CHAPTER II.

Concerning curvilinear motions in vacuo, under any hypothesis of the variability of gravity.

PROPOSITION XX. LEMMA .

153. Fig. 35. *If , with the magnitude* EF *increasing continually from that initial magnitude* AI : *the infinitesimal increment* Ff *always will be to the decrement* qr*, the decrease of the other magnitude* Dq*, from which initial magnitude* DR *clearly it begins to decrease : just as the increase* EF *will be to the decrease Dq ,* EF *will be to* Dq ; *the one increasing from the initial* AI, *the other decreasing from the initial* DR. [*i.e.* in Fig. 35, $Ff : qr = \mathrm{FE} : \mathrm{D}q$]

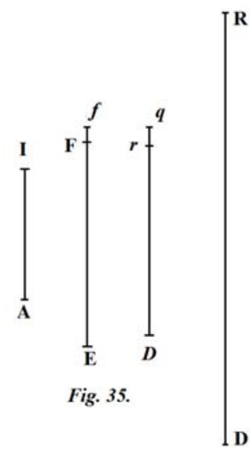


Fig. 35.

In Fig. 36. some curve IF shall be described about the axis DR, of which the abscissas refer to the decrease Dq, and its initial magnitude DR, truly the ordinates FE and IA, refer to its increase and to its initial magnitude, indeed in order that the labels q, E pertain to the same point, and likewise the labels A, R. Through the point F of the curve the tangent FO shall be drawn crossing the axis at O, and with the ordinate *fe* made infinitely close to the other FE, *fa* will be the increase of the increment FE, and either Fa, *qr* or *Ee* the infinitesimal decrement of the decreasing Dq. And therefore there will be (according to the hypothesis) $fa : qr = \mathrm{FE} : \mathrm{D}q$. Truly the similar triangles faF, FEO also produce $fa : qr = \mathrm{FE} : \mathrm{DO}$, therefore $\mathrm{FE} : \mathrm{D}q = \mathrm{FE} : \mathrm{EO}$, and thus $\mathrm{D}q = \mathrm{EO}$, hence with the tangent OF produced, also there will be $\mathrm{OF} = \mathrm{FS}$. Therefore the point of contact F is everywhere in the middle of the line OS, and of the subtended the right angle SDO, thus so that by the converse of Prop. III. of the Second Book of Conics of Apollonius, the curve IF shall be a hyperbola between the asymptotes SD and DO, and thus EF shall be to AI, just as DR to D*q*. Q.E.D.

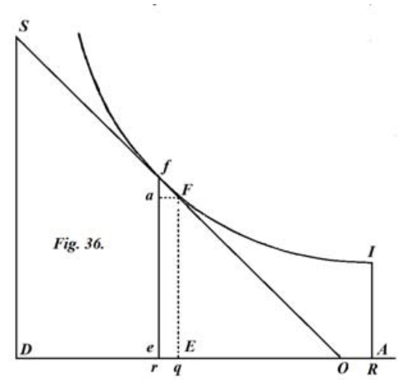
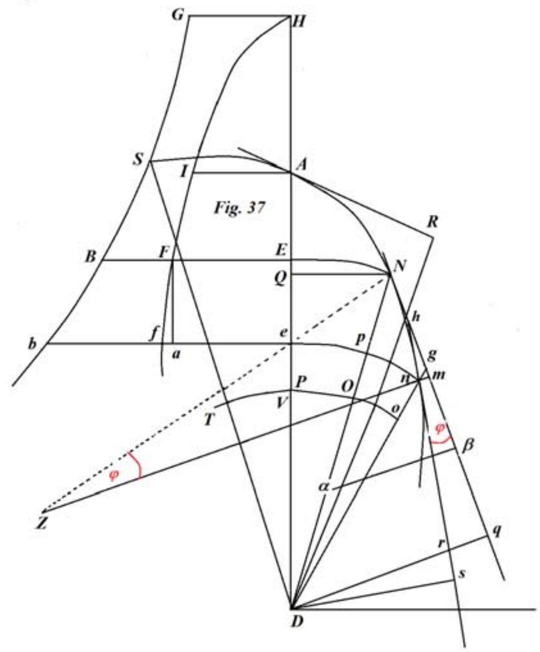


Fig. 36.

[Thus, for example in modern terms, if we take DR as the x-axis, and DS as the y-axis, then we can write $\frac{dy}{dx} = -\frac{y}{x}$, integrating to give $xy = x_o y_o$; or $\mathrm{D}q.\mathrm{EF} = \mathrm{DR.AI.}$]

PROPOSITION XXI. THEOREM.

154. *With the diagram* GBb *of the central forces acting, and* HIF *the diagram of the speed acquired of a body, beginning to fall along the right line* HD *from rest at* H ; *and this body itself propelled in the direction* AR *with the speed* AI*, which the body can acquire at* A *after falling from the height* HA*, with the forces acting from these centres indicated by the coordinates of the curve GB, will describe a certain curve AN in vacuo.* FE *will be the speed of the moving body at some point* N *of the curve or the force will be equal to the rectangle under the radius of osculation* Nz *at this point, and* $\beta\alpha$ *, which signifies the force* Nα *or* BE *derived perpendicular to the curve from the central force.* That is, $\mathrm{EF}^2 = n\mathrm{Z}.\beta\alpha$.



For the moving body A projected along AR with the speed AI describing the curve AN, it will acquire the speed EF at point N of this curve (§.138.), as also shall be the case for a body at the point E, equally distant from the centre D, as the point N of the curve [we would now regard such curves as equipotentials], and to be a consequence of falling from the height HE, and with this speed EF in the direction of the tangent N*q* to the curve the original body is trying to approach equality in the motion, truly resulting from the action of the central force Nα or EB borne out in the time increment, as by always acting uniformly in the direction Nq, it may be drawn back to be retained on the curve N*n*; thus indeed so that the incremental line *gn* parallel to ND shall itself be at an incremental distance from the force with respect to the uniform element of the curve N*n* arising from Nα or BE, and thus the time (§.151.), in which the weight may complete such a small distance from the force Nα, shall become $\sqrt{(2gn:\mathrm{N}\alpha)}$ with the mass of the object A assumed to be unity [*i.e.* essentially from the linear formula, applied to linear incremental changes: $\Delta s = \frac{1}{2} g\Delta t^2$]; but truly in that time, in which *gn* is generated by the action of the central force, the body will be described on the tangent with the speed EF equal to a motion as small distance N*g* or N*m* ; which time (§.128.) is also N*m*: EF [*i.e.* in a linear approximation] ; therefore $\sqrt{(2gn:\mathrm{N}\alpha)} = \mathrm{N}m:\mathrm{EF}$, and thus $2gn:\mathrm{N}\alpha = \mathrm{N}m^2:\mathrm{EF}^2$, or also $2gn.n\mathrm{Z}:\mathrm{N}\alpha.nZ = \mathrm{N}m^2:\mathrm{EF}^2$, and by interchanging, $2gn.n\mathrm{Z}:\mathrm{N}m^2 = \mathrm{N}\alpha.nZ:\mathrm{EF}^2$. But by considering an element of the curve N*n* in place of the arclet of the circle of osculation Z, of which the tangent is N*m*, there will be $\mathrm{N}m^2 = 2n\mathrm{Z}.nm$ or $= 2mn.n\mathrm{Z}$, [From similar triangles : $nm / \mathrm{N}m = \mathrm{N}m / 2n\mathrm{Z}$], therefore $2gn.\ n\mathrm{Z} : 2mn.\ nZ(\text{or } \mathrm{N}m^2) = gn.nZ : mn.n\mathrm{Z} = \mathrm{N}\alpha.nZ : \mathrm{EF}^2$; or with the right lines $\mathrm{N}\alpha, \beta\alpha$ put in place of the line increments *gn,* & *mn,* which by themselves, on account of the similar triangles $\mathrm{N}\alpha, \beta\alpha$ & *gnm* are proportional, there becomes $\mathrm{N}\alpha.n\mathrm{Z} : \beta\alpha.n\mathrm{Z} = \mathrm{N}\alpha.nZ : \mathrm{EF}^2$. Therefore $\mathrm{EF}^2 = n\mathrm{Z}.\beta\alpha$. Q.E.D. [Thus, the component of the central force normal to the curve at the point N is given by $\beta\alpha = \frac{\mathrm{EF}^2}{nZ} = v_{\perp}^2 / \rho$.]

PROPOSITIlON XXII. THEOREM .

155. *With the same in place, the speed of projection* AI *to the speed of the moving body at the point of the curve* N, *will be reciprocally as the perpendicular* D*q upon the tangent* N*q at the point* N *to the curve* N, *to the perpendicular* DR *drawn from the centre* D *to the projected direction* AR. *And thus the individual rectangles* D*q*.EF *will be equal to the given rectangle given* DR.AI.

For with the tangent *ns* drawn through the point *n* and with the arc of the circle *ne* described from the centre D, and finally NZ may be joined, with the ordinate *ef*, and the perpendicular D*s* may be dropped to the tangent of the curve *ns*. With which in place, Nβ will be the tangential force, and the moment of that (§.131) is equal to the moment of the velocity acquired at N, and thus $\mathrm{N}\beta.\ \mathrm{N}m$ or $\mathrm{N}n = \mathrm{EF}.af$; and because (§.154) $\mathrm{EF}^2 = n\mathrm{Z}.\beta\alpha$, there arises $\mathrm{EF}.af : \mathrm{EF}^2 = \mathrm{N}\beta.\mathrm{N}m : nZ.\beta\alpha$ or with the proportional magnitudes N*q* and D*q* substituted in place of $\mathrm{N}\beta$ and $\beta\alpha$, and with the proportionals *qr* and *hq* or N*q* in place of N*m* and *m*Z or *n*Z, there will be found $\mathrm{EF}.af : \mathrm{EF}^2 = \mathrm{N}q.qr : \mathrm{N}q.\mathrm{D}q$. For since ZN and Z*n* are perpendicular to the tangents Nq and *ns*, the angles NZ*n* and *ghs*, which the tangents contain, by necessity are equal [marked φ in Fig. 37] ; and thus the triangles NZ*m* and *qhr* are similar with right angles at N and *q*, and with the sides N*m* and NZ or *n*Z to be proportional to the homologous sides *qr* and *hq,* or N*q* in the triangle *qhr.* [There is no point *h* marked in the original diagram, which must refer to the intersection of the two tangent lines differing by an increment at N and *n* on the curve, and which I have inserted; hence we may refer to N as equivalent to *h.*] Truly in the last ratio with EF cancelled from the first and second, and N*q* from the third and fourth terms, there shall be $af : \mathrm{EF} = qr : \mathrm{D}q$ and by interchanging $af : qr = \mathrm{EF} : \mathrm{D}q$, that is the increment of the increase FE is to the decrement of the decrease D*q*, just as the increase and decrease themselves, and thus (§. 153) the increase FE will be to its initial magnitude AI, just as of the decrease *Dq* from the first magnitude DR to the decrease , and by inverting , the speed of the projectile AI is to EF the

speed of the body at any point N of the curve, as the perpendicular *Dq* to DR. Therefore the individual rectangles FE.D*q* will be equal to the given AI. DR. Q.E.D.
[This discussion amounts to the conservation of angular momentum of a unit mass in a plane trajectory attracted by a centre of force at D, for which we would now write $R\times V = r\times v = DR\times AI = Dq\times FE$; for which we can write $R\times\delta V + \delta R\times V = 0$ or $\frac{\delta V}{V}+\frac{\delta R}{R}=0$, or $af:\mathrm{EF} = qr:\mathrm{D}q$, etc., where *R, r* are the perpendicular distances to the tangent from D and *V*, *v* are the corresponding speed along the tangents at the respective points on the curve.]

COROLLARY I.

156. Because $\mathrm{EF:AI = DR:Dq}$, or $\mathrm{EF}^2:\mathrm{AI}^2 = \mathrm{DR}^2:\mathrm{D}q^2$, & $\mathrm{EF}^2 = \beta\alpha.n\mathrm{Z}$, there will be $\beta\alpha.n\mathrm{Z}:\mathrm{AI}^2 = \mathrm{DR}^2:\mathrm{D}q^2$, but $\beta\alpha.n\mathrm{Z}:\mathrm{N}\alpha.n\mathrm{Z} = \beta\alpha:\mathrm{N}\alpha = \mathrm{D}q:\mathrm{DN}$ [from similar triangles], or on inverting, $\mathrm{N}\alpha.n\mathrm{Z}:\beta\alpha.n\mathrm{Z} = \mathrm{DN}:\mathrm{D}q$: therefore from the equation and by the arrangement of the ratios there will be found $\mathrm{N}\alpha.n\mathrm{Z}:\mathrm{AI}^2 = \mathrm{DR}^2.\mathrm{DN}:Dq^2$,
and therefore $\mathrm{N}\alpha = \mathrm{AI}^2.\mathrm{DR}^2.\mathrm{DN}:n\mathrm{Z}.\mathrm{D}q^3$; therefore the central force at any point N of the curve is as $\mathrm{DN}:n\mathrm{Z}.\mathrm{D}q^3$.

$$[\frac{\beta\alpha.n\mathrm{Z}:\mathrm{AI}^2}{\beta\alpha.n\mathrm{Z}:\mathrm{N}\alpha.n\mathrm{Z}} = \frac{\mathrm{DR}^2:\mathrm{D}q^2}{\mathrm{D}q:\mathrm{DN}} \text{ or } \frac{\mathrm{N}\alpha.n\mathrm{Z}}{\mathrm{AI}^2} = \frac{\mathrm{DR}^2:\mathrm{D}q^2}{\mathrm{D}q:\mathrm{DN}}, \text{giving } \mathrm{N}\alpha = \frac{\mathrm{AI}^2.\mathrm{DN.DR}^2}{\mathrm{D}q^3.n\mathrm{Z}}, \text{ or } \mathrm{N}\alpha \propto \frac{\mathrm{DN}}{\mathrm{D}q^3.n\mathrm{Z}}.]$$

As the most celebrated of men Johann Bernoulli, Abraham de Moivre, and Guido Grandi have found, all maintaining this principle, that the times spent on curves are proportional to the areas, which we have not yet supposed; but we will deduce in the following Corollary from our basics put in place.

