CHAPTER X.

Concerning the Flow of Rivers.

DEFINITIONS.

I.

By the general designation of a *river* in this case water is indicated on the surface of the earth generally flowing incessantly in a journey with various bends from higher places to more lower lying places within its cavities and channels.

II.

The *channel* of a river is the cavity on the surface of the earth, within which the waters flow.

III.

The *section of a river* is the common section of the channel and of the plane cutting the channel at right angles to the base. A section of this kind is ordinarily a somewhat irregular shape and on that account is accustomed to be called the *natural* section to distinguish that from artificial ones : for

IV.

The *artificial section* is always a rectangular parallelogram, because it is understood to be the section of an artificial hollow, or having the form of a parallelepiped.

V.

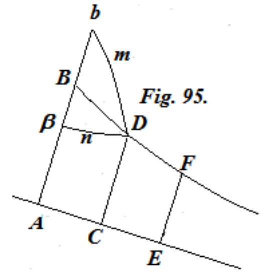
The *vital height* of the river is the distance of some point from the surface of the river to its bottom. And the *vital height* of any artificial section is that of the bottom.

V.

407. A river is said to remain in the same state, or to be in a state of constancy, when during the flow nowhere does its surface become raised and swollen, nor elsewhere does its surface become depressed or decreases, but always it keeps the flow enduring in the same steady course ; yet with accidental inequalities being removed forcibly, evidently by whirlpools &c. which usually arise from the bottom, and with other troublesome asperities.

PROPOSITION XXXVIII. THEOREM.

408. *With a river present in the permanent state, in equal times equal amounts of water will flow through all the sections of the river.* Fig. 95.



If it be denied, therefore more water may pass through the section AB than through the neighbouring section CD, and the water may well up between these sections at *bm*D, for the sake of an example : indeed if more water should pass through the section CD than through AB, water may decrease between these sections at *βnD* ; and thus the river will not remain in the same state, contrary to the hypothesis.

COROLLARY I

409. And conversely also, the surface of the river will be remaining, if through the individual sections AB, CD, EF, &c. of the river the same quantity of water will flow through in the same amounts of time.

PROPOSITION XXXIX. THEOREM.

410. *If a reservoir full of water* ABCD *may be joined with a lake ξωα indefinitely filled by an inclined channel IFθ closed above by the curved surface* IHFξ*, and with the exit of the water from the reservoir through the opening* CI*, thus water may flow into the channel, so that at some points* F, E *of the section* EF *the speeds may depend on the pressures of the incumbent water according to the heights sL, rO, by which these points remain apart on the surface of the reservoir* AD*, and so that those pressures at the same times make equal amounts of water flow, through the individual sections of the* GH, EF*. I say that it is the case, that in the same steady state water in the inclined channel will flow to the distant curved cover* IHFξ*, by which the cover of the channel is made indicated by that surface of the water, only as much water may flow out, as the amount continually passing through the individual sections of the channel.* Fig. 96.
[From the figure, we note that the diameter of the channel is made progressively narrower, as the speed of the water increases in travelling down the inclined plane.]

If you deny this, therefore water may be raised, with the covering moved away from the curve IFξ, as between the sections GH and EF, therefore less of the water will flow through the section EF than through the preceding one towards CI by the natural force of the incumbent pressure, contrary to the hypothesis.

SCHOLIUM

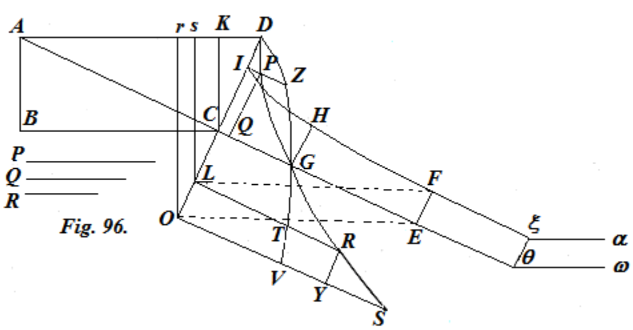


Fig. 96.

411. We understand these matters by calling them *natural pressures*, with the aid of which the velocities of the water, as at E & F, are in the square root proportion of the heights O*r*, L*s*, or of OD and LD themselves. For whatever the surface IHFξ shall be, the same abundance of water will flow through the individual sections GH, EF, &c. on account of the continuity of the parts of the water, as long as the cover remains with that curved surface of the channel, yet thus neither are the speeds of the waters yet governed only by these incumbent parts, according to the steady course and the law of the natural pressure ; that is, the velocities, e.g. at E & F, are not in this

case generally by necessity in the square root ratio *Or* to *Ls* [*i.e.* from the continuity equation, a narrower channel leads to higher speed, etc]. Thus it might happen that for a very distant covering the surface may be changed, while as far as water flowing down in the channel that situation shall be found, which is directed especially by the natural pressure of the water, and occurs constantly; and this therefore is that situation itself, as afforded by the cover IFξ, thus so that the water may be able to flow continually in the channel, if its surface may have the shape of this, whether the water shall be covered by this roof or not.

COROLLARY I.

412. Under this hypotenuse of the natural pressure, it can become known how much water must flow through the individual sections of the channel in a given time. With the first section IC at D produced as far as for water crossing the surface on the reservoir AD, and indefinitely downwards to O; and following the conditions of paragraph 404, with the vertex D, the parabola DTV is described around the axis DO, its ordinates LT and OV, which pass through the points L and O; in which the lines FL and EO are parallel to the surface of the water in the reservoir AD, and drawn through the end of any cross-section FE, crossing the axis of the parabola DO, will furnish the four-sided part of the parabola LTVO, which in the height of the devised section drawn will show the quantity of water flowing through the section FE in a time of one minute, because the calculation of proposition 37 is applicable to that time. An account of this rule is apparent from the review of that proposition.

COROLLARY II.

