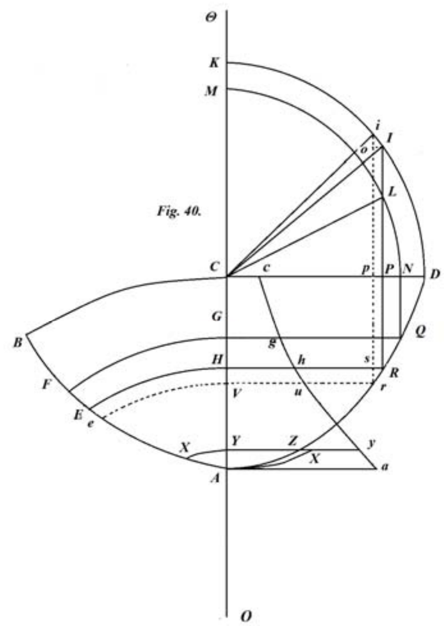
CHAPTER III.

Concerning the motion of bodies on isochrone curves of descent under the hypothesis of any variation of gravity, and with the directions of gravity converging towards a centre of gravitation ; and the motions of pendulums.

DEFINITION.

When a heavy body is moving by its own motion from rest along whatever unequal arcs of some curve or other, but yet all end at the lowest point of the curve at the same or in equal times, it is usual to call such curved lines *isochrones*. So that if the weight A begins to fall from rest at some point B, F, E &c. of the curve BFA , Fig. 40. by its motion with some acceleration, may resolve in the same or in an equal time any unequal arcs B*e*A, F*e*A, EXA, but all end at the lowest point of the curve A at the same time ; the curve BEA will be said to be an isochrone.



PROPOSITION XXVI. LEMMA.

171. Fig. 39. *In any rectilinear triangle, the base of which is divided into any two segments as desired, by a line drawn from the angle opposite which base, the volumes which may arise from the squares of the sides of each segment multiplied by the alternate segment of the base, added together, will be equal to the two volumes, of which one shall be from the square of the line dividing the base times by this base, the other formed from the rectangle of the segments of the base by the whole base together.*

ABC shall be some triangle, of which the base BC may be divided at D by the right line AD as desired drawn within the triangle. It must be proven that :

$$AB^2.DC + AC^2.BD = AD^2.BC + BD.DC.BC.$$

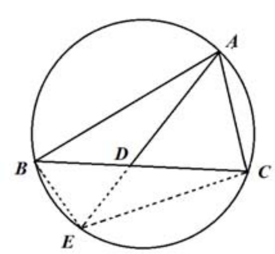


Fig. 39

Demonstration:

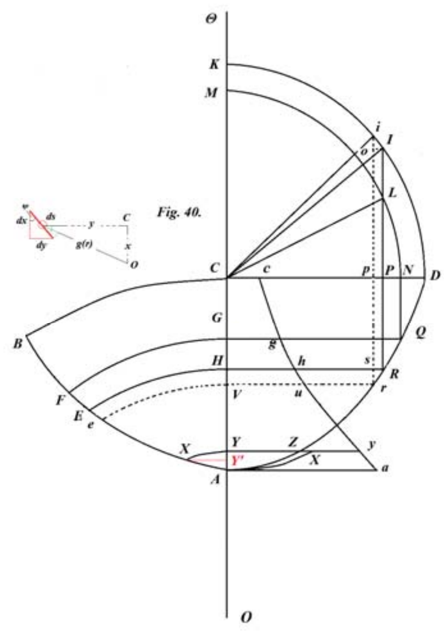
I. With the circle ABE described around the triangle, AD is produced to E, and BE, CE, may be joined & the similar triangles BAD and ECD produce the ratio $AB : AD = EC : DC$ and thus $AD.EC = AB.DC$. And from the similarity of the triangles ACD, BED it is deduced $AC : AD = BE : BD$, & the rectangle AD.BE is equal to the rectangle AC.BD.

II. With the quadrilateral ABEC inscribed in the circle, there is $AB.EC + AC.BE = AE.BC = AD.BC + DE.BC$. Or with the common altitude AD adopted, there becomes $AB.EC.AD + AC.BE.AD = AD^2.BC + AD.DE.BC$. And thus by demanding in place of the rectangles EC.AD & BE.AD these rectangles AB.DC & AC.BD which have been shown to be equal to these (no. I), and in place of the rectangle AD.DE aequal to the rectangle BD. DC, there will be $AB^2.DC + AC^2.BD = AD^2.BC + BD.DC.BC$. Q.E.D.

PROPOSITION XXVII. THEOREM.

172. Fig. 40. *If the directions of gravity converge to the centre* O, *and three curves are described around the axis* OC, *truly the first diagram cha is that of the attraction of variable gravity, then the diagram of the second curve* ARD, *the square of any ordinate of which,* HR, *shall be twice the homologous area* AH*ha from the gravity diagram, and finally for the third curve* AEB, *any arc of which* AXE, *with the circle* HE *described with the radius* OH, *the arc* AE *shall be to the homologous ordinate* HR *on the curve* ARQ, *as some given number* N *to unity, and this third curve* BEA *will be an* isochronous curve.

The fall by the weight beginning from rest at B is through the arc of the curve BEA ; E*e* is an element of the curve AE, and through its end *e* an arc *e*V is described with centre O crossing the axis at V, through which point the ordinate Vr may be drawn to the curve ARD, and with the quadrant of the circle CDIK described from the base CD and through the points R, r of this curve ARD, RI, and *ri* are put in place crossing the circle at I, *i*, and crossing its radius CD at the points P, *p*, and with the elemental line I*o* drawn through I parallel to CD, and finally CI, C*i* may be joined. With which in place:



I. Because (following the hypothesis) $CD^2 = 2.AC\mathit{cha}A$, & $HR^2 = 2.AH\mathit{ha}A$, there will be $CD^2 - HR^2$, that is $CD^2 - CP^2$, or $CI^2 - CP^2$, that is $IP^2 = 2.CH\mathit{hgc} = 2.AC\mathit{cha}A - 2.AH\mathit{ha}A$, therefore $IP = \sqrt{(2.CH\mathit{hgc})}$.

[To make this clearer, if we consider unit mass, CD^2 represents the final square of the speed of the body at A dropped from C ; HR^2 similarly for the body dropped from H to A; and likewise for the points B and E on the curve; IP is then the change in speed of the body in going from C to H. All of which follows at once in modern terms from the conservation of energy. Finally, the speed of the body at some point E on the curve is proportional to the length of the slope still to be traversed on the isochronous curve.]

Likewise (following the hypothesis) $\mathrm{AE : HR = N : 1}$, there will be $\mathrm{AE = N.HR}$, and thus

$\mathrm{E}e = \mathrm{N.}s\mathrm{R} = \mathrm{N.P}p = \mathrm{N.I}o.$ [*i.e.* the arc increment $ds = \mathrm{N.I}o = \mathrm{N.}dv.$ The relation $\mathrm{AE{=}N.HR}$ can be understood in two ways: either HR is the speed acquired by falling from rest along the arc EA, or it is the initial speed of projection up along the arc from A to E, where it comes to rest; and this applies to all the positions of E and HR on the curve. In any case, we have N as the time associated with the change in the increments. Thus the time is taken to be the same for the passage through any arc of the isochrone : this curve cannot yet be determined, as the source of the attraction has not been specified ; if the final speed is less than the maximum, then the arc traversed decreases in proportion. The upper quarter circle has a radius proportional to the final speed, and this circle is swept out at the same rate N, whatever the radius or speed, for a given situation. This is the isochrone condition used for the equality of the times to descent different arc lengths.]

II. It has been shown above (§.136.), for two moveable bodies with equal weights and masses at the two points E and H and equidistant from the centre, to acquire the same or equal speeds, if they may begin to move equally from rest from the two points B and C of equal height [above O], with one on the curve BEA and the other on the right line CA, and they will have described the distances BE & CH. And by calling the mass of the body A by the same letter A, the speed acquired at H from the descent through CH, or at E from the descent through BE (§. 144.) will be $\sqrt{(2.\mathrm{CH}hc)} : \sqrt{\mathrm{A}}$, or (by no. I of this section) $= \mathrm{IP} : \sqrt{\mathrm{A}}$.

[Thus, the work done by gravity of acceleration $g(r)$ centred at O, acting on the mobile body of mass m falling along the frictionless curve AB or by the straight descent CA, in both cases is equal the area CH*hc*, found by integrating the force curve $cha = \int mg(r)dr$ between these limits, to give the kinetic energy, as discussed in Sect. II, Ch. I, § 136 ; from which the speed at E along the tangent of the curve BE and at H for the straight descent CH, is given by $v = \frac{ds}{dt} = \sqrt{(2.\mathrm{CHhc})} : \sqrt{\mathrm{A}}$. Thus the author, in modern terms, makes use of the conservation of the sum of the kinetic and potential energy in the frictionless fall of the body along these two paths ; these results follow from Newton's second law of motion for any curve. It appears that Hermann made use of NII in the time independent form

$F = m\frac{dv}{dt} = m\frac{dv}{dr}\frac{dr}{dt} = m\frac{dv}{dr}v,$ so that $Fdr = mvdv,$ and for the vertical fall,

$\int mg(r)dr = \frac{1}{2}mv^2$; thus, for the motion along AEF, $F = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = m\frac{dv}{ds}v = mg(r)\sin\psi;$

where ψ is the angle of the slope at some point; in which case

$mvdv = mg(s)\sin\psi ds$ and $\frac{1}{2}mv^2 = \int mg(s)\sin\psi ds$,with suitable limits of integration,

representing the work done by the gravitational field on the mass being changed into kinetic energy .

Such energy concepts were of course unknown to Hermann at the time, and we have to consider that within each infinitesemal distance he used a linear approximation similar to that for uniform gravity, which he knew to be true. Newton was of course still very much alive and the second edition of the *Principia* came out around this time; and one might wonder as did Clifford Truesdell [see his Introduction to Euler's O.O., vol. 10 of the Second Series], whether a third party, such as the recently deceased James Bernoulli who had worked on the cycloid, had perhaps conveyed some ideas to Hermann through his posthumous unpublished work which were not made explicit in Newton's work, or if indeed Hermann had thought out the bulk of the ideas himself, as he leads us to understand ; Leibniz, for example, had not got this far with his ideas about natural philosophy and was enraptured when he read a proof copy of the work, as he indicated to Hermann by letter; unfortunately, Leibniz died soon afterwards, having just completed his main philosophical work, and so could make no new contributions. In any case, these rather remarkable developments were ignored, or not understood, and neglected by writers at the time ; Euler, some time later, did not make use of the correct formula for NII, which Hermann had made available, and which gave the correct results agreeing with experiments, and put a ' 2' into the formula occasionally, as he was influenced by Leibniz's Vis Viva idea, where mv^2 was considered as a conserved quantity in interactions between masses. Thus P.G. Tait, writing in the *Encyclopedia Britannica* in the late 19th Century on the subject *Mechanics*, ignored or did not even know about these developments, and no mention was made therein either of Hermann or of the *Phoronomia*: and that sums up more or less the attitude of physicists both then and now to this neglected work. Humanity had to await the genius of Thomas Young, who interpreted correctly anew Hermann's results, and introduced the idea of energy and its conservation ; even this people like Routh kept the V.V. idea alive, and presented it along with energy conservation in his earlier texts on dynamics.]

From which, because (by no.I) $\mathrm{E}e = \mathrm{N.I}o$
[*i.e.* $\mathrm{E}e = de = ds; \mathrm{I}o = s\mathrm{R} = dv$; and from the hypoth., $\frac{\mathrm{I}o}{\mathrm{E}e} = \frac{dv}{ds} = \frac{1}{\mathrm{N}}$ or $\mathrm{N}dv = ds$.]
and the incremental distance Ee applicable to the velocity with which it is traversed; or tEe denoting the time in which it is resolved, will on that account be
$\mathrm{tE}e = \mathrm{N.I}o\sqrt{(\mathrm{A})} : \ \mathrm{IP}$.
And on account of the similar triangles CIP, & I*io*, the ratio I*o* to IP is equal to the ratio I*i*: CI, that is (by §. 129) to the angle ICi; therefore $\mathrm{tEe} = \mathrm{N}.\sqrt{\mathrm{A}}.\mathrm{ang.\ ICi}$:
[*i.e.* $v = \frac{ds}{dt} = \sqrt{(2.\mathrm{CH}hc)} : \sqrt{\mathrm{A}} = \frac{\mathrm{IP}}{\sqrt{\mathrm{A}}}; \therefore dt = \frac{ds\sqrt{\mathrm{A}}}{\mathrm{IP}} = \frac{\mathrm{N}dv\sqrt{\mathrm{A}}}{\mathrm{IP}} = \mathrm{N}\sqrt{\mathrm{A}}\,\frac{\mathrm{I}o}{\mathrm{IP}} = \mathrm{N}\sqrt{\mathrm{A}}.\mathrm{ang.IC}i$].
therefore all of tEe, that is, $\mathrm{tBE} = \mathrm{N}.\sqrt{\mathrm{A}}$. integral of ICi or ICD, and thus
$\mathrm{tBE} = \mathrm{N}.\sqrt{\mathrm{A}}.\mathrm{ang.ICD}.$ & $\mathrm{tBEA} = \mathrm{N}.\sqrt{\mathrm{A}}.\mathrm{ang.KCD}.$

III. If now the mobile body no longer may begin descending at B, but at some other point F of the curve ; a circle FG with centre O may be drawn through the point F and crossing through G to the point Q, with GQ the ordinate of the curve ARD, through which QN acts parallel to the axis AC and finally with the quadrant MLN described by the radius CN, LC may be joined, and with these put in the same place, as in the preceding number, and with the argument put in place there comes about $tFE = N.\sqrt{A}.ang.LCN$. From your range of abscissas or the whole interval taken, which is between the different lines BCD & FGQ, and the fourth part MCN equally draws near to the ordinate GQ, and by this account this is the case, by which the mobile body begins to fall from F, of returning precisely to the preceding case ; since AEF now may be considered as just the whole curve from which the mobile object may depart from its starting point F and the base GQ has standing on it the quadrant of the analogous curve ARQ, just as the base CD has its quadrant KID, therefore the time to pass through FE, that is $tFE = N.\sqrt{A}.ang.LCN$, and thus $tFEA = N.\sqrt{A}.ang.MCN$. Therefore tBEA to tFEA, shall be as $N.\sqrt{A}.ang.KCD$ to $N.\sqrt{A}.ang.MCN$ that is, just as the right angle KCD to the right angle MCN, and thus in the ratio of equality ; therefore all the arcs BA, FA, XA &c. are transversed in an equal time, and thus the curve BEA of our proposition is isochronous. Q.E.D.

