CHAPTER XIII.

Concerned with the shapes, which flexible surfaces must adopt, when they receive the flow of the wind directly, or the curvature of sails.

Before the celebrated Bernoulli's, no one had applied themselves to discover what curvature a sail must adopt, swollen by the wind ; but these excellent geometers have found how a sail must be bent into that curve itself by the moving wind, as John Bernoulli, Leibniz and Huygens have shown to be in agreement for a loose flexible rope or chain hanging from both its ends, even if nowhere has the [detailed] analysis of the sail been deemed worthy of publication. But since catenaries and sails shall be one and the same curve, no one has yet considered, whether the investigation of the one likewise includes the other, thus so that [in this circumstance], whoever has found the catenary, shall be agreed to have given the solution of the sail also, which the most distinguished David Gregory is seen to suggest, thus writing in Cor. *7.* after Prop. 2. in his *Catenaria* :

Because if in place of gravity any other similar force acting on flexible line may exert its own forces, and will produce the same line; e.g. if the wind is considered uniform, and blowing along given parallel right lines, a line blown by the wind will be the same as for the catenary. [Assuming a weightless line secured at the ends.]

Indeed must I acknowledge in this thesis, as they say, the assertion of the most distinguished man to be true, for anyone to question whether or not some other force substituted in place of gravity and clearly acting in the same manner as gravity, will not be effecting the same, and thus shall not be producing the same curve as gravity? But the theory applied to the matter of sails is defective, since notable disparities intercede in the interaction of gravity on catenaries and the action of the wind on sails. For equal particles of the catenary are trying to descent along vertical directions with equal forces, whereas in the sail its particles, even if equal, still also undergo unequal forces from the filaments of the air with the same velocity striking these, and indeed not along the vertical direction, but perpendicular to the elements of the curve. Therefore since the circumstances, by which gravity acts on a catenary and the wind on a sail, will be different in the heavens, anyone can see, what has been demonstrated about catenaries, may no longer be able to be applied to the quadrature of the sail, even if the analysis of the linen curve may agree with the investigation for an elastic curve, with the observation by the most illus. James Bernoulli that the elastic curve shall be the same as for the linen curve. Therefore because the solution of the catenary problem does not involve the solution of the sail, and because the solution of this problem, or by requiring to say the correct analysis, nowhere has been demonstrated in a publication at this time, as far as I know, besides what has been shown in §.103 above, where we have elucidated the solution illustrating the most general solution of our theorem for the problem of the sail, where the physical part of the problem has been left to be performed now ; indeed by referring to the problem we have treated, the geometrical analysis of which is going to be treated according to that understanding, which is pleasing, with the premises from the first two lemmas still going to be of use in the other propositions.

PROPOSITION LI. LEMMA.

463. *With centre* O *and semi-lateral transverse distance OA, the quadrant of the circle* AGK *and the equilateral hyperbola* ABM *shall be described, as well also as the logarithmic curve* AC *about the asymptote* LK, *of which the subtangent* OL *shall be equal to the radius of the quadrant* OA, *which the logarithmic curve may pass through at the point* A, *and with the right line* BO *drawn from some point* B *of the hyperbola to the centre* O, *intersecting at the point* F *the tangent AP of the hyperbola at the vertex, and through this point F the right line* FH *is drawn parallel to the radius AO, intersecting the quadrant at* G, *and* OK *at* H. *If* KG *shall be the right line joining the points* K & G, *it may be produced until it coincides with the radius OA, likewise extended, at the point D, and through this point the right line* DC *were drawn parallel to* LK. *The rectangle under this line* DC *and the radius* OA *will be equal everywhere to twice the area of the corresponding sector of the hyperbola* ABO.

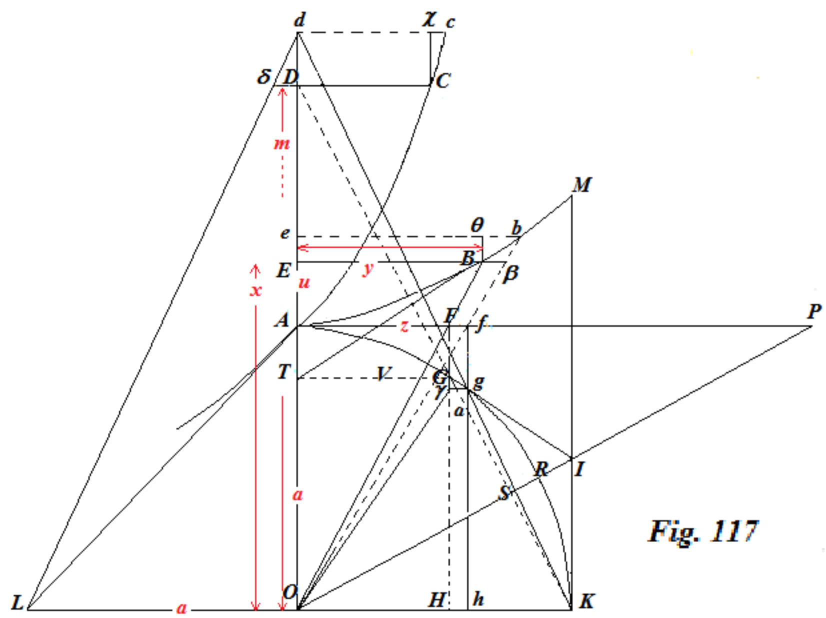


Fig. 117

From another point b taken on the hyperbola, indefinitely close to the other B, bO may be drawn cutting AP at f, likewise fh may be drawn, parallel to FH, intersecting the circle at g, the right line KG at a, & OK at the point h ; and Kg is joined and produced to d, through which point the other line dc shall be drawn parallel to the first DC; dL and OG are acting equally, likewise Cχ is parallel to Od, and γg is parallel to OK, the ordinate EB of the hyperbola may be continued as far as to β, and CD as far as to δ ; with which done the tangents GI and KI shall be drawn through the point G and K of the quadrant, which will be equal; with which put in place :

I. It is apparent dL to be parallel to the tangent of the logarithmic curve at the point C, since the subtangent of this logarithmic curve is equal to OL (following the hypothesis) ; and thus $d\delta$ will be parallel and equal to the increment of the logarithmic curve Cc, and also $c\chi = D\delta$.

II. The similar triangles OEB, OAF gives rise to $\mathrm{OE}^2 : \mathrm{EB}^2 = \mathrm{OA}^2 : \mathrm{AF}^2$. Or, because on the hyperbola the square OE is equal to the sum of the squares of OA and EB [*i.e.* $x^2 - y^2 = a^2$ for the equilateral hyperbola, and $x^2 + y^2 = a^2$ for the circle; see the Scholium below], also $\mathrm{OA}^2 + \mathrm{EB}^2 : \mathrm{EB}^2 = \mathrm{OG}^2 : \mathrm{OH}^2$ [As $\mathrm{OA} = \mathrm{OG} = a$] ; and on dividing, $\mathrm{OA}^2 : \mathrm{EB}^2 = \mathrm{GH}^2 : \mathrm{OH}^2$ or AF^2, or by inverting and interchanging

$EB^2 : AF^2 = OG^2 : GH^2$. And $EB^2 : AF^2 = OB^2 : OF^2 =$ 2.triang. $BO\beta$: 2.triang. FOf, that is, 2.triang. $BO\beta$: rect. $FH.Hh$, therefore 2.triangle $BO\beta : FH.Hh = OG^2 : GH^2$.

[Thus, the increment of the area of the hyperbolic sector is related to the corresponding increment of the area of the circle.]

III. The similar triangles *Gga* and GIK show $Gg : ga = GI : IK$, [*i.e.* G*g* is an increment of the tangent GI; thus *a* lies on the circle, but impossible to show in a diagram where all the increments are finite] thence, because the tangents IK and Gl are equal, *Gg* and *ga* are equal also.

IV. On account of the parallels O*d* and *hg,* [*i.e.* the triangles HGK and ODK are similar] there will be $Od : Dd = hg : ag$ or $HG : Gg$ (by no. III.), and by inverting and interchanging $Dd : Gg = Od : GH$, but $D\delta : Dd = OL$ or $OG : Od$, therefore from the equation $D\delta : Gg = OG : GH$, and on account of the similar triangles OGH and $Gg\gamma$, there shall be $Gg : \gamma g$ or $Ff = OG : GH$; therefore from the equation and by adding the ratios,

$$D\delta : Ff = OA.D\delta : FH.\ Ff = OG^2 : GH^2 \ (\text{no. II.})$$
$$= 2.\text{triangle } BO\beta : FH.Ff \text{ or } FH.Hh.$$

Therefore 2.triang. $AO\beta = OA.D\delta\,(\text{no. I.}) = OA : c\chi.$

therefore all the 2. BOβ that is twice the sector $BAO =$ all $c\chi$ by AO, that is to the rectangle DC.AO. Q.E.D.

Otherwise.

464. The tangent BT of the hyperbola at the point B shall be produced, and through the point T the right line TV parallel to OK crossing the line OB at the point V, and finally Bθ is acting parallel to AD.

I. I am going to show that TV produced will pass through the point G, thus so that $TO = GH$. For in the hyperbola the lines GH, OA and OF are in continued proportion, and on that account the tangent BT, also OT, OA and OE are in continued proportion, therefore $OT = GH$.

II. The element of the square G*g* will be equal to the increment of the line Bβ. For $Gg : \gamma g$ or $Ff = OG : GH = AO : TO = EO : AO$, and also $B\beta : Ff = EO : AO$, therefore $Gg : Ff = B\beta : Ff$, and thus $Gg = B\beta$.

III. The similar figures $b\theta B\beta$ and BETV, as well as those placed around the same right line *b*T, are composed from the similar triangle $b\theta$B, BET, and $bB\beta$, BTV, contain the ratio $b\theta : B\theta$ or (no. II. of this), *Gg* or (no. III. §.464.)
$ga = EB : TV = EO : TO$ and
$ga : Dd = GH.$(no. I. of this) $OT : Od$, & $Dd : D\delta = Od : UL$ or OA, therefore from the equation there is obtained $b\theta : D\delta = EO : AO = AO : TO$; and thus
$b\theta.TO = AO.D\delta = AO.c\chi$.

IV. Indeed twice the triangle $bTO = be.TO$, and twice the triangle

$\text{BTO} = \text{BE.TO}$, therefore twice the triangle $\text{BO}b = b\theta.\text{TO}$ (num. III of this) $= \text{AO}.c\chi$, and consequently also twice the area BAO will be equal to the rectangle under AO.DC. Q.E.D.

COROLLARY I.

465. With the right lines CA and CO drawn from the logarithmic curve at the point C to the points A and O, the rectilinear triangle CAO will be equal everywhere to the hyperbolic sector BAO.

COROLLARY II.

466. OI may be connected and produced to P, and there will be $\text{AP} = \text{OD}$, and $\text{OP} = \text{KD}$. For, because OS is normal to KD, the angles SOK, or of its alternate APO, and ODK will be equal, and thus the right angled triangle AOP and OKD are similar, indeed on account of the equality of the sides AO and OK the triangles will be equal, and therefore $\text{AP} = \text{OD}$ and $\text{OP} = \text{KD}$. But AP is the tangent of the arc AR composed from the arc AG, of which the sine OH is equal to the tangent AF of the angle of the sector AOB, and from the half of the complement of the same GR. And DC is the logarithm of the ratio DO to AO, that is, the logarithm of the ratio of the tangent AP of the aforesaid arc, or of the angle AOR composed from the angles AOG and GOR, to the radius of the whole sine AO. And thus the ratio AP to AO = ratio OK to IK, that is, to the ratio, which the OK radius has to the tangent IK of half the complement of the angle AOG, of which the sine is equal to the tangent of the angle of the sector AOB.

