SECTION II.

After having dealt with these matters, which are concerned with the equilibrium of forces acting on a body, likewise forces can be considered to be impressed on one and the same body moving : and these motions themselves of diverse accelerations and decelerations must be examined, which can result from forces applied continually, and with variations in these as it pleases. Therefore in this second section we will consider most generally the accelerated and retarded motions of bodies indeed as they arise from actions continually applied, yet not everywhere uniformly, as in the Galilean system, in which weights at any distance from the centre of gravity are considered to be acted on by the same force of gravity ; but with variability by some account, either the bodies may be carried straight towards the centre of gravity by the accelerative forces: or they may advance along curved lines. Therefore this section will include general matters pertaining to accelerated and retarded motions for the motions of projectiles in a vacuum, and which pertain to the isochronous motions of bodies along some curved lines, for the motions of pendulums, etc, under all kinds of gravity, or as to be transparent and easy from the hypothesis of a central force: none the less demonstrated by a universal method : for besides the general solution of the problem is required to be found from the law of gravity to be varied as it pleases, for the curved path of projectiles in regions without resistive forces, also this second examines a general rule, along which gravity is required to vary according to that, so that the curves of the trajectories shall be algebraic always, and thus geometrical constructions may be avoid, hence from which many curious and useful things can be deduced. The theory of the centre of oscillation proposed is much more general than before, in as much as for that to be applicable to bodies which oscillate in different fluids, or in some system of variable gravity, in place of that which Huygens, and those who have followed him, perform so much with the centre of oscillation in the special case of uniform gravity: with the single exception, as far as I know, of the celebrated Johann Bernoulli, who promised a more general rule for us in the Act. Lips. 1713, page 88, to be elicited from his principles, but which, lest I am mistaken has not yet seen the light of publication, and fundamentals of which differ from ours, or I judge perhaps to be found by a different method by the incomparable geometer, even if I have not yet seen the rule itself. Nor does this second section remain silent about the action of forces required for that case, so that moving bodies may be able to revolve in orbits, and about the motion of what are called the apses; but by a different account from that, which the illustrious Newton has used in Sect. IX . Book. I. Princip. Phil. Math. And finally the section closes with a dissertation upon the rules of motion arising from bodies colliding, which we have deduced from a new account from a single fundamental equality of the forces before and after the collision, as considered by us.

CHAPTER I.

Concerning the general actions of continuous forces, and with the motions thence arising in vacuo.

DEFINITIONS.

I.

114. By a vacuum every medium is described, which bodies are able to pass through freely without help of hindrance, by its motion alone, by an accepted motive force.

II.

115. If any moveable bodies, which may be carried in right lines or proceed along curved lines, may be speeded up by the forces acting, the directions of which are concurrent at some point with a given position, actions of this kind may be called central, or also the forces of variable gravity. And the point with the given position to which the weights may be drawn, the centre of the attraction. Following Newton, central attractions are called centripetal forces.

III.

116. Actions are said to be continuous or to be applied continually, when a body may be speeded up to a new motion in travelling through the individual points of space, or what amounts to the same thing, it may be action by a certain force.

IV.

117. Any curved line is called the scale of the central attraction or of the variation of gravity, related to some axis passing through the centre of attraction, of which the right ordinates to the axis set out the forces, by which the moving body at these points of the axis, through which the ordinates have been drawn, may be acted on towards the centre, evidently if the body may be drawn towards the axis ; or, if it may approach along some curve itself, at these points of the curve, which with the points on the axis through which the ordinates pass, to be equally distant from the centre of attraction.

V.

118. The action of gravity is called tangential however it may be derived from the central force, by which the moving body, carried around on some curve, is acted on along the direction of the tangent of the curve.

VI.

119. The perpendicular action of gravity on a curve being described by a moving body, is that, which is derived from a central force, the direction of which is perpendicular to the curve everywhere. This perpendicular action is equal everywhere to the attempt of the moving object striving to recede along the tangent, otherwise the body may not be moving along the curve that it is put to describe, and indeed if this trial by the

perpendicular force of receding from the curve may be greater, the moving object actually may recede from the curve; likewise it may move away from the curve by approaching more than an equal amount towards the centre of attraction, if it should become smaller, each being contrary to the hypothesis. Therefore so that the attempt of receding from the curve may be diminished and rendered useless, just as great an action of restraint is required to be applied to the moving body as to cancel out the attempt ; thus so that an attempt of this kind, and the force acting along the same perpendicular direction of the curve, but in opposite directions may be held in equilibrium and so that they may not hinder each other, the actions of one may be chosen not to have any effect on the other, and thus the mobile body abandons the curve along which it must begin.

VII.

120. The scale of the tangential force is some curve related to the axis passing through centre of the central force acting, of which the ordinates express the tangential forces at points of the curve, and the ordinates are at just as great distances from the centre of the actions.

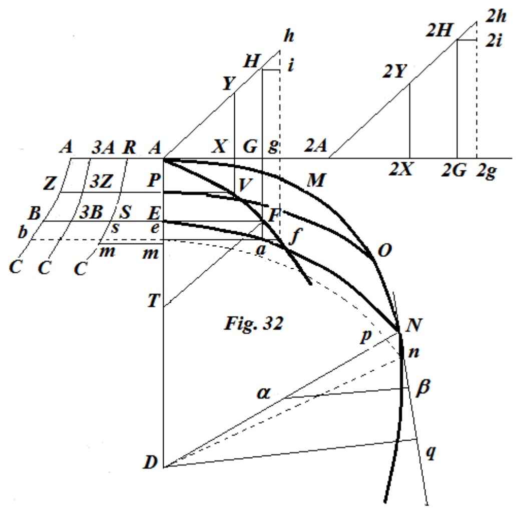


Fig. 32

121. Fig. 32. The mobile body M may be carried along the curve MON, and another mobile body at A on the right line AD, through the centre D, towards which the actions of the central force of gravity are directed; and the mobile body M at some point N of the curve may be forced to be conveyed towards D by the central force acting along Nα, and by this same force also a mobile body may also be urged to be conveyed at the point E of the right line AD, because the point E may be just as far from the centre D, as the point of the curve N, thus so that both the points N and E may be found somewhere on the arc of the circle EN described with centre D, and with the perpendicular EB to AD itself drawn through the individual points of the axis E, and everywhere being equal to the homologous lines $N\alpha$, the curve ABC will pass through the points of the individual B, which may be called the graph of the central forces acting, because any ordinate of that BE equal to the respective Nα (following the hypothesis) represents the central force at the points E and N on the line AD, and with the points E and N of the curve MO equidistant from the centre D. Again, because the action of the central force Nα at any point N of the curve MON (§. 39. 40.) is equivalent to the sides $N\beta$ and $\beta\alpha$, of which the first shall be in the direction of the tangent Nq to the curve, and with the other βα perpendicular to the same, through which it is clearly apparent $N\beta$ itself may be called the tangential force derived from the central force Nα; therefore if at any ordinate BE corresponding to each Nα there may be taken $SE = N\beta$, and the same may be understood to be done with respect of all the remaining points of the curve MON,

thence the curve RSC will result, which represents the graph of the tangential force of the mobile body M on the curve MON. The perpendicular force $\beta\alpha$ has no graph, or at any rate not one we need, since it may be considered at the individual points to be trying to destroy the recession of the mobile body M from the curve (§. 119.), nor hence does any other matter come to be considered in the reckoning.

VIII.

122. Because the central forces BE and the tangential forces SE equally may be continued, and with the effect of the preceding continually repeated in the succeeding without being removed, since the points of the ordinates of this kind BE pass through the individual points of the right line AD : it is necessary, so that with forces of this kind applied continuously indefinitely accelerated motions arise, since the moving bodies A and M descending on the right line AD and on the curve MON as they approach the centre of force D, truly to be retarded on beginning to rise from the points E and N on the right line and on the curve, receding from the same centre D, with some certain speeds which henceforth we will call the *initial speeds*. Therefore the actions of gravity either central or tangential are called *accelerative* with the mobile bodies descending along the right line or the curve ; and the same actions are called *retardative* with the mobile bodies rising, or receding from the centre of the forces.

IX

123. The moment of each force however the name may come about, is the rectangle under the right line which expresses the force, and with the element of the distance which the mobile body is sent across by the force continually impressed. Thus the infinitesimal rectangle BEe is the moment of the accelerative force BE or Nα, with the descent of the mobile body A enduring in the element of distance *Ee* of the adjoining ordinate BE; truly the elemental rectangle *beE* is the moment of the retardative force *be* of the same mobile body ascending through the element of distance eE. Thus also with Em = Nn, made equal to the element of curve MON, the elemental rectangle *SEm* or SE. N*n,* is the moment of the tangential force and of the accelerative force SE or Nβ , when the moving body descends along the curve by its arclet N*n*, and the elemental rectangle *se.n*N is the moment of the retardative tangential force *se,* when its moving body rises along the curve by the arclet *n*N.

X.

124. *The graph of the speeds* are expressed on a curve drawn about the axis of the scale of the central forces [*i.e.* AD], or at any rate described parallel to this, the ordinate of which is put in the direction of the ordinate in the scale of the forces, or at any rate corresponding to the same, the speeds are found for the moving body at these points of the right line AD, or for the points of the curve MON at which the ordinates of the scale of the forces are considered. Thus, if the ordinates PV, EF &c. of the scale of the of the ordinates of the forces ZP, BE in the direction are put in place, they signify the speed acquired or remaining, either of the moving body descending with an acceleration or rising with a retarded motion at the points P, E according to the right line AD , or at the

points of the curve O & N &c. The curve AVF will be the graph of the speeds of the moving body A on the right line AD, or of the moving body M descending or ascending on the curve MON.

XI.

125. *The moment of the speed* is the product from the speed of a body advancing either on the right line or on the curve, by the element of the increase or decrease of the speed. Thus in the graph of the speeds AVF, the rectangle EF.*af*, or the elemental rectangle *eaf* is the moment of the speed EF by the element *af* of the increase, with which the moving body passes across the element *Ee* ; and the rectangle *ef.fa* is the moment of the decreasing speed *ef*, by which the moving body A may be carried from the point *e* of the axis, or the moving body M may be carried to a greater height with a reduced motion from the point *n* on the curve MON, and *fa* the infinitesimal decrease in the speed *ef*, when the moving bodies rise through increments *eE* and *nN*, and EF will be the residual speed at the point E of the right line or at the point N of the curve.

XII.

126. Towards designating the time in which each motion may be completed, we will use a noted characteristic of the time *t* for the distance to be traversed put in place initially. Thus *t*AE will denote the time, in which the moving body *A* completes the distance AE with the acceleration from its falling motion, and *t*Ee will signify the time in which the element Ee of the AD may be ran through by a uniform motion. And so on with the others.

POSTULATED.

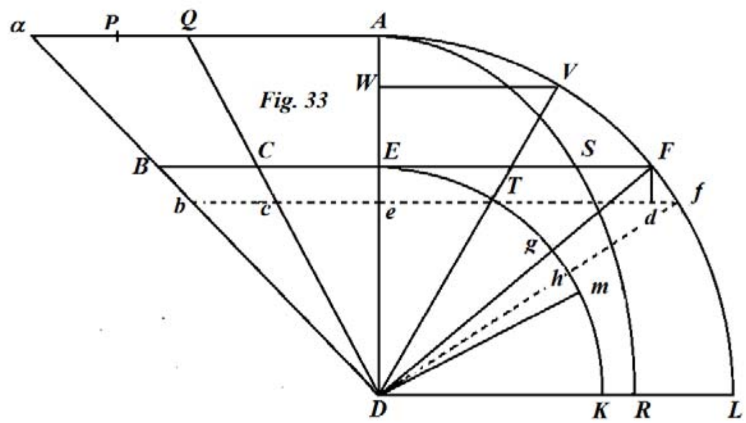
127. It may be desired as conceded that the motions of descent or ascent along the element of the axis *Ee*, or of the curve N*n*, can be taken as uniform and equal, to be conceded whenever the moving bodies A and M complete these elements of distance with a speed of finite magnitude, before they are able to begin to run through an infinitesimal increment or decrement of that, although in truth the motions in the same distances shall be varied, because the increments or decrements of the speeds are infinitesimal, added to or taken from the moving bodies A and B while in descent or ascent, they vanish before finite velocities, with which the bodies traverse the distances Ee and Nn.

COROLLARY.

128. Hence because the distances ran through with an equal motion shall be composed in the ratio of the times and of the velocities, these times, in which the aforementioned right or curved elements may be traversed, will be, as these elements correspond to the speeds, with which they are traversed ; thus $t\mathrm{E}e = \mathrm{E}e : \mathrm{EF}$, and $t\mathrm{N}n = \mathrm{N}n : \mathrm{EF}$; from which $\mathrm{E}e = \mathrm{EF}.t\mathrm{E}e$, and $\mathrm{N}n = \mathrm{EF}.t\mathrm{N}n.$

PROPOSITION XV. LEMMA.

