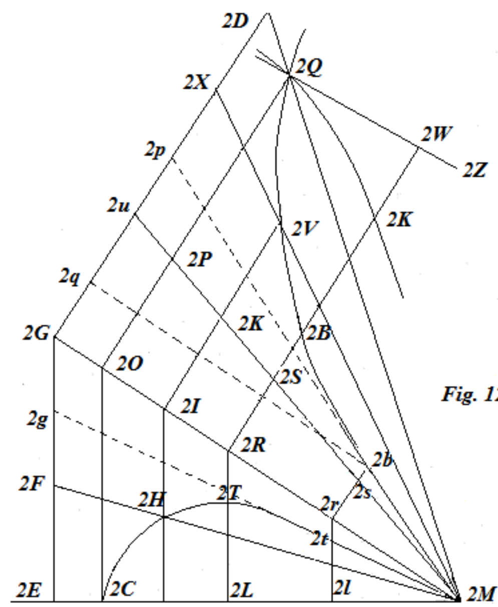


Fig. 123

Demonst. Intelligatur iterum quædam linea 2M2N, grave projectum secum deferens, motu sibi ipsi & 2G2E semper parallelo ferri, & quidem motu variato ex primitive uniformi in aëre secundum celeritatum rationem resistenti, existente celeritate initiali eadem cum celeritate jactus 2M2G*;* atque in hac linea deferente 2M2N grave libere cadere motu eadem ratione accelerato, quam qui in propositione antecedente definitus est; quibus positis, linea deferens 2M2N ex hoc suo primo situ veniet in situm 2*r*2*l*, describendo in directione jactus spatium 2M2*r*, (§. 495) tempore, quod exponitur per $\log.(2\mathrm{M}2\mathrm{G}:2r2\mathrm{G}):2\mathrm{G}2u=2r2b:2\mathrm{G}2u$, atqui hoc eodem tempore (§. 505) grave in linea deference perlabitur spatium æquale 2*s*2*b* seu (constr.) 2*r*2*t.* Ergo simulatque linea deferens 2*r*2*l* percurrit spatium 2M2*r*, grave perlabitur in ea spatium $2r2t=2s2b$, atque adeo id semper durante ejusmodi motu mixto invenietur incedere in curva 2M2T2C. Quod erat demonstrandum.

COROLLARIUM I.

511. Tangens 2*g*2*t* per quodlibet curvæ punctum 2*t* ducta, exponet celeritatem mobilis in eodem curvæ puncto. Nam elementa rectæ 2M2r & curvæ 2M2*t* duabus rectis indefinite vicinis & parallelis interjecta, quarum 2*r*2*t* esset altera, eodem tempore describentur, atque adeo velocitates, quibus dicta elementa percurruntur, erunt ut hæc ipsa elementa; sed hæc elementa sunt lineis 2*r*2G & 2*t*2*g* proportionalia, ergo hæ rectæ etiam velocitatibus, quibus prædicta elementa percurruntur, proportionari debent. Unde, quia (§.495) 2*r*2G exponit celeritatem, qua percurritur elementum rectæ 2M2*r*, ideo tangens 2*t*2*g* exponet celeritatem mobilis in curvæ 2M2T2C puncto 2*t*.

COROLLARIUM II.

512. Recta 2E2G asymptota erit curvæ projectionis 2M2T2C. Nam si 2O2Q cadat super 2.G1D, fiet 2P2Q infinita, ergo etiam 2O2C ipsi 2P2Q æqualis infinita erit, ubi ciciderit super lineam 2G2E.

COROLLARIUM III.

513. Si 2*u*2D par fiat 2G2E, jungaturque 2M2D log-micam secans in 2Q, ductaque ex 2Q recta ex 2Q2O parallela 2D2G, & per 2O recta 2O2C æquidistanti 2G2E; recta 2M2C erit amplitudo curvæ projectionis 2M2T2C.

COROLLARIUM IV.