SCHOLIUM.

467. This proposition leads for the most part to the reduction of the elements of hyperbolic sections to the more simple elements of the logarithmic curve. For if

$$\text{AO} = a;\ \text{OD} = m;\ \text{OE} = x;\ \text{EB} = y;\ \text{AF} = z\ \&\ \text{AE} = u.$$

And with these in place, there will be

I.

$y = \sqrt{(xx - aa)}, \text{AF} = [ay : x] = a\sqrt{(xx - aa : xx)}\ [= a\sqrt{\left(1 - \frac{a^2}{x^2}\right)}]$; and thus $\text{F}f = a^3dx : xx\sqrt{(xx - aa)}$,

$[i.e.\ dy = \frac{1}{2} \cdot \frac{a}{\sqrt{\left(1 - \frac{a^2}{x^2}\right)}} \cdot 2\frac{a^2}{x^3} \cdot dx]$ and $\text{FH.H}h = a^4dx :\ xx\sqrt{(xx - aa)}$.

From which because triangle $\text{BO}\beta$: triangle $\text{FO}f = \text{OE}^2 : \text{OA}^2$, twice the triangle $\text{BO}\beta$ will be found to be $= aadx : \sqrt{(xx - aa)}$. This likewise can be found more briefly with the aid of no. IV. §.466. as we have shown $2.\text{BO}b = \text{OT}.b\theta$; so that, therefore,

$\text{OT}.b\theta = aaxdx : x\sqrt{(xx-aa)} = aadx : \sqrt{(xx-aa)}$. Now $\text{KH} = \left(ax - a\sqrt{(xx-aa)}\right) : x$, and GH or $\text{OT} = aa : x$; and the triangles KHG and KOD are similar, therefore $m = aa : x - \sqrt{(xx-aa)}$ & $x = (aa+mm) : 2m$, which value substituted into the preceding expression of the element BOβ, produces $2.\text{BO}b = aadm : m$.

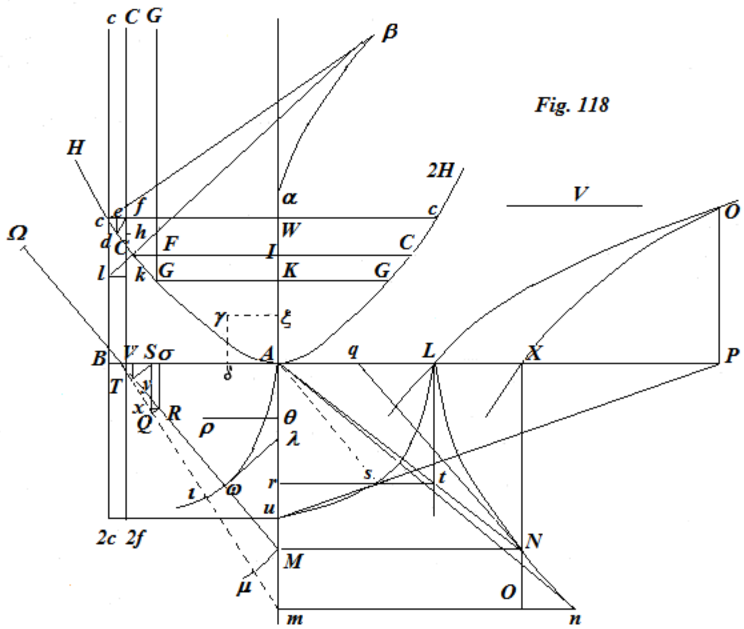
II. Because $x = a+u$ and $dx = du$, there will be $2\text{BO}b = aadx : \sqrt{(xx-aa)} = aadu : \sqrt{(2au+uu)}$, and $u + a = (aa+mm) : 2m$, or $u = (m-a)^2 : 2m$, which value substituted into the formula $aadu : \sqrt{(2au+uu)}$, again will give $2\text{BO}b = aadm : m$.

III. On account of the similar triangles OAF & OEB there will be found $ay = xz$, truly the hyperbola will give $y = \sqrt{(xx-aa)}$; from which $x = aa : \sqrt{(aa-zz)}$, truly from the similar triangles KHG & KOD there may be elicited $am - mz = a\sqrt{(aa-zz)}$ and $z = amm - a^3 : mm + aa$, which value substituted of z and of its element $a^3dz : aa - zz = 2.\text{BO}b$, even now will give $2.\text{BO}b = aadm : m$.

The most celebrated John Bernoulli, some time ago now in Com. Acad. Reg. Scient. Paris. 1702. d. 13 December. and in Act. Lips. 1703. pag. 30 & 31, elicited similar rules of reducing hyperbolic sectors to logarithms from his own general method of reducing the quadratures of curves, the ordinates of which were effected variously through the abscissa and given by the rational fractions shown. Furthermore, even if our preceding theorem applies only to the equilateral hyperbola, yet with a few changes it can be extended to any hyperbolas.

PROPOSITION LII. LEMMA.

468. *If another right line* AB *with a given magnitude may be situated perpendicularly to the given right line* AM *, through the end of which* B *some right lines* BM*,* B*m may be drawn to the other* AM*, and through the points* M, *m*, &c. *at which they meet this line, the perpendiculars* MN, *mn*, &c. *are equal respectively to these* BM , B*m*, &c.*, all the points* N, *n*, &c. *will be situated on the equilateral hyperbola curve* LN*n , of which the centre is* A, *and the transverse half line is* AL.



And with the points M, *m put in place indefinitely close together, and with the interval* BM *described with centre* B *with the arclet Mμ, and with* AN, A*n*, *joined, the rectangle under the radius* BA *and the arclet Mμ will be twice the trilinear figure* AN*n, or of the element of the hyperbolic sector* ANL. Fig. 118.

I. NX may be drawn parallel AM, and because (following the hypothesis) $\mathrm{MN} = \mathrm{BM}$, & $\mathrm{AB} = \mathrm{AL}$, there will be
$\mathrm{AX}^2 = \mathrm{XN}^2 + \mathrm{AL}^2$, or $\mathrm{XN}^2 = \mathrm{AX}^2 - \mathrm{AN}^2 = \text{rect. BXL}$, therefore the point N is on the equilateral hyperbola LN, of which the transverse line is BL and the centre A. The curve LN also will be a hyperbola, if MN to BM were in some given ratio.
II. N*q* shall be the tangent to the hyperbola drawn through the point N, and the three MN or BM, AL or BA, and A*q* shall be in continued proportion, and thus
$\mathrm{BM} : \mathrm{BA} = \mathrm{BM} : \mathrm{A}q$, and on account of the similar triangles ABM and μM*m*, there is also $\mathrm{BM} : \mathrm{BA} = \mathrm{M}m$ or $\mathrm{N}o : \mathrm{M}\mu$; therefore , $\mathrm{N}o : \mathrm{M}\mu = \mathrm{BA} : \mathrm{A}q$ and thus
$\mathrm{BA}.\mathrm{M}\mu = \mathrm{N}o.\mathrm{A}q$ (no. IV of §.464) $= 2.\mathrm{AN}n$.
Q.E.D.

COROLLARY I.

469. And thus, if the whole AM shall be composed of infinitely small parts such as M*m,* and they may be understood to be drawn through the individual ends of the right line BM, it will be made from all the arclets Mμ, which intercept BM in turn, and with the radius BA will be equal to the double of all the triangles AN*n*, which are contained in the hyperbolic sector ANL, that is, in twice that sector.

COROLLARY II.

470. Therefore by drawing the line *tsr* parallel to AL, and *us*P through the point of intersection *t* of the right line AN and the tangent to the hyperbola A*t* at the vertex, which itself produced cuts AL at the point P; and finally the logarithm LO through the point L, which shall have a subtangent equal to the radius AL, and PO through the point P is acting parallel to the logarithmic curve AM meeting at the point O ; all the arclets Mμ taken together will be equal to the right line PO. For since (§.469.) all the Mμ in $\mathrm{AB} = 2.\mathrm{are}a$ ANL and (§. 463.) 2.sect. $\mathrm{ANL} = \mathrm{AB}$ or AL by PO, all the $M\,\mu.\mathrm{AB} = \mathrm{AB.\ PO}$, and thus the sum of all $\mathrm{M}m = \mathrm{PO}$.

PROPOSITION LIII. PROBLEM.

471. *If a perfectly flexible filament* HA2H *with both its ends* H, 2H *fixed shall be exposed to the wind blowing along the direction cc*, CC, GG *parallel to the right line* AW*, the curve which the filament must adopt to be assigned.* Fig.118.

Geometrical Analysis. I. Because the wind exerts forces on the sail (§.249) along directions C*l* normal to the direction of the curve sought ACH , and thus (§.96.) it is agreed the retaining force of the sail at its individual points is required to be the same ; the constant tenacity [*i.e.* stress or tension] of such can be expressed by the given right line AB normal to the axis AW , if the velocity of the wind shall be 1, if indeed the velocity may be expressed by the right line V, (§.100.) the tenacity will be required to be represented by this factor $\mathrm{V}^2.\mathrm{AB}$, as we will see soon. For with equal and contiguous elements of the curve taken c*C* and CG, through the ends of these the ordinates *c*W, CI and GK are understood to be drawn to the axis of the curve AW, which the right lines CC and GG will cut at *f* and F*;* and with the perpendicular sent from the point *f* to the element of the curve *c*C*,* which shall be *fd,* through the point *d, de* is acting parallel to the axis AW, and finally let $\mathrm{C}h = \mathrm{GF}$; thus so that *fh* shall be the difference between *fC* and *Fg;* and with these in place the force, which the wide filament of air *cc*CC strikes the element of the curve C*c* with the speed V (§.429.) is expressed by the factor $\mathrm{V}^2.\mathrm{CD}$*;* and its direction C*l,* as now has been said to be perpendicular everywhere to the curve; on that account, and just as these which have been mentioned elsewhere (§.100.), the force of the filament must be expressed by the magnitude $\mathrm{V}^2.\mathrm{AB}$ of the same kind as $\mathrm{V}^2.cd$, or by the force that the element C*c* receives along C*l*, or along some other direction normal to it. Now C*l* may express the magnitude $V^2.cd$ or the force applied normally to the curve at the point C , and with C*k* and *lk* drawn parallel and normal to the axis AW, and the force *Ck* will be $V^2.ce$; for in the similar triangles C*lk* and *cde,* the line increment *ce* is homologous to the other C*k*, since there shall be $Cl : Ck = cd : ce$; and $Cl = V^2.cd$, therefore also $Ck = V^2.ce$. Now, as it may be shown elsewhere (§.93), C*k* is to *fh* as the force of the filament on the element C*c,* to that element C*c,* or by interchanging *Ck* for the given tenacity, as *fh* to *Cc;* there will be $V^2.ce : V^2.AB = ce : AB = fh : Cc$, and by interchanging $ce : fh = AB : Cc$.

