IX. *Concerning the algebraic curves required to be drawn through any given points.*

[This is one of the first publications on interpolation, predating that of Newton, Stirling, etc., but follows from hints in the *Principia* [Lemma V. Book Ill.], and work by Johan Bernoulli [*Acta Erud*. 1694], and evolves from the finite differences of generating polynomials or analytic curves, defined below the axis in the following diagram.]

Towards the end of Ch. VII. Book II. §.283. I drew attention to the problem of drawing algebraic curves through some given points in place, the solution of which the incomparable Newton first found : yet it did not seem to me that he produced the solution, except what he had outlined in Lemma V. Book. III. *Princ. Phil. Nat.* ,without any analysis or demonstration, where he shows how to draw the line of a parabolic nature through any number of given points. I attacked this problem in the years 1704 and 1705 and I found a solution, which I communicated to the most illus. Leibnitz by letter, which solution evidently pleased him, as can be gathered from the most civil of letters sent to me in this regard, in the years 1705 and 1706. The occasion of this problem arising brought to mind the letter from Newton to Oldenburg, in which he recalled this problem, and to be amongst the most pretty of these, he proclaimed, that he wished to solve. But on to the matter at hand :

If therefore an algebraic curve or one of an analytical kind CDA, shall be required to be drawn through some points A, 1A, 2A, 3A, &c. & C ; some line CB can be drawn through either of the end points C, which shall represent the axis, to which the ordinates AB, 1A1B, 2A2B, &c. may be sent from the individual given points, to be produced under the axis at L, 1L, 2L, &c. and some other curves CFL, CGM, CHN, CIO, CKP, &c. shall be sharing the axis. Now the individual lines AL, 1A1L, 2A2L, etc. may be considered as if they were just as many small levers for small weights attached at the points A, L, M, N, O, P, &c., and with these lines laden at the letters indicated, but of which the individual centres of equilibrium may be found on the axis CB, evidently at the points B, 1B, 2B, 3B, &c. on this line, and it will follow, that whatever line DF may be drawn parallel to the remaining AL1 &c., and crossing the other curves under the axis at the points F,

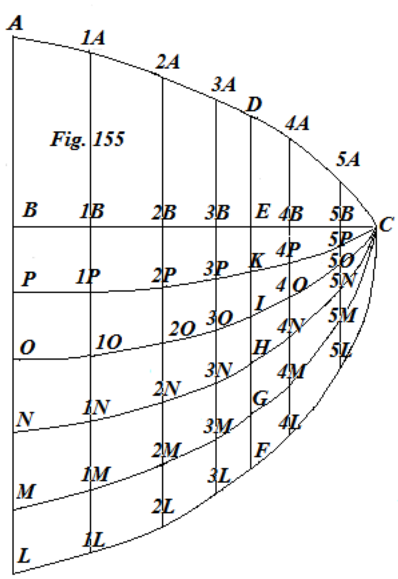


Fig. 155

G, H, I, K, &c. and the axis at E, of which the line or lever DF, by having the same tiny weights attached at the points D, F, G, H, I, K, &c. as at first at A, L, M, N, O, P, &c., the centre of equilibrium of these small weights shall be at E, and the branch ED is going to become the ordinate of the regular curve CDA, passing through the points A, 1A, 2A, 3A, &c. C, and hence the branches EF, EG, EH, &c. are the ordinates of the curves CL, CM, CN, &c. And from the given EF, EG, EH, El, EK; &c. and the small weights A, L, M, N, O , P, &c. the ordinate ED may be found on the curve sought CDA. Therefore the whole difficulty is reduced to finding the weights A, L, M, N, O, P, &c, or of the proportions between these. And the main levers give rise to the following equations [on taking moments] :

$$1.\quad \mathrm{A.AB = L.LB + M.MB + N.NB + O.OB + P.PB.}$$
$$2.\ \mathrm{A.1A1B = L.1L1B + M.1MIB + N.1N1B + O.1O1B + P.1P1B.}$$
$$3.\ \mathrm{A.2A2B = L.2L2B + M.2M2B + N.2N2B + O.2O2B + P.2P2B.}$$
$$4.\ \mathrm{A.3A3B = L.3L3B + M.3M3B + N.3N3B + O.3O3B + P.3P3B.}$$
$$5.\ \mathrm{A.4A4B = L.4L4B + M.4M4B + N.4N4B + O.4O4B + P.5P5B.}$$
$$6.\ \mathrm{A.5A5B = L.5L5B + M.5M5B + N.5N5B + O.5O5B + P.5P5B.}$$

These equations are taken from each other continually, clearly the second from the first, the third from the second, the fourth from the third, and thus henceforth, and the remainders divided by the respective differences of the ordinates AB, 1A1B, 1A1B, 2A2B, 2A2B*,* 3A3B, &c., which will give five equations, in which A will be from one single part, and for the remaining members, from the individual differences $\mathrm{BL - 1B1L}$; $\mathrm{BM - 1B1M}$; $\mathrm{BN - 1B1N}$, & and thus henceforth, divided by $\mathrm{AB - 1A1B}$, there may be written Q, R, S, T, V*,* in the five remaining equations, and these may be treated in the same way and the first six may be treated in the same way, evidently following from the preceding on being subtracted, four new will be left, which may be divided respectively by $\mathrm{Q - Q1}$, $\mathrm{Q1 - Q2}$, $\mathrm{Q2 - Q3}$, $\mathrm{Q3 - Q4}$ and for $\mathrm{R - R1 : Q - Q1}$; $\mathrm{S - S1 : Q - Q1}$; $\mathrm{T - TI : Q - Q1}$; $\mathrm{V - V1 : Q - Q1}$*,* there may be written *r,s,t,u;* likewise for $r1$, $s1, t1$, $t1$ for $\mathrm{R1 - R2 : Q1 - Q2; S1} - S2 : Q1 - \mathrm{Q2}$, &c. For $\mathrm{R2 - R3 : Q2 - Q3}$; $\mathrm{S2 - S3 : Q2 - Q3}$, &c. $r2$, $s2$, $t2$, $u2$; and thus again in the same order, and these subtractions and divisions will be continued to as far as only a single equation may remain, all the A, L, M, N, O, P, &c. in the given quantities will be able to be shown, or perhaps all except the final P, which is of an arbitrary magnitude, for with the given reductions there will be found

$$\mathrm{O} = -\mathrm{P}\alpha.$$
$$\mathrm{N} = -\mathrm{O}\tau - \mathrm{P}\eta.$$
$$\mathrm{M} = -\mathrm{N}\sigma - \mathrm{O}\theta - \mathrm{P}\omega.$$
$$\mathrm{L} = -\mathrm{M}r - \mathrm{N}s - \mathrm{O}t - \mathrm{P}u.$$
$$\mathrm{A = LQ + MR + NS + OT + PV.}$$

In these equations there shall be :

$$dd\mathrm{LB} : d\mathrm{AB} = \mathrm{Q};\ d\mathrm{MB} : d\mathrm{AB} = \mathrm{R};\ d\mathrm{NB} : d\mathrm{AB} = \mathrm{S};\ d\mathrm{OB} : d\mathrm{AB} = \mathrm{T}\ ;\ d\mathrm{PB} : d\mathrm{AB} = V.$$

Thus, $d\mathrm{1L1B} : d\mathrm{1A1B} = Q1; d\mathrm{1M1B} : d\mathrm{A1B} = \mathrm{R1}; d\mathrm{1N1B} : d\mathrm{1A1B} : \mathrm{S1}$, &c.

Again, $d\mathrm{2L2B} : d\mathrm{2A2B} = \mathrm{Q2}; d\mathrm{2M2B} : d\mathrm{2A2B} = \mathrm{Q2}$*;* &c.

Where it is to be noted the remaining quantities also thus can be defined readily, the letter *d* put in front of any order indicates the difference, between this order and of that nearer to C on the same curve, evidently

$r = d\mathrm{R} : d\mathrm{Q}; r1 = d\mathrm{R}1 : d\mathrm{Q}1, \&c.$	
$s = d\mathrm{S} : d\mathrm{Q}; s1 = d\mathrm{S}1 : d\mathrm{Q}1, \&c.$	$\sigma = ds : dr; \sigma 1 = ds1 : dr1, \&c.$
$t = d\mathrm{T} : d\mathrm{Q}; t1 = d\mathrm{T}1 : d\mathrm{Q}1, \&c.$	$\theta = dt : dr; \theta 1 = dt1 : dr1, \&c.$
$u = d\mathrm{V} : d\mathrm{Q}; u1 = d\mathrm{V}1 : d\mathrm{Q}1, \&c.$	$v = du : dr; v1 = du1 : dr1, \&c.$
$\tau = d\theta : d\sigma; \tau 1 = d\theta 1 : d\sigma 1$	
$\eta = dv : d\sigma; \eta 1 = dv1 : d\sigma 1$	$\omega = dv : d\tau.$

Here the letter *d* indicates the difference between a magnitude, to which it is joined, and the other indicated by the same letter, but with unity inserted , thus $d\mathrm{R} = \mathrm{R} - \mathrm{R}1,\ d\mathrm{Q} = \mathrm{Q} - \mathrm{Q}1;\ d\mathrm{R}1 = \mathrm{R}1 - \mathrm{R}2$, and thus with the rest. Thus with the values of the weights found A, L, M, N, O, &c, the same are required to be substituted into the equation $\mathrm{A.DE} = \mathrm{L.FE} + \mathrm{M.GE} + \mathrm{N.HE} + \mathrm{O.IE} + \mathrm{P.KE}$, and thus the ordinate DE of the curve CDA passing through the given points A, 1A, 2A, &c. can be designated everywhere by the ordinates FE, GE, HE, IE, KE, &, and the other given quantities. Q.E.D.

Coroll. I. The curve generated CDA is always of the same order as the curve of the highest order arising from the generating curves GFL, CGM, CHN, &c.

Coroll. II. The area of the generated curve CDA can always be found from the areas of the generating curves. There is generally $\mathrm{A.CDE} = \mathrm{L.CFE} + \mathrm{M.CGE} + \mathrm{N.CHE} + \mathrm{O.CI.E} + \mathrm{P.CKE}$. From which, if these areas of the generating curves are quadrable, the figure generated also will be quadrable.

Coroll. III. If the curves arise from parabolic generators, the areas of these are quadrable, and hence also the curve generated will be quadrable : therefore since any curves shall be able to be generating curves, and thus these are quadrable curves, from which it is clear that all curves by approximation are quadrable, indeed just as many as it may be wished, in that to be the assumed points, and through that a quadrable curve to be drawn, of which the area will be approximately equal to the area of the proposed curve.

Coroll. IV. Indeed any curve can be rectified approximately. For the rectification of curves is reduced to the quadratures, which by the preceding corollary, by approximation, will always be had. Likewise the understanding of the centres of gravity of plane figures and of volumes, as also from these volumes themselves can be understood from others. Indeed all these are able to be had as an approximation.

Coroll. V. From these also the maximum number of given points can be known, through which a curve of a given order can be drawn. Indeed this number *is half the product from the exponent of the order of the curve by the same exponent increased by* *three*. In this manner we know a line to be of the first order, or a right line, only if it can be drawn to through 2 given points, in a place given as it pleases ; indeed there is $2=\frac{1.4}{2}$, a conic section passes through five points, for $5=\frac{2.5}{2}$. A curve of the third order passes through nine, as $9=\frac{3.6}{2}$.