COROLLARY II.

157. Now the time will be, in which some arc of the curve AN will be described, that is: $\mathrm{tAN} = \text{area ADN}:\frac{1}{2}\mathrm{AI.DR}$. For there is (§.128.) :
$\mathrm{EF.\ tN}n = \mathrm{N}n$; and $\mathrm{D}q.\mathrm{EF.tN}n = \mathrm{D}q.\mathrm{N}n = 2.\text{triangle ND}n$, and (§.155) $\mathrm{D}q.\mathrm{EF} = \mathrm{AI.DR}$, therefore $\mathrm{AI.\ DR.\ tN}n = \text{triangle ND}n$, and thus $\int \mathrm{AI.\ DR.\ tN}n = \mathrm{AI.DR.tAN} = 2\int \mathrm{ND}n = 2.\text{area ADN}$,
hence $\mathrm{tAN} = \text{area ADN}:\frac{1}{2}\mathrm{AI.DR}$. And hence it is clear the times in which the different arcs AN, A*n* will be described, to be in proportion to the homologous areas AND, AN*n*D ; just as the most illustrious Newton first showed that in Prop. I. Book. I. *Prin, Phil. Nat. Math.* , but from the most different foundations.
[See *Reading the Principia*, by Niccolo Guicciardini, p. 211 onwards, and his article in *Historia Mathematica* **23** (1996), 167–181: An Episode in the History of Dynamics: Jakob Hermann's

Proof (1716–1717) of Proposition 1, Book 1, of Newton's *Principia* ; unfortunately, N.G. did not continue to examine the following propositions, esp. Prop. 41 of Sect. 8, which are related to the area law, and which consider numerous cases. Otherwise, Chandrasekhar vindicates Newton fully in his account of *Newton's Principia….*, and, a full translation of the 3rd Ed. of the *Principia* is available on this website, by this writer.]

SCHOLIUM.

158. This formula of the second corollary is elegant, because the value of the central force on any curve may be shown expressed by a finite quantity, but because the value of the radius [of curvature] enters into the same evaluation, it has the effect, that sometimes the calculation becomes a little more involved. Therefore I may prefer in practice to follow the rule $g = dp : p^3dz$, in which g signifies the gravity or the force of the central attraction, p the perpendicular from the centre of attraction to the tangent of the curve dropped at a given point, and z the right line radius, or the distance of the point of the curve, at which the magnitude of the central attraction is sought on a mobile body, from the centre. The demonstration of this is easy, and indeed by putting the rectangle $\mathrm{AI.DR} = 1$, there becomes also $\mathrm{FE.D}q = 1$, that is, by calling FE above, u, since Dq now shall be called p, there will be $pu = 1$ and $ppuu = 1$, or $uu = 1 : pp$, and thus $udu = -dp : p^3$. Now because the moment of the speed [*i.e. udu*] is equal to moment of the central force acting -gdz. (moreover I put − dz, because with u increasing p decreases, and as a consequence z, and thus +du itself must have a negative sign with the homologous dz) hence $-gdz = -dp : p^3$, & $g = dp : p^3dz$.

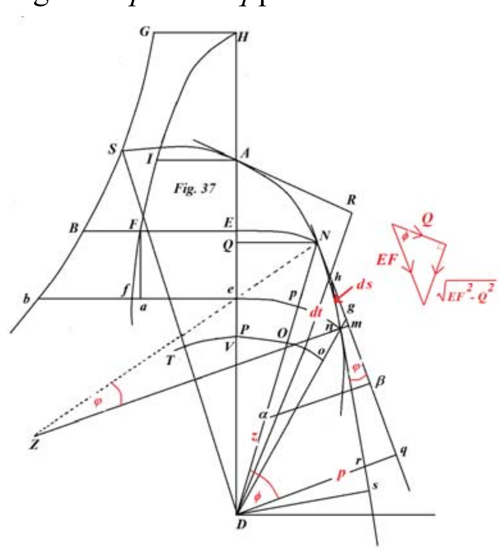
159. The use of this formula has been set out well enough; for from the equation of a given curve the value of p and z is sought and with constants, also there is a need in the formula praised above for a determination of this kind. For example the following equation is found for the hyperbola and the ellipse $pp = ccz : 2a \pm z$, where a denotes the transverse half-width, and b the distance from the centre of the section to each focus, and $cc = bb - aa$ in the hyperbola, and $cc = aa - bb$ in the ellipse. [In modern terms, where these curves are given by $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$, a retains its meaning, but Hermann's $b = ae$, where e is the eccentricity, a concept not yet known; hence Hermann's equation $cc = aa - bb$ becomes $cc = aa - aaee = b^2$ for the ellipse with $e < 1$, and similarly for the hyperbola, with $e > 1$.] In the denominator of the above fraction the upper sign of the fraction is with regard to the hyperbola, truly the lower for the ellipse. Therefore $1 : pp = (2a \pm z) : ccz = 2a : ccz, \pm 1 : cc, [i.e. \frac{c^2}{p^2} = \frac{2a}{z} \pm 1]$ and by differentiating, there will be $-2dp : p^3 = -2adz : cczz$ & $g = dp : p^3dz = a : cczz$; or with the given a:cc reduced to 1, there will

be g as 1: zz ; that is the equation of the central force, directed towards the focus of the conic sections, is everywhere inversely as the square of the distance of the moving body from the focus, which now agrees everywhere with the others.
[The pedal forms of the equations for the ellipse and hyperbola [see Lockwood, *A Book of Curves*] can be used to illustrate the motion of a particle in an inverse square gravitational field: these curves take the form $\frac{b^2}{p^2} = \frac{2a}{r} \pm 1$ with $+1$ for the hyperbola, and -1 for the ellipse. In modern terms we can write down an energy conservation equation :
$\frac{1}{2}mv^2 - \frac{GmM}{r} = \text{constant};$ while conservation of angular momentum gives $mvp = h,$ a constant , where p is the perpendicular distance from the attracting focus to the tangent to the curve, assuming the central mass of the sun M is much greater than that of the planet m, then on substitution of the latter into the former equation, we have $\frac{\frac{1}{2}mh^2}{p^2} - \frac{GmM}{r} = \text{constant};$ as above. On differentiation, the inverse square form of the central force can be found. Thus, motion arises from an inverse square law force with total energy less than or greater than zero, along elliptic or hyperbolic curves respectively. Here of course, Hermann is considering a more general form of the central force applicable to enumerable situations.]

160. Again if an element of the curve Nn may be called ds, with the arclet pn, dt, and with the rest remaining the same, as above, (§. 158). The similar triangles Nnp & NDq provide

$p = zdt : ds$ [*i.e.* $dp = dzdt : ds + zddt : ds - zdtdds : ds^2$] ,
and thus

$g = dp : p^3dz = dp : (zdt : ds)^3 \, dz$
$= (ds^2dtdz + zds^2ddt - zdsdtdds) : z^3dt^3$.



Which formula, except in the naming of terms, does not differ from the formula of Varignon, such as may be found in the Scientific Commentaries of the Royal Academy of Paris for 1701 and 1706, in which no more than at present is any of the differentials assumed constant except dt, dz, ds, but all to be variable.

161. The same may be found easily, or certainly found by insisting on the principles put in place above, and the formula can be set out quickly and generally from the radii of curvature of the motion. For the triangles NZm & $qhr,$ which have been shown above to be similar (§. 155.), give rise to hq or Nq : qr = NZ : Nm , and the similar triangles Nnp and NqD give rise to the other

ratio $\mathrm{ND}:\mathrm{N}q=\mathrm{N}m:\mathrm{N}p$, therefore from the equations $\mathrm{ND}:qr=\mathrm{NZ}:\mathrm{N}p$, and thus $\mathrm{NZ}=\mathrm{ND}.\mathrm{N}p:qr$, that is with the symbols assumed above ; and in addition by calling NZ, *r*; there will be $r=zdz:dp$, for *qr* in the figure is [the infinitesimal] *dp* and in terms of these symbols ; thus since *p* shall be $=zdt:ds$, there will be found $r=zdz:dp=zdzds^2:dsdtdz+zdsddt-zdtdds$, the general formula for the radius of osculation, in which no differential has been assumed constant, which hence can be transformed easily into an infinitude of others. And with the member *dsdtdz* deleted in the denominator, the formula will be had in the same manner $r=ds^2dz:dsddt-dtdds$ for curves the ordinates of which are parallel and perpendicular to the axis, in which *dt* are the elements of the abscissas.

PROPOSITION XXIII. PROBLEM.

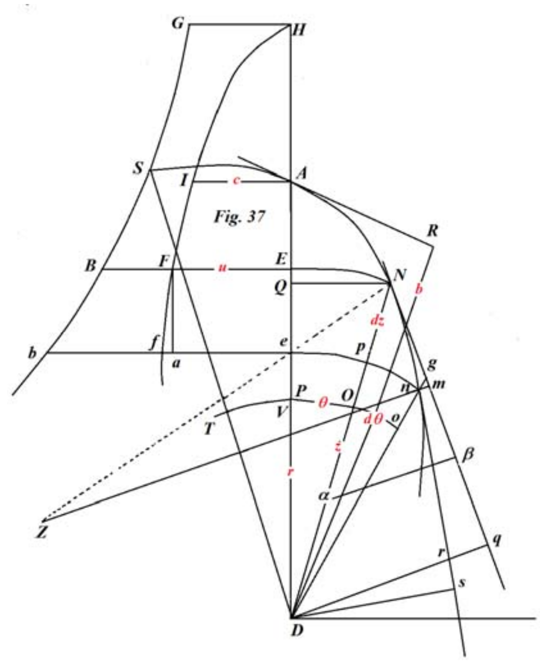
I62. Fig. 37. *From the given diagram of the central force* GBb, *with the speed initial* AI *given*, *and with* AR *the direction projected*, *to define and construct the curve that the body will describe in vacuo, with the quadratures of the figures granted.*
[Here the areas generated by the arcs traversed subtended at D are related in general to the previous results.]

I. The circle PO is described with centre D and with some arbitrary radius DP, cutting the radii DN, *Dn* at the points O, o ; and it is assumed Q to be the fourth proportional to DN, DR, & AI, [Thus, Q corresponds to the velocity perpendicular to DN ; the red force triangle, N*q*D, and N*pn* are similar, and $\mathrm{N}q:\mathrm{D}q=\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}:\mathrm{Q}=Np:pn$; clearly in the descent from A to N, the tangent to the curve goes through an angle corresponding to the arc PO.]
thus so that $\mathrm{DN.Q}(=\mathrm{DR.AI})=\mathrm{EF.D}q$, that is $\mathrm{DN}:\mathrm{D}q=\mathrm{EF}:\mathrm{Q}$, and $\mathrm{DN}^2:\mathrm{D}q^2=\mathrm{EF}^2:\mathrm{QQ}$; thus on dividing up [by subtracting 1 from both sides of the ratio] there may be had $\mathrm{Nq}^2:\mathrm{D}q^2=\mathrm{EF}^2-\mathrm{QQ}:\mathrm{QQ}$, and thus $\mathrm{N}q:\mathrm{D}q=\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}:\mathrm{Q}$, but just as N*q*:D*q* so the ratio of the radius to the tangent of the angle DN*q*, which tangent may be called T, and the radius is DP or DO, and hence $\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}:\mathrm{Q}=\mathrm{DP}:\mathrm{T}$ and thus $\mathrm{T}=\mathrm{Q.DP}:\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}$.
[See triangle added to Fig. 37 above in red; thus Q applied perpendicular to DP is equivalent $\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}$ applied to the radius T; note that the velocity EF acting along the tangent at N is resolved into components parallel and perpendicular to DN, where (z,ϕ) can be considered as the polar coordinates of N.]

II. On account of the similar arcs *np* & O*o*, there is $\mathrm{O}o:pn=\mathrm{DP}:\mathrm{DN}$, and $pn:\mathrm{N}p=\mathrm{T}:\mathrm{DP}$, therefore from the equation $\mathrm{O}o:\mathrm{N}p=\mathrm{T}:\mathrm{DP}$, hence

$\mathrm{O}o = \mathrm{T.N}p\mathrm{:DN}\left(\mathrm{no.I.}\right) = \mathrm{Q.DP.}Np : \mathrm{DN}\sqrt{(\mathrm{EF}^2 - \mathrm{QQ})},$

therefore $\mathrm{PO} = \int \mathrm{DP.Q.N}p : \mathrm{DN}\sqrt{\left(\mathrm{EF}^2 - \mathrm{QQ}\right)}$. That is, if it may be understood for some curve, of which the abscissa is DN, and of which the ordinate is $\mathrm{DP}^2.\mathrm{Q} : \mathrm{DN}\sqrt{\left(\mathrm{EF}^2 - \mathrm{QQ}\right)}$, the area of this curve placed between the ordinates of the abscissas DA and DE applied to the given line DP will provide a line, which if it were made equal to the arc PO everywhere, with the right line DON drawn through the end O of this, the arc being equal to the abscissa DE, will be the curve sought with its end at N. Now because the square of the ordinate EF is equal to twice the area GBEH given by the abscissa DE or DN and constants, either algebraic or transcendental, and Q is the fourth proportional to DE or ON, and given DR, AI, that equally will be given at DE and with constants, from which since all the quantities, which clearly are present in the ordinate of the figure being squared $\mathrm{DP}^2.\mathrm{Q} : \mathrm{DN}\sqrt{\left(\mathrm{EF}^2 - \mathrm{QQ}\right)}$, will be given at DN and with constants, and the quadrature of the figures shall be conceded ; thus it is made clear, what was required to be done.



