APPENDIX.

After the preceding treatise had been sent to the printer, a number of matters came to mind one after another, which may be seen both to be useful for the enlightenment of the work, as well also to improve some parts of the discussion. I have put in place a few of these in this appendix, to be added to the preceding books, when on being pressed for time and the haste of the bookseller, did not allow me the opportunity to deduce everything required to be published.

I. Concerning the most noteworthy law of nature, by which *for any action there is said to be an equal and opposite reaction,* as derived in §.11, I have returned to in the following §.12, they may be seen to be some of the most outstanding forces present, considering under no circumstances a motion must follow an action, if the reaction always shall be equal and opposite to this ; for if, as in the example introduced amongst others by the most Cel. Newton, a horse pulling a stone bound by a rope with a force is drawn backwards by an equal force towards the stone, how, one might ask, can the horse be able to move forwards and to move the stone, if the whole force acting is cancelled out completely by the equal and opposite resistance ? But this objection may be seen to have arisen from the ambiguity about the names of the *force* and *action* ; yet which two must be distinguished carefully. The force of bodies is not the action itself; for the action is the application of a certain force with an added usefulness, or to which it can be applied : truly it is agreed to be applied to that body alone, because it resists, struggles, and reacts. It may thus be said there is no particular action with material things, where there is no reaction ; for the horse, which draws no resisting body after itself, pulls nothing or does not pull, that is, it may be agreed, as simply conceded, nothing else to act on. Therefore in any action involving bodies, there is a collision between the force acting and the opposing struggle of the body enduring the impact, with the application of the force acting on the body sustaining the action, that is, the action itself is equal and opposite to the restraining force of the body acted on, which is the reaction, because this restraining force or this force of inertial of the suffering body must be taken up according to that, so that the body may be able to move from the action. Yet thus it does not follow that the whole force of the body acting is required to be devoted to overcoming the reaction experienced by the body, but only a part of that, and this part of the total force of the body, which is spent taking care of the resistance of the suffering body, that is the force, by which the action may properly emanate; for the remainder of this whole force, when it may have no resistance or restraining action requiring to be absorbed, goes into the least action. Therefore since whatever action is said to be equal and opposite to the reaction of body undergoing the interaction, this law indicates no more than this, *in all interactions just as much of the bodily strengths are to be decreased as the action of the undertaking body shall increase*.

[The author seems to have a confused understanding of NIII ; indeed, the latter statements seem to be about the loss of 'strength' , i.e. kinetic energy by the incident body and the gain by the other, a vindication of the vis viva idea, which of course applied only to elastic interactions.]

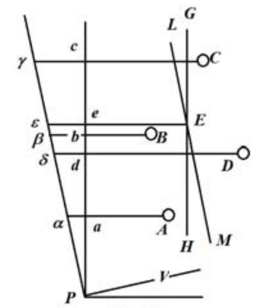


Fig. 150

II. Proposition IV. Lib. 1. §. 44. made available easily a way of demonstrating the centre of gravity *to be given* for each body, and this centre to be *single*, if the directions of gravity were parallel lines. For, if however many particles were to be the smallest possible A, B , C, D &c., Fig. 150, placed in some manner, and considered as the elements of any kind of body, from which the given perpendiculars A*a*, B*b*, C*c*, D*d*, &c. may be sent to the plane P*c* in the position given and another plane GH may be drawn parallel to the plane P*c*, of which the distance from that plane P*c* shall be E*e*, so that the product E*e* with the sum of the bodies $\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\&c.$ may equal the sum of the products $\mathrm{A.A}a+\mathrm{B.B}b+\mathrm{C.C}c+\mathrm{D.D}d+\&c.$ of the bodies A, B, C, D, *&c.* at their respective distances A*a*, B*b*, C*c*,*&c.* & from the plane P*c*, and at P*c* there is taken P*e*, in order that $\left(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\&\mathrm{c}\right).\mathrm{P}e$ shall be $=\mathrm{A.P}a+\mathrm{B.P}b+\mathrm{C.P}c+\mathrm{D.P}d+\&\mathrm{c.}$; and the right line *e*E drawn through *e* will cross GH at the point E, which is the common centre of gravity of the particles A, B, C, D, *&c.* or of the whole body, which is composed from these particles. This point E thus is called the centre of gravity of the whole body, because its parts or particles A, B, from one side of the line, or of the plane GH, are of equal moments with the particles C, D, &c. from the other side of the same plane or line GH, and because here the same happens with respect of any other line LM, which can be drawn through the same point E. For with the line $\mathrm{P}\gamma$ drawn parallel to the line LM through the point P, and with the products A*a*, B*b*, *Cc*, D*d*, &c. & E*e*, by $\alpha,\beta,\gamma,\delta,\&\mathrm{c.}\ \&\ \varepsilon$ there may be found