Otherwise. In place of the generators the right lines AF, 1A1F, 2A2F, &c. can be assumed parallel to each other, and placed at some angle with the axis CA, thus being produced above the axis by necessity as required, as the graph shows. Therefore the points shall be B, 1B, 2B,… &c. C, through which the curve must be drawn. BN is drawn acting parallel to AC through B, and from the individual given points B the perpendiculars BA, 1B1A, &c. are acting being produced upwards to 1O, 2O, &c., then with the right line MG drawn somewhere parallel to AB with the oblique line AF cut at the points G, H; I, K, L, &c. and the axis AC at E. With which carried out everywhere there will be

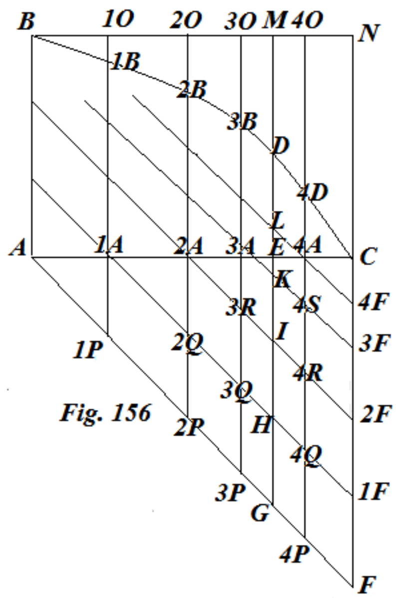


Fig. 156

$$\mathrm{MD}=\mathrm{G.EG}+\mathrm{H.EG.EH}+\mathrm{I.EG.EH.El}$$
$$+\mathrm{K.EG.EH.El.EK}-\mathrm{L.EG}\;....\mathrm{EL},$$

in which the values of the assumed G, H, I, K, L are required to be defined by the following reasoning. If the point E falls on A, the line EG shall vanish, and the remaining HE, IE, &c. will be above the axis, truly because EG now has contracted to nothing at the point A, and it merges into all the remaining members, MD will be zero at BA . But truly if E falls on the point 1A, then EG becomes $=1\mathrm{A}1\mathrm{P}$, & $\mathrm{EH}=0$ and thus in this case, since MD shall be 1O1B, there will be $1\mathrm{O}1\mathrm{B}=\mathrm{G}.1\mathrm{A}1\mathrm{P}$, or $\mathrm{G}=1\mathrm{O}1\mathrm{B}:1\mathrm{A}1\mathrm{P}$, calling $\mathrm{GI}=2\mathrm{O}2\mathrm{B}:1\mathrm{A}2\mathrm{P},\mathrm{G}2=3\mathrm{O}3\mathrm{B}:3\mathrm{A}3\mathrm{P},\&\mathrm{c}.$ Again if the point E falls on 2A, there becomes $\mathrm{MD}=2\mathrm{O}2\mathrm{B}$, & $\mathrm{EG}=2\mathrm{A}2\mathrm{P}$, $\mathrm{HE}=2\mathrm{A}2\mathrm{Q}$, & $\mathrm{IE}=0$, therefore

$$2\mathrm{O}2\mathrm{B}=\mathrm{G}.2\mathrm{A}2\mathrm{P}+\mathrm{H}.2\mathrm{A}2\mathrm{P}.2\mathrm{A}2\mathrm{Q},3\mathrm{O}3\mathrm{B}$$
$$=\mathrm{G}.3\mathrm{A}3\mathrm{P}+\mathrm{H}.3\mathrm{A}3\mathrm{P}.3\mathrm{A}3\mathrm{Q}+\mathrm{I}.3\mathrm{A}3\mathrm{P}.3\mathrm{A}3\mathrm{Q}.3\mathrm{A}3\mathrm{R},$$

and so on henceforth ; from which everything follows in the following table, clearly after the individual equations with respect to AP were applied:

$$\begin{aligned}
&\text{G}\ (=1\text{O}1\text{B}:1\text{A}1\text{P}) = \text{G}\\
&\text{G1}(=2\text{Q}2\text{B}:2\text{A}2\text{P}) = \text{G}+\text{H}.2\text{A}2\text{Q}\\
&\text{G2}(3\text{O}3\text{B}:3\text{A}3\text{P}) = \text{G}+\text{H}.3\text{A}3\text{Q}+\text{I}.\ 3\text{A}3\text{Q}.3\text{A}3\text{R}\\
&\text{G3}(=4\text{O}4\text{B}:4\text{A}4\text{P}) = \text{G}+\text{H}.4\text{A}4\text{Q}+\text{I}.4\text{A}4\text{Q}.4\text{A}4\text{R}\\
&\qquad\qquad +\text{K}.4\text{A}4\text{Q}.4\text{A}4\text{R}.4\text{A}4\text{S}.\\
&\&\text{c.}\qquad \&\text{c.}\qquad \&\text{c}\qquad \&\text{c.}
\end{aligned}$$

Again if the first may be taken from the second, the second from the third, the third from the fourth, and so on thus, there will be
$\text{G1}-\text{G}=\text{H}.2\text{A}2\text{Q};\ \text{G2}-\text{Gl}=\text{H}.3\text{A}3\text{R}+\text{I}.3\text{A}3\text{Q}.3\text{A}3\text{R};\ \&\text{c.}$ And by putting dG*,* dG1 , dG1 for $\text{G1}-\text{G},\ \text{G2}-\text{G1},\ \text{G3}-\text{G2},\ \&\text{c}$. there arises

$$\begin{aligned}
&\text{H}=d\text{G}:2\text{A}2\text{Q};\ \text{H1}=d\text{G1}:3\text{A}3\text{R};\ \text{H2}=d\text{G2}:4\text{A}4\text{S};\&\text{c.}\\
&\&\ \text{I}=d\text{H}:3\text{A}3\text{Q};\ \text{I1}=d\text{H1}:4\text{A}4\text{R};\&\text{c.}
\end{aligned}$$

and finally $\text{K}=d\text{I}=4\text{A}4\text{Q}.$ And thus so on ; for from these continuations the law appears to be satisfied. The values of the magnitudes G, H, I, K, &c. found in the above equality

$$\text{MD}=\text{G.GE}+\text{H.GE. HE}+\text{I.GE.HE.IE}+\text{K.GE.HE.IE.KE},\ \&\text{c.}$$

shall be substituted, thus MD itself will be given on the lines GE, HE, IE, KE, LE; &c. & with the magnitudes given, and thus the point D on the curve sought. Which was secondly required to be found.

Coroll. I. If the right lines AF, 1A1F, &c. are inclined at a semi-right angle to AC, the following case of this problem arises from the solution by the illus. Newton, Lemma V. Book Ill. *Princ. Phil. Nat.* in the solution of that namely *a, b, c, d, e., f,* &c.*,* the same therefore are as G, H, I, K., L, &c .

Coroll. II. With the same in place, as in the preceding corollary, if the individual intervals A1A, 1A2A, 1A3A, &c. shall be equal and they are called $p,$ and the differences of the ordinates AB, the first, second, third, &c. will be
$\text{G}=\delta:p;\ \text{H}=\delta2:2.pp;\ \text{I}=\delta3:2.3.p^3,\&c.$ because it agrees with the first case of the Newtonian solution in the lemma cited

Coroll. III. If in addition AE or EG may be called *z,* DE , *u*, & $\text{AB}=y$*,* there will be

$$\begin{aligned}
&u=y-\text{G}z-\text{H}.(zz-pz)-\text{I}.(z^3-3pzz+2ppz)\\
&-\text{K}.(z^4-6pz^3+11ppzz-6p^3z)-\&c.
\end{aligned}$$

and the area

$$\text{ABDE} = yz - \tfrac{1}{2}Gzz - \text{H}.(\tfrac{1}{3}z^3 - \tfrac{1}{2}pzz) - \text{I}.(\tfrac{1}{4}z^4 - pz^3 + ppzz)$$
$$-\text{K}.(\tfrac{1}{5}z^5 - \tfrac{3}{2}pz^4 + \tfrac{1}{3}ppz^3 - 3p^3zz) - \&\text{c}.$$

Indeed here y is constant, and both z and u are variables. Hence, if p is indefinitely small, in particular $p = dz$, and these $\delta, \delta 2, \ \delta 3, \ \&\text{c.}$ in this case become $dy, -ddy, +dddy, -d^4y, \&c.$ and in the preceding series all the members, which contain p vanish, and the whole series will be changed into
$yz - \tfrac{1}{2}\text{G}zz - \tfrac{1}{3}\text{H}z^3 - \tfrac{1}{4}\text{I}z^4 + \&\text{c.} = \text{area ABDE},$ and there becomes

$$\text{G} = \tfrac{dy}{dz}, \ \text{H} = \tfrac{-ddy}{2.dz^2}, \ \text{I} = \tfrac{+dddy}{2.3dz^3}; \&c.$$

therefore the area $\text{ABDE} = yz - \frac{zzdy}{2dz} + \frac{z^3ddy}{2.3dz^2} - \frac{z^4dddy}{2.3.4dz^3} + \&c.$ Which is the very general series for the quadratures, that the cel. Johan Bernoulli has shown in the *Actis Lips*. 1694, and which without doubt, he elicited from another principle.

Otherwise, in place of the right transversals AF, 1A1F, the parallels of any curve can be assumed, or the axis can be assumed for one and the same curve on the line CF, truly having the vertex successively at different points of CF.

X. *Concerning the velocity of a liquid erupting from a vessel through some hole.* Indeed in Proposition XXXII. Book II. we have shown after the most illustrious Varignon and others the speeds of liquids flowing out from a vessel to be in the square root ratio of the heights of the liquid above the opening of the vessel discharging the water, but we have not shown, nor have any of the others as far as I am aware*, how drops of water or of any other liquid erupting with that velocity can acquire the velocity near the opening by accelerating from the height of the water above the opening.* For this principle, in the form of a hypothesis, was assumed by Torricelli, Borelli, Gulielmi and others, and thence they deduced correctly, not only equal amounts of water flow out through equal holes in the same time, but also the speed of the efflux to be in the square root ratio of the heights above the openings. But this supposition may not seem to be so clear, that no demonstration may be needed ; for besides what the celebrated Newton tries to show, in Prop. XXXVII. Book II *Princ. Math*. of the first edition [see the comments made in Book II, Section 7 in my translation of the 3rd ed., Prop. XXXVI], water erupts from the vessel with that speed, by which with its motion conversely may be able to be carried from half its height, but which proposition has been omitted in the newest edition [*i.e.* 2nd] . Torricelli having assumed his principle, not on account of proof, but rather from the agreement it had acquired with experiments.

Let some vessel BSC be full of water or of some other homogeneous liquid, F its opening, and FA the height of the water above the opening F. Consider the water column Ap through an infinitely small part of time dt, to be acting constantly on the infinitesimal particle of water pF, clearly in that small time, in which this particle, that we will call p, by its own weight or gravity, the length Ff of the particle pF or p will be able to acquire the infinitesimal degree of the speed u. And g shall be the symbol of natural gravity, by which the individual bodies we are considering are disturbed, A the height of the water

AF, &c and finally V the velocity, with which it erupts through the opening F ; M the mass flowing out in the time *dt* and *m* the mass of the particle *p*F. With these in place, by §.31, the weight of the column A*p* or AF will be expressed by A.F.*g*; and the weight of the element *p*F by *p*.F.*g*. But these products are as the accelerative actions of the motions M.V and *m.u,* generated in the element of time *dt* ; therefore we have (§.130.) $\mathrm{A.F.}g.dt : p.\mathrm{F.}gdt = \mathrm{M.V} : m.u$, that is, $\mathrm{A}.g : p.q = \mathrm{M.V} : m.u$; and $\mathrm{M} : m = \frac{1}{2}\mathrm{V}dt : \frac{1}{2}udt = \mathrm{V} : u$ from the nature of the accelerating motion in fluids, therefore $\mathrm{A}g : pg = \mathrm{VV} : uu$, and $2pg$ (§150.) $= uu$, therefore $\mathrm{VV} = 2.\mathrm{A}.g$; this is the speed V, with which water erupts through the opening F (§.150.) that is, how an infinitesimal particle of water *p*F may be able to acquire naturally by the accelerated motion through the descent from the height A or AF. Q.E.D.

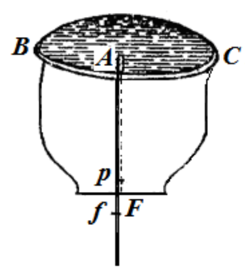


Fig. 157

[This explanation is not very convincing: the idea of a pressure gradient at the opening generating the accelerating force had not yet occurred to anyone; and of course neither had the conservation of energy principle been established.]

And hence now it is most carefully deduced, *the quantities of water flowing out from the same or equal openings, as well as the speeds of this, to be in the square root proportion of the height.*

XI. In CH. XX. Book II I have established a general theory of variable gravity, where clearly bodies are acted on along some directions yet proceeding according to a uniform law, therefore I have considered these directions to lie along the tangents of some curve, or, what amounts to the same, I have supposed the same directions to be normal to any curve as it pleases, according to that, so that the law of continuity may be observed in these. But, because the solution of the problem in proposition LXXVI is the evolute to that, on which I worked at first, thus, since generally as it usually happens, its geometrical analysis emerged a little longer and more difficult. Truly after this little work shall be in press, I have discovered, a little easier, and lest I am deceived, a more elegant solution of the same problem, which with the analysis itself that it pleases to communicate here, or rather by a demonstration for the benevolent reader. Let the curve of some moving body AMΩ require to be described, and the

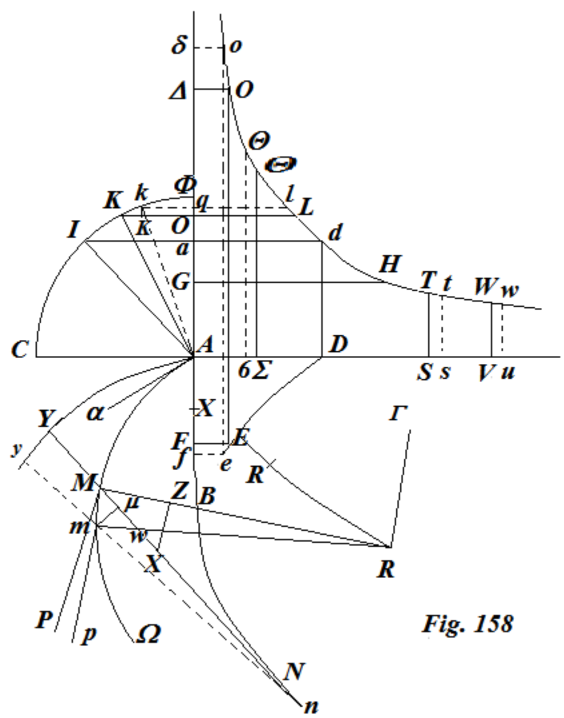


Fig. 158

directions MN of the actions of gravity MX shall be everywhere normal to any curve AY*y* and described by the evolute BN, since the directions MN everywhere may be along the tangent. To the tangent AB of the curve BN at B, produced upwards to Δ , the indefinite perpendicular CAV acts through A, and in the quadrant CKΦ described with centre A and with some radius AC, and the angle CAI described with the angles CAI, CAK contained by the radii AI and AK equal to the angles BAα, NMP, on which the lines AB, MN cross the curve AMΩ , the right lines Iα, KQ are acting parallel to AC and produced as far as to cross at *d* and L with the hyperbola O*d*T described between the asymptotes AA, AV. Truly near the axis AB a certain curve shall be described DE by this rule, that its abscissa AF everywhere shall be equal to the homologous intercept YM between the curve AY and the given curve being described by the moving body AM, truly for the ordinate FE, the ordinate ΔO of the abscissa AΔ on the hyperbola to be put $= \mathrm{MN}$. And, if the body shall be advancing in a resisting medium, of which the resistance shall be composed from the ratio of the densities and the squares of the velocities, on the hyperbola the four-sided figure *dDST* or *d*DΣΘ becomes equal to the surface of the cylinder, of which the base of the arc AM of the curve described, from which the surface of the cylinder arises, if at the individual points of the arc AM perpendiculars to its plane may be erected proportional to the density of the medium D, at the respective points of the curve, and it provides the four-sided figure *d*S for the descent of the body from A to M, and the other *d* Σ for the ascent of the same body from M to A. And with these present,