COROLLARY .

163. If now DR may be called b; AI, c; DE or DN, z , the velocity of the curve at the point N, or FE , u ; the arc of the circle PO , θ ; its radius DP, r; moreover, the element of the increasing arc Oo, $+\,\mathrm{d}\theta$, truly the decreasing element Np of the indeterminate ND, $-\mathrm{d}z$. And thus $\mathrm{Q} = cb : z$ and thus the whole formula $\mathrm{O}o = \mathrm{Q.DP.N}p : \mathrm{DN}\sqrt{(\mathrm{EF}^2 - \mathrm{QQ})}$, will be able to be changed with these substitutions into $\mathrm{d}\theta = -bcrdz : z\sqrt{(\,uuzz - bbcc)}$, which is the differential equation of the curve sought ANn.

SCHOLIUM.

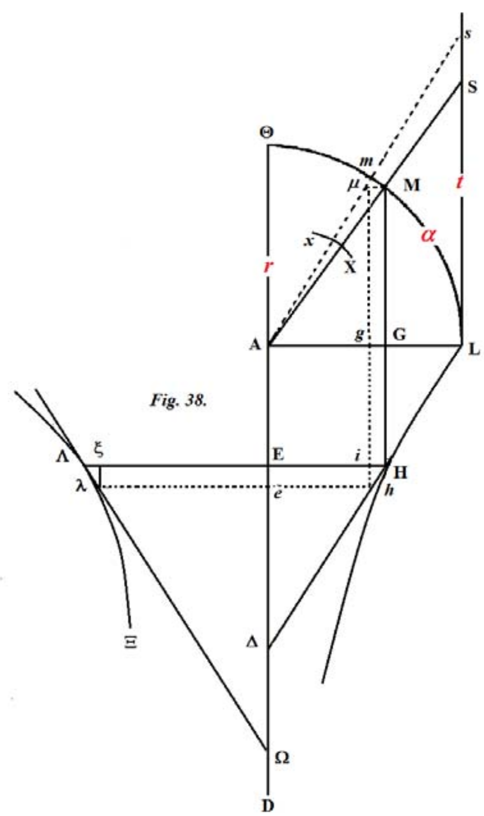
164. This problem has its initial solution in Prop. 41 Book I, *Princ. Phil. Nat. Math.* by the most celebrated Newton, and later from the most observant geometer Johann Bernoulli in two ways, while again by the most cel. Pierre Varignon in several way, where also our own solution can be referred to as shown in the Viennese Journal, Book V. p.318 et seq. with several other matters, and in Book VII. p. 194. [*Giornale de' Letterati d'Italia* : See Nagel's review in *Historia Math*.

(1991) for details of Hermann's publications.] Our present solution does not differ from that of Newton, except in trivial matters but not with the essential circumstances, for our EF indicates the side of the square of the area in Newton's work ABFD taken twice, as yet he takes that only once, and his Q for us is the rectangle with DR and AI given in our figure in, truly our Q is the same as for that Q:A or Z.

Otherwise, because this general solution presupposes the quadrature also of these curves, which are not quadrable, thus that problem generally taken is transcending, nor shall it be algebraic, except for certain laws of central attraction. Truly whatever these variable laws of gravity may be in general, so that from these in place algebraic curves of projection may arise, the problem is curious and elegant enough, but at first from the front certainly difficult, concerning which, as far as I know, no one at this stage has thought it out. Truly in whatever manner it may be set out, will become apparent in the following proposition, after the lemma being offered soon.

PROPOSITIlON XXIV. LEMMA.

165. Fig. 38. *The element* Ss *of the tangent* LS *of the circular arc* LM *is to the element of its arc* Mm, *as* AS^2 *the square of the secant to* AM^2 *the square of the radius, or also, as* AL^2 *the square of the radius, to* AG^2 *the square of the compliment of the sine of the arc* LM.



For, because the triangles SA*s* and MA*m* retain the common angle MA*m*, the area of triangle *SAs* will be to the area of triangle *Mam,* as the rectangle *SA.sA* to the rectangle MA.*m*A, that is, as the square SA, to the square MA. But because the altitude of triangle SA*s* is AL; with the presence of the base *Ss,* and the altitude MA of triangle Ma*m* having the base M*m* is equal to the other AL, it is clear the area of triangle SA*s* is itself had to the area of triangle MA*m*, as the base S*s* is to the base M*m,* therefore also Ss is to *Mm* as AS^2 ad AM^2, or on account of the parallel lines SL and MG, just as AL^2 to AG^2. Q.E.D.

$$\left[\frac{\Delta AmM}{\Delta AsS}=\frac{AM^2\delta A}{AS^2\delta A}=\frac{AM^2}{AS^2}=\frac{AM.mM}{AL.sS}=\frac{mM}{sS}\right]$$

COROLLARY.

166. Therefore, if the following may be called : the radius AL, r ; the tangent LS , t; the arc LM, α, the secant will be $\mathrm{AS} = \sqrt{(rr+tt)}$, and thus $d\alpha = rrdt : rr+tt$. And conversely from this equation converted into a series, the individual members of the series will be summable, thus so that from the sum of these an expression can actually be had of the arc of the circle α, for an infinite series expressed in terms of the tangent and radius.

[For, treating α as the angle, $\tan\alpha = \frac{t}{r}$; $\sec^2\alpha d\alpha = \frac{dt}{r}$; $d\alpha = \frac{dt}{r} \times \frac{rr}{tt+rr} = \frac{rdt}{tt+rr}$, and if α is now the arc, then $\frac{rrdt}{tt+rr}$.]

PROPOSITION XXV. PROBLEM.

167. *To find the general rule for the determination of variable gravity or the law of central forces, for indefinitely many algebraic curves expressed by finite quantities.*

I. In figure 38 make AD equal to AD in fig. 37, and in fig. 38 near the axis DA some algebraic curve LHh is understood to be described: and with the quadrant of the circle LMΘ described

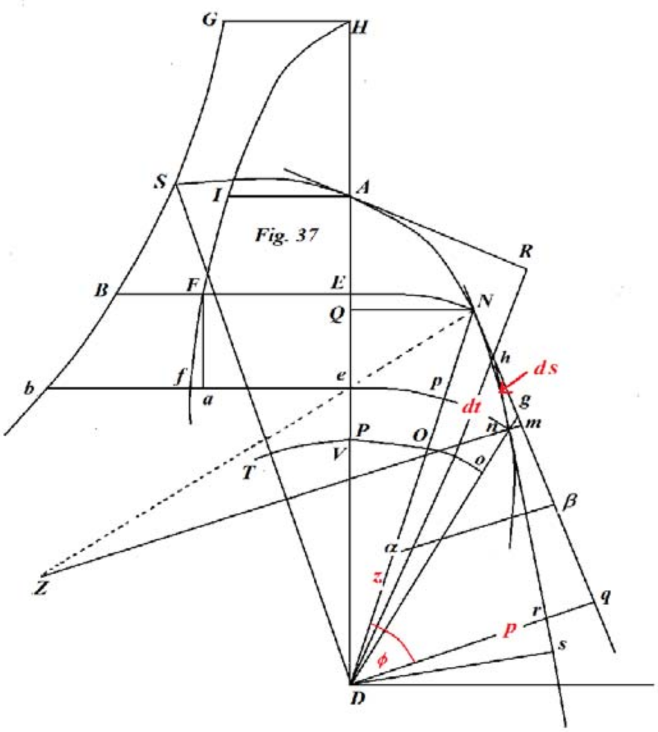


Fig. 37

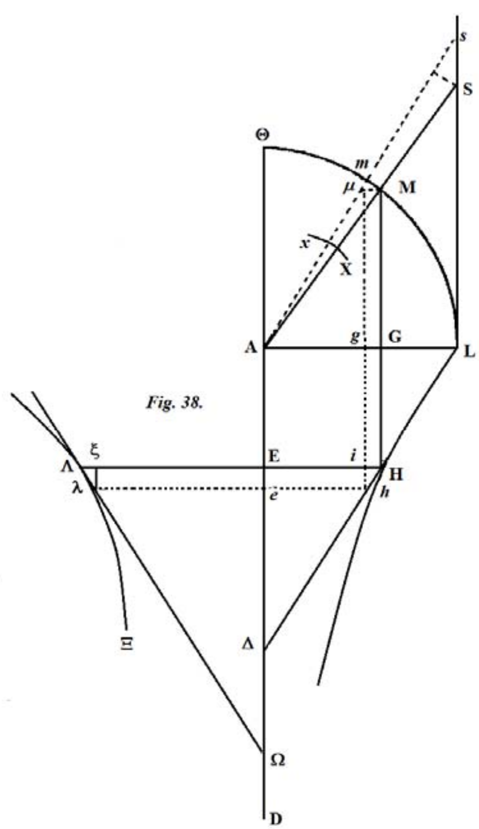


Fig. 38.

upon its ordinate AL, from any point M of this circumference some line MH may be acting parallel to the axis AD, crossing the curve LH somewhere at H under the ordinate AL, and through the point H of the curve LH the tangent HA is considered to be drawn crossing the axis at Δ, and with the ordinate HE perpendicular to the axis AD, at which with the ordinate HE extended to the other side of the axis, there is taken everywhere $E\Lambda$ equal to the subtangent $E\Delta$, thus so that the points Λ shall become a certain curve $\Lambda\Xi$, of which the subtangent at a point Λ shall become the line EΩ, evidently drawn from the tangent $\Lambda\Omega$. With which in place, if in the other figure (37) the angle ADN were everywhere to the angle MAL, as unity to some positive and rational number *n*, and in figure 37 the length DN always shall be equal to the length of the abscissa DE shown in figure 38, with the individual point located at N to have gone over to the algebraic curve AN*n,* when indeed all these points can be defined algebraically, since a section of the angle in the ratio of a number to number generally shall be able to be made geometrically. Now, just as from one or some other curve LH, yet another AN arises, thus any curve AN permits some other LH to be generated, thus as with this curve being considered as in general, without attending to any particular example, also it shall be possible to consider all possible curves AN*n*. Therefore, since the general rule will be sought, so that the law of variable gravity may be determined in all algebraic curves, the matter may be reduced to this, that a formula of the central attraction G may be found for this general curve AN, in as much as, that in the same manner, LH may result from a general algebraic curve. Thus it is required to proceed again with this inquiry.

II. Clearly with its element H*h* assumed on the curve LH, *hλ* and *hm* are acting through the point *h* to the equidistant lines ΛH & HM, and with the secants AS, A*s* drawn through the points M, *m* ; with centre A and with $AX = DP$ in figure 37, the element of the arc X*x* shall be described, where it will come about that, because (according to the hypothesis) $\text{ang.ADN} : \text{ang.MAL} = 1 : n$, also OD*o*:MA*m*, or X*ax*, that is the arclet O*o* shall be to the arclet X*x* just as 1 to *n*. Which thus I prove with these put in place.

III. The similar triangles H*ih* & HEΔ produce $\text{Ee (ih)} : \text{Gg}(\text{Hi}) = \text{E}\Delta : \text{AG}(\text{EH})$, and the similar triangles AMG & M*mμ* , give:

$\text{Gg}(\text{M}\mu) : \text{M}m = \text{GM} : \text{AL}(\text{AM})$ & $\text{M}m : \text{X}x = \text{AL}(\text{AM}) : \text{AX(DP)}$, and finally, from no. I of this section, $\text{X}x : \text{O}o = n : 1$, therefore from the equation and from the composition of the ratio, there becomes $\text{E}e : \text{O}o = n.\text{E}\Delta.\text{GM} : \text{AG.DP}$

[For

$(\text{Gg}(\text{M}\mu) : \text{M}m) \times \text{M}m : \text{X}x = (\text{GM} : \text{AL}(\text{AM})) \times \text{AL}(\text{AM}) : \text{AX(DP)}$ gives

$\text{Gg} : \text{X}x = \text{GM:AX} \therefore \text{X}x = \text{Gg.AX:GM}$ and

$\text{X}x : \text{E}e = \text{Gg.AX:GM.}Ee$. But $\text{E}e = \text{E}\Delta.\text{Gg} : \text{AG}$; $\therefore \text{X}x : \text{E}e = \text{AX:GM} \times \text{AG} : \text{E}\Delta$.]