$$(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\&\mathrm{c}).E\varepsilon=\mathrm{A.A}\alpha+\mathrm{B.B}\beta+\mathrm{C.C}\gamma+\mathrm{D.D}\delta+\&\mathrm{c.}$$

$$\&\ \left(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\&\mathrm{c}\right):\mathrm{P}\varepsilon=\mathrm{A.P}\alpha+\mathrm{B.P}\beta+\mathrm{C.P}\gamma+\mathrm{D.P}\delta+\&\mathrm{c}\,.$$

whenever all the triangles $\mathrm{P}a\alpha,\ \mathrm{P}b\beta,\ \mathrm{P}c\gamma,\&\mathrm{c.}$ are similar. Now from these two equalities it follows further, all the moments of the bodies A, B, C, D, &c. in respect of each of the planes $\mathrm{P}\gamma$ and of its normal PV likewise shall be equal to the respective products from the sum of the bodies by the distances of the centre E from each plane or line $\mathrm{P}\gamma$,PV, because since thus it may happen, also it is necessary that the particles, which are to one side of the right line LM shall be of equal moments with the small weights, which are on the other side of the same LM. And thus since all the lines, which can be drawn through the point E, shall be the axes of equilibrium of the small weights A, B, C, D, &c. it is clear the centre of gravity of the body constructed from these particles as its elements is to be the *single* centre of gravity at E. Q.E.D.
[Thus, the centre of gravity is a point fixed in the body, independent of the choice of axis, either rotated or translated, or both.]

Wallis also elicited this same proposition from other principles.

III. Concerning Guldini's rule, the demonstration of which has been provided above (§.47.), two items are to be noted.

1. That to be obtained not only for plane figures or round solids, but generally with all others, which can be generated by the motion of the figures generated made by this rule, in order that the plane of the figure may always remain perpendicular to the line, which the centre of gravity of the generating figure will describe by its motion. Indeed in this case the volume generated by the motion of the figure is equal always to the product of the [area] of the figure by the path of the centre of gravity of the same.

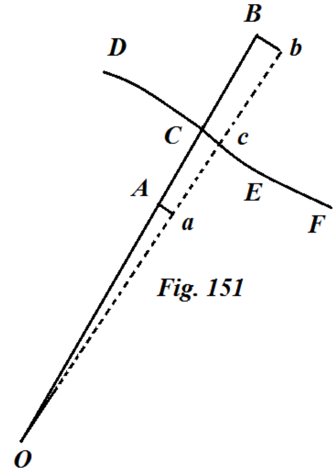
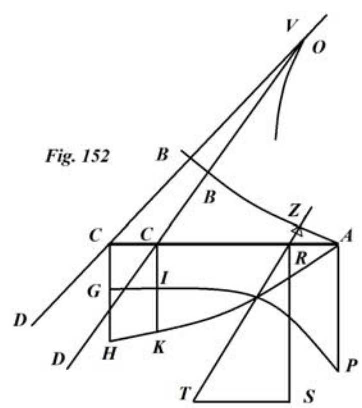


Fig. 151

Fig. 151. For if the centre of gravity C of some figure AB shall lie on some curve DEF, to which figure it shall be normal everywhere, the arclet of the curve C*c* to have been described, and from the position BA to have arrived at the position *ba* infinitely close to the former, thus so that the right lines CO and *cO,* which shall be in the planes of the figures AB and *ab,* and in the plane of the curve DEF, and O will be the centre of the osculating circle of the given curve DF at the point C of this or of *c* close to this. And thus the volume, which the figure AB will describe when its centre C runs through the arclet *Cc,* by §.47. is equal to the product from the figure AB by the arclet C*c*, therefore all the volume of this kind or the volume or figure, which is produced by the advance of the figure AB along the whole curve DEF, will be equal to that made from the same AB by all the arclets, which are contained in the curve DF, that is, with that made from the figure AB by the length of the curve DEF. Q.E.D.

2. Requiring to be noted, thus this is to be had only, if the weight arising in the figure shall be uniform, and to fail Guldini's rule, with the figure arising from dissimilar weight being present.

IV. Fig. 152. Proposition VIII. Book. I. (§.59.) also can be deduced, or at least itself equivalent, to the universal property of a right lever impelled by oblique forces at individual points, which it will be a pleasure to explain here. AC shall be an inflexible right line, which may be pressed on at the individual points C, C &c. by the forces CD, CD inclined to that according to a uniform law, and about that line as the axis, the figures ACGIP and ACHKA are understood to be put in place, with this agreed upon, that any ordinate CG (CI) shall be to the homologous force CD in a twofold proportion, as the sine of the respective angle BCA to the whole sine ; and the ordinate CH (CK) of the other figure ACH itself may be had to the ordinate CG (CI) of the first figure ACGP, as the sine of the complement of the angle BCA and of the same right sine, and thus with these also put in place everywhere, RS is acting at right angles to this line AC through the centre of gravity of the area ACGP, crossing AC at R, and the line RT with the line RS containing the angle TRS, of which the sine is to the sine of the complement, as the area AHC is to the area ACGP, the mean direction of all the forces CD, CD of the applied lines AC, and the weight by which the fulcrum will be

Fig. 152


acted on at R, will be to all the forces of the right line AC derived from the oblique lines CD, CD, as the whole sine to the sine of the complement of the angle TRS. The demonstration of this is left, for the sake of brevity, to the diligence of the reader.

