PROPOSITION L. PROBLEM.

451. *If some given trilinear figure* ABH*, defined by the curve* AVB*, and by its axis* AH *and ordinate* BH*, may be carried forwards in a fluid in some manner along the direction* SM *from* S *towards* M*, to find the mean direction Xω of the fluid moving [relative] to the curve* AVB*, and the force which the fluid will exert on the trilinear figure along the mean direction.*

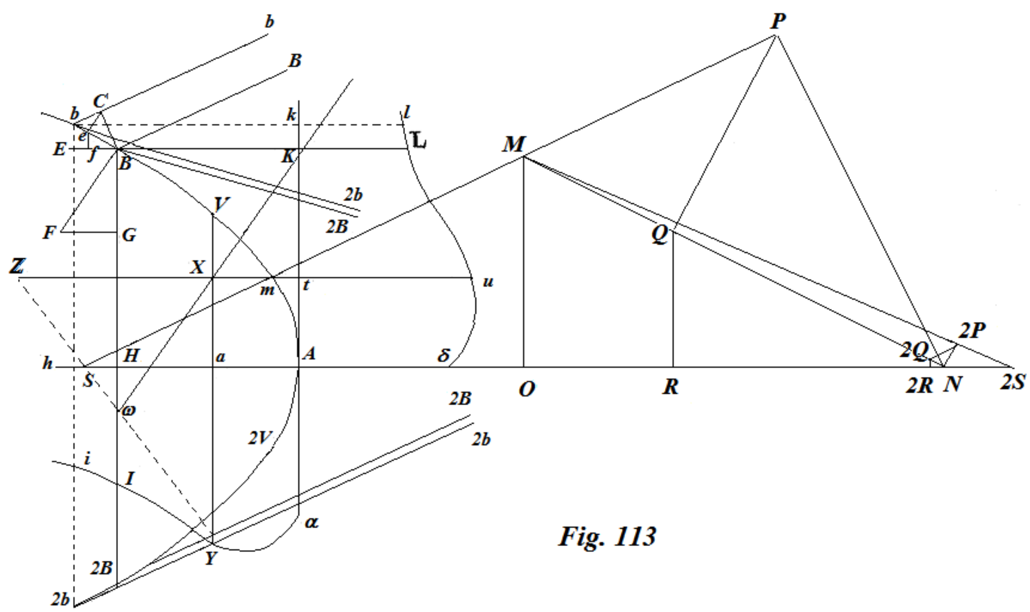


Fig. 113

Geometrical Analysis. I. The right lines BB, *bb* are acting through any [incremental] element B*b* of the curve AVB, parallel to the right line SM itself, and with BC drawn normal to the right line *bb,* and *Ce* drawn normal to the element of the curve B*b* , and the perpendicular MO, the square of the speed, may be put in place perpendicular to the axis of the curve AH , by which the trilinear figure advances in the fluid [along SM], or what amounts to the same, the square of the speed, by which the filament *bBBb* will slide past the element of the curve, and MO.B*e* (§.429.) will express the [size of the] force, which the filament *bb*BB exerts on the element B*b* along the [normal] direction BF ; moreover the components along BG and GF are equivalent to that force itself, of which the latter is parallel to the axis AH, and truly the former is perpendicular to that axis ; and because the forces along BF, BG and GF are as these lines respectively, or on account of the similarity of the triangles BFG and B*ef*, are as B*e*, *Bf* and *ef,* and the force along RS = MO and thus the force along GF = MO.*ef* , and the force along BG = MO.*Bf* . And thus respectively at any other element of the curve.
[Thus, the max. force is normal to the increment BC, and acts along C*b* in the incremental triangle, of which a component of magnitude *e*B.MO acts normal to the

increment B*b* along *e*C; the macroscopic triangle FBG, similar to *ef*B, has FB parallel to *e*C, and the sides of FBG are parallel to the axis of the curve, and so represent the components of the force parallel to the axes; and thus, in modern terms, $\text{GF} = \text{MO}.ef$ and $\text{BG} = \text{MO}.Bf$ are the x and y coordinates of the force on the curve AVB . Note that in Hermann's model, the direction of the force, even of the resolved components of the line increment *e*C, act along directions normal to the line increments. We may also note the lack of appropriate notation at the time, esp. for components of forces integrated over angles ; the function notation had not yet been introduced, so that most of the explanations are given in a heuristic manner, the integrals are represented by areas, etc. It is apparent also, that the problem is not symmetric on interchanging an object moving through a static fluid at a given rate, to the case of a static object subjected to a fluid moving with the same speed in the opposite direction , as the moving body loses energy and thus speed as it proceeds, unless it has some means of propulsion, which has not been indicate, at least at this stage.]

II. Again the right line MN is acting through M parallel to the element of the curve *Bb,* or to the tangent of the curve at B, the perpendicular NP is drawn from N to the line SM, produced as far as needed, PQ is normal to MN*,* and finally QR is acting parallel to MO ; and the figure NPMO will be similar to the figure BC*b*E*,* since the individual sides of the one shall be parallel to the individual sides of the other. Therefore the sides within these two figures similarly will be in proportion; this is, $b\text{E} : ef = \text{MO} : \text{QR}$, and thus $\text{MO}.ef = \text{QR}.b\text{E}$. Or also, with the indefinite right line BL drawn through B parallel to the axis AH, and on that KL taken everywhere to be equal to the respective QR, thus so that thence a new curve δuL may result, there will be $\text{MO}.ef = (\text{QR}.b\text{E}) = \text{KL}.\text{K}k$, or, for the inscribed incremental rectangle of the figure AδLK .

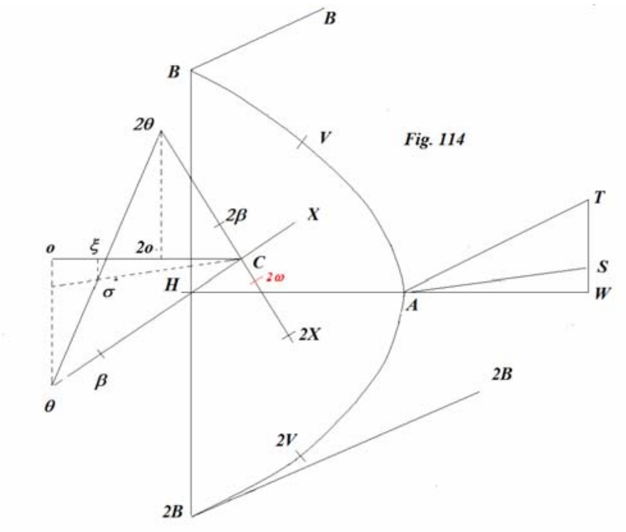
Likewise $\text{EB} : f\text{B} \ (= \text{ON} : \text{RN}) = \text{MO} : \text{QR}$, and consequently $\text{MO}.f\text{B} = \text{QR}.\text{EB}$, or also = to the inscribed incremental rectangle IH*h* of the area $\text{A}\alpha\text{IH}$, if, also at the individual ordinates BH produced beyond the axis AH, the segments HI will be taken equal to the homologous segments QR or KL.
[Again, $\text{MO}.ef = (\text{QR}.b\text{E}) = \text{KL}.\text{K}k$ and $\text{MO}.\text{B}f = \text{IH}h$ are the x and y coordinates of the force on the curve AVB .]

III. Therefore because the individual forces of the fluid acting on the curve AVB along directions perpendicular to the curve are equivalent to the sides [*i.e.* components] expressed by the elemental rectangles MO.*ef* (by no. I. of this), or by the elemental rectangles LK*k* (by no. II.) both parallel to the axis of the curve ; and with the perpendiculars to the axis of the curve expressed by the rectangle MO.B*f* (no. I.), or by the rectangles IH*h* (no. II.) : the force, which the whole curve AVB will be subjected to by the fluid, will be equivalent to all the LK*k* and to all the IH*h* [added together: *i.e.* the associated integrals]; and the mean direction *Zu* of all the forces *k*KL likewise parallel to the axis, (§ 56) will pass through the centre of gravity of the area AδLK [*i.e.* essentially the mean-value of the integral for the *x*-component, and likewise below for the *y*-components], which contains all the LK*k* ; and the mean direction VY of all the forces

IH*h* of the perpendiculars to AH, of the normal to the axis AH, and parallel to the ordinate HI of the curve αIY , will pass through the centre of gravity of the figure AαIH, which contains all the IH*h*. Therefore the force, which the fluid exerts on the curve AVB*,* gives rise to the same effect, as which the force substituted by the rectangle MO.XZ performs, equal to the area AαIH, acting on the curve along the direction XZ, likewise with the force MO.XY, equal to the area AαIH acting on the curve AVB along the direction XY*;* since these forces MO.XZ and MO.XY are equivalent to all the forces LK*k* and to all the forces IH*h*, and they are understood to be applied in the mean directions XZ and XY of all these acting on the curve AVB. But (§. 49) the mean direction of the forces MO.XZ and MO.XY is Xω, passing through the centre of gravity ω, of the points Z and Y, by which the right lines XZ and XY are terminated, and the [equivalent] weight of which points is displayed by the given MO, and the force, which the fluid hence exerts on the curve along this mean direction X*ω*, may be set out by the rectangle 2MO. X*ω*.

IV. But if indeed the curve A2V2B shall be put in place under the axis AH, which shall be carried along parallel in the same direction SM as in the preceding case, [*i.e.* the impeded curve now has its vertex at A and two parts] OS shall be required to be transferred to the opposite side O2S*,* and by drawing M2S, as from the point N the perpendicular N2P must be sent to M2S from the point N, and 2P2Q above MN, and finally 2Q2R parallel to MO; clearly with MN put parallel to the tangent of the curve A2V2B, translated to the other side of the axis AH at the point 2B, 2Q2R will be the common ordinate of the curves *δu*L and *α*YI, but pertaining to the lower curve 2V2B, which thereafter, for the sake of avoiding confusion, we will designate by 2*α*2*u*2L and 2A2*α*2Y2I , though they are not expressed on the diagram, because they are easily kept in mind. Therefore the force of the fluid on the lower curve A2V2B, will be along its mean direction 2X2*ω,*[not indicated on the diagrams] being expressed by 2MO.2X2*ω,* in short on account of the same reasoning, therefore as the force on the curve AVB in the mean direction X*ω* has been shown to be expressed by the rectangle 2MO by X*ω*. Because truly in place of SM now on the curve A2V2B there is required thence to take 2SM, because by the rotation of the figure A2V2B with the line 2B2B itself parallel to BB (following the hypothesis) about the axis AH, thus so that if the curve A2V2B were similar and equal to the other AVB, since it may be congruent with that, the lines 2B2B arise in the place B2B under the right line BL, or making equal angles with the angles BBL, which the lines BB extended above BL contain here with BL, thus so that any homologous angles BBL and LB2B or the angles MSO and M2SO are going to be equal.

V. If now the bilinear form BVA2V2B may advance in the fluid along the direction of BB, 2B2B, see Fig. 114,



or parallel to AT , and the mean direction of the force of the fluid Cσ is sought on the curve BA2B, thus the matter can be resolved with the aid of numbers four and three of this proposition. Also the mean direction Xβ will be had from number three, and the force of the fluid on the curve AVB, which may be expressed there by 2.MO.Xα. Therefore there will be $X\beta = 2M\omega$ in Fig.113 and MO.Xβ will be the force of the fluid, along the mean direction on the curve AVB, with MO designating everywhere the square of the velocity, with which speed it will be evident the fluid slides past the curve. Similarly, following number four of this, the mean direction 2X2β can be found and the force of the fluid, along this mean direction 2 in 2X2ω.Mo, or MO.2X2β (if 2X2β shall be twice 2X2ω). From the point of intersection C of the two Xβ and 2X2β and on these Cθ & C2θ may be taken equal, evidently $C\theta = X\beta$ and $C2\theta = 2X2\beta$, and $\theta 2\theta$ is joined, and thus by §.49 the mean direction of the forces MO.Cθ & MO.C2θ, which are equivalent to the forces of the fluid, which both curves AVB and A2V2B undergo, will pass through the centre of gravity of the points θ & 2θ, that is, through the middle point σ of the line $\theta 2\theta$, and through the point C and thus this mean direction is $C\sigma$, and the forces or the force of the fluid on the general curve BVA2V2B will be 2.MO.$C\sigma$. All of which were required to be found.

Otherwise and shorter.

Number three of this proposition, and consequently which follow that, all can be deduced as if from a corollary from proposition VIII of the first book, §.59. For the similar figures NPMO and BC*b*E present $bB : eB (= MN : QN) = MO : QR$ and thus $MO.Be = Bb.QR$.