413. And from the given section of the channel IC any other EF will be given. And indeed with the ordinates IZ, CG of the parabola IZ drawn through I, C, and with the right line EO parallel to AD through the point E, and with the ordinate OV, the quadrilinear figure VOLT becomes equal to the quadrilinear IZGC, and OL will be determined, the intercepts from the ordinates OV, LT, the height of the vital section EF. For, because these named quadrilinear parabolic figures are equal (from the construction), these quadrilinear figures multiplied by the height of the channel produce equal volumes, which (§. 404) set out the quantity of water flowing through any section of the channel IC or EF, &c. ; as by this ratio the same amount of water flows through the individual sections, it is apparent (§. 409) the surface of the water to be constant. In turn from any given section EF, the first IC can always be found.

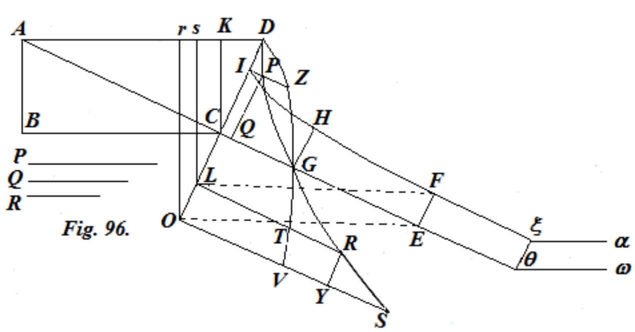


Fig. 96.

PROPOSITION XL. PROBLEM.

414. *With the surface of the water given in the reservoir* AD, *and with a single point* I *remaining on the surface of the water flowing out in the channel* IHξ, *to find any others.* Fig. 96.

Now the solution to this problem is in the final corollary of the preceding proposition ; but, because the construction by the preceding quadrilinear parabolic is inconvenient to some extent, with the aid of the following cubic parabola the matter can be made a little more elegant.

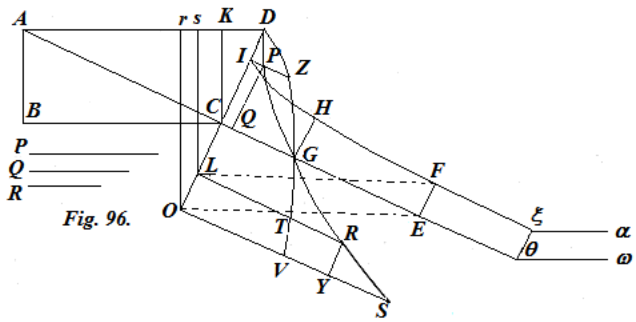


Fig. 96.

And thus a cubic parabola DPG can be described through the points D and G, in which the cube of the abscissa of any DL will be equal to the parallelepiped from the square of the ordinate of the corresponding LR by the parameter Q, thus so that everywhere there shall be $\mathrm{DL}^3 = \mathrm{Q.LR}^2$: And P shall be the parameter of the conical parabola DZT, & $\mathrm{R} = \frac{2}{3}\mathrm{DC}$. And with these in place, from the given first section IC, all the remaining can be found with the aid of the cubic parabola, and thus the curve IHF described through the point. For with the ordinates IP, CG of the cubic parabola drawn through the end P of the above IP, PQ is acting parallel to DC, and if the section of the channel at E may be required, with EO acting parallel to AD through the point O, and with the ordinate of the paraboloid OS drawn, in which SY may be assumed to equal GQ, is acting through Y with YR parallel to DO, this YR will show the height of the vital section EF passing through the point E, thus so that EF shall be $=\mathrm{YR}$; thus all the points F can be found geometrically.

Demonstration. I. The cubic parabola, which I have called a paraboloid, provides $\mathrm{DC}^3 = \mathrm{Q.CG}^2$; truly the conic parabola, $\mathrm{CG}^2 = \mathrm{P.DC}$, and $\mathrm{Q.CG}^2 = \mathrm{P.Q.DC}$; therefore $\mathrm{DC}^3 = \mathrm{P.Q.DC}$, or also $\mathrm{P.Q} = \mathrm{DC}^2$. Therefore DC is the geometric mean between the parameters of the parabola and of the paraboloid.

II. In the paraboloid there is $\mathrm{Q.LR}^2 = \mathrm{DL}^3$ or $\mathrm{P.Q.LR}^2$, that is (no.1. of this) $\mathrm{DC}^2.\mathrm{LR}^2 = \mathrm{P.DL}^3$, or because the parabola effects P.DL equal to $\mathrm{LT}^2, = \mathrm{LT}^2.\mathrm{DL}^2$ hence $\mathrm{DL}^2 : \mathrm{RL}^2 = \mathrm{DC}^2 : \mathrm{LT}^2$, and thus $\mathrm{DL} : \mathrm{LR} = \mathrm{DC} : \mathrm{LT}$, hence $\mathrm{DL.LT} = \mathrm{DC.LR}$, and as a consequence $\frac{2}{3}\mathrm{DL.LT}$ or the [area of the] parabola $\mathrm{DZTL} = \frac{2}{3}\mathrm{DC.LR}$ (following the hypothesis)
$=\mathrm{R.LR}$. Similarly the parabola, or the parabolic area may be found, $\mathrm{DGVO} = \mathrm{R.OS}$; therefore the quadrilineum $\mathrm{LTVO} = \mathrm{R.YS}$. Thus also the quadrilineum $\mathrm{IZGC} = \mathrm{R.QG}$;

so that because (by the construction) $\mathrm{YS} = \mathrm{QG}$, and thus the quadrilineum LTVO is equal to the quadrilineum IZGC, (§. 413) and the same amount of water will flow through the sections IC and EF, and thus the surface of the water IHFξ will be kept constant. Q.E.D.

