CHAPTER V.

Concerning the motions of weights connected to each other and falling jointly along the arcs of concentric circles ; or indeed the motion of composite pendulums with the centre of oscillation of these under all hypothesis of the variation of gravity.

As far as I may know, Huygens first treated the theory of the centre of oscillation, in the fourth part of that outstanding treatise concerning pendulum clocks. [*Horologium Oscillatorium sive de motu pendulorum* : there is a translation of the work on this website.] But because he pursued the work from this principle, which he supposed not to be in need of demonstration, the actual truth of this principle cannot be obtained in complete agreement with everyone. For he wished to concede, that the common centre of gravity of all the parts of which the pendulum was composed, would be unable to rise higher, with each bond in the restraining chain broken, by which each part was connected to the rest of the pendulum, so that each part could begin to rise freely with its own speed, which it had acquired by falling jointly with the rest of the pendulum, since the fall was by the descent of the whole composite pendulum. The Cel. James Bernoulli, by determining generally in another way the length of a simple composite isochronous pendulum, thence deduced a proof of the Huygens' postulate, but only under the hypothesis of uniform gravity acting on the individual parts of the pendulum. Truly we ourselves will consider a more general matter, imagining gravity to act in some different ways on these parts and we will demonstrate a remarkable law of nature, with this not hindering the direct method ; evidently the ascent of the centre of gravity of all the parts of the pendulum, mentioned briefly above, to be equal to the descent of the same, even if, with the change in the individual parts of the pendulum, the centre of gravity of these shall be greatly different from that, which is the more common hypothesis of uniform gravity, and with the proportionality of the weights to the masses of the bodies.
[This variation of gravity will be done by considering that parts of the pendulum of similar volumes be composed of materials with different specific gravities, and to be immersed in a fluid of yet another specific gravity, so that the upthrusts effectively change the strength of gravity acting on the parts of the pendulum, and so essentially the position of the centre of mass; here such an approach takes no account of the resistance offered by the fluid, terminal velocity of the falling pendulum, and so on, so that the whole chapter can be considered to be only of historical interest.]

DEFINITIONS.

I.

A pendulum may be called *composite*, because it has been constructed from several bodies or parts connected to each other. Truly a simple pendulum consists of a single weight.

II.

The Axis of a composite pendulum is a right line which passes through the common centre of gravity of all the parts of the pendulum, clearly about which the whole pendulum is oscillating to and fro in a reciprocal motion from the fixed point of suspension.

III.

The *axis of the oscillation* is a line perpendicular to the axis of the pendulum passing through the point of suspension, about which the pendulum performs its vibrations.

IV.

The *centre of oscillation* is a point on the axis of the pendulum, of which the distance from the axis of the oscillation is equal to the length of the simple synchronous pendulum put in place.

V.

A simple pendulum is said to be *synchronous with a composite one*, with its axis and the axis of the other placed vertically, or by being inclined similarly at some other angle to the horizontal, beginning to move at the same time constantly make equal angles by swinging to and fro.

197. All the lines CP, CM, CQ, PQ of the composite pendulum CPQ moveable about the point of suspension C shall be without weight ; truly at the ends of the right line PQ some small weights P, Q [or corpuscles] are to be attached, of which the common centre of gravity shall be M, with the right line CM drawn through the point of suspension C and the centre of gravity of the weights M, is called the *axis* of the composite pendulum PQC, indeed the right line CW drawn perpendicular to the axis CM through the point of suspension C, is the axis of the oscillation.

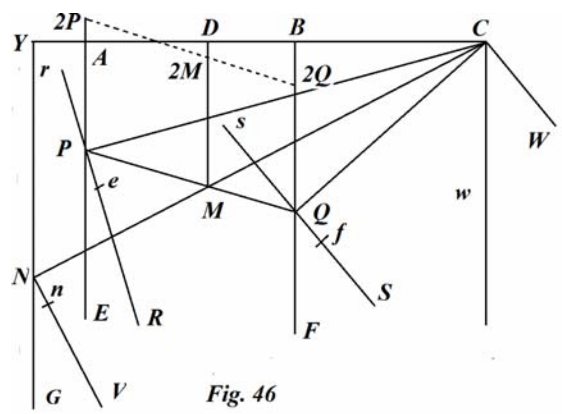

Fig. 46

Again if the axis CM of the composite pendulum arrived at the position CN falling from the horizontal position CA in the time in which a simple pendulum also falls from the horizontal position CN, and if the composite pendulum CPQ and the simple pendulum CN always have made the equal angles ACM & ACN, one may be called *synchronous* to the other, or which also on many occasions may be called *isochronous* ; truly the small weight in the simple pendulum N, is considered as the place of the centre of gravity, and as is applicable to the axis of the composite pendulum, and in that case signifies the *centre of oscillation* N.

VI.

198. *Vicarious actions* are the tangential solicitations [which we interpret here as the components of the gravitational accelerations, or forces per unit mass tangential to the circle at some point] required to be substituted in place of the forces of central gravity considered, and equivalent to these forces. Thus the tangential forces are expressed by the

product of the magnitudes R, S, &c. and V into the masses P, Q &c. & N, that is, the forces are R.P, S.Q, &c, & V.N in the directions PR, QS &c*:* & NV, which are the tangents of the arcs *Pe, Qf,* &c. *Nn* described by the oscillating bodies P, Q &c. & N, they are the vicarious forces acting on their respective bodies in place of the vertical forces of gravity AP, BQ &c., and YN acting on the bodies P, Q, &c. & N along vertical directions, if they prevail to produce the same effect as these, and thus the forces acting in the opposite directions along Pr, *Qs* &c. & *Nu* are in agreement with the central gravitational forces in equilibrium, which henceforth also I set out by the products E.P, F.Q &c & G.N from the magnitudes E, F, &c. & G acting on the masses P, Q, &c. & N. [Thus, R, S, V, etc. are taken as the components of the gravitational acceleration tangent to the circles at some point.]

VII.

199. And finally the vicarious forces will be said to be similar, as long as the magnitudes R, S, &c. & V, will be proportional to the homologous distances PC, QC &c. & NC. [This amounts to a linear tangential acceleration proportional to the radius.]

VIII.

200. The motion of the composite pendulum CPQ [considered as a rigid triangle] and of the simple pendulum CN are said to be accelerated similarly when infinitesimal increments of the speed from the action of central gravity in whatever minimum part of time are produced similarly on the individual parts of the pendulum P, Q &c. and on the corpuscle N of the simple pendulum, everywhere are proportional to the homologous distances PC, QC &c. and NC of the particles oscillating, and of the corpuscle N from the axe of oscillation. [That is, the tangential speeds also are proportional to the radii.]

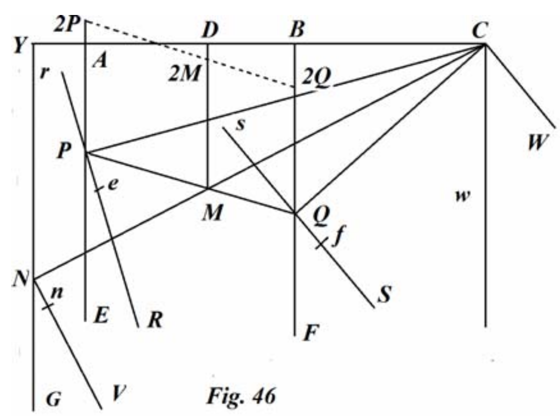

Fig. 46

201. And now the actual speeds acquired of the particles P, Q &c. of the composite pendulum, we will call henceforth *p, q* &c. respectively, and the momentary elements or increments of these *dp, dq* &c. Truly we will call the speed of the corpuscle N of the simple pendulum *n*, and its momentary increment *dn*. And finally the central force of gravity [here taken as the acceleration], by which the particle P is acted on in the direction AP, shall be E.P, truly the central force pushing the particle Q along the direction BQ is called F.Q ; and the force of gravity, which affects the particle N in the direction YN is called, G.N.