I say the velocity of the moving body acquired at M *after falling through the arc* AM *to be to the initial speed at* A *in the ratio composed from the these three ratios,* Aa *to* AQ*, or the reciprocals of the sine of the angle* NMP *to the sine of the angle* BAα, *from* Aa *to AG, or the abscissa of the minor to the major of the hyperbolic four-sided figure* aGHd *equal to the area* DEFA, *and finally in the ratio* AD *to AS or AΣ.*

Demon. 1st. Just as in §. 607, MR or *m*R express the radius of the evolute E of the curve AM at the point M or *m,* and MX the action of gravity urging MN along, and it is acting along XZ parallel to the tangent of the curve MP at the point M, and the triangles AKQ, MXZ and *Mμm* will be similar, after which with centre *v*, the arclet *mμ* would be described, since in the right angled triangles AKQ, MXZ, the other angle AKQ of KAC (const.) shall be equal to the angle PMX itself, or to its other angle MXZ *;* hence $\mathrm{ZX}:\mathrm{MZ}=\mathrm{KQ}:\mathrm{AQ}$.

2nd. With another tangent *mp* drawn through *m,* the right angles RMP & *Rmp* give

$$nmp+nm\mathrm{R}=\mathrm{NMP}+\mathrm{NMR},\ \text{and}\ \mathrm{KA}k\left(=nmp-\mathrm{NMP}\right)$$
$$=\mathrm{NMR}-nm\mathrm{R}=\mathrm{R}wv-nm\mathrm{R},\ \ -\mathrm{R}wv+\mathrm{NMR}=\mathrm{M}vm-\mathrm{MR}m.$$

Therefore (§.129.) $\frac{\mathrm{K}k}{\mathrm{KA}}$ (or on account of the similar triangles AKQ & K*kx*)
$\frac{\mathrm{Q}q}{\mathrm{KQ}}=\frac{m\mu}{mv}-\frac{\mathrm{M}m}{\mathrm{RM}}$, and thus $\frac{\mathrm{M}m}{\mathrm{MR}}=\frac{m\mu}{\mathrm{MN}}-\frac{\mathrm{Q}q}{\mathrm{KQ}}$. The ratios $Mm:\mathrm{MR};\ m\mu:\mathrm{MN},$ & $\mathrm{Q}q:\mathrm{KQ}$ may be multiplied as follows, which (no.1. of this) are all equal to each other
$\mathrm{ZX}:\mathrm{MZ};\ \mathrm{M}\mu:m\mu$ & $\mathrm{KQ}:\mathrm{AQ}$, first by the first, second by second, &c. and there becomes $\mathrm{ZX.M}m:\mathrm{MZ.MR}=\mathrm{M}\mu:\mathrm{MN},-\mathrm{Q}q:\mathrm{AQ}$. Truly these latter ratios also may be

multiplied in the same by the following rectangles $\mathrm{A}a.ad$; $\mathrm{A}\Delta.\Delta\mathrm{Q}$; $\mathrm{AQ.QL}$, which are all equal to each other on the hyperbola, and there will be :

$$\mathrm{A}a.ad.\mathrm{ZX.M}m:\mathrm{MZ.\ MR}$$
$$=\Delta\mathrm{O.M}\mu-\mathrm{QL.Q}q\left(\text{constr.}\right)=\mathrm{FE.\ F}f-\mathrm{QL.Q}q.$$

3rd. Again AV signifies the speed acquired at M, V*u* its element, R truly the resistance of the medium at M, and D as before its density; and with these in place (§. 607. no. II.), there will be had, $\mathrm{AV.V}u=\mathrm{ZX.M}m\mp\mathrm{R.M}m$, or by multiplying one part by A*a.ad,* and the other by AV.VW*,* there will be found

$$\mathrm{A}a.ad.\mathrm{ZX.M}m\mp\mathrm{A}a.ad.\mathrm{R.M}m=\mathrm{AV}^2.\mathrm{WV.V}u.$$

Truly, since (following the hypothesis) R shall be as $\mathrm{D.AV}^2$, there becomes $\mathrm{A}a.ad.\mathrm{R}=\mathrm{D.AV}^2$; and $\mathrm{AV}^2.\mathrm{WV.V}u=\mathrm{A}a.ad.\mathrm{ZX.M}m\mp\mathrm{D.AV}^2.\mathrm{M}m$; because truly (§.154.) $\mathrm{AV}^2=\mathrm{MZ.MR}$, by applying the preceding equation to this other, clearly retaining the parts of this AV^2 to that square AV^2, and the remainder to MZ.MR, there becomes $\mathrm{WV.V}u=\mathrm{A}a.ad.\mathrm{ZX.M}m:\mathrm{MZ.MR},\mp\mathrm{D.M}m$. And at the end no. 2 we have now found $\mathrm{A}a.ad.\mathrm{ZX.M}m:\mathrm{MZ.MR}=\mathrm{FE.F}f-\mathrm{QL.Q}q$; therefore $\mathrm{WV.V}u=\mathrm{FE.F}f-\mathrm{QL.Q}q\mp\mathrm{D.M}m$ (or, because D.M*m* by constr. is equal either to S*s*.ST as in the descent, or $-\Sigma\Theta.\Sigma\sigma$, as in the case of the ascent, since the individual four-sided figures *dS* and *dΣ* shall be equal, (constr.) for all D, M*m*, which may be present in the arc of the curve described AM) $\mathrm{WV.V}u:\mathrm{FE.F}f-\mathrm{QL.Q}q-\mathrm{ST.S}s$ (or $\Sigma\Theta.\Sigma\sigma$). Hence, since the same will arise with respect of any other element of the curve, there will be $d\mathrm{DVW}=\mathrm{AFED}-a\mathrm{QL}d=d\mathrm{DST}$ ($d\mathrm{D}\Sigma\Theta$), or, because (constr.) the four-sided figure $a\mathrm{GH}d$ is equal to the area $\mathrm{AFED},=a\mathrm{GH}d-a\mathrm{QL}d-d\mathrm{DST}$ ($d\mathrm{D}\Sigma\Theta$). For all these figures vanish when the point M falls on A, or the point K on I. Therefore the ratio AV to AD (§§.605, 606.) will be composed from the direct ratio *a*A to GA and from the two reciprocal ratios QA to *a*A, and AS or AΣ to AD, and therefore the velocity acquired at M is to the velocity at A or as $\mathrm{AV}:\mathrm{AD}=a\mathrm{A}.a\mathrm{A.AD}:\mathrm{GA.QA.SA}$ (AΣ). Q.e.d.

With the same in place, *the action of gravity* MX *along* MN *to the action* AX *at the point* A *along* AB *will be in the ratio composed from the direct ratio squared* AV *to* AD, *and inversely to the rectangle from* MR *by the sine of the angle* NMP *to the rectangle from* AR *by the sine of the angle* BAA*, that is,*

$\mathrm{MX}:\mathrm{AX}=(\mathrm{AV}^2:\mathrm{MR.AQ}):(\mathrm{AD}^2:\mathrm{AR.A}a)$.

Now, because {§.154) $\mathrm{AV}^2=\mathrm{MZ.MR}$, or $\mathrm{AK.AV}^2=\mathrm{AK.MZ.MR}$ (on account of the similar triangles AKQ & MXZ , the rectangle AQ.MX is equal to the rectangle $\mathrm{AK.MZ})=\mathrm{AQ.MX.MR},\&\ \mathrm{AK.AD}^2=\mathrm{A}a.\mathrm{AX.AR}$, there will be generally $\mathrm{AQ.MR.MX}:\mathrm{A}a.\mathrm{AR.AX}=\mathrm{AV}^2:\mathrm{AD}^2$; and thus $\mathrm{MX}:\mathrm{AX}=\left(\mathrm{AV}^2:\mathrm{MR.AQ}\right):\left(\mathrm{AD}^2:\mathrm{AR.A}a\right)$. Therefore there will be also $\mathrm{MX}:\mathrm{AX}=\mathrm{A}a^5.\mathrm{AD}^2.\mathrm{AR}:\mathrm{AQ}^3.\mathrm{AG}^2.\mathrm{AS}^2.(\mathrm{A}\Sigma^2).\ \mathrm{MR}$. [Since

$\text{AV} : \text{AD} = a\text{A}.a\text{A.AD} : \text{GA.QA.SA}$ from above.] Which is the general cannon, whenever the resistances are as D by AV^2; truly if these resistances shall be as D.AV^h, with h being some rational number, the ratio AD to AS will be obtained by the method now established in §.622. *In vacuo* this ratio AD to AS shall be one of equality, and there that shall be $\text{MX} : \text{AX} = \text{A}a^5.\text{AR} : \text{AQ}^3.\text{AG}^2.\text{MR}$.

XII. *With the same still in place* D *is taken generally, as* $(3\text{KQ} : 2\text{AQ.MR}) - (\text{QK} : 2\text{AK.MN}) - (\text{P} : 2\text{MX}) - (\text{R}\Gamma : 2\text{MR}^2)$ *and the ratio of the resistance to gravity,*

$$\pm\text{R} : \text{MX} = (3\text{KQ} : 2\text{AK}) - (\text{R}\Gamma.\text{AQ} : 2\text{AK.MR})$$
$$-(\text{KQ.AQ. MR} : 2\text{AK}^2.\text{MN}) - (\text{P.AQ.MR} : 2\text{AK.MX})$$

In which formulas, see Fig. 158, RΓ indicates the radius of osculation at the point R of the curve RR, from the evolute of which given curve AM it will be described, and P indicates the exponent of the ratio, which an element of magnitude MX of the weight at M has along MN, to an element M*m* of the curve of the body requiring to be described.

I will not add the demonstration of these two results, since that can be found easily according to the previous manner and rule. But, so that the agreement of these with the others found may be apparent, we will apply a few of the particular cases of the most celebrated Newton and Johan Bernoulli.

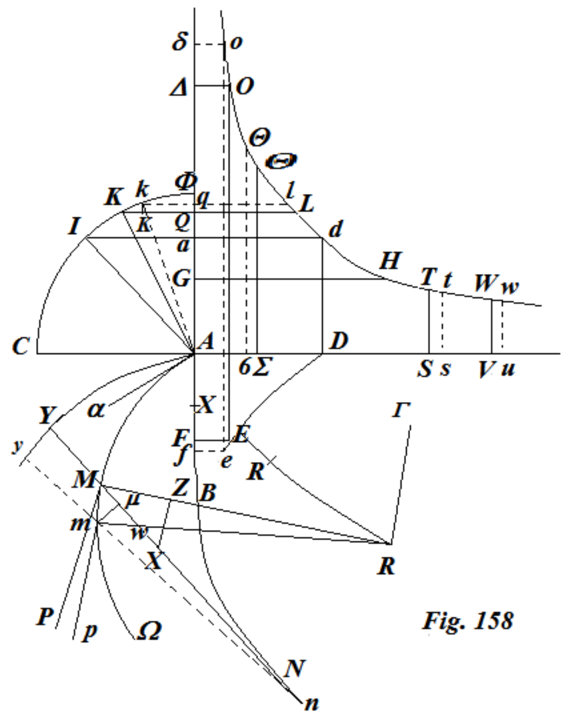


Fig. 158

1. Let the very curve AM being described by the moving body be the circle ΦKC with gravity acting uniformly along directions parallel to AQ , in which case P, and RΓ vanish, MN shall be infinite and $\text{MR} = \text{AK}$, therefore D will be as $3\text{KQ} : 2\text{AQ.AK}$, that is, thus as the tangent of the angle KAQ and the resistance to gravity, shall be as 3KQ to 2AK, clearly as the most praiseworthy author found.