SCHOLIUM

415. With the aid of the present theorems, and of that preceding, from the given inclination of the channel or of the angle CAD, for any section EF and its distance AE from the surface of the water in the reservoir AD, or from the beginning of the channel, the proportion of the velocities E and F will become known, and of all the remaining sections. For from the given angle DAC and from the distance AE, O*r* itself will become known, and from this and from the angle ADC, for the complement of the angle CAD, OD will be elicited ; from which, since the section EF shall be given and equal to OL from the hypothesis, with that taken from OD, LD remaining is known. Now the parabola DGV is described about the axis DO with the vertex D, the ordinates LT and OV will express the proportion of the speeds of the water at E and F. But, because there are many cases, in which the distance EA cannot be had for the actual dimension, most often here the way of determining the speeds in practice has minimal success, and it is convenient to investigate these velocities mechanically.

416. *To find the proportions of the velocities at E & F by mechanical means of any section of a river or channel EF*. Fig. 97.

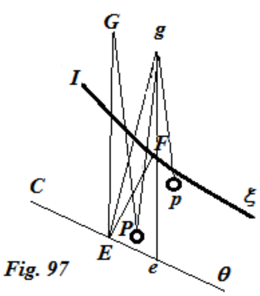


Fig. 97

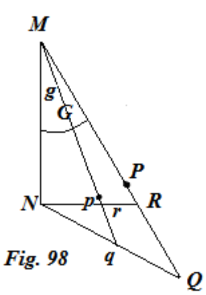


Fig. 98

The string GP may be had with the weight P, with specific gravity a little more than water, its other end fixed to the top, and with the weight P dropped as far as to the bottom of the channel Cθ, that weight will be washed away to some extent by the water flowing through the point D of the section FE, and the forces of the water on this weight will act so that the string GP will be turned aside from the vertical situation GE by a certain angle EGP. Later with the weight extracted from the bottom, and that being sent into the water in such a way, that it may remain near the surface at the point F as at *p*, with the declination of the string *gp* to the vertical *ge*, in which the weight with the string in any case may put itself in place, unless by the forces of the current of water near F it shall be moved away from the vertical situation, the angle shall be *egp,* which we will indicate by the simple letter *g*, and the former angle EGP by the similar letter G. The complements of these angles to the right angle are called C and *c*, the inclination of the channel, or the angle GEF or I; hence a certain N may be assumed, which shall be to the tangent of the angle G, as the sine of the complement C is to the sine of the angle $\mathrm{C} - \mathrm{I}$*;* likewise *n* to the tangent of the angle *g* in the ratio of the sine of the angle *c* to the sine of the angle $c - \mathrm{I}$, and the velocity of the water at E to the velocity at F will be in the square root ratio of the magnitudes N to *n*.

Demonstration. I. In the one figure 98 MN or MP, or *Mp* may represent the length of the string GP or *gp*, and MN vertical to the horizontal may contain with the right line NQ

the angle MNQ equal to the angle $GE\theta$ in Fig. 97, and the angles NMQ, NMq shall be equal to the angles EGP, *egp*, or to G and *g* respectively, and with NR drawn to the perpendicular MN, the tangent NR of the angle G, and N*r* the tangent of the angle *g:* And NRM will be C or the complement of the angle G, and N*r*M the complement of the other g, that is *c.* Indeed also the angle RNQ will be equal to the angle GEF, or to the angle I, for if from the equal angles GEθ and MNQ, the right angles EFθ & MNR are taken away the equal angles GEF, RNQ will remain. And thus the angle
$\mathrm{RNQ} = \mathrm{C} - \mathrm{I},$ and $rq\mathrm{N} = c - \mathrm{I}$.

II. Because the weights try to place themselves on a horizontal line, and in fact in such a way reducing by how much the attached string shall be lead away from the perpendicular situation, and they are composed for the second time according to that law. Therefore the string GP is unable to remain in this situation, unless by the forces of the water flowing through E impressed along the direction normal to the section EF, or parallel to the bottom Eθ, where it may be retained; therefore it is necessary, that besides the action of gravity on the body P, which exerts itself on this along a direction parallel to GE, the action of a certain another force must be added, which in this case we may consider to arise, as noted, along the direction parallel to Cθ itself. Now if in fig. 98. MN sets out the force of the weight P, which it has in water, the other force, which is exerted parallel to the base of the channel, will be represented by the line NQ, since (following the hypothesis) the angle MNQ is equal to the angle GEθ, and the action of the two forces side by side MN and NQ (§39,40) will be holding the weight P tied to the string at the position MP, with MN containing the angle NMQ or the angle G; similarly N*q* represents the force of the water drawing away the string and by making the angle g or *egp* the angle of deflection from the vertical *ge* : thus so that the forces drawing away shall be to each other in turn as NQ to Nq. And in the triangle NRQ, NR is to NQ as the sine of the angle NQR or $\mathrm{C} - \mathrm{I}$ to the sine of the angle NRQ or NRM, that is C, and from the hypothesis G of the triangle or NR is to the magnitude N, just as sine $\mathrm{C} - \mathrm{I}$ to sine C, therefore $\mathrm{N} = \mathrm{NQ}$. By the same argument it is agreed that $n = \mathrm{N}q$. And thus the force drawing the string GP away from the vertical position GE, is to the force drawing the string away *ge* as N to *n*.

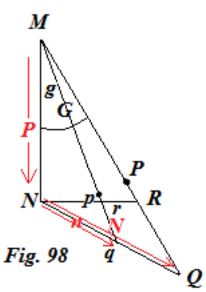


Fig. 98

III. Truly the deflecting forces are as the forces of the water exerted on the globe P, exerted along directions parallel to the bottom Cθ : truly these forces are in the square ratio of the velocities at E and F, as will be shown in the following, therefore the square of the velocity at E is to the square of the velocity at F, as N to *n,* and it follows the speed at E is to the speed at F, in the square root ratio of the magnitude N to the magnitude *n.* Q.E.D.

COROLLARY.

4I7. If the angle GEF, or the inclination of the channel to the horizontal, is zero, in that case N will be to *n*, as the tangent of the angle G to the tangent of the angle *g* ; and thus the velocities of the water at the highest and deepest sections SE will be in the square root ratio of the sines of the angles g and G.

SCHOLIUM.