COROLLARY I.

173. If ΘA were the radius of osculation of the isochronous curvature at the vertex A, there will be generally $N = \sqrt{(\Theta A.OA : \Theta O.Aa)}$, with the arclet AX taken indefinitely small adjacent to the vertical, and with the arclet XY drawn from the centre O, and through Y with the ordinate Y*y* cutting the curve AR at Z. Because now (following the hypothesis) $AX = N.YZ$, or $AX^2 = NN.YZ^2$ (following the hypothesis) $= NN.2.\text{ area } AYya = 2NN.AY.Aa$; and because AY is the sum of the versed sines of the arclets AX and XY required to be taken in place of the equality
[AY and AY' in diagram above in red; recall the versed sine of *z* is $a - a\cos z = 2a\sin^2\frac{z}{2} \doteq \frac{az^2}{2}$ for small angles and radius *a*.], the centres of which shall be Θ & O, there becomes [approximately, assuming the arcs are almost equal,]
$2AY = \frac{AX^2}{\Theta A} + \frac{XY^2}{OY} \doteq \frac{AX^2}{\Theta A} + \frac{(XY^2 \doteq AX^2)}{OA} = \frac{AX^2.\Theta O}{\Theta A.OA}$, and thus $2AY = AX^2.\Theta O : \Theta A.OA$, and hence with the substitution made, there comes about
$AX^2 = NN.AX^2.\Theta O.Aa : \Theta A.OA$, from which at once it may be deduced :

$$NN = \Theta A.OA : \Theta O.Aa, \ \ \& \ \ N = \sqrt{(\Theta A.OA : \Theta O.Aa)}.$$

COROLLARY II.

174. If now the smallest of the isochrone arclets XA adjacent to the vertex A may be considered to be in place, [*i.e.* we now consider small oscillations] the arc of the circle of radius ΘA to be described by a weight A appended by a wire ΘA; the time, in which the arclet XA or the whole of the curve BEA can be described, will be for half of one oscillation of the pendulum ΘA. From which, because (§. 171.)

$tXA = N.\sqrt{A}.\text{right angle}$, 4tXA will be the duration of two of these minimum oscillations, which henceforth may be called T, which, as I say, becomes $T = N.\sqrt{A}.4\text{ right angles}$ or by calling the periphery p, for which the radius shall be 1, then $T = pN.\sqrt{A}$, [*i.e.* $p = 2\pi$], or by substituting the value of N itself defined in the previous article, there may be found $T = p\sqrt{(A.\Theta A.AO : \Theta O.Aa)}$. Which is definitely the most general rule, in which ΘA indicates the length of the pendulum, AO the smallest distance of the weight A from the centre of gravity O, ΘO the distance of the point of suspension Θ from the same centre O; A*a* the weight of the body A oscillating at the lowest position A put in place, and T the time of two minimal oscillations of the pendulum ΘA [*i.e.* a complete oscillation back to the starting point]. This determination agrees properly with the somewhat more specialized assertions of Newton Prop. 52. *Book I. Pr.Ph. Nat.*

COROLLARY III.

175. If the centre O shall be infinitely distant from the point Θ, ΘO and AO themselves will be equal, and the formula of the preceding corollary will be changed into the following : $T = p\sqrt{(A.\Theta A : Aa)}$. Hence :

1st. The times of oscillations of different pendulums are in a ratio composed directly as the square roots of the masses and lengths of the pendulums, and likewise but inversely as the square roots of the weights of the pendulums.

2nd. The times of oscillation of equal pendulums are in a ratio composed directly as the square roots of the masses of the bodies and in the inverse square root ratio of the weights.

3rd. If the forces, by which the pendulums are made to move, that is the weights of the bodies shall be proportional to the lengths of the pendulums, the times of the oscillations will be in the square root ratio of the masses of the disturbed bodies, and these times shall be equal if, with the same in place, the above masses of the bodies were in proportion with the weights.

4th. Truly the masses, or the quantities of matter, will be in a ratio composed from the weights attached, on account of the equal lengths of the pendulums, and from the squared ratio of the times of the oscillations. And this is itself Prop. 27. *Lib.II. Pr. Ph. Nat.*, from which the celebrated man sets about investigating whether or not the weights of the bodies themselves shall be proportional to their masses ; and this the author found from experiments, with the help obtained from the most accurate of pendulums, the weights of bodies to be constantly in proportion to the masses. But here a word is to be said about the absolute weights of bodies; not truly about relative weights, such as they have when immersed in different fluids; for in this case the weight of a part of the same fluid with a volume equal to the volume of the immersed body must be taken away, so that the relative weight of this body may be found, which it has immersed in that fluid.
[*i.e.* Archimedes' Principle.]

COROLLARY IV.

176. Since a great many equal oscillations are required to be completed in equal times by different pendulums, they may be in a contrary ratio of the times, by which an individual pendulum completes a single oscillation, a multitude of oscillations in equal times hence are required to be completed in the ratio composed directly from the square root ratio of the weights and from the inverse square root ratio of the mass and the length of the pendulum. Or, if the weights may become proportional to the masses, as can be assumed without risk ; the periods of the aforementioned multitude of oscillations will become as the square roots of the lengths of the wires, which the weights or the forces of gravity establish, connected to the lengths of the pendulums, from which the pendulums are put into motion. And thus:

1st. of unequal pendulums, but with the same disturbing force of gravity, the number of vibrations being completed in the same time are in the inverse ratio of the square roots with the lengths of the pendulums.

2nd. The number of oscillations of one pendulum will be to the number of oscillations completed in the same time as the first, by another pendulum of the same length, in the square root ratio of the force of gravity acting, both according to the force of gravity for the first, as well as for the force acting on the other pendulum, as long as the disturbing lengths are equal. And this later agreed according to the precision with the rule that Bernoulli inserted in his most elegant work dashed off in the month of February 1713 in the *Acta Erud. Lips.* 1713, treated in paragraph 16, from which he instructs henceforth specific weights to be elicited from pendulum experiments plainly in a manner new nor known before.

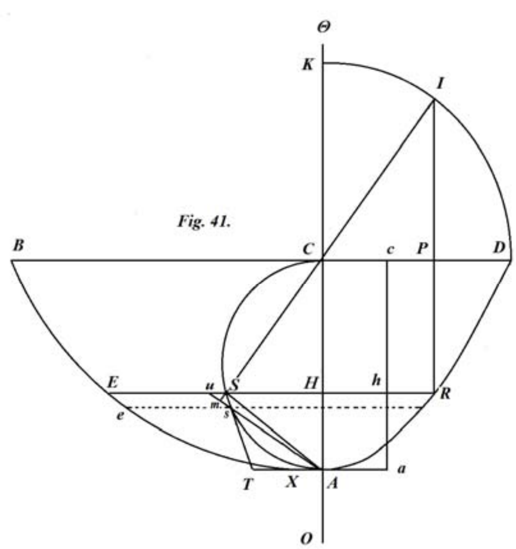
SCHOLIUM.

177. I consider from the nearby preceding corollaries to have shown well enough, of how much use our general theorem of isochronous bodies on curves, with the assigned law described of the descents, since from these everything, which they observe about the motion of pendulums, can be deduced with so much ease: meanwhile the oscillations of pendulums considered as minimal agree nearly with the ratios indicated in §.174, clearly because then finally the weight of the pendulum, or, of which we have measured the arclet of the isochrone, the time to traverse may be agreed when it will describe the minimum arclet itself, because in that case, such an arc is twice the arc of the osculating circle XA or the arc XAX ; with the arc AX put in place on the curve BEA, from the other axis AC with the part put in place to equal the arclet AX, and because geometrically to osculate and to be congruent at any rate signify one and the same thing in that side which the osculating circle touches. Truly if the arclets AX are not minimal [*i.e.* incremental], yet all of these may pause at this point, if the pendulums are collated between themselves, or the weights of these will have described similar curves.

178. Therefore so that what has been said diversely in the preceding corollaries may be gathered together into a summary ; the number of oscillations completed in some time by the first pendulum may be called N, its length L, which the line ΘA is going to represent in the diagram, the absolute weight of the body A may be called G, its mass M, and thus M now signifies the same as A, and G likewise the same as the line A*a* in the diagram ; and with T requiring to be kept for designating the time of two of the minimum oscillations of this pendulum. In the case of the second pendulum with these things the same as the first, but with the letters expressed in small type, thus Corollary 3 will produce these rules $T = p\sqrt{(M.L : G)}$ and $t = p\sqrt{(m.l : g)}$. Truly Corollary 4 shows $N : n = t : T$, or $NT = nt$. From which in just as many different ways as the letters T, G, L, M, N and the homologous *t, g, l, m, n* can be brought together, thus so that many more theorems will result, with all of which requiring to be reviewed in turn I refrain from doing so, for the sake of brevity. At this stage we have considered only isochrone lines in general adhering to no particular hypothesis of gravity. Truly what may result from one or other hypothesis of this kind, we will investigate in the following problems.

PROPOSITION XXVIII. PROBLEM.

I79. Fig. 41. *To find the isochrone curve under the hypothesis of uniform gravity and the direction of the weights are parallel to each other or uniting at an infinitely distant centre.*
[In this situation, an extra generating curve, here a semi-circle, is introduced into the above diagram to insure the arcs are swept out at the same rate, or that N is constant.]
Because the centre of attraction of gravity O stands at an infinite distance from the base CD of the curve ARD, the lines BC and EH, which before (§.172) were circular arcs described from the centre O, now become right lines parallel to each other and perpendicular to the axis AC: and because (following the hypothesis) of uniform gravity, its diagram will be the right line *cba* parallel to the axis AC, and because (§.172), $HR^2 = 2.HAah$, the curve ARD in this case will be a parabola, of which the parameter is 2Aa. Moreover, on account of O*o* [Fig. 40] and AO being equal, there becomes $N = \sqrt{(\Theta A : Aa)}$ [Note: If A*a* is considered as an acceleration, then N has the dimensions of time.]. Now because (§.172) $AE = N.HR = N.\sqrt{(2Aa.AH)} = \sqrt{(2\Theta A.AH)}$, there will be $\frac{1}{2}AE = \sqrt{(\frac{1}{2}\Theta A.AH)}$, or with ΘA bisected at C, there will be $\frac{1}{2}AE = \sqrt{(AC.AH)}$, that is, with the semi-circle ASC described on the diameter AC,



cutting the line EH at S, it will be equal to the chord AS, since the r.h.s. is the geometric mean between the diameter AC and the abscissa AH [*i.e.* $\frac{1}{2}$ the arc $\mathrm{AE} = \mathrm{AS}$]. Thus also $\frac{1}{2}\mathrm{A}e = \mathrm{A}s$, and therefore $\frac{1}{2}\mathrm{E}e = \mathrm{AS} - \mathrm{A}s$, that is with S*m* sent to the perpendicular on A*s* produced $= sm$, and thus $\mathrm{E}e = 2.sm$, on the supposition of an infinitely small arclet *Ee*, in which case *sm* is the difference of the greater chord AS over the lesser chord A*s*. As truly with the tangents AT and ST drawn through the points A and S crossing at T, these tangents will be equal, also *u*S will be equal to S*s*, and thus the perpendicular *Sm* divides the base *su* of the isosceles triangle *sSu* into two equal parts, thus so that *su* shall be equal to 2*sm*, and thus $\mathrm{E}e = su$ *;* thus the quadrilateral *Eesu*, in which the opposite sides E*u* and *su* are equidistant from and equal to the remaining opposite sides *es*, *Ee*, will be a parallelogram, and thus $\mathrm{ES} - es = u\mathrm{S} = \mathrm{S}s$, hence (§. 87.) the sum of the difference $\mathrm{ES} - es$ or the excess of the maximum ES over the minimum, which is zero at the vertex A, that is ES alone is equal to the sum of all the arclets S*s*, or to the arc of the circle AS and thus everywhere, on account of the isochrone sought BEA is the ordinary *Cycloid*, of which the semi-base BC is equal to the periphery of the semicircle ASC. Which was required to be found.