(or, on account of the similar triangles AGM & ALS, $\mathrm{GM : AG = LS : AL}) = n.\mathrm{E\Delta.SL : DP.AL}$. But (§. 162. no. II.) there was also $\mathrm{E}e$ (there $\mathrm{N}p) : \mathrm{O}o = \mathrm{DE : T}$, therefore $n.\mathrm{E\Delta.SL : DP.AL = DE : T}$, and thus there becomes

$\mathrm{T = DP.\ AL.DE} : n.\mathrm{E\Delta.SL}\ (\S.162.n.1.) = \mathrm{Q.DP} : \sqrt{\left(\mathrm{EF}^2 - \mathrm{QQ}\right)}$ and thus

$\sqrt{\left(\mathrm{EF}^2 - \mathrm{QQ}\right)} = n.\mathrm{Q.E\Delta.SL : AL.DE}$, and finally

$\mathrm{EF}^2 = \left(nn.\ \mathrm{E\Delta}^2.\ \mathrm{SL}^2 + \mathrm{AL}^2.\mathrm{DE}^2\right).\mathrm{QQ : AL}^2.\ \mathrm{DE}^2$, or because $\mathrm{Q = DR.AI : DE}$, and on replacing DR.AI by the factor one, there is $\mathrm{QQ} = 1 : \mathrm{DE}^2$, and there becomes also

$$\mathrm{EF}^2 = nn.\mathrm{E\Delta}^2.\mathrm{SL}^2 : \mathrm{AL}^2.\mathrm{DE}^4;\ \ +1 : \mathrm{DE}^2.$$

IV. The elements of the individual members of this final equality may be taken [*i.e.* we can differentiate], and with the individual elements divided by 2 (that twofold division is immediately apparent from the calculation) there will emerge:

$$\mathrm{EF}.af = (nn.\mathrm{E\Delta.SL}^2.\Delta\xi : \mathrm{AL}^2.\mathrm{DE}^4) + (nn.\mathrm{E\Delta}^2.\mathrm{SL.S}s : \mathrm{AL}^2.\mathrm{DE}^4)$$
$$+(2nn.\mathrm{E\Delta}^2.\mathrm{SL}^2.\ \mathrm{E}e : \mathrm{AL}^2.\mathrm{DE}^3) + \left(\mathrm{E}e : \mathrm{DE}^3\right).$$

Truly because (§. I65.) $\mathrm{S}s : \mathrm{M}m = \mathrm{AL}^2 : \mathrm{AG}^2$, indeed $\mathrm{M}m : \mathrm{G}g = \mathrm{AL : MG}$ and finally $\mathrm{G}g : \mathrm{E}e = \mathrm{AG : E\Delta}$, there will be $\mathrm{SL.S}s = \mathrm{AL}^4.\mathrm{E}e : \mathrm{AG}^2.\mathrm{E\Delta}$, and the similar triangles $\Delta\xi\lambda$, and $\Delta\mathrm{E}\Omega$ provide $\mathrm{E\Delta}\left(\Delta E\right).\Delta\xi = \Delta E^2.\mathrm{Ee : E\Omega}$, with which values substituted into the above equation $\mathrm{EF}.af = \left(nn.\mathrm{E\Delta.SL}^2.\Delta\xi : \mathrm{AL}^2.\mathrm{DE}^4\right) + \&\mathrm{c.}$, there will arise

$$\mathrm{EF}.af\,(\S.132) = \mathrm{BE.E}e = \mathrm{G.Ee} = (nn.\mathrm{E\Delta}^2.\mathrm{SL}^2.\mathrm{E}e : \mathrm{E\Omega.AL}^2.\mathrm{DE}^4)$$
$$+\ (nn.\mathrm{E\Delta.AL}^2.\mathrm{E}e : \mathrm{AG}^2.\ \mathrm{DE}^4) + \left(2nn : \mathrm{E\Delta}^2.\mathrm{SL}^2.\mathrm{E}e : \mathrm{AL}^2.\ \mathrm{DE}^5\right) + \left(\mathrm{E}e : \mathrm{DE}^3\right).$$

In which, if in place of the ratio $\mathrm{SL}^2 : \mathrm{AL}^2$ there may be substituted the equal ratio $\mathrm{MG}^2 : \mathrm{AG}^2$, and everything may be applied to Ee, there will be BE or

$$\mathrm{G} = \left(nn.\mathrm{E\Delta}^2.\mathrm{MG}^2 : \mathrm{E\Omega.AG}^2.\mathrm{DE}^4\right) + \left(\ nn.\ \mathrm{E\Delta.AL}^2 : \mathrm{AG}^2.\mathrm{DE}^4\right)$$
$$+\left(\ 2nn.\mathrm{E\Delta}^2.\mathrm{MG}^2 : \mathrm{AG}^2.\mathrm{DE}^5\right) + \left(1 : \mathrm{DE}^3\right),$$

the formula of the general central force acting at the point of the curve N, which duly the ordinate, clearly with the first three fractions reduced to the same kind, at last will be :

$$G=\frac{1}{DE^3}+\frac{\left(DE.E\Delta.MG^2+2.E\Omega.E\Delta.MG^2+DE.E\Omega.AL^3\right).\,nn.\Delta E}{E\Omega.EG^3.DE^5}.$$

In which with the curve LH proving to be algebraic, the other $\Delta\Xi$ will be algebraic, and thus all $E\Omega$, $E\Delta$, EH & GM will be given algebraically ; hence also, just as was said in no. I of this section, it will be necessary for the other curve AN to be algebraic ; and because for any imaginable algebraic curve AN some other LH corresponds, and it may represent generally all these curves LH, which are able to result from AN*n* ; thus the preceding rule shows the general formula of the central force for all algebraic curves which are considered possible, for infinitely many algebraic curves AN*n*. Which was required to be found.

COROLLARY I.

168. If now besides there may be put $1:n=\mu:v$, thus so that μ and v may indicate some whole positive numbers, and the tangent of the angle MAL, may be called T, its secant S, truly in fig. 37, the coordinates DQ and QN, x and y respectively, and $DN=z=\sqrt{(xx+yy)}$; with which in place the tangent of the angle $ADN=ry:x$, and the secant $=rz:x$, with the radius arising $DP=r$. And thus there will be by the general rule I have treated in the *Acta Erud. Lipz.* 1703, page 263, of the multi-sections of the angle, or the arc being cut, dividing up the angle : $v.ADN=rz^v:\left(x^v-02vx^{v-2}yy+04vx^{v-4}y^4-06vx^{v-6}y^6+\&c.\right)$, and the secant of the angle $\mu.MAL=S^\mu:(r^{\mu-1}-02\mu r^{\mu-3}T^2+04\mu r^{\mu-5}T^4-06\mu r^{\mu-7}T^6+\&c.$ in which the fictitious numbers 02*v*, 04*v*, 06*v*, &c. indicate $\frac{v.v-1}{1.2};\frac{v.v-1.v-2.v-3}{1.2.3.4};\frac{v.v-1.v-2.v-3.v-4.v-5}{1.2.3.4.5.6}$ or the coefficients of the binomial members raised to the power *v*, or at any rate by a few ; and the numbers $02\mu, 04\mu, 06\mu$, &c. are designated by the order, $\frac{\mu.\mu-1}{1.2};\frac{\mu.\mu-1.\mu-2.\mu-3}{1.2.3.4};\frac{\mu.\mu-1.\mu-2.\mu-3.\mu-4.\mu-5}{1.2.3.4.5.6}$; &c. the binomial coefficients of the same raised to the power μ, likewise by a few chosen terms. Now, because (following the hypothesis), the angle ADN is to the angle MAL, just as 1 to n or μ to v, generally there will be by multiplying the extremes and the means, $\mu.MAL=v.ADN$, therefore of these sections these angles will be equal. Therefore by multiplication in the cross put in place, there will be

$$S^\mu.\left(x^v-02vx^{v-2}yy+04vx^{v-4}y^4+\&c.\right)=rz^v.(r^{\mu-1}-02\mu r^{\mu-3}T^2+04\mu r^{\mu-5}T^4-06\mu r^{\mu-7}T^6+\&c.$$

the general equation of the algebraic curve AN , for with the whole positive numbers μ & v present, these indefinite series will always be terminated; and thus will contain a finite number of terms.

COROLLARY II.

169. Fig. 38. If in the rule of article 167, in place of the ordinate EH there may be put $...e\ ...$A, some quantity composed from a given or constant *e*, and with the variable A given, yet with *z* given as it pleases, and from constant quantities, thus so that A, composed in some manner with infinite different magnitudes from the number z, more correctly all designated, which shall be able to be known. The points with these predetermined magnitudes denote all possible variation of the signs $+$ & $-$ regarding these magnitudes , thus so that by the quantities added to the points in turn not only the sum of the magnitudes but also the difference of the same shall be understood in general, it is from the difference of these, either *e* may exceed the other A, of it may be less than the same. With the ordinate EH given in algebraic magnitudes, the subtangent $\mathrm{E}\Delta$ of the curve LH may be found readily, or the ordinate $\Delta\mathrm{E}$ of the other curve $\Delta\Xi$, of which the subtangent $\mathrm{E}\Omega$ also will be found in algebraic magnitudes. Therefore with the values of the subtangents $\mathrm{E}\Delta$, $\mathrm{E}\Omega$, of the ordinate EH substituted or of the equal AG, then also the value of the right line GM, which can be found easily from the other AG and the given AL cited in the above rule, and the general formula is found of the central attraction for all algebraic curves. Therefore by putting the element of A itself or dA to be Bdz, and the element dB of the magnitude B to be cdz, with $\mathrm{DE}=z$, arising, likewise $\mathrm{AL}=r,\&\ rr-ee=ss$. If the calculation were entered into correctly, there will be found for the central action *G* at any point N of the curve AN,

$$\mathrm{G}=\frac{1}{v}+\frac{\left(2ss\mathrm{B}+ss\mathit{C}z\mp e\mathrm{B}^{2}z\pm 4e\mathrm{AB}\pm 2e\mathrm{A}\mathit{C}z+\mathrm{AB}^{2}z-\mathrm{A}^{2}\mathrm{C}z\right)}{B^{3}z^{5}}.$$

As it may be widely apparent thence the use of this formula can be gathered, because, except what the quantity A as well as all the amount by *z* and given constants may designate, the number *n* may indicate all rational numbers as well as positive integers and fractions.

SCHOLIUM.

170. So that the use of our conspicuous formula may become apparent, it pleases to apply the same to a particular example. Thus if there were $A=z^{m}:a^{m-1}$ where *m* shall indicates some rational number, whether it may refer to a fraction or an integer, and there will be $\mathrm{B}=mz^{m-1}:a^{m-1}$,& $\mathrm{C}(=mm-m)z^{m-2}:a^{m-2}$. Which values, to be substituted into the formula of the above corollary, will give a formula which even if it is with respect to other particulars, from which it is deduced, yet it can give rise to an infinitude of diverse algebraic curves, more correctly an infinity of infinities. Moreover that formula will be such as follows :

$$\mathrm{G}=\frac{mm-nn}{mmz^{3}}\pm\frac{\overline{m\pm 2}.ea^{m-1}}{mmz^{m+3}}+\frac{\overline{m+1}.ea^{2m-2}}{mmz^{3m+3}}.$$

Hence 1st. If $m=-1,$ and $n=1;$ there will be $\mathrm{G}=\pm e:aazz$, and thus the customary force of gravity everywhere in the reciprocal square ratio of the distance of the attracted body from the centre D. We may see which curve must result from this hypothesis. In the first corollary we have put $1:n=\mu:v$ from which since in this case there shall be $n=1$, also there will be $\mu=v=1$, from which if these values are substituted into the general equation in §. 168, towards the end, all the coefficients $02\mu,\ 04\mu,$ &c. and all $02v,\ 04v,\ 06v,$ &c. vanish thus so that in this case the equation of the curve shall be $\mathrm{S}x=rz$. Now because $\mathrm{A}\left(=z^m:a^{m-1}\right)$ in this case is $=aa:z$, and thus in fig. 38, the ordinate EH, which is required to be equal to the difference of the magnitudes *e* & A and not of the sum, because we have kept the same above sign in the formula which indicates such a difference, and below the sum; I say the ordinate EH in this case will be $=\pm e\mp\mathrm{A}=\pm e\mp\frac{aa}{z}=\left(\pm ez\mp aa\right):z,$ but S, or the secant of the angle MAL, is the third proportional to AG or EH, and the radius AM or $\mathrm{AL}=r$, therefore $S=rrz:\pm ez\mp aa$, and thus the equation $Sx=rz$, is changed into $\frac{rrzx}{\pm ez\mp aa}=rz$, from which it is elicited $rx=+ez\mp aa$ or rather $ez-aa=+rx$, or $ez=aa+rx$, which is the equation of a conic section, in which the abscissas *x* are taken upwards and downwards from the focus. Therefore in this hypothesis the centre of the forces, or of the action of gravity, are the centres of conic sections, which now with everything understood to agree exceptionally well with these, which have been demonstrated by the illustrious Newton, Leibniz, Varignon and others, about the forces, which they call centripetal in the direct method of conic sections.

2°. Let there be $m=-2,$ & $n=2$, and *e* shall be greater than *r*, thus so that $ss=rr-ee$ shall be a negative quantity, there shall be $\mathrm{G}=-ssz:a^6$, and thus the actions shall be as the distances from the centre. The equation of the curve under this hypothesis will be investigated thus. Because $n=2,$ & $1:n=\mu:v$ there will be $v=2\mu$, and thus the general equation (§. 168.) applicable for this case will change into the following : $Sxx-Syy=rzz$ or , because $\mathrm{A}\left(=z^m:a^{m-1}\right)$ in this case $=a^3:zz,$ & $S=rr:e-A=rrzz:ezz-a^3$, making $rrxxzz-rryzz:ezz-a^3:rzz$, and thus $rxx-ryy=ezz-a^3=exx+eyy-a^3$; hence $a^3+\overline{r-e}.xx=\overline{e+r}.yy$. Now because *e* (following the hypothesis) is greater than *r*, the quantity $r-e$ will be negative, and thus the equation will be for an ellipse, of which the transverse axis is to the conjugate as $\sqrt{(e+r)}$ to $\sqrt{(e-r)}$; and in which the abscissas *x* have the origin at the centre. Therefore the actions of gravity which act on a body on an ellipse, acting as the distances from the centre, and directed towards the centre of the ellipse, as again the truth can be agreed from Newton's demonstration ; for if, as this excellent author and later the most celebrated Varignon worked out, if such a force of gravity is sought for a body moving around the perimeter

of the ellipse, it must follow the law, according to which the directions of these forces concur at the centre of the ellipse, and the forces of this kind found are proportional to the distances from the centre of the ellipse. But if truly *r* were greater than *e*, our equation found will be for a hyperbola, and the central action will be negative and hence centrifugal.
[Such physical orbits had to await the ingenious experiments of Rutherford scattering α particles from gold foil, demonstrating the existence of the nucleus of the atom.]