418. Fig. 96. All these which have been shown about the channel Iθ, will be applicable to rivers also, if these admit the same height of the artificial cross-sections through their whole length and the same height of the cross-sections, and water flowing freely and no resistances are allowed from the bank or bottom irregularities. Truly because such mathematical rivers do not exist in the natural state, it is required to be seen clearly, whether or not any of the preceding matters may be able to be produce a little aid, by which the flows of river waters may be recalled to measurement, and may be able to be explained in terms of these more general notions. In this regard, an elegant account for the mean heights and lengths has been devised by Castelli for rivers which the Cel. Gulielmini thence has perfected more.
Thus the method may be explained.

419. In Fig. 99, it is understood the sluice or regulator ABDA consists of two parallel and vertical tree trunks AB, AD joined to the horizontal BD, and constructed with notches, thus so that by these notches the board EH is inserted, which to be raised for the water to spring forth and may be lowered with the aid of the rope VS tied to the hook V, and thus the opening FBDH in this way may be able to be both diminished or enlarged. This sluice or regulator must be put in place at the most convenient place of the mean flow of the river, thus so that the tree-trunk BD shall be in contact with the bottom perpendicular to the bank, and finally the trunks AB and AD normal to the bottom, thus so that the whole plane ABDA shall be perpendicular to the bottom and with the banks.

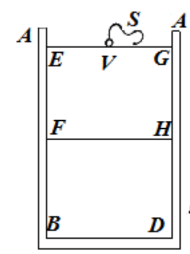


Fig. 99

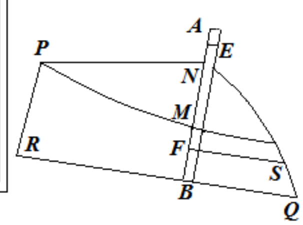


Fig. 100

If now it may be asked, how much water of the river PQ must flow through the section BM in a given time , see Fig. 100 ? In this section or in another more convenient in place the sluice is required to be put in place in that said manner, and the moveable board EF is required to be lowered, so that the section MB may be reduced to the smaller FB, with which done, so that with a smaller amount of water flowing through the diminished opening FB than before through MB, the water slowly swells up above the section MB, and gradually rises, as long as while the water surface shall be remaining at PN, in which case just as much water flows through the opening FB, as was flowing before through MB and just as much flowed through any other section PR, otherwise the surface shall not be remaining at PN. But we consider that to be remaining, so that reasonably in a little time it will compose itself by necessity into that state; in which case just as the body of water PRNB can be considered to be enclosed in the reservoir or receptacle, to which water flows in only from the top through PR, as much as flows out through the opening FB. And with these now put in place the calculation for obtaining the amount of water flowing out can be resolved in an easy, in a certain given time, through the opening FB. For if with the vertex N and NSQ may be understood to be the parabola described about the axis NB, whose parameter *p* shall be equal to $= ai:r$ with the symbols retained, which above (§. 401.) with *i* indicating the sine of the given angle from the observed PNB, *r* the whole sine and *a* that magnitude, whose logarithm now has been found already (§. 405). From which, because also BN and FN or *z* and r have been given from observation, the quantity of water Q flowing out through the opening FB along the normal to be led away may be found by the easy calculation of paragraph 406.

In the proceeding propositions we have considered that most motion of the water as brought about freely, and without any resistance or friction. Truly, because waters in the cavities of rivers running down encounter various resistances from the bottom and the banks, an account is required of resistances of this kind generally, and in certain hypothesis of the resistances the velocities of any individual points of the sections are required to be determined.

PROPOSITION XLI. PROBLEM.

420. *With the resistance present of water flowing proportional to the speed of the water at the individual points of the section, to find the velocity itself.* Fig. 101.

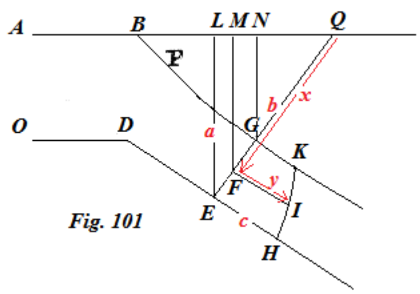


Fig. 101

The river shall be BHK, and EG its section, the surface AB remaining horizontal, or the beginning of the channel or of the river. EG is produced to Q as far as to its crossing with the plane AB likewise produced ; and finally from some points E, F, G of the section EG perpendiculars EL, FM, & GN &c. shall be sent to AQ, and the ordinates EH, FI, GK &c. of some curve KIH perpendicular to GE itself express the velocities of the water flowing through the points E, F, G . Now LE shall be *a*, QE shall be *b;* the ordinate $\text{EH} = c$, which may designate the amount of water flowing through the point E*;* the resistance of the bottom from the contact $= m$, the resistance of the banks $= n$ *;* QF, *x* and FI, *y* and this ordinate equally indicates the amount of water flowing through the point F and its speed is y:*c*, and the motion [*i.e.* momentum] $= yy:c$. And, because the resistance (following the hypothesis) at E, that is *m,* to the resistance at F itself is as EH to FI, the resistance at F, in as much as shares the resistance at the bottom, $= my:c$, and the resistance at the same place arising from the resistance of the bank $= ny:c$. Now with the resistances taken from the magnitude of the incumbent pressure at some point F, which quantity is expressed by MF, or *ax: b* is put in place, and the remainder $(ax:b)-(my+ny):c$, will express the force pushing the water through the physical point F, and since this extruding force shall be constantly proportional to the motion of the water generated, there will be $(ax:b)-(my+ny):c = yy:c$, that is $acx - bmy - bny = byy$, which is the equation for a parabola, of which the principle axis shall be different from EQ, lying closer to B, by the interval $\frac{1}{2}m+\frac{1}{2}n$, and the vertex on this axis from the base DH distant by the interval

$b+\frac{bpp}{4ac}$, evidently by making $p=m+n$; and finally the parameter of this parabola will be $\frac{ac}{b}$. Q.E.I.