COROLLARY.

180. The descent time along the arc BE of the cycloid, will be to the time along an equal part of the axis CH, as the intercepted arc CS of the generating circle between the parallel lines BC and EH, and the chord CS of the same arc. [The generating circle in this case does not roll along the upper line BC to the right, and the motion is provided by the weights E and H falling.] For on account of the parabola ARD and the circle ASC, there is $\mathrm{CI}(\mathrm{CD}):\mathrm{CP}(\mathrm{HR}) = \mathrm{CA}:\mathrm{AS}$, and thus the angles ICP and CAS or CSH are equal, thus so that SC and CI shall lie on a straight line. Now because (§.172. n. II.) $\mathrm{tBE} = \mathrm{N.ang.ICD}$ [regarded as an integration, and where N can be regarded as the inverse of the constant angular frequency, $N = \frac{T}{2\pi} = \frac{1}{\omega}$, unless the perimeter is expressed explicitly], also there will be $\mathrm{tBE} = \mathrm{N.ang.SAC}$, and (§.129.) the angle $\mathrm{SAC} = \frac{1}{2}\mathrm{arc.SC} : \frac{1}{2}\mathrm{CA} = \mathrm{arc.CS}:\mathrm{CA}$, therefore $\mathrm{tBE} = \mathrm{N.arc.CS}:\mathrm{CA}$, or because $\mathrm{N} = \sqrt{(\Theta\mathrm{A}:\mathrm{A}a)}$, there will be $\mathrm{tBE} = \mathrm{arc.CS}.\sqrt{\Theta\mathrm{A}} : \mathrm{CA}.\sqrt{\mathrm{Aa}}$, or because $\Theta\mathrm{A} = 2.\mathrm{CA}$, there becomes $\mathrm{tBE} = 2.\mathrm{arc.CS}:\sqrt{(2.\mathrm{CA.A}a)} = 2.\mathrm{arc.CS}:\mathrm{CD}$, on account of the parabola ARD; & tCH (§.151) $= \sqrt{(2.\mathrm{CH}:\mathrm{A}a)}$ with no account had of the mass of the body, since in this occasion the moving body is not carried with the others *;* but one and the same body only may be considered to be moved on the curved line BEA or on the axis CA, thus so that in place of M, which enters into the formula of the cited article 151, one may be put in place. [*i.e.* 1, in which case A*a* becomes the constant acceleration of gravity *g*, and we get the familiar formula in modern terms $s = \frac{1}{2}gt^2$.]

But $\text{tCH} = \sqrt{(2\text{CH} : \text{Aa})}$ is reduced to $\text{tCH} = \sqrt{(4\text{AC.CH} : 2\text{CA.Aa})} = 2.\text{CS} : \text{CD}$;
therefore tBE : tCH, becomes as the fraction $\frac{2.\text{arc.CS}}{CD}$ to $\frac{2.\text{chord CS}}{CD}$*;* and thus the time through BE is to the time through CH just as the arc CS to its chord CS.

Therefore the time for the whole semi cycloid BEA to the descent time of the body along its axis CA is itself found, as the semi-circumference CSA to the diameter CA as first was shown by Huygens, and thereafter by many others; but from fundamentals much different from these of ours.

PROPOSITION XXIX. PROBLEM.

181. Fig. 42. *If the graph of the central action of gravity were the right line ha, which produced downwards may pass through the centre of action of gravity* O*; to find the isochrone* BEA *under this hypothesis.*

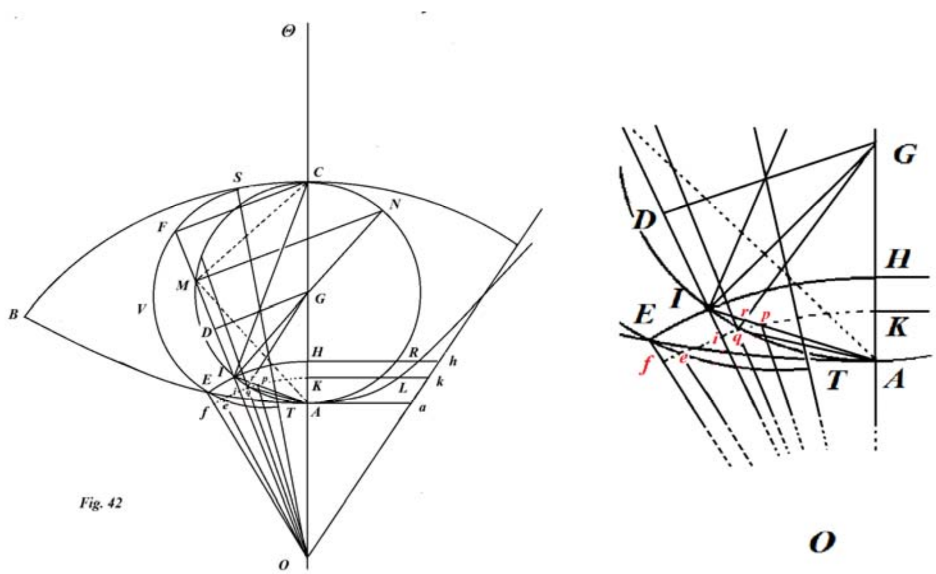


Fig. 42

I. Because (§.172.)
$\text{HR}^2 = 2.\text{HA}ah = \left(\text{HO}^2 - \text{AO}^2\right).\text{A}a : \text{AO}$; $[\text{HO} \times \text{H}h = \text{HO} \times \text{A}a \times \frac{\text{HO}}{\text{AO}} = \text{HO}^2 \times \text{A}a : \text{AO}.]$ on account of the trapezium HA*ah*, the curve AR is found to be the hyperbola, in which $\text{HR}^2.\text{AO} : \text{A}a = \text{HO}^2 - \text{AO}^2$.

[For we may write the equation in the form : $1 = \frac{\text{HO}^2}{\text{AO}^2} - \frac{\text{HR}^2}{\text{AO.A}a}$.]

II. From some point G of the axis as centre, the circle AMN is described ; from the concentric and indefinitely close arcs to EH, *eK* may be cut at the points I [$= i$] and *q* with these points having radii GI and *Gq* acting from the centre G, then also with the right lines AI and A*q*, the former AI cuts the arc *eK* at *p*, and the arclet *qr* shall be described with centre A, and finally IO, *q*O and *p*O are joined.

Now in the triangle IGO , if GO may be considered as the base in place, [*i.e.* if $A \to I$, $B \to G$, $C \to O$ & $D \to A$; then $AB^2.DC + AC^2.BD = AD^2.BC + BD.DC.BC$ becomes as shown;] there will be, as (§.171) has shown above : $IG^2.AO + IO^2.AG = AI^2.GO + GA.AO.GO[= GA.AO.(GA+AO)]$, and thus $AI^2.GO = IO^2.AG + \cancel{IG^2.AO} - \cancel{GA^2.AO} - AO^2.AG = HO^2.AG - AO^2.AG$, and consequently $HO^2 - AO^2 = Al^2.\,GO : AG$ (or no.1. of this section) $= HR^2.AO : Aa$; hence the ratio will become [$AI^2.GO : AG = HR^2.AO : Aa$, or] : $HR^2 : AI^2 = Aa.GO : AG.AO$, & (§172 [the isochrone condition for BEA]) $AE^2 : HR^2 (= NN : I) = \Theta A.OA : \Theta O.Aa$, [see §173], therefore from the equation and by the multiplication of the ratios there becomes $AE^2 : AI^2 = \cancel{Aa}.GO.\Theta A.\cancel{OA} : AG.\cancel{AO}.\Theta O.\cancel{Aa} = \Theta A.GO : \Theta O.AG$; hence $AE : AI = \sqrt{(\Theta A.GO)} : \sqrt{(\Theta O.AG)}$ and thus this curve AE is to the homologous AI in the ratio given, and therefore $Ee : Ir = \sqrt{(\Theta A.GO)} : \sqrt{(\Theta O.AG)}$.

III. Therefore between the concentric circles EH and *eq*K a certain line increment shall be required to be adapted for the point I, which shall be to Ir, the difference between the chords IA and *q*A, in the given ratio $\sqrt{(\Theta A.GO)}$ to $\sqrt{(\Theta O.AG)}$. In truth because it may readily be seen I*p* to be a small part of the chord AI from the intercept of the concentric circles constructed previously to the aforementioned difference I*r* of the chords, to be in the given ratio of twice GD to CF, clearly with the perpendiculars sent from the points G and C to the line OI produced, or of twice GO to CO [*i.e.* $Ip : Ir = 2.GD:CF$] ; therefore the element I*p* of the curve Ee can be put equal to that ratio itself, because I*p* and Ee may have a given ratio to I*r*, which thus is required to be demonstrated with respect to the ratio T*p* to I*r*. MC, IC, & MN are to be treated, with which done, in the first place the triangles I*pi* and CIF are similar, since the angle CIA is in effect a right angle in a semicircle, so that the two angles *p*I*i* and CIF likewise may be equal to right angles, thus so that the angle *p*I*i* shall be equal to the angle FCI and since the angle at *i* and F shall be right (following the hypothesis), it is necessary, that the third angle shall be equal to the third angle and thus the one triangle shall be similar to the other. In the second place, because the angle *q*I*i* under the tangent I*q* and with the secant IO, shall be equal to its vertically opposite equal angle MNI, in the other segment IM, and the angles *i* and IMN shall be right, the triangles INM and *q*I*i* prove to be similar. In the third place, also the triangles I*qr* and CAI will be similar, because the angle *q*IA under the tangent I*q* and with the secant IA shall be equal to the angle ICA in the alternate segment, and the angles for *r* and AIC are right ; and thus this triad of similar triangles will supply these ratios : $Ip : Ii = IC : CF$, likewise $Ii : Iq = MN : IN$, and finally $Iq : Ir = AC$ or $IN : IC$, therefore from these equation there becomes [on multiplying the three ratios together:

$$(Ip : \cancel{Ii}) \times (\cancel{Ii} : \cancel{Iq}) \times (\cancel{Iq} : Ir) = (\cancel{IC} : CF) \times (MN : \cancel{IN}) \times (\cancel{AC} \text{ or } \cancel{IN} : \cancel{IC})]$$

$$Ip : Ir = MN : CF = 2.DG : CF = 2.GO : CO.$$

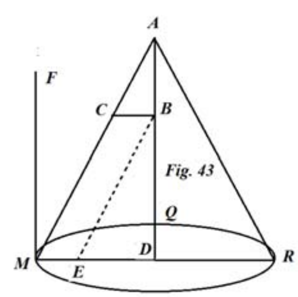
IV. And thus, since there shall be (by n. II) $\mathrm{E}e:\mathrm{I}r=\sqrt{(\Theta\mathrm{A.GO})}:\sqrt{(\Theta\mathrm{O.AG})}$ and (by no. III) $\mathrm{I}p:\mathrm{I}r=2\mathrm{GO}:\mathrm{CO}$, and there will be $\mathrm{E}e=\mathrm{I}p$, if there were $\sqrt{(\Theta\mathrm{A.GO}:\Theta\mathrm{O.AG})}=2.\mathrm{GO}:\mathrm{CO}$; therefore this equality of ratios is assumed, because the right line $\Theta\mathrm{A}$ at this point has not been restricted in magnitude, and its magnitude must also be related to the diameter of the circle CA; and in this case it may be written as $\Theta\mathrm{A}=2.\mathrm{OG.AC}:\mathrm{AO}$; and indeed $\mathrm{I}p=\mathrm{E}e$ and thus $ip=fe$ [on taking the limit]*;* again (§.119) there is $qp=p\mathrm{O.ang.}p\mathrm{O}q$, and $qr=\mathrm{A}q.\mathrm{ang.}\,q\mathrm{AI}=\mathrm{A}q.\frac{1}{2}.\mathrm{ang.IG}q$. Indeed because $pq:qr\,(=\mathrm{I}p:\mathrm{I}i=\mathrm{IC}:\mathrm{CF})=\mathrm{AC}:\mathrm{MC}$, thus $\mathrm{MC}.pq=\mathrm{AC}.qr$, or $p\mathrm{O.MC.ang.}\,p\mathrm{O}q=\mathrm{A}q.\mathrm{AC.ang.}q\mathrm{AI}=\mathrm{AI.AC}.\frac{1}{2}\mathrm{IG}q=\mathrm{AI.AG.IG}q$ [on taking the limit, and the angles are small], therefore $p\mathrm{O}q=\mathrm{AI.AG.IG}q:\mathrm{KO.MC}$; but on account of the similar triangles OIA and OCM [as AIMC is a cyclic quad. and a side has been extended], there will be $\mathrm{IA}:\mathrm{MC}=\mathrm{IO}:\mathrm{CO}=\mathrm{HO}$ vel $\mathrm{KO}:\mathrm{CO}$; therefore on putting in place of AI and MC, the homologous proportionals KO & CO; there becomes $\mathrm{ang.}p\mathrm{O}q=\mathrm{KO.AG.ang.IG}q:\mathrm{KO.CO}=\mathrm{AG.IG}q:\mathrm{CO}$, and thus the sum of all $p\mathrm{O}q$, which correspond to the individual $\mathrm{IG}q$ $=\mathrm{AG}.$ sum of all $\mathrm{IG}q:\mathrm{CO}=\mathrm{AG.ang.\ IGA}:\mathrm{CO}$, or on putting ang. $\mathrm{SOC}=$ the sum of all $q\mathrm{O}p$, there becomes $\mathrm{ang.SOC}=\mathrm{AG.ang.IGA{:}CO}$, or $\mathrm{CO.ang.SOC}=\mathrm{AG.ang.IGA}$, truly (§.129) CO.ang.SOC is equal to the arc SC, and $\mathrm{AG.ang.IGA}=$ arcui IA; therefore the arc $\mathrm{SC}=$ arc IA, and $\mathrm{BC}=\mathrm{AIMC}$. Now $\mathrm{EO}e=\mathrm{IO}p=\mathrm{IO}q+q\mathrm{O}p$; therefore $\int EOe=\int \mathrm{IO}q+\int q\mathrm{O}p$, that is, $\mathrm{EOA}=\mathrm{IOA}+\mathrm{SOC}$, but $\mathrm{EOA}=\mathrm{EOT}+\mathrm{SOC}$, therefore $\mathrm{EOT}+\mathrm{SOC}=\mathrm{IOA}+\mathrm{SOC}$, and thus $\mathrm{EOT}=\mathrm{IOA}$; therefore the semicircle SET described on the diameter $\mathrm{ST}=\mathrm{CA}$ will pass through the point E of the curve AB, and the arc $\mathrm{SVE}=$ arc SB, since now the arc AI or TE of CS itself, and AMC or TVS of the whole arc CSB have been shown to be equal. Therefore the curve sought under this hypothesis for the particular EAB is an *Epicycloid* which will be described by the motion of a point on the circumference of the circle SVT for fixed E, since clearly this circle is turned towards C in the concave part of the other circle BSC from B through S ; thus yet so that the initial motion of the point describing E in the circle of the moving body SVT will have touched the motionless point B, and it will begin to move from this point B, henceforth going on to describe the curve BEA by the rotation of the circle SVT on BSC. Which was required to be found.
[Thus the generating circle CMA with radius CG rotates on the inner arc BSC of the larger circle with the radius CO, and the point I on its circumference traces out the curve on moving to the point E, which lies on the epicycloid BEA.]