2. If in place of the curve AM the curve may be taken, in which the directions of constant gravity MN shall be parallel to each other, D will be as

$$(3\text{KQ} : 2\text{AQ.MR}) - (\text{R}\Gamma : 2\text{MR}^2), \ \&$$

$$\frac{\pm\text{R}}{\text{MX}} = (3\text{KQ} : 2\text{AK}) - (\text{R}\Gamma.\text{AQ} : 2\text{AK.MR}).$$

For, since the MN are infinite in these suppositions, the individual fractions in the denominator of which these MN may be found, will vanish. And thus the ordinates of the curve will be called *y*, along which uniform gravity is acting, the respective abscissas *x,* the radius of curvature at M or $\mathrm{MR} = r$, with $M\mu = dy$ and $m\mu = dx$, clearly with all the MN parallel to each other, truly the ratio $\mathrm{R\Gamma} : \mathrm{MR} = dr : ds$, with an element of the curve made $Mm = ds$. And thus with these symbols, D will be as

$$\frac{3dy}{2rdx} - \frac{dr}{2rds}, \text{and } \frac{\pm \mathrm{R}}{\mathrm{G}} = \frac{3dy}{2ds} - \frac{drdx}{2ds^2}.$$

But with the dx being constant, there will be

$$r = ds^3 : dxddy, \text{and } dr = \frac{3dsdy}{dx} - \frac{ds^3dddy}{dxddy^2},$$

with which values substituted into the formulas D becomes as

$$dddy : 2dsddy, \;\; \& \;\; \pm R : G = dsdddy : 2ddy^2$$

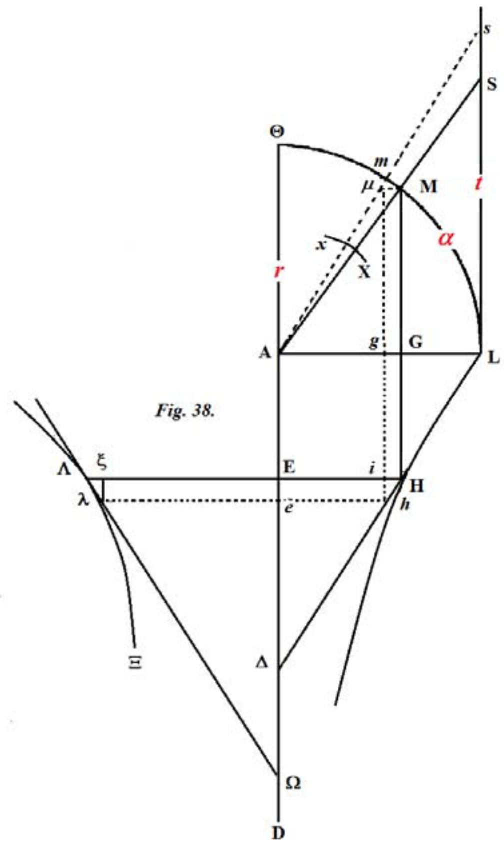
which two latter canons according to rule agree with the formulas of the cel. Newton Propos. X. Book II, latest edition *Princip. Phil. Nat*. For, if in these in place of these with our succession *dy, ds, ddy* & *dddy* there may be substituted

$\mathrm{Q}o,\ o\sqrt{1+\mathrm{QQ}},\ 2\mathrm{R}oo,\ \&\ 6\mathrm{S}o^3$, which are equal to these expressions in Newtonian elements, there will be generally D as $\frac{3\mathrm{S}}{2\mathrm{R}\sqrt{1+\mathrm{QQ}}}$, & $\frac{\pm \mathrm{R\ es.}}{\mathrm{Grav.}} = \frac{3\mathrm{S}\sqrt{1+\mathrm{QQ}}}{4\mathrm{RR}}$. Therefore as Newton has in the place mentioned.

XIII. In the demonstration Propos. XXV. Book I, §.167. an error has crept in, while we considered the element of the ordinate as positive, therefore in the progress of that demonstration the sign of $\Lambda\xi$ are to be changed from plus to minus everywhere, with which done there will be found

$$\mathrm{G} = \frac{1}{\mathrm{DE}^3} + \frac{(-\mathrm{DE.E\Delta.MG}^2 + 2.\mathrm{DE.E\Omega.MG}^2 + \mathrm{DE.E\Omega.AL}^2).xx.\Delta\mathrm{E}}{\mathrm{E\Omega.AG}^2\mathrm{DE}^5}$$

And on putting $\mathrm{HE} = A \mp e$, and thus $\Delta\mathrm{E} = \Lambda\mathrm{E} = \mathrm{HE} : \mathrm{B}$, & $\mathrm{E}\Omega = \mathrm{B.HE} : \mathrm{B}^2 - \mathrm{C.HE}$, where $B = dA : dz$; $C = dB : dz$ & $\mathrm{DE} = z$, likewise $\mathrm{AL} = r$, and $rr - ee = ss$; this appointed formula with the values of the lines substituted into DE, $\mathrm{E}\Delta$, $\mathrm{E}\Omega$, with that done the equation will be produced shown towards the end of §.169. Truly, because it is a little more elegant, as it appears to me, the solution of the same problem itself presented to me, that it is pleasing to find a place here. About the right line AC, as the axis, some curve $\Pi\mathrm{B}$ shall be described, and with AC taken also for the radius, and thus with the centre C a circle AG shall be described, of which the tangent at A will be the indefinite right line $\mathrm{OA}\Pi$, and with the tangent BV drawn through any point B of the curve $\Pi\mathrm{B}$ and with the ordinate BR being produced as far as for its crossing with the circle at G, the subtangent RV may be divided at S, so that RS shall be to the whole RV as I to some rational number *n,* and SG may be joined, while the right line AL may be drawn through A with CA*a* containing the PAQ to be equal everywhere with respect to the angle GSR, and this also if it were made from the tangent and the subtangent at the point Π on the curve , the right line $\mathrm{A}\lambda$ would result, since AL arrives in order at the point B of the curve $\Pi\mathrm{B}$. AI may express the speed of the moving body describing the curve AN at the point A with the centre of attraction of the centre D drawn through the point I on the line IK indefinitely close to the parallel line AC, the segment $\mathrm{A}\alpha = \mathrm{A}\Pi$ may be taken on $\mathrm{A}\lambda$ and through α the line αK is acting parallel to AI, meeting the right line IK at K**,** through which point the hyperbola KM may be considered to be drawn between the asymptotes A*a* and AI. Again at AL with AQ taken equal to the respective



ordinate RB of the curve ΠB, QP may be drawn parallel to AΠ, which produced shall meet the hyperbola at M, and finally at the ordinate MO drawn through this point M to the asymptote AO and produced downwards, OF may be made equal to the difference of the ordinates AΠ, RB and the point F will be on the graph of the speed of the moving body incident on the other curve AN. But this curve AN may be prepared thus, so that the carriers of the radius $\mathrm{AD} = \mathrm{A\Pi}$, & $\mathrm{ND} = \mathrm{RB}$, drawn through its ends, may contain the angle ADN, which shall be to the angle in the circle ACG, as 1 to the number which was before *n.* Let *Nq* be the tangent to the curve AN at N, upon which the perpendicular D*q,* and with the centre D the arc EN and the arclet *np* may be described*,* and by putting, Fig. 159 , the angle GC*g* to the angle $\mathrm{ND}n = n:1$, *gb* will be acting parallel to GB.

Demonst. Because the infinitesimal angle *NDn* is to GC*g* as 1 to *n,* or (constr.) $= \mathrm{RS} : \mathrm{RV}$, and (§.129.) the arclet *pn* to G*g* shall be composed in the ratio of the angle ND*n* to the angle GC*q,* or as 1 to *n*, or RS to RV, and of the radius DN or of the ordinate RB to the radius GC, there will be $pn : \mathrm{G}g = \mathrm{RS.RB} : \mathrm{GC.RV}$; but still there is

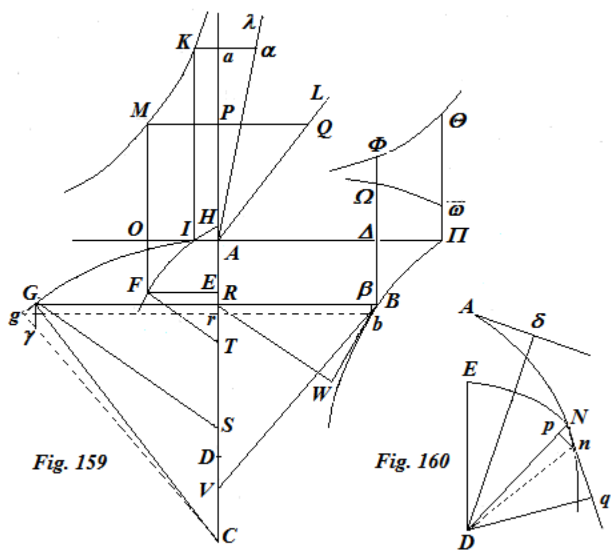


Fig. 159

Fig. 160

$$\mathrm{G}g : b\beta (= \mathrm{GC:GR}) = \mathrm{GC.RV:GR.RV};$$
$$\& \; b\beta : \mathrm{N}p (\text{or } \mathrm{B}\beta) = \mathrm{RV:RB} = \mathrm{GR.RV} : \mathrm{GR.RB}.$$

Therefore from the equation $pn : \mathrm{N}p = \mathrm{RS.RB} : \mathrm{GR.RB} = \mathrm{RS} : \mathrm{GR}$, or (because PAQ has been made equal to the angle GSR) $= \mathrm{AP} : \mathrm{PQ}$, indeed also there is $pn : \mathrm{N}p = \mathrm{D}q : \mathrm{N}q$, therefore $\mathrm{AP} : \mathrm{PQ} = \mathrm{D}q : \mathrm{N}q$; hence, because AQ (constr.) $=$ DN , AP or $\mathrm{MO} = \mathrm{D}q$, and $\mathrm{A}a = \mathrm{IK} = \mathrm{D}\delta$. From which, since (§. 155.) D*q* shall be to Dδ, as the velocity at A represented by AI to the velocity at N, and in the hyperbola KM, the ordinate MO shall be to KI as AI to AO*,* this AO sets out the speed at N completely, and since OF or AE (constr.) shall be equal to the difference of the ordinates AΠ, RB, that is, of the ordinates AD, NO, and in each figure they are equal to AE, and thus the curve HIF will be the graph of the speed of the moving body on the curve of AN descending or ascendenting, and thus with the normal of that ET drawn, the subnormal of ET (§.134) will express the action of gravity along ND on the curve AN at the point N. Quod erat demonstrandum .

Coroll. If D were the origin of the abscissas DR, and these abscissas may be called A, the respective ordinates RB; *z,* the radius AC; *r,* the distance of the origin of the abscissas of the curve ΠB from the centre of the circle AH, or DC, *e* ; there will be $\mathrm{CR} = e + \mathrm{A}$, if the point D falls between C and R, as in the figure, or $e - \mathrm{A}$, if between A & R, and finally $\mathrm{A} - e$, if the point D falls outside the radius AC; therefore we may put $\mathrm{CR} = e + \mathrm{A}$, and there will be $\mathrm{GR} = \sqrt{(ss \mp 2e\mathrm{A} - \mathrm{A}^2)}$ where $ss = rr - ee$, equally the element of the abscissas may be called DR : or $\mathrm{R}r = \mathrm{B}.dz$, and there will be $\mathrm{RV} = \mathrm{B}z$, and $\mathrm{RS} = \frac{1}{n}\mathrm{B}z$, and with all these symbols substituted into the preceding construction, there will be found that $\mathrm{AO}^2 = \mathrm{FE}^2 = \frac{1}{2z} + \frac{(ss \mp 2e\mathrm{A} - \mathrm{A}^2).nn}{B^2 z^4}$, and from this there will be elicited the very formula found at the end of §.169. Therefore it is apparent that the latter derivation agrees surprisingly well with the former.

Hence, if the points D and C coincide, and the curve ΠB is a hyperbola between the orthogonal asymptotes meeting at C, of which AC is one, the gravity at the point N of the curve AN , along ND, shall be as $\frac{1-nn}{z^3}$ and the curve AN itself will be that, of which Newton and Johan Bernoulli have shown many different kinds as examples, the matter still indeed agreeing, the one in Act. Lips. I 713. p. 129. truly the other in Prop. XLlV. Book I. *Princ. Phil. Nat*., as we have indicated above on p. 80. This curve AN has two asymptotes equally distant from the centre of forces D ; and indeed the distance which is allowed to be must be $\frac{1}{n}$ of the radius AC in fig.159 and 160. or $\frac{1}{n}$DP in fig. 37. and hence the other asymptote cannot go through the centre, as by misadventure I have written on p. 81. The remaining properties of the same curve AN, fig. 160, mentioned by Bernoulli have elicited no trouble from our constructions, on account of which I refrain from further explanations of the same.

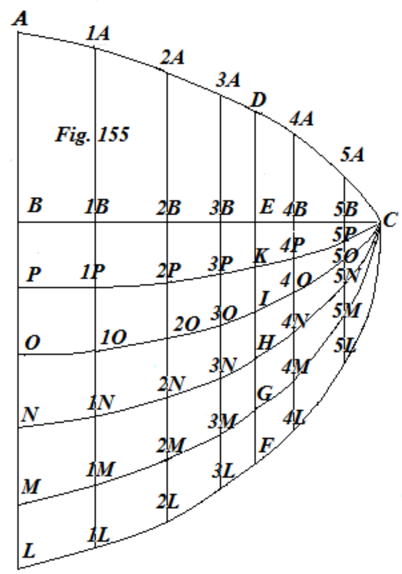
Concerning the rest, which pag. 80. no. 4 has with the exception of that case, where *p* is -1 , clearly is useless, and proposition mentioned there with this exception removed generally is obtained, if *e* or *s* is 0, but this exception only prevails in the contradiction of this proposition, evidently only in this case were $p = -1$, and with the two cases $p = +1$, and $= -2$ excepted, the curve AN can become algebraic at some time under the hypothesis that G shall be as z^p, with neither vanishing from the two *e* or *s*.