Perhaps other hypotheses of the resistance are able to be devised, which shall be truer than the present circumstances, for this is not the most certain assumption recommended by us, but by one example only it has pleased us to show, how the velocities of the water flowing must be assigned to agree, without ignoring the resistances, which arise from frictional forces.

CAPUT X.

De Cursu Fluminum.

DEFINITIONES.

I.

Generali *Fluminis* vocabulo indigitatur hoc loco aqua in superficie terrae itinere plerunque varie inflexo e locis altioribus intra alveum suum ad depressiora indefinenter fluens.

II.

Alveus fluminis est cavitas in superficie telluris, intra quam aquae decurrunt.

III.

Sectio fluminis est communis secto alvei & plani secantis alvei fundo perpendicularis. Ejusmodi sectio ordinarie est figura aliqua irregularis ac propterea vocari solet Sectio *naturalis* ad distinguendam eam ab artificiali : nam

IV.

Sectio artificialis est semper parallelogrammum rectangulum, quia intelligitur esse sectio alvei artificialis, seu parallelipipedi formam habentis.

V.

Altitudo viva fluminins est distantia cujusque puncti ist superficie fluminis a fundo ejusdem. Et *altitudo viva* est basis alicujus sectionis artificialis.

V.

407. Flumen in eodem dicitur statu manere, vel in statu manenti esse, cum inter fluendum nusquam attollitur ejus superficies & intumescit, nec alibi deprimitur vel decrescit, sed eodem semper tenore, durante fluxu, se habet; abstrahendo tamen ab inaequalitatibus accidentalibus, scilicet a vorticibus &c. quae a fundi & spondarum asperitatibus provenire solent.

PROPOSITIO XXXVIII. THEOREMA.

408. *Existente flumine in statu manenti, temporibus aequalibus aequales aquae copiae per omnes fluminis sectiones transfluent.* Fig. 95.

Si negas, transeat ergo plus aquae per sectionem AB quam per vicinam sectionem CD, & intumescet aqua inter has se sectiones in *bmD* exempl. gratia; sin vero plus aquae transiret per sectionem CD quam per AB, aqua inter has sectiones decresceret in *βnD* ; atque adeo flumen non maneret in eodem statu, contra hypothesin.

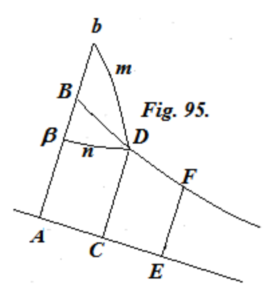


Fig. 95.

COROLLARIUM I

409. Et conversim etiam, superficies fluminis manens erit, si per singulas fluminis sectiones AB, CD, EF, &.c. eadem aquae quantitas temporibus aequalibus perfluit.

PROPOSITIO XXXIX. THEOREMA.

410. *Si castellum aquae plenun* ABCD *communicet cum lacu* $\xi\omega\alpha$ *indefinite amplo per canalem inclinatum* $IF\theta$ *superne opertum superficie curva* IHFξ, *atque exitu aquae ex castello per lumen* CI, *in canali aqua ita fluat, ut in punctis quibusvis* F, E *sectionis* EF *celerites pendeant a pressionibus aquae incumbentis juxta altitudines SL, rO, quibus puncta illa suprema aquae in castello superficie manente* AD *distant, faciantque pressiones illae, ut per singulas canalis sectiones* GH, EF, *&c. eadem temporibus aequalibus aquae copia fluat. Dico fore, ut eodem tenore aqua in canali inclinato fluat remoto operimento* IHFξ *curvilinea, quo faceret operta canalis aqua indicata ista superficie operiente, modo tantum aquae castello influat, quantum per singulas canalis sectiones jugiter transit.* Fig. 96.

Si negas, attollatur ergo aqua, amoto operimento curvo IFξ, quam inter sectiones GH & EF, ergo minus aquae fluit per sectionem EF quam per praecedentes versus CI vi pressionis naturalis aqae incumbentis, contra hypothesin.

SCHOLION

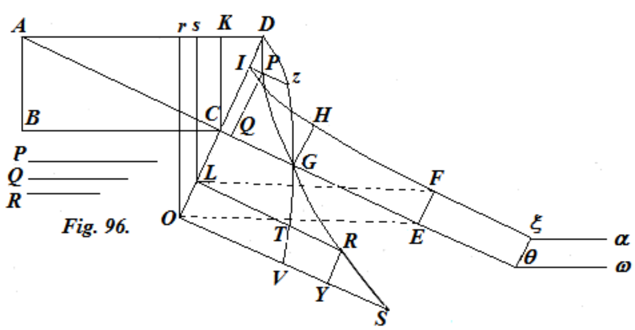


Fig. 96.

411. *Pressionum naturalium* nomine intelligimus eas, quarum ope velocitates aquae, ut in E & F, sunt in subduplicata proportione altitudinum O*r*, L*s*, vel ipsarum OD, & LD. Nam quaecunque sit superficies IHFξ, eadem aqae copia per singulas sectiones GH, EF, &c. fluet propter contiguitatem partium aquae, quandiu canalis superficie illa curva opertus manet, nec tamen ideo celeritates aquae reguntur a partibus incumbentibus, juxta tenorem & legem pressionum naturalium; hoc est, velocitates, ex. gr. in E & F, non sunt hoc casu generali ex necessitate in ratione subduplicata *Or,* ad *Ls.* Ac propterea contingeret ut remoto operimento canalis superficies tamdiu mutetur, usque dum aqua defluens in canali eam nacta sit, quae a pressionibus aquae naturalibus unice dirigatur, ac manens fiat; & haec proinde ea ipsa est, quam operimento IFξ, adeo ut aqua continue in canali fluere queat, si ejus superficies figuram habeat istius operimenti, sive aqua tecta sit isto operimento sive non.

COROLLARIUM I.