COROLLARY

182. If the diameter of the generating circle AC shall become infinite, cycloid BEA will be changed into a right line perpendicular to the axis OC ; therefore also on this straight line bodies may be called to the point A in the same time, either the initial motion shall be from the immediate vicinity of the point A, or also from the most distant point from that.

SCHOLIUM.

After Proposition XXI a suitable path would be from the centrifugal force arising from some circular motion requiring to be added but because the theory of this kind of centrifugal force presumes some matters about the motion of a pendulum, if indeed it may be needed to be examined fully, thus also in this case it required differentiation. After Huygens found the true laws of these attempts for the calculation of his oscillatory clocks [note: Huygens resorted to *reductio ad absurdum* proofs in the Archimedean manner], the Illus. Marquis de L'Hospital, instead of the demonstration proposed by their author, had given demonstrations of the Huygens theorems in the Proceedings of the Royal Academy of Science, Paris, in 1700, I am talking about all the Huygens theorems concerning centrifugal force: for some time before L'Hospital Newton had demonstrated some of these in *Philosoph: Nat. Princ*. *Math.* and had uncovered the way, by which all the others remaining could be put in place ; after these most praiseworthy authors several other authors have tried to demonstrate the 13 theorems of Huygens concerning centrifugal force, with some praiseworthy success, truly as some otherwise, who have spoiled their demonstrations with some faulty logic, as may be able to be shown easily, but only if it should arise with regard to that. But to the matter at hand : when a certain string fixed somewhere and having a small weight tied to its other end, will describe a circle turning around a fixed point, that will be said to be moving in a plane, since in truth it may be moving in a plane, but if indeed the string will describe the surface of a cone in its motion; that henceforth may be called a *conical pendulum*.



183. Fig. 43. AM therefore shall be the string fixed at A with a nail, [A and D have been interchanged from the text to agree with the diagram] at whose end the body M shall be joined, and the string describes the circle RQM about the point D by turning in the [horizontal] plane, of which the radius DM may simply be called R, C the centrifugal force, V the velocity by which the moving body M is revolving uniformly on its circumference, and this velocity shall be just as great as the mobile M would acquire likewise by its naturally accelerated motion from rest at F after a perpendicular fall from the height FM, which henceforth we will call the *determining height*, and we may designate by the letter D. The time, in which the moving body will resolve a complete circle by moving around the periphery RQM shall be T; and finally the ratio of the periphery to the semidiameter shall be as p to 1

[i.e. 2π], thus so that the periphery of the radius R, shall be pR . Truly the weight itself, so that now it may be made different, will be called G, which in the present discussion is required to be considered constant and not variable.

Now with these in place, above (§.119) it has been said the force of gravity normal to any curve tries to restrain the body from receding along the direction of the tangent to the curve, and thus generally to be an equal attempt of this kind in every curve ; and therefore in the circle. Generally likewise (§.154) it has been shown, with respect of any curve being described by a certain moving body, the square of the speed at some point of the curve to be equivalent to a rectangle under the radius of osculation of the curve or its curvature and of a right line, which force perpendicular to the curve is set out derived from the centre *;* it is necessary that here the same also may prevail by examining a circle ; truly in a circle its radius of curvature is its radius R, and the centrifugal force perpendicular to the circumference is equal to C, and the velocity at any point of the circumference is called V by us. Therefore we have eq. I for the force of the theorem (§.154) cited: $\mathrm{V}^2 = \mathrm{R.C.}$

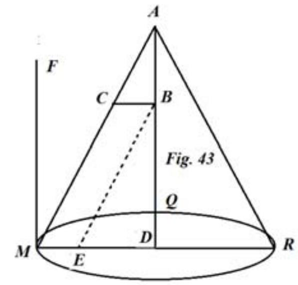
I. Then because the motion is uniform in the circle, there will be $\mathrm{T} = p\mathrm{R} : \mathrm{V}$, that is, the time becomes known by being applied to the periphery as the distance sent through to the velocity, which it has ran through, therefore also we have eq. II: $\mathrm{T} = p\mathrm{R} : \sqrt{\mathrm{RC}} = p.\sqrt{(\mathrm{R} : \mathrm{C})}$. Finally, if the centrifugal force may be brought together with the [acceleration of gravity] G, there is a need for a third equation, as article 150 puts in place : $2\ \mathrm{D.G} = \mathrm{V}^2$.

Hence 1st: the first and third formula bring about $\mathrm{R.C} = 2.\mathrm{D.G}$ and thus $\mathrm{C} : \mathrm{G} = 2\mathrm{D} : \mathrm{R}$ that is, the ratio is itself had of the centrifugal force to the gravitational force as twice the length of the determining line to the radius of the circle ; and thus everywhere this height determinator D will be equivalent to half the radius R, the centrifugal force will be equal to gravity. Which is Huygens' theorem 5.

2nd. If the periodic times T are equal in different circles also the expressions $p.\sqrt{(\mathrm{R} : \mathrm{C})}$ will be equal, and thus the centrifugal forces will be directly proportional to the radii.

3rd. If T were as R^n, there will be V or $p\mathrm{R} : \mathrm{T}$, as R^{n-1} inversely, and thus C or $pp\mathrm{R} : \mathrm{T}^2$ will be reciprocally as R^{2n-1}. Thus if the centrifugal force were inversely as R^2 [T^2 in the original] , the periodic time will be in the three on two ratio of the radius R, and in this rests the celebrated theorem of Kepler. Everything else, which can be elicited easily from the three preceding formulas, we leave to the diligence of readers, and thus without further delays we proceed to the contemplation of conical pendulums.

184. If the string AM may be moved in a conical motion around the axis AD at right angles to the horizontal, thus so that the weight M tied to it will describe the circumference of the circle MQR ; that cannot happen, unless also the direction of the force of gravity acting on the same moveable body M is present parallel to AD as well as the centrifugal force acting along the direction DM. For in order that the string AM may be kept in place at the angle inclined to the axis DAM , there is a need for the two forces



or actions from the sides AB & BC, the first of which expresses the force of gravity, and the other BC truly can only arise from the circular motion, from which central motion indeed the centrifugal force results, therefore BC represents the centrifugal force of this kind. [Following Newton rather than Huygens, we now of course consider the tension in the string as the centripetal force responsible for the motion, as any modern text relates.] Therefore BE acts parallel to the side AM; and there becomes $EM = BC$. In addition there may be put : $AD = A,\ AM = L,\ AB = G,\ \&\ BC = EM = C$, radius $DM = R$, the periphery $MRQ = pR$, and in the same way as above V represents the speed of the body on the circumference MQ ; with which suppositions put in place, again there will be, as in the preceding paragraph, the time of one circuit of the body M on the periphery MQR, while the string AM describes the conical surface, will be $T = p\sqrt{(R:C)}$ truly on account of the parallel lines AM and BE, there is $AD:AB = DM:EM$, that is, $A:G = R:C$, therefore also $T = p\sqrt{(A:G)}$, indeed this expression indicates also (§:175, 178.) the time of two oscillations of the minimum sides of the pendulum of which A shall be the length [*i.e.* the time for the to and fro swing of the simple pendulum, which is now called the period] ; truly the magnitude $\sqrt{(A:G)}$ indicates the time of the perpendicular descent of some weight from the height $\frac{1}{2}A$. Therefore the times of the circuits of conical pendulums are in the square root ratio of the heights of the cones A. I think the rest is superfluous, about pendulums of this kind indicated by Huygens, the theorems laborious to demonstrate, since they may emanate easily at once from these principles put in place, as anyone will see, who may wish to consider these. Yet before we may pass on to other matters we will consider for a short time a problem proposed by the most ingenious Johann Bernoulli and for which the solution arose previously from the most illustrious L'Hospital ; just as the form of the proof is most different from ours.

PROPOSITION XXX. PROBLEM.

185. Fig. 44. *To find the curve ABN of this condition, so that a weight A descending with a naturally accelerated motion, the same may press on the individual points with a force everywhere equal to the complete weight of the body*.

I. CD shall be the axis of the curve sought, CA the first ordinate in its continuation, AF may show the heaviness or absolute weight of the mobile A, and with the semicircle AGF described upon that, FG acting parallel to the tangent of the curve at the point B, and F*g* parallel to the tangent at *b*, and AG , A*g* may be joined. Then the radius of osculation of the curve at the point B, that is, BZ is produced to L, until $\mathrm{BL}=\mathrm{AF}=\mathrm{BI}$ which is assumed to be produced beyond the ordinate DB of the curve AG and A*g* themselves become equal to AM and A*m*, and finally from the point I the perpendicular IK may be dropped to the line BL, and there will be $\mathrm{BK}=\mathrm{AG}=\mathrm{AM}$, and thus $\mathrm{MF}=\mathrm{KL}$

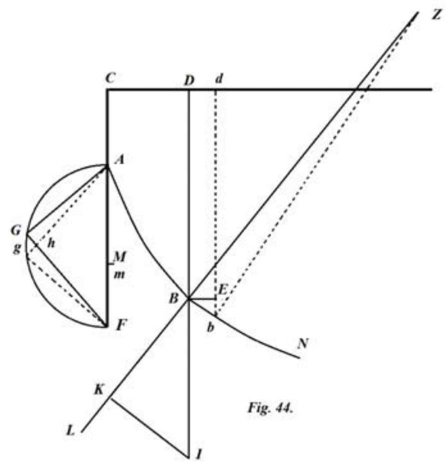


Fig. 44.