514. Sic etiam amplitudo curvæ projectionis, in quolibet plano ad horizontem inclinato, 2M2F invenietur, sumendo $2u2x = \mathrm{2F2G}$, & ducendo 2M2X log-micam secantem in puncto 2V. Nam, si ex hoc puncto agatur 2V2I parallela 2G2D, & 2I2H æquidistans rectæ 2E2G, intercepta 2M2H erit amplitudo quæsita in plano 2M2F ad horizontem inclinato. Erit enim $\mathrm{2X2}u : \mathrm{2V2K} = \mathrm{2G2F} : \mathrm{2I2H}$, unde cum (constr.) $\mathrm{2X2}u$ sit $= \mathrm{2G2F}$, erit etiam $\mathrm{2I2H} = \mathrm{2V2K}$.
Simili ferme ratione demonstratur præcedens corollarium.

COROLLARIUM V.

515. Quinimo datis amplitudine 2M2C atque celeritate jactus 2M2G, inveniri potest angulus 2G2M2E hisce datis conveniens. Nam si coordinatis 2M2O & 2O2Q, nominatis *x* & *y* respective, construatur curva 2Y2Q, cujus æquatio sit $y = \frac{cx}{a} + \sqrt{(xx - bb)}$, quam proinde liquet esse aliquam sectionem conicam, scilicet hyperbolam, in qua *a* significat 2M2G, dein *c* & *b* denotant log-micæ subtangentem datam 2G2*u*, & datam curvæ projectionis amplitudinem 2M2C. Communis intersectio curva 2Y2Q & log-micæ 2M2B2Q sit 2Q, per quam & per punctum 2M ducatur linea 2M2D rectæ 2G2D occurrens in puncto 2D, linea 2*u*D dabit sinum 2G2E anguli quæsiti 2G2M2E, existente 2G2M radio seu sinu toto.

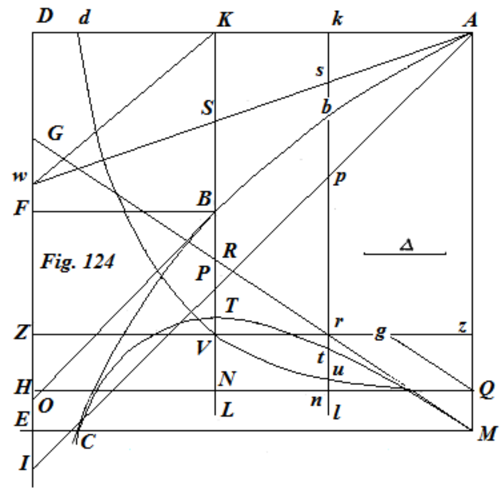
COROLLARIUM VI.

516. Nec amplius arduum erit determinatu, quinam angulus elevationis 2G2M2E conveniat maximæ omnium amplitudini 2M2C possibili. Nam, positis iisdem, quæ in coroll. præc. symbolis si hoc casu curvæ 2Y2Q æquatio fuerit $y=\frac{aax-axx}{cx-2ac}+\frac{cx}{w}$, quam proinde liquet esse aliquam Sectionem conicam, novæ curvæ hujus & log-micæ communis intersectio 2Q præbebit lineam 2*u*2D, quæ perpetuo æqualis est sinui anguli 2G2M2E quæsiti. Sed, si curva 2Y2Q log-micam nusquam intersecat, problema impossibile est, quod præsertim de corollorio antecedentc intelligendum, in quo sæpe contingere potest, ut *b*, seu amplitudo jactus, tanta assumatur, ut hyperbola inde resultans log-micam nusquam secare queat, atque adeo problema solutu impossibile sit, cum contra problema corollarii hujus sexti semper possibile existat. Nonnunquam etiam hyperbola corollarii V hujus log-micam in duobus punctus secare potest, quo fiet ut problema duas elevationes diversas 2G2M2E admittat. Horum duorum corollariorum fundamentum consisti in corollariis III. & lV. Calculum vero Lectoris industriæ relinquo.

COROLLARIUM VII.