412. In hac pressionum naturalium hypothesi innotescere potest; quantum aquae per singulas sectiones canalis dato tempore fluere debeat. Nam producta prima sectione IC in D usque ad occursum cum superficie aquae in castello AD , & indefinite deorsum in O, &

juxta conditiones paragraphi 404, vertice D circa axem DO descripta in parabola DTV, ordinatae ejus LT & OV, quae transeunt per puncta L & O; in quibus lineae FL & EO superficiei aquae in castello AD parallelae, & per terminos cujuscunque se sectionis FE ductae, axi parabolae DO occurrunt, exhibebunt quadrilineum parabolicum LTVO, quod in altitudinem sectionis artificialis ductum manifestabit quantitatem aquae per sectionem FE unius minuti horarii tempore fluentis, quia calculus propositionis 37 ad hoc tempus est aptatum. Ipsius regulae ratio patet ex hac ipsa propositione recensita.

COROLLARIUM II.

413. Et data prima canalis sectione IC dabitur quaelibet alia EF. Etenim ductis per I, C ordinatis parabolae IZ, CG, & per punctum E recta EO parallela AD, ac ordinata OV, fiat quadrilineum VOLT aequale quadrilineo IZGC, determinabitque OL, intercepta ab ordinatis: OV, LT, altitudinem vivam sectionis EF. Nam, quiae haec nominata quadrilinea parabolica (constr.) aequantur, haec quadrilinea ducta in canalis altitudinem producunt solida aequalia, quae (§. 404) quantitatem aquae fluentis per quamlibct sectionem canalis IC vel EF, &c. exponunt; unde cum hac ratione per singulas sectiones eadem aquae copia fluit, patet (§. 409) superficiem aquae manentem esse. Vicissim data quaelibet sectione EF, semper invenire licet primam IC.

PROPOSITIO XL. PROBLEMA.

414. *Datis superficie aquae in castello* AD, *& uno puncto* I *in superficie manente aquae in canali defluentis* IHξ, *invenire quotlibet alia.* Fig. 96.

Hoc problema jam solutum est in corollario ultimo praecedentis propositionis ; sed, quia constructo per quadrilinea parabolica procedens nonnihil incommoda est, ope parabolae cubicae secundae res paulo elegantius confici potest.

Describatur itaque per puncta D & G parabola cubica DPG, in qua cubus abscissae cujuscunque DL aequetur parallelepipedo ex quadrato ordinatae respondentis LR in parametrum Q, ita ut sit ubique $DL^3 = Q.LR^2$: Sitque P parameter parabolae conicae DZT, & $R = \frac{2}{3}DC$. Et hisce positis, data prima sectione IC, reliquae omnes inveniri possunt ope parabolae cubicae, atque adeo curva IHF per puncta describi. Nam ductis ordinatas IP, CG parabolae cubicae per superioris IP terminum P, agatur PQ parallela DC, & si sectio canalis in E expetatur, acta EO parallela AD per punctum O, & ducta ordinata paraboloidis OS, in qua sumta SY aequali GQ, agatur per Y recta YR parallela DO, haec YR exhibebit altitudinem vivam sectionis EF per punctam E transeuntis, ita ut EF sit $= YR$; ergo puncta omnia F geometrice inveniri possunt.

Demonst. I. Parabola cubica , quam paraboliodem dicam ; praebet $DC^3 = Q.CG^2$; parabola vero conica, $CG^2 = P.DC$, & $Q.CG^2 = P.Q.DC$; ergo $DC^3 = P.Q.DC$, vel etiam $P.Q = DC^2$. Est ergo DC media geometrica inter parametros parabolae & paraboloidis.

II. In paraboloide est $Q.LR^2 = DL^3$ vel $P.Q.LR^2$, id est (num.1. hujus) $DC^2.LR^2 = P.DL^3$, vel quia parabola efficit P.DL aequale $LT^2, = LT^2.DL^2$ hinc $DL^2 : RL^2 = DC^2 : LT^2$, atque adeo $DL : LR = DC : LT$, hinc $DL.LT = DC.LR$, & per consequens $\frac{2}{3}DL.LT$ seu parabola $DZTL = \frac{2}{3}DC.LR$ (secundum hypothesin) $= R.LR$. Similiter reperietur parabola, seu area parabolica, $DGVO = R.OS$; ergo quadrilineum $LTVO = R.YS$. Sic etiam quadrilineum $IZGC = R.QG$; unde quia (constr.) $YS = QG$, adeoque quadrilineum LTVO aequale quadrilineo IZGC, (§. 413) eadem aquae copia per sectiones IC & EF fluet, atque adeo superficies aqae IHFξ manens erit. Quod erat demonstrandum.

SCHOLION

415. Ope praesentis, ejusque quae eam praecedit, ex datis inclinatine canalae seu angulae CAD, sectione qualibet EF ejusque distantia AE a superficie aquae in castello AD, vel ab origine canalis, innotescent proportio velocitatum E & F, &c reliquae sectiones omnes. Nam ex dato angulo DAC & distantia AE, innotescet ipsa Or, & ex hac & angulo ABC, complemento anguli CAD, elicietur OD; unde, cum sectio EF eique aequalis OL data sit ex hypothesi, ea ex OD subtracta relinquet LD cognitam. Jam descripta circa axem DO & vertice D parabola DGV, ordinatae LT & OV exponent proportionem celeritatum aquae in E & F. Sed, quia multi sunt casus, quibus distantia EA actuali dimensione haberi nequit, saepissime hic modus determinandi praememoratas celeritates in praxi minime succedit, & mechanice eas velocitates investigare convenit.

416. *Invenire per observationes artificio mechanico proportionem velocitatum in E & F cujusque sectionis fluminis canalisve EF*. Fig.97.