PROPOSITION XXXIII: LEMMA.

202. *The vicarious forces* R, S, *&c. acting on the particles* P, Q, &c. *of the composite pendulum along the directions* PR, QS, &c. *prevail to impress the same increments of velocity, as the forces of gravity* E, F, &c. *acting along* AP, BQ, &c. *on the same*

particles P, Q, &c.*; and thus these vicarious forces will accelerate the composite pendulum in the same manner as these central forces.*

Because the parts of the composite pendulum are connected together, they will describe similar whole arcs, and their motion once conceived would continue without any change, unless gravity incessantly acting on these would accelerate the motion of the pendulum. Nothing of the actual motion need be brought to our attention, as long as in the present circumstances this motion acquired may be considered as common, by which the individual particles of the pendulum are moved away; but now we will consider the increments of the speeds for these particles newly impressed as the motion of the same, and this motion can arise likewise from the actions of the vicarious forces R, S, &c. As well as being considered from the central forces of gravity E, F, &c.

And indeed if this may be denied, if it were possible, the actions of the gravitational forces E, F, &c. may impress a greater speed on the particles P, Q, &c. than the vicarious forces R, S, &c ; therefore these forces, acting along Pr, Qs, &c., may not remain in equilibrium with the forces E, F, &c. applied to the same particles P , Q, &c., which is contrary to the hypothesis. Likewise it follows to be absurd, if the forces of gravity E, F, &c. can impress smaller speeds on the particles P, Q, &c. than the vicarious forces. Therefore etc.

PROPOSITION XXXIV. LEMMA.

203. *If the individual particles* P, Q, &c. *of any composite pendulum may be acted on by the forces* P.R, Q.S, &c. *of which* R, S, &c. *likewise shall be similar to* V *itself, with* V.N *denoting the tangential force of the moving body* N : P, Q, &c. & N *will be impressed along the directions* PR, QS &c. *with the infinitesimal speeds dp, dq,* &c. & *dn, that is, proportional to the radii* PC, QC, &c. & NC.

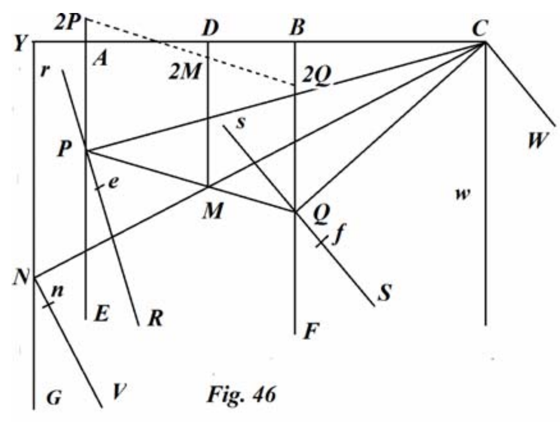

Fig. 46

I. The radii shall be NC, PC, QC &c. winding around the axis C freely and independently from each other, & because (§.131.) any force shall be equivalent to the motion arising applied for an infinitesimal time, by which this motion is produced, and because in our case all the vicarious forces R, S, V, are acting together or in the same time, these forces will be as the motions P.*dp,* Q.*dq* &c. & N.*dn*; and thus R, S, & V themselves shall be as *dp, dq* &c. & *dn* ; and R, S, &c. & N themselves, (following the hypothesis) are as the radii PC, QC, &c. & NC, and therefore the speeds arising *dp, dq,* &c. & *dn* are in proportion to the same PC, QC, &c., & NC.

II. Therefore since the speeds impressed *dp, dq*, &c. & *dn* shall be similar ; the angular motions PC*e,* QC*f*, &c. & NC*l* shall be equal, that is the angles described in the same time PCQ & *eCf* will be equal, and likewise the right lines PQ & *ef* joining the particles PQ shall be equal; thus so that by such a motion no change in the ratio of the mutual positions of the particles P, Q, &c. shall be able to happen. Therefore the tangential forces R, S; &c. & V will express a similar motion on the particles P, Q, &c., whether or not these particles P, Q shall be connected to each other or not, as with composite pendulums. Q.E.D.

[Hermann's analysis is basic, as the problem is exceedingly hard to solve exactly, and in the following he considers the centrifugal force as being proportional to the radius, the angular speeds being equal at any instant, and these centrifugal forces balance the weight of the pendulum at that instant ; the acceleration down the slope, as it were, he has considered already. As mentioned above, Hermann introduces later another source of wonder, as he considers his system immersed in a fluid of a certain specific gravity, and the masses to be made of different substances : this has the added effect of producing an apparent weights depending on the upthrust of the fluid displaced, so that the centre of oscillation will change, though the centre of percussion of the masses is left unchanged; it would of course be impossible to test this situation, as the fluid would become involved to a greater or lesser extent in any collision or oscillatory process.]

COROLLARY.

204. Hence because (§.202,) the similar vicarious forces R, S, &c. & V accelerate the motion in the directions PR, QS &c. & NV applied to the bodies P, Q, &c., & N and in the same manner for the connected pendulum CPQ and simple pendulums CN, from which it will occur that the forces of gravity E, F; &c. & G acting on the particles P, Q, &c., & N ; generally it is clear for both simple and composite pendulums, both being required to begin moving from a similar position with respect to the horizontal, henceforth to begin moving similarly and equally, thus so that equal angles YCN and ACM always may be made together, and thus (§. 197.) the one CN shall be *synchronous* or *isochronous* to the other CPQ; and conversely, if these pendulums shall be synchronous, the vicarious forces R, S, &c., & V will be similar.

PROPOSITION XXXV. THEOREM .

205. *If the individual particles of some composite pendulum suddenly somewhere were changed into a motion depending on the height with the bonds freed , each with that speed as it had acquired joined to the rest from the oscillation, for the common centre of gravity of all the parts of the pendulum will climb to that height, from which it had fallen by the descent of the whole composite pendulum, when evidently all the parts of the pendulum were connected.*

I. The axis of the composite pendulum falling from the horizontal position CA into the situation CM is put in place by an accelerated motion, truly the particles P, Q, &c. with the bonds freed rise along the right lines P_2P, Q_2Q, &c. each one with an initial speed, as it had acquired from the rest from the oscillation ; thus so that the common centre of gravity M shall rise along the line M_2M. It is agreed that this M_2M be equal to DM itself, which may denote the depth, from which the centre of gravity of all the parts of the pendulum has fallen by the motion of all of the composite pendulum. Because now P, Q, N denote the masses of the bodies and E.P, F.Q, &c. G.N the weights of these, of which

the sum, with the exception of the weight N, may be called M, that is,
$M = E.P + F.Q + \&c.$, and therefore (§.48.) the centre of gravity will be M;
$E.P.AC + F.Q.BC = M.DC$, also
$E.P.AP + F.Q.BQ + \&c. = M.DM$;
[*i.e*., these equations give the location of the centre of gravity (DM, M_2M) of the masses P and Q horizontally and vertically], and finally $E.P.P_2P + F.Q.Q_2Q + \&c. = M.M_2M.$