Fig. *33. The rectilinear angles* VDF *& TDm are directly as the arcs* VF *and* Tm *which subtend the angles, and reciprocally as the arcs of the radii* DV *and* DT.



That is the angle $\mathrm{DF : angle\ TDm} = \frac{\mathrm{VF}}{\mathrm{DV}} : \frac{\mathrm{T}m}{\mathrm{DT}}$. For there is

$\mathrm{ang.VDF : ang.TDm} = (\mathrm{Tg : DT}) : (\mathrm{Tm : DT})$; and on account of the similar arcs VF and TG, there will be $\mathrm{Tg : DT = VF : DV}$, therefore the

$\mathrm{angle\ VDF:\ angle\ FDM} = (\mathrm{VF : DV}) : (\mathrm{Tm : DT})$.

Q.E.D.

COROLLARY.

129. Hence :

I. any angle VDF can be expressed by the arc VF, which it subtends itself, divided by the radius DV, that is to the fraction VF:DF.

II. Any arc VF is as the product from the angle VDF and the radius DV, and thus can be expressed by this product.

PROPOSITION XVI. LEMMA.

130. *Every force acting uniformly is equivalent to the motion generated, depending on the time, in which this motion is produced.*

The mass of the body requiring to be moved may be called M, the speed being acquired V, and thus the motion being generated MV, so that the time to be produced must be T , from which acting uniformly in this time the force must be G , being shown to $\mathrm{G = M.V : T}$. Since the interaction arising from dead forces produces no motion, unless it were continued or repeated for some time in the body, and now the force is put to act uniformly, thus so that in equal times equal amounts of motion shall be produced in the body, and thus the motions produced shall be as the times in which they are generated, from that itself it is clear, that because in the time T the product of the motion produced M.V. is equivalent to the force G, so that with $\mathrm{M.V:T}$ multiplied by T, M.V may be produced, it follows that $\mathrm{G = MV:T}$. Q.E.D.

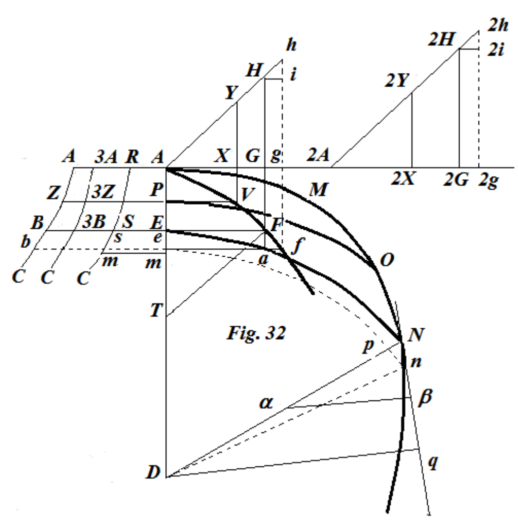
SCHOLIUM.

131. The matter may be considered generally, as the proposition indicates whether the velocity was acquired in a finite time or they were infinitely small, but the action G may act for the whole time uniformly. But truly that may be worked out in different ways, the proposition can no more be found with respect to any time ; but only with the time treated for an infinitely short time dt, so that only an infinitesimal speed dV is acquired by the moving body, because only through an infinitesimal time can its infinitesimal speed dV be considered as uniform by the action of the variable G (§.117.) ; therefore in this case there is $\mathrm{G = MdV:dT}$, and thus $\mathrm{dT = MdV:G}$. Where G indicates the weight or the variable force of gravity on a mass M.

PROPOSITION XVI. THEOREM.

132. *The moment of each force is equal to the moment of the speed of the mobile body multiplied by the mass of the body.*

With these matters in place, which have been discussed in §§. 121, 123 and 125, the mobile body M is placed to descend on the curve MON, [Note : this is a curve fixed in space, such as a frictionless wire, etc.] and through the starting point *A* of the graph of the speed AVF, an indefinite right line AH shall be drawn, containing the angle HAG of a half right angle, with the indefinite line AG normal to the axis AD, and through the points F, f &c. of the graph of the speeds, *a*H and *fh* are acting parallel to the same axis crossing the line AH at the points H, *h,* and indeed the other axis AX at the points G, *g;* and with H*i* drawn parallel to AG, and on account of the semi- right angle HAG, HG and AG, and thus EF are equal, and the elemental right rectangle to be $\mathrm{H}g = \mathrm{EF}.af$ or equal to the moment of the increasing speed. It is required to be proven, that $\mathrm{M.EF}.af = \mathrm{N}n.\mathrm{ES}$, or to the moment of the tangential force ES or $\mathrm{N}\beta$.



Demonstr. By §.131. there is $t\mathrm{N}n = \mathrm{M}.af:\mathrm{ES}$; for, which there are called dT, dV & G, in their place may be called $t\mathrm{N}n$, af and ES or $\mathrm{N}\beta$ respectv., therefore

$\text{ES.}t\text{N}n = \text{M.}af$, or also $\text{ES.EF.}t\text{N}n = \text{M.EF.}af$; and by (§.128) $\text{EF.}t\text{N}n = \text{N}n$, therefore $\text{ES.N}n = \text{ES. E}m = \text{M.EF.}af = \text{M.HG}g$. Q.E.D.

[In modern terms, we can express the time increment *t*N*n* as *dt*, the velocity increment dV or *af* as *dv*, and the weight or force G as F in which case, from $F = ma = m\frac{dv}{dt}$, we have $dt = m\frac{dv}{F}$ or $t\text{Nn} = \text{MdV:G}$; subsequently, the element of distance gone $ds = vdt$ or $\text{N}n = \text{ES.}t\text{N}n$, and

$$Fds = m \cdot \frac{dv}{dt} \cdot ds = m \cdot vdv \text{ or } \text{ES.N}n = \text{ES. E}m = \text{M.EF.}af = \text{M.HG}g.$$

Thus, the work done in falling integrates to give the increase in the kinetic energy.]

Similarly if the weight A may be carried downwards along the right line AD, there will be $\text{BE.Ee} = \text{A.}_2\text{H}_2\text{G}_2\text{g}$, by putting A to denote the mass of the body, $_2\text{H}_2\text{G}$ the speed at E acquired after falling from the height AE, and the rectangle $_2\text{H}_2\text{G}_2\text{g}$ inscribed in the right angled isosceles triangle $_2\text{H}_2\text{A}_2\text{G}$; the moment of the velocity $_2\text{A}_2\text{G}$ or $_2\text{H}_2\text{G}$. On account of the forces of retardation there will be had also :

$$se.n\text{N} = \text{M.}ef\,.\,fa = M\,.hg\text{G}, \;\&\; be\text{E} = A_2\text{h}_2\text{g}_2\text{G}.$$

PROPOSITION XVIII. THEOREM.

133. *If the two equal bodies* A *and* M, *the first of which may be carried along the curve* MON, *and the other along the right line* AD, *at the individual points* E & N *equally at the same distances from the centre* D, *may be acted on by the same force* BE, *everywhere the moment of the central action or the rectangle* BEe, *will be equal to the homologous rectangle* Nn.SE *or* Nn.Nβ, *which expresses the moment of the tangential action along the curve at any point* N.

The arc of the circle *epn* described with centre D makes the right line DN, at the point *p,* and the triangle N*np* as rectilinear on account of its infinite smallness can be considered as a right angle at *p,* and thus similar to the triangle Nαβ. Therefore there will be $\text{N}\alpha \text{ or BE} : \text{N}\beta(\text{SE}) = \text{N}n : \text{E}e(\text{N}p), \;\&\; \text{BE. E}e = \text{N}\alpha.\,\text{E}e = \text{SE.N}n = \text{N}\beta.\text{N}n.$ Q.E.D.

COROLLARY I.

134. Hence if the perpendicular FT of the graph of the speed AVF at the point F is drawn crossing the axis at T, the subnormal ET will be equal everywhere to the respective BE, which the central force expresses at the distances ED, ND equal from the centre D ; for because TFf is right, the triangles FTE and *Ffa* will be similar, and thus $\text{FE} : \text{ET} = Fa(Ee) : af$. Therefore $\text{EF.}af = \text{TE. E}e$; but (§ .133.) $\text{SE.N}n = \text{BE.E}e$ & (§.132.) $\text{SE.N}n = \text{M. EF. af}$, or with M taken in place of one, $= \text{EF.}af$, there will be $\text{TE.E}e = \text{BE.E}e$ and thus $\text{TE} = \text{BE}$.

COROLLARY II.

135. At any of the points E, N at equal distances ED and ND from the centre D, the moments of the speeds acquired by falling shall be equal. For just as (§.132.) AG or GH express the speeds acquired by the moving body M descending on the curve MON, and the rectangle Hg inscribed in the triangle HAG express the moment of the speed at N, thus also $_2A_2G$ and the rectangle $_2H_2g$ inscribed in the triangle $_2H_2A_2G$, express the speed acquired at the point E from the descent of the moving body A through the distance AE, and its moment. Truly (§.132.) there is $\text{SE.N}n = \text{M.EF.}af = \text{M.HG}g$, likewise $\text{BE.E}e = \text{A.}_2\text{H}_2\text{G}_2g$ and (§. 133) $\text{SE.N}n = \text{BE.E}e$; therefore there becomes $\text{M. HG}g = \text{A.}_2\text{H}_2\text{G}_2g$ or because (following the hypothesis) M and A themselves are equal : $\text{HG}g = \text{A.}_2\text{H}_2\text{G}_2g$.

PROPOSITION XIX. THEOREM.

Fig.32. 136. *If the mobile bodies* M and A *may begin to fall from rest at the points* M *and* A *on the curve* MON *and the right line* AD, *the speeds of these acquired at the points* N *and* E *will be equal with the distances from the centre* D *put equal, and the speed of each body will be twice the area* AZBEA.

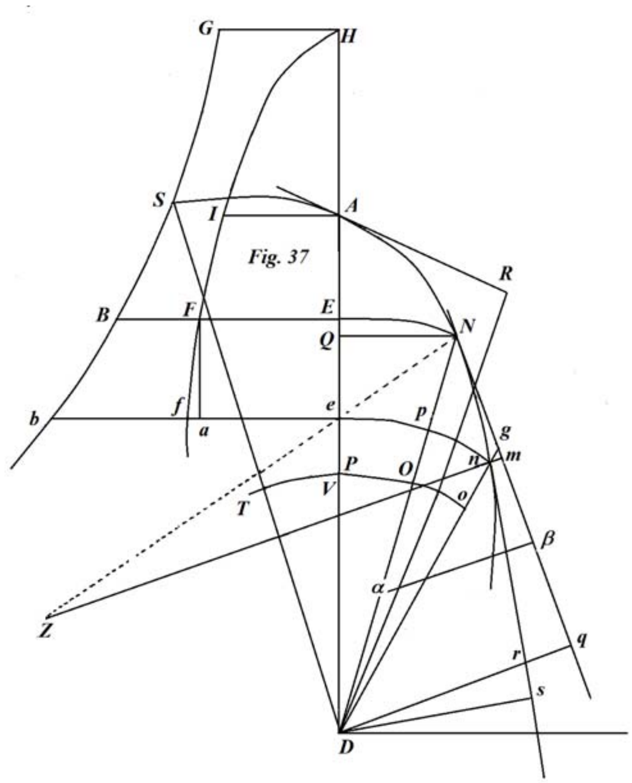
The whole distance AE may be considered to be divided up into innumerable elements, such as is the case for E*e*, and through the end of each an arc may be understood to be described from the centre D; thus so that all the elements of the curve may correspond to the individual elements of the right line AE, and to just as many elements of the curve MON. Now, since everywhere (§.135.), the moment of the speed at the point N of the curve may be equated to the moment of the speed at the point E on the right line, being equidistant to the other N from the centre D, always and everywhere there will be found $\text{HG}g = {}_2\text{H}_2\text{G}_2g$, therefore all the moments HG*g*, which are present in the triangle HAG are equal to all the moments $_2\text{H}_2\text{G}_2g$, which are present in the triangle $_2H_2A_2G$, that is the triangle HAG will be equal to the triangle $_2H_2A_2G$. From which since these triangles shall be equal, it follows the homologous sides in these are equal, and thus $\text{AG} = {}_2\text{A}_2\text{G}$; that is the speeds acquired at E and N are equal, which are expressed by the equal lengths $_2A_2G$ and AG.

