SECTION II.

Concerning the Motions of Water.

In the preceeding section we have considered the pressures of fluids from the weight only, which they exercise both on subjected planes as well as on the sides of vessels. Truly in this second section, which we will attend to shortly, it is concerned with the motion of liquids bursting forth through holes, cut out in any vessel as it pleases. This discussion has been investigated with its usefulness by the most celebrated geometers Castelli, Baliani, Torricelli, Borelli, Mariotte and others, and indeed not without reward from their careful examination, nor by the repeated method of exhaustion, since the Cel. Gulielmini added several of these items found [without integrating] ; and of these, truly Torricellii especially, has applied himself ingeniously to the theory of the motion of fluids. Yet no one has handled this material generally to a greater extent than the celebrated Varignon, who has shown the theorems most clearly and generally, applicable to any liquids, and who has been followed here in this matter, with the simple formulas included, just as with the Illiad in a nutshell.

CHAPTER IX.

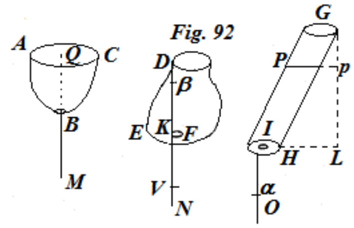
Concerning the motions of liquids erupting through small holes.

PROPOSITION XXX. THEOREM.

384. *Quantities or filaments of liquids, flowing out in equal times through any orifices with equal motions, are in a ratio composed from the ratio of the openings and the velocities, with which the liquids burst forth, and with each through its opening.* Fig. 92.

DEF and GH shall be vessels with some holes K and I bored through as it pleases, and in a given small time certain the filament KN may emerge with a certain uniform motion from the vessel DEF and the filament IO from the vessel GH, thus yet, so that on top so much liquid may be brought back to the vessel, as much as flowed out, and to such an extent that in each vessel the liquid is kept at the same level constantly ; and with these in place it is apparent on the other side the lengths of the filaments KN & IO as describe in equal time with an equal motion, with the velocities of the water emerging from the openings K and I to be in proportion, or rather these to represent the velocities. Truly it is agreed from the elements of geometry, the filaments themselves to be in a ratio composed from the ratio of the openings K and I, and of the prismatic basis, and the lengths KN and IO, that is, the speed with which these lengths are being described, whereby the proposition is agreed on.

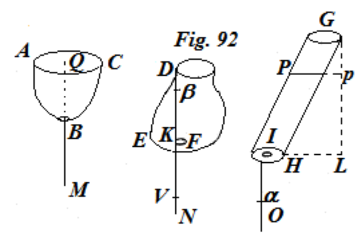
Fig. 92



PROPOSITION XXXI THEOREM.

385. *The quantities of motion, or the impetus of whatever liquids, erupting through whichever openings, are in a ratio composed from the ratios for the speed squared, the simple density or of the specific gravities of the liquids, and finally the simple ratio of the [areas of the] openings.* Fig. 92.

The filaments BM and KN erupt from the vessels ABC and DEF through the openings B & K in the same or equal times, the lengths of which express the speeds, with which these filaments are departing from the vessels. The specific gravities, or the densities of the liquids are expressed in the vessel ABC by the magnitude R, and the densities of the liquid DEF by S. On KN the third proportion for BM and KN is taken, and that shall be KV; it must be proven, the quantity of motion of the filament BM itself to be had to the quantity of motion of the filament KN as $B.BM^2.R$ to $K.KN^2.S$, or on account of the right line continued proportionals BM, KN, KV, just as B.BM.R to K.KV.S. [i.e. $KN^2 = BM \cdot KV \therefore KN^2 / BM = KV$.]



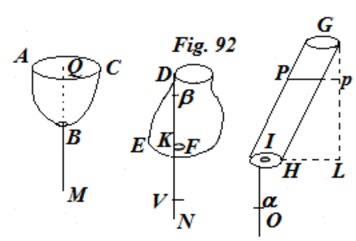
Demonstr. Because by the quantity of motion of some body is understood to be constructed from the mass of this body into its speed; from which that itself arises, the quantities of motion of the filaments BM and KN to be in a composite ratio from the ratios of the mass to the mass, and of the speed of one to the speed of the other; truly the masses BM and KN are between each other (§.18.) are as the volumes BM and KN, that is, as the products B.BM and K.KN multiplied by the densities of the liquids, or (§.33) by the specific gravities R and S of the same ; truly the velocities are as the lengths BM and KN, therefore the quantity of motion of the filament BM, that more simply we will call its motion henceforth, to the motion of the filament KN, as $R.B.\ BM^2$ to $S.K.KN^2$, that is, just as R.B.BM to S.K.KV. Which was required to be shown. The letters B and K denoting the [areas of the] openings of the vessels ABC & DEF cut out in some manner.

PROPOSITION XXXII THEOREM.

386. *The velocities of any liquids are in the square root ratio of the heights of the liquids above the openings, through which they burst out, if the surfaces of the liquids shall preserve the same position always, during the efflux.*

With the same in place, as in the preceding, BQ and KD shall be the heights of the liquids, and Kβ may be supposed to be the mean proportional between BQ and KD [i.e. $K\beta^2 = QB \cdot KD$], it must be shown that

$BM : KN = BQ : K\beta = \sqrt{BQ} : \sqrt{KD}$



Demonstr. The motions are generated from the same causes, and these causes are the pressures of the incumbent liquids for the openings, and the weight of the liquid QB (§. 255) is equivalent to the weight of the

column BQ.B, and (§. 33) its weight is as the product from its volume by the specific gravity R, and thus B.BQ.R is as the weight of the filament QB, and K.KD.S, on account of the same ratio, will be as the weight of the filament DK, so that
$\text{wt. QB} : \text{wt. DK} = \text{B.BQ.R} : \text{K.KD.S}$, from which, since the weight QB shall be to the weight or pressure DK, as the motion BM to the motion KN , there will be

$$\text{B.BQ.R} : \text{K.KD.S} = \text{mot. BM} : \text{mot.KN}\,(\S.385.) = \text{B.BM.R} : \text{K.KV.S},$$

and thus $\text{BQ} : \text{KD} = \text{BM} : \text{KV}$, and the ratio BQ to KD is as the square of the ratio BQ to Kβ, and BM: KV as the square of the ratio BM:KN,
[For $\text{BQ} : \text{KD} = \text{BQ} : (\text{K}\beta^2 / \text{BQ}) = \text{BQ}^2 : \text{K}\beta^2$ and
$\text{BM:KV} = \text{BM:}(\text{KN}^2 / \text{BM}) = \text{BM}^2\text{:KN}^2$.] therefore
$\text{BQ}^2 : \text{K}\beta^2 = \text{BM}^2 : \text{KN}^2$, & $\text{BQ} : \text{K}\beta = \text{BM} : \text{KN} = \sqrt{\text{BQ}} : \sqrt{\text{KD}}$. Q.E.D.

This proposition is obtained generally, wherever the holes B, K were cut in the base or the sides of the vessel, yet with friction being abstracted, which the liquids endure while moving out through the holes, and with other impediments of the motion, such as the resistance of the air, etc.

COROLLARY.

387. Because the quantities of liquids BM and KN (§.384) flowing out at the same time, are in a composite ratio from the ratios of the openings B and K and of the velocities, these quantities will be in the composite ratio from the ratio of the openings, and the square root ratio of the heights of the liquids in the vessels, with the densities of the liquids emerging being removed; for the absolute quantities of the filaments BM and KN or the masses of the same, are in a composite ratio from the square roots of the heights BQ and KD, from the simple densities R and S, and finally from the same simple ratio of the holes B and K.

SCHOLIUM.

388. The most celebrated Varignon shows in the *Comment. Acad. Reg. Scient.* 1703, 14 Nov. the velocities of the liquids bursting forth according to the circumstances of this proposition to be, as the roots from the product of the heights by the specific gravities, divided by the homologous densities, because it agrees properly with the proportion, since the (§. 33) densities shall be in direct proportion to the specific gravities.

PROPOSITION XXXIII THEOREM.

389. *To define how much of a liquid may flow out from a given vessel through a given hole in a given time, if the liquid may be kept constantly at the same given height.*
The given height of the liquid shall be *a*, the opening of the vessel *f,* the given time *t, m* the distance which a weight falls past with an accelerated motion, with another given

time d. Now, because the times of the descents of the weights (§. 151.) are in the square root ratio of the distances completed, there will be $\sqrt{m}:\sqrt{a}=d:n$,
where n denotes the time, in which a weight by falling with an accelerated motion may complete a descent a, and thus $n=d\sqrt{a}:\sqrt{m}$. Now because the liquid flows out with that velocity, which a weight can acquire in the case of falling freely from the height a, and, with this velocity once acquired, it will describe twice the distance equal to the motion from that in the time n, by which a was being traversed with an accelerated motion; the volume of the filament of liquid, flowing out with an equal uniform speed in this time n, as the weight acquired in the case by acceleration from the height a, will be $2af$. From which if the filament of liquid $2af$ flows out in the time n, in the time t the filament $2aft:n$ will flow out , and by substituting $d\sqrt{a}:\sqrt{m}$ for n, the quantity becomes $2aft\sqrt{m}:d\sqrt{a}=2ft\sqrt{(am)}:d$. Q.E.D.