Yet this requires to be said, the forces in the same directions BD, BD, applied to any curve AB, which shall be for the homologous CD pertaining to the line AC, as respectively CV at BV (by supposing the directions BD, BD to touch a certain curve OV) to have the mean direction ZT with the forces CD, CD, &c. applied to the right line AC, as that can be easily demonstrated. From which if some forces may be applied in the directions BC, BC, &c. to whatever curve AB, the mean direction of these can be found in an easy manner, by drawing the tangent AC through the peak of the curve A, or some other right line, and by inquiring which forces CD are required to be applied in the directions BC to this line AC according to that, so that the mean direction RT of these forces also may provide the mean direction of the forces applied in the directions BC, BC, &c. to the given curve ABB ; according to that CD is required to be made everywhere to the homologous force applied to the curve AB, thus as BV to CV.

V. The general use of proposition XII. Book. I, we have shown now in some manner with its attached corollaries and scholium. It will perhaps help to add these few items which follow to illustrate still more the same proposition and applications. Beside the symbols now set out in §.101 , we will call *t* the tension [firmness in text] of any element of the curve B*b* or Bβ, which there was indicated by the letter T, the forces along BH and BG, *dq* and *dp* respectively, the tangent and secant of the angle GBH, *h* and *k,* and finally the radius of osculation or of curvature B, *r;* likewise we will designate the total sine by *a* ; see Fig. 29. And with these in place, there will be (no.1, §. 93.)

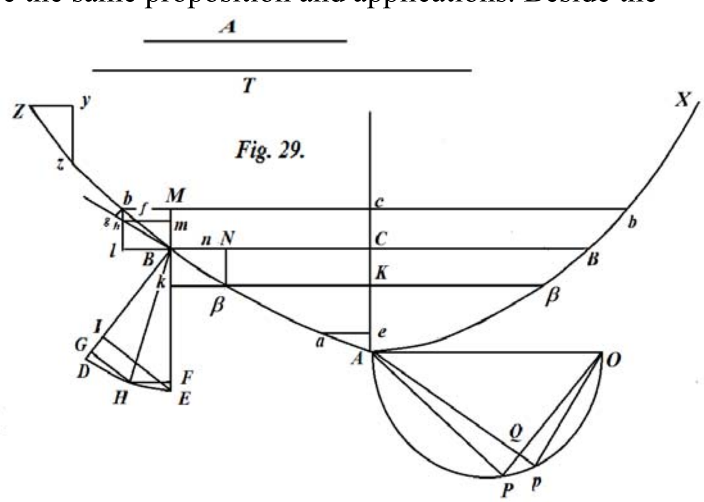


Fig. 29.

GH =*dt,* and the three elements *dp, dq,* & *dt,* will be proportional to *a* ,*k* & *h* themselves, thus also $\mathrm{AP}:\mathrm{PQ}=r:ds$, and (§.103.) $\mathrm{BG}(dp):\mathrm{T}(t)=\mathrm{PQ}:\mathrm{AP}$, therefore $dp:t=ds:r$, likewise $dp:dt=a:h$, therefore $dt:t=hds:ar$, & $\log t=\int(hds:ar)$, and thus from these four ratios the equations will be found, such as are seen in this small extra lateral note. Hence from the quantities *dp, dq, dt*, or *t* and *h* given as you wish at the indeterminate points of the curve *x, y dx, dy*, &c, and from the constant quantities, the remainder may be found ; if not algebraic, perhaps transcendental.

1. $dt=hdp:a=hdq:\ k.$
2. $t=rdp:\ ds.$
3. $\log.\ t=\int hds:ar.$
4. $h=ardt:tds.$

Example. The given curve ABZ shall be an equilateral hyperbola , of which the equation shall be $yy=xx-aa$, and in which the abscissas *x* are taken from the centre, with the transverse half line being $= a$, and the forces or tensions of the curve shall be proportional to the distances of the points of the curve from the centre, that is $t = \sqrt{(xx + yy)}$. And §.161. will produce $r = t^3 : aa$, and $h\left(= ardt : tds\right) = \frac{2xy}{a}$, and likewise $k = tt : a$; hence the first equation $dt = hdq : k$ will provide $dq = ttdt : 2xy = 2txdx : 2xy = tdx : y$; and $tdx = yds$, therefore $dq(= tdx : y) = yds : y = ds$. Therefore the forces BH, or *dq,* are as the elements of the curve *Bb,* and the directions of these come together at the centre of the hyperbola, because the tangent of the angle DBH has been found $= 2xy : a$, and this expression also will contain the tangent of the angle, which the normal to the hyperbola BD holds with the right line drawn from the point B to the centre of the hyperbola. Therefore the equilateral hyperbola is a *catenary,* if the directions of the weighs are assumed to concur at the centre of the hyperbola, as has been mentioned above near the end of §.105. without demonstration. Equally if I may put $dq = ds$ & $h = 2xy : a$ in the same hyperbola, we may find $t = \sqrt{(xx + yy)}$.