Again, because (§.132:) $\text{HG}g = \text{BE.E}e$: and thus everywhere, all the $\text{HG}g$ will be equal to all the $\text{BE.E}e$, and all the elementary rectangles HG*g* composing the triangle HAG, or *h*A*g*, since the motion at the points A and M is zero, and thus also the moments of the speeds at these points are zero, that is with the vertices of the triangles HAG, and $_2H_2A_2G$, and with A and $_2A$ combined with these, as thus I may say, truly with all the elementary rectangles BE*e* made zero, which are contained in the whole area AAZBE, to which area finally they are made equal, and thus the area AAZBE everywhere is equal to the homologous triangle HAG, and twice its area with twice the area of that triangle, but twice this triangle is equal to the square of side AG or GH, because this right angled triangle is isosceles ; therefore $\text{AG}^2 = 2.\text{AAZBE}$, or because

AG is equal to EF, $EF^2 = 2.AAZBE$, that is the speed acquired at E or N can be twice the area AAZBE. Which all were required to be demonstrated.

COROLLARY I.

137. Equally, if the bodies A and M may begin to fall with equal speeds from the points O and P on the curve and right line, and indeed there, that can be acquired by the body A gliding along from rest through the distance AP at the point P ; with the same bodies at some other points N and E, at the same distances from the centre of action, they will acquire an equal speed, evidently as much as with the body A, at this point A falling from beginning at rest, by falling with an acceleration through the distance AE may acquire at the point E. For since generally (§.135) there is $HGg = {}_2H_2G_2g$, now also all the HGg will be equal to all the ${}_2H_2G_2g$, truly in this case all the GHg are these, which only have been inscribed in the trapezium YXGH, evidently with the right line VY drawn through V parallel to the axis AD, because the speed, with which the body M begins to move from the point O of the curve, is not zero ; but (following the hypothesis) equals PV or AX or XY, and thus now all the HGg are equal to the trapezium YXGH , and the same by the same reason all the ${}_2H_2G_2g$ are equal to the trapezium ${}_2Y_2X_2G_2H$, because the body A begins its fall at the point P with the speed ${}_2X_2Y$ (following the hypothesis) to be equal to XY or AX; therefore these trapezia YXGH and ${}_2Y_2X_2G_2H$ are equal; truly because (following the hypothesis) the speed at P is the same as the speed at O, and that much is the amount the body is able to acquire by falling from A to that place P, it follows the triangles AXY and ${}_2A_2X_2Y$ likewise are equal, and by adding equal triangles the trapeziums YXGH and ${}_2Y_2X_2G_2H$ are equal, also the triangles AGH and ${}_2A_2G_2H$ become equal; from which since the same shall be similar, it is necessary that the sides AG and ${}_2A_2G$ shall be equal, and thus it is clear the speeds at N and E acquired by the mobile bodies O and P, at equal distances from the centre D by beginning to fall with equal speed PV or AG, to be equal between themselves, then for the speed EF acquired at E (§. 136.) after the fall from rest at A, through the distance AE, which potentially is equal to twice the area AAZBE.



COROLLARY II.

138. Fig. 37. Therefore, if a certain body A be given a certain impulse AR along some direction to that at A with that speed, as which the same body will acquire by gliding from rest at H, and with the actions from the ordinates of the curve GBb expressed continually at the centre D, it will describe the curve itself AN*n* in vacuo, its speed at any point N of the projected curve will be that, which the body beginning to fall from rest at H on the right line HD will be able to acquire at the point E accelerated by its own motion, with the other on the curve N to be equally distant from the centre D.

COROLLARY III.

139. Fig. 32. If bodies are present at the points *n* and *e* of the curve and of the right line, and with the same distances from the centre D with an initial speed *ef,* which motion they were able to acquire at *n* and *e* after descending along the curve MO*n* and the right line A*e,* on starting from rest with an accelerated motion at the points M and A ; those may be carried upwards to some other points O and P that may be equidistant from the centre D and they will maintain equal speeds [between each other], evidently of such a size, as what they would have acquired at these points O and P, as if by beginning with an accelerated motion from rest at M & A, so that they may descent by the arc of the curve MO and by the right line AP. For because the bodies rise, and thus they recede from this downwards centre of attraction BE, nevertheless these central forces BE, just as the tangential forces SE or Nβ, which may be derived from these (§.122.) now are required to be considered as *retarding forces.* Still the moments of these *be.e*E and *fe.n*N (§. 133.) will be equal, and by §.122. its moment shall be equal to this *hg.g*G of the speed of

descent *hg*, truly for that of the speed $_2h_2g$ the moment $_2h_2g._2g_2$G, therefore these are themselves the equal moments of the decreasing speed, and in this regard of the element *n*N any other curve of all of these which are contained in this right segment *eP* on the arc *n*O of any curve, and of the homologous element eE of the right line DA.

[The moments *be.e*E and *fe.n*N can be regarded as increments in the work done, or in the potential energy, positive or negative, and these are equated to the related changes in *vdv* of the common speed : it appears thus that Hermann had an intuitive concept of the conservation of potential + kinetic energy, arising from Newton's second law, that speed and height can be interchanged in this way for all masses, as demanded by Galileo. These ideas were present also in Leibniz's writings.]

Hence also the whole of *hg.g*G, which come together in the trapezium *hg*XY, will be equal to the whole $_2h_2g._2g_2G$, which come together in this circumscribed trapezium $_2h_2g_2X_2$Y, that is , these two trapezia will be equal.

[*i.e.* the area of the whole triangle is the total kinetic energy involved in falling or rising, while the area of the trapezium is the change in the kinetic energy in moving from one height to another.]

From which, if from the triangles *hgA* and $_2h_2g_2$A, which on account of the initial speeds Ag and $_2A_2g$ at *n* and *e* (following the hypothesis) are also equal to each other, these equal trapezia *hg*XY and $_2h_2g_2X_2$Y may be taken away, and they will leave the equal triangles YAX and $_2Y_2A_2$X, in which, as with the like, the homologous sides AX and $_2A_2$X are present equal, but these equal sides express the remaining speeds at O and P for the ascending bodies, which velocities hence are equal, and each of these is that (§. 136.), which will be acquired by the body at P beginning to fall from rest at A, as which [squared] can have twice the area AAZP, while the initial speed [squared] *ef* can be twice the area AAZ*be*. Therefore GX and $_2G_2$X are the parts of the initial velocity lost, or absorbed from the central forces acting B*e*, BE &c. in the ascent of the bodies, and thus so that the motion of the bodies at the points M and A for the curve and the right line by rising the motion of the reduced ascent shall be destroyed completely, since the absorbed speeds *g*A and $_2g_2$A shall be equal to the initial speeds.

COROLLARY IV.

140. Hence ascending bodies with any initial speed whatsoever maintain successively all the levels of the speed at these points of the line being run through, in which be falling it will have acquired these levels of the speed. Thus a body ascending with the speed *ef* beginning at *e*, at the remaining points E, P, A made of the distance *e*A in the ascent retains the same speeds, which it has acquired by beginning its fall from the point A by its acceleration, evidently at the same points, the speeds EF, PV &c. Which also is required to be understood about the body ascending along the curve MON.

COROLLARY V.

141. Hence also under all possible hypothesis of the central force; *bodies carried off to a height along right lines, or by ascending along any curved lines, rising to the same height in vacuo, acquire a speed, which they gather on descending, equal to the initial speed upwards*.

COROLLARY VI.

142. If the centre of the force of gravity D were infinitely distant from the right line AG, the concentric arcs AM, PO, EN &c. will be changed into right lines put in the direction by the ordinates AA, AEP , BE of the graph of the central forces, from which as has been shown generally in the preceding, they will be obtained in that case also, in which the directions of gravity are right lines perpendicular to AG , or are parallel to the axis AD. And thus *the speeds at different inclinations of planes* (having to be added to the continued curving in the case of the curve) *for the inclinations of the descent acquired, are equal under the all the hypothesis of uniform and varied gravity, if there were equal elevations of planes or curves, or what amounts to the same thing, if the distances passed through of the ends of the inclined planes were equal spaced horizontal planes.*

This corollary itself was demonstrated first by the geometer Torricelli in his manner, and later itself by Huygens in his *Horologio Oscillatorio* in the second part. Prop. VI, but both to some extent in the Galilean hypothesis of uniform gravity and by indirect demonstrations. Truly Galileo postulated that to be self-evident in the first edition of the Dialogues; but because afterwards he considered this perhaps to be a great postulate, that in the later edition of his works, 1656, Boulogne made some attempt to demonstrate ; but the demonstration of that, which was offered there, contained little as judged by Huygens.

SCHOLIUM I.

143. In this proposition, just as in the preceding XVIII moving bodies are considered equally both as in a ratio of amounts of matter as well as in a ratio of particles woven together, or what amounts to the same, the bodies M and A, as long as the volume and mass must be equal, as the propositions reviewed generally prevail, evidently also in that case, in which the weights are not proportional at equal distances from the centre of attraction D. For indeed truly the speeds acquired at E and N after the descent of the bodies from rest at A and M through the distances AE and MON shall be equal always, whether the masses of these bodies, or from the weights acquired from the attractions from the centres, shall be proportional to each other or not.

144. But if the mobile bodies M and A do not have weights proportionate to their masses, nor are they with equal weight and mass, the present proposition cannot be extended to such bodies ; for in this case the predicted speeds, acquired at these points N and A, are not equal, but the velocity at the point N of the curve itself will be had to the velocity of the right line acquired at the point E, as the side of a square of area AABE, of which the ordinate BE or the central force of the body M gravitates, in traversing the curve MON, are set out at the homologous points N, applied to the root of the moving mass M; to the side of the square of area A_3A_3BE, of which the ordinates E_3BE set out the weights of the body A at whatever points E, through which ordinates they pass, applied to the root of the mass or quantity of matter of the moving body A, which is carried on the right line AD. For if, as above, the side AG either is equal to GH in the right angled isosceles triangle HAG, the speed of the body M acquired at the point N from the descent through the arc of the curve MON, and the equal sides ${}_2A_2G$ or ${}_2H_2G$ of the isosceles right angle ${}_2H_2A_2G$ the speed of the body A are set out from the case of that acquired through the distance AE

at the point of the right line E , the moments of the velocities AG, ${}_2A_2G$ will be the rectangles HG*g* and ${}_2H_2G_2g$ inscribed in these similar triangles, and by §§.132, 133, there will be $BE.Ee = SE.Nn = M.HGg$, therefore all BE.E*e*, that is, so that in the demonstration of article 136 shown, the area AAZEBE the area is equal to all M.HGg, that has been made from the mass of the body M by the triangle, and thus $AG^2.M = 2.AAZBE$ and $AG : GH = EF = \sqrt{(2.AAZBE)} : \sqrt{M}$. And equally there is (§. 131.) ${}_3BE.Ee = A.{}_2H_2G_2g$, and thus all the ${}_3$BE.E*e*, or all E_3B.E*e*, that is, the area A_3A_3BE is equal to all the $A.{}_2H_2G_2g$ that is made from the triangle ${}_2H_2A_2G$ by the mass of the body A; therefore also ${}_2A_2G = {}_2H_2G = \sqrt{(2.A_3A_3BE)} : \sqrt{A}$. And thus AG is to ${}_2A_2G$, that is, the speed acquired by the body M at the point N of the curve is to the speed of the body *A* on the right line for the point E to the other from the centre D acquired to be equidistant, just as $\sqrt{(2.AAZBE)} : \sqrt{M}$ to $\sqrt{(2.A_3A_3BE)} : \sqrt{A}$ or also, just as $\sqrt{(AAZBE)} : \sqrt{M}$ to $\sqrt{(A_3A_3BE)} : \sqrt{A}$, that is, the said velocities are in a ratio composed directly from the square root of the area AAZBE of the area to the area A_3A_3BE, and reciprocally as the square root of the mass M to the mass A. And this ratio composed cannot become one of equality, unless the area AAZBE were to the area A_3A_3BE, as mass M to mass A*;* that is, unless the weights of the moving bodies themselves were proportional to the masses. And also the area AAZBE to the area $A_3A_3Z_3BE$ at the same height AE cannot be in the given ratio of the mass M to the mass A everywhere, unless the individual ordinates BE shall be to the homologous individual ordinates ${}_3$BE in the same ratio M to A, and thus with BE and ${}_3$BE expressed by the weights of the bodies M and A, at equal distances ND and ED from the centre D, it shall be apparent the speeds acquired at N & E cannot be equal, unless the weights of the bodies M and A at equal distances from the centre of gravity of the masses shall be in proportion.
[Thus, essentially, the equivalence of inertial and gravitational mass has been established.]

SCHOLIUM II.