II. Because BI exhibits the absolute weight of the moving body at A or at B , BK shows that pressure [here meaning a pressing force], that the weight exerts on the curve at the point B along the direction BL [*i.e.* the component of the weight normal to the slope at B], but besides this pressure it will sustain another likewise in addition at the point B from the centrifugal force of the body, and consequently here the centrifugal force will be set out by KL, since the total pressure BL must be equal (following the hypothesis) to AF or to BI itself. The speed acquired at B may be called V; and there will be (§. 154) $\mathrm{V}^2=\mathrm{BZ.KL}=\mathrm{BZ.MF}$, indeed (after §.150.) also there is $\mathrm{V}^2=2\mathrm{AF.DB}$, therefore

$$\mathrm{BZ.MF}=2.\mathrm{AF.DB}$$

[In modern terms we may write $\frac{mv^2}{\rho}\times\rho=2mg\times h$, where *g*, *h* etc. have their modern meanings, essentially energy conservation];

hence

$$\mathrm{B}b.\mathrm{MF}:\mathrm{BZ.\ MF}=\mathrm{B}b.\ \mathrm{MF}:2.\mathrm{AF.DB}.$$

And the similitude of the sector BZ*b*, GA*h* and *g*F*h*, provides $\mathrm{B}b:\mathrm{BZ}=\mathrm{M}m(gh):g\mathrm{F}$;

[Thus, as the body moves an incremental distance B*b* along the curve, the normal component of its weight increases from GA to *g*A, or by *gh*, equal to the difference of A*m* and AM, *i.e.* M*m*, and its tangential component decreases from GF to *g*F, or G*h*] and from the similarity of the triangles B*b*E and FAG there is elicited $\mathrm{B}b:\mathrm{AF}=\mathrm{E}b:\mathrm{FG}$; therefore by substituting in the preceding ratio, the proportionals M*m* and FG in place of B*b* and BZ, and in place of B*b* and AF, the proportionals E*b* and FG,

[i.e. $\mathrm{B}b\rightarrow\mathrm{M}m$; $\mathrm{BZ}\rightarrow\mathrm{FG}$; $\mathrm{B}b\rightarrow\mathrm{E}b$; $\mathrm{AF}\rightarrow\mathrm{FG}$;]

that proportionality will be changed into this other form

$$\text{M}m.\text{MF}:\text{FG.MF}=\text{E}b.\text{MF}:2.\text{DB.FG},$$

or

$$2.\text{MF.M}m:\text{FG.MF}=2.\text{E}b.\text{MF}:2.\text{DB.FG}$$

and on dividing consequently by MF: FG, there will be

$$2\text{MF.M}m:\text{MF}^2\left(=2\text{E}b.\text{MF}:2\text{DB.FG}\right)=\text{E}b:\text{DB}.$$

From which, since the decrement $2\text{MF.M}m$ is to the decrease of MF^2 as the increment E*b* of the increase of DB is to that increased, (§.153.) DB increasing to its first magnitude CA, will be as the decrease from the first magnitude, which is FA^2 to the decreased value MF^2 ; [*i.e.* on integrating the above ratio,

$$\int 2\text{MF.M}m:\text{MF}^2=\text{DB}:\int \text{E}b \; \textit{i.e.}\; \text{FA}^2:\text{MF}^2=\text{DB:CA} \;]$$

and thus with P taken to be the mean proportional between AC and DB, [*i.e.* P^2=AC.DB] there becomes

$$\text{P}^2:\text{AC}^2=\text{AF}^2:\text{MF}^2 \text{ or } \text{P}:\text{AC}=\text{AF}:\text{MF},$$

and by interchanging :

$$\text{P}-\text{AC}:\text{P}=\text{AG}:\text{AF}$$

$$[\text{i.e. } 1-\text{AC}:\text{P}=1-\text{MF}:\text{AF} \text{ or } \text{P}-\text{AC}:\text{P}=\text{AF}-\text{MF}:\text{AF}=\text{AG}:\text{AF}\;],$$

hence

$$\text{AF}^2:\text{AG}^2=\text{P}^2:\text{P}^2-2\text{AC.P}+\text{AC}^2,$$

and on dividing [*i.e.* taking 1 from each side as above]

$$\text{FG}^2:\text{AG}^2=2.\text{AC. P}-\text{AC}^2:\text{P}^2-2.\text{AC.P}+\text{AC}^2,$$

$$\text{and } \text{FG}:\text{AG}=\sqrt{\left(2.\text{AC.P}-\text{AC}^2\right)}:\text{P}-\text{AC}=\text{E}b:\text{BE};$$

and thus, if AC may be called *a*; CD, *x*; DB, *y*; E*b*, *dy*, there will be

$$p=\sqrt{ay},\; 2pdp=ady, \text{ and } \text{FG}:\text{AG}=\sqrt{(2.\text{AC.P}-\text{AC}^2)}:\text{P}-\text{AC},$$

becomes with these symbols $dy:dx=\sqrt{(2ap-aa)}:p-a$, therefore $pdy-ady=dx\sqrt{\left(2ap-aa\right)}$, or $apdy-aady=2ppdp-2apdp=adx\sqrt{\left(2ap-aa\right)}$, and thus $dx=\left(2ppdp-2apdp:a\sqrt{\left(2ap-aa\right)}\right)$ which may be changed into $dx=\left(q^4dq-a^4dq\right):2a^4$, if evidently there were put in place $q=\sqrt{(2ap-aa)}$, the integral of which is $10a^4x=q^5-5a^4q+4a^5$, and by working backwards there will be found $q=\sqrt{(2a\sqrt{ay}-aa)}$ and thus the equation of the curve sought will be $\left(aa+(y-a-\sqrt{ay}\right)$ into $\left(2a\sqrt{ay}-aa\right)=\frac{5}{2}ax$. Q.E.D.

$$[i.e.\ 10a^4x = \left(\sqrt{(2a\sqrt{ay}-aa)}\right)^5 - 5a^4\left(\sqrt{(2a\sqrt{ay}-aa)}\right) + 4a^5$$

$$= \left(\sqrt{(2a\sqrt{ay}-aa)}\right)\left(\left(2a\sqrt{ay}-aa\right)^2 - 5a^4\right) + 4a^5$$

$$= \left(\sqrt{(2a\sqrt{ay}-aa)}\right)\left(4a^3y - 4a^3\sqrt{ay} - 4a^4\right) + 4a^5;$$

$$\therefore \tfrac{5}{2}ax = \left(\sqrt{(2a\sqrt{ay}-aa)}\right)\left(y - \sqrt{ay} - a\right) + a^2.]$$

COROLLARY.

186. Because we have found
$\mathrm{BZ}:\mathrm{DB} = 2.\mathrm{AF}:\mathrm{MF}$ & $\mathrm{P}:\mathrm{CA} = \mathrm{AF}:\mathrm{MF}$, or $2\mathrm{P}:\mathrm{CA} = 2\mathrm{AF}:\mathrm{MF}$, there will be equally $2\mathrm{P}:\mathrm{CA} = \mathrm{BZ}:\mathrm{DB}$ and by inverting $\mathrm{CA}:\mathrm{P} = \mathrm{DB}:\mathrm{BZ}$, and thus the radius of the evolute at some point of the curve becomes known with the aid of its ratio, which ratio agrees with that, which the Illus. Marquis de L' Hospital treated in *Actis Acad. Reg. Scient. Paris.* 1700, from first principles.

CAPUT III.

De Motu Isochrono corporum in curvis descendentium juxta quamlibet gravitatis variabilis hypothesin, atque gravium directionibus etiam in centro gravium convergentibus; et de Motu Pendulorum.

DEFINITIO.

Cum corpus grave motu suo ex quiete arcus quoscunque inaequales alicujus curvae, sed terminatos tamen omnes ad infimum curvae punctum eodem vel aequali tempore, perlabitur, talis linea curva *isochrona* dici solet. Ut si grave A, ex quolibet curvae BFA Fig.40. puncto B, F, E &c. a quiete descensum incipiens motu suo quomodocunque accelerato, eodem seu pari tempore arcus utlibet inaequales BeA, FeA, EXA, sed termatos omnes ad infimum curvae punctum A, absolvat ; curva BEA Isochrona vocabitur.

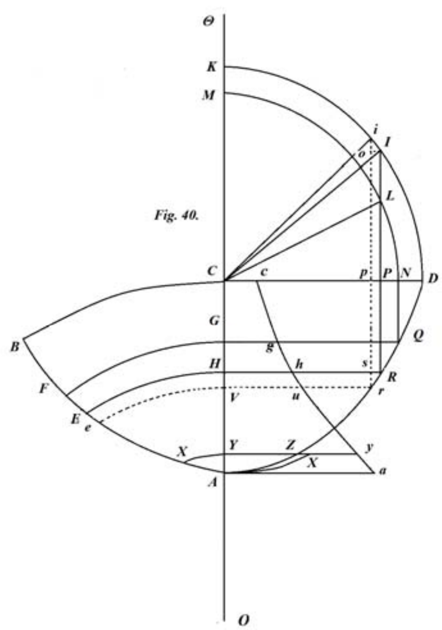


Fig. 40.

PROPOSITIO XXVI. LEMMA.

171. Fig. 39. *In quolibet triangulo rectilineo, cujus basis a linea intra triangulum ex angulo basi opposito ducta, pro libitu in duo quaecunque segmenta dividitur, solida quae fiunt ex quadratis laterum in alterna baseos segmenta simul sumpta aequantur duobus solidis, quorum unum sit ex quadrato lineae basin dividentis in hanc basin, alterum ex rectangulo segmentarum basis in totam pariter basim.*

Sit triangulum quodvis ABC, cujus basis BC dividatur in D a recta AD pro libitu intra triangulum ducta. Probari debet esse $AB^2.DC + AC^2.BD = AD^2.BC + BD.DC.BC.$

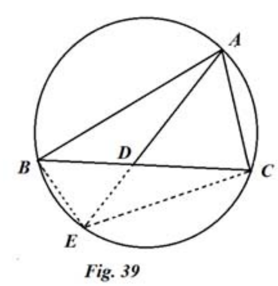


Fig. 39

Demonst. I. Descripto circa triangutum circulo ABE, productur AD in E, & jungantur BE, CE, & triangula similia BAD ac ECD praebent analogiam $AB : AD = EC : DC$ atque adeo $AD.EC = AB.DC$. Atque a triangulorum ACD, BED similitudine elicitur $AC : AD = BE : BD$, & rec-lum AD.BE aequale rec-lo AC.BD.

II. In quadrilatero ABEC circulo inscripto ,est $AB.EC + AC.BE = AE.BC = AD.BC + DE.BC$. Vel ascita communi altitudine AD, fiet $AB.EC.AD + AC.BE.AD = AD^2.BC + AD.DE.BC$. Adeoque subrogando loco rectangulorum EC.AD & BE.AD haec rec-la AB.DC & AC.BD quae illis (num.I) aequalia ostensa sunt, & loco rectanguli AD.DE aequale rec-lum BD. DC ; erit $AB^2.DC + AC^2.BD = AD^2.BC + BD.DC.BC$. Quod erat demonstrandum.

PROPOSITIO XXVII. THEOREMA.

172. Fig. 40. *Si gravium directiones convergant* in centro O, *atque circa axem* OC *descriptae sint tres curvae, scilicit scala solicationum gravitatis variabilis* cha, *deinde curvae* ARD, *cujus quaelibet ordinata* HR *possit duplum homologae areae AHha in scala gravitatis, ac denique tertia curva AEB, cujus arcus quilibet AXE, a circule HE radio OH descripto, terminatus sit ad homologam ordinatam HR in curvae ARQ, ut aliquis datus numerus N, ad unitatem, erit haec tertia curva BEA*, isochrona.

Ponatur grave delapsum esse per arcum curvae BEA motum a quiete in B incipiendo; atque Ee elementum esse curvae AE, per cujus terminum e descriptus centra O arcus eV axi occurrat in V, per quod punctum ducatur ordinata Vr in curva ARD, atque super hujus curvae basi CD descripto quadrante circuli CDIK, per punctra R, r agantur RI, & ri circulo occurrentes in I, i, ejusque radio DP in puntis P, p, ductaque per I lineola Io parallela CD, jungantur denique CI, Ci. Quibus positis

I. Quia (secundum hypothesin) $CD^2 = 2.ACchaA$, & $HR^2 = 2.AHhaA$, erit $CD^2 - HR^2$, hoc est $CD^2 - CP^2$, vel $CI^2 - CP^2$, id est $IP^2 = 2.CHhgc = 2.ACchaA - 2.AHhaA$, ergo $IP = \sqrt{(2.CHhgc)}$. Item quia (secundum hypothesin) $AE : HR = N : 1$, erit $AE = N.HR$, adeoque $Ee = N.rR = N.Pp = N.Io$.

II. Supra(§.136.) ostensum, eandem vel aequalem acquiere celeritatem duo mobilia pondere & massa aequalia in duobis punctis E, H centro aequidistantibus, si ex punctis pariter aeque altis B, C unum in curva BEA alterumque in recta CA a quiete moveri coeperint, atque spatia BE & CH descripserint. Et vocando corporis *A* massam per hanc eandem literam A, celeritas acquisita in H ex descensu per CH, vel in E ex descensu per BE (§. 144.) erit $\sqrt{(2.CHhc)} : \sqrt{A}$, seu (num.1 hujus) $= IP : \sqrt{A}$. Unde, quia (num.1) $Ee = N.Io$ atque spatiolum Ee applicatum ad velocitatem qua percurritur; denotat tempus quo absolvitur seu tEe; erit propterea $tEe = N.Io\sqrt{(A)} : IP$. Atqui propter triangula similia CIP, & I*io*, ratio Io ad IP aequalis est rationi Ii: CI, hoc est (§. 129) angulo ICi; idcirco $tEe = N.\sqrt{A}.ang. ICi$: ergo omn. tEe, hoc est, $tBE = N.\sqrt{A}$. omn. ICi seu ICD, atque adeo $tBE = N.\sqrt{A}.ang.ICD.$ & $tBEA = N.\sqrt{A}.ang.KCD.$

III. Si jam mobile non amplius in B, sed in alio quocunque curvae puncto F descensum incipiat ; centro O per hoc punctum F ducatur circulus FG & per G ordinata GQ curvae ARD occurrens in puncto Q, per quod agatur QN axi AC parallela ac denique radio CN descripto quadrante MLN, jungatur LC, atque hisce positis eodem, quo in antecedente numero, conficitur argumento existere $tFE = N.\sqrt{A}.ang.LCN$. Cogitatione tua abscinde vel aufer totum spatium, quod est inter lineas mixtas BCD & FGQ, ac quadrantem MCN admove pariter ordinatae GQ, & hac ratione hunc casum, quo mobile ex F descendere incipit, reducis praecise ad casum praecedentem; cum AEF nunc spectetur velut integra curva ex cujus principio F proficiscatur mobile & analogae curvae ARQ basis GQ. quadrantem habet sibi insistentem, quemadmodum basis CD suum quadrantem KID, idcirco tempus per FE, hoc est $tFE = N.\sqrt{A}.ang.LCN$, atque adeo $tFEA = N.\sqrt{A}.ang.MCN$. Est ergo tBEA ad tFEA, ut $N.\sqrt{A}.ang.KCD$ ad $N.\sqrt{A}.ang.MCN$ hoc est, sicut angulus rectus KCD ad angulum rectum MCN, atque adeo in ratione aequalitatis; ergo aequali tempore omnes arcus BA, FA, XA &c. percurrentur, atque adeo curva propositionis nostrae BEA est isochrona. Quod erat demonstrandum.