And MO.B*e* (number I of this prop.) expresses the force of the filament of the fluid *bbBB* on the element of the curve *Bb,* therefore this force can also be expressed by B*b*.QR, and thus the fluid exerts the same force on the element *Bb* , as if the individual points of the element were being acted on by forces QR perpendicular to the same, thus so that the whole matter may be reduced to the Coroll 1. of the aforesaid Prop. 8. Book I. Therefore the constructions can be re-read merely to be of this proposition and its corollary, and in place of the curves ABB, XF & XD in figure 15, AVB, αYI and δuL of figure 113 are required to be substituted, and finally, in place of the forces BG in that, the forces QR are required to be substituted in this figure, and we may arrive at the same final conclusions , which we have elicited in the preceding numbers of the present proposition.

COROLLARY I.

452. Therefore, with the right line C*o* drawn through the point C parallel to the axis AH, the whole sine may be found to the tangent of the angle $oC\sigma$, which the mean direction $C\sigma$ of the bilinear figure BA2B contains with the axis AH, or the line *Co* contains parallel to this axis, as the sum of the areas AδLK and 2A2δ2L2K to the difference of the areas AαH and 2A2α2I2H. For with the perpendiculars $\theta o, \sigma\xi, 2\theta 2o$ sent from the points $\theta, \sigma, 2\theta$ to *Co,* and because σ is the middle point of the line $\theta 2\theta$, or the centre of gravity of the points $\theta, 2\theta$, there will be (§. 46.) $2.C\xi = Co + C2o$, and $2.\xi\sigma = \theta o - 2\theta 2o$; and thus $C\xi : \xi\sigma$, or the whole sine to the tangent of the angle [*i.e.* 1 on tangent or cotangent, giving the direction of the resultant impressed force.]

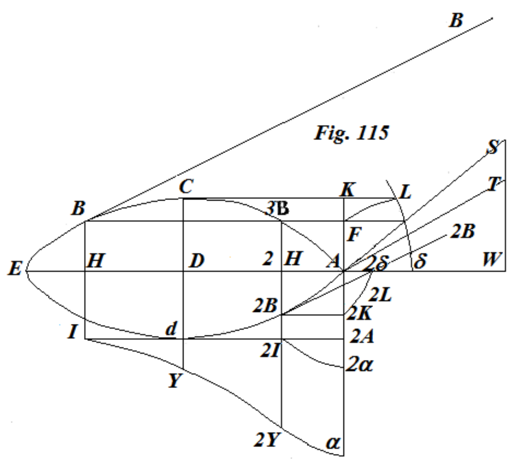
$$\xi C\sigma = Co + C2o : \theta o - 2\theta 2o = MO.Co + MO.C2o : MO.\theta o - MO.2\theta 2o$$
$$= A\delta LK + 2A2\delta 2L2K \text{ to } A\alpha IH - 2A2\alpha 2I2H.$$

For (by constr.) $MO.Co;\ MO.C2o;\ M0.\theta o\ \&\ MO.2\theta 2o$,
are equal to the areas
$A\delta LK,\ 2A2\delta 2L2K;\ A\alpha IH\ \&\ 2A2\alpha 2I2H$.

COROLLARY II.

453. See Fig. 115. Truly if the bilinear figure were ABE2BA, of which the parts shall be unequal, the upper ACB and the lower A2B*,* and the arc of the highest point C of the upper curve ACB were not to be situated at the end B of the arc [as originally], the whole sine will be to the tangent of the angle, which the mean direction of the force of the fluid on the bilinear figure B2B contains with the axis AE, as

$$A\delta LK + A2\delta 2L2K - FKL$$
$$\text{to } A\alpha YI - A2\alpha 2Y2I.$$



Where δL is the graph of the forces of the fluid parallel to the axis AE of the bilinear figure , which forces are derived from the perpendiculars to the curve AC , and FL is the graph of the forces also parallel to the same axis, but of those derived from the perpendiculars to the arc of the curve BC, and because these forces are parallel to the axis, by which the arc BC is affected, they are directly opposite to the parallels of the axis AE, which act on AC , thus the difference of the areas AδLK and FLK is required to be taken in order to have the expression of the forces parallel to the axis AE , by which the whole curve ACB extending above the axis is affected. Truly the curves αYI, 2α2Y2I for the axis IA, the curves of the forces normal to the axis AE are constructed from the perpendiculars of the curves ACB and A2B from the derived forces, and finally 2δ2L2K is the graph of the forces parallel to the axis pertaining to the curve A2B. The ordinates of the curves αYI and 2α2Y2I are zero at the points I and 2I on the axis, because the lines BB and 2B2B, which indicate the direction of the fluid sliding past the bilinear form, touching at the points B and 2B. On account of the same reason also the ordinates of the curve FL at the point F is zero, at which clearly it crosses the right line BF parallel to the other axis AK.

COROLLARY III.

See Fig. 113.

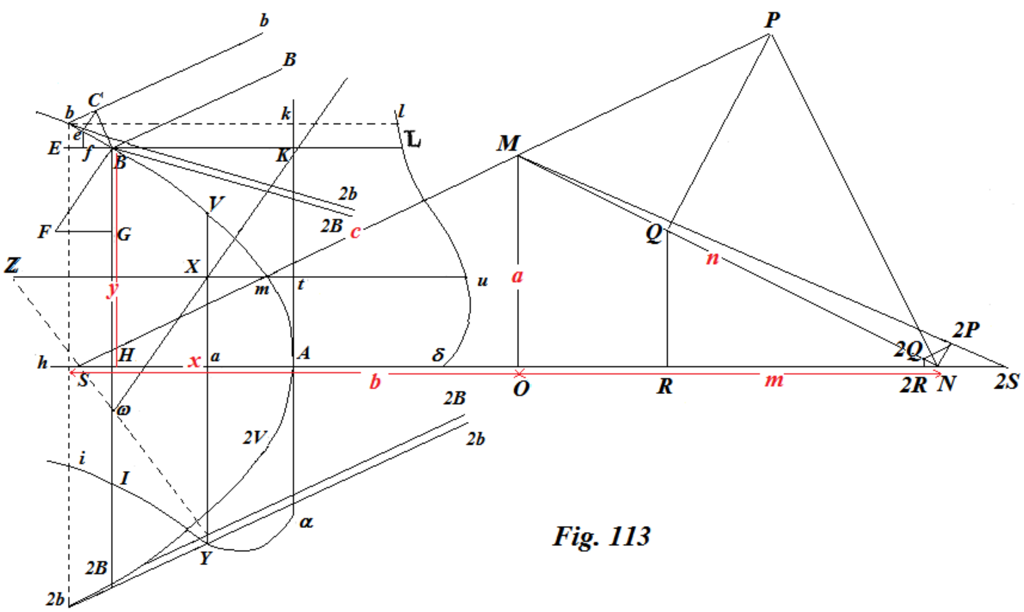


Fig. 113

454. If now MO may be called *a;* OS, *b*; MS, *c* ; the coordinates AH, *x*;
BH, *y*, & ON, *m* ; and MN, *n* : there will be $\mathrm{SN} = b + m$, or N2S with respect to the lower curve $= b - m$. Hence from the similarity of the triangles SMO, SNP there will be produced PN or $\mathrm{N2P} = (ab \pm am) : c$, where the upper sign shall be with respect to the upper curve AVB, truly the lower sign to the lower curve.

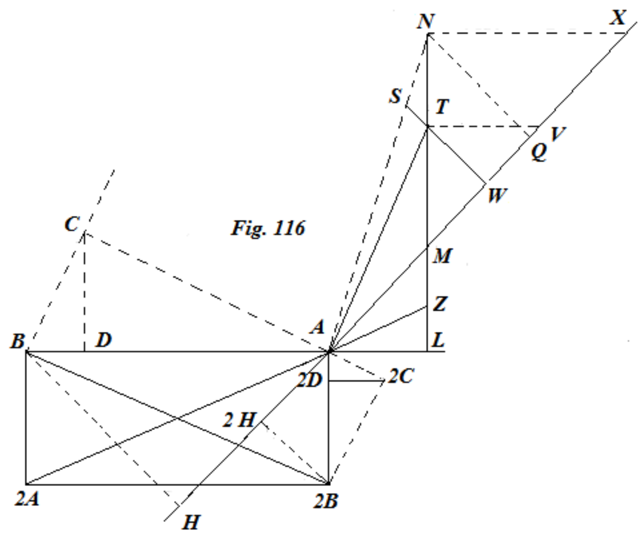
Again there will be found $\mathrm{MO} : \mathrm{QR}\ (= \mathrm{MN} : \mathrm{Q.N}) = \mathrm{MN}^2 : \mathrm{PN}^2$; and thus

$\mathrm{QR} = \mathrm{MO.\ PN}^2 : \mathrm{MN}^2 = (b \pm m)^2 .a^3 : aacc + ccmm$, or by assuming the magnitude *e*, and on putting *cce* equal to a^3, QR

$= (b \pm m)^2 .e : aa + mm = (bbe \pm 2bem + emm) : aa + mm = e, \pm 2bem : nn; + (bb - aa).e : nn,$

or on putting $bb - aa = ff$, the ordinate of the graph of the resistances will be found

finally, or of the common force of the fluid :
$\mathrm{QR} = \mathrm{KL} = \mathrm{Hl} = e\ ;\pm 2bem : nn;+eff$:
See Fig. 116 . Now with the given equation of the enduring curve AVB or A2V2B, and with that differentiated, if in place of the elements *dy* and *dx* the ordinates and abscissas of these proportionals to these *a* and *m* may be substituted, or MO and ON, an equation will be found expressed by the indeterminate quantities *x, y* and *m,* and by the quantities or constants given, from which the value of *m* at *x* & *y* will be elicited and by its constants, in which with the aid of the equation of the enduring curve AVB or A2V2B either from the indeterminates *x* or *y* can be eliminated always . Truly with the value of QR found in terms of *y* with constants, the areas AδLK, 2A2δ2L2K ; and AαIH, 2A2α2I2H will be found, if not algebraically then at least by a series transcendently, or also by approximations. But one more example ought to show the general use of these.



EXAMPLE I.

455. See Fig. 116 . Let the rectangle BA2B be proceeding along the direction AT in a resisting fluid, the mean direction ST is sought of the force of this fluid sliding past the rectangle. The right line AH shall bisect angle BA2B, and this line AH will be produced towards W to represent the axis of the lines AB and A2B, of which the latter represents the lower enduring curve, and truly the former the superior ; and with the perpendiculars BH, 2B2H and with TW cutting AN at the point S, sent from B, 2B and from the point T on the right line AT, taken as it pleases, there will be AH = BH and A2H = 2B2H, on account of the semi-right angles BAH and 2B2A2H. In addition there shall be AH = BH and A2H = 2B2H = x, and BH = Y, 2B2H = y , X = Y, and $x = y$; hence from that in each case there will be $a = m$, and thus $nn(= aa + mm) = 2aa$. Therefore with these determined values substituted into the formula of QR, the value of this line AB applicable for this case will be found
$\mathrm{QR}(= e,+2bem : nn;+eff : nn) = e;+be : a + eff : 2aa$, (or by restoring $a^3 : cc$, and $bb - aa$ the values of the quantities e and $ff) = (a^3 + 2aab + abb) : 2cc$. And thus the element of the area AδLK or the elemental rectangle $\mathrm{L}k = (aa + 2ab + bb).ad\mathrm{Y} : 2cc$, and the area itself

$=(aa+2ab+bb)a.\mathrm{AH}:2acc$. Similarly the area $2\mathrm{A}2\delta 2\mathrm{L}2.\mathrm{K}$ will be
$=(aa-2ab+bb).a.\mathrm{A2H}:2cc$. The area $\mathrm{A}\alpha\mathrm{IH}$ for the line
$\mathrm{AH}=(aa+2ab+bb).a.\mathrm{AH}:2cc$; and the area $2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H}=(aa-2ab+bb).a.\mathrm{A2H}:2cc$.
Hence (§.452) AW: SW or the whole sine to the tangent of the angle [i.e. the inverse tangent or cotangent, *dx:dy*] $\mathrm{SAW}=\mathrm{A}\delta\mathrm{LK}+2\mathrm{A}2\delta 2i2\mathrm{K}:\mathrm{A}\alpha i\mathrm{H}-2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H}$, and thus analytically there will be

$$\mathrm{AW{:}SW}=(aa+2ab+bb).a.\mathrm{AH},+(aa-2ab+bb.)a.\mathrm{A2H}\text{ to}$$
$$(aa+2ab+bb\ ).a.\mathrm{AH},-(aa-2ab+bb).a.\mathrm{A2H}=(2ab+cc).\mathrm{AH},+(cc+2ab).\mathrm{A2H}$$
$$\text{to }(cc+2ab).\mathrm{AH},-(cc-2ab).\mathrm{A2H}.$$