517. Per log-micæ punctum 2*b* ducta sit tangens 2*b*2*p*, eritque primo $2t2p=2\mathrm{G}2g$. Nam elementum lineæ 2*s*2*b* est ad elem. lineæ 2M2*s* ut $2n2p-2s2b$ ad $2u2s$, & elem. linæ 2M2*s* ad elem. 2M2*r* ut 2*u*2*s* ad 2G2*r*, ergo ex æquo elem. lineæ 2*s*2*b* ad elem. lineæ 2M2*r* se habet sicut $2u2p-2s2b$ ad 2G2*r*; atqui elementum lineæ 2*s*2*b* vel (constr.) æqualis 2*r*2*t* est ad element. 2M2*r* sicut $2\mathrm{G}2g-2r2t$ vel $2s2b$ ad $2\mathrm{G}2r$, ergo $2u2p-2s2b:2\mathrm{G}2r$ se habet ut $2\mathrm{G}2g-2s2b:2\mathrm{G}2r$,atque adeo $2\mathrm{G}2g-2s2b=2u2p-2s2b$, hoc est, $2\mathrm{G}2g=2u2p$, & sic ubique. Secunda est 2*u*2*p* ubique æqualis respectivæ $2r2b$. Nam ducta per *2*b recta 2*b*2*q*, parallela rectæ 2M2G, erit 2*p*2*q* subtangens, atque adeo æqualis ipsi 2G2*u*, hinc ablata (vel addita subinde) communi 2*q*2*u*, remanebunt æquales, 2*u*2*p* & 2*G*2*q* vel 2*r*2*b*, ergo etiam, $2\mathrm{G}2g=2r2b$, unde $2\mathrm{G}2g-2r2s=2r2b-2s2b=2r2s$. Propterea est elementum 2*t*2*r* ad elem. rectæ 2M2*r*, sicut 2*r*2*s* ad 2*r*2*G*. Jam $2s2r:2r2\mathrm{M}=2\mathrm{G}2u:2\mathrm{G}2\mathrm{M}$, & $2r2\mathrm{M}:2r2\mathrm{G}=2r2\mathrm{M}:2r2\mathrm{G}$, ergo per rationum compositionem , & ex æquo $2s2r:2r2G=2r2\mathrm{M}.\ 2\mathrm{G}2u:2\mathrm{G}2\mathrm{M}.\ 2r2\mathrm{G}$. Erit ergo etiam elementum rectæ 2*r*2*t* ad element. rectæ $2\mathrm{M}2r=2r2\mathrm{M}.\ 2\mathrm{G}2u:2\mathrm{G}2\mathrm{M}.2r2\mathrm{G}$. Hinc, si lineæ nominentur, ut sequitur, scilicet 2M2G, *b;* subtangens 2*G*2*u, a;* indeterminatæ 2M2*r, y* ; 2*s*2*t, x,* harum elementa *dy* & *dx,* & analogia modo reperta element. $2r2t:\text{elem.}2\mathrm{M}2r=2r2\mathrm{M}.2\mathrm{G}2u{:}2\mathrm{G}2\mathrm{M}.2r2\mathrm{G}$ præbebit hanc alteram, in terminis analyticis, $dx:dy=ay:bb-by$ *;* adeoque æquatio differentialis curvæ erit $dx=aydy:bb-by.$ Quam Celeb. Varignon primus reperit in Act.Acad. Scient. Paris. 1708. ad diem l8. Julii, coroll. III ; Clariss. hic Autor etiam in citato hoc schediasmate constructionem tradidit problematis in præsenti propositione exhibiti simillimam illi, quam supra (§.510) adduximus, sed aliter quam a nobis factum, demonstratam. Et