COROLLARIUM I.

173. Si ΘA fuerit radius osculi seu curvaturae isochronae in vertice A, erit generaliter $N = \sqrt{(\Theta A.OA : \Theta O.Aa)}$. Sumto enim arculo AX indefinite parvo vertici contiguo, ductisque ex centro O arculo XY, & per Y ordinata Yy curvam AR secante in z. Jam quia (secundum hypothesin) $AX = N.YZ$, vel $AX^2 = NN.YZ^2$ (secundum hypothesin) = NN.dupl. areae AYya $= 2NN.AY.Aa$; & quia AY est aggregatum sinuum versorum arculorum AX & XY instar aequalium accipiendorum, quorum centra sunt Θ & O, fiet $AY = \frac{AX^2}{2\Theta A} + \frac{AX^2(XY^2)}{2OA}$, atque adeo $2AY = AX^2.\Theta O : \Theta A.OA$, ac proinde substitutione facta, proveniet $AX^2 = NN.AX^2.\Theta O.Aa : \Theta A.OA$, ex qua facile elicitur $NN = \Theta A.OA{:}\Theta O.Aa$, & $N = \sqrt{\left(\Theta A.OA{:}\Theta O.Aa\right)}$.

COROLLARIUM II.

I74. Si jam minimus isochronae arculus XA vertici A adjacens consideretur instar arcus circularis radii ΘA a pondere A filo ΘA appenso descripi; erit tempus, quo arculus XA vel tota curva BEA describi potest, semissis unius oscillationis penduli ΘA. Unde, quia (§. 171.) $tXA = N.\sqrt{A}.ang.rectum$, erit 4tXA,seu duratio duarum penduli cujusque oscillationum minimarum, quod deinceps dicatur T, erit inquam $T = N.\sqrt{A}.4rectos$ seu nominando peripheriam p, cujus radius I erit $T = pN.\sqrt{A}$, vel substituendo valorem ipsius N articulo praecedenti definitum, reperietur $T = p\sqrt{\left(A.\Theta A.AO : \Theta O.Aa\right)}$. Qui est canon valde generalis, in quo ΘA significat longitudinem penduli, AO ponderis minimam distantiam a centro gravium O, ΘO distantiam puncti suspensionis Θ ab eodem centro O; Aa pondus corporis oscillantis A in infimo loco positi, & T tempus duarum penduli ΘA vibrationum minimarum. Haec determinatio probe consentit cum assertionibus paulo specialioribus Newtoni Prop.52. *Lib. Pr.Ph. Nat.*

COROLLARIUM III.

175. Si centrum O sit infinite distans a puncto Θ, aequabuntur ipsae ΘO & AO, atque formula praecedentis corollarii mutabitur in sequentem $T = p\sqrt{(A.\Theta A : Aa)}$. Hinc 1°. tempora oscillationum diversorum pendulorum sunt in composita ratione ex subduplicata massarum & longitudinis pendulorum directe & subduplicata itidem sed inversa ponderum. 2°. Tempora oscillationum pendulorum aequalium sunt in composita ratione ex subduplicata directa ratione massae corporum & subduplicata inversa ratione ponderum. 3°. Si solicitationes , quibus pendula agitantur, id est pondera corporum longitudini pendulorum proportionalia sint, tempora oscillationum erunt in subduplicata ratione massarum corporum agitatorum , & haec tempora erunt aequalia, si, iisdem positis, massae insuper corporum ponderibus proportionales fuerint. 4°. Massae vero, seu materiae quantitates, erunt in composita ratione ex ponderum ratione pendulis aequalis longitudinis appensorum, & ex duplicata ratione temporis oscillationum. Atque hoc ipsum est Prop. 27. *Lib.II. Pr. Ph. Nat.* qua usus est Cl. Vir ad explorandum utrum pondera corporum ipsorum massis proportionalia sint, nec ne *;* ac reperit hic Author experimentis, accuratissime pendulorum ope sumtis, pondera corporum massis constanter proportionata esse. Sed hoc loco sermo est de ponderibus absolutis corporum; non vero de relativis, qualia habent cum fluidis diversis demersa sunt; hoc enim casu pondus portionis cujusdam fluidi volumine aequalis corpori demerso a pondere absoluto hujus corporis debet auferri, ut habeatur ejus pondus relativum, quod intra fluidum, cui immersum est, habet.

COROLLARIUM IV.

176. Cum multitudines oscillationum aequalibus temporibus a divertis pendulis absolvendarum sint in contraria ratione temporum, quo unumquodque pendulum unam oscillationem peragit, multitudines oscillationum aequalibus temporibus peractarum erunt proinde in composita ratione ex ratione subduplicata directa ponderum & subduplicatis rationibus inversis massam & longitudinum pendulorum. Aut, si pondera massis proportionalia fiat, ut tuto assumi potest; erunt praedictae oscillationum multitudines, ut radices ex lineis, quae pondera seu vires gravitatis exponunt, quibus pendula agitantur, applicatis ad pendulorum longitudines. Atque adeo, 1°. pendulorum inaequalium, sed eadem gravitatis solicitatione agitatorum, vibrationes eodem tempore absolvendae sunt in reciproca subduplicata ratione longitudinis. pendulorum. 2°. Numerus oscillationum unius penduli erit ad numerum oscillationum eodem tempore peractarum in alio pendulo ejusdem longitudinis cum primo, in subduplicata ratione solicitationis gravitatis, qua primum ad solicitationem gravitatis, qua alterum pendulum primum, quoad longitudinem aequale agitatur. Atque hoc posterius ad amussim convenit cum regula quam Bernoullius in elegantissimo suo schediasmate Act. Lips. 1713. M. Februario inserto, tradit paragrapho 16, ex qua deinceps gravitates specificas eruere docet ex pendulorum experimentis modo plane novo nec antea cognito.

SCHOLION.

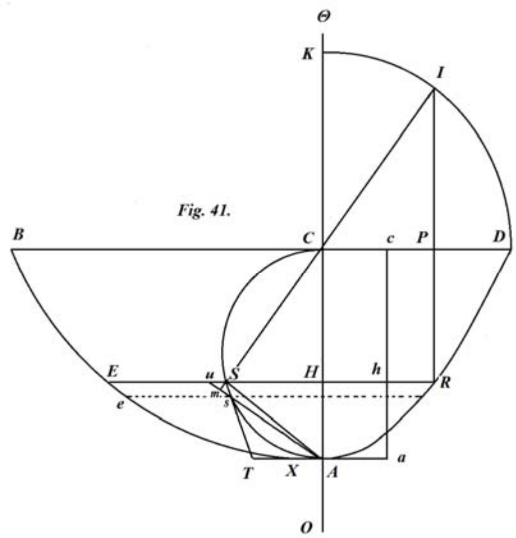
177. Ex corollariis proxime antecedentibus satis elucere existimo, quantae utilitatis sit theorema nostrum generale isochronismi corporum in curvis , assignata lege descriptis, descendentium, cum ex ea omnia, quae ad pendulorum motus spectant, tanta facilitate deducantur: interim oscillationes pendulorum quam minimal considerare convenit propter rationes §. 174 indicatas, scilicet quia tum demum pendulum vel penduli pondus arculum isochronae, cujus tempus dimensi sumus percurrere censetur cum minimum arculum circularem ipsum describit, quoniam , eo casu , talis arculus circularis osculatur arculum isochronae duplum ipsius XA seu arculum XAX ; posito arcu AX in curva BEA, ex altera axis AC parte constituta aequali arculo AX, & quia osculari atque congruere in Geometria unum idemque significant saltem in ea parte, in qua osculum contingit. Sin vero arculi AX non sunt minimi, haec tamen omnia adhuc subsistent, si pendula inter se collata vel eorum pondera curvas similes descripserint.

178. Ut igitur quae in antecedentibus corollariis sparsim dicta sunt in compendium colligantur ; nominentur numerus oscillationum aliquo tempore peractarum a primo pendulo N, ejus longitudo L, quae in schemate repraesentature linea ΘA, corporis A gravitas absoluta G, ejus massa M, adeoque M nunc significat idem ac A, & G idem ac linea A*a* in schemate ; atque retenta T pro designando tempore duarum oscillationum minimarum hujus penduli. In secundo pendulo eaedem res iisdem, ac in primo, literis, sed minusculis exprimantur, atque adeo corollorium 3 praebebit has regulas $\mathrm{T} = \mathrm{p}\sqrt{(\mathrm{M.L} : \mathrm{G})}$ & $\mathrm{t} = \mathrm{p}\sqrt{(\mathrm{m.l} : \mathrm{g})}$. Corollarium vero 4 exhibet $N : n = \mathrm{t} : \mathrm{T}$, vel $NT = nt$. Unde quot diversis modis singulae literae T, G, L, M, N & homologae *t, g, l, m, n* inter se conferri, tot inde resultabunt alia atque alia theoremata, a quibus omnibus sigillatim recensendis brevitatis gratia abstineo. Hactenus isochroniam tantum in genere consideravimus nulli particulari hypothesi gravitatis inhaerentes. Quid vero ex una alterave ejusmodi hypotheseon resultare debeat, indagibimus in sequentibus problematibus.

PROPOSITIO XXVIII. PROBLEMA.

I79. Fig. 41. *Invenire isochronam in hypothesi gravitatis uniformis atque directionum gravitatis inter se parallelarum seu in centro infinite distanti coeuntium.*

Quoniam centrum solititationum gravitatis O infinite distat a basi CD curvae ARD, lineae BC, EH, quae antea (§.172) erant arcus circulares centro O descripti, nunc fient rectae inter se parallelae & axi AC perpendiculares: & quia (secundum hypothesin) gravitas uniformis, ejus scala erit linea recta *cba* axi AC parallela, & quia (§.172), $\mathrm{HR}^2 = 2.\mathrm{HAah}$, curva ARD hoc casu parabola erit, cuius paramenter 2Aa. At propter Oo & AO aequales, fiet $\mathrm{N} = \sqrt{(\Theta\mathrm{A} : \mathrm{Aa})}$. Jam quia (§.172) $\mathrm{AE}=\mathrm{N.HR}=\mathrm{N}.\sqrt{(2\mathrm{Aa.AH})}=\sqrt{(2\Theta\mathrm{A.AH})}$, erit $\frac{1}{2}\mathrm{AE}=\sqrt{(\frac{1}{2}\Theta\mathrm{A.AH})}$, vel bisecta $\Theta\mathrm{A}$ in C, erit $\frac{1}{2}\mathrm{AE}=\sqrt{(\mathrm{AC.AH})}$, id est, super diametro AC descripto semicirculo ASC, lineam EH secante in S, $=$ subtensae AS, quandoquidem haec media est geometrica inter diametrum AC & abscissam AH. Sic etiam $\frac{1}{2}\mathrm{A}e = \mathrm{A}s$, atque adeo $\frac{1}{2}\mathrm{E}e = \mathrm{AS} - \mathrm{A}s$, id est demissa $\mathrm{S}m$ perpendiculari super *As* productam, $= sm$, adeoque $\mathrm{E}e = 2.sm$, in suppositione infinitae parvitatis arculi *Ee*, quo casu *sm* est differentia seu excessus subtensae AS supra subtensam minorem As. Verum quia ductis per puncta A & S tangentibus AT & ST occurrentibus in T, hae tangentes erunt aequales, etiam *u*S aequabitur S*s*, atque adeo perpendicularis *Sm* bifariam dividet basin *su* trianguli isoscelis *sSu* , adeo ut *su* sit $= 2sm$, atque adeo $\mathrm{E}e = su$ *;* idcirca quadrilaterum *Eesu,* in quo latera opposita E*u* & *su* sunt aequidistantia & reliqua opposita latera E*e*, *us* aequalia , erit parallelogrammum, atque adeo $\mathrm{ES} - es = u\mathrm{S} = \mathrm{S}s$, hinc (§. 87.) omnes differentiae $\mathrm{ES} - es$ seu excessus maximae ES supra minimam, quae in vertice A nulla est, hoc est sola ES aequatur omnibus S*s* seu arcui circuli AS & sic ubique, propterea Isochrona quaesita BEA est *Cyclois* ordinaria, cujus semibasis BC aequatur peripheriae semicirculi ASC. Quod erat inveniendum.



COROLLARIUM.