3rd. If $m = 1 = n = \mu = v,$ and $e = 0,$ there will be $\mathrm{G} = 2ss : z^5$. And the general equation (§.168.) will be changed into this particular form $Sx = rz$, in which there will be $S = rr : z,$ hence $rrx : z = rz,$ or $rx = zz = xx + yy$. Therefore in this case the centre of attraction is on the circumference of a circle, and with the curve sought itself the circumference of the circle which at this point for Newton and Varignon are to be praised for having shown the latter force. Vid: Book I. *Pr. Phil. Nat. Math.* Prop. VI. & *Comm. Acad. Reg. Sc.Par*. 1700. 31st March, art. XI.

4°. Generally, if G shall be as z^p, where *p* is any rational number, positive or negative, with the single exception -1, the algebraic curve always will be able to have a central force satisfying such a law, in which yet either *e* or *s* vanish or they become zero, otherwise no algebraic curves satisfying such hypotheses can be found, with the cases excepted with *p* signifying the numbers 1 and − 2, with which conic sections I have shown agreed upon in the preceding examples. If $p = 0$, the equation of the curve AN*n* will be found $x^3 - 3xyy + z^3 = \frac{2a^5}{rr}$, at the individual points of this curve a certain constant central action, indeed which is negative, repelling the body from the centre. But if truly there may be put into the formula §.170, $m = -1,$ & $e = 0,$ making G as $\frac{1-nn}{z^3}$, that is reciprocally as the cube of the distance of the moving body from the centre of attraction, and for this case an infinity of curves are in agreement, evidently an infinitely many geometrical curves and an infinity of mechanical or transcending curves, in short as the celebrated Johann Bernoulli pointed out in the *Acta Erud. Lips*. I713, March, p.119, where he showed infinitely many different kinds of spirals, of which one kind contained algebraic curves, truly the others transcending curves. For because (following the hypothesis) $m = -1$, there will be $\mathrm{A}\ (= z^m : a^{m-1}) = aa : z$, and therefore the curve LH will be in this case a hyperbola between the asymptotes through the centre of attraction, or rather through the point D drawn to the right angle; thus so that with one of these becoming DA the other shall be parallel to AL itself. And thus since the secant AS shall be the third proportional to AG, or $\mathrm{D\,EH} = aa : z$, and the radius AL, there will be $AS = rrz : aa = z$ if AL shall be $= a$. From which if in fig. 37 the angle ADN may be made to the angle MAD (§.167.) such as 1 to *n*, and on the right line Dp everywhere there may be taken DN equal to the secant AS, the point N will be on the curve sought AN*n*, which hence will be algebraic as often as *n* were a positive whole number, and such a construction almost coincides with that which the celebrated Newton gave in Corollary 6 after Prop. 44. Book I. *Princ. Phil. Nat. Math.* Indeed also with Bernoulli's works, which is found cited in place above, any curves of this kind present in the works of the sharpest of men may be put otherwise, as in the works of Newton or from our construction. For in figure 37 with the line Dd drawn normal to the axis DA, if now everywhere the angle *d*DN is made to the angle ΘAM, just as 1 to *n*, and as before, on the right line Dp, DN may be assumed equal to DS itself, the point N now also will be on the curve chosen, which will have an asymptote parallel to D*d*, of which from this the distance D*d* will be to DP, as 1 to *n* : truly the curve will not pass through the point A ; indeed in our other construction the curve does pass through the point A, and it will have an asymptote passing through D, being made since the angle DA which it has to a right angle, shall be as 1 ad *n.* And indeed these few examples towards illustrating the formulas from the beginning of this section may suffice to be introduced; from which the observant reader may note, that the general solution of the problem of section 167 proposed and solved shall hardly be of immense use, whenever only a particular formula of the particular section 170 mentioned a little previously, by just being written down, may be able to supply abundant material for consideration.

CAPUT II.

De Motibus curvilineis in Vacuo, in quacunque gravitatis variabilis Hypothesi.

PROPOSITIO XX. LEMMA .

153. *Si magnitudinis continue crescentis* (*cujus prima magnitudo* AI *ea dicitur*, *a qua crescere incipit*) *incrementum infinitesimum* Ff *semper fuerit ad decrementum* qr*, alterius decrescentis* Dq*, cujus prima magnitudo, a qua scilicet decrescere incipit, sit* DR*, sicut crescens* EF *ad decrescentem Dq , erit crescens* EF *ad suam primam magnitudinem* AI, *ut decrescentis prima magnitudo* DR *ad ipsam decrescentem* Dq .

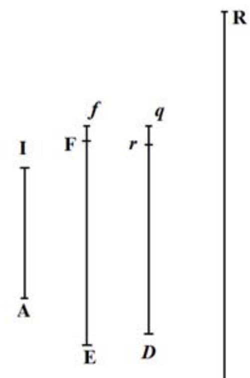


Fig. 35.

In Fig. 36. circa axem DR descripta sit aliqua curva IF, cujus abscissa Dq, DR decrescentem primamque ejus magnitudinem referant,ordinatae vero FE & IA, crescentem ejusque primam magnitudinem, ut adeo signa q, E ad idem punctum pertineant perinde ac signa A, R. Per curvae punctum F tangens FO ducta sit axi occurrens in O, factaque ordinata *fe* alteri FE infinite vicina, erit *fa* crescentis FE incrementum, & Fa vel *qr* seu *Ee* decrementum infinitesimum decrescentis Dq. Eritque adeo (secundum hypothesin) $fa : qr = \mathrm{FE} : \mathrm{D}q$. Verum triangula similia faF, FEO praebent etiam $fa : qr = \mathrm{FE} : \mathrm{DO}$, ergo $\mathrm{FE} : \mathrm{D}q = \mathrm{FE} : \mathrm{EO}$, atque adeo $\mathrm{D}q = \mathrm{EO}$, hinc producta tangente OF*,* erit etiam $\mathrm{OF} = \mathrm{FS}$. Est ergo punctum contactus F ubique in medio rectae OS, angulum rectum SDO subtendentis, ut adeo per conversam Prop. III. Secundi Conicorum Apollonii curva IF Hypebola sit inter asymptotas SD & DO, atque adeo EF sit ad AI, sicut DR ad D*q*. Quod erat demonstrandum.

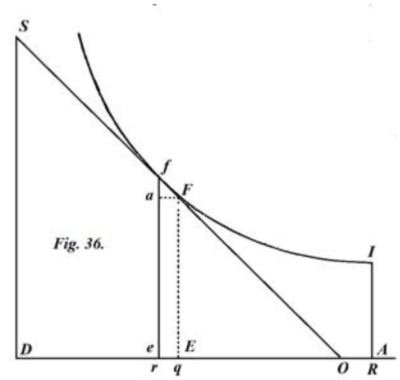


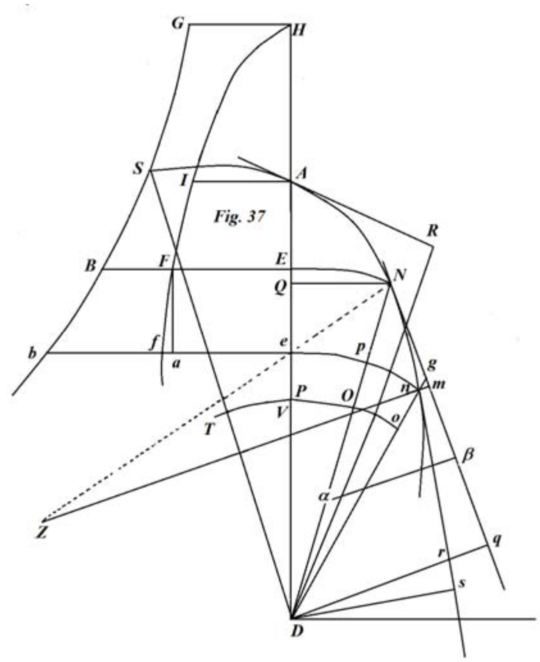
Fig. 36.

PROPOSITIO XXI. THEOREMA .

154. *Existentibus* GBb *scala solicitationum centralium, &* AIF *scala celeritatum mobili, ex quiete in* H *in recta* HD *cadere incipienti, acquisitarum; atque hoc ipsum mobile in directione* AR *celeritate* AI *propulsum, quam grave post casum ex altitudine* HA *in* A *acquirere potest , urgentibus solicitationibus illis centralibus per ordinatas curvae GB significatis, curvam quandam AN in Vacuo describit. Erit* FE *seu celeritas mobilis in quolibet curvae puncto* N

potentia aequalis rectangulo sub radio Nz *circuli osculatoris in hoc puncto, & sub* $\beta\alpha$ *quae solicitationem curvae perpendicularem ex centrali* Nα *vel* BE *derivatam signigicat.* Hoc est, $\mathrm{EF}^2 = n\mathrm{Z}.\beta\alpha$.

Mobile A projectum secundum AR celeritate AI describens curvam AN in ejus puncto N (§.138.) acquiret celeritatem EF, quam idem mobile in puncto E, aeque remoto a centro D, ac curvae punctum N, consequutum esset post lapsum ex altitudine HE, atque hac celeritate EF in directione tangentis curvae Nq aequabili motu incedere conatur, verum a solicitatione centrali Nα vel EB durante tempusculo quam minimo uniformiter agente indesinenter a directione Nq retractum in curva N*n* detinetur, adeo quidem ut lineola *gn* ipsi ND parallela sit spatiolum a solicitatione respectu curvae elementi N*n* uniformi Nα vel BE genitum, atque adeo (§.151.) tempus, quo grave quoddam tale spatulum a gravitate Nα conficeret, foret $\sqrt{(2gn:\mathrm{N}\alpha)}$ sumta mobilis massa A instar unitatis; vero eo tempore, quo *gn* a solicitatione centrali generatur, mobile in tangente describit aequabili motu celeritate EF spatiolum N*g* vel N*m* ; quod tempus (§.128.) est etiam N*m*: EF ; ergo $\sqrt{(2gn:\mathrm{N}\alpha)} = \mathrm{N}m:\mathrm{EF}$, atque adeo



$2gn:\mathrm{N}\alpha = \mathrm{N}m^2:\mathrm{EF}^2$, vel etiam $2gn.n\mathrm{Z}:\mathrm{N}\alpha.n\mathrm{Z} = \mathrm{N}m^2:\mathrm{EF}^2$, & permutando, $2gn.n\mathrm{Z}:\mathrm{N}m^2 = \mathrm{N}\alpha.nZ:\mathrm{EF}^2$. Atqui considerando elementum curvae N*n* instar arculi circuli osculatoris centri Z, cuis N*m* tangens, erit $\mathrm{N}m^2 = 2n\mathrm{Z}.nm$ vel $= 2mn.n\mathrm{Z}$, ergo $2gn.\,n\mathrm{Z}:2mn.\,n\mathrm{Z}(\,\mathrm{N}m^2) = gn.nZ:mn.n\mathrm{Z} = \mathrm{N}\alpha.nZ:\mathrm{EF}^2$; vel substitatis loco lineolarum *gn,* & *mn,* rectis $\mathrm{N}\alpha, \beta\alpha$, quae ipsis propter triangulorum $\mathrm{N}\alpha, \beta\alpha$ & *gnm* similitudinem proportionales sunt, fiet $\mathrm{N}\alpha.n\mathrm{Z}:\beta\alpha.n\mathrm{Z} = \mathrm{N}\alpha.nZ:\mathrm{EF}^2$. Ergo $\mathrm{EF}^2 = n\mathrm{Z}.\beta\alpha$. Quod erat demonstrandum.

PROPOSITIO XXII. THEOREMA .