Habeatur filum GP cum pondere P, aqua aliquantum specifice graviore, altero ejus capiti annexo, demissoque pondere P usque ad fundum canalis Gθ, id pondus ab aqua per punctum D sectionis FE fluente nonnihil abripietur, & aquae impressiones in hoc pondus efficient ut filum GP a situ verticali GE angulo quodam EGP declinet. Dehinc extracto pondere ex fundo, & aquae eum in modum immisso, ut prope superficiem ad punctum F ut in *p* consistat, declinatio fili *gp* a situ verticali *ge*, in quem pondus cum filo se alioqui composuisset, nisi aquae prope F currentis impressionibus a situ perpendiculari abductum fuisset, sit angulus *egp*, quem simplicili litera *g* indicabimus, & angulum priorem EGP litera simili G. Horum angulorum complementa ad rectum dicantur C & *c*, inclinatio canalis, seu angulus GEF , autem I; dehinc assumatur quaedam N, quae sit ad tangentem anguli G, ut sinus complementi C est ad sinum anguli $C - I$*;* item *n* ad tangentem anguli *g* in ratione sinus anguli *c* ad sinum anguli $c - I$, eritque velocitas aquae in E ad velocitatem in F in subduplicata ratione magnitudinis N ad *n*.

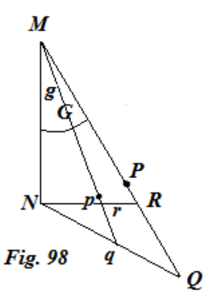


Fig. 98

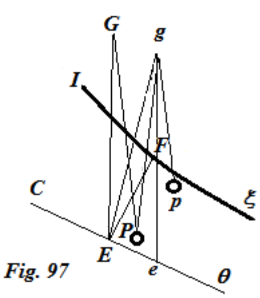


Fig. 97

Demonstr. I. In altera figura 98. repraesentent MN vel MP, aut *Mp* longitudinem fili GP vel *gp,* & MN horizonti verticalis contineat cum recta NQ angulum MNQ aequalem angulo $GE\theta$ in Fig. 97 , sintque anguli NMQ, NMq aequales angulis EGP, *egp,* seu G & *g* respective, ductaque NR perpendiculari ad MN, erunt NR tangens anguli G, & *Nr* tangens anguli *g:* Et NRM erit C seu complementum anguli G, ac N*r*M complementum alterius g, id est *c.* Quin etiam angulus RNQ aequabitur angulo GEF, seu angulo I, nam si ex aequalibus GEθ & MNQ auferentur recti EFθ & MNR remanebunt aequales GEF, RNQ. Eritque adeo angulus $\mathrm{RNQ} = \mathrm{C} - \mathrm{I}$, & $rq\mathrm{N} = c - \mathrm{I}$.

II. Quia gravia in situm horizonti rectum se componere affectant, atque in talem se reapse reducunt quoties filo annexa a perpendiculari situ abducta, atque sui iterum juris facta sunt. Idcirco filum GP non potest in hoc situ consistere, nisi impressionibus aquae fluentis per E juxta directionem sectioni EF normalem, vel fundo Eθ parallelam, in eo detineatur; necesse igitur est, ut praeter gravitatis actionem in corpus P, quae se in hoc exserit juxta directionem ipsi GE parallelam, alius cujusdam potentiae actio accedat, quam hoc casu concipimus fieri, ut dictum, juxta directionem ipsi Cθ parallelam. Jam si in figura 98. MN exponit gravitatem ponderis P, quam in aqua habet, altera potentia, quae fundo canalis parallela est, exponetur linea NQ, quandoquidem (secundum hypothesin) angulus MNQ aequalis est angulo GEθ, atque actione duarum potentiarum collateralium MN & NQ (§.39,40) detinetur pondus P filo annexum in situ MP, cum MN angulum NMQ seu angulum G continente; similiter repraesentat N*q* vim abducentem aquae filumque angulo g seu *egp* a verticali *ge* declinare facientis : adeo ut vires abducentes sint ad se invicem ut NQ ad Nq. Atqui in triangulo NRQ, est NR ad NQ ut sinus anguli NQR seu $\mathrm{C} - \mathrm{I}$ ad sinum anguli NRQ seu NRM, id est C ,& ex hypothesi est trianguli G seu NR ad magnitudinem N, sicut sinus $\mathrm{C} - \mathrm{I}$ ad sinum C, ergo $\mathrm{N} = \mathrm{NQ}$. Eodem probabitur argumento esse $n = \mathrm{N}q$. Itaque vis abducens filum GP a situ perpendiculari GE, est ad vim abducenem fili *ge* ut N ad *n*.

III. Sunt vero vires abducentes ut impressiones aquae in globum P, exsertae juxta directiones fundo Cθ parallelas : impressiones vero sunt in duplicata ratione velocitatum in E & in F, ut in sequentibus demonstrabitur, ergo quadratum velocitatis in E est ad quadratum velocitatis in F, ut N ad *n,* ac per consequens velocitas in E est ad velocitatem aquae in F, in subduplicata ratione magnitudinis N ad magnitudinem *n.* Quod erat demonstrandum.

COROLLARIUM.

4I7. Si angulus GEF, seu inclinatio canalis ad horizontem, nullus est, erit N ad *n* eo casu, ut tangens anguli G ad tangentem anguli *g* ; atque adeo velocitates aquae in summo & imo sectionis SE erunt in subduplicata ratione tangentium angulorum *g* & G.

SCHOLION.

Fig.96. 418. Haec omnae, quae de canali $\mathrm{I}\theta$ ostensa sunt, fluminibus quoque applicari possent, si haec per totam suam longitudinem eandem altitudinem sectionesque artificiales admitterent, & aquae fluentes a spondis & fundi inaequalitatibus nullam

resistentiam paterentur. Verum quia tales fluvii mathematici in rerum natura non existunt, dispiciendum est, num praescedentia nullum praebere queant adminiculum,quo fluminum aquae fluentes ad mensuram revocari, eorumque affectiones generaliores explicari queant. Hac in re non inelegantem rationem pro fluviis mediocris altitudinis & latitudinis excogitavit Castellus quam Celeb.Gulielminus deinceps magis perfecit.
Haec methodus ita habet.