The preceding construction thus can be returned more elegant by considering RW parallel to GS, and by dropping the perpendicular BW from the point B upon that, then with lines drawn through B & Π parallel to the axis AC, as BΦ and $\Pi\Theta$, and by taking in the first segment $\Delta\Phi$ equal to the perpendicular BW, and the curve $\Theta\Phi$ arises, in the ordinates of which, if $\Pi\pi = \mathrm{AI}$ were to $\Delta\Phi$ as $\Phi\Delta$ to $\Theta\Pi$; thus so that the curve $\pi\Omega$ would be the inverse of the other $\Theta\Phi$. This reciprocal $\pi\Omega$ now will be also the graph of the speed, and thus the same as the curve HIF, but related to the axis AΠ, in place of the axis HE, to which the other HIF has been constructed. In this construction neither is there a need for the line A*l* nor for the hyperbola KM , and thus nothing can be made simpler.

FINIS.

IX. *De Curvis Algebraicis per quotlibet data puncta ducendis.*

Circa finem capitis VII. Lib.II. §.283. mentionem feci problematis ducendæ curvæ algebraicæ per quocunque puncta positione data, cujus solutionem summus Newtonus primus invenit : ejus tamen solutionem mihi videre non contigit, excepta ea quam, in Lemmate V. Lib. III. Princ. Phil. Natur sine omni analysi & demonstratione tradit, ubi lineam parabolici generis per quotlibet data puncta ducere docet. Hujus problematis ego solutionem annis 1704 & 1705 aggressus sum & obtinui, quam cum Illust. Leibnitio per literas communicavi, quæ solutio summo viro non prorsus displicuit, ut ex literis ejus annis 1705 & 1706 perhumaniter ad me datis colligi potest. Occasionem de hoc problemate cogitandi mihi præbuit epistola Newtoni ad Oldenburgium, in qua hujus problematis meminit, & ex pulcherrimis id prædicat eorum, quæ solvere desiderasset. Sed ad rem:



Si ergo per puncta quotcunque A, 1A, 2A, 3A, &c. & C ducenda sit curva algebraica, seu analytici generis, CDA; per alterutrum C extremorum punctorum linea duci potest quæcunque CB, quæ instar axis sit, ad quem ex singulis datis punctis ordinatæ AB, 1A1B, 2A2B, &c. demittantur, producendæ subter axem in L, 1L, 2L, &c. & ad alteram axis partem sint curvæ quæcunque CFL, CGM, CHN, CIO, CKP ,&c. Considerentur jam singulæ AL, 1A1L, 2A2L, etc. tanquam totidem vectes ponduscul is A, L, M, N, O, P, &c. in punctis, hisce literis indicatis onusti, sed quarum centra æquilibrii singula reperiantur in axe CB, scilicet in punctis ejus B, 1B, 2B, 3B, &c. & sequetur, quod, ducti qualibet DF reliquis AL1 &c. parallela, atque curvis subter axem occurrente in punctis F, G, H, I, K, &c. axique in E, hujus lineæ seu vectis DF, in punctis D, F, G, H, I, K, &c. ponduscula eadem ac prius A, L, M, N, O, P, &c. appensa habentis, & centrum æquilibrii horum ponduscul orum in E, brachium ED futurum sit ordinata curvæ regularis CDA per singula puncta A, 1A, 2A, 3A, &c. C transeuntis, perinde ac brachia EF, EG, EH, &c. ordinatæ sunt curvarum CL, CM, CN, &c. Atqui ex datis EF, EG, EH, El, EK; &c. & ponderibus A, L, M, N, O , P, &c. invenietur ordinata ED in curva quæsita CDA. Tota ergo difficultas reducitur ad inventionem ponderum A, L, M, N, O, P, &c vel proportionis horum. Atqui principium vectis præbet sequentes æquationes

1. $\mathrm{A.AB = L.LB + M.MB + N.NB + O.OB + P.PB.}$

2. $\mathrm{A.1A1B = L.1L1B + M.1MIB + N.1N1B + O.1O1B + P.1P1B.}$

3. $\mathrm{A.2A2B = L.2L2B + M.2M2B + N.2N2B + O.2O2B + P.2P2B.}$

4. $\mathrm{A.3A3B = L.3L3B + M.3M3B + N.3N3B + O.3O3B + P.3P3B.}$

5. $\mathrm{A.4A4B = L.4L4B + M.4M4B + N.4N4B + O.4O4B + P.5P5B.}$

6. $\mathrm{A.5A5B = L.5L5B + M.5M5B + N.5N5B + O.5O5B + P.5P5B.}$

Hæ æquationes continuo a se invicem subductæ, scilicet secunda a prima, tertia a secunda, quarta a tertia & sic deinceps, atque residuæ divisæ per respectivas differentias ordinatarum AB, 1A1B, 1A1B, 2A2B, 2A2B*,* 3A3B, &c. habebuntur quinque æquationes, in quibus singulis A erit ex una parte sola, & si in reliquis membris, pro singulis $\mathrm{BL-1B1L}$; $\mathrm{BM-1B1M}$; $\mathrm{BN-1B1N}$, & sic deinceps, divisis per $\mathrm{AB-1A1B}$, scribantur Q, R, S, T, V*,* in quinque residuis æquationibus, & eæ eadem modo tractentur ac sex primæ, sequentes scilicet ab antecedentibus subducendo, relinquentur quatuor novæ, quæ per respectivas $\mathrm{Q-Q1}$, $\mathrm{Q1-Q2}$, $\mathrm{Q2-Q3}$, $\mathrm{Q3-Q4}$ dividantur & pro $\mathrm{R-R1:Q-Q1}$; $\mathrm{S-S1:Q-Q1}$; $\mathrm{T-TI:\ Q-Q1}$; $\mathrm{V-V1:Q-Q1}$*;* scribantur *r,s,t,u;* item pro $\mathrm{R1-R2:Q1-Q2; S1-}S2:Q1\mathrm{-Q2}$, &c. $r1$, $s1$, $t1$, $t1$. Pro $\mathrm{R2-R3:Q2-Q3}$; $\mathrm{S2-S3:Q2-Q3}$, &c. $r2$, $s2$, $t2$, $u2$; & sic porro eodem ordine, & hæ subductiones ac divisiones continentur usque dum unica tantum supersit æquatio, poterunt in quantitatibus datis omnes A, L, M, N, O, P, &c. exhiberi, aut saltem omnes excepta ultima P, quæ arbitrariæ magnitutudinis est, nam factis debitis reductionibus invenietur

$\mathrm{O} = -\mathrm{P}\alpha.$

$\mathrm{N} = -\mathrm{O}\tau - \mathrm{P}\eta.$

$\mathrm{M} = -\mathrm{N}\sigma - \mathrm{O}\theta - \mathrm{P}\omega.$

$\mathrm{L} = -\mathrm{M}r - \mathrm{N}s - \mathrm{O}t - \mathrm{P}u.$

$\mathrm{A = LQ + MR + NS + OT + PV.}$ In his æquationibus est
$dd\mathrm{LB}:d\mathrm{AB = Q}$; $d\mathrm{MB}:d\mathrm{AB = R}$; $d\mathrm{NB}:d\mathrm{AB = S}$; $d\mathrm{OB}:d\mathrm{AB = T}$; $d\mathrm{PB}:d\mathrm{AB} = V$.

Sic $d1\mathrm{L1B}:d1\mathrm{A1B} = Q1$; $d1\mathrm{M1B}:d\mathrm{A1B = R1}$; $d1\mathrm{N1B}:d1\mathrm{A1B:S1}$, &c.

Nec non $d2\mathrm{L2B}:d2\mathrm{A2B = Q2}$; $d2\mathrm{M2B}:d2\mathrm{A2B = Q2}$ *;* &c. Ubi notandum literam d cuililibet ordinatæ præpositam significare differentiam, inter hanc ordinatam ipsique proximam versus C in eadem curva reliquæ quantitates sic etiam facile definiuntur, scilicet

$r = dR : dQ; r1 = dR1 : dQ1, \&c.$	
$s = dS : dQ; s1 = dS1 : dQ1, \&c.$	$\sigma = ds : dr; \sigma 1 = ds1 : dr1, \&c.$
$t = dT : dQ; t1 = dT1 : dQ1, \&c.$	$\theta = dt : dr; \theta 1 = dt1 : dr1, \&c.$
$u = dV : dQ; u1 = dV1 : dQ1, \&c.$	$v = du : dr; v1 = du1 : dr1, \&c.$
$\tau = d\theta : d\sigma; \tau 1 = d\theta 1 : d\sigma 1$	
$\eta = dv : d\sigma; \eta 1 = dv1 : d\sigma 1$	$\omega = dv : d\tau.$

Hoc loco litera *d* signficat differentiam inter magnitudinem, cui profigitur & alteram eadem litera, sed cum unitate adscripta indicatam, sic $dR = R - R1,\ dQ = Q - Q1;\ dR1 = R1 - R2$, & sic de reliquis. Inventis ita valoribus ponderum A, L, M, N, O, &c iidem substituendi sunt in æqualitate $A.DE = L.FE + M.GE + N.HE + O.IE + P.KE$, atque sic per ordinatas FE, GE, HE, IE, KE, &, & alias datas quantitates assignari potest ubique ordinata DE curvæ CDA per data puncta A, 1A, 2A, &c. transeuntis. Quod erat inveniendum.

Coroll. I. Curva genita CDA ejusdem semper est gradus cum curva gradus altissimi ex generatricibus GFL, CGM, CHN, &c.

Coroll. II. Area genitæ CDA ex areis generatricum semper inveniri potest. Est enim generaliter $A.CDE = L.CFE + M.CGE + N.CHE + O.CI.E + P.CKE$. Unde, si hæ areæ curvarum generatricium sint quadrabiles, etiam figura genita quadribilis erit.

Coroll. III. Si curvæ genitrices sunt generis parabolici, earum areæ sunt quadrabiles, ac proinde etiam genita quadrabilis existet : cum igitur pro generatricibus curvæ quæcunque eligi possint, atque adeo curvæ quadrabiles, inde clarum est omnes curvas per appropinquationem quadrabiles esse, tot enim, quot libuærit, in ea possunt puncta assumi, & per ea curva quadrabilis duci, cujus area curvæ propositæ areæ quam proxime æqualis erit.

Coroll. IV. Imo omnis curva per, appropinquationem rectificari potest. Curvarum enim rectificatio ad quadraturas reducitur, quæ per præcedens corollarium, per appropinquationem, semper habentur. Idem intelligendum de centris gravitatis figurarum & solidorum, tum etiam de hisce solidis ipsis aliisque. Hæc enim omnia quam proxime vero haberi queunt.

Coroll. V. Ex hisce etiam cognoscitur *maximus* punctorum datorum numerus, per quæ curva dati gradus duci potest. Hic enim numerus *est semissis producti ex exponente gradus curvæ in eundem exponentem ternario auctum*. Hoc modo scimus lineam primi gradus, seu rectam, non nisi per 2. puncta, positione ut libet data, duci posse ; est enim $2 = \frac{1.4}{2}$, sectionem conicam per quinque, nam $5 = \frac{2.5}{2}$. Curvam tertii gradus per novem, nam $9 = \frac{3.6}{2}$.