145. At this point the general motions concerning gravity have been shown, so that from that not only everything may be understood about any motion which can be mentioned, whatever the acceleration ; but also they may show in an easy way which hypotheses shall be possible, and which in turn may be carried off to be rejected by nature. For, in order that I may begin from these, it is with all the impossible and imaginary hypothesis of interactions or of gravity, of which the graph passes through the start of the descent, thus so that its ordinate there shall be zero ; it is sufficient for me being silent in the accounting of this matter at the beginning. For the mobile body is resting, because it is not heavy, or is not set in motion by any central attraction (which happens at the point, at which meets the graph of the right line attraction, at which the moving body must begin to fall) cannot itself move. Truly this impossibility also shows itself in the consideration of the time, in which some distance must be traversed in imaginary hypotheses of this kind, since to be running through however small a distance may require an infinite time

in accordance with these hypotheses, which with the use of the formulas from the preceding mutually requiring to be taken whichever it will be able to test. The force of gravity as freely variable may be called *g*, the distance being described by the body falling perpendicularly being described by *x*, the time in which it has run through must be *t*, the velocity acquired at the end of this time *u*; the elements of the distance, of the time, and of the velocity *dx*, *dt* and *du*, with which in place (§. 132.) the first formula will be $gdx = udu$, and(§.131.) there will be obtained $dt = mdu : g$, where *m* denotes the mass of the moving body and *g* its weight : because indeed the body is compared with no other, but only its motion can be investigated, in place of *m* one itself can be used, and there will be $dt = du : g$. The second formula.

146. Now the speeds can be acquired. so that the distances can be done, just as Baliani wished [Baliani was a 17th century Italian physicist who corresponded with Galileo, and who first enunciated the law governing the acceleration of a body, and distinguished between mass and weight, in his book: *De motu naturali gravium, fluidorum et solidorum* : *Concerning the natural motion of weights, fluids and solids*], that is, $u = x$, $uu = xx$ and $udu = xdx$, and there will be on account of $gdx = udu$, also $gdx = xdx$ that is $g = x$, therefore with $x = 0$ present there will be also $g = 0$, therefore in principle there will be no descent of the moveable weight, and thus itself at no time may be able to fall through the distance *x*. Again, because the second formula $dt = du : g$ & $g = x$, gives rise to $dt = dx : x$, from which $t = \log.x$, now with $x = 0$ present, and log. of 0 itself infinite, which we will designate by ∞, there will be $t = \infty$, that is, an infinite time is required to traverse the initial zero distance, and thus a weight will remain there perpetually, and consequently the Baliani hypothesis is impossible and imaginary. Besides I have shown this by another account in the Actis Lips. I709. page 404. seq.

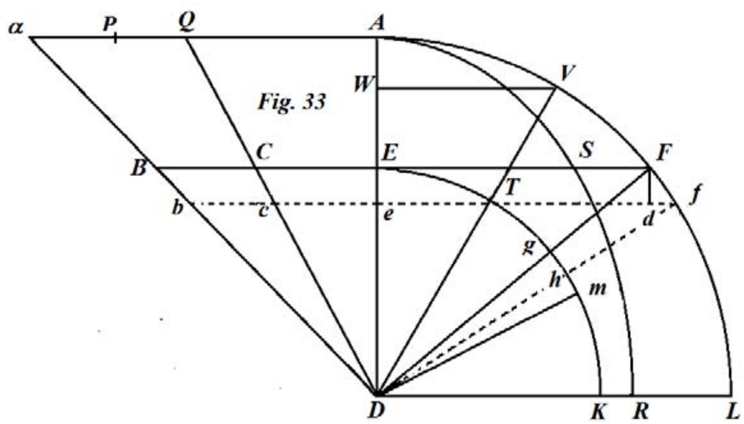
147. If the speeds acquired *u* shall be as x^m, where *m* is any whole number and a positive index of the power of the distance, to which the speed is proportional, there becomes $uu = x^{2m}$ *and* $udu = mx^{2m-1}dx = gdx$, and thus $g = mx^{2m-1}$, and with $x = 0$ present, also there will be $mx^{2m-1} = g = 0$, and therefore it is of this kind of impossible and imaginary hypothesis.

Again $dt(= du : g = mx^{m-1} : mx^{2m-1}) = dx : x^m$. Truly following article 92 by entering the summation, there is found $t : (\infty^{m-1} \ , -x^{1-m}) : m-1 = \infty^{m-1} : m-1$. That is, the descent time through some minimum height is according to this hypothesis, as the power of infinity ∞, of which the exponent is the whole number $m-1$. Therefore this hypothesis, much more than the preceding, is imaginary and impossible.

Thus far we have had the general motion of the acceleration, it remains to be seen clearly, what ought to follow from one or another hypothesis of gravity.

HYPOTHESIS I.

148. Fig.33. *If the actions of gravity shall be as the distances from the centre of gravity* D, *the graph of these will be a right line* QCD *drawn through the centre D containing some angle QDA with the axis AD, and the graph of the speed the elliptical curve* ASR, *whose transverse semiaxis is* AD, *and of which the conjugate semiaxis* DR *equal to the mean proportional between* DA *or its equal* $A\alpha$ *, and* QA.



For with the quadrant of the circle AVL described about the centre D, and αD drawn, and with BEF parallel to DL or αA*;* because DR (by the construction) is the mean proportional between DL or αA and QA, on that account there will be the ellipse ASR ; $\mathrm{EF}^2 : \mathrm{ES}^2 (= \alpha\mathrm{A} : \mathrm{QA}) = 2.\text{trapezium } \alpha\mathrm{BEA}$ to $2.\text{trapezium QCEA}$, and twice the trapezium αBEA is the excess of the square AD or DF over the square DE, and thus is equal to EF^2; therefore also twice the trapezium QCEA also can be equated, which is twice the area on the graph of the central attraction QCD, and the square ES, and thus (§. 136.) ES expresses the speed acquired at E from the descent through AE ; therefore the ellipse ASR is the graph of the speeds acquired. Q.E.D.

149. *If the angle* X *becomes in the ratio* DL *to* DR *to the angle* ADF, *the angle* X *itself sets out the time of descent of the mobile body through* AE , that is $\mathrm{X} = t\mathrm{AE}$. For if the element x of X to the homologous element of the angle ADF, which is FDf, surely *bf* drawn indefinitely near to the other BF. Therefore there will be
$\mathrm{x} : \mathrm{FDf} = \mathrm{DL} : \mathrm{DR} = \mathrm{EF} : \mathrm{ES}$, therefore $\mathrm{x.ES} = \mathrm{EF.ang.FDf}$, or
$\mathrm{x.ES.DF} = \mathrm{EF.DF.ang.FDf}$, and (§. 129.) DF by the angle FDE expresses the arclet Ff, therefore $\mathrm{x.ES.DF} = \mathrm{EF.Ff}$, but indeed $\mathrm{FE.Ff} = \mathrm{DF.Ee}$, which is readily deduced from the similarity of the triangles Ffa & FDE, therefore $\mathrm{x.ES.DF} = \mathrm{DF.Ee}$, or with DF deleted, there becomes $\mathrm{x.ES} = \mathrm{Ee}$, and $\mathrm{x} = \mathrm{Ee} : \mathrm{ES}$. Truly (§. 118.) $\mathrm{Ee} : \mathrm{ES} = \mathrm{tEe}$, therefore $\mathrm{x} = \mathrm{tEe}$, and thus the sum of all the increments x is
X = to the sum of all the increments of the time tEe, that is tAE. Q.E.D.

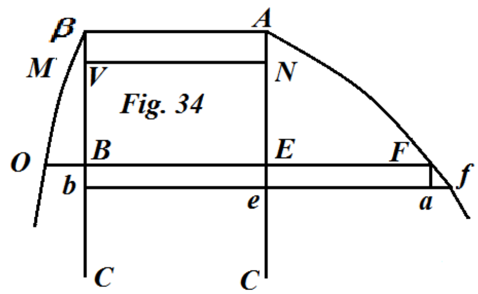
Hence with some radius DV drawn, which cuts the concentric quadrants AVL and ETK at V & T, and with the sines VW & T*e* of the arcs drawn AV & ET (it is by accident, that *Te* may fall on *bf* , the matter can be considered in any other way) the distances AW and E*e* are described, from the time in which equal mobile bodies A and E likewise begin to fall; and from that time both arrive at the centre D.

[This is an early discourse on simple harmonic motion: See, e.g. Newton's *Principia* : Book I, Section II. Prop. X, problem V ; Lamb's *Dynamics*, p. 76.]

HYPOTHESIS II.

150. Fig. 34. *If the weights of bodies is the same everywhere, as in Galileo's hypothesis, the graph of the speeds will be a conic parabola, of which the axis of the abscissas set out the distances requiring to be transversed by the moving body, and the ordinates the speed acquired. And therefore the speeds will be in the square root ratio of the distances completed.*

For the graph of the gravitational attraction will be, in this case, the right line βC parallel to the axis AC, and thus (§.144)



$2.\mathrm{A}\beta.\mathrm{AE} : \mathrm{M} = \mathrm{EF}^2$, where M denotes the mass of the body, Aβ its weight, and AE the distance completed by falling, and EF the speed acquired. From which on account of the constant Aβ the graph of the speed is a parabolic conic, and thus the distances described by the body are as the squares of the speeds acquired, and these velocities shall be in the square root ratio of the distances. Truly the parameter of the parabola is $2.\mathrm{A}\beta : \mathrm{M}$. [For $v^2 = 2mgs : m = 2gs$]

[Again, one is astonished that these are exactly the equations derived from kinetic/potential energy conservation; they have been derived without reference to the time parameter.]

151. *Truly the time in which some interval* AE *is completed by falling is* 1^{st}. *as twice the interval* AE *applicable to the speed acquired* EF *in this time is* 2^{nd}. *as the root from the product of the mass of the body by twice the distance applicable to the weight or gravitational attraction of the body. And finally* 3^{rd}, *as the product from the mass of the body by the speed acquired applicable to the weight of the same body.* As before, the time to pass through AE may be designated by tAE.

Now (§.131.), $\mathrm{tE}e = \mathrm{M.af} : \mathrm{A}\beta$, therefore on summing $\mathrm{tAE} = \mathrm{M.EF} : \beta$, which is the third result. [*i.e.* $dt = M.dv : F$; on summing, $\Delta t = M\Delta v : F$, or $F\Delta t = \Delta Mv.$ Or $F = M\Delta v : \Delta t = Ma.$]

Again because $\mathrm{tAE} = \mathrm{M.EF} : \mathrm{A}\beta$, again it will be $= \mathrm{M.EF}^2 : \mathrm{EF.A}\beta$, (§.150,)
$= 2.\mathrm{A}\beta.\mathrm{AE} : \mathrm{A}\beta.\mathrm{EF} = 2.\mathrm{AE} : \mathrm{EF}$. And that is the first result. [*i.e.* $t = mv^2 : mgv = 2h : v$.]

Finally, because the ordinate of the parabola is to half the parameter in the square root ratio of twice the abscissa to the half parameter, there will be $\mathrm{tAE}(=\mathrm{M.EF}:\mathrm{A\beta})=\sqrt{(\mathrm{M.2AE}:\mathrm{A\beta})}$. Which was the second result.

152. Therefore, by calling the two moveable masses M and *m* ; the forces of gravity or weights will be *G, g;* the distances described by falling S, *s* ; and finally the times of the descents T and *t.* There will be $\mathrm{T}=\sqrt{(2\mathrm{M.S}:\mathrm{G})}$ and $\mathrm{TT}=2\mathrm{M.S}:\mathrm{G}$, or $\mathrm{G.T}^2=2.\mathrm{M.S}$; & $gtt=2ms$. Thus, if the distances by both the different bodies are completed in different times shall be in the square ratio of the times, the weights of the bodies will be in proportion to their masses; for if $\mathrm{TT}:tt=\mathrm{S}:s$ there will be $\mathrm{T}^2s=tt.S$; and thus $\mathrm{G.T}^2=2.\mathrm{MS}$ becomes $tt.\mathrm{GS}:s=2.\mathrm{MS}$, or $Gtt=2.Ms$, or $gtt=\frac{2.g.M.s}{G}=2ms$ and thus $g\mathrm{M}=\mathrm{G}m$, or $\mathrm{G}:g=M:m$.

Therefore with the help of this paragraph the second proposition can be confirmed by experiment. See §. 30.

SECTION II.