II. BM and B*m* may be drawn through the end B of the given BA parallel to the elements of the curves CG, C*c,* and with centre B with the intervals $BQ = CG = Cc$ with the arclet QR described*,* QS*,* Rσ are acting parallel to AM*,* likewise R*x* is parallel to BA*;* ST is perpendicular to BQ, and finally TV is parallel to SQ, with which in place, it is further apparent, triangle BQS is similar and equal to the triangle *cCf,* and BRσ similar and equal to the triangle CGF, to be put in place similarly ; thus so that there shall be $xQ = fh,\ BV = ce,$ and $Cc = BQ$; and thus the ratio from the preceding numbers $ce : fh = AB : Cc$, is the same as $BV : xQ = AB : BQ$; indeed in the similar right-angled triangles *y*QR and BTS, the bases *y*Q and BS may be divided proportionally by the perpendiculars R*x* and TV, thus so that there shall be $xQ : yQ = BV : BS$, and by interchanging and inverting $BV : xQ = BS : yQ = BA : Mm$; truly before we had $BV : xQ = AB : BQ$*;* therefore also $AB : Mm = AB : BQ$, and thus $Mm = BQ = Cc$*;* which from the individual M*m,* and which in total A*m* from the individual C*c,* which all together contain the curve *c*GA, may be able respectively it is clear, to be equal either to all the *Mm*, or the whole A*m* or AM with all the C*c*, or to the whole *c*GA or CGA . From which, because BM is parallel to the tangent of the sail at the point C (following the hypothesis), BA to AM will be given or of the equivalent curve itself AGC, just as the ordinate CI to the respective subtangent of the point C of the sail. This property agrees with the catenary also, as now the identical nature of the sail and catenary has been shown (§.106.).

III. Moreover the construction of the curve at this point can be elucidated easily from what has been shown : for since M*m* shall be $= Cc$, and B*m* (following the hypothesis) parallel to C*c,* with centre B from the arclet Mμ, the triangle M$m\mu$ will be similar and equal to the triangle *cCf*, and hence $Cf = m\mu$*,* and all the C*f*, that is WA = omnibus $m\mu$ or *mi,* clearly with the circle Aω described with the centre B, and with the radius BA. Thus also the AI present will be equal to the Mω in place, as before, BM parallel to the tangent of the curve at C:

Indeed the element of the ordinate $cf = M\mu$ and all the *cf*, either *c*W or CI = sum of all the Mμ (§.470) = PO from the construction in place indicated in this paragraph 470. Therefore any ordinate CI is equal to the homologous PO. All of which were required to be shown.

COROLLARY I.

472. Because the three lines AX, AL and Aq are in continued proportion in the equilateral hyperbola LN, on account of the tangent Nq of the hyperbola , and from the nature of the hyperbola also the three lines AX, AL & *rs,* thus so that Aq and *rs* shall be equal, there will be, with As drawn, the similar triangles ABM and Asr ; for $\mathrm{AX} = \mathrm{MN}\ (\text{constr.}) = \mathrm{BM}$ is to AL or AB, as AL or As is to Aq or *rs,* and thus the triangles ABM and *rsA* are similar, and as a consequence As is parallel to BM or also to the tangent of the sail at C , and the angle uAs will be equal to that, which the tangent of the sail just indicated, and needs to be produced as far, that it may be constituted with the axis AW , or which half is the angle *rsu;* from which since PO shall be the logarithm of the ratio PA to LA, or of PA to Au, that is, of the ratio *rs* to *ru;* it follows that PO is the logarithm of the ratio, which the radius has to the tangent of the angle *rsu* or to half the angle uAs*,* in accordance with the logarithm of which the subtangent is equal to the radius AL ; and in any other logarithms, if the fourth proportion may be assumed proportional to the logarithm of the subtangent AL, and to the aforesaid ratio of the radius to the tangent of half the angle uAs*,* it will be evident that the ordinate of the sail CI at the point C, evidently if the angle uAs were equal to the angle, which the tangent to the sail at C produced makes with the axis AW.

COROLLARY II.

473. The circle with centre B and described with radius BA cuts the right line BM produced at Ω, and therefore the ratio of the tangent to the circle AM and the secant MΩ, $\mathrm{M\Omega} : \mathrm{MA} = \mathrm{MA} : \mathrm{M}\omega$, or $2\mathrm{AB} + \mathrm{M}\omega : \mathrm{AM} = \mathrm{AM} : \mathrm{M}\omega$, and thus $2\mathrm{AB} + \mathrm{M}\omega : \mathrm{M}\alpha = \mathrm{AM}^3 : \mathrm{M}\alpha^2$, and on dividing $2\mathrm{AB} : \mathrm{M}\alpha = 2.\mathrm{AB.M}\alpha : \mathrm{M}\omega^2 = \mathrm{AM}^2 - \mathrm{M}\alpha^2 : \mathrm{M}\omega^2$, and thus $2\mathrm{AB.M}\alpha = \mathrm{AM}^2 - \mathrm{M}\omega^2$. Therefore the parameter of the sail AB or Au will become known from the given AM or AC and Mω or AI.

Again, because (§. 472.)

$$\mathrm{A}r : r\mathrm{s} = \mathrm{AM} : \mathrm{AB} = 2\mathrm{AM.\,M}\omega : 2\mathrm{AB.M}\omega = 2\mathrm{AM.M}\omega : \mathrm{AM}^2 - \mathrm{M}\omega^2,$$

there will be As or $\mathrm{A}u : \mathrm{A}r = \mathrm{AM}^2 + \mathrm{M}\omega^2 : 2\mathrm{AM.M}\omega$,

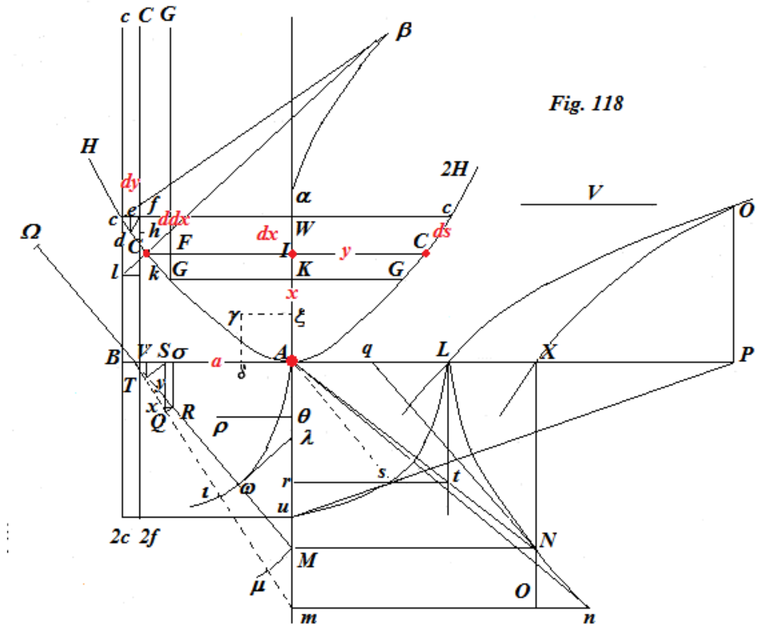
and on dividing $\mathrm{A}u - \mathrm{A}r$ or ru to $\mathrm{A}r = (\mathrm{AM} - \mathrm{M}\omega)^2 : 2\mathrm{AM.M}\omega$,

and $\mathrm{A}r : rs = 2\mathrm{AM.M}\omega : \mathrm{AM}^2 - \mathrm{M}\omega^2$, therefore from the equation $ru : rs = (\mathrm{AM} - \mathrm{M}\omega)^2 : \mathrm{AM}^2 - \mathrm{M}\omega^2 = \mathrm{AM} - \mathrm{M}\omega : \mathrm{AM} + \mathrm{M}\omega =$ the curve AC less the abscissa AI*,* to $\mathrm{AC} + \mathrm{AI}$. From which , since there shall be $\mathrm{PA} : \mathrm{LA} = rs : ru = \mathrm{AC} + \mathrm{AI} : \mathrm{AC} - \mathrm{AI}$, and PO the logarithm of the ratio PA to LA; also there will be the logarithm PO of the ratio $\mathrm{AC} + \mathrm{AI}$ to $\mathrm{AC} - \mathrm{AI}$, of which the ratio of the ends, evidently the sum and difference of the curve AC and of its abscissa or sagitta AI, shall be given. Therefore, if it may arise that 2Mω or 2AI to $\mathrm{AC} + \mathrm{AI}$ thus as $\mathrm{AC} - \mathrm{AI}$ to the quarter Q, then so that the subtangent of some logarithm to the logarithm of the ratio

$AC + AI$ to $AC - AI$ in this logarithm, thus the magnitude or the line Q to another R, this line, following the preceding corollary, will give the ordinate CI of our sail AC.

COROLLARY III.

474. Following §.110. the mean direction of the force of the wind flowing on the filament CAC will be axis AW, for this mean direction, the angle from the tangents of the sail drawn through the points C, C to each part of the axis shall be held in place similarly, is divided in two, as the axis AW is held also; and thus the force of the wind (§.110.) on the sail, along its mean direction WA, to the tenacity [*i.e.* stress, if we regard the force of the wind to be the pressure or force per unit area] of the sail at all of its points shall be the same, just as twice A*r* to A*s,* that is, as we had above (§.473) , just as $4AM.M\omega$ to $AM^2 + M\omega^2$, that is*,* just as $4.AC.AI$ to $AC^2 + AI^2$. From which, because the tenacity of the sail (no. I. §.471.) is $V^2.AB = (AC^2 - AI^2 : 2AI).V^2$, the force of the wind on the sail along its mean direction will be : $WA = V^2.2AC.\left(AC^2 - AI^2\right) : AC^2 + AI^2$.



COROLLARY IV.

475. Truly the differential equation of the sail will be most easily elicited from the construction brought forth; for if there may be called BA, a ; AI, x; CI, y ; Cf, dx; cf, dy ; $Cc = CG = ds$; $fh = ddx$, there will be $ce = dy^3 : ds^2$, and the ratio, in which above (no. I, §.471.) we have come upon ce or $dy^3 : ds^2$ to fh or ddx, as AB or a, to Cc or ds; which, with the extremes and means multiplied, will give the differentio-differential equation of the sail $dy^3 = adsddx$, which the celebrated Bernoulli's also came upon, as may be seen in the Act. Erudit. Lips. 1695. pag. 546. [Here Hermann has followed Bernoulli's exposition very closely.]

The differential equation of the sail may be elicited from the similarity of the triangles ABM and M$m\mu$; for there is $m\mu : M\mu = AM : AB$; thence because $AB = a$, $M\omega = AI = x$, and thus $AM = \sqrt{(2ax + xx)}, m\mu = dx$ and $M\mu = CF = dy$; there will be

$dy = adx : \sqrt{(2ax + xx)}$ a differential equation of the first order, expressing the nature of the sail.

But if indeed BM or uI were $= x$, then there will be

$AM = \sqrt{(xx - aa)}$, and $dy = adx : \sqrt{(xx - aa)}$, which is another differential equation of the sail. And these are the equations, which the praiseworthy Bernoullii's now first came upon, and with these derived by their own special method.

SCHOLIUM.