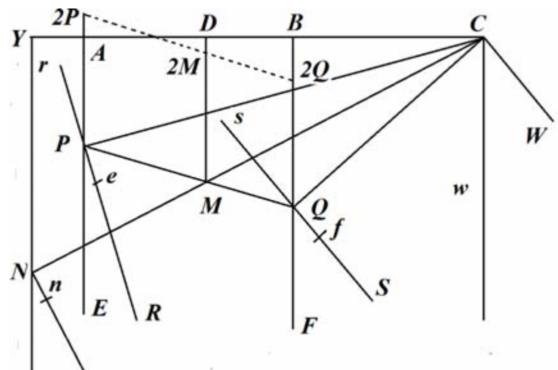

Fig. 46

II. In addition the simple pendulum CN shall be synchronous with the composite pendulum, thus so that it may be congruent with CM and the speeds of the corpuscles N, P, Q, &c. shall be similar; and thus the vicarious forces (§. 204.) R, S, &c. will be similar with V, truly because (§.198.) the vicarious forces of the centres of gravity E, F &c. are equivalent to G, and as they are able to be put in equilibrium with these, there will be (§.56.) $E.P.AC + F.Q.BC + \&c.$, that is (no.1.) $= M.DC = R.P.PC + S.Q.QC + \&c.$ & $G.N.YC = V.N.NC$,[we can think of these as the work done in the vertical and tangential situations by the two kinds of pendulum], and hence the ratio arises $M.DC : G.\ N.YC = R.P.PC + S.Q.QC + \&c. : V.N.NC$ (or because R, S, &c. : & V shall be proportional to the radii PC, QC, &c. and to NC) $M.DC : G.\ N.YC = P.PC^2 + Q.QC^2 + \&c. : N.NC^2$; but the speed *p*, *q* , &c. *n* of the corpuscles P, Q, N also shall be in proportion to the radii PC, QC, NC, therefore M.DE: G.N.YC, or (on account of the similar triangles CDM & CYN) the ratio shall become $M.DM : G.N.YN = P.pp + Q.qq + \&c. : N.nn$.
III. Since (following the hypothesis) the corpuscles P, Q, &c. are able to be measured rising the heights P_2P, Q_2Q with the initial speeds *p, q*, (§.141.) the same bodies are able to acquire their speeds *p, q* &c. by falling from rest through the distances $_2PP$, $_2QQ$, thus so that (§.150.) $2.E.P_2P = pp$; $2.F.Q_2Q = qq$, & $2.G.NY = nn$, by substituting the values found in place of *pp, qq, nn* in the last ratio of these following numbers, there will be $M.DM : G.N.YN = 2.E.P.P_2P + 2.F.Q.Q_2Q + \&c.$ that is (from no.1 if this section) $2M.M_2M : 2G.N.NY = M.M_2M : G.N.YN.$ Therefore since the latter first and last ratios shall be equal, the previous ones also will be equal; and thus
$M.DM = M.M_2M$, or $DM = M_2M$. That is, the centre of gravity has risen to the same height, from which it had fallen. Q.E.D.
[Essentially this is a conservation of kinetic and potential energy or work done in modern terms.]

COROLLARY I.

206. On resuming the ratio from no. II of the preceding section,
$M.DC : G.N.YC\ (= R.P.PC + S.Q.QC + \&c. : V.N.NC) = P.PC^2 + Q.QC^2 + \&c. : N.NC^2$,

if in place of the right lines DC, YC the proportions of these MC & NC may be substituted into the same ratio, there becomes

$\mathrm{M.MC : G.N.NC = P.PC^2 + Q.QC^2 + \&c : N.NC^2}$, from which the following general formula is elicited easily : $\mathrm{NC = (P.PC^2 + Q.QC^2 + \&c.).G : M.MC}$ for the determination of the simple pendulum NC isochronous with the compound pendulum CPQ, in which the formulas E.P, F.Q &c. denote the weights of the bodies P, Q &c.,& and the point M the weight at the centre of gravity $\mathrm{M = E.P + F.Q + \&c.}$ of these.

COROLLARY II.

207. If the individual terms E, F, &c. and G were equal, [corresponding to the weights per unit mass, or the acceleration of gravity at the points in question being the same], the formula of the preceding corollary shall
become

$\mathrm{NC = (P.PC^2 + Q.QC^2 + \&c.) : M.MC;}$[or $\mathrm{N.NC^2 : N.NC = (P.PC^2 + Q.QC^2 + \&c.) : M.MC,}$]

where now $\mathrm{M = P + Q + \&c.}$ And this will be the general formula for all the figures in the common system of uniform gravity in all the parts of the hanging weights, which have followed the most celebrated geometers Huygens and James Bernoulli from their particular method.

COROLLARY III.

208. With the same in place, as in the preceding corollary, if it may be considered to be composed of two particles P and Q only, of which even now M may designate the common centre of gravity of the sum. There will be had (§.171.) :
[Recall that ABC shall be some triangle, of which the base BC may be divided at D by the right line AD as desired drawn within the triangle. It can be shown that that :

$\mathrm{AB^2.DC + AC^2.BD = AD^2.BC + BD.DC.BC.}$]

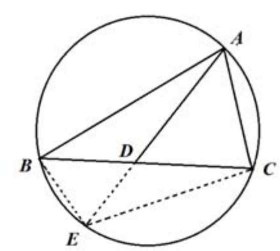

Fig. 39

$\mathrm{PC^2.MQ + QC^2.MP = MC^2.PQ + MP.MQ.PQ.}$ Or with the proportionals of MQ, MP, & PQ themselves substituted in place of the right lines, which on account of the centre of gravity M of the particles P and Q, and with these taken in the order, the proportionals are P, Q, & $\mathrm{P + Q}$ or M; there arises

$\mathrm{P.PC^2 + Q.QC^2 = M.MC^2 + MP.MQ.PQ}$,

[for $\mathrm{PC^2.MQ + QC^2.MP = MC^2.PQ + MP.MQ.PQ}$ gives

$\mathrm{PC^2.\frac{MQ}{PQ} + QC^2.\frac{MP}{PQ} = MC^2 + MP.MQ}$, or $\mathrm{PC^2.P + QC^2.Q = M.MC^2 + M.MP.MQ}$]

and thus the formula of the preceding corollary $\mathrm{CN = (P.PC^2 + Q.QC^2) : M.MC}$, now becomes $\mathrm{CN = (M.MC^2 + M.MP.MQ) : M.MC}$. And thus, if an innumerable number of equal particles P, Q may be taken, from which the whole pendulum may be made, the length of the simple pendulum CN isochronous with the composite pendulum then will be $\int(\mathrm{M.MC}^2 + \mathrm{M.MP.MQ}) : \int \mathrm{M.MC}$, where $\int$ indicates the sum or the integral of that quantity, to which the letter has been $\int$ prefixed . Therefore, if with the cel. James Bernoulli the particles P, Q are equal to each other, the incremental masses of which may be called *dp,* and CM, *x*; truly MP or MQ, *y*, these indeed will be equal to P and Q themselves, and the centre of gravity of these shall be at the point M, there will be $\mathrm{CN} = \int\left(2xxdp + 2yydp\right) : \int 2xdp = \int\left(xx + yy\right).dp : \int xdp$; which is the very formula, that the wisest man elicited by his method in the *Actis Academiae Reg. Paris. Scient*. 1703. for the 25 th of April. Just as truly the formula must be applied to these shapes and volumes, this is not the place to show how that may depend on the calculation to be summed, without our presentation becoming more lengthy, meanwhile they can be read, which matter the most praiseworthy man has indicated above in the place just indicated.