180. Tempus descensus per cycloidis arcum BE, erit ad tempus per axis partem aequalitatum CH, sicut arcus CS circuli generatoris interceptus inter parallelas BC & EH, ad subtensam CS ejusdem arcus. Nam propter parabolam ARD & circulum ASC, est $\mathrm{CI}(\mathrm{CD}) : \mathrm{CP}(\mathrm{HR}) = \mathrm{CA} : \mathrm{AS}$, atque adeo anguli ICP & CAS vel CSH aequales sunt,

adeo ut SC & CI in directum jaceant. Jam quia (§.172. n.II.) $tBE = N.ang.ICD$ erit etiam $tBE = N.ang.SAC$, atqui (§.129.) angulus $SAC = \frac{1}{2}arc.SC : \frac{1}{2}CA = arc.CS : CA$, ergo $tBE = N.arc.CS : CA$, vel quia $N = \sqrt{(\Theta A : Aa)}$, erit $tBE = arc.CS.\sqrt{\Theta A} : CA.\sqrt{Aa}$, aut quia $\Theta A = 2.CA$, fiet $tBE = 2.arc.CS : \sqrt{(CA.Aa)} = 2.arc.\ CS : CD$, propter parabolam ARD; & tCH (§.151) $= \sqrt{(2.CH : Aa)}$ nulla habita ratione massae corporis, cum hoc loco mobile non conferatur cum aliis *;* sed unum idemque corpus tantum in linea curva BEA vel in axe CA moveri intelligatur, adeo ut loco ipsius M, quae formulam articuli citati 151 ingreditur, poni possit unitas. Sed $tCH = \sqrt{(2CH : Aa)}$ reducitur ad $tCH = \sqrt{(4AC.CH : 2CA.Aa)} = 2.CS : CD$; ergo tBE:tCH, ut fractio $\frac{2.arc.CS}{CD}$ ad $\frac{2.subst.CS}{CD}$*;* atque adeo tempus per BE est ad tempus per CH sicut arcus CS ad ejus subtensam CS.

Propterea tempus per totam semicycloidem BEA ad tempus descensus mobilis per axem ejus CA se habet, ut semicircumferentia CSA ad diametrum CA prout primum ab Hugenio, dein a multls aliis demonstratum est; sed ex fundamentis ab hisce nostris multum diversis.

PROPOSITIO XXIX. PROBLEMA.

181. Fig.42. *Si scala solicitationum gravitatis centralium fuerit linea recta* ha, *quae deorsum producta transeat per centrum* O *solicitationum gravitatis; invenire isochronam* BEA *in hac hypothesi.*

I. Quia (§.172.)
$HR^2 = 2.HAah = \left(HO^2 - AO^2\right).Aa : AO$
propter trapezium HA*ah*, curva AR invenitur hyperbola esse, in qui
$HR^2.AO : Aa = HO^2 - AO^2$.

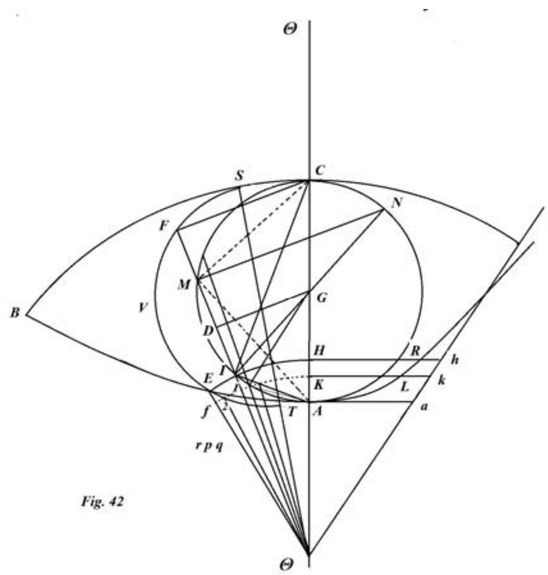


Fig. 42

II. Ex aliquo axis puncto G, tanquam centra, descriptus circulus AMN ab arcubus concentricis indefiniteque vicinis EH, *eK* secetur in punctis I & *q* actisque ex centra G ad haec puncta radiis GI & *Gq*, tum etiam rectis AI & *Aq*, illa seu AI secet arcum *eK* in *p*, sitque arculus *qr* centro A descriptus, junganturque demum IO, *qO* & *pO.*

Jam in triangulo IGO , si GO consideretur instar baseos, erit ut , supra (§.171) ostensum $IG^2.AO + IO^2.AG = AI^2.GO + GA.AO.GO$, atque adeo
$AI^2.GO = IO^2.AG + IG^2.AO - GA^2.AO - AO^2.AG = HO^2.AG - AO^2.AG$,

& per consequens $HO^2 - AO^2 = Al^2.\ GO:AG(\text{vel num. 1. hujus}) = HR^2.AO:Aa$; hinc fiet $HR^2:AI^2 = Aa.GO:AG.AO$, & (§172) $AE^2:HR^2(=NN:I) = \Theta A.OA:\Theta O.Aa$, ergo ex aequo & per compositionem rationum fiet
$AE^2:AI^2 = Aa.GO.\Theta A.OA:AG.AO.\Theta O.Aa = \Theta A.GO:\Theta O.AG$; hinc
$AE:AI = \sqrt{(\Theta A.GO)}:\sqrt{(\Theta O.AG)}$ atque adeo est curva AE ad homologam AI in data ratione, ac propterea $Ee:Ir = \sqrt{(\Theta A.GO)}:\sqrt{(\Theta O.AG)}$.

III. Ergo inter circulos concentricos EH & *eqK* ad punctum I aptanda esset quaedam lineola, quae sit ad Ir, differentiam inter subtensas IA & *q*A, in data ratione $\sqrt{(\Theta A.GO)}$ ad $\sqrt{(\Theta O.AG)}$. Verum quia facile videtur esse I*p* portiunculam subtensae AI a praefatis circulis concentricis interceptam ad praedictam subtensarum differentiam I*r* in data ratione duplae GD ad CF, demissis scilicet ex punctis G & C perpendicularibus ad lineam OI productam, vel duplae GO ad CO ; elementum ergo curvae Ee lineolae I*p* aequale poni potest ex eo ipso, quod I*p* & Ee habeant ad I*r* datam rationem, quod respectu rationis T*p* ad I*r* ita esse demonstrandum est. Agantur MC, IC, & MN, quo facto, erunt primum triangula I*pi &* CIF similia, quandoquidem angulus CIA in semicirculo rectus efficit, ut duo *p*I*i* & CIF simul rectum aequent, ut adeo angulus *p*I*i* ...., aequalis sit angulo FCI & cum anguli ad *i* & F sint (secundiun hypothesin) recti, necesse est, ut tertius tertio atque adeo triangulum triangulo simile sit. Secundo, quia angulus qIi subtangente Iq & secante IO, vel ejus verticaliter oppositus aequalis angulo MNI, in alterno segmento ipsius IM, angulique i & IMN recti sunt, triangula ipsa INM & *q*I*i* similia existent. Tertio, triangula etiam I*qr &* CAI similia erunt, quoniam, angulus *q*IA sub tangente I*q* & secante IA aequatur angulo ICA in alterno segmento, & anguli ad *r* & AIC recti; adeoque tria haec triangulorum similium paria suppeditabunt has analogias $Ip:Ii = IC:CF$, item $Ii:Iq = MN:IN$, & denique $Iq:Ir = AC \text{ vel } IN:IC$, ergo ex aequo fiet $Ip:Ir = MN:CF = 2.DG:CF = 2.GO:CO$.

IV. Adeoque, cum sit (num: 11.) $Ee:Ir = \sqrt{(\Theta A.GO)}:\sqrt{(\Theta O.AG)}$ & num. 111. $Ip:Ir = 2GO:CO$, erit $Ee = Ip$, si fuerit $\sqrt{(\Theta A.GO:\Theta O.AG)} = 2.GO:CO$; haec ergo rationum aequalitas assumatur, quoniam recta ΘA ad nullam adhuc magnitudinem est restricta, suamque magnitudinem etiam ad diametrum circuli CA relatam habere debet; fietque hoc casu $\Theta A = 2.OG.AC:AO$; nec non $Ip = Ee$ atque adeo $ip = fe$; porro (§.119) est $qp = pO.\text{ang.}pOq$, & $qr = Aq.\text{ang. }qAI = Aq.\frac{1}{2}.\text{ang.}IGq$. Verum quia $pq:qr(=Ip:Ii = IC:CF) = AC:MC$, ideo $MC.pq = AC.qr$, vel
$pO:MC.\text{ang. }pOq = Aq.AC.\text{ang.}qAI = AI.AC.\frac{1}{2}IGq = AI.AG.IGq$, ergo
$pOq = AI.AG.IGq:KO.MC$; sed propter triangula similia OIA & OCM, erit
$IA:MC = IO:CO = HO \text{ vel } KO:CO$; propterea ponendo loco AI & MC, homologas proportionales KO & CO; fiet $\text{ang.}pOq = KO.AG.\text{ang.}IGq:KO.CO = AG.IGq:CO$, atque adeo omnes *p*O*q*, qui singulis IG*q* respondent $= AG.\text{ omn. }IGq:CO = AG.\text{ang. }IGA:CO$,

aut posito angulo $\mathrm{SOC} = \text{omnibus } q\mathrm{O}p$, fiet $\mathrm{SOC} = \mathrm{AG.IGA:CO}$, seu $\mathrm{CO.ang.SOC} = \mathrm{AG.ang.IGA}$, verum (§.129) $\mathrm{CO.ang.SOC} = \text{arcui SC}$, & $\mathrm{AG.ang.IGC} = \text{arcui IA}$; ergo arcus $\mathrm{SC} = \text{arcui IA}$, atque $\mathrm{BC} = \mathrm{AIMC}$. Jam $\mathrm{EO}e = \mathrm{IO}p = \mathrm{IO}q + q\mathrm{O}p$; ergo $\int EOe = \int \mathrm{IO}q + \int q\mathrm{O}p$,id est, $\mathrm{EOA} = \mathrm{IOA} + \mathrm{SOC}$, sed $\mathrm{EOA} = \mathrm{EOT} + \mathrm{SOC}$, ergo $\mathrm{EOT} + \mathrm{SOC} = \mathrm{IOA} + \mathrm{SOC}$, atque adeo $\mathrm{EOT} = \mathrm{IOA}$; idcirco super diametro $\mathrm{ST} = \mathrm{CA}$ descriptus semicirculus SET transibit per curvae AB punctum E, eritque arcus $\mathrm{SVE} = \text{arcui SB}$, quandoquidem jam arcus AI vel TE ipsi CS, & AMC vel TVS toti CSB aequales sunt ostensi. Propterea curva, quaesita in hac hypothesi particulari EAB est *Epycyclois* quae describitur motu puncti in circumferentia circuli SVT fixi E, cum scilicet hic circulus in cava parte alterius circuli BSC ex B per S versus C volvitur; ita tamen ut initio motus punctum describens E in circulo mobili SVT punctum B immobilis tetigerit, atque ab hoc puncto B moveri coeperit, curvam BEA deinceps descripturum rotatione circuli SVT super BSC. Quod erat inveniendum

COROLLARIUM

182. Si diameter circuli generatoris AC fiat infinita, cyclois BEA mutabitur in lineam rectam axi OC perpendicularem; idcirco etiam in hac recta corpora codem tempore ad punctum A appellent, sive initium motus puncto A vicinissimum sit, sive etiam ab eadem remotissimum fuerit.

SCHOLION.

Post Propositionem XXI commodus locus fuisset de conatu centrifugo ex motu circulari oriundo aliqua adjiciendi sed quia ejusmodi conatuum centrifugorum theoria nonnulla circa motum pendulorum praesupponit, siquidem plene sit tractanda, ideo etiam in hunc locum erat differenda. Postquam Hugenius veras horum conatuum leges aperuit ad calcem Horologii sui Oscillatorii, Illustr. Marchio Hospitalius Hugeniana theoremata, absque demonstratione ab Autore suo proposita, demonstrata dedit in Actis Acad. Reg. Par. Scient. 1700, loquor de omnibus Hugenianis theorematibus de vi centrifuga : nam diu ante Hospitalium Newtonus nonnulla eorum demonstraverat in *Philosoph: Nat. Princ. Math.* viamque aperuerat, cui insistendo reliqua omnia possent expediri ; post laudatissimos hosce Geometras plures alii Autores Hugenii 13. theoremata de vi centrifuga demonstrate conati sunt, nonnulli laudabili successu, alii vero non item, utpote qui paralogismis nonnullis demonstrationes suas foedarunt, ut facile ostendi posset, si modo id e re esset. Sed ad rem : cum filum quoddam alicubi affixum alterique suo capiti annexum habens pondusculum, circa punctum fixum conversum describit circulum, id in plano moveri dicetur, cum reapse in plano circuli moveatur; sin vero filum motu suo superfiem conicam describit; id *Pendulum conicum* deinceps dicetur.