More generally, if the directions of the weights converge at a point, and the abscissas are taken on the axis from this centre of gravity, there will be $h = axdx + aydy : xdy - ydx$, and with the aid of this and the other $dt = hdq : k$, on substituting $dq = ds$ we may arrive at $t = \sqrt{(xx + yy)}$ & $yy = xx - aa$.

If the directions of the weights are parallel to the axis AC, there will be $h = adx : dy$, and by the substitution of these proportionals *n* and *m* in place of *dx* and *dy*, there becomes $h\left(= adx : dy\right) = an : m$. Likewise in the fourth equation, by replacing r & ds by m & dn, to which they are proportional, there becomes $an : m = amdt : tdn,$ & $dt : t = ndn : mm$ (or, because $mm + nn = aa$, & $ndn = -mdm) = -mdm : mm = -dm : m$, therefore $lt = l\left(aa : m\right)$, that is, $t = aa : m = ads : dy$, so that we have equation nine. Therefore by substituting the value of *t* found in the second equation, the seventh equation will be found ; and from this with the aid of the ratio $dp : dq = a : k, = dy : ds$, the eighth equation will be elicited.

5. $h = adx : dy$
6. $k = ads : dy$
7. $dp = ads^2 : rdy$
8. $dq = ads^3 : rdy^2$
9. $t = ads : dy$

Hence, from these three *dp, dq* & *t* with one of the indeterminates of the curve given and with the constants in turn added as desired, the remaining two may be found, if not algebraic, perhaps transcendental.

If the directions of the forces BH are normal to the curve, there will be $h = 0,$ & $k = a$, and the second equation alone suffices; the celebrated Johan Bernoulli gave the solution to this particular case now some time ago in the Commentaries of the Royal Acad. of Sc., 1706, and recently besides in the book to which he gave the title *Essay d'une Nouvelle Theorie de la Manoeuvre des Vaisseaix.*

VI. *Concerned with the change of gravity with height, by which projectiles describe conic sections in vacuo.* Because a moving body describing the perimeter of some conic section AQ*a* is acted on by forces, which are inversely proportional to the square of the distances of the body from the centre, the *graph* of these forces GB will be a squared hyperbola, the applied lines of which HG, AB, which express the action of gravity at H and A, are inversely proportional to the squares of the abscissas DH, DA, with the centre of the hyperbola being at D. Now how great a *change* in height required to be taken, that is, such as along the line HA, so that the moving weight begins to fall along that line from rest at H, and by being acted on by forces which are expressed by the ordinates of the hyperbola in the four-sided figure HGBA, thus the speed that may acquire at A is sought, by which the propelled body describes the given conic section AQ*a* along the line Aα acting along the direction BA.