SCHOLIUM.

209. The universality of the rule in the first corollary (§.206.) cannot be approved to be shown better, than if that may be shown now to contain the rule required defining the centre of oscillation in some composite pendulum, the parts of which may oscillate in a certain homogeneous fluid of different specific gravity, or, what amounts to the same, the particles of which from the same material or of the same specific gravity perform their vibrations in a fluid of a different density such as the rule promises for us from the cel. Joh. Bernoulli on p. 88 of the *Acta Erud. Lipz.* I7l3. For if the ratio of the specific gravity of the weight P to the fluid, in which it is swinging, shall be as G to $\mathrm{G} - \mathrm{E}$, and the ratio of the specific gravity of the weight Q and of the liquid shall be as G to $\mathrm{G} - \mathrm{F}$, the aforementioned formula of section 206 will show the length of the simple pendulum CN swinging in a vacuum and isochronous with the composite pendulum CPQ, which will perform its vibrations in the fluid.

For the sake of an example, P shall be a gold sphere, Q an iron one, and the pendulum CPQ may swing in water ; and because the weights of gold, of iron, and of water are as the numbers 100, 42 & 5, just as approximately in turn there will be $\mathrm{G} : \mathrm{G} - \mathrm{E} = 100 : 5$, and thus $\mathrm{E} = \frac{95}{100}\mathrm{G}$; & $\mathrm{G} : \mathrm{G} - \mathrm{F} = 42 : 5$, and thus $\mathrm{F} = \frac{37}{42}\mathrm{G}$; hence because (following the hypothesis) $\mathrm{M} = \mathrm{E.P} + \mathrm{F.Q}$, in this case M will be equal to $\left(\frac{95}{100}\mathrm{P} + \frac{37}{42}\mathrm{Q}\right).\mathrm{G}$; and thus by the substitution of the value M into the general formula of the first corollary, there becomes $\mathrm{NC} = (\mathrm{P.PC}^2 + \mathrm{Q.QC}^2).\mathrm{G} : \left(\frac{95}{100}\mathrm{P} + \frac{37}{42}\mathrm{Q}\right).\mathrm{G} = (\mathrm{P.PC}^2 + \mathrm{Q.QC}^2) : \left(\frac{95}{100}\mathrm{P} + \frac{37}{42}\mathrm{Q}\right)$. In which P & Q denote the masses of these particles, or also the absolute weights of these, which (§§.30, 151, 175.) always are the proportions of the masses. And the point M is the centre of gravity, not of the corpuscles P and Q themselves, but of their fractional parts $\frac{95}{100}\mathrm{P}$ and $\frac{37}{42}\mathrm{Q}$. Hence therefore it is clear the centre of percussion differs in this case from the centre of oscillation. Therefore they err, who have not indentified gravity as entering into the reasoning, and who have confused these two centres of oscillation and percussion.

If the formula of the first corollary may be applied to the bodies P and Q, which are for the lighter liquid specified, the whole denominator of the formula will be transported into a negative quantity, which indicates the pendulum in place inverted with respect of the preceding case, free to perform its oscillations within the liquid by describing an arc towards the bottom of the hollow vessel with the axis near the bottom. Just as the aforementioned formula of the composite pendulum may be applied, in which the individual parts have been arranged on one and the same right line, I consider it quite clear, so that it may not need any explanation.

PROPOSITION XXXVI. THEOREM.

210. Fig. 47. *The identity of the centre of oscillation and of percussion in that case, in which the weights of the individual parts of the composite pendulum are proportional to the masses of the same.*

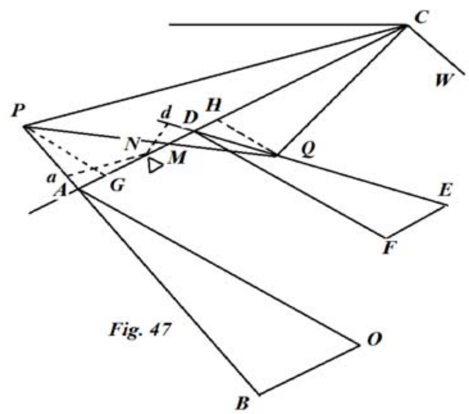

Fig. 47

If the particles P, Q. &c., connected to each other by the inflexible rods CP, CQ, PQ may be turned about the axis CW, the point N on the axis CN, at which the maximum force may be exerted on an object in the way, is called the centre of percussion, and it is required to show this point shall be at just as great a distance from the axis of oscillation CW, as the centre of oscillation, as shown in the case of Coroll. 2 of equal gravity in the preceding proposition (§. 207). With the radii CP and CQ ; PB and QE act normal to the axis CM crossing at the points A and D, through which and through the centres of the bodies P and Q the normals AO, DF, PG & QH are drawn to the axis CM. And with these in place it is evident at once from the rotation of the whole figure CPQ about the axis CW, the motion of the particles P and Q is going to be with speeds proportional to the distances of these from this axis, PC and QC ; and along that, which have been said above (§.53), the same axis of the pendulum CM may take an impressed force from the particles P, Q acting in a circular course about C, as if the points A & D of that, in which evidently the directions of the weights cross the axis in some arc being described by some point trying to follow its motion, they are impressed along the directions AB & DE by these forces with the speeds which the particles P & Q &c. have ; therefore with AB, DE &c. made equal or perhaps in proportion to the homologous radii CP, CQ which the velocities exhibit, with which the particles P, Q are acted on in rotating, the question is reduced to this, so that the centre of equilibrium of the forces P.AB & Q.DE may be found ; N shall be this centre, which thus must be prepared, so that with the perpendiculars N*a*, N*d* &c. drawn from that to the individual PB, DE, &c., all the products P.AB.N*a* shall be equal to all the products Q.DE.Nd. But with BO and EF drawn parallel to CM, the similar triangles AOB and N*a*A produce $\mathrm{AB.N}a = \mathrm{AO.AN}$ and the similar triangles DSE & N*d*D accomplish $\mathrm{DE.N}d = \mathrm{DF.DN}$; therefore it is required that all the P.AO.AN shall be equal to all the Q.DF.DN; only the two P, & Q need be examined, and there will be $\mathrm{P.AO.AN = Q.DF.DN}$, or $\mathrm{P.AO.AC - P.AO.NC = Q.DF.NC - Q.DF.DE}$, and thus $\mathrm{P.AO.AC + Q.DF.DC = (P.AO + Q.DF).NC}$; in truth because $\mathrm{AO = CG}$ & $\mathrm{DF = CH}$ on account of the similar and equal triangles ABO & CPG, as in which the hypotenuses AB and CP shall be equal, and therefore the similar and equal triangles DEF and CQH, in which equally DE and CQ are equal, there becomes

$$\mathrm{P.AO.AC + Q.DF.DE\left(= P.GC.AC + Q.HC.DC\right) = P.PC^2 + Q.QC^2};$$

$$\text{\& } \mathrm{P.AO + Q.DF = P.GC + Q.HC}, \text{ that is } (\S.44) = \mathrm{M.MC};$$

with the point M present for the centre of gravity of the particles P, Q and $\mathrm{M = P + Q}$. Therefore with these substitutions made in $\mathrm{P.AO.AC + Q.DF.DE = (P.AO + Q.DF).NC}$, there will be obtained : $\mathrm{P.PC^2 + Q.QC^2 = M.MC.NC}$, and thus $\mathrm{NC = \left(P.PC^2 + Q.QC^2\right) : M.MC}$. Which expression in short is the same as that, which we have found above (§. 207) for the centre of oscillation under the common hypothesis, as the weights of the particular masses P, Q themselves are in proportion, or, what amounts to the same, in which the individual E, F, G &c. are equal, such as we have used above (§.198.) for signifying the nearby masses P, Q, N for central forces of gravity acting. Truly in all other hypothesis of gravity the centres of oscillation and percussion are different. Q.E.D.