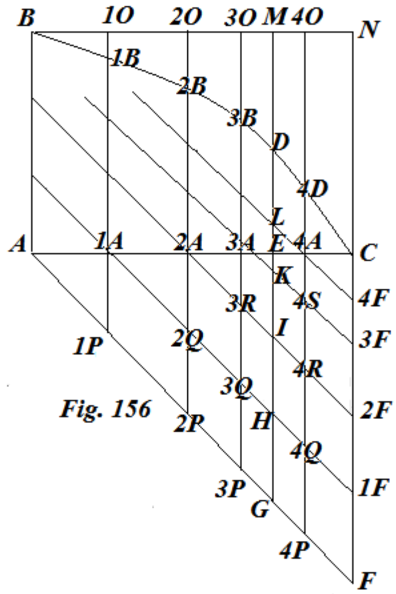


Fig. 156

Aliter. Loco generatricium assumi possunt lineæ rectæ AF, 1A1F, 2A2F, &c. inter se parallelæ, & quemlibet angulum cum axe CA continentes atque, necessitate ita postulante, supra axem producendæ, ut schema ostendit. Sint ergo puncta, per quæ curva duci debet B, 1B, 2B, &c. C. Per B agatur BN æquidistans AC, & ex singulis punctis datis B agatitur perpendicula BA, 1B1A, &c. producenda sursum in 1O, 2O, &c. deinde ducta ubilibet recta MG parallela AB obliquas AF secante in punctis, G, H; I, K, L,&c. & axem AC in E. Quibus peractis fiat ubilibet

$$\mathrm{MD} = \mathrm{G.EG} + \mathrm{H.EG.EH} + \mathrm{I.EG.EH.El}$$
$$+\mathrm{K.EG.EH.El.EK} - \mathrm{L.EG} \ldots\mathrm{EL},$$

in qua valores assumtarum G, H, I, K, L sunt definiendi sequenti ratione. Si punctum E cadit in A, linea EG evanescet, & reliquæ HE, IE, &c. supra axem erunt, verum quoniam *EG* jam in punctum A contracta nullescit, & in omnia reliqua membra influit, erit MD in BA nulla. Sin vero E cadit in punctum 1A, fiet EG tunc $= 1\mathrm{A}1\mathrm{P}$, & $\mathrm{EH} = 0$ adeoque eo casu, quo MD sit 1O1B, erit $1\mathrm{O}1\mathrm{B} = \mathrm{G}.1\mathrm{A}1\mathrm{P}$, seu $\mathrm{G} = 1\mathrm{O}1\mathrm{B}:1\mathrm{A}1\mathrm{P}$. dicatur $\mathrm{GI} = 2\mathrm{O}2\mathrm{B}:1\mathrm{A}2\mathrm{P}, \mathrm{G}2 = 3\mathrm{O}3\mathrm{B}:3\mathrm{A}3\mathrm{P}, \&\mathrm{c}.$ Porro si punctum E cadit in 2A, fiet $\mathrm{MD} = 2\mathrm{O}2\mathrm{B}$, & $\mathrm{EG} = 2\mathrm{A}2\mathrm{P}$, $\mathrm{HE} = 2\mathrm{A}2\mathrm{Q}$, & $\mathrm{IE} = 0$, ergo

$$2\mathrm{O}2\mathrm{B} = \mathrm{G}.2\mathrm{A}2\mathrm{P} + \mathrm{H}.2\mathrm{A}2\mathrm{P}.2\mathrm{A}2\mathrm{Q}, 3\mathrm{O}3\mathrm{B}$$
$$= \mathrm{G}.3\mathrm{A}3\mathrm{P} + \mathrm{H}.3\mathrm{A}3\mathrm{P}.3\mathrm{A}3\mathrm{Q} + \mathrm{I}.3\mathrm{A}3\mathrm{P}.3\mathrm{A}3\mathrm{Q}.3\mathrm{A}3\mathrm{R},$$

atque ita deinceps; ex quibus omnibus sequens resultat tabella, postquam scilicet singulæ æquationes ad respectivas AP applicatæ fuerunt:

$$\mathrm{G}\ \left(=1\mathrm{O}1\mathrm{B}:1\mathrm{A}1\mathrm{P}\right) = \mathrm{G}$$
$$\mathrm{G}1\left(=2\mathrm{Q}2\mathrm{B}:2\mathrm{A}2\mathrm{P}\right) = \mathrm{G} + \mathrm{H}.2\mathrm{A}2\mathrm{Q}$$
$$\mathrm{G}2\left(3\mathrm{O}3\mathrm{B}:3\mathrm{A}3\mathrm{P}\right) = \mathrm{G} + \mathrm{H}.3\mathrm{A}3\mathrm{Q} + \mathrm{I}.\ 3\mathrm{A}3\mathrm{Q}.3\mathrm{A}3\mathrm{R}$$
$$\mathrm{G}3(=4\mathrm{O}4\mathrm{B}:4\mathrm{A}4\mathrm{P}) = \mathrm{G} + \mathrm{H}.4\mathrm{A}4\mathrm{Q} + \mathrm{I}.4\mathrm{A}4\mathrm{Q}.4\mathrm{A}4\mathrm{R}$$
$$+\mathrm{K}.4\mathrm{A}4\mathrm{Q}.4\mathrm{A}4\mathrm{R}.4\mathrm{A}4\mathrm{S}.$$
$$\&\mathrm{c}. \qquad \&\mathrm{c}. \qquad \&\mathrm{c} \qquad \&\mathrm{c}.$$

Si porro prima a secunda, secunda a tertia, tertia a quarta, atque ita porro subducantur, erit $\mathrm{G}1 - \mathrm{G} = \mathrm{H}.2\mathrm{A}2\mathrm{Q}$; $\mathrm{G}2 - \mathrm{Gl} = \mathrm{H}.3\mathrm{A}3\mathrm{R} + \mathrm{I}.3\mathrm{A}3\mathrm{Q}.3\mathrm{A}3\mathrm{R}$; $\&\mathrm{c}.$. Et ponendo $d\mathrm{G}, d\mathrm{G}1$, $d\mathrm{G}1$ pro $\mathrm{G}1 - \mathrm{G}$, $\mathrm{G}2 - \mathrm{G}1$, $\mathrm{G}3 - \mathrm{G}2$, $\&\mathrm{c}$. fient

$$\text{H} = \text{dG} : 2\text{A}2\text{Q};\ \text{H}1 = d\text{G}1 : 3\text{A}3\text{R};\ \text{H}2 = d\text{G}2 : 4\text{A}4\text{S}; \&\text{c.}$$
$$\&\ \text{I} = d\text{H} : 3\text{A}3\text{Q};\ \text{I}1 = d\text{H}1 : 4\text{A}4\text{R}; \&\text{c.}$$

denique $\text{K} = d\text{I} = 4\text{A}4\text{Q}$. Et sic porro; nam ex hisce continuationis lex satis patet. Inventi valores magnitudinum G, H, I, K, &c. in superiori æqualitate

$$\text{MD} = \text{G.GE} + \text{H.GE. HE} + \text{I.GE.HE.IE} + \text{K.GE.HE.IE.KE}, \ \ \&\text{c.}$$

substituantur, dabitur sic ipsa MD in lineis GE, HE, IE, KE, LE; &c. & magnitudinibus datis, atque adeo punctum D in curva quæsita. Quod erat secundo inveniendum.

Coroll. I. Si rectæ AF, 1A1F, &c. angulo semirecto ad AC inclinatæ.sunt, provenit casus secundus hujus problematis ab Illustr. Newtono solutus Lemm. V. Lib. Ill. Princ. Phil. Nat. in ejus solutione enim *a, b, c, d, e., f,* &c. idem prorsus sunt cum G, H, I, K., L, &c .

Coroll. II. Iisdem positis, quæ in præcedenti corollario, si intervalla singula A1A, 1A2A, 1A3A, &c. sint æqualia & dicantur *p, &* differentiæ ordinatæ AB, prima, secunda, tertia, &c. erunt $\text{G} = \delta : p;\ \text{H} = \delta 2 : 2.pp;\ \text{I} = \delta 3 : 2.3.p^3, \&c.$ quod congruit solutioni Newtonianæ primi casus in Lemmate citato.

Coroll. III. Si insuper AE vel EG dicatur *z,* DE , *u*, & $\text{AB} = y$, erit

$$u = y - \text{G}z - \text{H.}\left(zz - pz\right) - \text{I.}(z^3 - 3pzz + 2ppz)$$
$$-\text{K.}(z^4 - 6pz^3 + 11ppzz - 6p^3z) - \&c.$$

& area

$$\text{ABDE} = yz - \tfrac{1}{2}Gzz - \text{H. }(\tfrac{1}{3}z^3 - \tfrac{1}{2}pzz) - \text{I.}(\tfrac{1}{4}\text{z}^4 - pz^3 + ppzz)$$
$$-\text{K.}(\tfrac{1}{5}z^5 - \tfrac{3}{2}pz^4 + \tfrac{1}{3}ppz^3 - 3p^3zz) - \&\text{c.}$$

Hoc loco enim *y* est constans, & *z* ac *u* sunt variabiles. Hinc, si *p* sunt indefinite parvæ, erunt singulæ $p = dz$, & ipsæ $\delta, \delta 2,\ \delta 3,\ \ \&\text{c.}$ hoc casu fient $dy, -ddy, +dddy, -d^4y, \&c.$ & in præcedenti serie membra omnia, quæ *p* continent , evanescent, totaque series abibit in $yz - \frac{1}{2}\text{G}zz - \frac{1}{3}\text{H}z^3 - \frac{1}{4}\text{I}z^4 + \&\text{c.} = \text{area ABDE},$ atqui sunt

$$\text{G} = \tfrac{dy}{dx},\ \text{H} = \tfrac{-ddy}{2.dz^2},\ \text{I} = \tfrac{+dddy}{2.3dz^3}; \&c.$$

ergo area $\text{ABDE} = yz - \frac{zzdy}{2dz} + \frac{z^3ddy}{2.3dz^2} + \frac{z^4dddy}{2.3.4dz^3} + \&c.$ Quæ est ipsissima series universalis pro quadraturis, quam Celeb. Joh. Bernoulli in Act is Lips. 1694 exhibuit, quamque haud dubie ex alio fundamento elicuit.

Cæterum, loco rectarum transversalium AF, 1A1F, curvæ quæcunque parallelæ assumi possunt, vel una eademque curva axem in linea CF, verticem vero successive in punctis diversis ipsius CF habens.

X. *De velocitate liquidum per foramina quæcunque ex vasis erumpentium.* In Propositione XXXII. Lib. II. demonstravimus quidem post Clar. Varignonium aliosque celeritates liquorum ex vasis effluentium in subduplicata esse ratione altitudinis liquorum supra foramina vasis aquam emittentia, sed non demonstravimus, nec quisquam alius quod sciam*, aquam aliumve liquorem ea velocitate ex vase erumpere,* eadem constanter manente aquæ altitudine supra foramen, *quam aquæ guttula orificio proxima acquire potest casu acceleratio ex altitudine liquoris supra orificiam.* Nam hoc principium, instar hypotheseos, supposuerunt Torricellius, Borellus, Gulielminus aliique, atque exinde recte deduxerunt, aquas eodem tempore per foramina æqualia effluentes, seu etiam celeritates effluxus in subduplicata esse ratione altitudinis aquarum supra foramina. Sed hæc suppositio tantæ non videtur evidentiæ, ut nulla demonstratione egeat; nam præterquam quod Celeb. Newtonus, Propos. XXXVII. Lib. II Princ. Math. primæ editionis, demonstrare conatur, aquam ea cum velocitate erumpere ex vasis, qua motu suo in altum converso ad dimidiam altitudinem aquæ supra foramen evehi possit, sed quam propositionem in novissimis editionibus omisit. Torricellius suppositioni suæ, non propter evidentiam, sed potius propter consensum ejus, cum experientia acquievit.

Esto vas quodcunque BSC aquæ vel cujusvis alius liquoris homogenei plenum, F foramen ejus, FA altitudo aquæ supra foramen F. Concipio columnam aqueam A*p* per temporis tractum indefinite parvum *dt,* uniformiter urgere particulam aquæ infinitesimam *p*F, eo scilicet tempusculo, quo hæc particula, quam *p* nominabimus, proprio pondere seu gravitate, longitudinem F*f* particulæ *p*F seu *p* æqualem acquirere potest celeritatis gradum infinitesimum *u.* Et sint *g* signum gravitatis naturalis, qua singula corpora apud nos agitantur, A altitudinis aquæ AF, &c deniquc V velocitatis, qua per foramen F erumpit; M massæ tempusculo *dt* effluentis ac *m* massa particulæ *p*F. Hisce positis per §.31. pondus columnæ A*p* vel AF exponetur per A.F.*g*; & pondus elementi *p*F per *p*.F.*g.* Sed hæc facta sunt ut solicitationes acceleratrices motus M.V & *m.u,* tempusculo *dt* generantes ; idcirco habemus (§.130.) $\mathrm{A.F.}g.dt : p.\mathrm{F}.gdt = \mathrm{M.V} : m.u$, id est, $\mathrm{A}.g : p.q = \mathrm{M.V} : m.u$; atqui $\mathrm{M} : m = \frac{1}{2}\mathrm{V}dt : \frac{1}{2}udt = \mathrm{V} : u$ ex natura motus accelerati in fluidis, ergo $\mathrm{A}g : pg = \mathrm{VV} : uu$, atqui $2pg$ (§150.) $= uu$, ergo $\mathrm{VV} = 2.\mathrm{A}.g$; hoc, est velocitas V, quacum aqua per foramen F erumpit (§.150.) ea est, quam particula infinitesima aquæ *p*F acquirere posset motu naturaliter accelerato per descensum ex altitudine A seu AF. Quod erat demonstrandum.

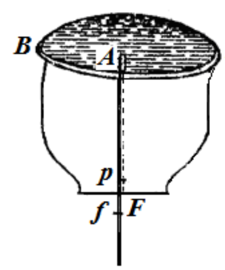


Fig. 157

Atque hinc jam tutissime deducitur, *quantitates aquæ per idem vel æqualia foramina effluentis, vel etiam celeritates ejus, esse in subduplicata proportione altitudinum*.