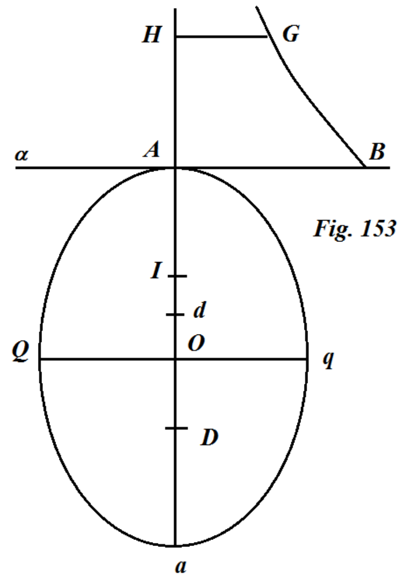


Fig. 153

The point A is put for the first apse, and therefore DA will be perpendicular to the section at A, and thus this point A will be one vertex of the conic section AQ*a*, and the radius of curvature at A will be equal to $\frac{1}{2}\mathrm{L}$, that is, to half the latus rectum of the conic section, and the normal action of the curve at A derived from this centre AB ; therefore by indicating the speed of the moving body at the point A, by V, from the force, §154, there will be had $\mathrm{V}^2=\frac{1}{2}\mathrm{L.AB}$. And the same velocity V, but (following the hypothesis) acquired in descent on HA, also is expressed (§.136.) with the side squared from twice the area HGBA, therefore $\mathrm{V}^2=2.\mathrm{HGBA}$, and $\mathrm{AHGB}=\mathrm{AD.AB}-\mathrm{HD.HG}$, therefore also $\frac{1}{2}\mathrm{L.AB}=2.\mathrm{AD.AB}-2.\mathrm{HD.HG}$, or, on account of the hyperbola, by substituting in place of AB and HG the proportional squares of these $\mathrm{HD}^2,\mathrm{AD}^2$, to become $\frac{1}{2}\mathrm{L.HD}^2=2\mathrm{AD.\ HD}^2-2\mathrm{HD.AD}^2=2\mathrm{HD.AD.HA}$, hence also $\mathrm{L.HD}=4.\mathrm{HA.AD}$, or $\mathrm{L.A}a.\mathrm{HD}=4\mathrm{A}a.\mathrm{HA.AD}$. Truly from conics there is $4\mathrm{QO}^2=\mathrm{L.A}a=4\mathrm{AD.A}d$, therefore $4\mathrm{A}a.\mathrm{HA.AD}=4\mathrm{AD.A}d.\mathrm{HD}$, that is, $\mathrm{A}a.\mathrm{HA}=\mathrm{A}d.\mathrm{HD}$, and $\mathrm{HD}:\mathrm{HA}=\mathrm{A}a:\mathrm{A}d$, and by dividing, $\mathrm{AD}:\mathrm{HA}=\mathrm{AD}\ (\text{or } ad):\mathrm{A}d$, therefore $\mathrm{HA}=\mathrm{A}d$, and thus $\mathrm{HD}=\mathrm{A}a$. *Therefore the height in conic sections is equal to the distance of the more distant focus from the centre of force, to the distance of the same apse to its nearer focus* . Hence various things can be deduced, a few of which follow.

1°. Indeed an infinite number of conic sections AQ*a* can be agreed to come from one and the same height HA *;* since with the same distance kept A*d* the distance between the foci *dD* shall be able to be of any magnitude; but not thus with all agreeing to have been projected along A*a* with one and the same speed . For, because $\mathrm{V}^2=\frac{1}{2}\mathrm{L.AB}=\frac{1}{2}\mathrm{L.AD}^2.\ \mathrm{AB}:\mathrm{AD}^2$ (or, because $\mathrm{AD}^2.\mathrm{AB}$ is a constant volume) is

$\frac{1}{2}\mathrm{L}:\mathrm{AD}^2$, V will be as $\sqrt{\frac{1}{2}\mathrm{L}}:\mathrm{AD}$. That is, in diverse conic sections, the velocities of the moving body at either apse will be composed in the square root ratio of the latus rectum of the sections, and inversely in the ratio of the distances of the apses from the centre of the attraction.

2°. If the distance D*d* between the foci were infinite, the conic section A*qa* is a parabola having the focus at *d*, and hence the height HA:A*d* shall be equal to the fourth part of the latus rectum L, as Galileo and many others have demonstrated from other principles. Therefore the general principles of the motion of projectiles, according to the hypothesis of uniform parallel gravitation, is included in this discussion.

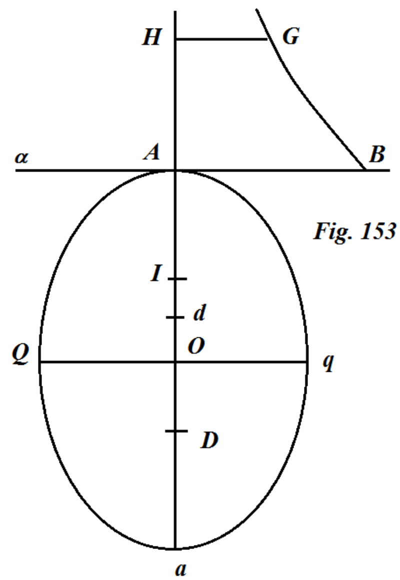


Fig. 153

3°. If the body, with the same velocity V acquired at A after falling along HA, shall describe a circle with centre I, and having the radius $\mathrm{AI}=\frac{1}{2}\mathrm{L}$, the centrifugal force of the moving body in the circle will be the same as from the central force AB at the apse A of the conic section AQ*a*.

4°. The uniform gravity, such as we have here on earth, shall be to the centrifugal force in the said circle, or to the central action AB at A as $\frac{1}{2}\mathrm{AI}$ or $\frac{1}{4}\mathrm{L}$ to HA or A*d*. And thus for a parabola described from the focus *d* and vertex A, the weight of the projectile at its vertex A, will be equal to the centrifugal force in the afore-mentioned circle.

[*i.e.* $v^2=2g.\mathrm{HA};\therefore g:v^2/R=gR:2g.\mathrm{HA}=(\mathrm{AI}/2):\mathrm{HA}$;
see e.g. French, *Newtonian Mechanics*, p. 592.]