SCHOLIUM.

390. The quantity of liquid flowing out in a given time t through the opening f may be called x, and it shall be a circle of diameter e; and there will be $f=\pi ee:4\delta$, which value substituted into $x=2ft\sqrt{(am)}:d$ produces $x=\pi ftee\sqrt{(am)}:2\delta d$, on putting $\delta:\pi$ for the ratio of the diameter to the circumference of the circle. [In modern terms, we take δ to be 1.]
Example. It is required to find the quantity of water flowing out from a vessel kept full at a height of 15 ft. 5 in. 7 lin., through a circular hole in the base one inch in diameter, in a time of 6 seconds. Therefore there will be :

$$a=15\text{ ft. }5\text{ in. }7\text{ lin.};\ t=6'';\ e=1\text{ in.}=12.\text{ lin.};\ \delta=113\ \&\ \pi=355.$$

Now, since Huygens observed any weight to fall through a height of 15 ft. and 1 in. in a time of one second [roughly 1 ft. too short, depending on how long the foot used actually was in modern terms], d can indicate 1" and m, 15 ft. 1 in. or 1172 lin. In place of the letters by using the numbers indicated at this stage, and the operation being completed with logarithms, the logarithm of x will be found, or the logarithm of the number sought $\pi ftee\sqrt{(am)}:2\delta d=6.4811958$ in cubic lines ; therefore if from this logarithm the three times the logarithm of 12 or the logarithm of 12 cubed may be taken away, $\log.x=3{\cdot}2436522$ will remain, the logarithm of the number sought expressed in cubic inches, to which logarithm the number 1752 agrees approximately, therefore just as many cubic inches water will flow out from the vessel in a time of 6 seconds, or one cubic foot and 24 cubic inches.

391. In turn from the quantity of water flowing out in a given time and from the observation given the height of the vessel will be elicited, according to the requirement, so that for any other time a certain amount of water will flow out, for from the above formula $x=\pi ftee\sqrt{(am)}:2\delta d$ there may be elicited $a=4\delta\delta ddxx:\pi\pi ffe^4ttm$, in which all the quantities have been given except a.

PROPOSITION XXXIV. THEOREM.

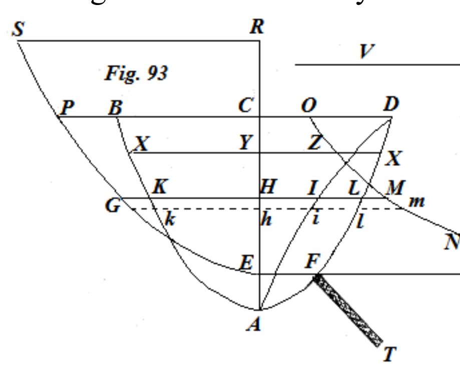
392. *If the liquid may flow out through the opening* F *from some vessel filled with liquid* BAD, *and the parabola* EGS *may be described about the line* AC *normal to the surface of the liquid* BD, *having its vertex at the point* E *just as far from the surface of the liquid as the hole of the vessel* F, *and the curves* AID *and* OMN *are of such a kind, that the individual rectangles under the given right line* V *and with the ordinate* HI *of the curve* AID, *shall be equal everywhere to the respective section of the vessel* KL, *as well also as the rectangle of the ordinate GH of the parabola and the ordinate* HM *of the curve* OMN *shall be equal everywhere to the respective section of the vessel* KL, *I say* :

[i.e. (*ord*.) HI = KL (area of surface)/V; (*ord*.) HM = KL/GH (velocity) .]

I°. *That the descent time of the surface* BD *at* XX *through the distance CY to the descent time at* KL *through the distance* CH *shall be, as the area* COZY *to the area* COMH.

II°. *That the velocity of the surface KL shall be in the composite ratio from the direct ratio of the ordinate of the parabola* GH, *and from the inverse ratio of the ordinate* HI *of the curve* AID, *that is, as* GH: HI. Fig. 93.

Demonstration. I°. Because on the parabola there is $\mathrm{PC}:\mathrm{GH}=\sqrt{\mathrm{EC}}:\sqrt{\mathrm{EH}}=$ ratio of the velocity of the liquid flowing out from the height EC to the velocity flowing out from the height HE (§.386.) ; thus, if PC may express the speed of that liquid, which has the height EC above the opening F, the ordinate GH expresses the speed with which an infinitely small filament of liquid FT emerges likewise in an indefinitely small time, with the height of liquid above the opening F to be the line EH. But in that very short time, in which the filament FT flows out, the surface KL descending by the infinitesimal distance H*h* itself will drop to *kl*, indeed thus so that the two prisms, the one under the section KL and with the height H*h*, and FT [by F] by necessity shall be equal, from which because (from the constr.) the section [area] $\mathrm{KL}=\mathrm{GH.HM}$, there will be $\mathrm{GH.HM.H}h=\mathrm{F.FT}$, or $\mathrm{HM.H}h:\mathrm{F}=\mathrm{FT}:\mathrm{GH}$. And FT, or the length of the filament, applicable to the speed, with which that flows out with a uniform motion, indicates the time of the flow of this filament, or the synchronous descent of the surface KI from that position into *kl* through the tiny distance H*h*, therefore HM.H*h*: F also expresses this same small descent time, and all the rectangles HM.H*h*, which are contained in the area CHMO : or this area itself, divided by F, expresses the time of descent of the surface KL from BD to KL through the distance CH; and by the same argument the time of descent from BD by XX through the interval CY is deduced to express the area CYZO, divided by F; therefore the $(\text{time through CH}):\mathrm{tCY}=\mathrm{CHMO}:\mathrm{CYZO}$. Which is the first part.



II°. Because the velocity of the surface KL is to the velocity GH of the filament FT, as H*h* to FT, that is, on account of the equality of the [volumes of the] prisms from KL by H*h* and of the prism from F by FT, just as F is to the plane KL, which (by const.) is equal to the rectangle V.HI, the velocity of the surface KL to GH = F : V.HI , and accordingly the speed of the surface KL = F.GH : V.HI , that is, with the constant ratio F:V omitted, which does not change the proportions of the indeterminate quantities, the speed of the surface KL is as GH: HI. Which was the second part.

COROLLARY I.

393. Therefore if the whole area COMNNE may be divided into the equal quadrilinear figures CYZO, YHMZ, HMNNE, or into some other having a ratio between each other, the times of the descent of the surface of the liquid BD through the distances CY, YH, HE or equally, by some other given ratio between themselves, as they have homologous areas. And hence it is clear, any vessel can be used to take the place of a water clock, provided the figure ECOMN of the shape of the vessel were divided by lines BD parallel to the surface of the water into equal agreeing parts, and the points of division at the wall of the vessel were duly marked. But I shall not linger further with these, since the cel. Varignon is seen to have quite fully exhausted this discussion in *Comment. Reg. Scient. Paris. Academ.* 1699, 29 Apr.

COROLLARY II.

394. Equally with the curves EGP, OMN given, the third curve AID will be given, and thus the curve BKALD the generator of the vessel, and that from the section KL = V.HI = GH.HM . And thus, if the vessel shall be a round solid from the rotation of the figure BAD about the axis CA, or also if the individual sections of the vessel BD, KL, &c. were similar and similarly placed figures, the ordinates HI will be, as the squares of the homologous orders KL.

COROLLARY III.

395. Therefore, if such a vessel BAD is sought, so that the descent CY of the surface of the liquid BD shall be as the times of descent, the curve OMN in this case will be a right line parallel to the axis AR, from which when the plane KL, or the circle from the diameter KL is equal to the rectangle GHM, the square from KL will be as the rectangle from HG by the given HM, or simply as GH, or the square-squared of the right line KL as the square HG, from which since this square shall be as the abscissa HE, the square-squared will be as HE, and thus the curve sought BAD will be a biquadratic parabola. As from others now has been shown by others more often.

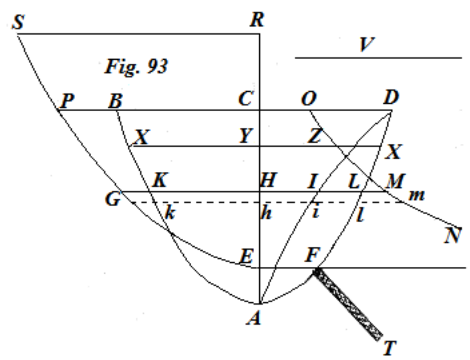
And in these the areas CYZO, CHMO only indicate the proportion of the times, by which the surface of the liquid BD by descending will describe the distances CY, CH. But if the absolute time shall require to be defined, in whatever manner the descent happens, the following must be consulted.