COROLLARY.

211. Hence if there were an infinitude of equal particles P, Q, then there becomes $\mathrm{CN} = \int\left(\mathrm{P.PC^2 + Q.QC^2}\right) : \int \mathrm{M.MC}$, and thus by calling CN, *t*, and by retaining the symbols of paragraph 208, there becomes $t = \int (xx + yy)dp : \int x.dp$, which is again that very formula, in which we have indicated the place, as by another method the cel. James Bernoulli has found the *Actis. Acad. Reg. Paris. Scient. Acad*. 1704 for the 14th of April.

CAPUT V.

De Motibus gravium inter se connexorum atque in arcubus circularibus concentricis junctim delabentium ;seu de motu Pendulorum compositorum eorumque centro oscillationis in omni possibili gravitatis variabilis hypothesi.

Theoriam centri oscillationis Hugenius primus, quod sciam, aperuit in parte quarta eximii Tractibus de Horologio Oscillatorio. Sed quia is principio institit, quod indemonstratum supposuit, verissima ejus doctrina ab omnibus integrum assensum non impetravit. Petebat enim, ut sibi concedatur, fore ut centrum commune gravitatis omnium, compositi cujusque penduli, partium non possit altius assurgere, cum unaquaeque particula diffracto vinculo, quo cum reliquis connexa erat, ea celeritate ascensum suum liberum incipiet, quam junctim cum reliquis descendendo acquisiverat, quin unde delapsum erat descensu totius penduli compositi. Celeb. Jac. Bernoulli alia via longitudinem penduli simplicis composito isochroni generaliter determinans, demonstrationem postulati Hugeniani inde deduxit, sed tantum in hypothesi gravitatis in singulas penduli partes uniformiter agentis. Nos vero rem generalius concipiemus, fingentes gravitatem in partes illas utcunque difformiter agere atque hoc non obstante methodo directa & facili memorabilem naturae legem demonstrabimus; ascensum scilicet centri gravitatis omnium penduli compositi partium, modo paulo supra memorato, aequalem esse descensui ejusdem, tametsi, pro variante in singulis penduli partibus gravitate, centrum earum gravitatis multum diversum sit ab eo, quod est in hypothesi communiori gravitatis uniformis & proportionalitatis ponderum cum massis corporum.

DEFINITIONES.

I.

Pendulum compositum vocatur, quod pluribus corporibus seu partibus inter se connexis instructum est. Simplex vero quod unico pondere constat.

II.

Axis penduli compositi est recta, quae ex puncto suspensionis immobili, circa quod scilicet totum pendulum reciproco motu eundo & redeundo oscillatur, per commune centrum gravitatis omnium partium penduli compositi transit.

III.

Axis oscillationis est linea penduli axi perpendicularis per punctum suspensionis transiens, circa quam pendulum vibrationes suas peragit.

IV.

Centrum oscillationis est punctum in axe penduli, cujus distantia ab axe oscillationis aequatur longitudini penduli simplicis composito synchroni.

V.

Pendulum simplex composito synchronum dicitur, cum hujus axis & illud ex situ horizontali, vel quolibet alio ad horizontem similiter inclinato, simul exire incipientes aequales constanter angulos simul oscillando conficiunt.

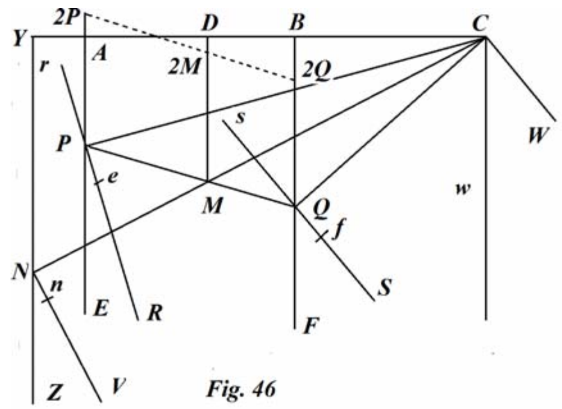

Fig. 46

197. Penduli compositi CPQ circa punctum suspensionis C convertibilis, lineae omnes CP, CM, CQ, PQ gravitatis expertes sint; in terminis vero rectae PQ ponduscula quaecunque P, Q affixa existant, quorum commune centrum gravitatis sit M, recta CM per punctum suspensionis C & centrum gravitatis ponderum M ducta, vocatur *axis* penduli compositi PQC, recta vero CW axi CM perpendicularis per suspensionis punctum C ducta, est axis oscillationis. Porro si penduli compositi axis CM ex situ horizontali CA delapsus venerit in situm CN eo tempore, quo pendulum simplex etiam ex situ horizontali in CN ceciderit, angulosque perpetuo aequales ACM & ACN confecerint compositum CPQ & simplex pendulum CN, hoc illi *synchronum*, vel subinde etiam *isochronum* dicetur; ponduscolum vero in pendulo simplici N, instar puncti gravis consideratum, ad axem penduli compositi adductum atque applicatum, in eo signat *centrum oscillationis* N.

VI.

198. *Solicitationes vicariae* solicitationum gravitatis centralium sunt solicitationes tangentiales loco centralium gravitatis cogitatione substituendae, atque solicitationibus hisce aequipollentes. Sic solicitationes tangentiales expositae per facta ex magnitudinibus R, S, &c. & *V* in massas P, Q &c. & N, hoc est solicitationes R.P, S.Q, &c, & V.N in directionibus PR, QS &c*:* & NV, quae tangentes sunt arcuum *Pe, Qf*, &c. *Nn* a corporibus P, Q &c. & N oscillando descriptorum, in sua respectiva corpora agentes, sunt vicariae solicitationum gravitatis centralium secundum directiones horizonti normales AP, BQ &c. YN in corpora P, Q, &c. & N agentium, si eundem cum hisce effectum producere valent, atque adeo in oppositas partes secundum Pr, *Qs* &c. & *Nu* agentes in aequilibrio consistunt cum solicitationibus gravitatis centralibus, quas deinceps etiam exponam per facta E.P, F.Q &c & G.N ex magnitudinibus E, F, &c. & G in massas P, Q, &c. & N.

VII.

199. Ac denique *Solicitationes vicariae similes* dicentur, quoties magnitudines R, S, &c. & V, homologis distantiis PC, QC &c. & NC proportionales erunt.

VIII.

200. Motus penduli compositi CPQ & simplicis CN similiter accelerari dicuntur cum celeritatis incrementa infinitesima a solicitationibus gravitatis centralibus quolibet

tempusculo minimo in singulis penduli particulis P, Q &c. & simplicis pondusculo N simul producta, ubique proportionalia sunt homologis distantiis PC, QC &c. & NC particularum oscillantium & corpusculi N ab axe oscillationis.