Postea quam ea, quae ad aequilibria solicitationum, quibus unum idemque mobile urgeri potest, spectant, excussimus ipsi etiam motus varie accelerati atque retardati examinari debent, qui ex solicitationibus continue replicatis, & ut libet variabilibus, resultare possunt . Propterea in hac secunda Sectione Motus Corporum acceleratos & retardatos generalissime contemplabimur tanquam oriundos ex solicitationibus continue quidem replicatis , non tamen utique uniformiter, ut in Galilaei systemate, in quo gravia in quacunque distantia a centro gravium eadem gravitatis solicitatione urgeri intelliguntur; sed quacunque ratione variabilibus, sive corpora recta ad centrum gravium seu solicitationum acceleratricium ferantur : sive in lineis curvis incedant . Haec igitur Sectio complectetur quaecunque ad motus acceleratos, retardatos, ad motus projectorum in vacuo, quiae ad isochronismum motus corporum in lineis quibuscunque curvis, quae ad motus pendulorum &c. pertinent, in omni possibili gravitatis, seu solicitationum centralium hypothesi, demonstrata, methodo nonminus universali, quam perspicua atque facili: nam praeter solutionem generalem Problematis inveniendi ex lege gravitatis, ut libet variabilis, semitam curvilineam projectorum in spatiis resistendi vi carentibus, tradet etiam haec secunda Sectio regulam generalem , secundum quam gravitatem variare oportet ad id, ut curvae projectorum semper algebraicae, atque adeo geometrica construibiles evadant, ex qua multa deinceps curiosa atque utilia deducentur Theoriam centri oscillationis proponet multo magis quam antea generalem, iis namque corporibus applicabilem, quae in diversis fluidis oscillantur , vel in quocunque systemate gravitatis variabilis, loco ejus, quod Huygenius, & qui eum sequuti sunt, tantum egerunt de centro oscillationis in casu speciali gravitatis uniformis, excepto solo, quod sciam, Cel. Joh. Bernoulli qui nobis in Act. Lips. 1713, pag. 88, regulum generaliorum ex suis principiis eliciendam promittit, sed quae nondum, ni fallor lucem publicam aspexit, quamque ex fundamentis, a nostris diversis, aut saltem diversa methodo ab incomparabili Geometra erutam existimo, tametsi ipsam regulam nondum vidi. Nec reticendum hanc Sectionem secundam etiam acturam de solicitationibus requisitis ad id, ut mobilia in orbibus mobilibus revolvi queant, & de motu qui dicitur Apsidum; sed diversa ratione ab ea, qua Ill Newtonus Sect.lX . Lib. I. Princip. Phil. Math. usus est. Sectionem denique claudet dissertatio de regulis motus ex percussione, quas ex unico fundamento aequalitatis virium ante & post conflictum corporum, nova, ut nobis videtur, ratione deducemus.

CAPUT I.

De generalibus Solicitationum continuatarum affectionibus, et de motibus in Vacuo inde oriundis.

DEFINITIONES.

I.

114. Per vacuum designatur omne medium, quod corpora absque impedimento aut adjumento libere trajicere possunt, solo suo motu , a vi motrice accepto.

II.

115. Si mobilia quaecunque, quae in lineis rectis feruntur, aut in curvis lineis incedunt, solicitationibus citantur, quarum directiones in aliquo puncto, positione dato, concurrunt, solicitationes ejusmodi vocentur centrales, vel etiam solicitationes gravitatis variabilis. Et punctum positione datum ad quod gravia solicitantur, centrum solicitationum. Solicitationes centrales ab Newtono vires centripetae vocantur.

III.

116. Solicitationes continuari vel continue mobile applicari dicuntur, cum corpus in singulis spatii percurrendi punctis ad motum recens citatur, seu, quod idem est, a quadam solicitatione urgetur.

IV.

117. *Scala solicitationum centralium* seu *gravitatis variabilis* appellatur quaelibet linea curva, ad aliquem axem per centrum solicitationum transeuntem relata, cujus ordinatae axi rectae solicitationes exponunt, quibus mobile in illis axis punctis, per quae ordinatae ductae sunt, versus centrum urgetur, si scilicet corpus in axe feratur; vel, si ipsum in curva aliqua incedat , in illis curvae punctis, quae cum punctis in axe, per quae ordinatae transeunt, a centro solicitationum aeque distant.

V.

118. Solicitatio gravitatis *tangentialis* vocatur quaelibet solicitationis ex centrali derivata, qua mobile, in qualibet curva delatum, juxta directionem tangentis curvae solicitatur.

VI.

119. *Solicitatio* gravitatis curvae mobili describendae *perpendicularis,* est ea, quae ex centrali derivatur, cujusque directio ubi que curvae perpendicularis est. Haec solicitatio perpendicularis conatui mobilis a directione tangentis recedere nitentis ubique aequalis est, alioqui corpus non moveratur in curva, quam describere ponitur, etenim si hac vi perpendiculari conatus a curva recessorsius, major esset mobile, revera a curva recederet, recederet itidem accedendo magis quam par est ad centrum solicitationum si minor existeret utrumque contra hypothesin. Igitur ut conatus recessorius a curva retundatur & inutilis reddatur, tam hae solicitationem coercentem mobile applicari oportet, quantus est conatus destruendus; adeo ut ejusmodi conatus, & solicitatio, juxta eandem directionem curvae perpendicularem, sed in oppositas partes agentes in aequilibrio detineantur impedianturque quominus alterutra ejusmodi solicitationum super alterum effectum aliquem sortiatur, atque adeo mobile curvam, in qua incedere debet, deserat

VII.

120. *Scala* solicitationum *tangentialium* est curva quaelibet ad axem per centrum solicitationum centralium transeuntem relata, cujus ordinatae solicitationes tangentiales exponunt in curvae punctis, a centro solicitationum tantundem distantibus, ac ordinatae.

121. Fig. 32. Feratur mobile M in curva MON, alterumque A in recta AD per centrum D, ad quod solicitationes gravitatis centrales diriguntur, atque mobile M ad quodlibet curvae punctum N delatum urgeatur versus D solicitatione centrali Nα, & hac eadem solicitatione urgeatur etiam mobile A delatum in punctum E rectae , AD, quod punctum E tantum distet a centro D, quantum curvae punctum N, adeo ut ambo puncta N & E reperiantur ubique in aliquo arcu circulari EN centra D descripto, ductisque per singula puncta axis E, perpendicularibus EB ipsi AD & aequalibus ubique homologis lineis Nα , per singularum puncta B transibit curva ABC, quae vocatur scala solicitationum centralium, quoniam quaelibet ejus ordinata BE respectivae Nα (secundum hypothesin) aequalis repraesentat solicitationem centralem in rectae AD & curvae MO punctis E & N centro D aequidistantibus. Porro, quia solicitatio centralis Nα in quolibet curvae MO puncto N (§. 39. 40.) aequipollet lateralibus Nβ & $\beta\alpha$, quarum haec sit in directione tangentis curvae Nq, alteraque eidem perpendicularis $\beta\alpha$, per se patet illam, scilicet Nβ, vocandam esse solicitationem tangentialem ex centrali Nα derivatam; propterea si in qualibet ordinata BE respondi cuique Nα sumatur SE = Nβ , & idem factum intelligatur respectu omnium reliquorum curvae MON punctorum, resultabit inde curva RSC, quae scala solicitationum tangentialium mobilis M in curva MON audit. Solicitatio perpendicularis $\beta\alpha$ nullam habet scalam, vel saltem nulla indigemus, quandoquidem in singulis punctis destruendo conatui recessorio mobilis M a curva (§. 119.) adhibetur, nec proinde alia sub ratione in considerationem venit.

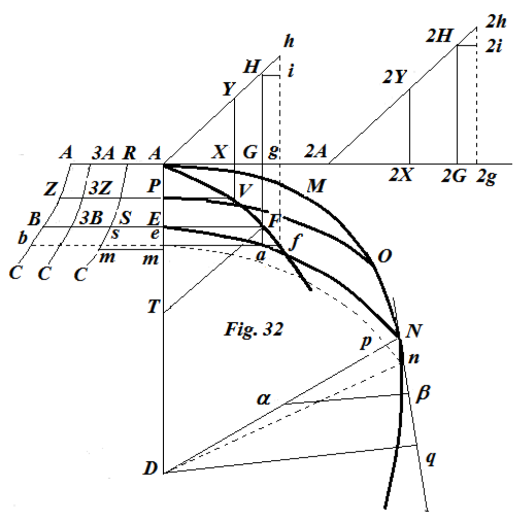


Fig. 32

VIII.

122. Quoniam solicitationes centrales BE aeque ac tangentiales SE continuantur, atque in mobili continue replicantur effectu praecedentium in succedentibus non sublato, quandoquidem per singula rectae AD puncta ordinatae ejusmodi BE transeunt : necesse est, ut ab hujusmodi solicitationibus. indefinenter replicatis proveniant *motus* continue *accelerati,* cum mobilia A, M in recta AD & curva MON descendentia ad solicitationum centrum D accedunt, *retardati* vera cum certis quibusdam celeritatibus, quas *initiales* deinceps nominabimus, ex punctis E & N ascendere incipientia in recta & curva ab eodem centro D recedunt. Propterea solicitationes gravitatis centrales vel tangentiales

vocantur *acceleratrices* mobilis in recta vel curva descendentis ; eaedemque solicitationes vocantur *retardatrices* mobilis ascendentis, vel a solicitationum centro recedentis.

IX

123. *Momentum cujusque* solicitationis quocunque haec nomine veniat, est rectangulum sub recta, quae solicitationem exponit, & elemento spatii quod mobile ejusmodi solicitatione continae urgente transmittit. Sic rec-lum BEe est momentum solicitationis acceleratricis BE vel Nα , durante descensu mobilis *A* in spatiolo *Ee* ordinatae BE adjacente; rec-lum vero *beE* momentum solicitationis retardatricis *be* mobilis ejusdem in spatiolo eE ascendentis. Sic etiam facta E*m* = N*n* elemento curvae MON, rec-lum *SEm* vel SE. *Nn,* est momentum solicitationis tangentialis & acceleratricis SE vel Na , cum mobile in curva per ejus arculum *Nn* descendit, & rec-lum *se. n*N est momentum solicitationis tangentialis retardatricis *se ,* cum mobile in curvae arculo *n*N ascendit.

X.

124. *Scala celeritatum* est curva circa axem scalae solicitationum centralium, vel saltem huic parallelum descripta, cujus ordinatae ordinatis in scala solicitationum in directum positae , vel iisdem saltem respondentes, celeritates mobili in illis rectae AD, vel curvae MON punctis ad quae ordinatae scalae solicitationum respiciunt, acquisitas exponunt. Sic, si ordinatae PV, EF &c. scalae solicitationum ordinatis ZP, BE in directum positae, significant celeritates acquisitas, vel residuas mobilis descendentis accelerato vel ascendentis retardato motu in punctis P, E rectae AD , vel in punctis curvae O & N &c. Curvae AVF erit scala celeritatum mobili A in recta AD, vel Mobilis M in curva MON descendentis vel ascendentis.

XI.

125. *Momentum celeritatis* est factum ex celeritate mobilis in recta vel curva incedentis in celeritatis crescentis vel decrescentis elementum. Si in scala celeritatum AVF, rectangulum EF.af, vel rec-lum *eaf* est momentum celeritatis EF elemento *af* crestentis , qua mobile spatiolum *Ee,* transmittit, & rectangulum *ef.fa* est momentum celeratis decrescentis *ef,* qua mobile A ex puncto axis *e,* vel in curva MON mobile M ex puncta *n* in altum feruntur retardato motu, & *fa* decrementum infinitesimum celeritatis decrescentis *ef,* cum mobilia ascendunt in spatiolis *eE* & *nN,* & EF celeritas residua erit in rectae puncto E aut curvae puncto N.

XII.

126. Ad designandum tempus quo unusquisque motus absolvitur, utemur nota characteristica temporis *t* spatio percurso aut conficiendo· praefigenda. Sic *t*AE denotabit tempus, quo mobile *A* spatium aE accelerato suo motu cadendo absolvit, & *t*Ee significabit tempus quo elementum Ee axis AD uniformi motu percurritur . Et sic de reliquis.

POSTULATUM.