476. Now at once special and the most remarkable properties of sails arise from our construction and analysis. For :

1°. The curve AC is equal everywhere to the respective ordinate XN of the equilateral hyperbola LN, since this ordinate shall be equal to MA, to which the curve AC has been shown to be equal.

2°. On making everywhere AM = to the length of the curve AC, if through the point C of the sail a line may be drawn parallel to BM, that will touch the sail at the point C.

3°. The area $u\mathrm{AC}2f$ is equal everywhere to the rectangle BAM or to the rectangle under the curve AC and with the right line AB, which is an instance of the parameter; and thus the areas $u\mathrm{AC}2f$ are proportional to their homologous curves AC . For the element of the before mentioned is $f2f.fc = \mathrm{B}m.\ \mathrm{M}\mu = 2.\text{triangle BM}m$; for indeed $f2f$ or $u\mathrm{W} = \mathrm{B}m, \&\ cf = \mathrm{M}\mu$, therefore all the $f2f.cf$ or the area $u\mathrm{AC}2f$ = to all the $2.\mathrm{MB}m$, that is, to the double of the triangle BA*m,* that is, to the rectangle AB by A*m,* or AB.AM or AB.AC.

4°. If on MI there may be taken $u\xi$ of this magnitude, so that the rectangle $u\xi$.AM may be equal to the hyperbolic area ALNM, then the point ξ will be the centre of gravity of the curve CAC. For the centre of gravity of this curve shall be on some right line $\xi\gamma$, parallel to AB, and the heavy curve AC may be placed to hang on the weighing axis $u2c$; there will be (§.44.) :

$\mathrm{AC}.u\xi$ = all the $u\mathrm{I.C}c$ = all the $\mathrm{BM.M}m$ = all the MN.Mm = area ALNM,

and AC = AM ; therefore $\mathrm{AC}.u\xi$ = ALNM,

so that since the centre of gravity of the curve CAC also shall be on the axis AW, generally it will be at the point ξ.

5°. But if indeed the centre of gravity of the curve AC were on the right line $\delta\gamma$, there will be (§.44.) $\mathrm{A}\delta.\mathrm{AC}$ = all the $\mathrm{CI.C}c$. And all the $\mathrm{CI.C}c = \mathrm{AC.CI} -$ all the $\mathrm{AC}.cf = AC.\mathrm{CI} -$ all the $\mathrm{AM.M}\mu$, and the similar triangles ABM and $Mm\mu$ give $\mathrm{AM.M}\mu = \mathrm{AB}.m\mu$, and thus all the $\mathrm{AM.M}\mu = \mathrm{AB}.\omega : \mathrm{AB.AI}$; therefore all the $\mathrm{CI.C}c = \mathrm{AC.CI} - \mathrm{AB.AI} = \mathrm{AM.CI} - \mathrm{AB.AI}$; and thus $\mathrm{A}\delta$ or $\xi\gamma$ into $\mathrm{AC} = \mathrm{AC.\ CI} - \mathrm{AB.\ AI} = \mathrm{AC.CI} - \mathrm{AC.A}\lambda$, clearly with the tangent $\omega\lambda$ drawn; and thus $\xi\gamma = \mathrm{CI} - \mathrm{A}\lambda$; and thus the centre of gravity of the curve AC is distant from the right line $c2c$ by the interval Aλ or $\omega\lambda$.

6°. And thus the surface arising from the rotation of the curve AC about the axis AI is equal to a circle, whose radius is able to double that of the rectangle $\xi\gamma$.AC or $2.\mathrm{AC.CI} - 2\mathrm{AB.AI}$. For it has been shown above (§.47.), the magnitude arising from the rotation of any generating circle about a given axis in place, can be made equally from the magnitude arising from the path of its centre of gravity ; and the paths of the centre of gravity are circular circumferences described from the centre, and thus are proportional to their radii. Therefore the magnitudes generated are in a ratio composed from the generating magnitude and of the distances of their centre of gravity from the axis of rotation. Therefore the surface from the rotation of the curve AC about AI will be to some circle whose radius is R, in the ratio composed of the magnitude AC of the generating surface to the magnitude R of the generating circle, and of the distance of the centre of gravity γ of that magnitude from the axis AI or $\gamma\xi$ to the distance of the centre of gravity of this, or $\frac{1}{2}\mathrm{R}$, that is, just as $\mathrm{AC}.\gamma\xi$ to $\mathrm{R}.\frac{1}{2}\mathrm{R}$, that is, just as $2\mathrm{AC}.\gamma\xi$, or $2\mathrm{AC.CI}-2\mathrm{AB.AI}$ to R^2; and thus if $\mathrm{R}^2=2\mathrm{AC.CI}-2\mathrm{AB.AI}$, the surface generated will be equal to the circle.

7°. By the same argument the circle, whose radius R is able to double the area of the hyperbola AMNL, is equal to the surface generated from the rotation AC about the axis *u*2*c*. For the said surface will be to the circle, as $\mathrm{AC}.u\xi$ to $\frac{1}{2}\mathrm{R}^2$ or as $2\mathrm{AC}.u\xi$ to R^2; and (no. 4) $\mathrm{AC}.u\xi=\mathrm{AMNI}$, therefore the surface arising from the curve AC about *u*2*c* is to the circle of radius $\mathrm{R}=\sqrt{2\mathrm{AMNL}}$, as 2AMNL to 2AMNL, that is in the ratio of equality.

8°. But if ρ were the centre of the area *u*AC2*f*, $\rho\theta$ may be drawn parallel to AB, and (§.44.) $u\theta$ will be by the area $u\mathrm{AC}2f=$ to the sum of the moments of the rectangles *f*2*c* for the axis of suspension *u*2*c*; that is $u\theta.\mathrm{AM.AB}=$ the sum of all the $\frac{1}{2}.c2c.c2c.cf$; for (no. 3.) the area is $u\mathrm{AC}2f=\mathrm{AM.AB}$. And
$\frac{1}{2}.c2c.c2c.cf=\frac{1}{2}\mathrm{BM.BM.M}\mu=\frac{1}{2}\mathrm{AB.BM.M}m=\frac{1}{2}\mathrm{AB.MN}.Mm$, therefore the sum of all $\frac{1}{2}.c2c.c2c.cf=\frac{1}{2}\mathrm{AB}$ by the area ALNM ; and thus
$u\theta.\mathrm{AM.AB}=\frac{1}{2}.\mathrm{AB.\ ALNM}$, or $2u\theta.\mathrm{AM}=\mathrm{ALNM}$. But (no. 4.) we had also $u\xi.\mathrm{AM}=\mathrm{ALNM}$, therefore $u\xi.\mathrm{AM}=2.u\theta.\mathrm{AM}$, or $u\xi=2.u\theta$, and thus the centre of gravity of the curve AC will be twice as far from the line *u*2*c,* as the centre of gravity of the area *u*AC2*f.*

[The following make use of the Pappus Centroid Theorems for surfaces and volumes.]

9°. Hence the volume from the rotation of the figure *u*AC2*f* about the axis *u*2*f*, is of a half cylinder, whose height is AB, and whose base radius is able to double the area of the hyperbola ALNM. For this solid (§.47.) can be made equally from the *u*AC2*f* into the circumference of radius $u\theta$ since ρ shall be the centre of gravity of the area, that is $=\mathrm{AB.AM}$ into the circumference of radius $u\theta$ (no. 8.)$=\frac{1}{2}\mathrm{AB.\ AM}$ into the circumference $u\xi$. And AC or AM into the circumference $u\xi$ is the surface arising from AC about *u*2*c* (no. 7.) equals the circle whose radius $=\sqrt{2\mathrm{ALMN}}$; therefore the volume of the area *u*AC2*f* rotated about *u*2*f* equals the cylinder, whose radius is 2ALMN , and whose height is $\frac{1}{2}\mathrm{AB}$; and thus the aforementioned volume can be that of a half cylinder, whose height is AB and the base radius can be 2ALNM.

10°. Hence it follows further, the volumes of revolution for the figures *u*AC2*f* and *u*A*c2c* arising from the rotation about the *u*2*c* to be proportional to the surfaces from the homologous curves AC and Ac about the same axis *u*2*c,* just as the areas *u*AC2*f* and *u*AC*2c* are proportional to the curves AC, A*c* ; which property, which first was found by the illustrious Leibniz, is deservedly of note ; as well as that where the centre of gravity of the curve AC shall be more than twice the distance from the right line *u*2*c* than the centre of gravity ρ of the area *u*AC2*f*, and both these centres γ and ρ to be equidistant from the axis AI.

11°. The centres of gravity ρ and γ of the area *u*AC2*f* and of the curve AC shall be equidistant from the axis of the sail. For (§.44) $\theta\rho$ into *u*AC*2f*

$$= \theta\rho.\mathrm{AB.AM} = \text{sum of all } \mathrm{CI}.f2f.\,cf = \text{sum CI.BM.M}\mu = \text{sum CI.AB.}Mm$$
$$= \mathrm{AB.CI.AM} - \text{sum AB.AM.}cf = \mathrm{AB.CI.AM} - \text{sum AB.AM.}m\mu$$
$$= \mathrm{AB.CI.AM} - \text{sum AB}^2.m\mu = \mathrm{AB.CI.AM} - \mathrm{AB}^2.\mathrm{M}\omega,$$

therefore also $\theta\rho.\mathrm{AM} = \mathrm{AM.CI} - \mathrm{AB}$ in $\mathrm{M}\omega$. But (no. 5.) we had also $\mathrm{AM}.\xi\gamma = \mathrm{AM.CI} - \mathrm{AB}$ into $\mathrm{M}\omega$ or AI, therefore $\theta\rho.\mathrm{AM} = \mathrm{AM}.\xi\gamma$, or $\theta\rho = \xi\gamma$.

12°. And thus the volume from the rotation of the figure *u*AC2*f* about *u*I is equal to the cylinder, whose radius is able to double the area $2\mathrm{CI.AC} - 2\mathrm{AB.AI}$ and the height is AB. Hence again, the volumes from the figures from the revolution of *u*AC2*f* and *u*A*c2c* about the axis AI shall be proportional to the surfaces arising from the curves AC, A*c* rotated about the same axes.

13°. The radius of osculation of the circle $\mathrm{C}\beta$ on the curve at the point C, is the third proportional to *u*A and *u*I. For, since B*m* and BM are parallel to the elements of the curve C*c* and CG; the evanescent sectors MBμ and Cβ*c* will be similar , from which $\mathrm{M}\mu : \mathrm{BM} = Cc : \mathrm{C}\beta$ or on permuting $\mathrm{M}\mu : Mm$ or $\mathrm{C}c = \mathrm{BM} : \mathrm{C}\beta$; and $\mathrm{M}\mu : \mathrm{M}m = \mathrm{AB} : \mathrm{BM}$, therefore $\mathrm{AB} : \mathrm{BM} = \mathrm{BM} : \mathrm{C}\beta$, or, because $\mathrm{BM} = u\mathrm{I}$, there will be $u\mathrm{A} : u\mathrm{I} = u\mathrm{I} : \mathrm{C}\beta$. Hence, for the curve $\alpha\beta$ from whose evolute the sail will be described, the start α from the vertex of the sail A shall be equidistant from the point *u* on the quadrant of the circle *u*SL.

CAPUT XIII.

De Figuris, quas superficies flexiles induere debent, cum Venti allapsus directe excipiunt, seu de curva

VELARIA.