PROPOSITION XXXV. THEOREM.

396. *If the parameter of the parabola* EGS *were equal to the altitude* RE, *which the weight, in the case of beginning from rest at* R, *can descend in a time of one second with the naturally accelerated motion, and with everything put in place, as in the preceding*

Proposition, the number of seconds taken, in which the surface BD *falls a distance* CH, *will be equal to the number, which arises from the division of the area* CHMO *by twice the [area of the] opening F.*

If the liquid may remain constantly at the height ER, its first drop, next to the opening F, may be released with that speed, which by converting the motion into a height can ascend through the height ER in one second, so that by falling it can acquire the same speed, and the emerging drop pursuing its uniform motion describes a distance twice that of ER, by which since the particles shall be contiguous with the outflow of the liquid, in a time of one second the filament 2F.RE will flow out through the opening F, and if the liquid may remain at the same height HE it will flow out through the same opening in a time, which the filament 2F.HE itself has to one second in the square root ratio HE to RE, that is, in the ratio GH to SR; from which if the mass of the liquid 2F.HE flowed out in the time (GH: SR) , in how long a time would the mass $\mathrm{KL.H}h\ (\text{constr.}) = \mathrm{GH.HM.H}h$ flow out ? That will be found by the golden rule to happen in the time $\mathrm{GH}^2.\mathrm{HM.H}h : \mathrm{2F.HE.SR}$, because now RS or RE is the parameter of the parabola, and thus with GH equal to the rectangle HE.SR, it will be $\mathrm{tH}h = \mathrm{HM.H}h : 2\mathrm{F}$. therefore $\mathrm{tCH} = \text{area CHMO} : 2\mathrm{F}$, that is the number of seconds of the time, in which the amount of liquid BKLD flows out through the opening F will be found, freed from friction and other impediments to motion, by applying the homologous area CHMO to twice the opening F. Which was to be found and demonstrated.



Otherwise and shorter.

397. Above (§.392.*n*.1.) we came upon this ratio $\mathrm{HM.H}h : \mathrm{F} = \mathrm{FT} : \mathrm{GH}$, from which with the second dividing parts doubled there arises $\mathrm{HM.\ H}h : 2\mathrm{F} = \mathrm{FT} : 2\mathrm{GH}$. And FT:2GH indicates a small part of a second of time, in which the filament ST either with a mass equal to $\mathrm{KL.H}h$ or K*kl*L flows out. For 2RE or 2RS must express the velocity acquired in a time of one second from the fall of the weight through the height RE, because this height doubled 2RE divided by the speed 2RE, gives 1, or one second, and because both the velocities are acquired in descending through RE and HE and these with equal speeds, with which the water bursts out through the hole F, with that present in the vessel remaining at the heights RE and HE, are in the square root ratio RE to HE or SR to GH, that is, 2SR or 2RE to 2GH; for these SR and ER are equal, since RE (following the hypothesis) shall be the parameter of the parabola, from which with 2SR the speed of the water erupting may be expressed under the height RE, the other 2GH expresses the speed of the water erupting from the height HE, hence the filament ST flowing out with a uniform motion, the application to its speed 2GH, expresses the efflux time of this filament, or the efflux time of the mass of liquid H*kl*L, that is, $\mathrm{tH}h = \mathrm{FT} : 2\mathrm{GH}$, and since

above there shall be found $\mathrm{HM.H}h : 2F = \mathrm{FT} : 2\mathrm{GH}$, there will be $\mathrm{tH}h = \mathrm{HM.H}h : 2\mathrm{F}$; and thus $\mathrm{tCH} = \mathrm{CHMO} : 2\mathrm{F}$, as above. Q.E.D.

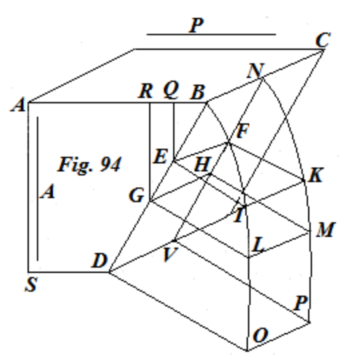
COROLLARY.

398. Therefore the present proposition may be elicited easily from the preceding as long as the value of that may be designated by the parameter of the parabola EGS, and it pertains always, that the area CYZO, &c. shall be provided in the ratio of twice the area of the opening, [and this ratio] shall express the times in seconds, in which homologous amounts of liquids must flow out.

PROPOSITION XXXVI. THEOREM.

399. *If the liquid from a full and constantly filled vessel* ADC *may flow out through the opening* EGHF, *constructed in the plane* BDIC *at some oblique angle to the horizontal, with the parabola* BLO *described about the axis* BD, *of which the parameter* P *shall be to the line A, but which a certain weight can fall with a natural acceleration in a motion beginning from rest in a given time t, as the sine of the angle of inclination ABD of the planes* ABC *and* BDI *to the whole sine,* [*i.e.* $\mathrm{P} : \mathrm{A} = \sin \mathrm{ABD} : 1 = \mathrm{RG{:}BG}$] *that is, just as* RG *to* BG; *the quantity Q of liquid flowing out in some given time* T *through that opening* EGHF *will be expressed by the frustum* EIKMLGHFE *of the parabolic prism* BDOPMNV *multiplied by double of the ratio of the times expressed, that is by* 2T: *t*. Fig. 94

Because with the vessel remaining full of liquid, the filament of liquid 2RG must flow out with a uniform motion through some physical point G of the section or of the opening EGHF, in that time in which a certain weight can be made to fall with an accelerated motion through the height RG, and because the times of descent of weights are in the square root proportion of the distances, following that which has been shown in article 151, the time of descent of the weight at the height *A*, which time is called *t*, will be to the time it descends through the height RG, as $\sqrt{\mathrm{A}}$ to $\sqrt{\mathrm{RG}}$; or just as A to $\sqrt{(\mathrm{A.RG})} = \sqrt{(\mathrm{P.BG})}$ since $\mathrm{A} : \mathrm{P} = \mathrm{BG} : \mathrm{RG}$, or



just as A to GL on account of the parabola BLO, on which there is $\mathrm{GL} = \sqrt{(\mathrm{P.BG})}$, and thus the time of descent [T] through the height RG of any weight is found to be GL.*t*:A. Now, because in this time the point G will pour out the filament of liquid 2RG, the same point in the time T will pour out the filament $2\mathrm{A.T.RG} : \mathrm{GL.}t = 2\mathrm{T.GL}^2 : \mathrm{GL.}t = 2\mathrm{T.GL} : t$. Similarly the physical point F will pour out, in the same time T, the filament 2T.FK:*t*, and thus for the remaining points of the opening respectively. Therefore the quantity of liquid Q pouring out through the whole opening GEFH in the given time T is made up from all the GI, FK, &c. which are contained in 2T:*t* , i.e. contained by the volume EILMKFHGE ; that is, made from this volume multiplied by 2T:*t*. Q.E.D.

COROLLARY I.

400. If the plane BDI with the opening EGH cut out from the right lines BG and RG, were at right angles to the horizontal, and thus P and A become equal; in order that the magnitude itself, or the distance A, because the weight in the time *t* with the accelerated motion beginning from rest may traverse the parameter of the parabola BLO.

COROLLARY II.

401. Again, the sine of the angle RBG may be called *i*; the whole sine *r*, the parameter of the parabola BLO *p*, which before was called P, and similarly the letter *a* may indicate the length A, and there becomes in addition $\mathrm{GE}=b,\ \mathrm{GH}=c,\ \mathrm{BG}=z,\ \mathrm{BE}=x$, and there will be $\mathrm{EI}=\sqrt{px}$ and $\mathrm{GL}=\sqrt{pz}$, and thus the four-lined parabola EGLI, or tri-lineam BLG − the tri-lineam BIE $=\frac{2}{3}z\sqrt{pz}-\frac{2}{3}x\sqrt{px}$ (that is, if *u* may be put for $\frac{2}{3}z\sqrt{z}-\frac{2}{3}x\sqrt{x}$ for factorizing) $=u\sqrt{p}$; and thus the volume EKLGF $=cu\sqrt{p}$, thus so that there shall be $\mathrm{Q}=2\mathrm{T}cu\sqrt{p}:t$.

COROLLARY III.