VII. *Concerning the periodic times in conic sections.* These may be called the time T of one circuit at the section AQ*a*, also the diameter A*a*, D; the conjugate diameter of which *Qq*, *d;* the latus rectum *l*, the distance or height AD, A, the central attraction AB at the apse A, G. And there will be $d=\sqrt{lD}$,and the area of the whole ellipse, that is, $2.\mathrm{AQ}a=\frac{1}{2}p\mathrm{D}d$, where p is the expression of the ratio of the circumference of a circle to the radius. Indeed (§.157.) $\mathrm{T}=2.\mathrm{AQ}a\mathrm{A}:\frac{1}{2}\mathrm{AD.V}$, therefore by substituting analytical values in place of areas and of the curve AD, V*;* there will be had $\mathrm{T}=p\mathrm{D}d:\mathrm{A}u$, or $\mathrm{T}^2=pp\mathrm{D}^2d^2:\mathrm{A}^2uu$, that is $dd=l\mathrm{D}$, and $uu=\frac{1}{2}l\mathrm{G}$, therefore $\mathrm{T}^2\left(=pp\mathrm{D}^3l:\frac{1}{2}l\mathrm{A}^2\mathrm{G}\right)=2pp\mathrm{D}^3:\mathrm{A}^2.\mathrm{G}$. Hence *the square from the periodic time is in direct proportion to the cubic ratio of the transverse side, and in an inverse ratio composed from the square of the distance of the apse from the focus, or from the centre of attraction, and from the simple force G at the apse.* From which, because the actions of gravity are in the inverse square ratio of the distances of the moving body from the centre, the ratios composed from the square of the distances of the apse from the centre and with simple gravity at the apses are in an equal ratio, therefore the squares of the periodic times are as the cubes of the principle diameters in conic sections. And this is a demonstration of the canon of the celebrated Kepler.

VIII. *Concerning the centre of oscillation.* In the theory I have set out briefly of the centre of oscillation in Ch. V. Book I., the centre of oscillation can be found, but not necessarily on the line through the centre, that is, on the right line from the point of suspension drawn through the centre of gravity of the oscillating figure. For indeed our method prevails for any line drawn through the point of suspension, on which the centre of oscillation is present, it will be found always this line must always pass through the centre of gravity of the oscillating figure, just as also the celebrated Johan Bernoulli has shown in his paper in §.28 of the June issue of *Act. Lips*. 1714. For if placed together with fig. 46., the simple isochrone pendulum CN can pass through some other point *m* rather than through the centre of gravity M of the oscillating figure, so that it must still be prepared, or put in place, so that the line from this point *m* (since indeed it is not indicated in the figure, but can easily supplied by a pen or imagined) drawn through the centre of gravity M always will cross the horizontal line CY at right angles, it will be found, always, as in §.206, Fig. 46. $\mathrm{NC} = (\mathrm{P.PC}^2 + \mathrm{Q.QC}^2 + \&\mathrm{c.}).\mathrm{G} : \mathrm{M}.m\mathrm{C}$, where M, as indicated there, signifies $\mathrm{E.P} + \mathrm{F.Q} + \&\mathrm{c.}$ or the sum of all the weights of the oscillating parts of the figure. Now indeed, with the angle YCN changed in any way, likewise the magnitude of *m*C itself will be changed; and therefore the quantity $\mathrm{NC} = \left(\mathrm{P.\ PC}^2 + \mathrm{Q.QC}^2 + \&\mathrm{c.}\right).\mathrm{G} : \mathrm{M}.m\mathrm{C}$ cannot be constant and invariant; unless *m*C coincides with MC, and thus with the points *m* and M placed together everywhere, thus so that hence it shall be apparent the centre of oscillation by necessity is present in the line of the centre. Q.E.D.

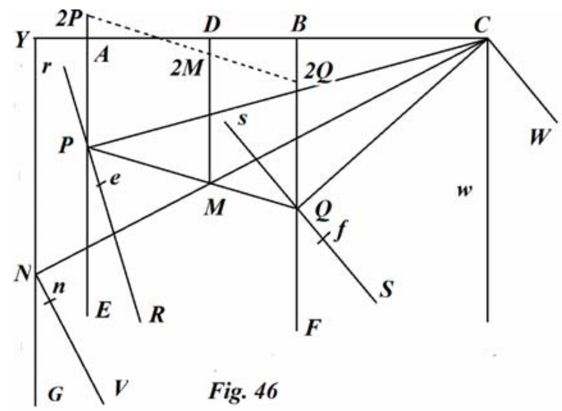


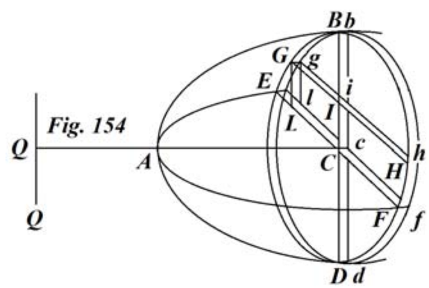
Fig. 46

IX. The application of this new theory of the centre of oscillation has now been set out a little more abundantly, than in §. 208, where the formula is set out for figures of uniform gravity only. By designating the particles P, Q, &c. as there, by *dp*, and the weight of these by βdp, thus so that β may denote the degree of the weight, and if *g* may signify the weight, by which a simple pendulum is acted on, and *z* the distance of an oscillating particle from the axis of oscillation, according to the rule §. 206,