Quamnam curvam induere debeat *velum* vento tumidum ante Celeberrimos Bernoullios nemo assignavit; sed eximii hi geometræ invenerunt velum ab allabente vento in eam ipsam curvam flecti debere, quam Joh. Bernoullius & Leibnitius atque Hugenius funi laxo atque flexili, vel catenulæ ab ambobus sui terminis pendenti convenire docuerunt, etsi eorum analysis velariæ nusquam publici facta sit juris. Sed quanquam *catenaria* atque *velaria* una eademque sint curva, non tamen ideo putandum, unius investigationem simul alterius quoque inventionem includere, adeo ut, qui catenariam geometrice invenerit, etiam velariæ problema solutum dedisse censendus sit, quod tamen Clariss. Vir David Gregorius insinuare videtur, in sua Catenaria Coroll. *7.* post Prop. 2. sic scribens:

Quod si loco gravitatis alia quælibet vis similiter agens in lineuam flexilem vires suas exerat, eadem producetur linea v. g. Si ventus æquabilis supponatur, & secundum rectas datæ positione rectæ parallelas spirans, linea vento inflata eadem erit cum catenaria.

Fateor quidem in thesi, ut ajunt, verom esse Egregii Viri assertionem, ecquis enim ambiget, quin alia quæcunque vis, loco gravitatis substituta, similiter applicata, ac eodem prorsus ac gravitas modo agens, eundem esse effectum, eandemque adeo curvam ac gravitas, productura sit? Sed theseos applicatio in negotio velariæ claudicat, cum notabilis disparitas intercedat interactionem gravitatis in catenaria, & venti actionem in velaria. Nam æquales catenariæ particulæ æqualibus nisibus descendere conantur juxta directiones horizonti perpendiculares, secus quam in velaria cujus particulæ etiamsi æquales a filamentis aereis eadem etiam velocitate in eas impingentibus inæquales tamen impressiones subeunt, & quidem juxta directiones non horizonti, sed curvæ elementis perpendiculares. Cum igitur circumstantiæ, quibus gravitas in catenariam ventusque in velariam agunt, toto coelo differant, nemo non videt, quod ea, quæ de catenaria demonstrata sunt, non magis velariæ quadrare queant,quam analysis curvæ lintei convenire possit indagini curvæ elasticæ, etsi notante Clar. Jac. Bernoullio elastica etiam eadem fit cum curva lintei. Quoniam igitur problematis catenariæ solutio non involvit solutionem velariæ, & quoniam hujus problematis solutio, vel rectus dicendo analysis, nusquam adhuc publice, quod sciam, exhibita est præterquam §. 103. ubi ad illustrationem generalissimi nostri theorematis problematis de velaria solutionem obiter elicuimus, partem physicam problematis alibi excutiendam illic relinquentes; nunc vero ex professo problema tractabimus, analysin ejus geometricam tradituri intelligentibus, ut opinor, non displicituram, præmissis tamen prius duobos lemmatis in aliis etiam usui futuris.

PROPOSITIO LI. LEMMA.

463. *Centro* O *& semilatere transverso O A descripti fint quadrans circuli* AGK, *& hyperbola æquilatera* ABM, *tum etiam logarithmica* AC *circa asymptotam* LK, *cujus subtangens fit* OL *æqualis radio quadrantis* OA, *quæ log-mica per punctum* A *transeat, ductisque ex quolibet hyperbolæ puncto* B *ad centrum* O *recta* BO, *tangente hyperbolam in vertice AP; secante in puncto* F, *& per hoc punctum F recta* FH *parallela radio AO, quadrantem secante in* G, & OK *in* H. *Si* KG *recta jungens puncta* K & G *producatur usque dum cum radio OA itidem protenso concurrat in puncto D, atque per hoc punctum ducta fuerit recta* DC *ipsi* LK *æquidistans. Rectangulum sub hac* DC *& radio* OA *æquabit ubique duplum respondentis sectoris hyperbolici* ABO.

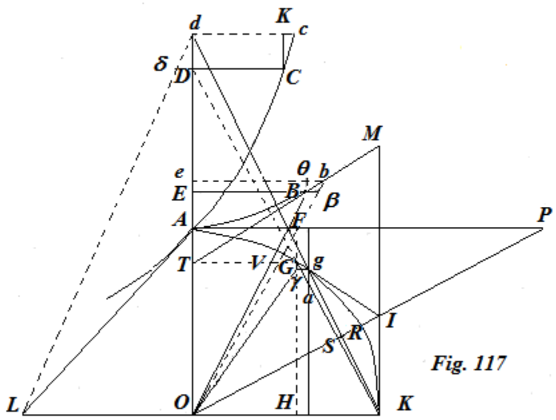


Fig. 117

Sumto alio in hyperbola puncto *b,* alteri B indefinite vicino, ducantur *bO* secans AP in *f,* item *fh,* parallela FH, circulum in *g*, rectam KG in *a,* & OK in puncto *h* intersecans ; jungaturque *Kg* & producatur in *d*, per quod punctum alia *de* ducta fit priori DC æquidistans: agantur pariter *d*L & OG, item C*x* parallela *Od,* & *γg* parallela OK, hyperbolæ ordinata EB continuetur usque in *β*, & CD usque in δ ; quibus factis per quadrantis puncta G & K ductæ sint tangentes GI & KI, quæ æquales erunt; quibus præparatis

I. Liquet fore *dL* parallelam tangenti log-micæ in puncto C, quandoquidem hujus log-micæ subtangens (secundum hypothesin) æqualis est OL; atque adeo $d\delta$ parallela & æqualis erit particulæ log-micæ C*c*, ac etiam $cx = D\chi$.

II. Triangula similia OEB, OAF præbent $\mathrm{OE}^3 : \mathrm{EB}^3 = \mathrm{OA}^2 : \mathrm{AF}^2$. Vel, quia in hyperbola quadratum OE æquatur binis quadratis OA & EB collective sumtis, erit etiam $\mathrm{OA}^2 + \mathrm{EB}^2 : \mathrm{EB}^2 = \mathrm{OG}^2 : \mathrm{OH}^2$; & dividendo $\mathrm{OA}^2 : \mathrm{EB}^2 = \mathrm{GH}^2 : \mathrm{OH}^2$ vel AF^2, vel invertendo ac permutando $\mathrm{EB}^2 : \mathrm{AF}^2 = \mathrm{OG}^2 : \mathrm{GH}^2$. Atqui $\mathrm{EB}^2 . \mathrm{AF}^2 = \mathrm{OB}^2 : \mathrm{OF}^2 = 2.\mathrm{triang.BO}\beta$ ad $2.\mathrm{triang.FO}f$, id est, ad rec-lum FH.H*h,* ergo $2.\text{triangulum } \mathrm{BO}\beta : \mathrm{FH.H}h = \mathrm{OG}^2 : \mathrm{GH}^2$.

III. Triangula similia *Gga* & GIK exhibent $Gg : ga = \mathrm{GI} : \mathrm{IK}$, unde, quia tangentes IK & Gl æquantur, etiam *Gg* & *ga* æquales erunt.

IV. Propter parallelas O*d* & *hg,* erit $\mathrm{O}d : \mathrm{D}d = hg$ vel HG : *ag* vel HG *:ag* vel (num.111.) G*g*, & invertendo ac permutando $\mathrm{D}d : \mathrm{G}g = \mathrm{O}d : \mathrm{GH}$, at $\mathrm{D}\delta : \mathrm{D}d = \mathrm{OL}$ vel $\mathrm{OG} : \mathrm{O}d$, ergo ex æquo $\mathrm{D}\delta : \mathrm{G}g = \mathrm{OG} : \mathrm{GH}$, & propter triangula similia OGH & $Gg\gamma$, sit $Gg:\gamma g$ vel $Ff = OG:GH$ ergo ex æquo & per rationum compositionem,

$$D\delta : Ff = OA.D\delta : FH.\ Ff = OG^2 : GH^2 \left(\text{num.II.}\right)$$
$$= 2.\text{triangulum } BO\beta : FH.Ff \text{ vel } FH.Hh.$$

Adeoque $2.\text{triang.}AO\beta = OA.D\delta\left(\text{num. I.}\right) = OA : c\chi$.

ergo omnia 2.BOβ id est duplus sector $BAO =$ omnibus cx in AO,
id est rec-lo DC in AO. Quod erat demonstrandum.

Aliter.

264. Per hyperbolæ punctum B producta sit tangens BT, ac per punctum T recta TV parallela OK rectæ OB occurrens in puncto V, ac denique agatur Bθ parallela AD.

I. Ostendam, quod TV producta transibit per punctum G, adeo ut $TO = GH$. Nam in hyperbola lineæ GH, OA & OF sunt in continua ratione, atqui propter tangentem BT, etiam OT, O A & OE sunt in continua ratione, ergo $OT = GH$.

II. Elementum quadrantis G*g* æquabitur lineolæ Bβ. Nam $Gg:\gamma g$ vel $Ff = OG:GH = AO:TO = EO:AO$, atqui etiam $B\beta : Ff = EO:AO$, ergo $Gg : Ff = B\beta : Ff$, atque adeo $Gg = B\beta$.

III. Figuræ similes $b\theta B\beta$ & BETV,utpote quæ circa eandem rectam *b*T constitutæ, & e triangulis similibus $b\theta$B, BET, & $bB\beta$, BTV compositæ sunt, præbent analogiam $b\theta : B\theta$ vel (num. II. hujus) Gg aut (num. 111. §.463.) $ga = EB:TV = EO:TO$ atqui $ga : Dd = GH.$(num. I. hujus) $OT:Od$, & $Dd : D\delta = Od : UL$ vel OA, ergo ex æquo habetur $b\theta : D\delta = EO:AO = AO:TO$; atque adeo $b\theta.TO = AO.D\delta = AO.c\chi$.

IV. Est vero duplum trianguli $bTO = be:TO$, duplumque trianguli $BTO = BE.TO$, ergo duplum trianguli $BOb = b\theta.TO$ (num.III. hujus) $= AO.c\chi$, & per conseqens etiam areæ BAO duplum æquabit rectangulum sub AO.DC. Quod erat demonstrandum.

COROLLARIUM I.

465. Ductis ex logarithmicæ puncto C ad puncta A & O rectis CA & CO, triangulum rectilineum CAO æquabitur ubique homologo sectori hyperbolico BAO.

COROLLARIUM II.

466. Jungatur OI, eaque producatur in P, eruntque $AP = OD$, & $OP = KD$. Nam, quia OS ipsi KD normalis est, erunt anguli SOK ejusve alternus APO & ODK æquales, atque adeo triangula rectangula AOP & OKD similia, ob latera vero AO & OK æqualia erunt etiam hæc triangula æqualia, ac proinde $AP = OD$ & $OP = KD$. AP autem est tangenti arcus AR compositi ex arcu AG, cujus sinus OH æquatur tangenti AF anguli sectoris AOB, & ex semissi GR complementi ejusdem. Et DC est log-us rationis DO

ad AO, id est, log-us rationis tangentis AP prædicti arcus seu anguli compositi AOR ex angula AOG & GOR ad radium seu sinum totum AO. Atqui ratio AP ad AO $=$ rationi OK ad IK, id est, rationi, quam habet radius OK ad tangentem IK semissis complementi anguli AOG, cujus sinus æquatur tangenti anguli sectoris AOB.