402. But, if BG, BE may not be given themselves, but only the proportion of these, which shall be the same as *m* to 1, the preceding paraboloid volume found in the previous paragraph will be had in terms of the quantities *b, c,* & *m* only. For because (following the hypothesis) $z:x=m:1$ there will be $z=mx$ &, $z\sqrt{z}-x\sqrt{x}=mx\sqrt{mx}-x\sqrt{x}=(m\sqrt{m},-1)$ into $x\sqrt{x}$. But $z-x=mx-x=b$, or $x=b:m-1$, & $x\sqrt{x}=b\sqrt{b}:m-1\sqrt{(m-1)}=b\sqrt{b}:n\sqrt{n}$, evidently if *n* may be put in place of *m*-1. Therefore $z\sqrt{z}-x\sqrt{x}=(m\sqrt{m},-1).b\sqrt{b}:n\sqrt{n}$. Hence *u* or $\frac{2}{3}z\sqrt{z}-\frac{2}{3}x\sqrt{x}=(\frac{2}{3}x\sqrt{x}-\frac{2}{3}).b\sqrt{b}:n\sqrt{n}=(2m\sqrt{m},-2).b\sqrt{b}:3n\sqrt{n}$.

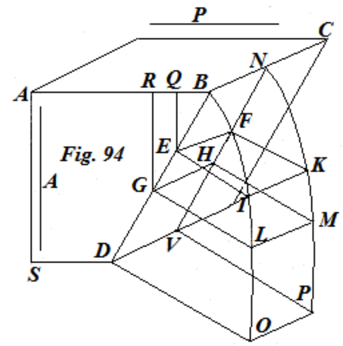
SCHOLIUM.

403. But because A or *a* specifies the height, which a certain weight by accelerating beginning from rest is able to describe in a given time *t* with a uniform motion, and because on account of friction and other impediments unable to happen, so that the filaments of fluid flowing out through the physical points G, E, F, &c. shall be precisely double of the height GR, EQ, EQ, &c. thus the proposition is more mathematical than physical. Just as also it is known how great the height A must be for a time of one second, or whatever other distance is required to be describe by a weight falling in a vacuum, it is not yet allowed in practice to use such a height, since it is not convenient t to remove all the kinds of, but the said height A can be elicited from the phenomena in the following manner :

PROPOSITION XXXVII. PROBLEM.

404. *With the same in place, as in the preceding proposition, to find the magnitude A, and thus the parameter P of the parabola BIO from observations or from the phenomena.* Fig. 94.

Because (§.§.384 & 386.) the quantities of fluid flowing through any physical points G, E, F, &c are in the square root ratio of the homologous GR, EQ, &c. or of GB, EB, &c. themselves, these quantities can be expressed correctly by the ordinates of the parabola GL, EI, &c., thus so that the quantity of fluid flowing out through the whole opening EGHF, can be expressed by the volume ELKFG. And with the above symbols retained, this same volume $= cu\sqrt{p}$, as found above, therefore



$Q = cu\sqrt{p}$, and $QQ = ccuup$, that is $p = QQ : ccuu$, and (§.399.) there is $p = ai : r$, therefore $a = rQQ : iccuu$.
Now, because (following the hypothesis) Q and the remainder *r*, *i*, *c*, *u* ; also are given from observation, the height itself sought *a* can be found readily from these quantities given with the aid of the equations found. Which was required to be found.

SCHOLIUM I.

405. Towards the illustration of this rule, and the finding of the height *a*, nothing more outstanding new is required to be observed, than what has been related near the end of the treatise *De Mensura aquarum fluentium* [in an Appendix on p. 191] of the celebrated Gulielmini, who by his own merit occupies the first place in the *Miscellaneis ltalicis Physico-Mathematicis* edited by P. Gaudentio Roberti. The most sharp-minded Gulielmini used a cylindrical vessel of some two feet diameter and somewhere on the side at right angles to the horizontal he cut out a square opening, of which the individual sides were of 3 lines of a Bolognian ft. [*i.e.* $\frac{1}{4}$ in.], and the base of the same opening, parallel to the horizontal, was 3 ft. 11 in. distant from the surface of the water in the vessel. In an experiment repeated eight times without any change, the most learned man found 32 lbs. &10 ounces of water to be flowing out through the opening in a time of one minute ; and a cubic inch of water containing one ounce and 146 grains, that is, 786 grains altogether. [1 cu. in. of water weighs 0.03606 lb. = 0.577 oz. = 252 grains in modern units.] After 32 lbs. &10 ounces, that is, 252160 grains is divided by 786 grains of one cubic inch, and $320\frac{320}{393}$ inches of this kind is found to flow out in that predicted time. Now, if we wish to apply the above formula to these observations, knowing *i* and *r* to be equal in this case, because the opening to the plane or the surface of the cylinder has been cut at right angles to the horizontal ; and therefore in this case there will be $p = a$. As the height sought *a* in the numbers we have found we may perform the whole operation with logarithms, putting

$\mathrm{Q} = 320\frac{320}{393}$ cu.in. ; $c = 3$ lin. ; $z = 3\frac{11}{12}$ ft. = 564 lines & $x = 561$ lines. With which in place, there will be :

Log. $(\frac{2}{3}z\sqrt{z})$.	3·9508274	Subtr.
Log. $(\frac{2}{3}x\sqrt{x})$.	3·9473531	

Log. $\left((\frac{2}{3}z\sqrt{z} : \frac{2}{3}x\sqrt{x})\right) =$ log. of the number 1.008033, 0·0034743

Therefore $(\frac{2}{3}z\sqrt{z} : \frac{2}{3}x\sqrt{x}) = 1.008033 : 1000000$, and thus $(\frac{2}{3}z\sqrt{z} : \frac{2}{3}x\sqrt{x})$ or $u : \frac{2}{3}x\sqrt{x} = 8033 : 1000000$, & $\log.u = \log.(\frac{2}{3}x\sqrt{x}) + \log.8035 - \log.1000000$. Now

Log. $(\frac{2}{3}x\sqrt{x})$.	3·9473531	Add.
Log. 8033	3·9048778	
	7.8522309	Subtr.
Log. 1000000	6.0000000	
Log. n	1.8522309	Add.
Log.$c = 3$ lin.	0.4771212	
Log.cu	2.3293521	
Log. Q cubic lines . . .	5.7437973	Subtr.
Log.cu	2.3293521	
Log.$(\mathrm{Q} : cu)$	3.4144452	Dupl.
Log.$(\mathrm{QQ} : ccuu) = \log. a$ in lines . . .	6.8288904	Subtr.
Log.144, ...	2.1583625	
Log.a or $(\mathrm{QQ} : ccuu)$ in feet, . . .	4.6705279	

Here the logarithm found is of the fundamental number a in the calculation of the amount of water flowing out through a given opening henceforth to be used, in place of the table, at the end of the same work, completed by the praiseworthy Guilielmini condemned to tedious labour. Indeed we have now shown, always to be able by logarithms to resolve a calculation, and besides by this one example we will show more clearly that we do not need Guilielmini's table; nor that the most distinguished author should have a need of the same, and he would have spared himself much tedium with these, if he might have considered matters in that direction, calculations in that way and by that method can be removed, as we have set out in this place.

SCHOLIUM II.

406. Now we may show the use of the preceding in another example, see Fig. 94. Therefore we may put with the section FG of Guilielmini, or the plane of the opening to be inclined at an angle 88 degrees to the horizontal plane, and the base of the rectangular opening GH to be 50 ft. high, indeed GE to be 10 ft. and finally the proportion of the

speeds at G and E shall be the same as 4 to 1. Now because (§.386.) the velocities at G and E are in the square root proportion of GR and EQ or of GB and EB themselves, these lines themselves will be as the squares of the velocities at G and E; that is, $\mathrm{GB}:\mathrm{EB}=16:1$, and above (§. 422.) there was put $z:x$ or $\mathrm{GB}:\mathrm{EB}=m:r$, therefore $m=16$, and thus $n=m-1=15$. Likewise $b=10$ ft. & $c=50$ ft..
Therefore $u=\left(2m\sqrt{m},-2\right).b\sqrt{b}:3n\sqrt{n}=28\sqrt{10}:\sqrt{15}$ therefore $cu=1400\sqrt{10}:\sqrt{15}$.

And with these put in place

Because Log$a$ . .	4.6705279	Add.
Log.$i$sin.88 deg.	9.9997354	
	14.6702633	
$\mathrm{Log.}(ai:r)=\log.p$.	4.6702633	
$\mathrm{Log.}\sqrt{p}$, . .	2.3351316	Add.
$\mathrm{Log.}(cu)=1400\sqrt{10}:\sqrt{15}$	3.0580824	
$\mathrm{Log.}\left(cu\sqrt{p}\right)=\mathrm{Q}$.	5·3932140.	

The number 247285 agrees closely with this logarithm as an approximation. Gulielminus found 247321 in his calculation [Example I, p. 198], the difference 36 is negligible besides the number 247285, by denoting how much cubic feet of water must flow out through that opening or section inclined to the horizontal in a time of one minute. And thus we proceed to other matters.

SECTIO II.
De Motibus Aquarum.