$\mathrm{CN} = \left(\mathrm{P.PC}^2 + \&\mathrm{c.}\right).\mathrm{G} : \mathrm{M.MC}$ will give $t = g\int zzdp : \int \beta xdp$. Hence, if

$\beta = gzz : ax,$ there becomes $t = ag\int zzdp : g\int zzdp = a.$

Now the figure BAD may be oscillating in a plane ; or thus so that the axis of oscillation in Fig. 154 may remain always in the plane of the figure of the oscillation, and thus the ordinates of the figure BD shall be constantly parallel to the axis of oscillation QQ ; in this case these will be *z*, or the ordinates of the distances of the points, $\mathrm{BD} = \mathrm{QC} = x$, and thus $\int zzdp = \int xxdp$, and, because $dp = \mathrm{BC.C}c = ydx$, there will be $\int zzdp = \int xxydx$, & $t(= g\int zzdp : \int \beta xdp) = \int gxxydx : \int \beta xdp$, or, if the places of all the ordinates BD shall be of uniform gravity, $= g\int xxydx : \int \beta xydp$.



If indeed the figure may be moved about a line, that is, if the plane of the oscillating figure may be at right angles to the axis of the oscillation; any part of the ordinate BC shall be $\mathrm{CI} = u$, & $dp = dudx$, there will be $z = \sqrt{(\mathrm{QC}^2 + \mathrm{CI}^2)} = \sqrt{xx + uu}$, hence $zzdp = xxdudx + uududx$, and on integrating, and with putting *x* and *dx* constant, there will be found $\int zzdp = uxxdx + \frac{1}{3}u^3dx$, or (by making $x = y$) $= xxydx + \frac{1}{3}y^3dx$, but this is only an element with respect to the whole figure ABD. And $\int \beta xdp = \int \beta xydx$, and thus $t(= g\int zzdp : \int \beta xdp) = \left(\int xxydx + \int \frac{1}{3}y^3dx\right).g : \int \beta xydx.$

The formula can be established for a solid of revolution thus: BEAD shall be a solid of this kind, the base of which shall be the circle BEDH, of which the diameter EF is assumed to be parallel to the axis of oscillation QQ . With the perpendiculars GL, *gl*, sent to the diameter EF there becomes $\mathrm{CL} = t$, and as before LG or $\mathrm{CI} = u$, first all of *zzdp*, which are contained in the rectangle GL*l* are required. Because all the *zzdp*, which have been found at the ordinate GL are $xxu + \frac{1}{3}u^3$, all the *zzdp*, which are in the rectangle GL*l*, will be $= xxudt + \frac{1}{3}u^3dt$. Now there may be put $udt = d\alpha$, & $u^3t = \alpha$, and thus $d\alpha = u^3dt + 3uutdu$. But the circle EBF gives rise to $tt + uu = yy$: or by multiplying this by udt, the equation becomes $ttudt + u^3dt$, or (because $tdt = -udu$) $= -uutdu + u^3dt = yyudt = yyd\alpha$, or also $3u^3dt - 3uutdu = 3yyd\alpha$, and this equation is added to the other $u^3dt + 3uutdu = d\alpha$, there becomes $4u^3dt = 3yyd\alpha + d\alpha$ or $u^3dt = \frac{3}{4}yyd\alpha + \frac{1}{4}d\alpha$, which value may be substituted into $xxudt + \frac{1}{3}u^3dt$, and $d\alpha$ in place of udt, and there becomes

$$zzdp(= xxudt + \frac{1}{3}u^3dt) = xxd\alpha + \frac{1}{4}yyd\alpha + \frac{1}{12}d\alpha.$$

And thus all the *zzdp*, which may be contained in the segment of the circle CBGL, or α, are $= xx\alpha + \frac{1}{4}yy\alpha + \frac{1}{12}\alpha$, for QC and CB, which are themselves invariant or constants, remain, with whatever the variations of CL & GL, or t and u, thus so that they shall be constants with respect of these indeterminates x, y. Now, because α signifies the segment CBGL, and α the product CL.GL^3, all the $zzdp$ which are present in the quadrant CBE will be $=(xx+\frac{1}{4}yy)$.CBEC, with GL vanishing at E, also the product CL.GL^3 or u vanishes. Hence, if π designates the setting out of the ratio of the periphery of any circle to the radius, there will be found $\text{CBEC}=\frac{1}{2}\pi yy$, and the volume $\text{CBEC.C}c=\frac{1}{2}\pi yydx$, and thus $\int zzdp$ in this volume, $=\left(\frac{1}{2}xxyydx+\frac{1}{12}y^4dx\right).\pi g$. Therefore there becomes $t=\left(\frac{1}{2}\int xxyydx+\frac{1}{2}\int y^4dx\right).\pi g:\int \beta xdp$. And now from these and from everything else, all the most general rules emerge, which is shown in the following table : if there may be put