SCHOLION.

467. Haec propositio plurimum conducit reductioni elementorum sectorum hyperbolicorum ad simpliciora elementa log-mica. Nam si

$$\mathrm{AO}=a;\ \mathrm{OD}=m;\ \mathrm{OE}=x;\ \mathrm{EB}=y;\ \mathrm{AF}=z\ \ \&\ \ \mathrm{AE}=u.$$

Atque his positis, erit

I.

$$y=\sqrt{(xx-aa)},\mathrm{AF}=a\sqrt{(xx-aa:xx)};\text{adeoque }\mathrm{F}f=a^3dx:xx\sqrt{(xx-aa)},$$

$$\&\ \mathrm{FH.H}h=a^4dx:\ xx\sqrt{(xx-aa)}.$$

Unde quia triang.BOβ : triang.FOf $=\mathrm{OE}^3:\mathrm{OA}^2$, reperietur duplum trianguli $\mathrm{BO}\beta=aadx:\sqrt{(xx-aa)}$. Hoc idem brevius inventum suisset ope num. IV. §.466. uti ostendimus $2.\mathrm{BO}b=\mathrm{OT.b}\theta$; nam, ergo $\mathrm{OT}.b\theta=aaxdx:x\sqrt{(xx-aa)}=aadx:\sqrt{(xx-aa)}$. Jam $\mathrm{KH}=\left(ax-a\sqrt{(xx-aa)}\right):x$, & GH vel OT $=aa:x$; atque triangula KHG& KOD similia sunt, ergo $m=aa:x-\sqrt{(xx-aa)}$ & $x=(aa+mm):2m$, qui valor substitutus in præcedenti expressione elementi BOβ, producet $2\mathrm{BO}b=aadm:m$.

II. Quia $x=a+u$ & $dx=du$, erit $2BOb=aadx:\sqrt{(xx-aa)}=aadu:\sqrt{(2au+uu)}$, & $u+a=(aa+mm):2m$, vel $u=(m-a)^2:2m$, qui valor in formula $aadu:\sqrt{(2au+uu)}$ suffectus, dabit iterum $2\mathrm{BO}b=aadm:m$.

III. Propter triangula similia OAF & OEB invenietur $ay=xz$, hyperbola vero præbet $y=\sqrt{(xx-aa)}$; unde $x=\sqrt{(aa-zz)}$, ex triangulis vero similibus KHG & KOD elicitur $am-mz=a\sqrt{(aa-zz)}$ & $z=amm-a^3:mm+aa$, qui valor ipsius z ejusque elementi $a^3dz:aa-zz=2\mathrm{BO}b$ substitutus, dabit etiamnum $2\mathrm{BO}b=aadm:m$.

Similes regulas reducendorum sectorum hyperbolicorum ad logarithmos ex sua methodo generali reducendi quadraturas curvarum, quarum ordinatæ fractonibus rationalibus per abscissam varie affectam & datas, exhibitis exprimuntur, ad logarithmos, jam pridem elicuit Vir Celeb. Joh. Bernoulli in Comm. Acad. Reg. Scient. Paris. 1702. d. 13 Decemb. & in Act. Lips.1703. pag. 30 & 31. Cæterum, etsi præcedens theorema nostrum de hyperbola tantum æquilatera agit, paucis tamen mutatis· ad quascunque hyperbolas potest extendi.

PROPOSITIO LII. LEMMA.

468. *Si rectæ* AM *positione datæ alia* AB *magnitudine data perpendiculariter insista, per cujus terminum* B *rectæ quæcunque* BM, B*m ad alteram* AM *ducantur, & per puncta* M, *m*, &c. *in quibus huic occurrunt, perpendiculares* MN, *mn*, &c. *ipsis* BM , B*m*, &c. *respective æquales, omnia puncta* N, n, &c. *in curva* LNn *hyperbolæ æqualateræ sita erunt, cujus centrum* A , *& semilatus transversum* AL.
Positisque punctis M, *m indefinite vicinis, ac descripto centro* B *intervallo* BM *arculo* Mμ*, junctisque* AN, A*n*, *rectangulum sub radio* BA *& arculo* Mμ *duplum erit trilinei* AN*n seu elementi sectoris hyperbolici* ANL. Fig. 118.

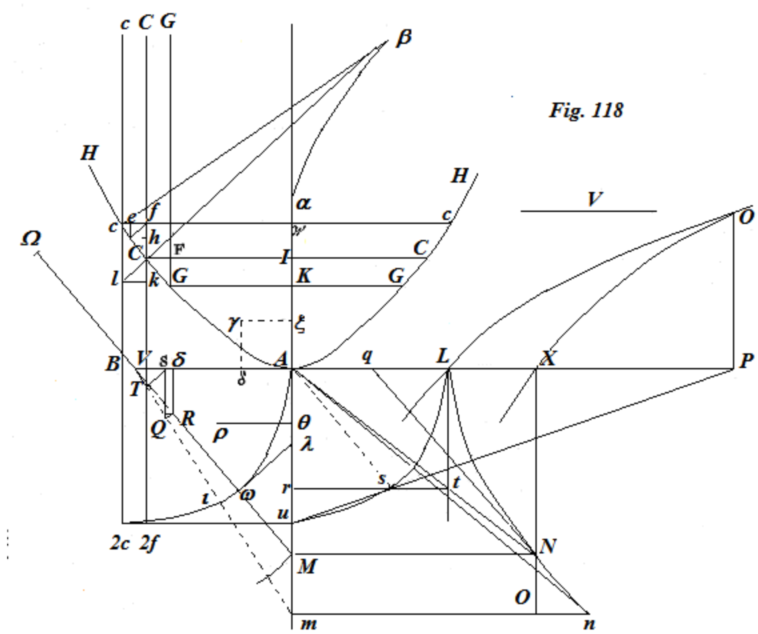


Fig. 118

I. Ducatur NX parallela AM, & quia (secundum hypothesin) $\mathrm{MN} = \mathrm{BM}$, & $\mathrm{AB} = \mathrm{AL}$, erit $\mathrm{AX}^2 = \mathrm{XN}^2 + \mathrm{AL}^2$, vel $\mathrm{XN}^2 = \mathrm{AX}^2 - \mathrm{AN}^2 = \mathrm{rec-lo\ BXL}$, ergo punctum N est in hyperbola æquilatera LN, cujus latus transversum est BL & centrum A. Curva LN etiam hyperbola erit, si MN ad BM fuerit in quacunque data ratione.
II. Per punctum N ducta sit tangens hyperbolæ N*q*, eruntque tres MN vel BM, AL vel BA & A*q* in continua ratione, atque adeo $\mathrm{BM} : \mathrm{BA} = \mathrm{BM} : \mathrm{A}q$, atqui propter triangula similia. ABM & μM*m*, est etiam $\mathrm{BM} : \mathrm{BA} = \mathrm{M}m$ vel $\mathrm{N}o : \mathrm{M}\mu$; ergo , $\mathrm{N}o : \mathrm{M}\mu = \mathrm{BA} : \mathrm{A}q$ atque adeo $\mathrm{BA.M}\mu = \mathrm{N}o.\mathrm{A}q$ (num. IV. §.464) $= 2.\mathrm{AN}n.$.
Quod erat demonstrandum.

COROLLARIUM I.

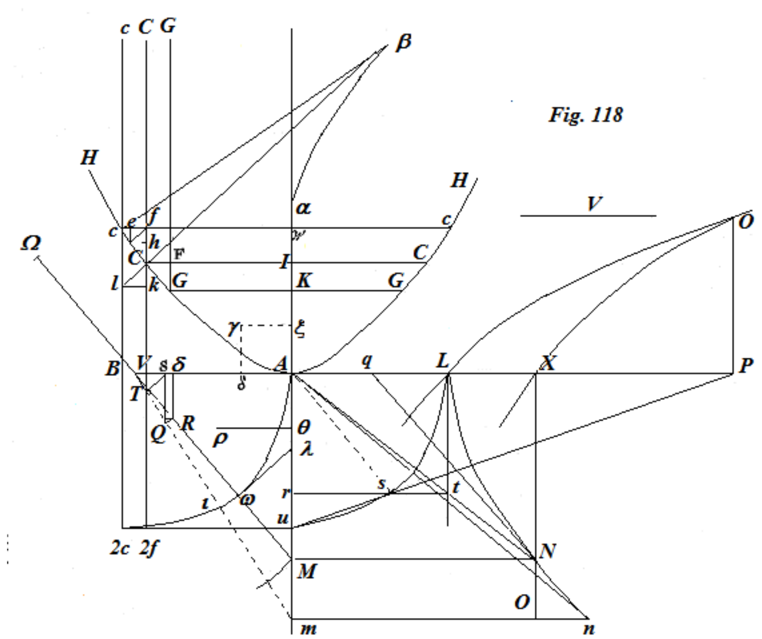
469. Adeoque, si tota AM infinitis composita sit particulis infinitesimis qualis M*m,* & per singularum terminos rectæ BM ductæ intelligantur, erit factum ex omnibus arculis Mμ, quos vicinæ quæque BM intercipiunt, & radio BA æquale duplo omnium triangulorum AN*n*, quæ in sectore hyperbolico ANL continentur, id est, duplo ipsius sectoris.

COROLLARIUM II.

470. Ducendo igitur per punctum intersectionis *t* rectæ AN & tangentis hyperbolæ A*t* in vertice, lineam *tsr* parallelam AL, & *us*L, quæ ipsam AL productum secet in puncto P; ac denique per punctum L logarithmicam LO, quæ subtangentem habeat æqualem radio AL, & per P agatur PO parallela AM log-micæ occurrens in puncto O ; erunt omnes arculi Mμ collective sumpti, æquales rectæ PO. Nam quia (§.469.) omnes Mμ in $\text{AB} = 2.\text{are}\textit{æ}\ \text{ANL}$ & (§. 463.) $2.\text{sect. ANL} = \text{AB vel AL in PO}$, erunt omnes $M\,\mu.\text{AB} = \text{AB. PO}$, atque adeo omnes $\text{M}m = \text{PO}$.

PROPOSITIO LIII. PROBLEMA.