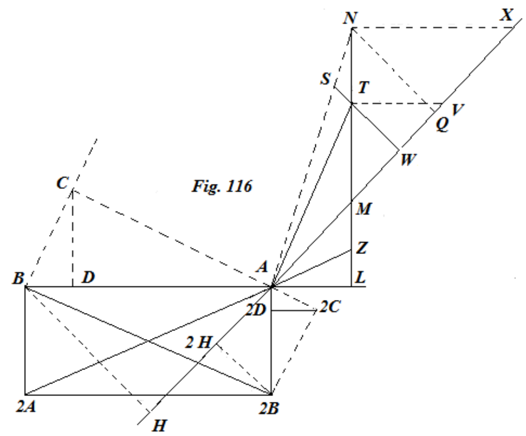
TV may be drawn parallel to AL, and on account of the semi-right angles MTW and WTV, there will be :
$\mathrm{MW}=\mathrm{TW}=\mathrm{WV}$, and thus since AW (§.454) may be called *b* and TW, *a;* there will be $a+b=\mathrm{AV}$, or $aa+2ab+bb=\mathrm{AV}^2=cc+2ab, \& \ aa-2ab+bb=cc-2ab=\mathrm{AM}^2$; hence

$$\mathrm{AW}:\mathrm{SW}=\mathrm{AV}^2.\mathrm{AH}+\mathrm{AM}^2.\mathrm{A2H}:\mathrm{AV}^2.\mathrm{AH}-\mathrm{AM}^2.\mathrm{A2H}$$
$$=\mathrm{TL}^2.\mathrm{AH}+\mathrm{ML}^2.\mathrm{A2H}:\mathrm{TL}^2.\mathrm{AH}-\mathrm{ML}^2.\mathrm{A2H}=\mathrm{R.T}^2+\mathrm{R}^2t:\mathrm{R.T}^2-\mathrm{R}^2.t$$
$$=\mathrm{T}^2+\mathrm{Rt}:\mathrm{T}^2-\mathrm{R}t.$$

Putting $\mathrm{AL}:\mathrm{TL}=\mathrm{R}:\mathrm{T}$ and $\mathrm{AH}:\mathrm{A2H}=\mathrm{R}:t$; thus so that T and *t* shall be the tangents of the given angles TAL & AB.2B, evidently with the diameter drawn to the rectangle B2B, with the whole sine or radius R. Thus NX may be drawn parallel to TV or AL, and also MQ, NQ, and QX will be equal ; since now $\mathrm{AW}:\mathrm{SW}=\mathrm{AQ}:\mathrm{NQ}$ or QX, also it will be $=\mathrm{AX}+\mathrm{AM}:\mathrm{MX}=\mathrm{LN}+\mathrm{LM}:\mathrm{LN}-\mathrm{LM}$, and thus,
$\mathrm{AW}+\mathrm{SW}:\mathrm{AW}-\mathrm{SW}=2\mathrm{LN}:2\mathrm{LM}=\mathrm{LN}:\mathrm{AL}$ or LM . Truly, because a little before there was found $\mathrm{AW}:\mathrm{SW}=\mathrm{T}^2+\mathrm{R}t:\mathrm{T}^2-\mathrm{R}t$, also there will be
$\mathrm{AW}+\mathrm{SW}:\mathrm{AW}-\mathrm{SW}=\mathrm{T}^2{:}\mathrm{R}t$; therefore also $\mathrm{LN}:\mathrm{AL}=\mathrm{T}^2:\mathrm{R}t$; but there is
$\mathrm{AL}:\mathrm{LT}=\mathrm{R}:\mathrm{T}=\mathrm{Rt}:\mathrm{Tt}$; therefore from the equation $\mathrm{LN}:\ \mathrm{LT}=\mathrm{T}^2:\mathrm{T}t=\mathrm{T}:t$. The diagonal 2AA of the rectangle may be drawn , and that may be produced to Z, and with the radius AL present, or with the whole sine, TL and ZL will be the tangents of the angles TAL and ZAL or A2A2B , which were being called the tangents T, & *t*; and thus $\mathrm{LN}:\mathrm{LT}=\mathrm{LT}:\mathrm{LZ}$; therefore NL, TL and ZL are in continued proportion, as consequently NL to TL shall be as the square root ratio of NL to ZL, but this ratio NL to ZL is composed from the ratios NL to AL, and AL to ZL, that is, 2A2B to A2B; therefore the ratio NL to TL is as the square root, which is composed from NL to AL and from the length of the rectangle AB to the width A2B ; and this is the same thing, as the Celeb. Jacob Bernoulli had shown without giving any analysis in the Act. Erud. Lips. 1696, page 336, and he had arrived at that doubtlessly in a much shorter way. For this particular case cannot be shorter than done by us here in accordance with our general method, yet I am aware by the consideration of other matters I have not examined, I shall

illustrate how such a case may be treated further, from which its truth may be confirmed otherwise.

456. The present example thus will be set out shorter also: the right lines BC, 2B2C drawn through the right angles B and 2B are parallel to AT, and C2C shall be drawn through the point A perpendicular to the same, and finally from the points C, 2C the perpendiculars CD and 2C2D may be sent to the sides of the rectangle, and with unity assumed for the designated speed, with which the rectangle proceeds through the fluid, or what amounts to the same, for the velocity, with which the fluid moves past the sides of the rectangle AB, A2B , and (§. 429.) A2D expresses the force if the fluid, which the side A2B receives in the direction of the normal itself, and AD expresses the force, which the side AB receives along the direction parallel to the side A2B, or perpendicular to the side AB. From which since the mean direction AN of the force of the fluid on the sides AB and A2B arises from the forces on the sides AL and LN, it is necessary that there shall be, $\mathrm{AL:LN = A2D:AD}$; and



$\mathrm{A2D:A2B = (A2B)^2:(A2C)^2 = AL^2:AT^2}$, & $\mathrm{A2B:2A2B}$ or $\mathrm{AB = ZL:AL = AL.\ ZL:AL^2}$ therefore from the equation $\mathrm{A2D:AB = AL.\ ZL:AT^2}$; and again there shall be $\mathrm{AB:AD = AB^2:AC^2 = AT^2:TL^2}$, and thus again from the equation $\mathrm{A2D:AD = AL.\ ZL:TL^2}$. And (following the hypothesis) $\mathrm{AL:LN = A2D:AD}$, therefore $\mathrm{AL:LN = AL.ZL:TL^2}$; and because $\mathrm{TL:AL = TL^2:AL.TL}$, and finally from the equation $\mathrm{TL:NL = AL.ZL:AL.TL = ZL:TL}$, and thus TL is the mean proportional between the tangents of the given angles ZAL or A2A2B & NAL , as we have found above. Q.E.D.

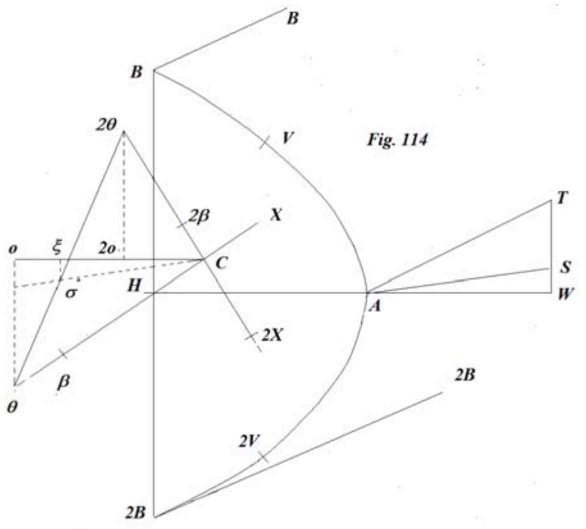
Fig.113 & 114.

EXAMPLE II.

457. The bilinear form BA2B may be resumed; which shall be put to be a parabola, the equation of which shall be $2ax = yy$; with the coordinates present for the axis AH, x ; and HB or 2H2B, y ; and with the parameter $2a$. The differential equation of the bilinear figure produces $adx = ydy$, or with the proportional lines m and a substituted in place of the elements dx & dy ; $am = ay$ and thus $m = y$; hence $nn = aa + yy$. Therefore

$$\mathrm{QR = KL = HI}(= e; \pm 2bm : nn; + eff : nn) = e; \pm 2bey : aa + yy; + effaa + yy.$$

Hence (§. 454) the elemental rectangle $Lk = edy;\ \pm 2beydy : aa + yy;\ + effdy : aa + yy$. And all the *edy* of the sum $edy = ey$*;* and all the $ydy : aa + yy$ is equal to the logarithm of the ratio $aa + yy$ to aa which may be called the logarithm of *z:a* ; and with regard to the logarithm, of which the subtangent is unity. And finally all the $dy : aa + yy$ are equal to the arc of a circle ω applied to the square of the radius aa , of which arc the tangent shall be y ; for the arc of the element of this kind, or $d\omega = aady : aa + yy.$ Therefore the area will be found $\mathrm{A}\delta\mathrm{LK}(2\mathrm{A}2\delta 2\mathrm{L}2\mathrm{K}) = ey;\pm 2bez : a\ ;+ e$; and thus by adding: $\mathrm{A}\delta\mathrm{LK} + 2\mathrm{A}2\delta 2\mathrm{L}2\mathrm{K}; = 2ey;+2.eff\,\omega : aa$ (or in place of ff , by restoring its value $bb - aa$)



$= 2ey;+2bbe\omega : aa;-\ 2e\omega = 2eg;+2bbe\omega : aa$; on putting $g = y - \omega$ for brevity.
The element of the area $\mathrm{A}\alpha\mathrm{IH}\left(2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H}\right)$ is

$= edx;\pm 2beydx : aa + yy;\ \ effdx : aa + yy = eydy : a;\pm\ 2bey^2dy : a^3 + ayy;+effydy : a^3 + ayy;$
because $ydy = adx$. and $yydy : aa + yy = dy,-aady : aa + yy = dy - d\omega$, and as before, $ydy : aa + yy = dz : a$, and therefore.
$\mathrm{A}\alpha\mathrm{IH}(2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H}) = eyy : 2a;\pm(2bey - 2be\omega) : a;+effz : aa$; and thus
$\mathrm{A}\alpha\mathrm{IH} - 2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H} = (4by - 3be\omega) : a = 4beg : a$, clearly on putting $y - \omega = g$.

The right lines AT parallel to BB and AS parallel to Cσ are drawn through A, and there will be, see Fig. 114,
$\mathrm{AW} : \mathrm{WS} = \mathrm{C}\xi : \xi\sigma = \mathrm{A}\delta\mathrm{LK} + 2\mathrm{A}2\delta 2\mathrm{L}2\mathrm{K} : \mathrm{A}\alpha\mathrm{IH} - 2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H}$
$= (2eg;+2bbe\omega : aa) : \left(4beg : a\right) = aag +\ bb\omega : 2abg;\&\ \mathrm{TW} : \mathrm{AW} = a : b = 2abg : 2bbg;$
therefore from the equation there becomes $\mathrm{TW} : \mathrm{WS} = aag + bb\omega : 2bbg.$ In addition there may be called TW, θ & SW, t ; and the radius AW, *r;* and there will be $\theta : t = aag + bb\omega : 2bbg$, and because $r : \theta = b : a$, with the proportionals $r \,\&\, \theta$ substituted in place of b & a , there arises $\theta : t = \theta\theta\mathrm{g} + rr\omega : 2rrg$ *;* from which $\theta\theta = \frac{2rr\theta}{t} - \frac{rr\omega}{g}$ is elicited; and therefore the roots of this equation will be

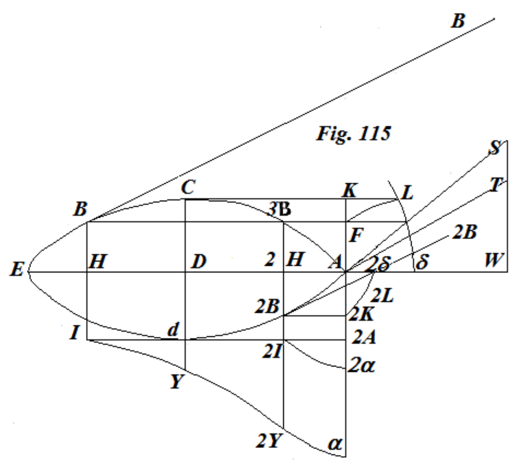
$$\theta = \frac{rr}{t} \pm \sqrt{\left(\frac{r^4}{tt} - \frac{rr\omega}{g}\right)}$$

Now because the individual *r, g,* ω & *t* are given, also θ will be given, which is the tangent of the angle TAW, which contains the direction of the figure BA2B advancing in the fluid with the axis AH, with the figure continually being carried forwards without rotating about itself ; which angle TAW agrees with the angle of declination of the figure, and its tangent, with that declination ATW ; truly the line σC is the direction, along which the figure thus must be impelled in the fluid, in order to travel parallel to the directions BB, AT, themselves, etc. or it may be able to be carried in the fluid by that remaining way ; which all will be explained in their place more fully.