155. *Iisdem positis, erit celeritas projectionis* AI *ad celeritatem mobilis in curvae puncto* N, *reciproce, ut perpendicularis* D*q* *super tangente* N*q in curvae puncto* N, *ad perpendicularem* DR *ex centro* D *ad directionem jactus* AR *ductam. Atque adeo singula rectangula* D*q*.EF, *dato rec-lo* DR.AI *aequabuntur.*

Nam ductis per punctum *n* tangente *ns* & arcu circulari *ne* ex centro D descripto, ac denique ordinata *ef*, jungatur NZ, & ad tangetem curvae *ns* demittatur perpendicularis D*s*. Quibus positis, erit Nβ solicitatio tangentialis ,ejusque momentum (§.131) aequatur momento velocitatis in N acquisitae, atque adeo $\mathrm{N}\beta.\ \mathrm{N}m$ vel $\mathrm{N}n = \mathrm{EF}.af$; & quia (§.154) $\mathrm{EF}^2 = n\mathrm{Z}.\beta\alpha$,

fiet $\mathrm{EF}.af : \mathrm{EF}^2 = \mathrm{N}\beta.\mathrm{N}m : nZ.\beta\alpha$ vel substitutis loco $\mathrm{N}\beta$ & $\beta\alpha$ proportionalibus N*q* & D*q* & loco

N*m* & *m*Z vel *n*Z proportionalibus *qr & hq* vel N*q*, invenietur $\mathrm{EF}.af : \mathrm{EF}^2 = \mathrm{N}q.qr : \mathrm{N}q.\mathrm{D}q$. Nam quia ZN & Z*n* tangentibus Nq & *ns* perpendiculares sunt, anguli NZ*n* & *ghs*, quem tangentes continent, necessario aequales sunt ; atquc adeo trianguli NZ*m* & *qhr* ad N & *q* rectangula similia sunt, ac lateribus N*m* & NZ vel *n*Z proportionalia , latera homologa *qr* & *hq,* vel N*q* in triangulo *qhr.* Verum in postrema analogia deletis EF exprimo & secundo, & N*q* ex tertio & quarto terminis, sit $af : \mathrm{EF} = qr : \mathrm{D}q$ & permutando $af : qr = \mathrm{EF} : \mathrm{D}q$, id est incrementum crescentis FE est ad decrementum decrescentis D*q*, sicut ipsa crescens ad decrescentem, adeoque erit (§. 153) crescens FE ad suam primam magnitudinem AI, sicut decrescentis *Dq* prima magnitudo DR ad decrescentem , & invertendo , AI celeritas jactus est ad EF celeritatem mobilis in quolibet curvae puncto N, ut perpendicularis *Dq* ad DR. Propterea erunt singula rectangula FE.D*q* aequalia dato AI. DR. Quae erant demonstranda.

COROLLARIUM I.

156. Quia $\mathrm{EF} : \mathrm{AI} = \mathrm{DR} : \mathrm{Dq}$, vel $\mathrm{EF}^2 : \mathrm{AI}^2 = \mathrm{DR}^2 : \mathrm{D}q^2$, & $\mathrm{EF}^2 = \beta\alpha.n\mathrm{Z}$, erit

$\beta\alpha.n\mathrm{Z} : \mathrm{AI}^2 = \mathrm{DR}^2 : \mathrm{Dq}^2$. sed $\beta\alpha.n\mathrm{Z} : \mathrm{N}\alpha.n\mathrm{Z} = \beta\alpha : \mathrm{N}\alpha = \mathrm{D}q : \mathrm{DN}$ vel invertendo,

$\mathrm{N}\alpha.n\mathrm{Z} : \beta\alpha.n\mathrm{Z} = \mathrm{DN} : \mathrm{D}q$: ergo ex aequo & per compositionem rationum

invenietur $\mathrm{N}\alpha.n\mathrm{Z} : \mathrm{AI}^2 = \mathrm{DR}^2.\mathrm{DN} : Dq^2,$ atque adeo $\mathrm{N}\alpha = \mathrm{AI}^2.\mathrm{DR}^2.\mathrm{DN} : n\mathrm{Z}.\mathrm{D}q^3$; ergo solicitatio

centralis in quolibet curvae puncto N est ut $\mathrm{DN} : n\mathrm{Z}.\mathrm{D}q^3$. Ut Celeberrimi viri Joh. Bernoulli, Abr. Moyvraeus , & Guido Grandus invenerunt, insistentes omnes principio isti, quod tempora lationum in curvis sunt areis proportionalia, quod nos nondum supposuimus; sed ia sequenti Corollario ex positis fundamentis nostris deducemus.

COROLLARIUM II.

157. Erit jam tempus, quo quilibet arcus curvae AN describetur, id est
$\mathrm{tAN} = \text{areae ADN} : \frac{1}{2}\mathrm{AI.DR}.$ Nam (§.128.) est
$\mathrm{EF.\ tN}n = \mathrm{N}n; \&\ \mathrm{D}q.\mathrm{EF.tN}n = \mathrm{Dq.N}n = 2.\text{trianguli } \mathrm{ND}n$,atqui(§.155) $\mathrm{D}q.\mathrm{EF} = \mathrm{AI.DR}$, ergo

$\mathrm{AI.\ DR.\ tN}n = \text{trian. } \mathrm{ND}n$ atque adeo $\int A\mathrm{I.\ DR.\ tN}n = \mathrm{AI.DR.tAN} = 2\int \mathrm{ND}n = 2.\text{areae ADN}$,

hinc $\mathrm{tAN} = \text{areae ADN} : \frac{1}{2}\mathrm{AI.DR}$. Atque hinc liquet tempora quibus diversi Arcus AN, A*n* describuntur; areis homologis AND, ANnD proportionalia esse ; prout Illustris Newtonus id primus demonstravit Prop. I. Lib. I. *Prin, Phil. Nat. Math.* sed ex diversissimo fundamento.

SCHOLION.

158. Elegans est formula corollarii secundi, quod valorem solicitationis centralis in qualibet curva exhibeat quantitatibus finitis expressum, sed quia valor radii evolutae eandem ingreditur, id efficit, ut nonnunquam paulo prolixior evadat calculus. Propterea mallem in praxi sequi canonem $g = dp : p^3dz$, in quo g significat gravitatem seu solicitationem centralem, p perpendicularem ex centro solicitationum ad tangentem curvae in dato puncto demissam, & z radium rectorem, seu distantiam puncti curvae, in quo solicitationis centrali quantitas quaeitur a centro. Demonstratio ejus facilis est, etenim ponendo rectanangulum AI. DR $= 1$, fiet etiam FE.D$q = 1$, id est vocando insuper FE, u, cum Dq, jam dicta sit p, erit $pu = 1$ & $ppuu = 1$, vel $uu = 1 : pp$, atque adeo $udu = -dp : p^3$. Jam quia momentum celeritatis aequatur memento solicitationis centralis -gdz. (pono autem - dz, quia crescente u decrescit p, & per consequens z atque adeo ipsi $+du$ homologa dz debet habere signum privativum) hinc $-gdz = -dp : p^3$, & $g = dp : p^3dz$.

159.Usus hujus formulae satis expeditus est; nam ex aequatione curvae datae quaeritur valor ipsius p in z & constantibus, cujusmodi determinatione etiam opus est in formula supra laudata. Exempli gratia in hyperbola & ellipsi reperitur sequens aequatio $pp = ccz : 2a \pm z$, ubi a denotat semilatus transversum, b distantiam centri sectionis ab alterutro foco, & $cc = bb - aa$ in hyperbola, & $cc : aa - bb$ in ellipsi. In denominatore fractionis superius signum hyperbolam, ellipipsin vero inferius respicit. Igitur $1 : pp = (2a \pm z) : ccz = 2a : ccz, \pm 1 : cc$, & differentiando, erit $p - 2dp : p^3 = -2adz : cczz$ & $g = 2dp : 2p^3dz = a : cczz$; vel omissa data a:cc, erit g ut 1: zz ; hoc est solicitatio centralis, ad focum sectionum Conicorum directa, est ubique ut quadratum distantiae mobilis a foco inverse, quod jam passim constat ex aliis.

160. Porro si elementum curvae Nn dicatur ds, arculis pn, dt, reliquis manentibus, ut supra, (§. 158). Triangula similia Nnp & NDq suppeditant $p = zdt : ds$, atque adeo $g = dp : p^3dz = (ds^2dtdz + zds^2ddt - zdsdtdds) : z^3dt^3$. Quae formula non, nisi in nominibus, differt a formulis Varignonianis, quales in Commentariis Academtae Regiae Paris. Scientiarum 1701 & 1706 habentur, in quibus non magis quam in praesenti ullum differentialium dt, dz, ds constans assumtum est, sed omnia variabilia.

161. Eadem facilitate habetur, aut saltem invenitur insistendo principiis supra positis, formula admodum expedita & generalis pro radiis evolutarum. Nam triangula NZ*m* & *qhr,* quae supra (§. 155.) similia esse ostensa sunt, praebent *hq* vel $\mathrm{N}q:qr=\mathrm{NZ}:\mathrm{N}m$, & triangula similia N*np*, ac N*q*D hanc alteram analogiam $\mathrm{ND}:\mathrm{N}q=\mathrm{N}m:\mathrm{N}p$, ergo ex aequo $\mathrm{ND}:qr=\mathrm{NZ}:\mathrm{N}p$, atque adeo $\mathrm{NZ}=\mathrm{ND.N}p:qr$, id est in symbolis supra assumtis; & vocando insuper NZ, *r*; erit $r=zdz:dp$, nam *qr* in figura est *dp* in hisce symbolis ; unde cum p sit $=zdt:ds$, invenietur $r=zdz:dp=zdzds^2:dsdtdz+zdsddt-zdtdds$ formula generalis radii osculatoris, in qua nullum differentiale constans assumtum est, quae proinde in infinitas alias facile transformari potest. Atque deleto membro in denominatore *dsdtdz*, habebitur $r=ds^2dz:dsddt-dtdds$ formula itidem generalis pro curvis quarum ordinatae parallelae. axique perpendiculares sunt, inquibus *dt* sunt elementa abscissarum.

PROPOSITlO XXIII. PROBLEMA.

I62. *Datis scala solicitationum centralium* GBb, *celeritate* AI *&* *directione jactus* AR, *definire & construere curvam, quam missile in vacuo describet, concessis figurarum quadraturis.*

I. Centro D intervalloque arbitrario DP describatur circulus PO, radios DN, *Dn* secans in punctis O, o; assumaturque Q quarta proportionalis ad DN, DR, & AI, ita ut $\mathrm{DN.Q}(=\mathrm{DR.AI})=\mathrm{EF.D}q$,· hoc est $\mathrm{DN}:\mathrm{Q}q=\mathrm{EF}:\mathrm{Q}$, $\&\,\mathrm{DN}^2:\mathrm{D}q^2=\mathrm{EF}^2:\mathrm{QQ}$; ut dividendo habeatur $\mathrm{Nq}^2:\mathrm{D}q^2=\mathrm{EF}^2-\mathrm{QQ}:\mathrm{QQ}$, atque adeo $\mathrm{N}q:\mathrm{D}q=\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}:\mathrm{Q}$, sed N*q*:D*q* sicut radius ad tangentem anguli DN*q*, quae tangens dicatur T, & radius DP vel DO, hinc $\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}:\mathrm{Q}=\mathrm{DP}:\mathrm{T}$ atque adeo $\mathrm{T}=\mathrm{Q.DP}:\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}$.

II. Propter arcus similes *np* & O*o*, est $\mathrm{O}o:pn=\mathrm{DP}:\mathrm{DN}$, & $pn:\mathrm{N}p=\mathrm{T}:\mathrm{DP}$, ergo ex aequo $\mathrm{O}o:\mathrm{N}p=\mathrm{T:DP}$, hinc $\mathrm{O}o=\mathrm{T.N}p\mathrm{:DN}(\mathrm{num.1.})=\mathrm{Q.DP.}Np:\mathrm{DN}\sqrt{(\mathrm{EF}^2-\mathrm{QQ})}$, ergo $\mathrm{PO}=\int\mathrm{DP.Q.N}p:\mathrm{DN}\sqrt{\left(\mathrm{EF}^2-\mathrm{QQ}\right)}$. Hoc est, si intelligatur aliqua curva, cujus abscissa DN, ordinati $\mathrm{DP}^2.\mathrm{Q}:\mathrm{DN}\sqrt{\left(\mathrm{EF}^2-\mathrm{QQ}\right)}$ area, hujus curvae ordinatis abscissarum DA & DE interjecta ad datam lineam DP applicata praebebit lineam, cui si aequalis ubique factus fuerit arcus PO, recta DON per hujus arcus terminum O ducta aequalis abscissae DE sua extremitate N in curva quaesita erit. Jam quia quadratum ordinatae EF aequatur duplo areae GBEH datae per abscissam DE vel DN & constantes, si non algebraice, saltem transcendenter, atque Q est quarta proportionalis ad DE vel ON, & datas DR, AI, ea pariter data erit in DE & constantibus, unde cum omnes quantitates, quae ipsam $DP^2.Q : DN\sqrt{\left(EF^2 - QQ\right)}$ ordinatam scilicet figurae quadrandae ingrediuntur, dentur in DN & constantibus, & figurarum quadraturae concessae sint; factum esse liquet, quod erat faciendum.

COROLLARIUM.

163. Si nunc dicantur DR, *b*; AI, *c*; DE vel DN; *z*, velocitas in curvae puncta N, seu FE , *u* ; arcus circuli PO , θ ; ejus radius DP, *r*; at elementum arcus crescens OO, $+d\theta$, elementum vero decrescens N*p;* indeterminatae ND, -dz. Adeoque $Q = cb : z$ & tota formula $Oo = Q.DP.Np : DN\sqrt{(EF^2 - QQ)}$, mutabitur factis debitis substitutionibus in $d\theta = -bcrdz : z\sqrt{(uuzz - bbcc)}$, quae est aequatio differentialis curvae quaesitae AN*n*.

SCHOLION.