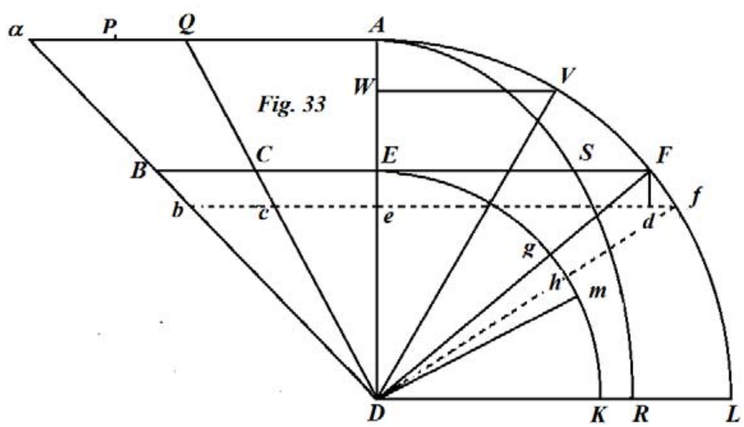
127. Petitur ut concedatur , motum descensus ascensusve per axis elementum *Ee,* vel curvae N*n,* sumi posse tanquam uniformem & aequabilem, quoties mobilia A & M haec spatiorum elementa celeritate finitae magnitudinis, prae incremento vel decremento ejus infinitesimo percurrere incipiunt, licet motus in iisdem spatiis reapse variati sint, quoniam celeritatum incrementa vel decrementa infinitesima, durante descensu vel ascensu mobilibus A & M superaddita vel ademta, evanescunt prae velocitatibus finitis, quibus corpora spatiola Ee, Nn transmittunt.

COROLLARIUM.

128. Hinc quia spatia aequabili motu percursa sunt in composita ratione temporum & velocitatum, ipsa tempora, quibus praedicta rectae vel curvae elementa percurruntur, erunt, ut haec elementa applicata ad celeritatem, qui percurruntur ; sic
$t\mathrm{E}e = \mathrm{E}e : \mathrm{EF}$, & $t\mathrm{N}n : \mathrm{N}n : \mathrm{EF}$; unde $\mathrm{E}e = \mathrm{EF}.t\mathrm{E}e$, & $\mathrm{N}n = \mathrm{EF}.t\mathrm{N}n.$

PROPOSITIO XV. LEMMA.

Fig. *33. Anguli rectilinei* VDF *& TDm sunt directe ut arcus* VF *&* Tm *qui angulos*



subtendunt, & reciproce ut arcuum radii DV *&* DT.
Hoc est angulus VDF*:* ang. $\mathrm{TDm} = \frac{\mathrm{VF}}{\mathrm{DV}} : \frac{\mathrm{T}m}{\mathrm{DT}}$. Est enim
$\mathrm{ang.VDF} : \mathrm{ang.TDm} = (\mathrm{Tg} : \mathrm{DT}) : (\mathrm{Tm} : \mathrm{DT})$; atqui ob arcus similes VF *&* TG*,* erit
$\mathrm{Tg} : \mathrm{DT} = \mathrm{VF} : \mathrm{DV}$, ergo angulus VDF: angulum FDM= (VF*;* DV): (Tm*:* DT). Quod erat demonstrandum.

COROLLARIUM.

129. Hinc 1°. quilibet angulus VDF exprimi potest arcu VF, qui ipsum subtendit, diviso per radium DV, id est fractioni VF:DF. 2°. Quilibet arcus VF est ut factum ex angulo VDF in radium DV, atque adeo hoc facto exprimi potest.

PROPOSITIO XVI. LEMMA.

130. *Omnis solicitatio uniformiter agens aequivalet motui genito, applicato ad tempus, quo motus iste producitur.*

Dicantur massa corporis movendi M, celeritas acquirenda *V,* adeoque motus generandus MV, tempus quo produci debet T , solicitatio, a qua uniformiter hoc tempore agente generari debet G, ostendendum, fore $G = M.V : T$. Cum solicitatio (§. 9.) utpote ex genere vis mortuae nullum motum producat, nisi aliquo tempore in corpere continuata vel replicata fuerit, atque nunc solicitatio uniformiter agere ponatur, ita ut temporibus aequalibus aequales motus quantitates in corpore producat, atque adeo motus geniti sint ut tempora, quibus generantur, ex se ipso clarum est , id quod in tempus T ductum producit motum M.V. aequivalere solicitationi G, unde cum $M.V : T$ ductum in T producat M.V, sequitur $G = MV : T$. Quod erat demonstrandum.

SCHOLION.

131. Res universaliter se habet, ut propositio indicat sive velocitas acquisita & tempus finita aut infinite parva fuerint, modo solicitatio G toto tempore uniformiter agat. Sin vero ea difformiter operetur, propositio generaliter respectu cujuscunque temporis non amplius obtinet; sed duntaxat per temporis tractum indefinite parvum dT, quo mobili tantum celeritas infinitesima dV acquiritur, quia non nisi per tempusculum ejus celeritas infinitesimum dV solicitatio variabilis G (§.117.) tanquam uniformis spectari potest; est ergo hoc casu $G = MdV : dT$, atque adeo $dT = MdV : G$. Ubi G significat pondus seu gravitate utcunque variabilem massa M.

PROPOSITIO XVI. THEOREM.

132. *Momentum solicitationis cujusque aequale est momento celeritatis mobilis ducto in corporis massam.*

Positis iis, quae in §§. 121, 123 & 125 dicta sunt, ponatur mobile M descendere super curva MON, atque per initium *A* scalae celeritatum AVF ducta sit recta AH indefinita, angulum semirectum HAG continens, cum indefinita AG axi AD normali, & per puncta F, f &c. scalae celeritatum agantur aH, *fh* eidem axi aequidistantes rectae AH, occurrentes in punctis H, *h,* alteri vero AX in punctis G, *g;* ductaque H*i* parallela AG, propter angulum semi- rectum HAG erunt HG & AG; atque adeo EF aequales, & rec-lum $Hg = EF.af$ seu momento celeritatis crescentis. Probandum est, fore $M.EF.af = Nn.ES$ seu momento solicitationis tangentialis ES vel $N\beta$.

Demonstr. Per §. 131. est tNn=M.af:ES; nam, quae ibi sunt dT, dV & G, hoc loco dicuntur tNn, af & ES vel Nβ, ergo $\mathrm{ES}.t\mathrm{N}n = \mathrm{M}.af$, vel etiam ES. $EF.tNn=M$. EF. af; atqui (§.128) $EF.tNn=Nn$, ergo $\mathrm{ES.N}n = \mathrm{ES.\ E}m = \mathrm{M.EF.}af = \mathrm{M.HG}g$. Quad erat demonstrandum.

Similiter si grave A in recta AD deorsum feratur, erit BE.Ee = A. ${}_2\mathrm{H}_2\mathrm{G}_2\mathrm{g}$, ponendo A denotare massam corporis, ${}_2\mathrm{H}_2\mathrm{G}$ celeritatem in E acquisitam post casum ex altitudine AE, & rectangulum ${}_2\mathrm{H}_2\mathrm{G}_2\mathrm{g}$ triangulo rectangulo isosceli ${}_2\mathrm{H}_2\mathrm{A}_2\mathrm{G}$ inscriptum; momentum velocitatis ${}_2\mathrm{A}_2\mathrm{G}$ vel ${}_2\mathrm{H}_2\mathrm{G}$.

Ratione solicitationum retardatricium habetur etiam $se.n$N = M. $ef.fa=M.hg$G, & beE=A_2h$_2$g$_2$G.

PROPOSITIO XVIII. THEOREMA.

133. *Si duo mobilia* A, M *aequalia, quorum hoc in curva* MON, *illud in recta* AD *feratur, in singulis punctis* E & N *a centro* D *aequaliter distantibus eadem solicitatione centrali* BE *urgentur, erit ubique momentum solicitationis centralis seu rectangulum* BEe *aequale homologo rectangulo* Nn.SE *vel* Nn.Nβ , *quod exponit momentum solicitationis tangentialis in curvae puncto quodlibet* N.

Arcus circuli epn centra D descripti fecet rectam DN, in puncto p, & triangulum Nnp tanquam rectilineum ob suam infinitam parvitatem spectari potest rectangulum ad p, atque adeo simile triangulo Nαβ. Idcirco erit Nα vel $\mathrm{BE}:\mathrm{N}\beta(\mathrm{SE}) = \mathrm{Nn}:\mathrm{E}e(\mathrm{N}p)$, & $\mathrm{BE.\ E}e = N\beta.\ \mathrm{E}e = \mathrm{SE.N}n = \mathrm{N}\beta.Nn.$ Quod erat demonstrandum.

COROLLARIUM I.

134. Hinc si scalae celeritatum AVF ad punctum F perpendicularis FT ducatur axi occurrens in T, subnormalis ET aequalis ubique erit respectivae BE, quae solicitationem centralem exponit in distantiis ED, ND a centra D aequalibus; nam quia TFf rectus est, triangula FTE & Ffa similia erunt, atque adeo $\mathrm{FE}:\mathrm{ET} = Fa(Ee):qf$. Ergo $\mathrm{EF.}af = \mathrm{TE.\ E}e$; atqui (§ .133.) $\mathrm{SE.N}n = \mathrm{BE.E}e$ & (§.132.) $\mathrm{SE.N}n = \mathrm{M.\ EF.\ af}$, aut sumta M instar unitatis, $= \mathrm{EF.}af$, erit $\mathrm{TE.E}e = \mathrm{BE.E}e$ atque adeo $\mathrm{TE} = \mathrm{BE}$.

COROLLARIUM II.

135. In quibuslibet punctorum E, N distantiis aequalibus ED & ND a centra D, momenta celeritatum cadendo acquisitarum erunt aequalia. Nam sicut (§.132.) AG vel GH celeritatem acquisitam mobilis M in curva MON descendentis, & rectangulum Hg triangulo HAG inscriptum celeritatis momentum in N exponunt , ita etiam ${}_2\mathrm{A}_2\mathrm{G}$ & rectangulum ${}_2\mathrm{H}_2g$ triangulo ${}_2\mathrm{H}_2\mathrm{A}_2\mathrm{G}$ inscriptum, exponent celeritatem ex descensu mobilis A per spatium AE in puncto E acquisitam, ejusque momentum. Verum (§.132.) est $\mathrm{SE.N}n = \mathrm{M.EF.}af = \mathrm{M.HG}g$, item $\mathrm{BE.E}e = \mathrm{A.}_2\mathrm{H}_2\mathrm{G}_2g$. atque (§. 133) $\mathrm{SE.N}n = \mathrm{BE.E}e$; propterea fiet $\mathrm{M.\ HG}g = \mathrm{A.}_2\mathrm{H}_2\mathrm{G}_2g$ vel quia (secundum hypothesin) ipsae M & A aequales $\mathrm{HG}g = \mathrm{A.}_2\mathrm{H}_2\mathrm{G}_2g$.

PROPOSITIO XIX. THEOREMA.

Fig.32. 136. *Si mobilia* M & A *ex punctis* M & A *in curva* MON & *recta* AD *a quiete cadere incipiant, celeritates ipsorum in punctis &* N & E *acquisitae erunt aequales distantiis punctorum a centro* D *positis aequalibus, celeritasque uniuscujusque mobilis poterit duplum areae* AZBEA.

Totum spatium AE in innumera elementa, quale est E*e*, divisum cogitetur, & per singulorum terminos arcus ex centri D descripti intelligantur; adeo ut singulis elementis rectae AE totidem elementa curvae MON respondeant. Jam cum (§.135.) ubique momentum celeritatis in curvae puncto N aequetur momento celeritatis in rectae puncto E, alteri N a centro D aequidistanti, habebitur semper & ubique $HGg = {}_2H_2G_2g$, ergo omnia HG*g*, quae in triangulo HAG continentur aequalia omnibus ${}_2H_2G_2g$, quae in triangulo ${}_2H_2A_2G$, id est triangulum HAG triangulo ${}_2H_2A_2G$ aequabitur . Unde cum haec aequalia triangula similia sint, sequitur latera in illis homologa aequalia esse, atque adeo $AG = {}_2A_2G$; id est celeritates in E & N acquisitae, quae per aequales ${}_2A_2G$ & AG exponuntur, aequales sunt.

Porro, quia (§.132:) $HGg = BE.Ee$: & sic ubique, erunt omnia $HGg =$ omnibus $BE.Ee$, atqui omnia rec-la HG*g* componunt triangulum HAG, vel *h*A*g*, quandoquidem motus in A & M punctis nullus est, atque adeo etiam momenta celeritatum in illis punctis nulla sunt, id est cum triangulorum HAG, & ${}_2H_2A_2G$ verticibus A & ${}_2A$ confunduntur & in illis, ut ita dicam, nullescunt omnia vero rectangula elementaria BE*e*, quae in tota areae AAZBE continentur, huic areae ultimo aequalia fiunt, atque adeo area AAZBE aequatur ubique triangulo homologo HAG, & duplum areae illius duplo hujus trianguli, sed hoc trianguli duplum aequale est quadrato lateris AG vel GH, quoniam hoc triangulum rectangulum isosceles est; ergo $AG^2 = 2.AAZBE$, vel quia AG aequalis est EF, $EF^2 = 2.AAZBE$, id est celeritas in E vel N acquisita potest duplum areae AAZBE. Quae omnia erant demonstranda.

COROLLARIUM I.