The remaining examples of these cases, are those in which the curves, above AB and below 2A2B, are similar and equal, thus so that these curves exposed to the forces of the fluid may have a common abscissa AH , and they may have equal ordinates BH and 2B2H. But these avoid a more complicated account of the calculation, if the curves AB and A2B, as in fig. 115, are unequal and yet similar, or their sums ACE, and A2BE are equal, then also the position of the points B and 2B, and thus of the abscissas AH, A2H, and of the ordinates BH & 2BEH depend on the magnitude from the position of the right line AT, or on the angle sought TAW. Truly the manner in which the calculation must be set out for these cases, will be elucidated from the following corollary.

COROLLARY IV.

458. For the curves AC, A2B and CB there shall be $\int mdy : nn = \text{A}1 : a;\ \text{A}2 : a;$ or $\text{A}3 : a$, likewise $\int dy : nn = \text{B}1 : aa, \text{B}2 : aa;$ or $\text{B}3 : aa$, with respect to the same curves AC, A2B, & CB. $\int mdx = nn = \text{C}1 : a,\ \text{C}2 : a,$ or $\text{C}3 : a$; with respect of the same curves, and finally $\int dx : nn = \text{D}1 : aa,\ \text{D}2 : aa,$ or $\text{D}3 : aa$. Where the numbers put after the letters A, B, C, D are not the indices of the powers, but only indicate the order for the curves AC, A2B & BC in which they are enumerated, with respect to Fig. 115. From which , because in Fig. 113, QR, KL, or HI



$= e;\ \pm 2bem : nn\ ;+eff : nn = e;\pm 2bem : nn;+\left(bb-ad\right).e : nn;$ because $ff = bb-aa$, as we have found above (§.545) . Hence the area

$\text{A}\delta\text{LK} = e.\text{CD};+2be.\text{A}1 : a;+\left(bb-aa\right).e\text{B}1 : aa$, with regard to the curve AC.

The area $2\text{A}2\delta 2\text{L}2\text{K} = e.2\text{B}2\text{H};-2be.\text{A}2 : a;+\left(bb-aa\right).e\text{B}2 : aa,$ with regard to the curve A2B. The area $\text{FKL} = e.\text{KF};-2be.\text{A}3 : a;+\left(bb-aa\right).eB3 : aa$,with regard to the curve CB. The area $\text{A}\alpha\text{Y}d = e.\ \text{AD};+2be.\text{C}1 : a;+\left(bb-aa\right).e\text{D}1 : aa$, with regard to the curve AC. The area $2\text{A}2\alpha 2\text{Y}2\text{I} = e.\text{A}2\text{H};-2be.\text{C}2 : a,+\left(bb-aa\right).eD2 : aa;$ with regard to the curve A2B, and the area $d\text{YI} = e.\text{DH};-2be.\text{C}3 : a;+\left(bb-aa\right).e\text{D}3 : aa$; with regard to the curve BC. Therefore

$$\mathrm{A}\delta\mathrm{LK}+2\mathrm{A}2\delta2\mathrm{L}2\mathrm{K}-\mathrm{FLK}=e.\ \mathrm{F}2\mathrm{K};+2be.\left(\mathrm{A}1-\mathrm{A}2+\mathrm{A}3\right):a;$$
$$+bb-aa.\left(\mathrm{B}1+\mathrm{B}2-\mathrm{B}3\right)e:aa,\ \ \&\ A\alpha Yd+\mathrm{Y}d\mathrm{I}-2\mathrm{A}2\alpha2\mathrm{Y}2\mathrm{I}=e.\mathrm{H}2\mathrm{H};$$
$$+2be.(\mathrm{C}1+\mathrm{C}1-\mathrm{C}3):a;+bb-aa.\left(\mathrm{D}1-\mathrm{D}2+\mathrm{D}3\right)e:aa.$$

And (§. 453) there is
$\mathrm{AW}:\mathrm{WS}=\mathrm{A}\delta\mathrm{LK}+2\mathrm{A}2\delta2\mathrm{L}2\mathrm{K}-\mathrm{FLK}:\mathrm{A}\alpha\mathrm{Y}d+\mathrm{Y}d\mathrm{I}-2\mathrm{A}2\alpha2\mathrm{Y}2\mathrm{I},$
therefore also

$$\mathrm{AW}:\mathrm{WS}=e.\mathrm{F}2\mathrm{K};+2be.(\mathrm{A}1-\mathrm{A}2+\mathrm{A}3):a;+bb-aa.(\mathrm{B}1+\mathrm{B}2-\mathrm{B}3)e:aa,\ \text{ad}$$
$$e.\mathrm{H}2\mathrm{H};+2be(\mathrm{C}1+\mathrm{C}2-\mathrm{C}3):a;+bb-aa.\left(\mathrm{D}1-\mathrm{D}2+\mathrm{D}3\right)e:aa.$$

From which, since TW shall be to $\mathrm{AW}=a:b$; it will be found from the analogous equation, which makes available the equation sought of the tangent of the angle TAW going to shown ; but which themselves A2H, 2H2B, EH, HB and as consequently these too HD and 2HD depend on the quantities *a* and *b,* of which this at this point is unknown ; therefore these also A2, A3, B2, B3, C2 ,C3; & D2 & D3 depend on the same; which renders the calculation intricate; especially if the curves ACE and A2BE were dissimilar or different.

459. Truly we may consider the named curves ACE and A2BE to be similar and equal, from which it arises that the arcs EB and A2B shall be equal, since the lines BB and 2B2B, parallel to AT itself (following the hypothesis), may touch the arcs at B and 2B, therefore there will be $\mathrm{EH}=\mathrm{A}2\mathrm{H}$; likewise $\mathrm{BH}=2\mathrm{B}2\mathrm{H}$, and thus $\mathrm{HD}=\mathrm{D}2\mathrm{H}$, also 2B2H produced as far as to the crossing of that curve ECA at 3B, making $2\mathrm{H}3\mathrm{B}=2\mathrm{B}2\mathrm{H}=\mathrm{BH}$, and as a consequence the line BF, parallel to EA itself , will cut the curve ECA at the point 3B, thus so that the arcs B and C3B are going to be similar and equal. With these in place there will be found :

$$\mathrm{A}1=\mathrm{A}2+\mathrm{A}3\ ;\ \mathrm{B}1=\mathrm{B}2+\mathrm{B}3;\ \mathrm{C}1=\mathrm{C}2+\mathrm{C}3\ \ \&\ \ \mathrm{D}1=\mathrm{D}1+\mathrm{D}3;$$

and hence in this case we will have :

$$\mathrm{A}1-\mathrm{A}2+\mathrm{A}3=2.\mathrm{A}3\ \text{or twice A3.}$$
$$\mathrm{B}1+\mathrm{B}2-\mathrm{B}3=2.\mathrm{B}2.$$
$$\mathrm{C}1+\mathrm{C}2-\mathrm{C}3=2.\mathrm{C}2.$$
$$\mathrm{D}1-\mathrm{D}2+\mathrm{D}3=2.\mathrm{D}3.$$

And thus theses values, in analogy with these substituted in the preceding paragraph, will produce

$$\mathrm{AW}:\mathrm{WS}=e.\mathrm{F}2\mathrm{K}+4be.\mathrm{A}3:a;+\ (2bbe-2a\text{æ}\ ).\mathrm{B}2:a^2,$$
$$\text{to } e.\mathrm{H}2\mathrm{H};+\ 4be.\mathrm{C}2:a;+\ (2bbe-2a\text{æ}):\mathrm{D}3:aa,$$

or on dividing the terms by 2*e,* as it were :

$$\mathrm{FA},+\ 2b\mathrm{A}3:a;+\left(bb-aa\right).\mathrm{B}2:aa;\ \text{to DH};+2bC2:a,+\left(bb-aa\right).\mathrm{D}3:aa;$$
$$\&\ \mathrm{WT}:\mathrm{AW}=a:b\ ;$$

therefore from the equation, there will be

$$\text{WT} : \text{WS} = a.\text{FA} + \ 2b.\text{A3} \ ;+(bb-aa).\text{B2} : a,$$
$$\text{to } b.\text{DH};+2bb.\text{C2} : a;+\left(b^3-aab\right).\text{D3} : aa;$$

or by calling the tangent of the give angle SAW, *t*; and the tangent of the angle sought TAW, θ ; we will have:

$$\theta : t = a^3.\text{FA} + 2aab.\ \text{A3} \ ;+\left(abb-a^3\right).\text{B2} : aab.\text{DH},+2aab\text{C2},+\left(b^3-aab\right)\text{D3} \ ;$$

and thus, by multiplying the extremes and the means, we will have the following general equation :

$$a^3t.\text{FA} + 2aabt.\text{A3} + (abbt - a^3t).\text{B2} = aab\theta.\text{DH},+2abb\theta.\text{C2},+ \ (b^3\theta - aab\theta).\text{D3}.$$

In which equation in place of the magnitudes FA, DH, A3, B2, C2 & D3 the values of the same are to be substituted , which will provide the nature of the curve A3BC expressed by the quantities *a,b*, and with other constants put in place of *b*, of the radius or the whole sine called *r*, and in place of *a* the name of the tangent sought θ, an equation will be had in terms of θ only and by constant quantities or given known values, the roots of which equation will show the value of the tangent of the angle sought TAW.

EXAMPLE.

460. The curves ACE and A2BE shall be equal arcs of some circle, whose radius $= e$, and the sine of the complement of the arc AC, which is half of the whole of ACE or A2BE, may be called *h,* and there shall be $h^3 : 3ee = k$ & $k - h = l$, and there will be found :

$$\text{A3} = a^3e : 3c^3,\ \text{B2} = l \ ;+be : c \ ;-b^3e : 3c^3, \text{C2} = b^3e : 3c^3;-k,$$

and finally $\text{D3} = a^3e : 3c^3$. Again there will be AF or $\text{BH} = be : c;-h$, & DH or D2H $= ae : c$; which values substituted into the general equation, with the reductions of the fractions to the name $3c^3$ duly performed , will produce :

$$bblt - aalt - aaht;+2.(\ aa+bb)^2.bet : 3c^3 = -2bbk\theta;+2.(aa+bb)^2.be\theta : 3c^3.$$

And (following the hypothesis) there is $l + h = k$, & (§.454) $cc = aa + bb$, therefore these values substituted into the last equation, there will be produced

$$bblt - aakt + \tfrac{2}{3}hcet = -2bk\theta \ + \tfrac{2}{3}bce\theta. \text{ Hinc etiam } 3bblt + 6bbd\theta - 3aakt = 2bce\theta - 2bcet,$$

and on squaring,

$$9b^4lltt + 36b^4lkt\theta - 18aabblktt + 36b^4kk\theta\theta - 36aabbkktt + 9a^4kktt$$
$$= (4bbee\theta\theta - 8bbeet\theta + 4bbeett) \text{ in } cc.$$

From which on substituting $aa + bb$ in place of *cc,* there becomes :

$$4aabbee\theta\theta - 8aabbeet\theta + 4aabbaeettt + 4b^3ee\theta\theta - 8b^4eet\theta + 4b^4eett =$$
$$9b^3lltt + 36b^3lkt\theta - 18aabbtlktt + 36b^4kk\theta\theta - 36aabbkkktt + 9a^4kktt.$$

Or with the proportionals r and θ put in place of b and a, in this last equation, there comes about :

$$44rree\theta^4 - 8rreet\theta^3 + 4rreett\theta\theta + 4r^4ee\theta\theta - 8r^4eet\theta + 4r^4eett =$$
$$9r^4lltt + 36r^4kt\theta - 18r^4lktt\theta\theta + 36r^4kk\theta\theta - 36rrkktt\theta\theta + 9kktt\theta^4.$$

From which it is apparent the problem is to be a solid, for the roots of this equation are shown to be biquadratics, the value of the tangent sought θ, or of the tangent of the angle TAW, or of the angle of declination of the bilinear form 2BAC.

COROLLARY V.