201. Celeritates actuales & jam acquisitas particularum P, Q &c. penduli compositi, deinceps vocabimus *p, q* &c. respective, earumque elementa seu incrementa momentanea *dp*, *dq* &c. Celeritem vero corpusculi N penduli simplicis nominabimus *n*, ejusque
incrementum momentaneum *dn*. Ac tandem solicitatio gravitatis centralis, qua particula P in directione AP urgetur, sit E.P, solicitatio vero centralis particulam Q secundum directionem BQ urgens dicatur F.Q ; ac solicitatio gravitatis, quae particula N in directione YN afficitur, G.N.

PROPOSITIO XXXIII: LEMMA.

202. *Solicitationtes vicariae* R, S, *&c. juxta directiones* PR, QS, &c. *in particulas* P, Q, &c. *penduli compositi agens eadem velocitatis incrementa singulis hisce particulis imprimere valent, quae solicitationes gravitatis E, F, &c. juxta AP, BQ, &c. in easdem particulas P, Q, &c. agens; atque adeo vicariae illae eodem, quo hae centrales solicitationes , modo penduli compositi motum accelerabunt.*

Quia compositi penduli partes inter se connexae sunt, similes arcus simul cunctae describent, atque suum motum semel conceptum absque ulla mutatione continuarent, nisi gravitas indesinenter in eas agens motum penduli acceleraret. Ad rem nostram motus actualis consideratio nihil confert, quandoquidem in praesenti negotio hic motus jam acquisitus tanquam motus *communis* considerari debet, quo singulae penduli particulae abripiuntur; sed celeritaris incrementa particulis illis recens imprimenda tanquam motum earundem particularem nunc spectabimus, & hunc motum nascentem perinde oriri posse a solicitationibus vicariis R, S, &c. Ut a centralibus gravitatis E, F, &c. probandum.

Etenim si hoc negetur, imprimant, si fieri potest, gravitatis solicitationes
E, F, &c. particulis P, Q, *&c:* majorem celeritatem quam solicitationes vicariae R, S, &c ergo hae solicitationes, secundum *Pr, Qs,* &c. agentes, non consisterent in aequilibrio cum solicitationibus E, F, &c. iisdem particulis P , Q, &c. applicatis, quod est contra hypothesin. Idem absurdum sequetur, si dicatur solicitationes gravitatis E, F, &c. minores particulis P, Q, &c. celeritates imprimere posse quam vicariae. Ergo &c.

PROPOSITIO XXXIV. LEMMA.

203. *Si singulae particulae* P, Q, &c. *alicujus penduli compositi a solicitationibus* P.R, S.Q, *&c. quarum R, S,* &c. *similes sint ipsi* V, *denotante* V. N *solicitatione tangentiale mobilis* N, *juxta directiones* PR, QS *&c: urgentur, celeritates infinitesime* dp, dq, &c. & dn *ipis* P, Q, &c. & N *impresse erunt similes, hoc est radiis* PC, QC, &c. & NC *proportionales.*
I. Sint radii NC, PC, QC &c. libere atque independenter uni ab aliis volubiles circa axem C, & quia (§. 131.) quaelibet solicitatio aequivalet motui genito applicato ad tempusculum infinitesimum, quo motus iste producitur, & quia in casu nostro omnes solicitationes vicariae R, S, V, simul seu eodem tempore agunt, erunt hae solicitationes ut

motus P.*dp,* Q.*dq* &c. & N.*dn*; atque adeo ipsae R, S, *p* & V, ut *dp, dq* &c. & *dn,* atqui ipsae R; S, &c.
& N, (secundum hypothesin) sunt, ut radii PC, QC, &c. & NC, ergo & celeritates nascentes *dp, dq*, &c. & *dn* iisdem PC, QC, &c. & NC proportionales sunt.

II. Cum igitur celeritates *dp, dq,* &c. & *dn* similes fint ; motus angulares *PCe, QCF,* &c. & NC*l* aequales, hoc est anguli simul descripti PCQ & *eCf* aequales erunt, perinde ac rectae PQ & *ef* particulas PQ connectentes; aqeo ut tali motu nulla variatio ratione mutuae poitionis particularum P, Q, &c. accidere possit. Propterea solicitationes tangentiales similes R, S; &c. & V particulis P, Q, &c. similes motus impriment, sive particulae illae P, Q, inter se connexae sint, ut in pendulis compositis, sive non. Quod erat demonstrandum.

COROLLARIUM.

204. Hinc quia (§.202,*)* solicitationes vicariae similes R, S, &c. &V in directionibus PR, QS &c. & NV corporibus P, Q, &c.& N applicatae eodem modo pendulorum simplicis CN &. compositi CPQ motus accelerant, quo fieret a solicitationibus gravitatis E, F; &c. & G in particulas P, Q, &c. & N agentibus ; liquet omnino ambo pendula simplex & compositum, ex simili situ respectu horizontis moveri incipientia, similiter deinceps & constanter motum iri, ita ut aequales semper angulos YCN, ACM simul conficiant, atque adeo (§. 197.) unum CN alteri CPQ *synchronum* aut *isochronum* sit ; & conversim, si haec pendula synchrona sint, solicitationes vicariae R, S, &c. & V similes erunt.

PROPOSITIO XXXV. THEOREM .

205. *Si singulae particulae cujusque penduli compositi derepente alicubi vinculo solutae motum in altum convertant, unaquaeque ea celeritate quam cum reliquis connexa oscillando acquisivit, commune omnium penduli partium gravitatis centrum ad eam ipsam altitudinem ascendet, ex qua delapsum erat descensu totius penduli compositi, cum scilicet omnes penduli partes athuc connexae essent.*

I. Axis penduli compositi ex situ horizontali CA in situm CM delapsum esse ponatur motu accelerato, particulas vero P, *Q,* &c. vinculis solutae ascendere in rectis P_2P, Q_2Q, &c. unamquamque celeritate initiali, quam cum reliquis oscillando acquisivit; ita ut commune omnium gravitatis centrum M ascendat per lineam M_1M. Probandum est hanc M1M aequalem fore ipsi DM, quae profunditatem denotat, ex qua omnium penduli partium centrum gravitatis cecidit motu totius penduli compositi. Quia nunc P, Q, N denotant massas corporum & E.P, F.Q, &c. G.N eorundem pondera, horum summa, excepto pondere N, dicatur M, hoc est, $M = E.P + F.Q + \&c.$, eritque (§.48.) propter centrum gravitatis M; $E.P.AC + F.Q.BC = M.DC$, nec non $E.P.AP + F.Q.BQ + \&c. = M.DM$; ac denique $E.P.P_2P + F.Q.Q_2Q + \&c. : M.M_2M.$

II. Sit insuper pendulum simplex CN composito *synchronum,* adeo ut id cum CM congruat atque celeritates pondusculorum N, P, Q, &c. similes sint; eruntque adeo (§. 204.) solicitationes vicariae R, S, &c. V similes, verum quia (§. 198.) solicitationes vicariae solicitationibus gravitatis centralibus E, F &c. G aequipollent, atque cum hisce in aequilibrio consistere queunt, erit (§.56.) $\mathrm{E.P.AC}+\mathrm{F.Q.BC}+\&c.$ id est (num. 1.) $=\mathrm{M.DC}=\mathrm{R.P.PC}+\mathrm{S.Q.QC}+\&c.$ & $\mathrm{G.N.YC}=\mathrm{V.N.NC}$, atque hinc resultat analogia $\mathrm{M.DC}:\mathrm{G.\,N.YC}=\mathrm{R.P.PC}+\mathrm{S.Q.QC}+\&c.:\mathrm{V.N.NC}$ (vel quia R, S, &c*:* & V radiis PC, QC, &c. NC proportionales sunt) $=\mathrm{P.PC}^2+\mathrm{Q.QC}^2+\&c.:\mathrm{N.NC}^2$; at celeritates p, q , &c. n corpusculorum P, Q, N etiam radiis PC, QC, NC proportionantur, ergo M.DE: G.N.YC, vel (propter triangula similia CDM & CYN) ratio $\mathrm{M.DM}:\mathrm{G.N.YN}=\mathrm{P.}pp+\mathrm{Q.}qq+\&c.:\mathrm{N.}nn$.