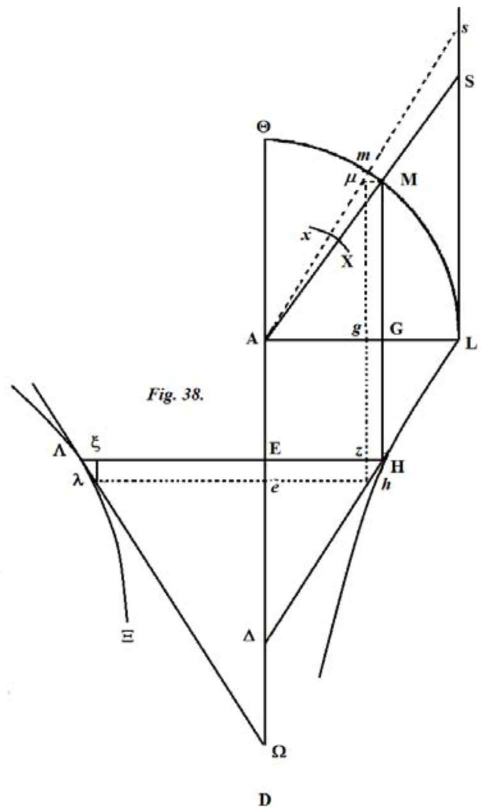
164. Hoc problema primum solutionem accepit a Cel. Newtono Prop. 41 Lib. Princ. Phil. Nat. Math. & postea a Perspicacissimo Geometra Joh. Bernoulli gemino modo, tum etiam a Cl. Viro P. Varignon diversis modis, quo etiam referri posset & nostra solutio quam in Diario Veneto Tom. V. pag.318. seq. exhibui cum nonnullis aliis, &Tom. VII. pag. 194. Praesens nostra solutio a Newtoniana non differt, nisi in levibus nec essentialibus circumstantiis, nam nostra EF significat latus quadratum areae apud Newtonum ABFD bis sumtae, quam tamen ille semel tantum accipit, ejusque Q nobis est rectangulum sub datis DR & AI in nostra figura, Q vero nobis est idem quod illi Q:A seu Z.

Caeterum, quia generalis haec solutio quodraturas praesupponit earum etiam curvarum, quae quadrabiles non sunt, ideo problema istud generaliter sumtum est transcendens, nec algebraicum fit, nisi pro certis legibus solicitationum centralium. Quaenam vero debeant esse in genere hae leges gravitatis variabilis, ut illis positis curvae projectorum algebraicae evadant, problema est satis curiosum & elegans, sed prima fronte admodum difficile, de quo, quod sciam, nemo adhuc cogitavit. Quomodo vero debeat expediri, id in sequenti apparebit propositione, post lemma mox afferendum.

PROPOSITIO XXIV. LEMMA.

165. *Elementum* Ss *tangentis* LS *arcus circularis* LM *est ad elementum hujus arcus* Mm, *ut* AS^2 *quadratum secantis ad* AM^2 *quadratum radii, vel etiam, ut* AL^2 *quadratum radii, ad* AG^2 *quadratum sinus complementi arcus* LM.

Nam, quia triangula SA*s* & MA*m* communem angulum MA*m* continent, erit triangulum *SAs* ad triangulum *Mam,* ut rec-lum *SA.sA* ad rec-lum MA.*m*A,id est, ut quadratum SA, ad quadratum MA. At quia trianguli SA*s* altitudo est AL; existente basi *Ss,* & trianguli Ma*m* basin M*m* habentis altitudo est MA aequalis alteri AL, liquet triangulum SA*s* se habere ad triangulum MA*m*, ut basis S*s* ad basin M*m,* ergo etiam Ss ad *Mm* ut AS^2 ad AM^2, vel propter parallelas SL & MG, sicut AL^2 ad AG^2. Quod erat demonstrandum.



CORLLARIUM.

166. Idcirco, si dicantur radius AL, *r,* tangens LS , *t*; arcus LM, α, erit secans $AS = \sqrt{(rr + tt)}$, atque adeo $d\alpha = rrdt : rr + tt$. Et conversa hac aequatione in seriem, singula seriei membra erunt summabilia, adeo ut summatione eorum actu instituta habeatur expressio arcus circularis α*,* serie infinita tangente & radio expressa.

PROPOSITlO XXV. PROBLEMA.

167. *Inveniri canonem generalem determinandae gravitatis variabilis seu legis solicitationum centralium, pro omnibus curvis algebraicis in infinitum, quantitatibus finitis expressum.*

I. In figura 38 fiat AD aequalis AD in fig. 37, atque in fig. 38 circa axem DA descripta intelligatur curva quaecunque algebraica LHh: descriptoque super ordinata ejus AL quadrante LMΘ circuli, ex quolibet hujus circumferentiae puncto M agatur quaedam MH parallela axi AD, curvae LH alicubi occurrens in H subter ordinatam AL, per curvae LH punctum H tangens HA ducta intelligatur axi occurrens in Δ, & ordinata HE axi AD perpendicularis, in qua, ordinata HE ad alternam axis partem protracta, sumatur ubique $E\Delta =$ subtangenti $E\Delta$, adeo ut puncta Λ futura sint in quadam curva $\Delta\Xi$, cujus subtangens sit in ejus puncto Δ linea EΩ, ducta scilicct tangente ΛΩ. Quibus positis, si in altera figura (37.) angulus ADN ubique fuerit ad angulum MAL, ut unitas ad quemlibet numerum positivum & rationalem *n*, atque in figura 37. longitudo DN semper aequalis fuerit abscissae DE in figure 38. manifestum, singula puncta N locatum iri in curva algebraica AN*n* quando quidem omnia haec puncta algebraice definiri possunt, cum sectio anguli in ratione numeri ad numerum omnino geometrice fieri possit. Jam, sicut ex alia atque alia curva LH alia atque alia AN oritur, ita quaelibet curva AN aliquam generatricem LH admittet, adeo ut hac curva considerata duntaxat in genere, absque attentione ad particularem aliquam speciem, etiam suppeditare possit omnes possibilies curvas AN*n*. Idcirco, cum canon generalis quaeritur, que lex gravitatis variabilis in omnibus curvis algebraicis determinetur, res eo deducetur, ut inveniatur formula solicitationem centralium G pro hac curva generali AN, quatenus, ea ex curva itidem generali & algebraica LH resultat. In hac vero disquisitione ita porro est procedendum.

II. Assumto scilicet in curva LH ejus elemento H*h*, per punctum *h*, agantur *hλ & hm* rectis ΔH & HM aequidistantibus, ac ductis per puncta M, *m* secantibus AS, A*s;* centro A & radio $\mathrm{AX} = \mathrm{DP}$ in figura 37, descriptus sit arculus X*x*, quo fiet ut, quoniam (secundum hypothesin) $\mathrm{ang.ADN} : \mathrm{ang.\ MAL} = 1 : n$, etiam OD*o*:MA*m*, vel X*ax*, hoc est arculus O*o* sit ad arculum X*x* sicut 1 ad *n*. Quibus positis sic arguo.

III. Triangula similia H*ih* & HIΔ praebent $\mathrm{Ee\ (ih)} : \mathrm{Gg}(\mathrm{Hi}) = \mathrm{E\Delta} : \mathrm{AG}(\mathrm{EH})$, & triangula simila AMG & Mmμ , dant $\mathrm{Gg}(\mathrm{M}\mu) : \mathrm{M}m = \mathrm{GM} : \mathrm{AL}(\mathrm{AM})$ & $\mathrm{M}m : \mathrm{X}x = \mathrm{AL}(\mathrm{AM}) : \mathrm{AX(DP)}$, ac denique, num. 1. hujus $\mathrm{X}x : \mathrm{O}o = n : 1$, ergo ex aequo & per compositionem rationum, fiet $\mathrm{E}e : \mathrm{O}o = \mathrm{n.\ E\Delta.\ GM} : \mathrm{AG.DP}$ (vel quia ob triangulorum AGM & ALS similitudinem $\mathrm{GM} : \mathrm{AG} = \mathrm{LS} : \mathrm{AL}) = \mathrm{n.E\Delta.SL} : \mathrm{DP.AL}$. Atqui (§. 162. n. II.) erat etiam $\mathrm{Ee\ (ibi\ Np)} : \mathrm{O}o = \mathrm{DE} : \mathrm{T}$, ergo $\mathrm{n.E\Delta.SL} : \mathrm{DP.AL} = \mathrm{DE} : \mathrm{T}$, atque adeo fiet $\mathrm{T} = \mathrm{DP.\ AL.DE} : \mathrm{n.E\Delta.SL}\ (\S.162.\mathrm{n.1.}) = \mathrm{Q.DP} : \sqrt{\left(\mathrm{EF}^2 - \mathrm{QQ}\right)}$ atque adeo $\sqrt{\left(\mathrm{EF}^2 - \mathrm{QQ}\right)} = \mathrm{n.Q.E\Delta.SL} : \mathrm{AL.DE}$, ac denique $\mathrm{EF}^2 = \left(\mathrm{nn.\ E\Delta}^2.\ \mathrm{SL}^2 + \mathrm{AL}^2.\mathrm{DE}^2\right).\mathrm{QQ} : \mathrm{AL}^2.\ \mathrm{DE}^2$,vel quia $\mathrm{Q} = \mathrm{DR.AI} : \mathrm{DE}$, & omittendo instar unitatis factum constans DR, AI, est $\mathrm{QQ} = 1 : \mathrm{DE}^2$, fiet etiam $\mathrm{EF}^2 = \mathrm{nn.E\Delta}^2.\mathrm{SL}^2 : \mathrm{AL}^2.\mathrm{DE}^4;\ +1 : \mathrm{DE}^2$.

IV. Sumantur elementa singulorum membrorum hujus ultimae aequalitatis, atque singulis elementis per 2 divisis (bifariam ea dividi posse ex calculo ilico patet) proveniet

$$\mathrm{EF.af} = (\mathrm{nn.E\Delta.SL}^2.\Delta\xi : \mathrm{AL}^2.\mathrm{DE}^4) + (\mathrm{nn.E\Delta}^2.\mathrm{SL.S}s : \mathrm{AL}^2.\mathrm{DE}^4)$$
$$+(2\mathrm{nn.E\Delta}^2.\mathrm{SL}^2.\ \mathrm{Ee} : \mathrm{AL}^2.\mathrm{DE}^3) + \left(\mathrm{Ee} : \mathrm{DE}^3\right).$$

Verum quia (§. I65.) $\mathrm{S}s : \mathrm{M}m = \mathrm{AL}^2 : \mathrm{AG}^2$, nec non $\mathrm{M}m : \mathrm{G}g = \mathrm{AL} : \mathrm{MG}$ ac denique $\mathrm{G}g : \mathrm{E}e = \mathrm{AG} : \mathrm{E\Delta}$, erit $\mathrm{SL.S}s = \mathrm{AL}^4.\mathrm{E}e : \mathrm{AG}^2.\mathrm{E\Delta}$, triangulaque similia $\Delta\xi\lambda$, ac ΔEΩ suppeditant $E\Delta(\Delta E).\Delta\xi = \Delta E^2.Ee : E\Omega$, quibus valoribus in superiore aequalitate $EF.af = \left(nn.E\Delta.SL^2.\Delta\xi : AL^2.DE^4\right) + \&c.$ substitutis, proveniet

$$EF.af\,(\S.132) = BE.Ee = G.Ee = (nn.E\Delta^2.SL^2.Ee : E\Omega.AL^2.DE^4)$$
$$+\ (nn.E\Delta.AL^2.Ee : AG^2.\ DE^4) + \left(2nn : E\Delta^2.SL^2.Ee : AL^2.\ DE^5\right) + \left(Ee : DE^3\right).$$

In qua, si loco rationis $SL^2 : AL^2$ substituetur aequalis ratio $MG^2 : AG^2$, atque omnia ad E*e* applicentur, erit BE seu

$$G = \left(nn.E\Delta^2.MG^2 : E\Omega.AG^2.DE^4\right) + \left(\ nn.\ E\Delta.AL^2 : AG^2.DE^4\right)$$
$$+\left(\ 2nn.E\Delta^2.MG^2 : AG^2.DE^4\right) + \left(1 : DE^3\right),$$

formula generalis solicitationis centralis in puncto curvae N, quae rite ordinata reductis scilicet tribus primis fractionibus ad idem nomen , demum erit

$$G = \frac{1}{DE^3} + \frac{\left(DE.E\Delta.MG^2 + 2.E\Omega.E\Delta.MG^2 + DE.E\Omega.AL^3\right).\ nn.\Delta E}{E\Omega.EG^3.DE^5}.$$

In qua existente curva LH algebraica, altera $\Delta\Xi$ algebraica erit, atque adeo omnes $E\Omega,\ E\Delta,\ EH\ \&\ GM$ algebraice dabuntur ; hinc etiam, juxta dicta numero 1. hujus, altera curva AN algebraica sit, necesse est; & quia cuilibet curvae algebraicae imaginabili AN aliqua LH respondet, & haec LH omnes curvas generaliter repraesentat, quae ex omnibus AN*n* resultare queunt; ideo praecedens canon exhibet formulam generalem solicitationum centralium pro omnibus, quae concipi possunt, curvis algebriaicis Ann in infinitum. Quae erat invenienda.

COROLLARIUM I.