XI. Capite XX. Lib. II. exposui generalem theoriam gravitatis variabilis, cum scilicet corpora solicitantur secundum directiones quascunque uniformi tamen lege procedentes , propterea consideravi has directiones contingere curvam quamlibet, vel, quod eodem recidit, supposui easdem directiones curvæ cuicunque pro libitu assumtæ normales esse, ad id ut lex continuitatis in hisce observaretur. Sed, quia solutio problematis in propositione LXXVI. evoluti ea est, in quam primum incidi, ideo, quod plerumque accidere solet, ejus analysis geometrica paulo longior atque difficilior evasit. Verum posteaquam hoc opusculum jam sub prælo esset, faciliorem paulo &, ni fallor, elegantiorem ejusdem problematis nactus sum solutionem, quam cum analysi ipsa vel potius demonstratione cum Benevolo Lectore hoc loco communicare placet. Esto curva quæcunque. AMΩ mobili describenda, ac solicitationum gravitatis MX directiones MN ubique normales sint curvæ cuicunque AY*y* descriptæ evolutione BN, quam proinde directiones MN ubique contingent. Tangenti AB sursum productæ in Δ curvæ BN in B, agatur per A perpendicularis indefinita CAV, & in quadrante CKΦ centro A & radio quocunque descripto ductis AI & AK radiis cum AC angulos CAI, CAK continentibus æquales angulis BAα, NMP, in quibus lineæ AB, MN curvæ AMΩ occurrunt, agantur rectæ Iα, KQ parallelæ AC & producantur usque ad occursum in *d* & L cum hyperbola O*d*T inter asymptotas AA, AV descripta. Circa axem vera AB descripta sit curva quædam DE hac lege, ut abscissa ejus AF æquet ubique homologam interceptam YM inter curvam AY & datam curvam AM mobili describendam, ordinatæ vero FE ordinatam ΔO in hyperbola abscissa ejus AΔ posita = MN . Et, si mobile incedat in medio resistente, cujus resistentiæ sint in composita ratione densitatum & duplicatæ celeritatum, fiat in hyperbola quadrilineum *dDST* vel *d*D ΣΘ æquale superficiei cylindricæ, cujus basis arcus AM curvæ describendæ, quæ cylindrica superficies oritur, si in singulis punctis arcus AM ad ejus planum perpendiculares erigantur proportionales densitati medii D, in respectivis curvæ punctis, inservietque quadrilineum *d*S descensui mobilis ex A in M, alterumque *d* Σ ascensui ejusdem mobilis ex M in *A*. Hisce præparatis,

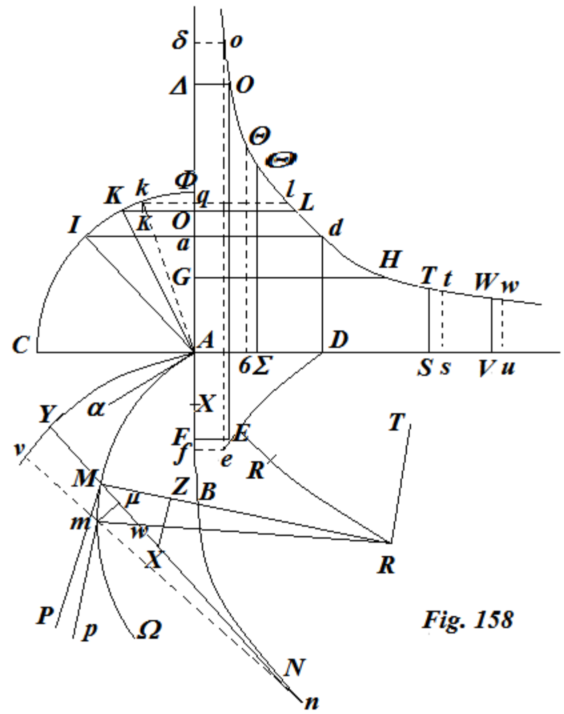


Fig. 158

Dico velocitatem mobili in M *acquisitam post descensum per arcum* AM *fore ad celeritatem initialem in* A *in composita ratione, ex hisce tribus* Aa *ad AQ, seu reciproca sinus anguli* NMP *ad sinum anguli* BAα, *ex* Aa *ad AG, seu abscissa minoris ad majorem quadrilinei hyperbolici* aGHd *æqualis areæ* DEFA, *ac denique ratione* AD *ad AS vel AΣ.*

Demonstr. 1°. Exponant, sicut in §. 607. MR vel *m*R radium evolutæ E curvæ AM in puncto M vel *m* & MX solicitationem gravitatis mobile secundum MN urgentem, & agatur XZ parallela tangenti curvæ MP in puncto M, eruntque triangula AKQ, MXZ & *Mμm* similia, posteaquam centro *v* arculus *mμ* descriptus fuerit, cum in triangulis rectangulis AKQ, MXZ, angulus *AKQ* alternus ipsius KAC (constr.) æqualis sit angulo PMX, vel ejus alterno MXZ *;* hinc $\mathrm{ZX : MZ = KQ : AQ}$.

2°. Ducta per *m* alia tangente *mp,* anguli recti RMP & *Rmp* præbent

$$nmp + nm\mathrm{R} = \mathrm{NMP} + \mathrm{NMR}, \;\; \& \, \mathrm{KA}k\left(= nmp - \mathrm{NMP}\right)$$
$$= \mathrm{NMR} - nm\mathrm{R} = \mathrm{R}wv - nm\mathrm{R}, \;\mathrm{R}wv + \mathrm{NMR} = \mathrm{M}vm - \mathrm{MR}m.$$

Ergo (§.129.) $\frac{\mathrm{K}k}{\mathrm{KA}}$ (vel propter trianguluum AKQ & K*kx* similitudinem)
$\frac{\mathrm{Q}q}{\mathrm{KQ}} = \frac{m\mu}{mv} - \frac{\mathrm{M}m}{\mathrm{RM}}$, atque adeo $\frac{\mathrm{M}m}{\mathrm{MR}} = \frac{m\mu}{\mathrm{MN}} - \frac{\mathrm{Q}q}{\mathrm{KQ}}$. Ducantur rationes $\mathrm{M}m : \mathrm{MR};\; m\mu : \mathrm{MN}$, & $\mathrm{Q}q$:KQ in sequentes, quæ (num. *1.hujus)* omnes inter se æquales sunt $\mathrm{ZX : MZ};\; \mathrm{M}\mu : m\mu$ & $\mathrm{KQ : AQ}$, prima in priman, secunda in secundam, &c. fietque $\mathrm{ZX.M}m : \mathrm{MZ.MR} = \mathrm{M}\mu : \mathrm{MN}, -\mathrm{Q}q : \mathrm{AQ}$. Hæ vero postremæ rationes ducantur etiam pari ordine in hæc sequentia rectangula $\mathrm{A}a.ad;\; \mathrm{A\Delta.\Delta Q};\; \mathrm{AQ.QL}$, quæ in hyperbola omnia inter se æquantur, eritque:

$$\mathrm{A}a.ad.\mathrm{ZX.M}m : \mathrm{MZ.\ MR}$$
$$= \Delta\mathrm{O.M}\mu - \mathrm{QL.Q}q\left(\text{ constr.}\right) = \mathrm{FE.\ F}f - \mathrm{QL.Q}q.$$

3°. Significent porro AV celeritatem in M acquisitam, V*u* ejus elementum, R vero resistentiam medii in M & D, ut antea ejus densitatem; & hisce positis.(§. 607. num. 11.), habetur, $\mathrm{AV.V}u = \mathrm{ZX.M}m \pm \mathrm{R.M}m$*,* seu ducendo unam partem in A*a.ad,* alteramque in AV.VW*,* reperietur

$$\mathrm{A}a.ad.\mathrm{ZX.M}m \pm \mathrm{A}a.ad.\mathrm{R.M}m = \mathrm{AV}^2.\mathrm{WV.V}u.$$

Verum, cum. (secundum hypothesin) R sit ut $\mathrm{D.AV}^2$, fiat $\mathrm{A}a.ad.\mathrm{R} = \mathrm{D.AV}^2$; eritque $\mathrm{AV}^2.\mathrm{WV.V}u = Aa.ad.\mathrm{ZX.M}m \pm \mathrm{D.AV}^2.\mathrm{M}m;$ quia vero (§. 154.) $\mathrm{AV}^2 = \mathrm{MZ.MR}$, applicando præcædentem æquationem ad hanc alteram, scilicet membra ejus AV^2 continentia ad hoc quadratum AV^2, & reliquum ad MZ.MR, fiet
$\mathrm{WV.V}u = \mathrm{A}a.ad.\mathrm{ZX.M}m : \mathrm{MZ.MR}, \pm \mathrm{D.M}m$. Atqui num.2 in fine reperimus jam $\mathrm{A}a.ad.\mathrm{ZX.M}m : \mathrm{MZ.MR} = \mathrm{FE.F}f - \mathrm{QL.Q}q$; ergo $\mathrm{WV.V}u = \mathrm{FE.F}f - \mathrm{QL.Q}q \mp \mathrm{D.M}m$ (vel, quia D.M*m* per constr. æquatur vel S*s*.ST ut in descensu, aut $-\Sigma\Theta.\Sigma\sigma$, ut in casu ascensionis, cum quadrilinea *dS & dΣ* singula æquentur, (constr.) omnibus. D.M*m*, quæ in arcu curvæ describendæ AM continentur) $\mathrm{WV.V}u : \mathrm{FE.F}f - \mathrm{QL.Q}q - \mathrm{ST.S}s$ (vel $\Sigma\Theta.\Sigma\sigma$). Unde, cum hoc idem eveniat respectu cujusvis alius curvæ elementi, erit $d\mathrm{DVW} = \mathrm{AFED} - a\mathrm{QL}d = d\mathrm{DST}$ ($d\mathrm{D}\Sigma\Theta$), vel, quia (constr.) quadrilineum $a\mathrm{GH}d$ æquatur areæ $\mathrm{AFED}, = a\mathrm{GH}d - a\mathrm{QL}d - d\mathrm{DST}$ ($d\mathrm{D}\Sigma\Theta$). Hæc enim quadrilinea omnia evanescunt cum punctum M cadit in A, vel punctum K in I. Idcirco ratio AV ad AD (§§.605, 606.) componetur ex directa *a*A ad GA & duabus reciprocis rationum QA ad *a*A, & AS vel AΣ ad AD, ac propterea velocitas acquisita in M est ad velocitatem in A seu $\mathrm{AV : AD} = a\mathrm{A}.a\mathrm{A.AD : GA.QA.SA}$ (AΣ). Quod erat demonstrandum.

Iisdem positis *erit solicitatio gravitatis* MX *secundum* MN *ad solicitationem* AX *in puncto* A *secundum* AB *in composita ratione ex directa duplicata ratione* AV *ad* AD, *& inversa rectanguli ex* MR *in sinum anguli* NMP *ad* rectangulum *ex* AR *in sinum anguli* BAA*, id est,* $MX : AX = (AV^2 : MR.AQ) : (AD^2 : AR.Aa)$.

Nam, quia {§.154) $AV^2 = MZ.MR$, vel $AK.AV^2 = AK.MZ.MR$ (vel propter triangula similia AKQ & MXZ rectangulum AQ.MX æquatur rec-lo $AK.MZ) = AQ.MX.MR, \& AK.AD^2 = Aa.AX.AR$, erit omnino $AQ.MR.MX : Aa.AR.AX = AV^2 : AD^2$; atque adeo $MX : AX = \left(AV^2 : MR.AQ\right) : \left(AD^2 : AR.Aa\right)$. Idcirco erit etiam $MX : AX = Aa^3.AD^2.AR : AQ^3.AG^2.AS^2.(A\sum{}^3). MR$. Qui est canon generalis, quoties resistentiæ sunt ut D in AV^2, sin vero hæ resistentiæ sint generaliores, ut $D.AV^h$, existente *h* numero quolibet rationali, ratio AD ad AS obtinebitur methodo jam §.622 exposita. In vacuo sit hæc ratio AD ad AS æqualitatis, eoque ea fuerit $MX : AX = Aa^5.AR : AQ^3.AG^2.MR$.

XII. *Iisdem adhuc positis* D *est generaliter, ut* $\left(3KQ : 2AQ.MR\right) - \left(QK : 2AK.MN\right) - \left(P : 2MX\right) - \left(R\Gamma : 2MR^2\right)$ *& resistentia ad gravitatem, seu*

$$\pm R : MX = \left(3KQ : 2AK\right) - \left(R\Gamma.AQ : 2AK.MR\right)$$
$$-\left(KQ.AQ. MR : 2AK^2.MN\right) - \left(P.AQ.MR : 2AK.MX\right)$$

In quibus Fig.158 formulis $R\Gamma$ significat radium osculi in puncto R curvæ RR, cujus evolutione curva data AM describitur, & P significat exponentem rationis, quam elementum magnitudinis MX gravitatis in M secundum MN, habet ad elementum M*m* curvæ mobili describendæ.

Demonstrationem horum duorum canonum non addo, cum ea ad præcedentium imitationem & normam facile haberi queat. Sed, ut eorum consensus cum aliorum inventis appareat, eos casibus particularibus nonnullis Celeberrimorum virorum Newtoni & Joh. Bernoullii applicabimus.

1°. Sit curva AM ipse circulus ΦKC mobili describendus a gravitate uniformiter agente secundum directiones ipsi AQ parallelas, quo casu P, & $R\Gamma$ evanescunt, MN sit infinitæ & $MR = AK$, propterea erit D, ut $3KQ : 2AQ.AK$, hoc est, sicut tangens anguli KAQ & resistentia ad gravitatem, ut 3KQ ad 2AK*,* plane ut laudatissimi Autores invenerunt.