471. *Si filum perfecte flexile* HA2H *ambobus suis terminis* H, *2*H Fig.118. *fixum, spiranti vento juxta directiones cc*, CC, GG *datæ positione rectæ* AW *parallas expositum sit, assignare curvam, quam filum induere debet.*



Analysis Geometrica. I. Quia ventus impressiones in velariam exserit (§. 249.) juxta directiones C*l* curvæ quæsitæ ACH normales, atque adeo (§.96.) constat tenacitatem veli in singulis ejus punctis eandem esse oportere; talis tenacitas constans per datam rectam AB axi AW normalem exponi potest, si venti velocitas sit 1, sin vero velocitas exponatur linea recta V, §.100.) tenacitatem per hoc factum $V^2.AB$ repræsentare oportebit, ut mox videbimus. Nam sumtis æqualibus & contiguis curvæ elementis C*c*, & CG, per eorum terminos ordinatæ ad axem curvæ AW ductæ intelligantur *c*W, CI & GK, quæ rectas CC & GG fecabunt in *f* & F*;* & demisssa ex puncto *f* perpendiculari ad elementum curvæ *c*C*,* quæ fit *fd,* per punctum *d,* agatur *de* æquidistans axi AW, & denique esto $Ch = GF$; adeo ut *fh* differentia sit inter *fC* & *Fg;* & his positis impressio, quam filamentum aeris *cc*CC celeritate V lati & in curvam impingentis in elementum C*c* exseret (§. 429.) exponitur par factum $V^2.CD$*;* ejusque directio C*l,* ut jam dictum est ubique curvæ perpendicularis; propterea, & juxta ea quæ alibi (§.100.) dictæ sunt, tenacitas fili exponi debet magnitudine $V^2.AB$ ejusdem generis cum $V^2.cd$, seu impressione quam elementum C*c* excipit juxta Cl, vel quodlibet aliud juxta directionem ipsi normalem. Exponat jam C*l* magnitudinem $V^2.cd$ vel potentiam curvæ in puncto C normaliter applicatam, ductsque C*k* & *lk* axi AW parallela & normali, & potentia *Ck* erit $V^2.ce$; nam in triangulis similibus C*lk* & *cde,* lineola *ce* alteri C*k* homologa est, cum sit $Cl : Ck = cd : ce$*;* atqui $Cl = V^2.cd$, ergo etiam $Ck = V^2.ce$. Jam, cum alibi (§. 93) ostensum sit, C*k* esse ad *fh* ut tenacitas fili in elemento C*c,* ad hoc elementum C*c,* vel permutando *Ck* ad dictam tenacitatem ut *fh* ad *Cc;* erit $V^2.ce : V^2.AB = ce : AB = fh : Cc$, & permutando: $ce : fh = AB : Cc$.

II. Per terminum B datæ BA ducantur BM, B*m* parallelæ elementis curvæ CG, C*c,* centroque B intervallo $BQ = CG = Cc$ descripto arculo QR*,* agantur QS*,* Rσ parallelæ AM*,* item R*x* æquidistans BA*;* ST perpendicularis ipsi BQ, & denique TV parallela SQ, quibus positis, ultra liquet, triangulum BQS triangulo *cCf,* & BRσ triangulo CGF similia, æqualia, & similiter posita esse; adeo ut sint $xQ = fh$, $BV = ce$, & $Cc = BQ$; adeoque analogia præcedentis numeri $ce : fh = AB : Cc$, eadem est cum $BV : xQ = AB : BQ$; in triangulis vero rectangulis similibus *y*QR & BTS, bases *y*Q & BS a perpendiculis R*x* & TV proportionaliter dividentur, adeo ut sit $xQ : yQ = BV : BS$, & permutando ac invertendo $BV : xQ = BS : yQ = BA : Mm$; antea vero habuimus $BV : xQ = AB : BQ$*;* ergo etiam $AB : Mm = AB : BQ$, atque adeo $Mm = BQ = Cc$*;* quod cum de singulis M*m,* quæ in tota A*m* & de singulis C*c,* quæ in tota curva *c*GA continentur, respective valeat, liquet omnes *Mm*, seu tota Am vel AM omnibus C*c*, seu toti *c*GA vel CGA æquari. Unde, quia BM tangenti velariæ in puncto C (secundum hypothesin) parallela est, erit data BA ad AM seu ipsi æqualem curvam AGC, sicut ordinata Cl ad subtangentem velariæ respicientem ejus punctum C. Hæc proprietas etiam catenariæ convenit, ut alibi velariæ atque catenariæ identitas jam (§.106.) ostensa est.

III. Constructio autem curvae hactenus ex ostensis facile elicietur: nam cum M*m* sit $= Cc$, & B*m* (secundum hypothesin) parallela C*c,* descripto centro B arculo Mμ, triangulum M$m\mu$, triangulo *cCf* simile & æquale erit, ac proinde $Cf = m\mu$, & omnes C*f*, id est, WA=omnibus $m\mu$ seu *mi,* descripto scilicet centro B, radioque BA circulo Aω. Sic etiam abscissa AI æqualis existet Mω posita, ut antea, BM parallela tangenti curvæ in C:

Elementum vero ordinatæ $cf = M\mu$ & omn. *cf*, seu *c*W aut $\mathrm{CI} =$ omnibus Mμ (§.470) $= \mathrm{PO}$ posita constructione in citato hoc paragrapho 470. Est ergo quælibet ordinata CI æqualis homologæ PO. Quæ omnia erant invenienda.

COROLLARIUM I.

472. Quia in hyperbola æquilatera LN tres AX, AL & A*q* sunt continue proportionales propter tangentem hyperbolæ A*q*, & ex natura hyperbolæ etiam tres AX, AL & *rs,* adeo ut N*q* & *rs* æquentur, erunt, ducta A*s*, triangula ABM & *Asr* similia; nam $\mathrm{AX} = \mathrm{MN}\ (\text{constr.}) = \mathrm{BM}$ est ad AL vel AB, sicut AL vel A*s* ad A*f* vel *rs,* atque adeo triangula ABM & *rsA* similia sunt, ac per consequens A*s* ipsi BM seu etiam tangenti velariæ in C parallela est, angulusque *u*A*s* æquabitur illi, quem tangens velariæ modo nominata, & quantum opus est producta, cum axe AW constituit, cujus semissis est angulus *rsu;* unde cum PO sit log-us rationis PA ad LA, seu PA ad A*u*, hoc est, rationis *rs* ad *ru;* sequitur PO esse log-mum rationis, quam radius habet ad tangentem anguli *rsu* vel dimidii anguli *u*A*s,* in logarithmica, cujus subtangens par est radio AL ; & in omni alia logarithmica, si sumatur quarta proportionalis ad log-micæ subtangentem AL, & ad log-um prædictæ rationis radii ad tangentem semissis anguli *u*A*s,* manifestabit ea ordinatam velariæ CI in puncto C, si scilicet angulus *u*A*s* æqualis fuerit angulo, quem tangens velariæ in C producta cum axe AW continet.

COROLLARIUM II.

473. Circulus centro B radioque BA descriptus rectam BM productam secet in Ω, eritque propter tangentem circuli AM & secantem MΩ, $\mathrm{M\Omega : MA = MA : M\omega}$, seu $\mathrm{2AB + M\omega : AM = AM : M\omega}$, adeoque $\mathrm{2AB + M\omega : M\alpha = AM^3 : M\alpha^2}$, & dividendo $\mathrm{2AB : M\alpha = 2.AB.M\alpha : M\omega^2 = AM^2 - M\alpha^2 : M\omega^2}$, atque adeo $\mathrm{2AB.M\alpha = AM^2 - M\omega^2}$. Innotescit ergo ex datis AM vel AC & Mω vel AI parameter velariæ AB seu A*u.*

Porro quia (§. 472.) $\mathrm{A}r : rs = \mathrm{AM : AB = 2AM.\ M\omega : 2AB.M\omega = 2AM.M\omega : AM^2 - M\omega^2}$, erit A*s* vel $\mathrm{A}u : \mathrm{A}r = \mathrm{AM^2 + M\omega^2 : 2AM.M\omega}$,

& dividendo $\mathrm{A}u - \mathrm{A}r$ vel *ru* ad $\mathrm{A}r = \mathrm{(AM - M\omega)^2 : 2AM.M\omega}$,

& $\mathrm{A}r : rs = \mathrm{2AM.M\omega : AM^2 - M\omega^2}$, ergo ex æquo

$ru : rs = \mathrm{(AM - M\omega)^2 : AM^2 - M\omega^2 = AM - M\omega : AM + M\omega} =$ curva AC abscissa ejus AI, ad $\mathrm{AC + AI}$. Unde, cum sit $\mathrm{PA : LA} = rs : ru = \mathrm{AC + AI : AC - AI}$, & PO log-us rationis PA ad LA; erit etiam PO log-us rationis $\mathrm{AC + AI}$ ad $\mathrm{AC - AI}$, cujus rationis termini, scilicet aggregatun & differentia curvæ AC & ejus abscissæ vel sagittæ AT, dati sint. Idcirco,si fiat ut 2Mα seu 2AI ad $\mathrm{AC + AI}$ ita $\mathrm{AC - AI}$ ad quartam magnitudinem Q, deinde ut subtangeni log-micæ cujuscunque ad log-mum rationis $\mathrm{AC + AI}$ ad $\mathrm{AC - AI}$ in hac logarithmica, ita magnitudo vel linea Q ad aliam R, hæc linea, juxta præcedens corollarium, dabit ordinatam CI nostræ velariæ AC.

COROLLARIUM III.

474. Juxta §.110. erit media directio impressionis venti filum CAC inflantis in axe AW, nam hæc media directio angulum a tangentibus velariæ ductis per puncta C, C ad utramque axis partem similiter sita contentum, bifariam dividit, quod etiam axis AW præstat; estque adeo (§.110.) impressio venti in velaria, juxta ejus mediam directionem WA, ad tenacitatem velariæ in omnibus ejus punctis eandem, sicut dupla A*r* ad A*s,* id est, ut supra (§. 473) habuimus, sicut $4\text{AM.M}\omega,$ ad $\text{AM}^2+\text{M}\omega^2$, id est*,* sicut $4.\text{AC.AI}$ ad AC^2+AI^2. Unde, quia tenacitas velariæ (num.I. §.471.) est $\text{V}^2.\text{AB}=(\text{AC}^2-\text{AI}^2:2\text{AI}).\text{V}^2$, erit impressio venti in velariam juxta ejus mediam, directionem $\text{WA}=\text{V}^2.2\text{AC}.\left(\text{AC}^2-\text{AI}^2\right):\text{AC}^2+\text{AI}^2$.

COROLLARIUM IV.

475. Velariæ vero æquatio differentialis ex allata constructione facillime elicietur; nam si dicantur BA, a ; AI, $x;$ CI, y ; C*f*, $dx;$ cf, dy ; $\text{C}c=\text{CG}=ds$; $\text{fh}=ddx$, erit $ce=dy^3:ds^2$, & analogia, in quam supra (num. I.§.471.) incidimus *ce* vel $dy^3:ds^2$ ad fh seu *ddx,* ut AB vel *a,* ad *Cc* seu *ds;* quæ, multiplicatis extremis & mediis, dabit æquationem velariæ differentio-differentialem $dy^3=adsddx$, in quam Celeberrimi Bernoullii etiam inciderunt, ut videlecit in Act. Erudit.Lips. 1695. pag. 546.
Æquatio differentialis velariæ elicitur ex similitudine triangulorum ABM & M$m\mu$; est enim $m\mu:\text{M}\mu=\text{AM}:\text{AB}$; unde quoniam $\text{AB}=a,\ \text{M}\omega=\text{AI}=x$, atque adeo $\text{AM}=\sqrt{(2ax+xx)}, m\mu=dx \,\&\, \text{M}\mu=\text{CF}=dy$; erit $dy=adx:\sqrt{(2ax+xx)}$ prima æquatio differentialis primi gradus, naturam velariæ explicans.
Sin vero BM vel *u*I fuerit $=x$, tunc erit $\text{AM}=\sqrt{(xx-aa)},\ \& \,dy=adx:\sqrt{(xx-aa)}$, quæ est altera velariæ æquatio differentialis. Et hæ sunt æquationes, in quas laudati Bernoullii jam pridem inciderunt, quisque eorum sua methodo propria usus.

SCHOLION.