137. Pariter si corpora A & M in curva & recta ex punctis O & P celeritate aequali cadere incipiant, & quidem ea, quam mobile A spatium AP a quiete perlabens in puncto P acquirere. potest; eadem corpora in quibusvis aliis punctis N & E, a centro sollicitationum aequidistantibus, aequalem celeritatem acquirent, tam hae scilicet quantam mobile A, in hoc puncto A casum a quiete incipiens lapsu accelerato per spatium AE in puncto E acquirere posset. Nam quia (§.135) generaliter est $HGg = {}_2H_2G_2g$, erunt etiam nunc omnia HG*g* aequalia omnibus ${}_2H_2G_2g$, verum hoc casu omnia GH*g* sunt ea, quae soli trapezio YXGH inscripta sunt, ducta scilicet per V recta VY axi AD parallela, quia celeritas, qua mobile M ex curvae puncto O moveri incipit, non est nulla; sed (secundum hypothesin) aequalis PV seu AX vel XY, atque adeo nunc omnia HGg aequantur trapezio YXGH , & eandem ob ratione omnia ${}_2H_2G_2g$ aequantur trapezio ${}_2Y_2X_2G_2H$, quoniam mobile A casum suum in puncta P cum celeritate ${}_2X_2Y$ (secundum hypothesis) aequali XY vel AX incipit; aequantur, igitur haec trapezia

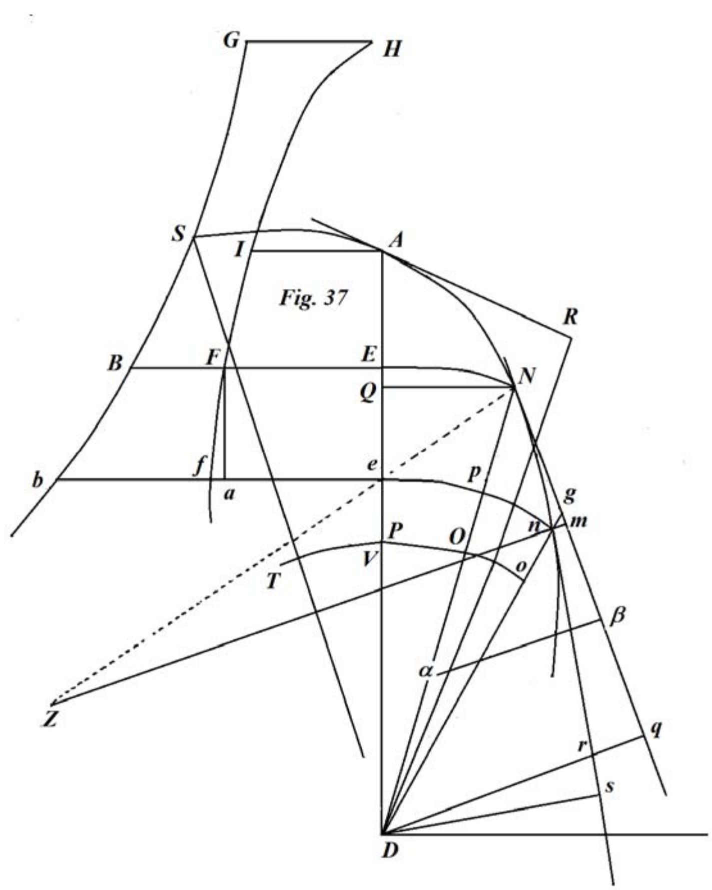
YXGH & ${}_2Y_2X_2G_2H$; verum quia (secundum hypothesin) celeritas in P eadem est cum celeritate in O, & illa tanta est quanta mobili et A in P cadenti illic in P acquiri potest, sequitur triangula AXY & ${}_2A_2X_2Y$ itidem aequalia esse, & addendo hisce aequalibus triangulis trapezia illa aequalia YXGH, & ${}_2Y_2X_2G_2H$, fient etiam triangula AGH & ${}_2A_2G_2H$ aequalia; unde cum eadem & Similia sint, necessum est, ut latera AG & ${}_2A_2G$ aequentur, atque adeo liquet celeritates in N & E acqusitas mobilibus ex puntis O & P, a centro D aequidistantibus cum celeritate PV vel AG casum incipientibus, aequales esse cum inter se, tum celeritati EF in E acquisitae (§. 136.) post casum a quiete in A, per spatium AE, quae potentia aeqaetur duplo areae AAZBE.

COROLLARIUM II.

138. Fig. 37. Propterea, si mobile quoddam A secundum quamlibet directionem datam AR impulsum celeritate ea, quam idem mobile spatium HA a quiete in H perlabens acquireret in A, & solicitat ionibus ordinatis curvae GBb expressis ad centrum D continue ipsum urgentibus curvam AN*n* in vacuo describat, celeritas ejus in quolibet curvae projectionis puncto N ea erit, quam corpus ex H a quiete cadere incipiens in linea recta HD accelerato suo motu acquirere posset in puncto E, cum altero in curva N aequali a centro D distanti.

COROLLARIUM III.

139. Fig. 32. Si corpora ex curvae & rectae punctis *n* & *e* a centra D aequidistantibus



celeritate initiali *ef,* quam post descensum per curvam MO*n* & rectam A*e* in punctis M & A motum a quiete incipientia accelerato motu illic in *n* & *e* acquirere queunt, in altum ferantur; illa in quibusvis aliis punctis O & P eidem centro D aequidistantibus celeritates residuas aequales habebunt, tam hae scilicet, quantam in illis punctis O & P acquisivissent, si acceletato motu a quiete in M & A incipiente in curvae arcu MO & recta AP descendissent. Nam quia mobilia ascendunt, atque adeo a centro solicitationum BE deorsum ad hoc centrum urgentium recedunt, tam hae solicitationes centrales BE ; quam tangentiales SE vel Nβ, quae ex illis derivantur (§.122.) nunc spectandae sunt tanquam solicitationes *retardatrices.* Earum tamen momenta
*be.e*E & *fe.n*N (§. 133.) erunt aequalia, & per §. 122. huic aequatur suum momentum *hg.g*G celeritatis decrescentis *hg,* illi vero celeritatis ${}_2h_2g$ momentum ${}_2h_2g.{}_2g_2$G, propterea haec ipsa celeritatis decresentis momenta sunt aequalia, & hoc respectu cujuslibet alius curvae elementi *n*N omnium eorum, quae in curvae arcu *n*O, & homologi elementi eE rectae DA, omnium eorum, quae in hujus recto segmento *eP* continentur. Hinc etiam universa *hg.g*G, quae in trapezium *hg*XY desinunt, aequabuntur universis ${}_2h_2g.{}_2g_2G,$ quae trapezia ${}_2h_2g_2X_2Y$ circumscripta in hoc trapezium desinunt, id est, haec duo trapezia aequalia erunt. Unde, si ex triangulis *hgA* & ${}_2h_2g_2$A, quae ob celeritates initiales Ag & ${}_2A_2g$ in *n* ac *e* (secundum hypothesin) inter se aequales etiam aequalia sunt, auferentur trapezia illa aequalia *hg*XY & ${}_2h_2g_2X_2Y$, remanebunt triangula aequalia YAX & ${}_2Y_2A_2X$, in quibus, utpote similibus, latera homologa AX & ${}_2A_2X$ aequalia existent, atqui haec aequalia latera celeritates in O & P mobilibus ascendentibus residuas exponunt, quae velocitates proinde aequales sunt, & unaquaeque earum (§. 136.) ea est, quae mobili ex A a quiete descensum incipienti acquireretur in P, utpote quae potest duplum areae AAZP, dum celeritas initialis *ef* potest duplum area AAZ*be.* Propterea GX & ${}_2G_2X$ sunt initialis velocitatis partes *amissae,* vel a solicitationibus centralibus B*e,* BE &c. in ascensu mobilium *absorptae,* adeo ut corporum ad curvae & rectae puncta M & A ascendendo delatorum motus ascensionalis penitus extinguendus sit, cum celeritates *absorptae g*A & ${}_2g_2$A initialibus aequales sint.

COROLLARIUM IV.

140. Hinc mobilia ascendentia cum celeritatibus initialibus quibuscunque successive omnes celeritatis gradus servant in illis ipsis lineae percurrendae punctis, in quibus cadendo hos celeritatis gradus acquisiverunt. Sic mobile cum celeritate *ef* ascensum in *e* incipiens, in reliquis spatii *e*A ascensu conficiendi punctis E, P, A easdem celeritates retinet, quas in puncto A casum inchoans descensu suo accelerato acquisivisset, in iisdem punctis scilicet, celeritates EF, PV &c.. Quod etiam de mobili in curva MON ascendenti intelligendum est.

COROLLARIUM V.

141. Hinc etiam in omni possibili solicitationum centralium hypothesi; *corpora recta in altum lata, vel in curvis quibuscunque ascendentia, ad eam ipsam altitudinem in vacuo assurgunt, quam, ubi descendentio percurrunt celeritatem acquirunt initiali aequalem.*

COROLLARIUM VI.

142. Si centrum solicitationum gravitatis D infinite distat a recta AG, arcus concentrici AM, PO, EN &c. abibunt in lineas rectas in directum positas ordinatis AA, AEP , BE scalae solicitationum centralium, unde quae universaliter in antecedentibus ostensa sunt, etiam obtinebunt in hoc casu, quo gravium directiones rectae AG perpendiculares vel axi AD parallelae sunt. Adeoque *celeritates in diversis planorum* (adde & *curvarum* continuam curvaturam habentium) *inclinationibus descensu acquisitae, aequates sunt in omni gravitatis variabilis & uniformis hypothesi, si planorum vel curvarum elevationes aequales fuerint, vel quod idem est, si planorum horizontalium per terminos planorum inclinatorum vel curvarum transeuntium distantae* a*equales fuerint.*

Hoc ipsum corollarium Torricellius primus geometrice suo more demonstravit, & post ipsum Hugenius in suo Horologio Oscillatorio Parte secunda Prop. VI. sed ambo tantum in Galilaeana hypothesi gravitatis uniformis atque demonstrationibus indirectis. Galilaeus vero id in prima Dialogorum editione tanquam per se evidens sibi concedi postulavit; sed quia hoc forte magnum postulatum ipsi postea visum, id in posteriori operum suorum editione 1656. Bononiae facta utcunque demonstrare tentavit; sed ejus demonstratio, quae ibi affertur, Hugenii judicio pariun concludit.

SCHOLION I.

143. In hac propositione, perinde ac in antecedente XVIII mobilia aequalia ponuntur tam ratione quantitatis materiae quam ratione texturae particularum, vel quod idem est, corpora M & A, quoad volumen & massam, aequalia esse debent, ut recensitae propositiones generaliter obtineant, scilicet eo etiam casu, quo eorum pondera in aequalibus a centro solicitatiomum D distantiis proportionlia non sunt. Semper emim verum erit celeritates in E & N acquisitas post descensus corporum a quiete in A & M, per spatia AE & MON aequales esse, sive massae corporum eorum gravitatibus seu solicitationibus centralibus aut ponderibus proportionales sint, sive non.