III. Quoniam (secundum hypothesin) corpuscula P, Q, &c. celeritatibus initialibus *p*, *q*, ascendentia spatia $\mathrm{P_2P}$, $\mathrm{Q_2Q}$ ascendendo emetiri possunt, (§.141.) eadem corpuscula spatia $_2\mathrm{PP}$, $_2\mathrm{QQ}$ a quiete perlabentia celeritates suas *p, q* &c. acquirere queunt, adeo ut (§.150.) $2.\mathrm{E.P_2P}=pp$; $2.\mathrm{F.Q_2Q}=qq$, & $2.\mathrm{G.NY}=nn$, loco *pp, qq, nn* substituendo valores inventos in ultima analogia numeri secundi hujus, erit $\mathrm{M.DM}:\mathrm{G.N.YN}=2.\mathrm{E.P.P_2P}+2.\mathrm{F.Q.Q_2Q}+\&c.$ hoc est (num.1. hujus) $2\mathrm{M.M_2M}:2\mathrm{G.N.NY}=\mathrm{M.M_2M}:\mathrm{G.N.YN}.$ Ergo cum consequentes primae & ultimae rationis aequales sint, antecedentes etiam aequabuntur ; atque adeo $\mathrm{M.DM}=\mathrm{M.M_2M}$, vel $\mathrm{DM}=\mathrm{M_2M}$. Id est centrum gravitatis ad eandem altitudinem ascendit, ex qua delapsum erat. Quod erat demonstrandum.

COROLLARIUM I.

206. Resumendo ex num.11 articuli praecedentis analogiam,
$\mathrm{M.DC}:\mathrm{G.N.YC}\ (=\mathrm{R.P.PC}+\mathrm{S.Q.QC}+\&c.:\mathrm{V.N.NC})=\mathrm{P.PC}^2+\mathrm{Q.QC}^2+\&c.:\mathrm{N.NC}^2$, si loco rectarum DC, YC earum proportionales MC & NC in eadem analogia substituantur, fiet $\mathrm{M.MC}:\mathrm{G.N.NC}=\mathrm{P.PC}^2+\mathrm{Q.QC}^2+\&c:\mathrm{N.NC}^2$, ex qua facile elicitur sequens formula generalis. $\mathrm{NC}=(\ \mathrm{P.PC}^2+\mathrm{Q.QC}^2+\&c.).\mathrm{G}:\mathrm{M.MC}$ pro determinatione penduli simplicis CN composito CPQ *isochroni* , in qua formula $\mathrm{M}=\mathrm{E.P}+\mathrm{F.Q}+\&c.$ & E.P, F.Q &c. pondera corporum P, Q &c. denotant, ac punctum M ipsorum centrum gravitas.

COROLLARIUM II.

207. Si singulae E, F, &c. &. G aeqales fuerint, formula praecedentis corollarii sit $\mathrm{NC}=(\mathrm{P.PC}^2+\mathrm{Q.QC}^2+\&c.):\mathrm{M.MC}$, ubi nunc $\mathrm{M}=\mathrm{P}+\mathrm{Q}+\&c.$ Atque haec foret generalis formula pro omnibus figuris in systemate vulgari gravitatis uniformis in omnibus penduli partibus, quod secuti sunt Celeberrimi Geometrae Hugenius & Jac. Bernoullius sua methodo particulari

COROLLARIUM III.

208. Iisdem positis, quae in corollario antecedenti, si duae tantum particulae P, Q penduli compositi spectentur, quarum M etiamnunc centrun gravitatis & aggregatum designet.

Habetur (§.171.) $PC^2.MQ + QC^2.MP = MC^2.PQ + MP.MQ.PQ.$ Vel substitutis loco rectarum MQ, MP, & PQ ipsarum proportionalibus , quae propter centrum gravitatis M particularum P , Q, eodem ac illae ordine sumta, sunt P, Q, & P + Q seu M; fiet $P.PC^2 + Q.QC^2 = M.MC^2 + MP.MQ.PQ.$, atque adeo formula praecedentis corollarii $CN = (P.PC^2 + Q.QC^2) : M.MC$, nunc fiet $CN = (M.MC^2 + M.MP.MQ) : M.MC$. Adeoque, si innumera sumantur particularum P, Q paria, quibus totum pendulum compositum sit, longitudo penduli simplicis CN composito isochroni tunc erit $\int (M.MC^2 + M.MP.MQ) : \int M.MC$, ubi $\int$ significat summam seu integrale illius quantitatis, cui litera $\int$ praefixea est. Idcirco, si cum Cel. Jac. Bernoullio particulae P, Q inter se aequales , dicantur *dp*, & CM, *x*; MP vera vel MQ ,*y*, hae erum aequales erunt cum ipsae P & Q aequentur, earumque centrum gravitatis sit in puncto M, erit $CN = \int \left(2xxdp + 2yydp\right) : \int 2xdp = \int \left(xx + yy\right).dp : \int xdp$; quae est ipsissima formula, quam Accutiss. Vir in Actis Academiae Reg. Paris. Scient. 1703. ad diem 25 Apr. ex sua methodo elicuit. Quomodo vero formula haec figuris & solidis debeat applicari, non est hujus loci ostendere cum id pendeat a calculo summatorio, quem non patitur institutum nostrum prolixius exponere, legi interim possunt, quae habet super hanc rem Vir laudatissimus loco modo indicato.

SCHOLION.

209. Universalitas Canonis in corollario primo (§.206.) exhibiti melius non poterit probari, quam si ostendatur illum jam continere regulam definiendi centrum oscillationis in quolibet pendulo composito, cujus partes diversae specificae gravitatis oscillentur in fluido quodam homogeneo, aut, quod eodem recidit, cujus particulae ex materia eadem ejusdemve specificae gravitatis vibrationes suas in fluido diversae densitatis peragant qualem regulam nobis promittit Celeb. Joh. Bernoullius Acta. Lips. I7l3. pag. 88. Nam si ratio gravitatis specificae pondusculi P ad fluidum, in quo oscillatur, sit ut G ad $G - E$, & ratio gravitatis specificae pondusculi Q liquorisque ut G ad $G - F$, praedicta formula articuli 206 exhibebit longitudinem penduli simplicis CN in vacuo oscillantis atque composito CPQ, quod in fluido vibrationes suas peragit, isochroni.