In Sectione praecedenti pressiones tantum fluidorum ex gravitate, quas tum in subjecta plana tum etiam in vasorum latera exercent, contemplati sumus. In hac vero secunda Sectione, quae ad motus liquorum per foramina, utlibet vasis insculpta, erumpentium pertinent, breviter excutiemus. Hoc argumentum utilitate sua celeberrimis Geometris Castello, Baliano, Torricellio, Borellio, Mariotto aliisque se probavit, atque ab ipsis diligenter nec sine fructu examinatum est quidem, non itero exhaustum, quandoquidem Cel. Gulielminus ipsorum repertis nonnulla adjecit; ipsorumque, maxime vera Torricellii, doctrinam motui fluminum ingeniose applicuit. Nemo tamen hanc materiam adeo generaliter pertractavit ac Celeb. Varignonius, qui praeeclara atque universalia, exhibuit theoremata, quibusvis liquoribus applicabilia, & quae hac in re assequutus est, velut Iliada nuci, simplicibus formulis inclusit.

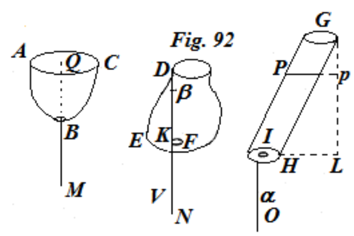
CAPUT IX.

De motibus Liquorum per minor foramina erumpentium.

PROPOSITIO XXX. THEOREMA.

384. *Quantitates, seu filamenta liquorum, aequali tempore per qualibet orificia aequabili motu effluentia, sunt in composita ratione ex rationibus orificiorum & velocitatum, quibus liquores, quisque per suum foramem,erumpunt.* Fig. 92.

Sint DEF, GH vasa quaelibet foraminibus quibuslibet K & I pertusa, exeantque data quodam tempusculo motu uniformi filamentum KN ex vase DEF & filamentum IO ex vase GH, ita tamen, ut superne tantum liquoris restituatur, quantum effluit, atque adeo in utroque vase liquor in eadem altitudine constanter conservatur; & hisce positis ultro liquet longitudines filamentorum KN & IO utpote aequali tempore uniformi motu descriptas, velocitatibus aquae ex foraminibus K & I erumpentis proportionari, vel potius has velocitates repraesentare. Ex Geometriae vera elementis constat, ipsa filamenta esse in composita ratione foraminum K & I tanquam basium prismatum, & longitudinum KN & IO, id est, velocitatum, quibus hae longitudines describuntur, quare constat Propositum.



PROPOSITIO XXXI THEOREMA.

385. *Quantitates motus, seu impetus liquorum quorumcunque & per quaevis formina erumpentium, sunt in ratione composita ex rationibus duplicata celeritatum, simpla densitatum seu gravitatum specificarum liquorum, & denique simpla foraminum.* Fig. 92.

Ex vasis ABC & DEF erumpant eodem vel aequali tempore, per foramina B & K filamenta BM & KN, quorum longitudines celeritates exponent, quibus haec filamenta e vasis egrediuntur. Gravitas specifica, seu densitas liquoris in vase ABC exponatur per magnitudinem R, densitasque liquoris DEF per S. In KN sumatur tertia proportionalis ad BM & KN & sit ea KV, & probari debet, quantitatem motum filamenti BM se habere ad quantitatem motus filamenti KN ut $B.BM^2.R$ ad $K.KN^2.S$, aut propter rectas BM, KN, KV continue proportionales, sicut B.BM.R ad K.KV.S.

Demonstr. Quoniam per quantitatem motus alicujus corporis intelligitur factum ex massa hujus corporis in celeritatem; qua id se fertur, erunt quantitates motus filamentorum BM & KN in composita ratione ex rationibus massae ad massam, & celeritatis unius ad celeritatem alterius, sunt vero massae BM & KN inter se (§.18.) ut volumina BM & KN, id est, ut facta B.BM & K.KN ducta in densitates liquorum, seu (§.33) in gravitatem eorundem specificam R & S ; velocitates vero sunt ut longitudines BM & KN, est ergo quantitas motus filamenti BM, quam simpliciter ejus motum deinceps dicemus, ad motum filamenti KN, ut $R.B.\ BM^2$ & ad $S.K.KN^2$, hoc est, sicut R.B.BM ad S.K.KV. Quod erat demonstrandum.

Literae B & K denotant orifica Vasis ABC & DEF quomodocunque insculpta.

PROPOSITIO XXXII THEOREMA.

386. *Velocitatus liquorum quorumlibet sunt in subduplicata ratione altitudinis liquorum super orificiis, per quae erumpunt, si liquorum superficies eandem semper, durante effluxu, positionem servarint.*

Iisdem positis, quae in praecedenti, sint BQ & KD liquorum altitudincs, & sumatur Kβ media proportionalis inter QB & DK, probari debet fore

$$BM : KN = BQ : K\beta = \sqrt{BQ} : \sqrt{KD}$$

Demonstr. Motus sunt ut eorundem causae genitrices, & hae causae sunt pressiones liquorum foraminibus incumbentium, atqui gravitatio liquoris QB (§. 255) aequivalet ponderi columnae BQ. B, & (§. 33) ejus pondus est ut factum ex volumine in gravitatem specificam R, adeoque B.BQ.R est ut gravitatio filamenti QB, & K.KD.S propter eandem rationem erit ut gravitatio filamenti DK, adeoque gravitatio $QB : \text{gravit. } DK = B.BQ.R : K.KD.S$, unde , cum gravitatio QB sit ad gravitationem seu pressionem DK, ut motus BM ad motum KN ,erit

$$B.BQ.R : K.KD.S = \text{motus } BM : \text{mot.}KN\,(\S.385.) = B.BM.R : K.KV.S,$$

adeoque $BQ : KD = BM : KV$, atqui est ratio BQ ad KD duplicata rationis BQ ad Kβ, & BM: KV duplicata rationis BM:KN, ergo

$BQ^2 : K\beta^2 = BM^2 : KN^2$, & $BQ : K\beta = BM : KN = \sqrt{BQ} : \sqrt{KD}$. Quod erat demonstrandum.

Haec propositio generaliter obtinet, ubicunque foramina B, K vasis insculpta fuerint in fundo vel ad latera vasis, abstrahendo tamen a frictionibus, quas liquores in foraminibus exeundo subeunt, allisque motus impedimentis, ut resistentia aëris, &c.

COROLLARIUM.

387. Quoniam quantitates liquorum simul effluentes BM & KN (§.384). sunt in composita ratione ex rationibus foraminum B & K & velocitatum, hae quantitates erunt in

composita ratione ex ratione foraminum, & subduplicata ratione altitudinis liquorum in vasis, abstrahendo a densitatibus liquorum exeuntium; nam quantitates absolutae filamentorum BM & KN seu massae eorundem, sunt in composita ratione ex subduplicata altitudinum BQ & KD, ex simpla densitatum R & S, & denique ex simpla itidem foraminum B, K.

SCHOLION.

388. Celeberrimus Varignon in Comment. Acad. Reg. Scient. 1703, 14 Nov. velocitates liquorum erumpentium in circumstantiis hujus propositionis demonstrat esse, ut radices ex factis altitudinum in specificas gravitates liquorum, divisis per densitates homologas, quod cum praesenti proportione probe conspirat, cum (§. 33) densitates gravitatibus specificis liquorum directe proportionales sint.

PROPOSITIO XXXIII THEOREMA.

389. *Definire quantum liquoris effluere debeat ex dato vase per datum foramen tempore dato, si liquor vasis constanter in eadem altitudine data conservetur.*

Sint data altitudo liquoris a , foramen vasis *f*, tempus datum *t*, *m* spatium quod grave accelerato motu, alio dato tempore *d*, perlabitur. Jam, quia tempora descensuum gravium (§. 151.) sunt in subduplicata ratione spatiorum confectorum, erit $\sqrt{m}:\sqrt{a}=d:n$, ubi *n* denotat tempus, quo grave altitudinem *a* accelerato motu cadendo absolveret, adeoque $n=d\sqrt{a}:\sqrt{m}$. Jam quia liquor ea velocitate effluit, quam grave casu ex altitudine *a* acquirere potest, &, hac velocitate semel acquisita, duplum spatium describit aequabili motu eo tempore *n*, quo accelerato motu simplex *a* percurrebatur; filamentum liquoris hoc tempore *n*, aequabiliter effluens celeritate aequali, quam grave acquirit ex casu accelerato ex altitudine *a* erit *2af*. Unde si tempore *n* effluit filamentum liquoris 2*af*, tempore *t* effluet filamentum $2aFT{:}n$ vel substituendo $d\sqrt{a}:\sqrt{m}$ pro *n*, quantitas $2aft\sqrt{m}:d\sqrt{a}=2\,ft\sqrt{(am)}:d$. Quod erat inveniendum.

SCHOLION.