144. Sed si mobilia M & A massas non habent ponderibus suis proportionatas, nec pondere & massa aequalia sunt, propositio praesens ad talia corpora non est extendenda ; hoc enim casu praedictae celeritates, in punctis illis N & A acquisitae, non sunt aequales, sed velocitas in curvae puncto N se habebit ad velocitatem in rectae puncto E acquisitam, ut latus quadratum areae AABE, cujus ordinatae BE gravitates seu solicitationes centrales mobilis M, in curva MON discurrentis, exponunt in homologis punctis N, applicatum ad radicem massae mobilis M; ad latus quadratum areae A_3A_3BE, cujus ordinatae E_3BE exponunt gravitates corporis A in locis quibusvis E, per quae ordinatae transeunt, applicatum ad radicem massae seu quantitatis materiae mobilis A, quod in recta AD fertur. Nam si, ut supra, latus AG vel aequale GH in triangulo rectangulo isosceli HAG, celeritatem mobili M ex descensu per arcum curvae MON in puncto N acquisitam, & latera aequalia $_2A_2G$ vel $_2H_2G$ trianguli rectanguli isoscelis $_2H_2A_2G$ celeritatem mobilis A ex casu ejus per spatium AE in puncto rectae E acquisitam , exponant, erunt velocitatum AG, $_2A_2G$ momenta rectangula HGg & $_2H_2G_2g$ triangulis illis inscripta &, per §§.132.133, erit $BE.Ee = SE.Nn = M.HGg$, ergo omma BE. *Ee,* hoc est, ut in demonstratione articuli 136 ostensum, area AAZEBE aequatur omnibus M.HGg, id est

facto ex massa corporis M in triangulum, atque adeo $AG^2.M = 2.AAZBE$ & $AG : GH = EF = \sqrt{(2.AAZBE)} : \sqrt{M}$. Et (§. 131.) est pariter $_3BE.Ee = A._2H_2G_2g$, atque adeo omnia $E_3B.Ee$ vel omnia $_3EB.\ Ee$, hoc est, area A_3A_3BE aequatur omnibus $A._2H_2G_2g$ id est facto ex triangulo $_2H_2A_2G$ in corporis massam A; ergo etiam $_2A_2G = {_2H_2G} = \sqrt{(2.A_3A_3BE)} : \sqrt{A}$. Adeoque AG est ad $_2A_2G$, hoc est, celeritas acquisita mobili M in curvae puncto N est ad celeritatem mobili *A* in rectae puncto E alteri a centro D aequidistanti acquisitam, sicut , $\sqrt{(2.AAZBE)} : \sqrt{M}$ ad $\sqrt{(2.A_3A_3BE)} : \sqrt{A}$ vel etiam, sicut $\sqrt{(AAZBE)} : \sqrt{M}$ ad $\sqrt{(A_3A_3BE)} : \sqrt{A}$, hoc est, dictae velocitates sunt in composita ratione ex directa subduplicata proportione areae AAZBE ad aream A_3A_3BE, & ex reciproca subduplicata ratione massae M ad massam A. Atqui haec ratio composita aequalitatis ratio fieri non potest, nisi area AAZBE fuerit ad aream A_3A_3BE, ut massa M ad massam *A;* id est, nisi pondera mobilium ipsorum massis proportionalia fuerint. Et enim area AAZBE ad aream $A_3A_3Z_3BE$ in eadem altitudine AE non potest esse in data ubique ratione massae M ad massam A, nisi singulae ordinatae BE sint ad singulas homologas ordinatas $_3BE$ in eadem ratione M ad A, atque adeo cum BE & $_3BE$ exponant pondera corporum M & A, in distantiis ND & ED a centro D aequalibus, liquet celeritates in N & E acquisitas aequari non posse, nisi pondera corporum M & A in aequalibus a centro gravium distantiis massis ipsorum proportionalia sint.

SCHOLION II.

145. Hactenus ostensa circa motus gravium adeo generalia sunt, ut ea non solum omnia, quae circa motus quomodocunque acceleratos excogitari possunt, attingant; sed etiam facili negotio ostendant quaenam hypotheses possibiles sint, & quas vice versa natura ferre recuset. Nam, ut ab his incipiam, omnis solicitationum seu gravitatis hypothesis impossibilis & imaginaria est, cujus scala per descensus initium transit, adeo ut illic ordinata ejus nulla sit *;* cujus rei ratio, me etiam tacente, satis in propatulo est. Mobile enim quiescens, quod grave non est, seu a nulla solicitatione centrali citatur (quod contingit in puncto, in quo scala solicitationum rectae, in qua mobile cadere deberet, occurrit) non potest se ipsum movere. Verum haec ipsa impossibilitas etiam in consideratione temporis se manifestat, quo aliquod spatium deberet percurri in imaginariis ejusmodi hypothesibus, quandoquidem ad percurrendum spatium quantumlibet parvum requireretur in hisce hypothesibus tempus infinitum, quod usu formularum ex praecedentibus mutuo sumendarum quilibet poterit experiri. Gravitas ut libet variabilis dicatur *g,* spatium perpendiculariter a corpore cedente describendum *x*, tempus quo percurri debet *t,* velocitas in fine hujus temporis acquisita *u*; elementa spatii, temporis & velocitatis *dx, dt* & *du*, quibus positis (§. 132.) erit prima formula $gdx = udu$, & (§.131.) habetur $dt = mdu : g$, ubi *m* significat massam mobilis & *g* ejus pondus : quoniam vero mobile cum nullo alio comparatur, sed ejus tantum motus exquiruntur, loco ipsius *m* unitas poni potest, eritque $dt = du : g$ .Secunda formula.

146. Sint nunc celeritates acquisitae, ut spatia confecta, ut vult Balianus , id est $u = x$, & $uu = xx$ ac $udu = xdx$, eritque propter, $gdx = udu$, etiam $gdx = xdx$ id est $g = x$, ·ergo existente $x = 0$ erit etiam $g = 0$, ergo in principio descensus gravitas mobilis nulla foret, atque adeo ipsum per spatium *x* nunquam delabi posset. Porro, quia secunda formula $dt = du : g$ & $g = x$, praebent $dt = dx : x$, unde $t = \log.x$, jam existente $x = 0$, & log. ipsius 0 infinito, quod per ∞ significabimus, erit $t = \infty$, hoc est, infinitum tempus requiritur ad percurrendum ab initio nullum spatium, adeoque grave ibi perpetuo quiescet, & consequenter Baliani hypothesis impossibilis & imaginaria est. Hoc ipsum alia adhuc ratione ostendi in Actis Lipf. I709. pag. 404. seq.

147. Si celeritates acquisitae *u* sunt ut x^m, ubi *m* est quilibet numerus integer & positivus index potestatis spatii, cui celeritas proportionatur, fiet $uu = x^{2m}$ & $udu = mx^{2m-1}dx = gdx$, atque adeo $g = mx^{2m-1}$, atqui existente $x = 0$, erit etiam $mx^{2m-1} = g = 0$, ergo ejusmodi hypothesis impossibilis & imaginaria est.

Porro $dt(= du : g = mx^{m-1} : mx^{2m-1}) = dx : x^m$. Verum juxta articulum 92. summationem ineundo, invenietur $t : (\infty^{m-1} \ , -x^{1-m}) : m-1 = \infty^{m-1} : m-1$. Hoc est, tempus descensus per minimam quamcunque altitudinem est in hac hypothesi, ut potestas infiniti ∞, cujus exponens est numerus integer $m-1$. Ergo haec hypothesis, multo magis quam praecedens, imaginaria est & impossibilis.

Hactenus generalia motuum acceleratorum habuimus, dispiciendum restat, quid ex una alteraque gravitatis hypothesi sequi debeat

HYPOTHESIS I.

148. Fig.33. *Si solicitationes gravitatis sint ut distantiae a centro gravium* D, *earum scala erit linea recta* QCD *per centrum* D *ducta cum axe AD angulum quemlibet* QDA *continens, & scala celeritatum linea elliptica* ASR, *cujus semiaxis transversus est* AD, & *semissis axis conjugati* DR *par mediae proportionali inter* DA *vel aequalem* $A\alpha$ *, &* QA.

Nam descripto circa centrum D quadrante circuli AVL, ductusque αD, & BEF parallela DL vel αA*;* quia DR (constr.) media est proportionalis inter DL vel αA & QA, erit propter Ellipsin ASR ; $\mathrm{EF}^2 : \mathrm{ES}^2 (= \alpha\mathrm{A} : \mathrm{QA}) = 2.\text{trapezii } \alpha\mathrm{BEA} \text{ ad } 2.\text{trap. QCEA}$, atqui duplum trapezii αBEA est excessus quadrati AD vel DF supra quadratum DE, atque adeo aequatur EF^2; ergo aequantur etiam duplum trapezii QCEA, quod est duplum areae in scala solicitationum centralium QCD, & quadratum ES, atque adeo (§. 136.) ES exponit celeritatem acquisitam in E ex descensu per AE ; ergo ellipsis ASR est scala celeritatum acquisitarum. Quod erat demonstrandum.

149. *Si fiat angulus* X *ad angulum* ADF, *in ratione* DL a*d* DR, *ipsi angulus* X *exponet tempus discensus mobilis per* AE , hoc est $\mathrm{X} = t\mathrm{AE}$. Sit enim x elementum ipsius X homologum elemento anguli ADF, quod est FDf, ducta scilicct *bf* alteri BF indefinito vicina. Erit ergo $\mathrm{x} : \mathrm{FDf} = \mathrm{DL} : \mathrm{DR} = \mathrm{EF} : \mathrm{ES}$, ergo $\mathrm{x.ES} = \mathrm{EF.ang.FDf}$, vel $\mathrm{x.ES.DF} = \mathrm{EF.DF.ang.FDf}$, atqui (§. 129.) DF in ang. FDE exprimit arculum Ff, ergo $\mathrm{x.ES.DF} = \mathrm{EF.Ff}$, at vero $\mathrm{FE.Ff} = \mathrm{DF.Ee}$, quod ex similitudine triangulorum Ffa & FDE

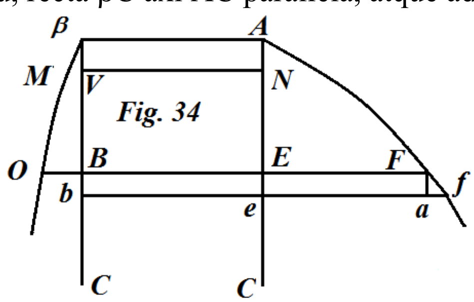
facile colligitur, igitur $x.ES.DF = DF.Ee$, vel deleta DF, erit $x.ES = Ee$, & $x = Ee : ES$. Verum (§. 118.) $Ee : ES = tEe$, ergo $x = tEe$, atque adeo omnes x hoc est $X =$ omnibus tEe, hoc est tAE. Quod erat demonstrandum.

Hinc ducto quolibet radio DV, qui quadrantes concentricos AVL & ETK secet in V & T, ductisque arcuum AV & ET sinibus VW & T*e* (per accidens est, quod *Te* cadat super *bf*, res utcunque aliter se habere potest) spatia AW & E*e* tempore a mobilibus aequalibus A & E simul cadere incipientibus, describentur; eodemque tempore ambo ad centrum D pervenient.

HYPOTHESIS II.

150. Fig. 34. *Si Gravitas corpororum ubique uniformis est, ut in Galilei hypothesi, scala Celeritatum erit Parabola conica, cujus abscissae axis spatia mobili percurenda, &c. ordinatae celeritates acquisitas exponunt. Eruntque propterea celeritates in subduplicata ratione spatiorum confectorum.*

Nam scala solicitationum gravitatis erit, hoc casu, recta βC axi AC parallela, atque adeo (§.144) $2.A\beta.aE : M = EF^2$, ubi M denotat massam mobilis, Aβ ejus pondus, & AE spatium cadendo confectum, atque EF celeritatem acquisitam. Unde ob constantem Aβ scala celeritatum est parabola conica, atque adeo spatia mobili descripta sunt ut quadrata celeritatum acquisitarum, & hae velocitates in subduplicata ratione spatiorum. Parameter vero Parabolae est $2.A\beta : M$.



151. *Tempus vero quo spatium quodlibet* AE *cadendo conficitur est* 1°. *ut duplum spatium* AE *applicatum ad celeritatem* EF *hoc tempore acquisitam.* 2°. *Ut radix ex facto massae corporis in duplum spatium applicato ad mobilis pondus seu gravitatem. Et denique* 3°. *ut factum ex Massa corporis in celeritatem acquisitam applicatum ad gravitatem ejusdem corporis.* Signetur, ut antea, tempus per AE, per tAE.

Jam (§.131.), $tEe = M.af : A\beta$, ergo summando $tAE = M.EF : \beta$, quod est tertium.

Porro quia $tAE = M.EF : A\beta$, erit etiam $= M.EF^2 : EF.A\beta$, (§.150,) $= 2.A\beta.AE : A\beta.EF = 2.AE : EF$. Et hoc est primum. Denique, quia ordinata parabolae est ad dimidium parametrum in subduplicata ratione dupli abscissae ad semissem parametri, erit $tAE(= M.EF : A\beta) = \sqrt{(M.2AE : A\beta)}$. Quod erat secundum.

152. Idcirco, vocando duorum mobilium massas M, *m* ; gravitates seu pondera *G, g;* spatia cadendo descripta S, *s*; & denique tempera descensuum T, *t*. Erit $T = \sqrt{(2M.S : G)}$ & $TT = 2M.S : G$, vel $G.T^2 = 2.M.S$; & $gtt = 2ms$. Adeoque, si spatia ab ambobus mobilibus diversis temporibus confecta sunt in duplicata ratione temporum, corporum pondera massis eorum proportionalia erunt; nam si $TT : tt = S : s$ erit $T^2 s = tt.S$; adeoque $G.T^2 = 2.MS$ fiet

$tt.\mathrm{GS} : s = 2.\mathrm{MS}$, vel $Gtt = 2.Ms$, *seu* $gtt = \frac{2.g.M.s}{G} = 2ms$ adeoque
$g\mathrm{M} = \mathrm{G}m$, vel $\mathrm{G}{:}g = M : m.$

Propterea hujus paragraphi ope secunda Propositio per experimenta confirmari potest. Videatur §. 30.