Exempli causa sit P globulus aureus, Q ferreus, atque pendulum CPQ oscilletur in aqua; & quia gravitates auri, ferri & aquae sunt ut numeri 100, 42 & 5 ad invicem quam proxime, erit $G : G - E = 100 : 5$, atque adeo $E = \frac{95}{100}G$; & $G : G - F = 42 : 5$, adeoque $F = \frac{37}{42}G$; hinc quia (secundum hypothesin) $M = E.P + F.Q$, erit M hoc casu aequale $\left(\frac{95}{100}P + \frac{37}{42}Q\right).G$; atque adeo formula generalis corollarii primi substitutione valoris M, fiet $NC = (P.PC^2 + Q.QC^2).G : \left(\frac{95}{100}P + \frac{37}{42}Q\right).G = (P.PC^2 + Q.QC^2) : \left(\frac{95}{100}P + \frac{37}{42}Q\right).$ In qua P & *Q* denotant massas harum particularum, seu etiam pondera earum absoluta, quae (§§. 30, 151, 175.) massis semper proportionalia sunt. Punctumque M est centrum gravitatis, non ipsorum corpusculorum P, Q sed eorum partium $\frac{95}{100}P$ & $\frac{37}{42}Q$. Hinc ergo liquet centrum percussionis differre in hoc casu a centro oscillationis. Errant ergo, qui nulla habita ratione gravitatis subito identificant, atque confundunt haec duo centra oscillationis & percussionis.

Si formula corollarii primi applicetur corporibus P, Q, quae liquore specifice leviora sunt, totus formulae denominator migrabit in quantitatem negativam, quod indicat pendulum situ inverso respectu praecedentis casus oscillationes suas intra liquorem absolvere describendo arcus versus fundum vasis cavos existente oscillationis axe prope fundum. Quomodo praedicta formula pendulis compositis debeat applicari, in quibus singulae partes in una eademque linea recta dispositae sunt, nimis planum esse existimo, quam ut ulla explicatione indigeat.

PROPOSITIO XXXVI. THEOREMA.

210. Fig. 47. *Identitas centri oscillationis & percussionis eo casu, quo singularum penduli compositi partium pondera massis earundem proportionalia sunt.*

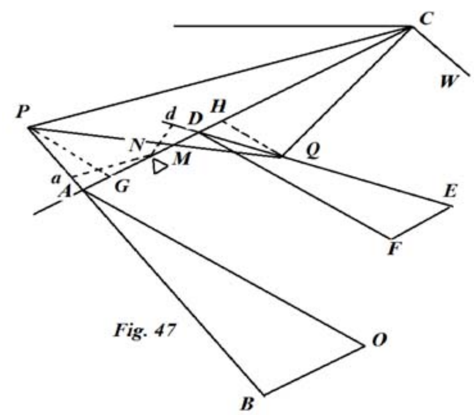

Fig. 47

Si particulae P, Q. &c. virgis inflexilibus CP, CQ, PQ inter se connexae convertantur circa axem CW, punctum in axe CN penduli N, in quo maxima vis in obstaculum ipsi objectum eo exeretur, vocatur centrum percussionis , ostendendumque est hoc punctum tantumdem ab axe oscillationis CW distare, quantum centrum oscillationis in casu Coroll. 2 Prop. anteced. (§. 207). Radiis CP, CQ agantur normales PB, QE axi CM occurrentes in punctis A & D, per quae & per centra corpusculorum P, Q ducantur normales AO, DF, PG & QH ad axem CM. Atque hisce positis statim manifestum est conversione totius figurae CPQ circa axem CW, particulas P, Q motum iri celeritatibus proportionalibus earum distantiis PC, QC ab hoc axe; atqui juxta ea, quae supra (§.53) dicta sunt, axis penduli CM eandem a particulis P, Q circa C in gyrum actis impressionem accipiet, quam si ejus puncta A & D, in quibus scilicet directiones ponderum in quolibet arcus ab ipsis describendi puncto motum suum prosequi nitentium axi occurrunt, juxta directiones AB & DE illis ipsis celeritatibus impellantur, quas particulae P & Q &c. habent; propterea factis AB, DE &c. aequalibus vel saltem proportionalibus homologe radiis CP, CQ qui velocitates exponunt, quibus particulae P, Q in gyrum aguntur, quaesitio reducetur ad id, ut inveniatur centrum aequilibrii potentiarum P.AB & Q.DE; sit N hoc centrum, quod ita comparatum esse debet, ut ductis ex eo ad singulas PB, DE, &c. perpendicularibus N*a*, N*d* &c. omnia facta P.AB.N*a* aequalia sint omnibus factis Q.DE.Nd. Sed ductis BO & EF parallelis CM·, triangula AOB, N*a*A similia praebent AB.N*a* = AO.AN & triangula similia DSE & N*d*D efficiunt $\mathrm{DE.N}d=\mathrm{DF.DN}$; oportet ergo ut omnia P.AO.AN sint aequalia omnibus Q.DF.DN; considerentur tantum duae P, & Q, eritque $\mathrm{P.AO.AN}=\mathrm{Q.DF.DN}$, vel $\mathrm{P.AO.AC}-\mathrm{P.AO.NC}=\mathrm{Q.DF.NC}-\mathrm{Q.DF.DE}$, atque adeo $\mathrm{P.AO.AC}+\mathrm{Q.DF.DC}=\left(\mathrm{P.AO}+\mathrm{Q.DF}\right)\mathrm{.NC}$; verum quia $\mathrm{AO}=\mathrm{CG}$ & $\mathrm{DF}=\mathrm{CH}$ propter triangula similia & aequalia ABO & CPG, utpote in quibus hypothenusae AB & CP aequantur, ac propter similia & aequalia triangula DEF & CQH, in quibus pariter DE & CQ sunt aequales, fiet

$$\mathrm{P.AO.AC}+\mathrm{Q.DF.DE}\left(=\mathrm{P.GC.AC}+\mathrm{Q.HC.DC}\right)=\mathrm{P.PC}^2+\mathrm{Q.QC}^2;$$

$$\&\ \mathrm{P.AO}+\mathrm{Q.DF}=\mathrm{P.GC}+\mathrm{Q.HC},\ \text{hoc est}\ \left(\S.44\right)=\mathrm{M.MC};$$

existente puncto M centro gravitatis particularum P, Q.&. $\mathrm{M}=\mathrm{P}+\mathrm{Q}$. Idcirco factis substitutionibus debitis in $\mathrm{P.AO.AC}+\mathrm{Q.DF.DE}=(\mathrm{P.AO}+\mathrm{Q.DF})\mathrm{.NC}$, habebitur $\mathrm{P.PC}^2+\mathrm{Q.QC}^2=\mathrm{M.MC.NC}$, atque adeo $\mathrm{NC}=\left(\mathrm{P.PC}^2+\mathrm{Q.QC}^2\right):\mathrm{M.MC}$. Quae expressio prorsus eadem est cum ea, quam supra (§. 207*)* reperimus pro centro oscillationis in hypothesi communi, quam particularum P, Q pondera massis ipsarum proportionalia sunt, vel, quod eodem recidit, in qua singulae E, F, G &c. quales supra (§.198.) pro significandis juxta massas P, Q, N gravitatis solicitationibus centralibus adhibuimus, aequales sunt. In omni vero alia gravitatis hypothesi different ab invicem centra oscillationis & percussionis. Quod erat demonstrandum.

COROLLARIUM.

211. Hinc si fuerint infinita particularum aequalium P, Q paria, tunc fiet $\mathrm{CN}=\int\left(\mathrm{P.PC}^2+\mathrm{Q.QC}^2\right):\int\mathrm{M.MC}$, adeoque nomimando CN, *t*, atque retentis symbolis paragraphi 208. fiet $t=\int\left(xx+yy\right)dp:\int x.dp$, quae est ipsissima iterum formula, in quam dicto loco incidimus, quamque alia methodo etiam reperit Cl. Jac. Bernoulli in Actis. Acad. Reg. Paris. Scient. Acad. 1704 ad diem 14 Apr.