168. Si nunc praeterea ponatur $1 : n = \mu : v$, ita ut μ & v significent quoslibet numeros integros & positivos, dicanturque tangens anguli MAL, T, ejus secans S, in fig. vero 37, coordinatae DQ & QN, x & y respective, & $DN = z = \sqrt{(xx + yy)}$; quibus positis erit tangens anguli $ADN = ry : x$, & secans $= rz : x$, existente radio $DP = r$. Eritque adeo per regulam generalem multisectionis anguli vel arcus per secantis in Act. Erud. Lips. 1706. pag. 163. traditam, secans anguli v, $ADN = rz^v : \left(x^v - o2vx^{v-2}yy + o4vx^{v-4}y^4 - o6vx^{v-6}y^6 + \&c.\right)$ & secans anguli μ, $MAL = S^\mu : (r^{\mu-1} - 02\mu r^{\mu-3}T^2 + 04\mu r^{\mu-5}T^4 - 06\mu r^{\mu-7}T^6 + \&c.$ in quibus numeri ficti o2*v*, o4*v*,o6*v*, &c. significant $\frac{v.v-1}{1.2}; \frac{v.v-1.v-2.v-3}{1.2.3.4}; \frac{v.v-1.v-2.v-3.v-4.v-5}{1.2.3.4.5.6}$ seu coefficientes membrorum

binomii ad potestatem v sublati, per saltum excerptas; & numeri $02\mu, 04\mu\ , 06\mu,\ \&\text{c.}$ significant ordine, $\frac{\mu.\mu-1}{1.2}; \frac{\mu.\mu-1.\mu-2.\mu-3}{1.2.3.4}; \frac{\mu.\mu-1.\mu-2.\mu-3.\mu-4.\mu-5}{1.2.3.4.5.6}$; &c. coefficientes binomii cujusdam ad potestatem μ evecti , itidem per saltum excerptas.

Jam, quia (secundum hypothesin) ang.ADN est ad ang.MAL, sicut 1 ad n seu μ ad v, erit omnino multiplicando extrema & media $\mu.\text{MAL} = v.\text{ADN}$, ergo horum secantis aequales erunt. Propterea multiplicatione in crucem instituta, erit

$$S^{\mu}.\left(x^{v} - \text{o}2vx^{v-2}yy + \text{o}4vx^{v-4}y^{4} + \&c.\right) = rz^{v}.(r^{\mu-1} - 02\mu r^{\mu-3}\text{T}^{2} + 04\mu r^{\mu-5}\text{T}^{4} - \text{o}6\mu r^{\mu-7}\text{T}^{6} + \&\text{c.}$$

generalis aequatio curvarum algebraicarum AN , nam existentibus $\mu \,\&\, v$ numeris integris & positivis, hae series indefinitae semper abrumpentes erunt; atque adeo finito terminorum numero constabunt.

COROLLARIUM II.

169. Si in Canone articuli 167. loco ordinatae EH ponatur $...e\ ...$A, quaelibet quantitas composita ex data seu constante e, & variabile A, data tamen utlibet in z, & quantitatibus constantibus, adeo ut A, infinitas numero diversas quantitates ex z & datis quomodocunque compositas, immo omnes, quae cogitari possint significet. Puncta quantitatibus praefixa omnem possibilem signorum $+$ & $-$ ad quantitates illas respicientium variationem denotant, adeo ut per quantitates punctis invicem adjunctas non solum summa quantitatum sed etiam differentia earundem intelligenda sit in genere, hoc est differentia ipsarum, sive e excedat alteram A, sive ab eadem deficiat. Data ordinata EH, in quantitatibus algebraicis, facile invenietur subtangens $\text{E}\Delta$ curvae LH, seu ordinata ΔE alterius curvae $\Delta\Xi$, cujus subtangens $\text{E}\Omega$ etiam habebitur in quantitatibus algebraicis. Idcirco substitutis valoribus subtangentium $\text{E}\Delta$, $\text{E}\Omega$, ordinatae EH vel aequalis AG, tum etiam rectae GM, quae ex altera AG & data AL facile reperitur, in canone supra citato & habebitur generalis formula solicitationum centralium in omnibus curvis algebraicis. Ponendo igitur elementum ipsius A seu dA esse Bdz, & elementum dB quantitatis B esse cdz, existente $\text{DE} = z$, item $\text{AL} = r, \& \ rr - ee = ss$. Si calculus recte initas fuerit, reperietur pro solicitatione centrali G in quolibet curvae AN puncto N,

$$\text{G} = \frac{1}{v} + \frac{\left(2\text{ssB} + \text{ss}C\text{z} \mp \text{eB}^{2}z \pm 4\text{eAB} \pm 2\text{eA}C\text{z} + \text{AB}^{2}\text{z} - \text{A}^{2}\text{Cz}\right)}{B^{3}z^{5}}.$$

Quam late pateat usus hujus formulae exinde potest colligi, quod, praeterquam quod quantitas A omnem quantitatem per z & constantes datam designet, numerus n omnes numeros rationales atque positivos integros & fractos significet.

SCHOLIUM.

170. Ut appareat usus insignis nostrae formulae, exemplo cuidam particulari eandem applicare libet. Sic si fuerit $A = z^m : a^{m-1}$ ubi *m* significet quemlibet numerum rationalem, fractus ne sit an integer nil refert, eritque $\mathrm{B} = mz^{m-1} : a^{m-1}$,& $\mathrm{C}(= mm - m)z^{m-2} : a^{m-2}$. Qui valores, in formula superioris Corollarii substituti, dabunt formulam quae etsi particularis est alterius respectu, ex qua deducta est, infinitas tamen diversas curvas algebraicas suppeditare potest, immo infinities infinitas. Formula autem ipsa erit talis quae sequitur

$$\mathrm{G} = \frac{mm-nn}{mmz^3} \pm \frac{\overline{m\pm2}.ea^{m-1}}{mmz^{m+3}} + \frac{\overline{m+1}.ea^{2m-2}}{mmz^{3m+3}}.$$

Hinc 1°. Si $m = -1,$& $n = 1;$ erit $\mathrm{G} = \pm e : aazz$, atque adeo solitatio gravitatis ubique in reciproca duplicata ratione distantiae mobilis a centro D. Videamus quaenam curva ex ista hypothesi debeat resultare. In corollario primo fecimus $1 : \mathrm{n} = \mu : v$ unde cum hoc casu sit $\mathrm{n} = 1$, erit etiam $\mu = v = 1$, unde si valores hosce in aequatione generali §. 168. circa finem, substituas, evanescent omnes coefficientes $02\mu,\ 04\mu,$ &c. & omnes $02v,\ 04v,\ 06v,$ &c. adeo ut hoc casu curvae aequatio fit $\mathrm{S}x = rz$. Jam quia $\mathrm{A}\left(= z^m : a^{m-1}\right)$ hoc casu est $= aa : z$, atque adeo in fig.38, ordinata EH, quae assumenda est aequalis differentiae quantitatum *e* & A non summae, quia in ambiguo signo superius in formula retinuimus quod indicat talem differentiam, atque inferius summam; ordinata inquam EH hoc casu erit $= \pm e \mp \mathrm{A} = \pm e \mp \frac{aa}{z} = \left(\pm ez \mp aa\right) : z,$ atqui S, seu secans anguli MAL, est tertia proportionalis ad AG seu EH, & radium AM vel $\mathrm{AL} = r$, ergo $S = rrz : \pm ez \mp aa$, adeoque aequatio $Sx = rz$, mutatur in $\frac{rrzx}{\pm ez \mp aa} = rz$, ex qua elicitur $rx = +ez \mp aa$ vel potius $ez - aa = +rx$, aut $ez = aa + rx$, quae est aequatio *Sectionum Conicarum,* in quibus abscissae *x* sumuntur sursum & deorsum a foco. Ergo in hac hypothesi centrum virium, seu solicitationum gravitatis, sunt umbilici sectionum conicarum , quod jam omnibus constat egregie conspirare cum iis, quae demonstrata sunt ab Illustr. Newtono, Leibnitio, Varignonio & aliis, circa vires, quas vocant centripetas in sectionibus conicis methodis directis.

2°. Sit $m = -2,$ & $n = 2$, atque *e* major quam *r*, ita ut $ss = rr - ee$ sit quantitas negativa, erit $\mathrm{G} = -ssz : a^6$, atque adeo solicitationes erunt ut distantiae a centro. Aequatio curvae pro hac hypothesi sic indagabitur. Quoniam $n = 2,$ & $1 : n = \mu : v$ erit $v = 2\mu$, atque adeo aequatio generalis (§. 168.) huic casui applicata mutabitur in hanc sequentem $Sxx - Syy = rzz$ seu, quia $\mathrm{A}\left(= z^m : a^{m-1}\right)$ hoc casu est $= a^3 : zz,$ & $S = rr : e - A = rrzz : ezz - a^3$, fiet

$= a^3 : zz$, & $S = rr : e - A = rrzz : ezz - a^3$, atque adeo $rxx - ryy = ezz - a^3 = exx + eyy - a^3$; hinc $a^3 + \overline{r-e}.xx = \overline{e+r}.yy$. Jam quia *e* (secundum hypothesin) major est quam *r*, quantitas $r - e$ erit negativa, atque adeo aequatio erit ad ellipsin, cujus axis transversus est ad conjugatum sicut $\sqrt{(e+r)}$ ad $\sqrt{(e-r)}$; & in qua abscissae x originem in centro habent. Idcirco solicitationes gravitatis quae in ellipsi sunt ut distantiae a centro solicitationum, diriguntur ad centrum ellipseos, quod iterum ex demonstratis Newtonianis constat verum esse ; nam si, ut Excell. hic Autor, & postea Cl. Varignon fecerunt, quaeratur qualem gravitates mobilis, perimetrum ellipseos circumeuntis, legem sequi debeant, si earum directiones in centro ellipseos concurrant, reperietur ejusmodi solicitationes distantiis mobilis a centro ellipseos proportionales esse. Sin vero *r* major quam *e*, inventa nostra aequatio erit ad hyperbolam, & solicitatio centralis erit negativa atque adeo centrifuga.

3°. Si $m = 1 = n = \mu = v$, & $e = 0$, erit $G = 2ss : z^5$. Atque aequatio generalis (§.168.) mutabitur in hanc particularem $Sx = rz$, in qua erit $S = rr : z$, hinc $rrx : z = rz$, vel $rx = zz = xx + yy$. Ergo hoc casu centrum solicitationum est in circumferentia circuli, curvaque quaesita ipsa circuli circumferentia quod a posteriori hactenus laudati viri Newtonus atque Varignonius ostenderunt. Vid: Lib. I. *Pr. Phil. Nat. Math.* Prop. VI. & Comm. Acad. Reg. Sc.Par. 1700. die 31. Martii art. XI.

4°. Generaliter, si G sit ut z^p, ubi *p* est quilibet numerus rationalis positivus vel negativus, excepto solo -1, curva semper algebraica haberi potest tali solicitationum centralium legi satisfaciens, in qua tamen vel *e* aut *s* evanescunt seu nihil sunt, alioqui nullae curvae algebraicae talibus hypothesibus inveniri possent satisfacientes, exceptis casibus ipsius *p* significantis numeros 1 & - 2, quibus sectiones conicas convenire in praecedentibus exemplis ostensum. Si $p = 0$, reperietur curvae AN*n* aequatio $x^3 - 3xyy + z^3 = \frac{2a^5}{rr}$, in cujus curvae singulis punctis solicitatio centralis constans quidem sed negativa est, mobile a centra repellens. Sin vero ponatur in formula §.170, $m = -1$, & $e = 0$, fiet G ut $\frac{1-nn}{z^3}$, hoc est reciproce, ut cubus distantiae mobilis a centra solicitationum, & huic casui infinitae conveniunt curvae, infinitae scilicet geometricae infinitiesque infinitae Mechanicae seu transcendentes, prorsus ut a Cel.Joh. Bernoullio animadversum in Act. Lips. I713. Mens. Mart. p.119, ubi infinitas exhibet diversas spiralium species, quarum unae continent curvas algebraicas, aliae vero transcendentes. Nam quia (secundum hypothesin) $m = -1$, erit A $(= z^m : a^{m-1}) = aa : z$, ideoque curva LH erit hoc casu hyperbola inter asymptotas per centrum solicitationum , vel potius per punctum D ad angulos rectos ductas; ita ut una earum existence DA altera sit ipsi AL parallela. Adeoque cum secans AS sit tertia proportionalis ad AG,vel $EH = aa : z$,& radium AL, erit $AS = rrz : aa = z$ si AL sit $= a$. Unde si in fig. 37 fiat angulus ADN ad angulum MAD (§.167.) sicut 1 ad *n*, atque in recta Dp sumatur ubique DN aequalis secanti AS, punctum N erit in curva quaesita ANn, quae proinde erit algebraica quoties *n* numerus fuerit rationalis & affirmativus, atque talis constructio coincideret fere cum ea quam Celeb. Newtonus dedit in Corollario 6. post Prop.44. Lib. I. *Princ. Phil. Nat.Math.* Imo etiam cum Bcrnoulliana, quae habetur loco supra citato, quanquam ejusmodi curvae apud Acutissimum Virum aliter positae sint, quam in Newtoniana vel nostra constructione. Nam ducta in figura 37. 1inea Dd axi DA normali, si jam ubique fiat angulus dDN ad angulum ΘAM; sicut 1 ad n, atque ut antea, in recta Dp sumatur DN aequalis ipsi DS, punctum N etiam nunc erit in curva optata, quae asymptotam habebit ipsi Dd parallelam, cujus ab hac Dd distantia erit ad DP, ut 1 ad *n* curva vero per punctum A non transibit; in altera vero constructione nostra curva transit per A, atque asymptotam habebit per D transeuntem, facientem cum DA angulum, qui se habet ad angulum rectum, ut 1 ad *n.* Et quidem haec pauca exempla ad illustrationem formulae ab initio hujus articuli allatae adduxisse sufficiat; ex quibus satis jam perspicax Lector animadvertere potest, quam immensi pene usus sit generalis solutio problematis articulo 167 propositi & soluti, quandoquidem sola formula particularis articuli 170 paulo ante memorata justo tractatui conscribendo abundantem materiam subministrare posset.