2°.Si loco curvæ AM sumatur curva, in qua directiones gravitatis uniformis MN parallelæ sint inter se, erit D ut

$$\left(3KQ : 2AQ.MR\right) - \left(R\Gamma : 2MR^2\right), \ \&$$
$$\tfrac{\pm R}{MX} = \left(3KQ : 2AK\right) - (R\Gamma.AQ : 2AK.MR).$$

Nam, quia MN in hisce suppositionibus sunt infinitæ, singulæ fractiones, in quarum denominatoribus hæ MN reperiuntur, evanescent. Vocentur itaque ordinatæ curvæ datæ y, secundum quas gravitas uniformis operatur, abscissæ respectivæ x, radius curvaturæ in M seu $\mathrm{MR} = r$, existentibus $M\mu = dy \,\&\, m\mu = dx$, positis scilicet omnibus MN inter se parallelis, ratio vero $\mathrm{R\Gamma} : \mathrm{MR} = dr : ds$, facto elemento curvæ $Mm = ds$. Adeoque in hisce symbolis erit D ut

$$\frac{3dy}{2rdx} - \frac{dr}{2rds}, \& \ \frac{\pm \mathrm{R}}{\mathrm{G}} = \frac{3dy}{2ds} - \frac{drdx}{2ds^2}.$$

Sed existentibus dx constantibus erit

$$r = ds^2 : dxddy, \& \ dr = \frac{3dsdy}{dx} - \frac{ds^3dddy}{dxddy^2},$$

quibus valoribus in formulis substitutis resultat D, ut

$$dddy : 2dsddy, \ \& \ \pm R : G = dsdddy : 2ddy^2$$

qui duo postremi canones ad amussim conspirant cum formulis Celeb. Newtoni Propos. X. Lib. II. Edit novissimæ Princip. Phil.Nat. Nam, si in hisce nostris loco *dy, ds, ddy* & *dddy* ordine substituantur $\mathrm{Q}o,\ o\sqrt{1+\mathrm{QQ}}, 2\mathrm{R}oo,\ \&\ 6\mathrm{S}o^3$, quæ in expressionibus Newtonianis elementis illis æqualia sunt, erit omnino D, ut

$\frac{3\mathrm{S}}{2\mathrm{R}\sqrt{1+\mathrm{QQ}}}, \ \& \ \frac{\pm \mathrm{R\ es.}}{\mathrm{Grav.}} = \frac{3\mathrm{S}\sqrt{1+\mathrm{QQ}}}{4\mathrm{RR}}$. Prorsus ut habet Newtonus loco citato.

XIII. In demonstratione Propos. XXV. Lib. I.§.167. irrepsit error, dum elementum ordinatæ tanquam positivum spectavimus, idcirco in progressu demonstrationis illius signa ipsius $\Lambda\xi$ ubique sunt invertenda, quo facto invenietur

$$\mathrm{G} = \frac{1}{\mathrm{DE}^3} + \frac{(-\mathrm{DE.E\Delta.MG}^2 + 2.\mathrm{DE.E\Omega.MG}^2 + \mathrm{DE.E\Omega.AL}^2).xx.\Delta\mathrm{E}}{\mathrm{E\Omega.AG}^2\mathrm{DE}^5}$$

Ponendoque $\mathrm{HE} = A \mp e$, atque adeo $\Delta\mathrm{E} = \Lambda\mathrm{E} = \mathrm{HE} : \mathrm{B},\ \&\ \mathrm{E\Omega} = \mathrm{B.HE} : \mathrm{B}^2 - \mathrm{C.HE}$, ubi $B = dA : dz;\ C = dB : dz\ \&\ \mathrm{DE} = z$, item $AL = r$, ac $rr - ee = ss$ hæc apposita formula factis in ea substitutionibus valorum linearum DE, $\mathrm{E\Delta}$, $\mathrm{E\Omega}$, prodibit æquatio circa finem §.169 exhibita. Verum, quia nonnihil elegantior, ut mihi videtur, solutio ejusdem problematis sese mihi obtulit, eam hoc loco apponere placet. Circa rectam AC, tanquam axem, descripta sit curva quæcunque ΠB, radioque etiam pro lubitu sumto AC, atque adeo centro C circulus AG, cujus tangens in A erit linea recta indefinita $\mathrm{OA\Pi}$, ac per quodlibet punctum B curvæ ΠB ductis tangente BV ordinataque BR producenda usque ad occursum ejus cum circulo in G, dividatur subtangens RV in S, ut RS sit ad totam RV ut I ad numerum quemcumque rationalem *n,* ac jungatur SG, tum ducatur per A recta AL cum CA*a* continens angulum PAQ æqualem, ubique respectivo angulo GSR, atque hoc idem si factum fuerit cum tangente & subtangente ad punctum curvæ Π, resultet recta $\mathrm{A\Lambda}$, quemadmodum in ordine ad punctum B curvæ ΠB provenit AL. Exponat AI celeritatem mobilis curvam AN describentis in puncto A existente centro solicitationum centralium D, per punctum I ducta indefinita IK parallela AC, sumatur in $\mathrm{A\Lambda}$ segmentum $\mathrm{A}\alpha = \mathrm{A\Pi}$ & per α agatur αK parallela AI, rectæ IK conveniens in K, per quod punctum descripta intelligatur hyperbola KM inter asymptotas A*a* & AI. Porro in AL sumta AQ æquali respectivæ ordinatæ RB curvæ ΠB ducatur QP parallela $\mathrm{A\Pi}$, quæ producta ocurrat hyperbolæ in M, ac denique in ordinata MO per hoc punctum M ad asymptotam AO ducta ac deorsum producta, fiat OF æqualis differentiæ ordinatarum $\mathrm{A\Pi}$, RB & punctum F erit in scala celeritatum mobilis in curva altera AN incedentis. Hæc autem curva AN ita comparata est, ut radii vectores $\mathrm{AD} = \mathrm{A\Pi}$, & $\mathrm{ND} = \mathrm{RB}$, per ejus terminos ducti, angulum ADN contineat, qui sit ad angulum in circulo ACG, ut 1 ad numerum qui antea *n.* Esto *Nq* curvæ AN tangens in N, super quam cadat perpendicularis D*q,* ac centro D descripti sint arcus EN arculusque *np,* positoque in fig. 159 angulo GC*g* ad ang. $\mathrm{ND}n = n:1$ agatur *gb* parallela GB.

Demonstr. Quia angulus infinitesimus *NDn* est ad GC*g* ut 1 ad *n,* seu (constr.) $= \mathrm{RS}:\mathrm{RV}$, & (§.129.) arculus *pn* ad G*g* in composita ratione anguli ND*n* ad angulum GC*q,* vel 1 ad *n*, seu RS ad RV, & radii DN vel ordinatæ RB ad radium GC, erit $pn:\mathrm{G}g = \mathrm{RS.RB}:\mathrm{GC.RV}$; atqui est

$$\mathrm{G}g : b\beta (= \mathrm{GC:GR}) = \mathrm{GC.RV:GR.RV};$$
$$\& \ b\beta : \mathrm{N}p(\text{vel } \mathrm{B}\beta) = \mathrm{RV:RB} = \mathrm{GR.RV}:\mathrm{GR.RB}.$$

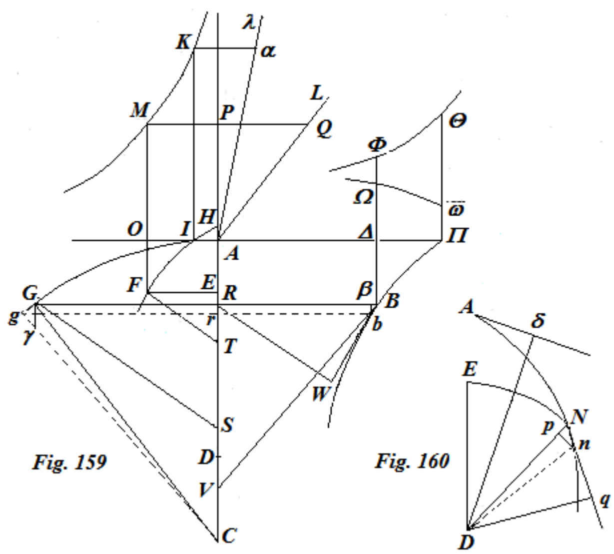


Fig. 159
Fig. 160

Ergo ex æquo $pn:\mathrm{N}p = \mathrm{RS.RB}:\mathrm{GR.RB} = \mathrm{RS}:\mathrm{GR}$,vel (quia PAQ angulo GSR æqualis factus est) $= \mathrm{AP}:\mathrm{PQ}$, est vero etiam $pn:\mathrm{N}p = \mathrm{D}q:\mathrm{N}q$, ergo $\mathrm{AP}:\mathrm{PQ} = \mathrm{D}q:\mathrm{N}q$; hinc, quia AQ (constr.) $= \mathrm{DN}$, erit AP vel $\mathrm{MO} = \mathrm{D}q$, & $\mathrm{A}a = \mathrm{IK} = \mathrm{D}\delta$. Unde, cum (§. 155.) D*q* sit ad Dδ, ut velocitas in A repræsentata per AI ad velocitatem in N, & in hyperbola KM, ordinata MO sit ad KI ut AI ad AO*,* hæc AO omnino exponit celeritatem in N, & cum OF vel AE (constr.) æquet differentiam ordinatarum $\mathrm{A\Pi}$, RB, id est, ordinatarum AD, NO, in utraque figura erunt æquales ipsæ AE, atque adeo curva HIF erit scala celeritatum mobilis in curva AN descendentis vel ascendentis, adeoque ducta ejus normali ET exponet subnormalis ET (§.134) solicitationem gravitatis secundum ND in curvæ AN puncto N. Quod erat demonstrandum .

Coroll. Si D fuerit origo abscissarum DR, atque hæ abscissæ dicantur A, ordinatæ respectivæ RB, *z,* radius AC, *r,* distantia originis abscissarum curvæ ΠB a centro circuli AH, seu DC, *e* ; erit $\mathrm{CR} = e + \mathrm{A}$, si punctum D cadit inter C & R, ut in figura, vel $e = \mathrm{A}$, si inter A & R, ac denique $\mathrm{A} - e$, si punctum D cadit extra radium AC; ponamus ergo $\mathrm{CR} = e + \mathrm{A}$, eritque $\mathrm{GR} = \sqrt{(ss \mp 2e\mathrm{A} - \mathrm{A}^2)}$ ubi $ss = rr - ee$, dicatur pariter elementum abscissæ DR : seu $\mathrm{R}r = \mathrm{B}.dz$, eritque $\mathrm{RV} = \mathrm{B}z$, & $\mathrm{RS} = \frac{1}{n}\mathrm{B}z$, substitutisque omnibus hisce symbolis in præcedenti constructione, reperietur $\mathrm{AO}^2 = \mathrm{FE}^2 = \frac{1}{2z} + \frac{(ss \mp 2e\mathrm{A} - \mathrm{A}^2).nn}{B^2 z^4}$, & ex hac elicietur ipsissima formula circa finem §.169. Patet ergo hanc posteriorem cum priori egregie conscentire.

Hinc, si puncta D, C coincidunt, curvaque ΠB est hyperbola inter asymptotas orthogonales in C convenientes, quarum AC una, erit in curvæ AN puncto N gravitas, secundum ND, ut $\frac{1-nn}{z^3}$ & ipsa curva AN erit ea, cujus Newtonus & Joh. Bernoullius constructiones diversas in speciem, re vera tamen conspirarites, exhibuerunt, hic in Act. Lips. I 713. pag. 129. ille vero Propos. XLlV. Lib. I. Princ. Phil. Nat., ut supra pag. 80. indicavimus. Hæc curva AN duas asymptotas habet a centro virium D æqualiter distantes ; & quidem distantia quæ libet esse debet $\frac{1}{n}$ radii AC in fig.159, & 160. vel $\frac{1}{n}$DP in fig. 37. ac proinde altera asymptota per centrum ire non potest, ut per inadvertentia pag. 81 scripseram. Reliquæ proprietates ejusdem curvæ AN fig. 160. a Bernoullio commemoratæ nullo negotio ex nostris constructionibus eliciuntur, propterea iisdem fusius explicandis supersedeo.

Cæterum, quæ pag. 80. num. 4 habetur exceptio ipsius casus, quo *p* est -1, prorsus inutilis est, & propositio illic memorata absque hac exceptione generaliter obtinet, si *e* vel *s* sunt 0, sed hæc exceptio tantum valet in contradictoria ejusdem propositionis, scilicet non nisi hoc casu $p = -1$, duobusque reliquis $p = +1$, $\mathrm{ac} = -2$ exceptis, curvam AN unquam fieri posse algebraicam in hypothesi quod G ut z^p, neutra ex duabus *e* vel *s* evanescente.

Constructio præcedens adhuc elegantior reddi potest ducendo RW parallelam GS, & demittendo super eam ex puncto B perpendicularem BW, tum ductis per B & Π parallelis axi AC, ut $\mathrm{B}\Phi$ & $\Pi\Theta$, sumendoque in priore segmentum $\Delta\Phi$ æquale perpendiculo BW, orieturque curva $\Theta\Phi$, in cujus ordinatis, si $\Pi\pi = \mathrm{AI}$ fuerit ad $\Delta\Phi$ ut $\Phi\Delta$ ad $\Theta\Pi$; ita ut curva $\pi\Omega$ reciproca fuerit alterius $\Theta\Phi$. Hæc reciproca $\pi\Omega$ erit etiam nunc scala celeritatum, atque adeo eadem cum curva HIF, sed ad axem $\mathrm{A}\Pi$ relata, loco axis HE, ad quem altera HIF exstructa est. In hac constructione neque lineis A*l* neque hyperbola KM opus est, atque adeo nonnihil simplicior facta est.

FINIS.