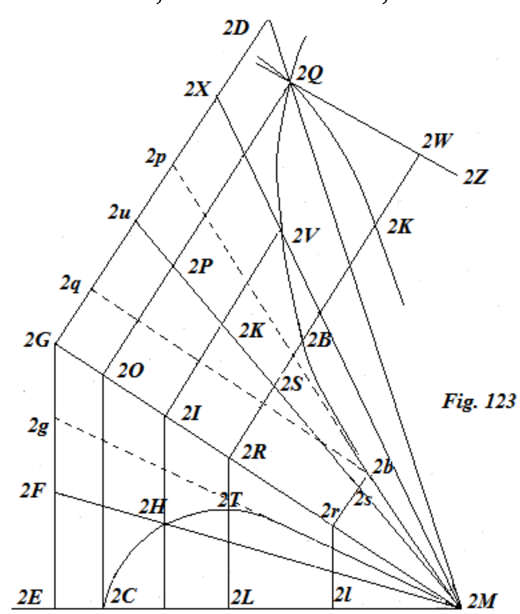
denique in schediasmate die 22. Augusti ejusdem anni 1708. coram societatæ prælecto, identitatem curvarum projectionis a summis geometris Newtono & Hugenio constructarum cum effectione a se adducta (quam ab ea, quam nos attulimus, non nisi in demonstratione differre jam diximos) analytico calculo eleganter commonstravit. Verum, quia eadem identitas curvarum, quas constructiones Newtoni atque Hugenii cum inter se, tum etiam a constructione Varignovii prima fronte non parum discrepantes præbent, absque calculo eleganter etiam demonstrari potest, eandem identitatem geometrica demonstratione hoc loco confirmare non gravabor. Ad demonstrationis facilitatem a constructione Hugeniana rem ordiar.



PROPOSITIO LXI. THEOREMA.

518. *Curva projectionis, cujus constructio in propositione præcedenti exhibita est, similis & æqualis est curvæ, quæ resultat ex constructione Hugeniana (vide* Discours de la Cause de la Pesanteur pag.171.) *in hypothesi resistentiarum mediis velocitatibus actualibus mobilis proportionalium.*

Hugenii constructio ita habet. Sint MG directio jactus, ME horizontalis, DE horizonti perpendicularis, circa quam velut axem descripta sit log-mica ABC subtangentem habens D*w*, vel FO, divisaque qualibet AD alteri EM parallela in K, ut AK sit ad KD, sicut celeritas jactus verticalis ex obliqua MG derivata, ad celeritatem terminalum, seu maximam exclusivc, quam scilicet grave nunquam attingere potest, etsi ei magis magisque semper cadendo accedat. Per K agatur KBL æquidistans DE log-micam in B, & horizontalem in L secans; ductisque porro per log-micæ puncta A, B tangentibus A*w*, BO, tum etiam AC tangenti BO parallela, quæ KL secet in P, & logarithmicam in C; si fiat in qualibet KL, rectus A*w*, AC, MG & ME secunte in punctis S, P, R & L, logarithmicam vero in B, ut RL ad TL ita SP ad BP, & $rl : tl = sp : bp$, erunt puncta T, *t*, &c. in curva optata MTC, quam dico eandem, esse cum curva 2M2T2C in propositione LX. §.510 constructa, si anguli GME, 2G2M2E, subtangentes logarithmicarum D*u* & 2G2*u* & denique celeritates jactus MG, & 2M2G, æquales fuerint.