461. If the curves AB and A2B are equal, thus so that they may have a common abscissa AH and the equal ordinates BH and 2B2H, as in figure 113, there will be $\mathrm{A1} = \mathrm{A2}$ & $\mathrm{A3} = 0$; $\mathrm{B1} = \mathrm{B2}$, & $\mathrm{C1} = \mathrm{C2}$; $\mathrm{D1} = \mathrm{D2}$, truly the individual B3, C3 & D3, and likewise A3 are equal to 0 ; likewise the line vanishes, which is represented by DH or D2H in Fig. 115, and formerly, what was FA in this same Fig. 115 , clearly in the former Fig. 113 will be BH. In this case the general formula of paragraph 459 will be changed into : $aat.\mathrm{BH} + (b\mathrm{B}t - \mathrm{AA}t).\mathrm{B2} = 2bb\theta.\mathrm{C2}$. Therefore in the example above, of the parabolic curve with these symbols retained in that place, there will be $\mathrm{BH} = y$; $\mathrm{B2} = \omega$ & $\mathrm{C2} = y - \omega = g$, and thus the equation $aat.\mathrm{BH} + (bbt - aat).\mathrm{B2} = 2bb\theta.C2$, will be changed into $aaty + bbtu - aatu = 2bb\theta g$, or $aatg + bbt\omega = 2bb\theta g$; and thus by substituting the proportionals θ and r in place of a and b, there will be $tg\theta\theta + rrt\omega = 2rrg\theta$, and hence $\theta\theta = \frac{2rr\theta}{t} - \frac{rr\omega}{g}$; which is the same equation, that we have found in the place cited. But knowing this solution to be imperfect, because the equation cannot be of further use, if the ratio of the tangent y to the arc ω were less than twice the ratio of the secant of the angle SAW to the radius in Fig. 114. But therefore in that case θ itself, as indeed it cannot be supposed to be from the rectification of circular arcs, cannot be shown to be algebraic from any equation.

SCHOLIUM.

462. From the examples introduced it can be agreed well enough, that the solution of the problem for the present proposition broadly has been shown. But truly and as far as that may agree with the solution of the same problem, as the great geometer Joh. Bernoulli

has shown in the tract, in French, under the title *Essay d'une nouvelle Theorie de la Manoeuvrœ des Vaisseaux,* published most recently, a collation of each.

If the angle TAW shall be given, and the other angle SAW sought, the problem is easy; for then in the above, θ and *r,* together with the remaining evidently *a, b*, *c* , are known magnitudes, and *t* is sought.

PROPOSITIO L. PROBLEMA.

451. *Si datum trilineum quodcunque* ABH*, curvæ* AVB *ejus axe* AH & *ordinata* BH *terminatum, quomodocunque in fluido feratur juxta directionem* SM *ex* S *versus* M, *invenire mediam directionem Xω fluidi curvæ* AVB *allabentis, & impressionem, quam fluidum juxta inventam mediam directionem in trilineum exseret.*

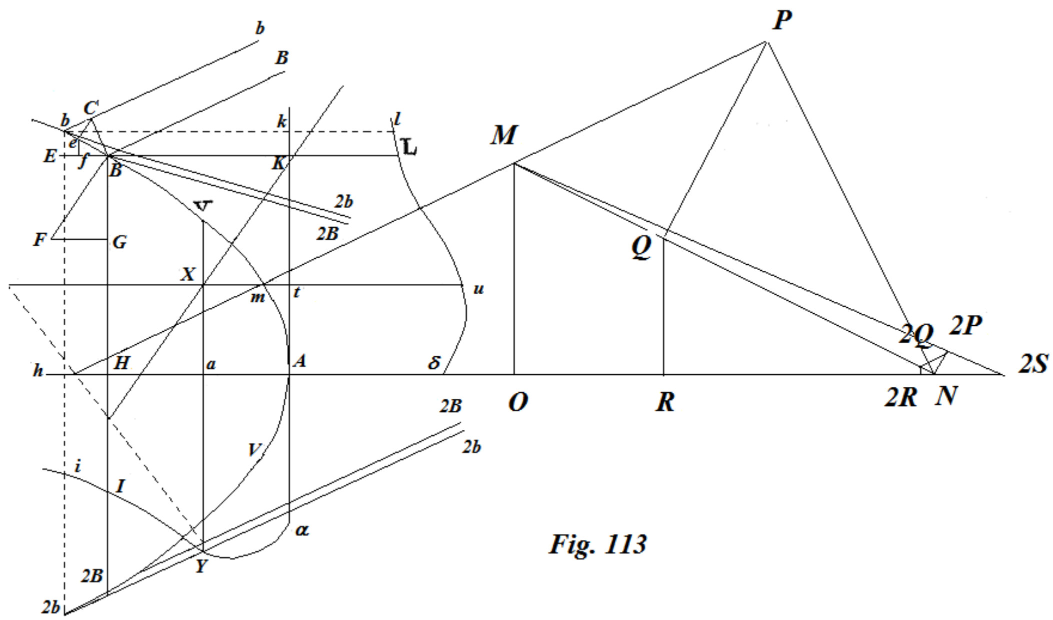


Fig. 113

Analysis Geometrica. I. Per quodlibet curvæ AVB elementum B*b* agantur BB, *bb* rectæ ipsi SM parallelæ, ductisque BC & *Ce* rectæ *bb* & elemento curvæ B*b* normalibus, exponatque MO perpendicularis ad axem curvæ AH quadratum celeritatis, qua trilineum in fluido incedit, vel quod idem est, quadratum velocitatis, qua filamentum *bBBb* elemento curvæ allabitur, exponetque (§.429.) MO.B*e* impressionem, quam filamentum *bb*BB, juxta directionem BF in elementum B*b* exseret, sed impressioni isti æquipollent laterales juxta BG & GF, quarum hæc axi AH parallela, illa vero perpendicularis est; & quia impressiones juxta BF, BG & GF sunt ut hae lineæ respective , vel propter triangulorum BFG & *Bef* similitudinem, sicut B*e*, *Bf* & *ef,* atque impressio juxta $\mathrm{RS} = \mathrm{MO}$ & erit impressio juxta $\mathrm{GF} = \mathrm{MO}.ef$ impressio juxta $\mathrm{BG}, = \mathrm{MO}.Bf$. Et sic respective in quolibet alio curvæ elemento.

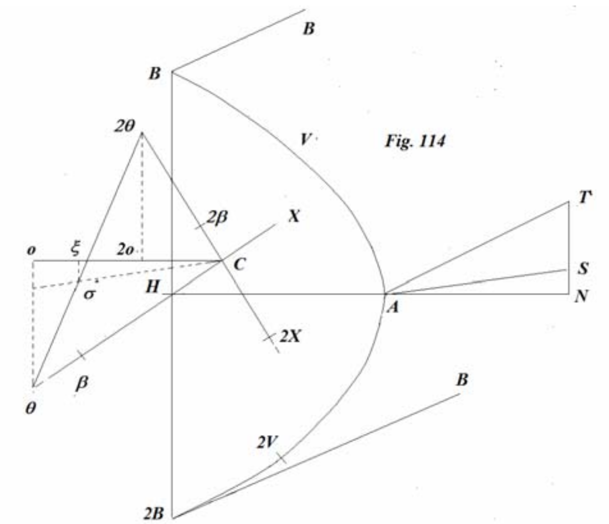
II. Agantur porro per M recta MN parallela elemento curvæ *Bb,* vel tangenti curvæ in B ,ex N perpendicularis NP ad lineam SM, quantum opus est productam, PQ normalis MN*,* & denique QR parallela MO; eritque figura NPMO similis figuræ BC*b*E*,* cum singula latera unius, figuræ parallela sint singulis alterius. Propterea latera in hisce duabus figuris similiter posita proportionalia erunt; hoc est, $b\mathrm{E}:ef=\mathrm{MO}:\mathrm{QR}$, atque adeo $\mathrm{MO}.ef=\mathrm{QR}.b\mathrm{E}$. Vel etiam, ducta per B recta indefinita BL parallela axi AH, atque in ea sumta KL ubique æquali respectivæ QR, ita ut inde nova curva δuL resultet, erit $\mathrm{MO}.ef=(\mathrm{QR}.b\mathrm{E}):\mathrm{KL}.\mathrm{K}k$ seu rec-lo figuræ AδLK inscripto.

Item $\mathrm{EB}:f\mathrm{B}\ (=\mathrm{ON}:\mathrm{RN})=\mathrm{MO}:\mathrm{QR}$,ac consequenter $\mathrm{MO.B}f=\mathrm{QR.\ EB}$, vel etiam = rec-lo IH*h* areæ AαIH inscripto, si, etiam in singulis ordinatis BH ultra axem AH productis, sumta fuerint segmenta HI æqualia homologis QR seu KL.

III. Quoniam igitur singulæ fluidi impressiones in curva AVB juxta directiones curvæ perpendiculares æquipollent lateralibus axi curvæ parallelis exponendis (num. 1. hujus) rec-lis MO.*ef,* vel (num. II.) rec-lis LK*k*, & perpendicularibus axi exponendis (num. I.) per rec-la MO.B*f*, seu (num. II.) per rec.la IH*h*, impressio, quam universa curva AVB a fluido subibit, æquipollebit omnibus LK*k* & omnibus IH*h* ; atqui omnium potentiarum *k*KL axi parallelarum media. directio *Zu,* axi itidem parallela, (§ 56) transit per centrum gravitatis areæ AδLK, quæ omnia LK*k* continet ; & omnium potentiarum IH*h* axi AH perpendicularium media directio VY, axi AH normalis, seu ordinatis HI curvæ αIY æquidistans, transit per centrum gravitatis figuræ: AαIH, quæ omnia IHh continet. Ergo impression, quam fluidum in curvam AVB exerit*,* eundem effectum præstat, quem præstaret potentia quam exponit rec-lum MO.XZ, æquale areæ AαIH, agente in curvam juxta directionem XZ, simul cum potentia MO.XY, æquali areæ AαIH agente in curvam AVB juxta directionem XY*;* quandoquidem hæ potenuæ MO.XZ & MO.XY æquivalent omnibus LK*k* & omnibus IH*h*, & in harum omnium mediis directionibus XZ & XY curvæ AVB applicatæ intelliguntur. Sed (§. 49) media directio potentiarum MO.XZ & MO.XY est Xω, transiens per centrum gravitatis ω, punctorum Z & Y, quibus rectæ XZ & XY terminantur, & quorum punctorum gravitas exponitur per datam MO, & impressio, quam proinde fluidum juxta hanc mediam directionem Xω in curvam AVB exeret, exponetur rec-lo *2*MO. Xω.

IV. Sin vero curva A2V2B sit subter axem AH constituta, quæ juxta directionem eidem ac in præcedenti casu SM, parallelam feratur, transferenda erit OS ad oppositam partem O2S*,* & ducenda M2S, tum ex puncto N demitti debet super M2S perpendicularis N2P, & 2P2Q super MN, ac denique 2Q2R parallela MO; posita scilicet MN parallela tangenti curvæ A2V2B ad alteram axis AH partem translatæ in puncto IB , erit 2Q2R communis ordinate curvarum δuL & αYI, sed ad curvam inferiorem A2V2B pertinentium, quas deinceps, confusionis vitandæ gratia, per $2\alpha 2u$2L & $2\mathrm{A}2\alpha 2\mathrm{Y}2\mathrm{I}$, insigniemus, quanquam in schemate expressæ non sunt, quia mente facile supplentur. Erit ergo impressio fluidi in curva inferiore A2V2B, juxta mediam directionem ejus $2\mathrm{X}2\omega$*,* exponenda per $2\mathrm{MO}.2\mathrm{X}2\omega$*,* eandem prorsus ob rationem, propter quam impressio in curva AVB juxta mediam directionem $\mathrm{X}\omega$ ostensa est exponi rec-lo 2MO in $\mathrm{X}\omega$. Quod vero loco ipsius SM in curva A2V2B nunc sumenda sit 2SM inde est, quia revolutione figuræ A2V2B cum lineis 2B2B ipsis BB (secundum hypothesin) parallelis circa axem AH, adeo ut si curva A2V2B alteri AVB similis & æqualis fuerit, cum ea congruat, lineæ 2B2B veniunt in situm B2B subter rectam BL, eosdem vel æquales angulos constituentes cum angulis BBL, quos lineæ BB supra BL extantes cum hac BL continent, adeo ut anguli quilibet homologi BBL & LB2B vel anguli MSO & M2SO æquales futuri sint.