390. Dicatur quantitas liquoris dato tempore *t* per foramen *f* effluens *x*, sitque foramen circulus diametri *e;* eritque $f=\pi ee:4\delta$, qui valor substitutus in $x=2\,ft\sqrt{(am)}:d$ exhibet $x=\pi\,ftee\sqrt{(am)}:2\delta d$, ponendo $\delta:\pi$ pro ratione diametri ad circumferentiam circuli.

Exempl. Quaeritur quantum aquae effluere debeat ex vase cylindrico indefinenter pleno 15 ped. 5 poll. 7 lin. alto, per foramen circulare in fundo pollicis unius in diametro, tempore 6 secundorum. Erunt ergo

$$a=15 \text{ ped. } 5 \text{ poll. } 7 \text{ lin.};\ t=6'';\ e=1 \text{ poll.}=12. \text{ lin.};\ \delta=113\ \&\ \pi=355.$$

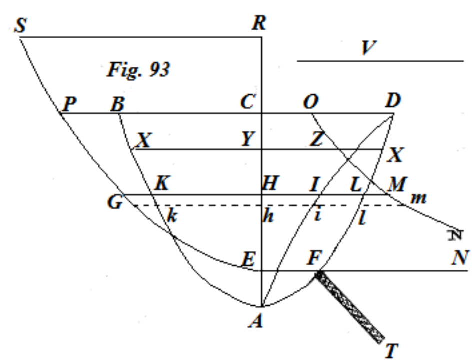
Jam, cum Hugenius observarit grave aliquod altitudinem 15 ped. & 1 toll. tempore minuti secundi perlabi, *d* significare potest 1" & *m,* 15 ped. 1 poll. seu 1172 lin. Loco literarum adbibendo numeros hactenus indicatos & operationem logarithmis perficiendo invenietur log-mus *x*, seu quantitatis $\pi\, ftee\sqrt{(am)} : 2\delta d = 6.4811958$ log-us numeri quaesiti in lineis cubicis; idcirco si ex hoc log-mo log-us ipsius 12 triplicatus, seu log-us cubi ex 12, auferatur, remanebit $\log.x = 3{\cdot}2436522$ log-us numeri quaesiti in pollicibus cubicis expresii, cui log-mo convenit proxime numerus 1752, ergo tot pollicum cubicorum erit aqua effluens ex vase 6 secundis temporis, vel unius pedis cubici, & 24 pollicum.

391. Vicissim ex aquae quantitate tempore dato effluente & observatione data elicietur altitudo vasis, ad requisita, ut quocunque alio tempore determinata quaedam aquae copia effluat, nam ex formula superiore $x = \pi\, ftee\sqrt{(am)} : 2\delta d$ elicietur $a = 4\delta\delta ddxx : \pi\pi\, ffe^4 ttm$, in qua excepta *a* omnes reliquae sunt quantitatcs datae.

PROPOSITIO XXXIV. THEOREMA.

392. *Si ex vase quocunque liquore pleno* BAD *liquor per foramen* F *effluat, & circa lineam* AC *liquoris superficiei* BD *normalem descripta sit parabola* EGS, *verticem suum habens in puncto* E *tantum distantia a liquoris superficie quantum foramen vasis* F, *& curvae* AID *ac* OMN *ejus indolis, ut singula rectangula sub data recta* V *& ordinata* HI *curvae* AID, *tum etiam sub ordinatis GH parabolae &* HM *curvae* OMN *aequalia ubique sint sectioni vasis respectivae* KL, *dico fore* I°. *Ut tempus descensus* BD *in* XX *per spatium CY ad tempus descensus in* KL *per spatium* CH *sit, ut area* COZY *ad aream* COMH. II°. *Ut velocitas superficiei KL sit in composita ratione ex directa ordinatae parabolae* GH, *& reciproca ordinatae* HI *curvae* AID, *id est, ut* GH: HI. Fig. 93

Demonstr. I°. Quia in parabola est $\mathrm{PC} : \mathrm{GH} = \sqrt{\mathrm{EC}} : \sqrt{\mathrm{EH}}$ (§.386.) = velocitas liquoris effluentis sub altitudine EC ad celeritatem effluentis sub altitudine HE; adeo, si PC exponat celeritatem illius liquoris, qui habet altitudinem EC super foramen F, ordinata GH exponet celeritatem qua filamentum liquoris indefinite parvum FH tempusculo itidem indefinite exiguo erumpit, existente liquoris altitudine super foramen F linea EH. Sed eo tempusculo, quo effluit filamentum FT, superficies KL descendens spatiolo infinitesimo Hh se demittet in kL, ita quidem ut duo prismata sub sectione KL & altitudine Hh, & FT necessario aequalia sint, unde



quia (constr.) sectio $\mathrm{KL} = \mathrm{GH.HM}$, erit $\mathrm{GH.HM.Hh} = \mathrm{F.FT}$, vel $\mathrm{HM.Hh} : \mathrm{F} = \mathrm{FT} : \mathrm{GH}$. Atqui FT, seu longitudo filamenti, applicata ad celeritatem, qua id effluit aequabili motu, exponit tempus effluxus hujus filamenti, seu synchroni descensus superficiei KI ex hoc situ in kl per spatiolum Hh, ergo etiam HM.Hh: F exponit hoc idem tempusculum descensus, & omnia rectangula HM.Hh, quae in area CHMO continentur: seu haec ipsa area, divisa per F, exponet tempus descensus superfiei KL ex BD in KL per spatium CH; eodemque argumento colligitur tempus descensus BD in XX per spatium CY exponi area CYZO, divisa per F; ergo tempus per $\mathrm{CH} : \mathrm{tCY} = \mathrm{CHMO} : \mathrm{CYZO}$. Quod est primum.

II°. Quia velocitas, superficiei KL est ad velocitatem GH filamenti FT, ut Hh ad FT, hoc est, ob aequalitatem prismatis ex KL in Hh & prismatis ex F in FT, sicut F ad planum KL, quod (constr.) aequatur rectangulo V.HI, erit velocitas superficiei KL ad $GH = F : V.HI$, atque adeo cel: superficiei $KL = F.GH : V.HI$, id est, omissa ratione constanti F: V , quae proportiones indeterminatarum magnitudinum non alterat, velocitas superficiei KLest ut GH: HI. Quod erat secundum.

COROLLARIUM I.

393. Si itaque area tota COMNNE dividatur in quadrilinea aequa CYZO, YHMZ, HMNNE, aut quamcunque aliam inter se rationem habentia, tempora descensus superficiei liquoris BD per spatia CY, YH, HE aut aequalia, vel aliam inter se datam rationem, quam habent homologae areae. Atque hinc liquet, quodlibet vas adhiberi posse loco clepsydrae, dummodo figura ECOMN figurae vasis conveniens in suas partea aequales divisa fuerit per lineas superficiei horizontali aquae BD parallelas, atque divisionum puncta in latere vasis rite notata fuerint. Sed hisce diutius non immorabor, quandoquidem Celeb. Varignon hoc argumentum pene exhausisse videtur in Comment. Reg. Scient. Paris. Academ. 1699, 29 Apr .

COROLLARIUM II.

394. Pariter datis curvis EGP, OMN dabitur tertia AID, atque adeo curva BKALD genitrix vasis, idque ex sectione $KL = V.HI = GH.HM$. Adeoque, si vas sit solidum rotundum ex conversione figurae BAD circa axem CA, vel etiam si singulae vasis sectiones BD, KL, &c. figurae similes & similiter positae fuerint, ordinatae HI erunt, ut quadrata ordinatarum homologarum KL.

COROLLARIUM III.

395. Idcirco, si quaeratur vas BAD tale, ut descensus CY superficiei liquoris BD sint ut tempora descensus, erit curva OMN hoc casu linea recta axi AR parallela, unde cum planum KL, seu circulus ex diametro KL aequatur rec-lo GHM, erit quadratum ex KL ut rec-lum ex HG in datam HM, vel simpliciter ut GH, vel quadrato-quadratum rectae KL ut quadratum HG, unde cum hoc quadratum sit ut abscissa HE, erit quadrato-quadratum ut HE, atque adeo curva quaesita BAD erit parabola biquadratica. Ut ab aliis saepius jam demonstratum est.

In hisce areae CYZO, CHMO tantum proportionem temporum indicant, quibus superficies liquoris BD spatia CY,CH descendendo describit. Sed si tempus absolutum sit definiendum, quoquilibet descensus contingit, consuli debet sequens.

PROPOSITIO XXXV. THEOREMA.