Demonstr. Sint præterea M*r* & 2M2*r* æquales, & æquabuntur pariter *rl* ac 2*r*2*l*. Jam, quia MG exponit celeritatem jactus, EG denotabit velocitatem verticalem ex obliqua MG derivatam, & D*u* vel FO (§.498) exponit celeritatem terminalem, erit (constr.) $\mathrm{AK}:\mathrm{DK}=\mathrm{EG}:\mathrm{D}w$; &, quia si æqualibus D*w* & FO communis *wF* addatur, provenientes inde DF seu KB & *w*O æquales, atque adeo rectæ *w*K, HB & AI parallelae sunt, erit etiam $\mathrm{AK}:\mathrm{DK}=w\mathrm{I}:\mathrm{D}u=\mathrm{EG}:\mathrm{D}w$; & per consequens $w\mathrm{I}=\mathrm{EG}$, & sic ubique $sp=rl=2r2l$, cum M*r* & 2M2*r* angulique GME, 2G2M2E (secundum hypothesin) æquæntur; porro $\mathrm{AD}:k\mathrm{D}\ (=\mathrm{MG}:rG)=2\mathrm{M}2\mathrm{G}:2r2\mathrm{G}$, ergo (§.492) $kb:\mathrm{D}w=2r2b:2\mathrm{G}2u$, sunt enim *kb* log-mus rationis AD ad *k*D in log-mica ABC, & 2*r*2*b* log-mus rationis alteri æqualis, 2M2G ad 2*r*2G in log-mica 2M2B2Q. Vel, quia (secundum hypothesin) $\mathrm{D}w=2\mathrm{G}2u$, fiet $kb=2r2b$, & quia $2\mathrm{G}2u:2r2s(=2\mathrm{G}2\mathrm{M}:2\mathrm{r}2\mathrm{M}=\mathrm{GM}:r\mathrm{M}=\mathrm{DA}:k\mathrm{A})=Dw:ks$ atque $2\mathrm{G}2u=\mathrm{D}w$, erit $2r2s=ks\ \&\ sb=2s2b$; hinc etiam ex æqualibus *sp* & 2*r*2*l* ablatis æqualibus *sb* & *2s & 2b* vel 2*r*2*t*, remanet *bp= 2t2l;* sed constructio præbet $rl:tl=sp:bp$; seu quia $rl=sp$, sit etiam $tl=bp$, ergo pariter habebitur $tl=2t2l$, & sic ubique, ergo curva Hugeniana MTC eadem est cum altera 2M2T2C. Quod erat demonstrandum.

COROLLARIUM I.

519. Hinc liquet Hugenium suam constructionem nonnihil simpliciorem traditurum fuisse, si loco analogiæ $\mathrm{RL}:\mathrm{TL}=\mathrm{SP}:\mathrm{BP}$ simpliciter jussiffet sumere ubique applicatam TL æqualem homologæ BP interceptæ inter log-micam ABC & ejus subtensam AC.

COROLLARIUM II.

520. Parameter parabolæ, quam missile juxta directionem MG & celeritate hac recta expressa in vacuo describeret, foret $2.\mathrm{EM}^2:\mathrm{D}w$. Nam, quia celeritas verticalis ex obliqua derivata est EG, & gravitas uniformis exponitur per subtangentem log-micæ D*u*, maxima altitudo, ad quam grave celeritate initiali EG pervenire potest, erit (§.150) $\mathrm{EG}^2:2\cdot\mathrm{D}w$. Sed GE est ad EM, ut dupla parabolæ altitudo ad semissem amplitudinis, ac propterea hæc dimidia amplitudo est $\mathrm{EM}.\ \mathrm{EG}:\mathrm{D}w$. Atqui quadratum dimidiæ amplitudinis parabolæ applicatum ad ejus altitudinem præbet parametrum, ergo hic parameter est $2\cdot\mathrm{EM}^2:\mathrm{D}w$. Quod Hugenianæ & Varignonianæ determinationibus, utut aliis terminis expressis, consonum est.

PROPOSITIO LXII. THEOREMI.

521. *Eadem adhuc resistentiæ hypothesi posita, curva projectionis, quæ ex constructione a Cel. Newtono* (Princ. Phil. Nat. Math. Lib. II. Prop. IV.) *exhibita nascitur, eadem est cum Hugeniana, de qua in propositione proxime antecedenti egimus, aut cum curva propositionis nostræ sexagesimæ primæ.*

Ne Lectori Newtoni constructionem alibi quærere opus sit, eam, sed aliis quam apud Autorem litteris signatam, huc afferre libet.
Eadem vero ita habet:

Assumpta MG pro celeritate & directione jactus, ductisque per terminos ejus horizontali ME, ac verticali ED, in qua, ejus segmentum EG celeritatem verticalem ex obliqua MG derivatam exponit; inter asymptotas DEM hyperbola quæcunque QV*d* descripta intelligatur rectam AM in Q secans, per quod punctum agatur insuper QH æquidistans horizontali ME, divisaque hac ME in L, ut ML sit ad LE sicut resistantia medii renitens motui in altum initio projectionis, seu velocitas verticalis EG ad gravitatem, seu celeritatem terminalem, antea expositam per subtangentem FO log-micæ ABC, ducatur LK parallela AM, hyperbolam in V secans, quæ log-micræ occurret in B & AD in K, quandoquidem in constructione Hugeniana.(§. 518) AK etiam est ad KD sicut EG ad FO vel D*w*. Per hyperbolæ punctum V agatur *Zz* parallela EM, sumaturque in ea portio *gz*, quam Newtonus per litteram N designat, quæ sit ad *z*Q vel VN*r*, sicut ME ad EG, aut simplicius, ducatur Q*g* æquidistans ipsi MG. Quibus positis, si in qualibet *rl* parallela KL capiatur *rt* æqualis trilineo hyperbolico Q*un* applicato ad datam *gz*, punctum *t*. erit in curva, quæ eadem prorsus est cum ea, quam constructiones propositionum antecedentium LX & LXI, præbent.

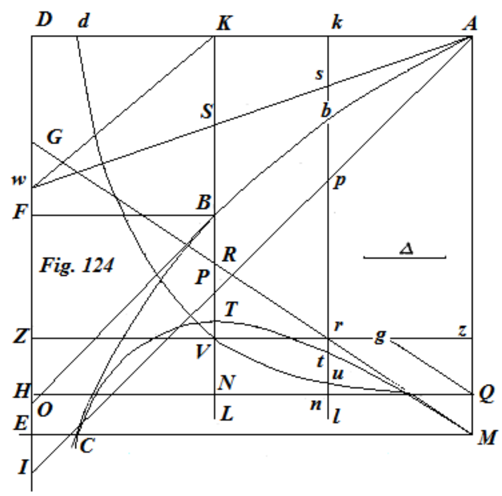


Fig. 124

Demonstr. I. Hyperbola QV*d* præbet, $\mathrm{EM}:\mathrm{EL}=\mathrm{VL}:\mathrm{QM}$ & dividendo $\mathrm{LM}:\mathrm{EL}=z\mathrm{Q};\mathrm{QM}$, & (constr.) $\mathrm{LM}:\mathrm{EL}=\mathrm{EG}:\mathrm{D}u=rl:ks$, ergo etiam $z\mathrm{Q}:\mathrm{QM}=rl:ks$ tum etiam (constr.) $gz:z\mathrm{Q}\left(=\mathrm{ME}:\mathrm{EG}\right)=\mathrm{M}l:rl$, ergo ex æquo $gz:\mathrm{QM}=\mathrm{M}l:ks$; atque adeo $\mathrm{M}l.\mathrm{QM}=gz.ks$.

II. Resumatur analogia $z\mathrm{Q}:\mathrm{QM}=\mathrm{GE}:\mathrm{D}w$, & $gz:z\mathrm{Q}=\mathrm{EM}:\mathrm{GE}$, ergo ex æquo $gz:\mathrm{QM}=\mathrm{EM}:\mathrm{D}w$. Atque adeo $\mathrm{EM.QM}=\mathrm{D}w.gz$.

III. Constructione est trilineam hyperbolicum $Qnu=gz.rt$, additisque $\mathrm{M}l.\mathrm{QM}(\text{num.1.hujus})=gh.ks$, fiet $\mathrm{Q}ul\mathrm{M}=rt.gz+ks.gz$. Atqui (§.368) $\mathrm{QM}lu:\mathrm{EM.MQ}=kb:\mathrm{D}u$ & $\left(\text{num. II.hujus}\right)\mathrm{EM.QM}=\mathrm{D}w.gz$; ergo $rt.gz+ks.gz:\mathrm{D}u.gz=rt+ks:\mathrm{D}u=kb:\mathrm{D}u$; atque adeo $rt+ks=kb=sb+ks$ *:* hinc tandem $rt=sh$, ut habent constructiones propositionum LX & LXI; propterea tres diversæ constructiones propositionum trium postremarum unam eandemque curvam suppeditant. Quod erat demonstrandum.