V. Si jam bilineum BVA2V2B juxta directionem ipsis BB, 2B2B, Fig. 114,



vel AT parallelas in fluido incedat, & quæratur media directio Cσ impressionum fluidi in curva BA2B, sic res expediri potest beneficio numerorum quarti & tertii hujus propositionis. Per numerum tertium etiam habentur media directio Xβ & impressio fluidi in curva AVB, quæ illic exponebatur per 2.MO.Xα. Sit igitur $X\beta = 2M\omega$ in fig.113 eritque MO.Xβ impressio fluidi, juxta mediam directionem in curva AVB, designante MO ubique velocitatis quadratum, qua celeritate videlicet fluidum curvæ prædictæ alliditur. Similiter, juxta numerum quartum hujus, invenientur media directio 2X2β & impressio fluidi, juxta hanc mediam directionem 2 in 2X2ω.Mo, vel (si 2X2β sit dupla ipsius 2X2ω) MO.2X2β. A puncto intersectionis C duarum Xβ & 2X2β sumantur in hisce æquales Cθ & C2θ, scilicet $C\theta = X\beta$ & $C2\theta = 2X2\beta$, jungaturque $\theta 2\theta$, adeoque per §.49 media directio potentiarum MO.Cθ & MO.C2θ, quæ æquipollent fluidi impressionibus; quas ambæ curvæ AVB & A2V2B subeunt, transit per centrum gravitatis punctorum θ & 2θ, id est, per punctum medium σ rectæ $\theta 2\theta$, & per punctum C atque adeo hæc media directio est *C*σ, potentiaque seu impressio fluidi in universa curva BVA2V2B erit 2.MO.*C*σ. Quod Omnia erant invenienda.

Aliter & brevius.

Numeros tertius hujus propositionis, ac consequenter quæ post eum sequuntur, omnia velut corollarium duntaxat deduci possunt ex Propositione VIII. Libri Primi §. 59. Nam figuræ similes NPMO & HCbE præbent $b\mathrm{B} : e\mathrm{B} (= \mathrm{MN} = \mathrm{QN}) = \mathrm{MO} : \mathrm{QR}$ atque adeo $\mathrm{MO.B}e = \mathrm{B}b.\mathrm{QR}$. Atqui MO. Be (num. I. hujus) exponit impressionem filamenti fluidi *bbBB* in curvæ elemento *Bb,* ergo hæc impressio etiam exponi potest, per B*b*.QR, atque adeo fluidum eandem vim in elementum *Bb* exserit, quam si singula elementi puncta juxta directiones eidem perpendiculares urgerentur potentia QR, adeo ut tota res reducatur ad Coroll 1. prædictæ Prop. 8. Lib. 1. Idcirco duntaxat propositionis ejusque corollarii constructiones forent relegendæ, atque loco curvarum ABB, XS & XD in figura 15 substituendæ AVB, αYI & δuL figuræ 113 ac denique, loco potentiarum BG in illa, subrogandae potentiæ QR in hac figura, incideremusque in easdem penitus conclusiones, quas in præcedentibus numeris præsentis propositionis elicuimus.

COROLLARIUM I.

452. Idcirco, ducta per punctum C recta *Co* parallela axi AH, invenietur sinus totus ad tangentem anguli $oC\sigma$, quem media directio $C\sigma$ bilinei BA2B cum axe AH, seu linea huic axi parallela *Co* continet, ut aggregarum arearum AδLK & 2A2δ2L2K ad differentiam arearum AαH & 2A2α2I2H. Nam demissis ex punctis $\theta, \sigma, 2\theta$ perpendicularibus $\theta o, \sigma\xi, 2\theta 2o$ ad *Co,* & quia σ est punctum medium rectae $\theta 2\theta$, seu centrum gravitatis punctorum $\theta, 2\theta$, erunt (§. 46.)

$2.C\xi = Co + C2o, \& \ 2.\xi\sigma = \theta o - 2\theta 2o$; adeoque $C\xi : \xi\sigma$, vel sinus totus ad tangentem anguli

$\xi C\sigma = Co + C2o : \theta o - 2\theta 2o = MO : Co + MO.C2o : MO.\theta o - MO.2\theta 2o$

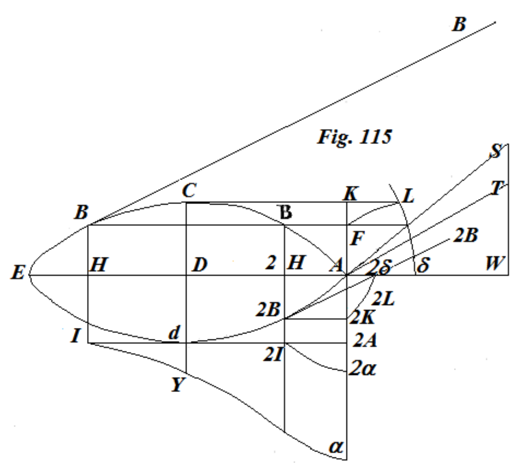
$= A\delta LK + 2A2\delta 2L2K$ ad $A\alpha IH - 2A2\alpha 2I2H$.

Nam (constr.) $MO.Co$; $MO.C2o$; $M0.\theta o$ & $MO.2\theta 2o$ æquantur areis $A\delta LK$, $2A2\delta 2L2K$; $A\alpha IH$ & $2A2\alpha 2I2H$.

COROLLARIUM II.

Fig. 115.

453. Sin vero bilineum fuerit ABE2BA, cuius partes, superior ACB & inferior A2B, inæquales sint, punctumque altissimum C arcus curvæ superioris ACB non existat in arcus termino B, sinus totus erit ad tangentem anguli, quem media directio impressionis fluidi in bilineo B2B cum axe AE continet, ut



$A\delta LK + A2\delta 2L2K - FK.L$
ad $A\alpha YI - A2\alpha 2Y2I$.

Ubi δL est scala impressionum fluidi axi bilinei AE parallelarum, quæ ex perpendicularibus curvæ arcui AC derivantur, & FL est scala impressionum axi etiam parallelarum, sed derivatarum ex perpendicularibus arcui curvæ BC, & quia impressiones illæ axi parallelæ, quibus arcus BC afficitur, directe contrariæ sunt parallelis axi AE, quibus AC urgetur, ideo differentia arearum AδLK & FLK sumenda est ad habendam expressionem impressionum axi AE parallelarum, quibus tota curva ACB supra axem extans afficitur. Curvæ vero αYI, 2α2Y2I ad axem IA, exstructæ sunt scalæ impressionum axi AE normalium ex perpendicularibus curvarum ACB & A2B impressionibus derivatarum, ac denique 2δ2L2K est scala impressionum axi parallelarum pertinentium ad curvam A2B. Curvarum αYI & 2α2Y2I ordinatæ in punctum axis I & 2I nullæ sunt, quia lineæ BB & 2B2B, quæ fluidi bilineo allidentis directionem denotant, bilineum in punctis B & 2B contingunt. Propter eandem rationem etiam curvæ FL ordinata in puncto F, in quo scilicet recta BF axi parallela alteri AK occurrit, nulla est.

COROLLARIUM III.

Fig. 113.

454. Si jam dicantur MO ,*a;* OS, *b*; MS, *c* ; coordinatæ AH, *x*; BH, *y*, & ON*, m* ; ac MN, *u* : erunt $\mathrm{SN} = b + m$, vel N2S respectu curvæ inferioris $= b - m$. Hinc similitudo triangulorum SMO, SNP præbebit PN vel $\mathrm{N2P} = (ab \pm am) : c$, ubi signum superius respicit curvam superiorem AVB, inferius vero inferiorem. Porro habetur $\mathrm{MO} : \mathrm{QR}\ \left(= \mathrm{MN} : \mathrm{Q.N}\right) = \mathrm{MN}^2 : \mathrm{PN}^2$; atque adeo

$\mathrm{QR} = \mathrm{MO.\ PN}^2 : \mathrm{MN}^2 = \left(b \pm m\right)^2 .a^3 : aacc + ccmm$, vel assumendo magnitudinem *e*, atque ponendo cce æquale

$a^3, = (b \pm m)^2 .e : aa + mm = \left(bbe \pm 2bem + emm\right).aa + mm = e, \pm 2bem : nu; + \left(bb - aa\right).e : uu$,

vel ponendo denique $bb - aa = ff$, invenietur tandem scalarum resistentiæ seu fluidi impressionum communis ordinata

$$\mathrm{QR} = \mathrm{KL} = \mathrm{Hl} = e\ ; \pm 2bem : nn; + eff : nn.$$

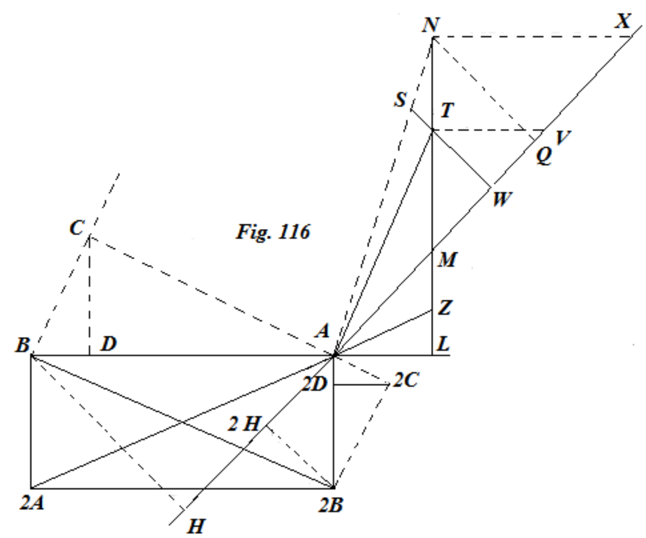


Fig. 116 . Jam data æquatione curvæ patientis AVB vel A2V2B, eaque differentiata, si loco elementorum *dy* & *dx* ordinatæ & abscissæ substituentur eorum proportionales *a* & *m,* seu MO & ON, habebitur æquatio expressa indeterminatis *x, y* & *m,* & quantitatibus datis seu constantibus, ex qua elicietur valor ipsius *m* in *x* & *y* ac constantibus,in quo ope æquationis curvæ patientis AVB vel A2V2B alterutra ex indeterminatis *x* vel *y* semper eliminari potest. Invento vero valore ipsius QR in *y* & constantibus, areæ AδLK, 2A2δ2L2K ; & AαIH, 2A2α2I2H inveniri poterunt, si non algebraice saltem transcendenter per series, aut etiam approximationibus. Sed horum omnium usus atque uno altero exemplo illustrari debet.

EXEMPLUM I.

455. Esto rectangulum BA2B incedens juxta directionem AT in fluido resistente, quæritur media directio ST impressionum hujus fluidi rectangulo allabentis. Angulus BA2B recta AH bisectus sit, & hæc linea AH versus W producta consideretur instar axis linearum AB & A2B, quarum hæc repræsentat curvam patientem inferiorem, illa vero superiorem atque demissis ex B, 2B & ex puncto T in recta AT, pro libitu accepto, perpendicularibus BH, 2B2H ac TW secante AN in puncto S, erunt $\mathrm{AH}=\mathrm{BH}$ & $\mathrm{A2H}=\mathrm{2B2H}$, propter angulos semirectos BAH & 2B2A2H. Sint insuper $\mathrm{AH}=\mathrm{BH}$ & $\mathrm{A2H}=\mathrm{2B2H}=x$, & $\mathrm{BH}=\mathrm{Y}, \mathrm{2B2H}=y$, unde $\mathrm{X}=\mathrm{Y}$, & $x=y$; hinc unde utroque casu erit $a=m$, atque adeo $nn(=aa+mm)=2aa$. Ergo substitutis hisce valoribus in formula ipsius QR valores determinante, invenietur pro linea AB ejus valor huic casui applicatus $\mathrm{QR}(=e,+2bem:nn;+eff:nn)=e;+be:a+eff:2aa$, (seu restituendo $a^3:cc$, & $bb-aa$ valores quantitatum e & ff) $=(a^3+2aab+abb):2cc$. Adeoque elementum areæ AδLK seu rec-lum $Lk=(aa+2ab+bb).adY:2cc$, & area ipsa $=(aa+2ab+bb)a.\mathrm{AH}:2acc$. Similiter area 2A2$\delta$2L2.K erit $=(aa-2ab+bb).a.\mathrm{A2H}:2cc$. Area A$\alpha$IH pro linea $\mathrm{AH}=(aa+2ab+bb).a.\mathrm{AH}:2cc$; & area $2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H}=(aa-2ab+bb).a.\mathrm{A2H}:2cc$. Hinc (§.452) AW: SW seu sinus totus ad tangentem anguli $\mathrm{SAW}=\mathrm{A}\delta\mathrm{LK}+2\mathrm{A}2\delta 2i2\mathrm{K}:\mathrm{A}\alpha\mathrm{iH}-2\mathrm{A}2\alpha 2\mathrm{I}2\mathrm{H}$, atque adeo analytice erit

$$AW:SW=(aa+2ab+bb).a.\mathrm{AH},+(aa-2ab+bb.)a.\mathrm{A2H} \text{ ad}$$
$$(aa+2ab+bb).a.\mathrm{AH},-(aa-2ab+bb).a.\mathrm{A2H}=(2ab+cc).\mathrm{AH},+(cc+2ab).\mathrm{A2H}$$
$$\text{ad } (cc+2ab).\mathrm{AH},-(cc-2ab).\mathrm{A2H}.$$