396. *Si parabolae EGS parameter aequalis fuerit altitudini RE, quam grave, casum* a *quiete in* R *incipiens, motu naturaliter accelerato perlabi potest uno minuto secundo temporis, positisque omnibus, quae in Propositione praecedenti, numerus minutorum*

secundorum, quibus superficies BD *spatium* CH *cadendo conficit, aequabitur numero, qui oritur ex divisione areae* CHMO *per duplum foraminis F.*

Si liquor constanter maneret in altitudine ER, ejus prima gutta, prope foramen F, ea celeritate erumperet, qua motum in altum convertens tempore eodem minuti secundi per altitudinem ER ascendere potest, quo cadendo eandem celeritatem acquirit, atque gutta erumpens motum suum uniformem prosequens hoc tempore describet spatium duplum ipsius ER, unde cum effluentis liquoris particulae contiguae sint, tempore unius minuti secundi effluet per foramen F filamentum 2F.RE & si liquor maneret in altitudine HE per idem foramen effluet tempore, quod se habet ad unum minutum secundum in subduplicata ratione HE ad RE , id est, in ratione GH ad SR, filamentum 2F.HE; unde si massa liquoris 2F.HE effluit tempore (GH: SR) quanto tempore effluet massa $\mathrm{KL.Hh}\ (\mathrm{constr.}) = \mathrm{GH.HM.H}h$? Id per regulam auream invenietur fieri tempore $\mathrm{GH}^2.\mathrm{HM.H}h : \mathrm{2F.HE.SR}$, jam quia RS vel RE est parameter parabolae, atque adeo GH aequale rec-lo HE.SR, erit $\mathrm{tH}h = \mathrm{HM.H}h : 2\mathrm{F}$. ergo $\mathrm{tCH} = \text{areae } \mathrm{CHMO} : 2\mathrm{F}$, id est numerus secundorum temporis, quo liquoris quantitas BKLD per foramen F effluit, praescindendo a frictionibus aliisque motus impedimentis, reperietur, applicando aream homologam CHMO ad duplum foraminis F. Quod erat inveniendum & demonstrandum.

Aliter & brevius.

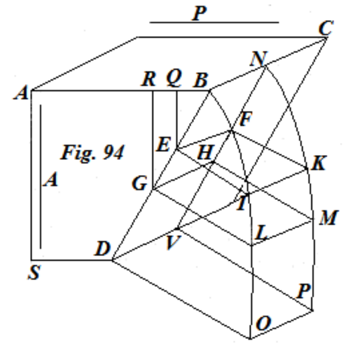
397. Supra (§.392.n.1.) incidimus in hanc analogiam $\mathrm{HM.H}h : \mathrm{F} = \mathrm{FT} : \mathrm{GH}$, unde duplicatis consequentibus provenit $\mathrm{HM.\ H}h : 2\mathrm{F} = \mathrm{FT} : 2\mathrm{GH}$. Atqui FT:2GH indicat particulam minuti secundi temporis, qua filamentum ST vel si aequalis massa KL.H*h* vel K*kl*L effluit. Nam 2RE vel 2RS exponere debet velocitatem acquisitam tempore unius minuti secundi ex lapsu gravis per altitudinem RE, quia haec altitudo duplicata 2RE divisa per celeritatem 2RE, dat I, seu unum minutum secundum, & quia velocitates & descensu per RE & HE acquisitae & hisce aequales celeritates, quibus aqua per foramen F erumpit, existente ea in vase in altitudinibus RE & HE manentibus, sunt in subduplicata ratione RE ad HE seu SR ad GH, id est, 2SR vel 2RE ad 2GH; hae enim SR & ER aequeales sunt, cum RE (secundum hypothesin) parameter sit parabolae, unde cum 2SR exponat celeritatem aquae erumpentis sub altitudine RE, altera 2GH exponet celeritatem aquae erumpentis sub altitudne HE, hinc filamentum ST aequabili motu effluens, applicatum ad suam celeritatem 2GH, exponit tempus effluxus hujus filamenti, seu tempus effluxus massae liquidae H*kl*L, id est, $\mathrm{tH}h = \mathrm{FT} : 2\mathrm{GH}$, & cum supra inventum sit $\mathrm{HM.H}h : 2F = \mathrm{FT} : 2\mathrm{GH}$, erit $\mathrm{tH}h = \mathrm{HM.H}h : 2\mathrm{F}$; atque adeo $\mathrm{tCH} = \mathrm{CHMO} : 2\mathrm{F}$, ut supra. Quod erat demonstrandum.

COROLLARIUM.

398. Ergo praesens propositio ex antecedenti facile elicitur parametro parabolae EGS tantum eum valorem assignando, quem Propositio indicat, & semper continget, ut area CYZO, &c. applicatae ad duplum foraminis praebiturae sint tempora minutis secundis expressa, quibus homologae liquoris quantitates effluere debent.

PROPOSITIO XXXVI. THEOREMA.

399. *Si liquor ex vase amplo & constanter pleno* ADC *effluat per lumen* EGHF *plano* BDIC *horizonti utcunque obliquo insculptum, descripta circa axem* BD *parabola* BLO, *cujus parameter* P *sit ad lineam A, quam grave quoddam dato tempore* t *motu a quiete incipiente, sed naturaliter accelerato, perlabi potest, ut sinus anguli ABD inclinationis planorum ABC, &* BDI *ad sinum totum, hoc est, sicut* RG *ad* BG; *quantitas Q liquoris tempore quolibet dato* T *per lumen illud* EGHF *effluentis exponetur frusto* EIKMLGHFE *prismatis parabolici* BDOPMNV *ducto in duplum exponentis rationis temporum, hoc est in* 2T:t. Fig.94



Quia manente vase liquoris pleno, per quodlibet punctum physicum G sectionis seu luminis EGHF filamentum liquoris 2RG effluere debet aequabili motu, tempore eo, quo grave quoddam altitudinem RG accelerato motu descendens conficere potest, & quia tempora descensus gravium sunt in subduplicata proportione spatiorum, juxta ea quae articulo 151 ostensa sunt, erit tempus descensus gravis in altitudine *A,* quod tempus nominatum est *t,* ad tempus per altitudinem RG, sicut $\sqrt{\mathrm{A}}$ ad $\sqrt{\mathrm{RG}}$; vel sicut A ad $\sqrt{(\mathrm{A.RG})} = \sqrt{(\mathrm{P.BG})}$ quandoquidem $\mathrm{A:P = BG:RG}$, vel sicut A ad GL ob parabolam BLO, in qua est $\mathrm{GL} = \sqrt{(\mathrm{P.BG})}$, adeoque tempus descensus per altitudinem RG alicujus gravis reperitur esse GL.t:A. Jam, quia hoc tempore punctum G effundit filamentum liquoris 2RG, idem punctum effundet tempore T, filamentum $\mathrm{2A.T.RG : GL.t = 2T.GL^2 : GL.t = 2T.GL : t}$. Similiter punctum physicum F effundet, eodem tempore T, filamentum 2T.FK:t, atque sic respective reliqua luminis puncta. Igitur quantitae Q liquoris per universum lumen GEFH dato tempore T effluentis est factum ex omnibus GI, FK, &c. quae in solido EILMKFHGE continentur in 2T:t; id est, factum ex hoc solido in 2T:t. Quod erat demonstrandum.

COROLLARIUM I.

400. Si planum BDI lumine EGH pertusum, rectum fuerit horizonti, rectae BG & RG, atque adeo P ac A aequales fient; adeo ut magnitudo ipsa, seu spatium A, quod grave tempore *t* accelerato motu a quiete incepto percurrere potest parameter sit parabolae BLO.

COROLLARIUM II.

401. Dicantur porro sinus anguli RBG, *i;* sinus totus *r*, parabolae BLO parameter *p,* qui antea P nominabatur, similiterque longitudinem A litera *a* insigniat, fiantque insuper $\mathrm{GE} = b,\ \mathrm{GH} = c,\ \mathrm{BG} = z,\ \mathrm{BE} = x$, eruntque $\mathrm{EI} = \sqrt{px}$ & $GL = \sqrt{pz}$, adeoque quadrilineum parabolicum EGLI, seu trilineum BLG-trilineum BIE

$=\frac{2}{3}z\sqrt{pz}-\frac{2}{3}x\sqrt{px}$ (id est, si ad contrahendam formulam pro $\frac{2}{3}z\sqrt{z}-\frac{2}{3}x\sqrt{x}$ ponatur *u*) $=u\sqrt{p}$; adeoque solidum EKLGF $=cu\sqrt{p}$, adeo ut sit $Q=2Tcu\sqrt{p}:t$.

COROLLARIUM III.

402. Sed, si ipsae BG, BE datae non sint, sed earum tantum proportio, quae sit eadem cum *m* ad 1, solidum parabolicum praecedente paragrapho inventum habebitur in solis quantitatibus *b, c,* & *m*. Nam quia (secundum hypothesin) $z:x=m:1$ erit $z=mx$ &, $z\sqrt{z}-x\sqrt{x}=mx\sqrt{mx}-x\sqrt{x}=(m\sqrt{m},-1)$ in $x\sqrt{x}$. Sed $z-x=mx-x=b$, seu $x=b:m-1$, & $x\sqrt{x}=b\sqrt{b}:m-1\sqrt{(m-1)}=b\sqrt{b}:n\sqrt{n}$, si scilicet loco *m*-1 ponatur *n*. ergo $z\sqrt{z}-x\sqrt{x}=(m\sqrt{m},-1).b\sqrt{b}:n\sqrt{n}$. Hinc *u* seu $\frac{2}{3}z\sqrt{z}-\frac{2}{3}x\sqrt{x}=(\frac{2}{3}x\sqrt{x}-\frac{2}{3}).b\sqrt{b}:n\sqrt{n}=(2m\sqrt{m},-2).b\sqrt{b}:3n\sqrt{n}.$

SCHOLION.