476. Ex nostra constructione atque analysi jam sponte nascuntur præcipuæ proprietates velariæ, & maxime notabiles. Nam

1°. Curva AC æqualis est ubique respectivæ ordinatæ XN hyperbolæ æquilateræ LN*,* cum hæc ordinata sit æqualis MA*,* cui curva AC æqualis ostensa.

2°. Facta ubique $\text{AM}=$ curvæ AC, si per velariæ punctum C parallela ducatur rectæ BM, ea velariam in puncto C continget.

3°. Area $u\text{AC}2f$ æquatur ubique rec-lo BAM seu rec-lo sub curva AC & recta AB, quæ instar parametri est; atque adeo areæ $u\text{AC}2f$ suis homologis curvis AC proportionantur. Nam elementum areæ prædictæ est $f2f.fc=\text{B}m.\ \text{M}\mu=2.$trianguli B$\text{M}m$ *;* sunt enim $f2f$ vel $u\mathrm{W} = \mathrm{B}m, \&\ cf = \mathrm{M}\mu$, ergo omnia $f2f.cf$ seu area $u\mathrm{AC}2f =$ omnibus $2.\mathrm{MB}m$, hoc est, duplo trianguli BA*m,* id est, rec-lo AB in A*m,* vel AB.AM seu AB.AC.

4°. Si in MI sumatur $u\xi$ ejus magnitudinis, ut rec-lum $u\xi$.AM æquet aream hyperbolicam ALNM, punctum ξ erit centrum gravitatis curvæ CAC. Sit enim centrum hujus curvæ in aliqua recta $\xi\gamma$, parallela ipsi AB, ponaturque curvam AC libratam pendere ad axem librationis $u2c$; erit (§.44.)

$$\mathrm{AC}.u\xi = \text{omnbus } u\mathrm{I.C}c = \text{omn. BM.M}m = \text{omn.MN.Mm} = \text{areæ ALNM},$$

$$\text{atque AC} = \text{AM ; ergo AC}.u\xi = \text{ALNM},$$

unde cum centrum gravitatis curvæ CAC sit etiam in axe AW*,* erit omnino in puncto ξ.

5°. Sin vero centrum gravitatis curvæ AC fuerit in recta $\delta\gamma$, erit (§.44.) $\mathrm{A}\delta.\mathrm{AC} = \text{omn.CI.C}c$. Atqui omn. $\mathrm{CI.C}c = \mathrm{AC.CI} - \text{omn.AC}.cf = AC.\mathrm{CI} - \text{omn. AM.M}\mu$, & triangula similia ABM & $Mm\mu$ præbent $\mathrm{AM.M}\mu = \mathrm{AB}.m\mu$, atque adeo omn. $\mathrm{AM.M}\mu = \mathrm{AB}.\omega : \mathrm{AB.AI}$; ergo omn. $\mathrm{CI.C}c = \mathrm{AC.CI} - \mathrm{AB.AI} = \mathrm{AM.CI} - \mathrm{AB.AI}$; atque adeo $\mathrm{A}\delta$ vel $\xi\gamma$ in $\mathrm{AC} = \mathrm{AC.\ CI} - \mathrm{AB.\ AI} = \mathrm{AC.CI} - \mathrm{AC.A}\lambda$, ducta scilicet tangente $\omega\lambda$; adeoque $\xi\gamma = \mathrm{CI} - \mathrm{A}\lambda$; adeoque centr. grav. curvæ AC distat a recta $c2c$ intervallo Aλ vel $\omega\lambda$.

5°. Adeoque superfices genita ex convertione curvæ AC circa axem AI æquatur circulo, cujus radius potest duplum rec-li $\xi\gamma$.AC seu $2.\mathrm{AC.CI} - 2\mathrm{AB.AI}$. Nam supra (§.47.) demonstratum est, magnitudinem *genitam* ex rotatione alicujus *genitricis* circa axem positione datum, æquari facto ex magnitudine genitrice in viam centri ejus gravitatis ; & viæ centri gravitatis sunt circumferentiæ circulares a centro descriptæ, atque adeo radiis suis proportionales. Propterea magnitudines genitæ sunt in composita ratione magnitudinum genitricium & distantiarum centri earum gravitatis ab axe rotationis. Idcirco erit superficies ex conversione curvæ AC circa AI ad circulum quemcunque, cujus radius est R, in composita ratione magnitudinis AC supeficiei genitricis ad magnitudinem R circuli genitricem, & distantiæ centri gravitatis γ illius magnitudinis ab axe AI seu $\gamma\xi$ ad distantiam centri gravitatis hujus, seu $\frac{1}{2}\mathrm{R}$, hoc est, sicut $\mathrm{AC}.\gamma\xi$ ad $\mathrm{R}.\frac{1}{2}\mathrm{R}$, id est, sicut $2\mathrm{AC}.\gamma\xi$, vel $2\mathrm{AC.CI} - 2\mathrm{AB.AI}$ ad R^2; adeoque si $\mathrm{R}^2 = 2\mathrm{AC.CI} - 2\mathrm{AB.AI}$, erit superficies genita circulo æqualis.

7°. Eodem argumento circulus, cujus radius R potest duplum areæ hyperbolicæ, AMNL, æquatur superficiei ex conversione AC circa axem $u2c$ genitæ. Nam dicta superficies erit ad circulum, ut $\mathrm{AC}.u\xi$ ad $\frac{1}{2}\mathrm{R}^2$ vel ut $2\mathrm{AC}.u\xi$ ad R^2; atqui (num.4) $\mathrm{AC}.u\xi = \mathrm{AMNI}$, ergo superficies ex curva AC circa $u2c$ est ad circulum radii $\mathrm{R} = \sqrt{2\mathrm{AMNL}}$, ut 2AMNL ad 2AMNL, id est in ratione æqualitatis.

8°. Sin vero ρ centrum fuerit gravitatis areæ uAC2*f,* ducatur $\rho\theta$ parallela AB, eritque (§.44.) $u\theta$ in aream $u\mathrm{AC}2f =$ omnibus momentis rec-lorum $f2c$ ad axem $u2c$ appensorum; hoc est $u\theta.\mathrm{AM.AB} = \text{omnibus } \frac{1}{2}.c2c.c2c.cf$ *;* nam (num.3.) est area $u\mathrm{AC}2f = \mathrm{AM.AB}$. Atqui $\frac{1}{2}.c2c.c2c.cf = \frac{1}{2}\mathrm{BM.BM.M}\mu = \frac{1}{2}\mathrm{AB.BM.M}m = \frac{1}{2}\mathrm{AB.MN}.Mm$, , ergo omni $\frac{1}{2}.c2c.c2c.cf = \frac{1}{2}\text{AB}$ in aream ALNM ; atque adeo $u\theta.\text{AM.AB} = \frac{1}{2}.\text{AB. ALNM}$, vel $2u\theta.\text{AM} = \text{ALNM}$. Sed (num.4.) habuimus etiam $u\xi.\text{AM} = \text{ALNM}$, ergo $u\xi.\text{AM} = 2.u\theta.\text{AM}$, vel $u\xi = 2.u\theta$, adeoque centrum gravitatis curvæ AC duplo longius distat a linea *u*2*c,* quam centrum gravitatis areæ *u*AC*2f.*

9°. Hinc solidum ex rotatione figuræ *u*AC2*f* circa axem *u*2*f*, semissis est cylindri, cujus altitudo AB, & baseos radius potest duplum areæ hyperbolicæ ALNM. Nam hoc solidum (§.47.) æquatur facto ex area *u*AC*2f* in circumferentiam radii $u\theta$ cum ρ sit centrum gravitatis areæ, hoc est $= \text{AB.AM}$ in circumf. radii $u\theta$ (num. 8.) $= \frac{1}{2}\text{AB. AM}$ in circumf. $u\xi$. Atqui AC vel AM in circumf. $u\xi$ est superf. ex AC circa *u*2*c* (num. 7.) $=$ circulo cujus radius $= \sqrt{2\text{ALMN}}$;ergo solidum areæ *u*AC*2f* circa *u*2*f* rotatæ $=$ cylindro, cujus radius potest 2ALMN , & altitudo est $\frac{1}{2}\text{AB}$; adeoque prædictum solidum dimidium est cylindri, cujus altitudo AB baseosque radius potest 2ALNM.

10°. Hinc ultra sequitur, solida a figuris *u*AC2*f*, *u*A*c*2*c* circa *u*2*c* rotatis genita proportionari superficiebus ex curvis homologis AC, Ac circa eundem axem *u*2*c,* ficut areæ *u*AC2*f*, *u*AC*2c* curvis AC, A*c* proportionales sunt; quæ proprietas, & ea qua centrum gravitatis curvæ AC duplo magis a recta *u*2*c* distat quam centrum gravitatis ρ areæ *u*AC2*f*, & ambo hæc centra γ & ρ ab axe AI æqualiter distant, lllustri Leibnitio, qui primus eam advertit, merito memorabilis visa est.

11°. Centra gravitatis ρ & γ areæ *u*AC2*f* & curvæ AC æqualiter distant ab axe velariæ Nam. (§.44) $\theta\rho$ in *u*AC*2f*

$$= \theta\rho.\text{AB.AM} = \text{omnibus CI.}f2f.\,cf = \text{omn. CI.BM.M}\mu = \text{omn. CI.AB.}Mm$$

$$= \text{AB.CI.AM} - \text{omn.AB.AM.}cf = \text{AB.CI.AM} - \text{omn.AB.AM.}m\mu$$

$$= \text{AB.CI.AM} - \text{omn.AB}^2.m\mu = \text{AB.CI.AM} - \text{AB}^2.\text{M}\omega,$$

ergo etiam $\theta\rho.\text{AM} = \text{AM.CI} - \text{AB in M}\omega$. Sed (num.5.) etiam habuimus $\text{AM.}\xi\gamma = \text{AM.CI} - \text{AB in M}\omega$ vel AI , ergo $\theta\rho.\text{AM} = \text{AM.}\xi\gamma$, vel $\theta\rho = \xi\gamma$.

12°. Adeoque solidum ex conversione figuræ *u*AC2*f* circa *u*I æquatur cylindro, cujus radius potest duplum spatium $2\text{CI.AC} - 2\text{AB.AI}$ & altitudo est AB. Hinc iterum, solida ex figuris *u*AC2*f*, *u*A*c2c* circa axem AI rotatis proportionalia sunt superficiebus ex curvis AC, A*c* circa eundem axem in gyrum actis.

13°. Radius circuli osculatoris Cβ curvæ puncto C, est tertius proportionalis ad *u*A & *u*I. Nam, quia B*m* & BM: parallelæ sunt particulis curvæ C*c*, & CG; sectores evanescentes MBμ & Cβ*c* similes erunt , unde $\text{M}\mu : \text{BM} = Cc : \text{C}\beta$ vel permutando $\text{M}\mu : Mm$ vel $\text{C}c = \text{BM} : \text{C}\beta$; atqui $\text{M}\mu : \text{M}m = \text{AB} : \text{BM}$, ergo $\text{AB} : \text{BM} = \text{BM} : \text{C}\beta$, seu, quia $\text{BM} = u\text{I}$, erit $u\text{A} : u\text{I} = u\text{I} : \text{C}\beta$. Hinc, curvæ $\alpha\beta$, ex cujus evolutione velaria describitur, initium α a vertice velariæ *A* æque distat ac punctum *u* in circuli quadrante *u*SL.