Ducatur TV parallela AL, eritque ob angulos semirectos MTW & WTV, $\mathrm{MW}=\mathrm{TW}=\mathrm{WV}$, atque adeo cum A.W (§.454) dicatur b & TW, a; erunt $a+b=\mathrm{AV}$, seu $aa+2ab+bb=\mathrm{AV}^2=cc+2ab$, & $aa-2ab+bb=cc-2ab=\mathrm{AM}^2$; hinc

$$\mathrm{AW}:\mathrm{SW}=\mathrm{AV}^2.\mathrm{AH}+\mathrm{AM}^2.\mathrm{A2H}:\mathrm{AV}^2.\mathrm{AH}-\mathrm{AM}^2.\mathrm{A2H}$$
$$=\mathrm{TL}^2.\mathrm{AH}+\mathrm{ML}^2.\mathrm{A2H}:\mathrm{TL}^2.\mathrm{AH}-\mathrm{ML}^2.\mathrm{A2H}=\mathrm{R.T}^2+\mathrm{R}^2\mathrm{t}:\mathrm{R.T}^2-\mathrm{R}^2.\mathrm{t}$$
$$=\mathrm{T}^2+\mathrm{Rt}:\mathrm{T}^2-\mathrm{Rt}.$$

Positis $\mathrm{AL}:\mathrm{TL}=\mathrm{R}:\mathrm{T}$ & $\mathrm{AH}:\mathrm{A2H}=\mathrm{R}:\mathrm{t}$; ita ut T & t tangentes sint angulorum datorum TAL & AB.2B, ducta scilicet rectanguli diametro B2B, existente sinu toto seu radio R. Ducatur adhuc NX parallela TV vel AL, eruntque etiam MQ, NQ, & QX æquales; jam quia $\mathrm{AW}:\mathrm{SW}=\mathrm{AQ}:\mathrm{NQ}$ vel QX, erit etiam $=\mathrm{AX}+\mathrm{AM}:\mathrm{MX}=\mathrm{LN}+\mathrm{LM}:\mathrm{LN}-\mathrm{LM}$, atque adeo, $\mathrm{AW}+\mathrm{SW}:\mathrm{AW}-\mathrm{SW}=2\mathrm{LN}:2\mathrm{LM}=\mathrm{LN}:\mathrm{AL}$ vel LM. Verum, quia paulo ante reperiebatur $\mathrm{AW}:\mathrm{SW}=\mathrm{T}^2+\mathrm{Rt}:\mathrm{T}^2-\mathrm{Rt}$, erit etiam $\mathrm{AW}+\mathrm{SW}:\mathrm{AW}-\mathrm{SW}=\mathrm{T}^2:\mathrm{Rt}$; ergo etiam $\mathrm{LN}:\mathrm{AL}=\mathrm{T}^2:\mathrm{R}t$; sed est $\mathrm{AL}:\mathrm{LT}=\mathrm{R}:\mathrm{T}=\mathrm{Rt}:\mathrm{Tt}$; ergo ex æquo $\mathrm{LN}:\mathrm{LT}=\mathrm{T}^2:\mathrm{T}t=\mathrm{T}:t$. Ducatur diagonalis rectanguli 2AA, eaque producatur in Z, & existente AL radio, seu sinu toto, erunt TL & ZL tangentes angulorum TAL & ZAL vel A2A2B , quæ tangentes dicebantur T, & *t;* atque adeo $LN : LT = LT : LZ$; sunt ergo NL, TL & ZL in continua ratione, ac per consequens NL ad TL in subduplicata ratione NL ad ZL, sed hæc ratio NL ad ZL componitur rationibus NL ad AL, & AL ad ZL, id est, 2A2B ad A2B; ergo ratio NL ad TL est subduplicata ejus, quæ componitur ex NL ad AL & longitudinis rectanguli AB ad latitudinem A2B ; atque hæc est analogia; quam Celeb. Jac. Bernoullius in Actis Lips. 1696, pag. 336. sine ulla analysi & demonstratione exhibuit, & in quam haud dubie via multo breviore incidit. Nam licet hunc casum particularem non parum brevius, quam hoc loco a nobis factum sit, solvi posse non ignoro, non abs re tamen fore arbitratus sum, si methodum nostram generalem talia tractandi etiam in aliquo casu fusius illustrarem, de cujus veritate aliunde constare possit.

456. Præsens cxemplum particulare sic brevius etiam expedietur: per angulos rectanguli B & 2B agantur BC, 2B2C æquidistantes ipsi AT, & per punctum A eidem perpendicularis C2C, ac denique ex punctis C, 2C demittantur ad latera rectanguli perpendiculares CD2C2D, ac assumta unitate pro designanda celeritate, qua rectangulum in fluido incedit, vel quod idem est, pro velocitate, qua fluidum lateribus rectanguli AB, A2B allabitur, & (§. 429.) A2D exponet impressionem fluidi, quam latus A2B in directione ipsi normali excipiet, & AD exponet impressionem, quam latus AB excipiet juxta directionem lateri A2B parallelam, seu AB perpendicularem. Unde cum media directio AN impressionum fluidi in lateribus AB & A2B nascatur a potentiis lateralibus AL & LN, necesse est ut habeatur, $AL : LN = A2D : AD$; atqui

$A2D : A2B = (A2B)^2 : (A2C)^2 = AL^2 : AT^2$, & $A2B : 2A2B \text{ vel } AB = ZL : AL = AL.\ ZL : AL^2$

ergo ex æquo $A2D : AB = AL.\ ZL : AT^2$; estque porro $AB : AD = AB^2 : AC^2 = AT^2 : TL^2$, adeoque iterum ex æquo $A2D : AD = AL.\ ZL : TL^2$. Atqui (secundum hypothesin) $AL : LN = A2D : AD$, ergo $AL : LN = AL.ZL : TL^2$; & quia $TL : AL = TL^2 : AL.TL$, erit denique ex æquo $TL : NL = AL.ZL : AL.TL = ZL : TL$, atque adeo TL est media proportionalis inter tangentes angulorum datorum ZAL vel A2A2B & NAL , ut supra invenimus. Quod erat demonstrandum.

Fig.113 & 114.

EXEMPL. II.

457. Resumatur bilineum BA2B; quod ponatur esse parabolicum, cujus æquatio $2ax = yy$; existentibus coordinatis ad axem AH, x & HB vel 2H2B, y ac parametro $2a$. Æquatio bilinei differentiata præbet $adx = ydy$, vel substitutis loco elementorum dx & dy lineis proportionalibus m & a ; $am = ay$ atquo adeo $m = y$; hinc $nn = aa + yy$. Idcirco

$$QR = KL = HI(= e; \pm 2bm : nn; + eff : nn) = e; \pm 2bey : aa + yy; + effaa + yy.$$

Hinc (§. 454) rec-lum $Lk = edy; \ \pm 2beydy : aa + yy; \ + effdy : aa + yy$.

Atqui omnia edy seu summa $edy = ey$; ac omnnia ydy ; $aa + yy$ æquantur log-mo rationis $aa + yy$ ad aa qui log-us dicatur $z{:}a$; respicitque logarithmicam, cujus

subtangens est unitas. Et denique omnia $dy:aa+yy$ æquantur arcui circulari ω applicato ad aa quadratum radii, cujus arcus tangens sit y ; nam ejusmodi arcus elementum: seu $d\omega = aady:aa+yy.$ Propterea invenietur area
$\mathrm{A}\delta\mathrm{LK}(2\mathrm{A}2\delta2\mathrm{L}2\mathrm{K}) = ey;\pm 2bez:a\ ;+\ eff\,\omega:aa$; adeoque addendo
$\mathrm{A}\delta\mathrm{LK}+2\mathrm{A}2\delta2\mathrm{L}2\mathrm{K}; = 2ey;+2.eff\,\omega:aa$ (vel loco *ff* restituendo suum valorem $bb-aa$) $= 2ey;+2bbe\omega:aa;-\ 2e\omega = 2eg;+2bbe\omega:aa$; posita ad abbreviandum $g = y-\omega.$
Arearum $\mathrm{A}\alpha\mathrm{IH}\left(2\mathrm{A}2\alpha2\mathrm{I}2\mathrm{H}\right)$ elementum est
$= edx;\pm 2beydx:aa+yy;\ effdx:aa+yy = eydy:a;\pm\ 2bey^2dy:a^3+ayy;+effydy:a^3+ayy;$ quia $ydy = adx$. Atqui $yydy:aa+yy = dy,-aady:aa+yy = dy-d\omega$, & ut antea, $ydy:aa+yy = dz:a$, ac propterea.
$\mathrm{A}\alpha\mathrm{IH}(2\mathrm{A}2\alpha2\mathrm{I}2\mathrm{H}) = eyy:2a;\pm(2bey-2be\omega):a;+effz:aa$; adeoque
$\mathrm{A}\alpha\mathrm{IH}-2\mathrm{A}2\alpha2\mathrm{I}2\mathrm{H} = (4by-3be\omega):a = 4beg:a$, posita scilicet $y-\omega = g$.

Ducantur per A rectæ AT parallela BB, & AS æquidistans $\mathrm{C}\sigma$, eritque Fig. 114.
$\mathrm{AW}:\mathrm{WS} = \mathrm{C}\xi:\xi\sigma = \mathrm{A}\delta\mathrm{LK}+2\mathrm{A}2\delta2\mathrm{L}2\mathrm{K}:\mathrm{A}\alpha\mathrm{IH}-2\mathrm{A}2\alpha2\mathrm{I}2\mathrm{H}$
$= (2eg;+2bbe\omega:aa):\left(4beg:a\right) = aag+\ bb\omega:2abg;\&\ \mathrm{TW}:\mathrm{A}W = a:b = 2abg:2bbg;$
ergo ex æquo fiet $\mathrm{TW}:\mathrm{WS} = aag+bb\omega:2bbg.$ Dicantur insuper TW,θ & SW,t ; radiusque AW, r; eritque $\theta:t = aag+bb\omega:2bbg$, & quia $r:\theta = b:a$, substitutis loco b & a proportionalibus $r\ \&\ \theta$, proveniet $\theta:t = \theta\theta\mathrm{g}+rr\omega:2rrg$; ex qua elicitur
$\theta\theta = \frac{2rr\theta}{t}-\frac{rr\omega}{g}$; adeoque ipsae aequationis radices erunt $\theta = \frac{rr}{t}\pm\sqrt{\left(\frac{r^4}{tt}-\frac{rr\omega}{g}\right)}$
Jam quia singulæ $r,\ g,\ \omega$ & t datæ sunt, etiam θ data & erit, quæ est tangens anguli TAW, quem directio figuræ BA2B in fluido incedentis cum axe AH continet, cum figura itinere permanente fertur absque conversionibus circa seipsam; qui angulus TAW angulus declinationis figuræ audit, ejusque tangens, TW declinatio ipsa ; linea vero σC est directio, juxta quam figura in fluido ideo impelli debet, ut in directione ipsis BB, AT, &c. parallela itinere seu via manenti ea in fluido ferri queat ; quæ omnia suo loco uberius explicabuntur.

Cæterum exempla eorum casuum, in quibus curvæ, superior AB & inferior 2A2B, sunt similes & æquales, adeo ut hæ curvæ fluidi impressionibus expositæ communem abscissam AH, ordinatasque æquales BH & 2B2H habeant. Sed ea, ratione calculi, intricatiora evadunt, si curvæ AB & A2B, ut fig. 115 inæquales & tamen similes seu totæ ACE, A2BE æquales sunt, tunc enim punctorum B & 2B positio, atque adeo abscissarum AH, A2H, & ordinatarum BH & 2BEH magnitudo pendet a positione rectæ AT, seu ab angulo quæsito TAW. Quomodo vero pro casibus hisce calculus debeat subduci, ex sequenti elucescet corollario.

COROLLARIUM IV.

Fig. 115.
458. Sint $\int mdy:nn = \mathrm{A}1:a;$ vel $\mathrm{A}2:a;$ vel $\mathrm{A}3:a$ pro curvis A*C,* A2B & CB, item $\int dy:nn = \mathrm{B}1:aa$ vel $\mathrm{B}2:aa;$ aut $\mathrm{B}3:aa$, respectu earundem curvarum AC, A2B, & CB.