403. Sed, quia A vel *a* denotat altitudinem, quam grave quoddam accelerato motu a quiete incepto cadendo scribere potest tempore dato *t,* & quia propter frictiones aliaque impedimenta fieri nequit, ut filamenta fluidi eodem tempore effluentia per puncta physica G, E, F, &c. praecise dupla sint altitudinum GR, EQ, EQ, &c. ideo propositio Mathematica est potius quam Physica. Etsi enim scitur quantam oporteat esse altitudinem A tempore unius minuti secundi, vel quolibet alio a gravi decidenti in vacuo describendam, non tamen in praxi talem altitudinem adhibere licet, quandoquidem ab omnis generis resistentiis abstrahere non semper convenit, sed dicta altitudo A ex phaenomenis est elicienda sequenti modo :

PROPOSITIO XXXVII. PROBLEMA.

404. *Iisdem positis, quae in praecedenti Propositione, invenire magnitudenem A, atque adeo parametrum P parabolae BIO ex observationibus seu phaenomenis.* Fig. 94.

Quia (§.§.384 & 386.) quantitates fluidi per puncta quaelibet physica G, E, F, &c fluentia sunt in subduplicata ratione homologarum GR, EQ., &c. seu ipsarum GB, EB, &c. quantitates illae per ordinatas parabolae GL, EI, &c. recte exponi possunt, adeo ut quantitas fluidi, per universum lumen EGHF effluentis, debeat exponi solido ELKFG. Atquae retentis symbolis superioribus, erit solidum istud $=cu\sqrt{p}$, ut supra inventum, ergo $Q=cu\sqrt{p}$,& $QQ=ccuup$, id est $p=QQ:ccuu$, atqui (§.399.) est $p=ai:r$, ergo $a=rQQ:iccuu$. Jam, quia (secundum hypothesin) Q ex observatione, & reliquae *r, i, c, u* ; etiam datae sunt, ipsa altitudo quaesita *a* ex hisce datis ope repertae aequationis facili negotio haberi potest. Quod erat inveniendum.

SCHOLION I.

405. Ad illustrationem hujus canonis, & inventionem altitudinis *a*, nullam praestantiorem novi observationem, quam quae relata est circa finem Tractatus *De Mensura aquarum fluentium* Celeberrimi Gulielmini, qui merito suo primum locum in Miscellaneis Italicis Physico-Mathematicis a P. Gaudentio Roberti editis, occupat. Adhibuerat sagacissimus Gulielminus vas cylindricum bipedalis diametri alicubi ad latus horizonti rectum lumine quadrato pertusum, cujus singula latera erant 3 linearum pedis Bononiensis, & ejusdem luminiae basis horizonti parallela 3 pedibus cum 11 unciis (pollicibus) ab aquae superficie distabat in vase. Experimento octies repetito sine ulla variatione invenerat Doctissimus Vir, 32 libr. & 10 uncias per lumen illud tempore unius minuti horarii effluxisse ; & pollicem cubicum continere unciam unam aquae cum granis 146, id est, in universum grana 786. Divisit postea 32 libr. 10 uncias, id est, 252160 grana per 786 grana unius pollicis cubici, invenitque $320\frac{320}{393}$ pollices ejusdem generis praedicto illo tempore effluxisse . Jam, si formulam superiorem observationi isti velimus applicare, sciendum *i* & *r* hoc casu aequari, quia lumen plano aut superficiei cylindri horizonti rectae insculptum est; ac proinde erit hoc casu $p = a$. Ut altitudinem quaesitam *a* in numeris inveniamus totam operationem logarithmis perficiemus, ponentes $\mathrm{Q} = 320\frac{320}{393}$ poll. cub. $c = 3$ lin. $z = 3\frac{11}{12}$ ped. $= 564$ lineis & $x = 561$ lineis. Quibus positis, erunt

$$\left.\begin{array}{ll}\text{Log. } (\tfrac{2}{3}z\sqrt{z}). \quad . \quad . \quad . \quad . \quad . & 3{\cdot}9508274 \\ \text{Log. } (\tfrac{2}{3}x\sqrt{x}). \quad . \quad . \quad . \quad . \quad . & \underline{3{\cdot}9473531}\end{array}\right\}\text{Subtr.}$$

$$\text{Log. } \left((\tfrac{2}{3}z\sqrt{z} - \tfrac{2}{3}x\sqrt{x})\right) = \text{log.numeri } 1.008033.... \quad 0.0034743$$

Ergo $(\frac{2}{3}z\sqrt{z} - \frac{2}{3}x\sqrt{x}) = 1.008033 : 1000000$, adeoque $(\frac{2}{3}z\sqrt{z} - \frac{2}{3}x\sqrt{x})$ seu $u : \frac{2}{3}x\sqrt{x} = 8033 : 1000000$, & $\log.u = \log.(\frac{2}{3}x\sqrt{x}) + \log.8035 - \log.1000000$. Jam

Log. $(\frac{2}{3}x\sqrt{x})$.	3·9473531	Add.
Log. 8033	3·9048778	
	7.8522309	Subtr.
Log. 1000000	6.0000000	
Log. n	1.8522309	Add.
Log.$c = 3$ lin	0.4771212	
Log.cu	2.3293521	
Log. Q lineis cubicis expressae . .	5.7437973	Subtr.
Log.cu	2.3293521	
Log.$(\mathrm{Q}:cu)$	3.4144452	Dupl.
Log.$(\mathrm{QQ}:ccuu) = \log.\, a$ in lineis . . .	6.8288904	Subtr.
Log.144, ...	2.1583625	
Log.a seu $(\mathrm{QQ}:ccuu)$ in pedibus, . . .	4.6705279	

Hic inventus log-us ipsius a est fundamentalis numeras in calculo quantitatis aquae per datum foramen effluentis deinceps adhibendus, loco Tabulae, eundem in finem, a supra laudato Guilielmino improbo laboris taedio confectae. Nos enim, qui logarithmis semper calculum absolvi posse jam ostendimus , & uno adhuc exemplo clarius id probabimus, Tabula Guilielminea plane non indigemus; nec eadem Clariss. ejusdem Autor opus habuisset, sibique ipsi multo taedio pepercisset, si animum ad vertisset, calculam eo modo eaque methodo subduci posse, quam hoc loco exposuimus.

SCHOLION II.

406. Videamus nunc praecedentium usum in aliquo exemplo Fig. 94. Ponamus igitur cum Guilielmino sectionem FG, seu luminis planum angulo 88 grad. inclinatum esse ad horizontale planum, atque luminis rectanguli basin GH esse 50 ped. altitudinem vero GE, 10 ped. ac denique proportio celeritatum in G & E sit eadem quae 4 ad 1. Jam quia (§.386.) velocitates in G & E sunt in subduplicata proportione ipsarum GR & EQ seu ipsarum GB & EB, hae linae ipsae erunt ut quadrata velocitatum in G & E; id est, $\mathrm{GB}:\mathrm{EB} = 16:1$, atqui supra (§. 422.) erat posita $z : x$ seu $\mathrm{GB}:\mathrm{EB} = m:r$, ergo $m = 16$, adeoque $n = m - 1 = 15$. Item $b = 10$ ped. & $c = 50$ ped.
Ergo $u = \left(2m\sqrt{m}, -2\right).b\sqrt{b} : 3n\sqrt{n} = 28\sqrt{10} : \sqrt{15}$ ergo $cu = 1400\sqrt{10} : \sqrt{15}$.
Hisce positis

Quia Log$a$ . .	4.6705279	Add.
Log.isin.88 grad.	9.9997354	
	14.6702633	

$\text{Log.}(ai:r)=\log.p$	.	4.6702633	
$\text{Log.}\sqrt{p}$,	. .	2.3351316	Add.
$\text{Log.}(cu)=1400\sqrt{10}:\sqrt{15}$		3.0580824	
$\text{Log.}\left(cu\sqrt{p}\right)=\text{Q}$	.	5·3932140.	

Huic log-mo numerus 247285 quam proxime convenit. Gulielminus suo calculo invenit 247321, differentia. 36 est insensibilis prae numero 247285, denotante quot pedes cubici tempore unius-minuti horarii effluerae debeant per lumen illud seu sectionem horizonti inclinatam.

Atque sic in aliis procedendum.