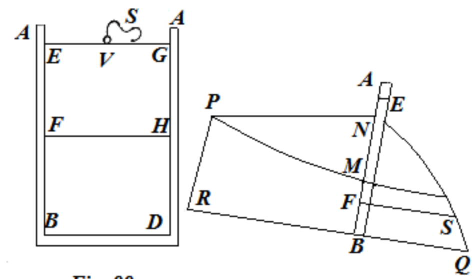


Fig. 99 Fig. 100

Fig. 99. 419. Intelligatur *Cataracta* seu *Regulator* ABDA constans duobus tignis parallelis & verticalibus AB, AD horizontali BD conjunctis, & crenis suis instructis, adeo ut his crenis tabula EH inseri, quae pro re nata attolli & demitti queat ope funis VS unco V alligati, atque adeo lumen FBDH modo arctari modo etiam ampliari queat. Haec Cataracta seu Regulator in loco commodo fluvii mediocris transversim debet aptari, ita ut tignum BD ripis perpendiculare sit fundoque contiguum, ac denique tigna AB & AD fundo normalia, adeo ut totum planum ABDA fundo ripisque perpendiculare sit.

Si jam quaeratur, quantum aquae per sectionem BM fluminis PQ, Fig.100 dato tempore fluere debeat ? In hac sectione vel alio commodiore loco collocandus est Regulator eo modo, ut dictum, & tabula EF mobilis demittenda est, ut sectio MB reducatur ad minorem FB, quo fiet, ut minore aquae copia fluente per lumen arctum FB quam antea per MB, aqua supra sectionem MB intumescat sensim, atque sensim attollatur, usque dum aqua superficiem manentem PN adepta sit, quo casu tantum aquae fluet per lumens FB, quantum ante fluxerat per MB & quantum per quamlibet aliam sectionem PR fluit, alioqui superficies PN manens non esset. Sed eam ponimus esse manentem, ut sane ad talem statum de necessitate se aliquando componet;
quo casu corpus aquae PRNB tanquam amplo castello seu receptacula inclusa considerari potest, cui tantum aquae influit superne per PR, quantum effluit per lumen FB. Atque hisce jam positis calculus facili negotio absolvetur pro obtinenda quantitate aquae effluentis, dato quodam tempore, per lumen FB. Nam si vertice N & circa axem NB parabola descripta intelligatur NSQ, cujus parameter *p* sit $= ai : r$ retentis symbolis, quae supra (§. 401.) significante *i* sinum anguli observatione dati PNB, *r* sinum totum & *a* eam magnitudinem, cujus log-us jam antea (§. 405)repertus est. Unde, quia etiam BN & FN seu *z* & r ex observatione datae sunt, quantitas Q aquae per lumen FB effluentis juxta normam paragraphi 406 facile calculo subducetur.

In Propositionibus praecedentibus consideravimus ut plurimum motus aquarum tanquam liberrime factos absque ulla resistentia ex frictionibus. Verum, quia aquae in fluviorum alveis decurrentes varias resistentias a fundo & spondis subeunt, ejusmodi resistentiarum omnino ratio habenda, atque in certis resistentiaruin hypothesibus velocitates singulorum alicujus sectionis punctorum determinandae sunt.

PROPOSITIO XLI. PROBLEMA.

420. *Resistentiis aquae fluentis existentibus proportionalibus aquae velocitatibus in singulis sectionis punctis, invenire ipsas velocitates.* Fig. 101.

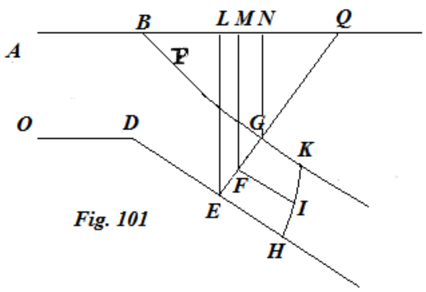


Fig. 101

Sit flumen BHK, atque EG sectio ejus, AB superficies manens horizontalis, seu initium canalis aut fluminis. Producatur EG in Q usque ad occursum ejus cum plano AB itidem producto; & denique ex quibuslibet punctis E, F, G sectionis EG ad AQ demissae sint perpendiculares EL, FM, & GN , &c. atque ordinatae EH, FI, GK &c. alicujus curvae KIH ipsi GE perpendiculares exponant velocitates aquae per puncta E, F, G fluentis. Sint jam LE, QE, *b;* ordinata EH, quae aquae per punctum E fluentis quantitatem designat $=c$ *;* resistentia fundi ex contactu $=m$, resistentia spondae $=n$ *;* QF, *x* & FI, *y* atque haec ordinata pariter quantitatem aquae per punctum F fluentis indicat ejusque celeritas est y:*c*, ac motus $= yy:c$. Et, quia (secundum hypothesin) resistentia in E, hoc est *m,* ad resistentiam in F se habet ut EH ad FI, erit resistentia in F, quatenus haec participat resistentiam in fundo, $= my:c$, & resistentia in eodem loco proveniens a resistentia spondae $= ny:c$. Detractis nunc resistentiis a quantitate pressionis aquae puncto cuilibet F incumbentis, quae quantitas per MF, seu *ax: b* exponitur, & reliquum $(ax:b)-(my-ny):c$, exponet vim extrudentem aquam per punctum physicum F, & cum haec vis extrudens constanter proportionalis sit motui aquae genito, erit $(ax:b)-(my-ny):c = yy:c$, id est $acx - bmy - bny = byy$, quae est aequatio ad parabolam, cujus axis principalis ab EQ distat, propius accedens ad B, intervallo $\frac{1}{2}m + \frac{1}{2}n$, & vertex in hoc axe a fundo DH distat intervallo $b + \frac{bpp}{4ac}$, facta scilicet $p = m + n$ *;* ac denique parabolae hujus parameter erit $\frac{ac}{b}$. Quod erat inveniendum.

Fortasse aliae possunt excogitari resistentiae hypotheses, quae praesente veriores sint, hanc enim a nobis assumptam non pro certissima vendito, sed duntaxat uno exemplo facili ostendere placuit, quo pacto velocitates aquarum fluentium assignari debeant, non neglectis resistentiis, quae a frictionibus proveniunt.