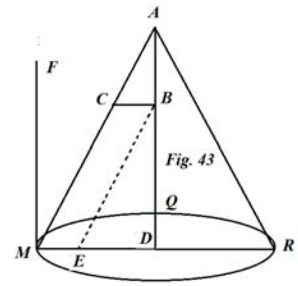


Fig. 43

183. Fig. 43 Sit ergo filum DM clavo in D affixum, in cujus extremitate annexum sit corpus M, atque filum circa punctum D in plano conversum describat circulum RQM, cujus radius DM dicatur simpliciter R, conatus centrifugus C, velocitas qua mobile M in sua circumferentia aequabiliter revolvitur, V, & haec velocitas, tanta sit quantam idem mobile M acquireret motu naturaliter accelerato a quiete in F post casum perpendicularem ex altitudine FM; quam *Altitudinem determinatricem* posthac vocabimus, atque litera D insigniemus. Tempus, quo mobil e circumeundo peripheriam RQM unum circuitum absolvit, sit T; ac denique ratio peripheriae ad semidiametrum sit ut p ad 1, adeo ut peripheria radii R, sit *p*R . Ipsa vero gravitas, ut jam alibi factum, dicetur G, quae in praesenti materia non variabilis sed uniformis est consideranda.

Hisce jam positis, supra (§.119) dictum est solicitationem gravitatis cuilibet curvae normalem coercere conatum mobilis a curva juxta directionem tangentis recedendi, atque adeo ejusmodi conatui generaliter aequalem esse in omni curva ; ac propterea etiam in circulo. Generaliter itidem (§.154) demonstratum est, respectu cujuslibet curvae a mobili quodam describendae, quadratum celeritatis in quolibet curvae puncto aequivalere rectangulo sub radio osculi seu curvaturae & rectae, quae solicitationem curvae perpendicularem ex centrali derivatam exponit *;* necesse est ut hoc idem etiam valeat in circulo in *specie* ; verum in circulo radius curvaturae est ejus radius R, & solicitatio . circumferentiae perpendicularis conatui centrifugo est aeqitalis est, ac velocitas in quolibet peripheriae puncto nobis dicitur V. Propterea vi citati theorematis (§.154)
aeq.I. $V^2 = R.C.$

I. Deinde quia motus aequabilis est in circulo erit $T = pR : V$, hoc est, tempus innotescit applicando peripheriam tanquam spatium transmissum ad velocitatem, qua id percurritur, ergo etiam aeq.II. $T = pR : \sqrt{RC} = p.\sqrt{(R:C)}$. Denique, Si conatus centrifugi conferendi sint cum gravitate G, tertia formula opus est quam articulus 150 suppeditat
aeq.III $2\,D.G = V^2$.

Hinc 1°. formulae prima & tertia efficiunt $R.C = 2.D.G$ atque adeo $C : G = 2D : R$ hoc est, eo natus centrifugus se habet ad gravitatem, ut dupla lineae determinatricis ad radium circuli ; atque adeo ubi haec determinatrix altitudo D semissem radii R aequaverit, conatus centrifugus gravitati aequalis erit. Quod est theor.5 Hugenii.

2°. Si in diversis circulis tempora periodica T sunt aequalia etiam ipsae $p.\sqrt{(R:C)}$ aequabuntur, atque adeo conatus centrifugi radiis directae proportionales erunt.

3°. Si T ut R^n, erit V seu $pR : T$, ut R^{n-1} inverse, atque adeo C seu $ppR : T^2$ erit reciproce ut R^{2n-1}. Proinde si conatus centrifugus fuerit reciproce ut T^2, erit tempus periodicum in ratione sesquiplicata radii R, & in hoc consistit celebre Kepleri theorema. Reliqua, quae eadem facilitate ex tribus praecedentibus formulis elici possunt, Lectoris industriae relinquimus, atque adeo ad contemplationem pendulorum conicorum absque ulterioribus ambagibus accedimus.

184. Si filum AM motu conico moveatur circa axem AD horizonti rectum, adeo ut pondus ipsi annexum M circumferentiam circuli MQR describat; id fieri non potest, quin praeter gravitatem secundum directionem ipsi AD parallelam in corpus M agentem eidem mobili insit conatus alius secundum directionem DM agens. Nam ut filum AM in situ hoc sub angulo DAM ad axem inclinato detineatur duabus viribus aut solicitationibus lateralibus AB & BC opus est, quarum prior gravitatem exponit, altera vera BC nonnisi a motu circulari provenire potest, ex motu vero centrali resultat conatus centrifugus, ergo BC repraesentat ejusmodi conatum centrifugum. Agatur igitur BE aequidistans lateri AM; fietque $EM = BC$. Ponantur insuper $AD = A$, $AM = L$, $AB = G$, & $BC = EM = C$, radius $DM = R$, peripheria $MRQ = pR$, & perinde ac supra V celeritas mobilis in circumferentia MQ ; quibus praesuppositis, erit iterum, ut in antecedenti paragrapho, tempus unius circuitus T mobilis M in peripheria MQR, dum filum AM superficiem conicam describit $= p\sqrt{(R:C)}$ verum ob parallelas AM & BE, est $AD:AB = DM:EM$, hoc est; $A:G = R:C$, ergo etiam $T = p\sqrt{(A:G)}$, haec vero expressio etiam significat (§:175, 178.) tempus duarum oscillationum minimarum lateralium penduli cujus A sit longitudo ; magnitudo vero $\sqrt{(A:G)}$ significat tempus descensus perpendicularis alicujus gravis ex altitudine $\frac{1}{2}A$. Propterea tempora circuitus pendulorum conicorum sunt in subduplicata ratione altitudinum *A* conorum. Superfluum duco reliqua, circa ejusmodi pendula ab Hugenio indicata, theoremata operose demonstrare, cum ex hisce positis principiis sponte sua facillime fluant, ceu quilibet videbit, qui animum iis advertere velit. Priusquam tamen ad alia transeam contemplabimur paulisper problema ab ingeniosissimo Joh. Bernoullio olim propositum quodque ab lllustr. Hospitalio solutionem nactum est; sed quoad arguendi formam a nostra differentissimam.

PROPOSITIO XXX. PROBLEMA.

185. Fig. 44. *Invenire curvam ABN ejus conditionis, ut in ea descendens grave A motu naturaliter accelerato, eandem in singulis punctis premat vi ubique aequali ponderi corporis absoluto*.

I. Sint CD axis curvae quaesitae, CA prima ordinata in cujus continuation AF exponat gravitatem seu pondus absolutum mobilis A, atque super descripto semicirculo AGF, agantur FG tangenti curvae in puncto B, & F*g* tangenti in *b* aequidistantes, junganturque AG, A*g*. Radius deinde circuli osculatoris curvae in puncto B, id est, BZ producatur in L, usquedum $\mathrm{BL} = \mathrm{AF} = \mathrm{BI}$ quae sumta est in ordinata curvae DB ultra curvam prolongata. Ipsis *AG*, A*g* fient aequales AM, A*m*, &. denique ex puncto I demittatur, perpendicularis IK ad lineam BL, eritque $\mathrm{BK} = \mathrm{AG} = \mathrm{AM}$, atque adeo $\mathrm{MF} = \mathrm{KL}$

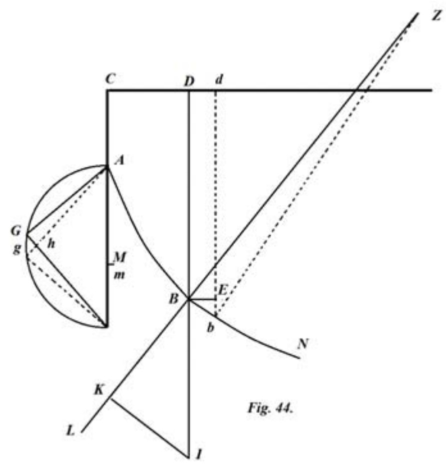


Fig. 44.

II. Quoniam BI exponit, pondus absolutum mobilis A vel B , ipsa BK exponet pressuram, quam pondus in curvae punctum B exeret secundum directionem BL, sed praeter hanc pressionem aliam insuper sustinebit idem curvae punctum B a conatu centrifugo mobilis, ac per consequens hic conatus exponi debebit per KL, quandoquidem pressio totalis BL aequari debet (secundum hypothesin) ipsi AF vel BI. Dicatur celeritas acquisita in B, V; eritque (§. 154) $\mathrm{V}^2 = \mathrm{BZ.KL} = \mathrm{BZ.MF}$,verum (§.150.) est etiam $\mathrm{V}^2 = 2\mathrm{AF.DB}$, ergo $\mathrm{BZ.MF} = 2.\mathrm{AF.DB}$; hinc $\mathrm{B}b.\mathrm{MF} : \mathrm{BZ.\ MF} = \mathrm{Bb.\ MF} : 2.\mathrm{AF.DB}$. Atqui similitudo sectorum BZb, GA*b*, & *g*F*h* praebet $\mathrm{B}b : \mathrm{BZ} = \mathrm{M}m(gh) : g\mathrm{F}$*;* atque ex similitudine triangulorum B*b*E, & FAG elicitur $\mathrm{B}b : \mathrm{AF} = \mathrm{E}b : \mathrm{FG}$; propterea subrogando in antecedenti analogia loco B*b*, & BZ proportionales M*m* & FG, atque loco B*b* & AF, proportionales E*b* & FG, eaque mutabitur in hanc alteram $\mathrm{M}m.\mathrm{MF} : \mathrm{FG.MF} = \mathrm{E}b.\mathrm{MF} : 2.\mathrm{DB.FG}$, vel $2.\mathrm{MF.M}m : \mathrm{FG.MF} = 2.\mathrm{Eb.MF} : 2.\mathrm{DB.FG}$ vel ducis consequentibus in MF: FG, erit $2\mathrm{MF.M}m : \mathrm{MF}^2 \left(= 2\mathrm{E}b.\mathrm{MF} : 2\mathrm{DB.MF}\right) = \mathrm{Eb} : \mathrm{DB}$. Unde, quoniam decrementum 2MF.M*m* est ad decrescentem MF^2 sicut incrementum E*b* crescentis DB ad hanc crescentem, erit (§.153.) crescens DB ad suam primam magnitudinem CA, ut decrescentis prima magnitudo, quae est FA^2 ad decrescentem MF^2 ; adeoque sumta P media proportionali inter AC & DB, fiet $\mathrm{P}^2 : \mathrm{AC}^2 = \mathrm{AF}^2 : \mathrm{MF}^2$ vel $\mathrm{P} : \mathrm{AC} = \mathrm{AF} : \mathrm{MF}$, & convertendo $\mathrm{P} - \mathrm{AC} : \mathrm{P} = \mathrm{AG} : \mathrm{AF}$, hinc $\mathrm{AF}^2 : \mathrm{AG}^2 = \mathrm{P}^2 : \mathrm{P}^2 - 2\mathrm{AC.P} + \mathrm{AC}^2$, & dividendo $\mathrm{FG}^2 : \mathrm{AG}^2 = 2.\mathrm{AC.\ P} - \mathrm{AC}^2 : \mathrm{P}^2 - 2.\mathrm{AC.P} + \mathrm{AC}^2$, & $\mathrm{FG} : \mathrm{AG} = \sqrt{\left(2.\mathrm{AC.P} - \mathrm{AC}^2\right)} : \mathrm{P} - \mathrm{AC} = \mathrm{E}b{:}\mathrm{BE}$; adeoque, si dicantur AC, *a*; CD, *x*; DB,

y; Eb, dy, erit $p=\sqrt{ay}$, & $2.pdp=ady$, & $\mathrm{FG}:\mathrm{AG}=\sqrt{(2.\mathrm{AC.P}-\mathrm{AC}^2)}:\mathrm{P\text{-}AC}$, fiet in his symbolis $dy:dx=\sqrt{(2ap-aa)}:p-a$, ergo $pdy-ady=dx\sqrt{(2ap-aa)}$, vel $apdy-aady=2ppdp-2apdp=adx\sqrt{(2ap-aa)}$, atque adeo $dx=\left(2ppdp-2apdp:a\sqrt{(2ap-aa)}\right)$ quae mutatur in $dx=\left(q^4dq-a^4dq\right):2a^4$, cujus integralis est $10a^4x=q^5-5a^4q+4a^5$ si scilicet positum fuerit $q=\sqrt{(2ap-aa)}$, atque retrogradiendo invenietur $q=\sqrt{(2a\sqrt{ay}-aa)}$ atque adeo aequatio curvae quaesitae$\left(aa+(y-a-\sqrt{ay}\right)$ in $\left(2a\sqrt{ay}-aa\right)=\frac{5}{2}ax$. Quod erat demonstratum.

COROLLARIUM.

186. Quoniam invenimus $\mathrm{BZ}:\mathrm{DB}=2.\mathrm{AF}:\mathrm{MF}$ & $\mathrm{P}:\mathrm{CA}=\mathrm{AF}:\mathrm{MF}$, vel $2\mathrm{P}:\mathrm{CA}=2\mathrm{AF}:\mathrm{MF}$, erit pariter $2\mathrm{P}:\mathrm{CA}=\mathrm{BZ}:\mathrm{DB}$ & invertendo $\mathrm{CA}:\mathrm{P}=\mathrm{DB}:\mathrm{BZ}$, atque adeo radius evolutae in quolibet curvae puncto ope hujus analogiae innotescit, quae analogia consentit cum ea, quam Ill. Marchio Hospitalius tradidit in Actis Acad. Reg. Scient. Paris. 1700, ab initio.