$\int mdx = nn = \mathrm{C1} : a$, vel $\mathrm{C2} : a$, vel $\mathrm{C3} : a$; respectu earundem curvarum, & denique $\int dx : nn = \mathrm{D1} : aa$, vel $\mathrm{D2} : aa$, vel $\mathrm{D3} : aa$. Ubi numeri, literis A, B, C, D postpositi non sunt potestatum indices, sed tantum indicant quamnam ex curvis AC, A2B & BC hoc ordine, quo recensentur, respiciant in figura 115. Unde, quia in fig. 113, QR vel KL aut HI $= e$; $\pm 2bem : nn$;$+eff : nn = e; \pm 2bem : nn; +\left(bb-ad\right).e : nn$; quia $ff = bb-aa$ ut supra(§.545) invenimus. Hinc area $\mathrm{A}\delta\mathrm{LK} = e.\mathrm{CD}; +2be.\mathrm{A1} : a; +\left(bb-aa\right).e\mathrm{B1} : aa$, respectu curvæ AC fig.115. Area $2\mathrm{A}2\delta 2\mathrm{L}2\mathrm{K} = e.2\mathrm{B}2\mathrm{H}; -2be.\mathrm{A2} : a; +\left(bb-aa\right).e\mathrm{B2} : aa$, respectu curvæ A2B.Area $\mathrm{FKL} = e.\mathrm{KF}; -2be.\mathrm{A3} : a; +\left(bb-aa\right).eB3 : aa$, respectu curvæ CB. Area $\mathrm{A}\alpha\mathrm{Y}d = e.\ \mathrm{AD}; +2be.\mathrm{C1} : a; +\left(bb-aa\right).e\mathrm{D1} : aa$, respectu curvæ AC. Area $2\mathrm{A}2\alpha 2\mathrm{Y}2\mathrm{I} = e.\mathrm{A2H}; -2be.\mathrm{C2} : a, +\left(bb-aa\right).eD2 : aa$; respectu curvæ A2B, & Area $d\mathrm{YI} = e.\mathrm{DH}; -2be.\mathrm{C3} : a; +\left(bb-aa\right).e\mathrm{D3} : aa$; respectu curvæ BC. Ergo

$\mathrm{A}\delta\mathrm{LK} + 2\mathrm{A}2\delta 2\mathrm{L}2\mathrm{K} - \mathrm{FLK} = e.\ \mathrm{F2K}; +2be.\left(\mathrm{A1}-\mathrm{A2}+\mathrm{A3}\right) : a;$
$+bb-aa.\left(\mathrm{B1}+\mathrm{B2}-\mathrm{B3}\right)e : aa$, & $A\alpha Yd + \mathrm{Y}d\mathrm{I} - 2\mathrm{A}2\alpha 2\mathrm{Y}2\mathrm{I} = e.\mathrm{H2H};$
$+2be.(\mathrm{C1}+\mathrm{C1}-\mathrm{C3}) : a; +bb-aa.\left(\mathrm{D1}-\mathrm{D2}-\mathrm{D3}\right)e : aa.$

Atqui (§. 453) est
$\mathrm{AW} : \mathrm{WS} = \mathrm{A}\delta\mathrm{LK} + 2\mathrm{A}2\delta 2\mathrm{L}2\mathrm{K} - \mathrm{FLK} : \mathrm{A}\alpha\mathrm{Y}d + \mathrm{Y}d\mathrm{I} - 2\mathrm{A}2\alpha 2\mathrm{Y}2\mathrm{I},$
ergo etiam

$$\mathrm{AW} : \mathrm{WS} = e.\mathrm{F2K}; +2be.(\mathrm{A1}-\mathrm{A2}+\mathrm{A3}) : a; +bb-aa.(\mathrm{B1}+\mathrm{B2}-\mathrm{B3})e : aa, \text{ ad}$$
$$e.\mathrm{H2H}; +2be(\mathrm{C1}+\mathrm{C2}-\mathrm{C3}) : a; +bb-aa.\left(\mathrm{D1}-\mathrm{D2}+\mathrm{D3}\right)e : aa.$$

Unde, cum TW sit ad $\mathrm{AW} = a : b$; habebitur ex æquo analogia, quæ suppeditabit æquationem quæsitam anguli TAW tangentem manifestaturam; sed qui ipsæ A2H, 2H2B, EH, HB ac per consequens ipsæ quoque HD & 2HD pendent a quantitatibus *a & b,* quarum hæc adhuc incognita est; ideo etiam ipsæ A2, A3, B2, B3, C2 ,C3; & D2 & D3 ab iisdem pendent; quod calculum perplexum reddit; præsertim si curvæ ACE & A2BE fuerint dissimiles seu diversæ.

459. Ponamus vero nominatas curvas ACE & A2BE similes & æquales esse, quo fiet ut arcus EB & A2B æquales sint, cum lineæ BB & 2B2B, ipsi AT (secundum hypothesin) parallelæ,arcus in B & 2B contingant, proptera erunt $\mathrm{EH} = \mathrm{A2H}$; item $\mathrm{BH} = \mathrm{2B2H}$, atque adeo $\mathrm{HD} = \mathrm{D2H}$, nec non producta 2B2H usque ad occursum ejus curvæ ECA in 3B, fiet $\mathrm{2H3B} = \mathrm{2B2H} = \mathrm{BH}$, ac per consequens linea BF, ipsi EA parallela, secabit curvam ECA in puncto 3B, adeo ut arcus B & C3B similes & æquales futuri sint. Hisce positis reperietur

$$\mathrm{A1} = \mathrm{A2} + \mathrm{A3}\ ;\ \mathrm{B1} = \mathrm{B2} + \mathrm{B3};\ \mathrm{C1} = \mathrm{C2} + \mathrm{C3}\ \ \&\ \ \mathrm{D1} = \mathrm{D1} + \mathrm{D3};$$

ac proinde hoc casu habebimus

$A1 - A2 + A3 = 2.A3$ seu dupla A3.

$B1 + B2 - B3 = 2.B2.$

$C1 + C2 - C3 = 2.C2.$

$D1 - D2 + D3 = 2.D3.$

Adeoque hi valores,in analogia præcedentis paragraphi subrogati, præbebunt
$AW : WS = e.F2K + 4be.A3 : a; + (2bbe - 2aæ).B2 : a^2,$
ad $e.H2H; + 4be.C2 : a; + (2bbe - 2aæ) : D3 : aa,$
vel dividendo terminos per $2e$, sicut
$FA, + 2bA3 : a; + \left(bb - aa\right).B2 : aa;$ ad $DH; +2bC2 : a, + \left(bb - aa\right).D3 : aa;$
& $WT : AW = a : b$;
ergo ex æquo erit
$WT : WS = a.FA + 2b.A3 ; +(bb - aa).B2 : a,$
ad $b.DH; +2bb.C2 : a; + \left(b^3 - aab\right).D3 : aa;$
vel vocando tangentem anguli dati SAW, t; & tangentem quæsiti TAW, θ habebimus
$\theta : t = a^3.FA + 2aab.\ A3 ; + \left(abb - a^3\right).B2 : aab.DH, +2aabC2, + \left(b^3 - aab\right)D3$;
adeoque, multiplicando extrema & media, habebimos æquationem sequentem generalem
$a^3t.FA + 2aabt.A3 + (abbt - a^3t).B2 = aab\theta.DH, +2abb\theta.C2, + (b^3\theta - aab\theta).D3.$
In qua æquatione loco magnitudinum FA, DH, A3, B2, C2 & D3 subrogandi sunt earum valores, quos præbebit natura curvæ A3BC expressi quantitatibus *a,b*, aliisqæ constantibus, positisque loco *b*, radii seu sinus totius nomineæ *r*, & loco ipsius *a* nomino tangentis quæsitæ θ, habebitur æquatiô in sola θ & quantitatibus constantibus seu cognitis data, cujus æquationis radices manifestabunt valorem tangentis anguli quæsiti TAW.

460. EXEMPL. Sint curvæ ACE & A2BE arcus æquales alicujus circuli, cujus radius $= e$, ac sinus complementi arcus AC, qui est semissis totius ACE vel A2BE, dicatur *h,* sintque $h^3 : 3ee = k$ & $k - h = l$, invenienturque
$A3 = a^3e : 3c^3$, $B2 = l\ ; +be : c\ ; -b^3e : 3c^3, C2 = b^3e : 3c^3; -k,$ ac denique $D3 = a^3e : 3c^3.$
Erunt porro AF vel $BH = be : c; -h,$ & DH vel $D2H = ae : c$; qui valores in æquatione generali substituti, fractionum reductionibus ad nomen $3c^3$ rite peractis , præbebunt
$bblt - aalt - aaht; +2.\left(aa + bb\right)^2 .bet : 3c^3 = -2bbk\theta; +2.(aa + bb)^2 .be\theta : 3c^3.$
Atqui (secundum hypothesin) est $l + h = k$, & (§.454) $cc = aa + bb$, ergo hi valores, in postrema æquatione suffecti, præbebunt
$bblt - aakt + \frac{2}{3}hcet = -2bk\theta + \frac{2}{3}bce\theta.$ Hinc etiam $3bblt + 6bbd\theta - 3aakt = 2bce\theta - 2bcet,$
& quadrando
$9b^4lltt + 36b^4lkt\theta - 18aabblktt + 36b^4kk\theta\theta - 36aabbkktt + 9a^4kktt$
$= (4bbee\theta\theta - 8bbeet\theta + 4bbeett)$ in $cc.$

Unde substituendo $aa+bb$ loco *cc,* fiet

$$4aabbee\theta\theta - 8aabbeet\theta + 4aabbaeettt + 4b^3ee\theta\theta - 8b^4eet\theta + 4b^4eett =$$
$$9b^3lltt + 36b^3lkt\theta - 18aabbtlktt + 36b^4kk\theta\theta - 36aabbkkktt + 9a^4kktt.$$

Vel positis loco *b* & *a* proportionalibus *r* & *θ*, in hac postrema æquatione, proveniet

$$44rree\theta^4 - 8rreet\theta^3 + 4rreett\theta\theta + 4r^4ee\theta\theta - 8r^4eet\theta + 4r^4eett =$$
$$9r^4lltt + 36r^4kt\theta - 18r^4lktt\theta\theta + 36r^4kk\theta\theta - 36rrkktt\theta\theta + 9kktt\theta^4.$$

Ex qua liquet problema esse solidum, nam radices hujus æquationis biquadraticæ exhibent, valorem tangentis quæsitæ *θ*, seu tangentis anguli TAW, seu anguli declinationis bilinei 2BAC.

COROLLARIUM V.

461. Si curvæ AB & A2B sunt æquales, adeo ut communem abscissam AH habeant & ordinatas æquales BH & 2B2H, ut in figura 113. erunt
$\mathrm{A1} = \mathrm{A2}$ & $\mathrm{A3} = 0$; $\mathrm{B1} = \mathrm{B2}$, & $\mathrm{C1} = \mathrm{C2}$; $\mathrm{D1} = \mathrm{D2}$, singulæ vero B3, C3 & D3, perinde ac A3 æquales 0 ; evanescet: pariter linea, quæ in figura 115 repræsentatur per DH vel D2H, alteraque, quæ erat in hac eadem figura 115 FA , in priore scilicet fig.113 erit BH. Hoc ergo casu formula generalis paragraphi 459 mutabitur in
$aat.\mathrm{BH} + (b\mathrm{B}t - \mathrm{AA}t).\mathrm{B2} = 2bb\theta.\mathrm{C2}$. In exempla proinde superiore (§.457.) curvæ parabolicæ retentis symbolis illic positis, erunt $\mathrm{BH} = y$;$\mathrm{B2} = \omega$ & $\mathrm{C2} = y - \omega = g$, atque adeo æquatio $aat.\mathrm{BH} + (bbt - aat).\mathrm{B2} = 2bb\theta.\mathrm{C2}$, abibit in
$aaty + bbtu - aatu = 2bb\theta g$, vel $aatg + bbt\omega = 2bb\theta g$; acque adeo subrogatis loco *a* & *b* proportionalibus θ & r , erit $tg\theta\theta + rrt\omega = 2rrg\theta$, ac proinde $\theta\theta = \frac{2rr\theta}{t} - \frac{rr\omega}{g}$;
quæ est eadem æquatio, quam citato loco reperimus.
At sciendum hunc solutionem imperfectam esse, quoniam æquatio amplius non inservit, si ratio tangentis *y* ad arcum *ω* minor fuerit duplicata ratione secantis anguli SAW ad radium in fig. 114. Sed eo in casu ipsa *θ*, ne quidem supposita arcuum circularium rectificatione, nulla æquatione algebraica exhiberi potest.

SCHOLION.

462. Ex allatis exemplis satis constare potest, quam late pateat solutio problematis in præsenti propositione exhibita. An vero & quousque ea cum solutione ejusdem problematis conveniat, quam summus geometra Joh. Bernoullius in tractatu, Gallico idiomate sub titulo *Essay d'une nouvelle Theorie de la Manoeuvræ des Vaisseaux,* nuperrime edito dedit, utriusque collatio ostendet.

Si angulus TAW datus sit, & alter angulus SAW quæratur, problema facile est ; tunc enim in superioribus *θ* & *r* cum reliquis scilicet *a, b*, *c* , sunt magnitudines cognitæ, & quæsita est *t.*