$$\text{A}=\int ydx, \qquad \text{I}=\int xyydx.$$

$$\text{B}=\int xydx, \qquad \text{K}=\int xxyydx.$$

$$\text{C}=\int xxydx \qquad \text{L}=\int y^4dx.$$

$$\text{D}=\int y^3dx. \qquad \text{N}=y^3x.$$

With figures oscillating $\begin{cases}\text{in a plane } t=gC:\int \beta xdp. \\ \text{about a line } t=\left(gC+\frac{1}{2}gD\right):\int \beta xdp.\end{cases}$

In solids of revolution $t=\left(\frac{1}{3}\pi g\text{K}+\frac{1}{4}\pi g\text{L}\right):\int \beta xdp.$

These rules are general, because they serve both in any case of uniform gravity as well as for variable gravity as it pleases, for, as has been said now above, βdp generally indicates the absolute weight of any element dp of the oscillating figure, from which since β shall be able to be varied in an infinite number of ways, thence it is evident enough, that endless rules embrace an infinitude of different cases.

If β is constant, the preceding rules will be changed into those, which are shown in the other table.

$t=gC:\beta\text{B}$ . . for plane figures oscillating;

$t=(g\text{C}+\frac{1}{2}g\text{D}).\beta\text{B}$ for figures oscillating about a line;

$t=\left(g\text{K}+\frac{1}{4}g\text{L}\right):\beta\text{I}.$ for solids of revolution oscillating.

So that at least I may show the use of these latter rules, let the figure BAD be a conic section oscillating about the axis QQ, of which the general equation shall be

$\pm aa \mp ee \pm 2ex \mp xx = \frac{aa}{bb}yy$, in which *a* indicates the transverse semi-latus rectum, *b* the conjugate semi-axes, *e* the distance of the point of suspension Q from the centre of the section, and finally *x*, *y* the coordinates QC, CB. If the differential equation of the curve $\pm edx \mp xdx = \frac{aa}{bb}yy$, may be multiplied by *y*, there will be had $\pm edx \mp xdx$ by *y*

$$= \frac{aa}{bb}yydy = \pm eydx \mp xydx = \frac{aa}{bb}yydy = \pm ed\text{A} \mp d\text{B}, \text{ and}$$

and by substitution . . . . $\pm d\text{B} = \mp ed\text{A} - \frac{aa}{bb}yy$. And

by integrating, ... $\pm \text{B} = \mp e\text{A} - \frac{aa}{3bb}y^3$.

Again, because (following the hypothesis) $N = y^3 x$, of which the differential multiplied by $\frac{aa}{bb}$ gives. . . . . . . $\frac{aa}{bb}d\text{N} = \frac{aa}{bb}y^3 dx + \frac{3aa}{bb}xyy$.

The equation of the curve may be multiplied by *ydx*, and becomes

$$\left.\begin{aligned} \frac{aa}{bb}y^3dx &= (\pm aa \mp ee).d\text{A} - \frac{2aae}{bb}yydy \mp d\text{C} \\ \text{Truly the diff. eq. multiplied by } 3xy \text{ gives } \frac{3aae}{bb}xyydy &= \pm 33eed\text{A} - \frac{3aae}{bb}yydy \mp 3d\text{C} \end{aligned}\right\} \text{Add.}$$

And the sum will be $\frac{aa}{bb}y^3dx + \frac{3aa}{bb}xyydy = \frac{aa}{bb}d\text{N} = (\pm aa \mp 4ee).d\text{A} - \frac{5aae}{bb}yydy \mp 4d\text{C}$. Of which the integral divided by 4 gives $C = \left(aa + \frac{1}{4}ee\right).\text{A} \mp \frac{5aa}{12bb}y^3 \mp \frac{aa}{4bb}xy^3$ (or N).

And B has been found above, clearly $\text{B} = e\text{A} \mp \frac{aa}{3bb}y^3$.

Truly above there was $\frac{aa}{bb}d\text{D}(=\frac{aa}{bb}y^3dx) = (\pm aa \mp ee).d\text{A} - \frac{2aae}{bb}yydy \mp d\text{C}$.

And by integrating there will be found $\frac{aa}{bb}\text{D} = (\pm aa \mp ee).\text{A} - \frac{2aae}{3bb}y^3 \mp \text{C}$ (or by substituting this value of C) $= \pm\frac{1}{4}aa\text{A} - \frac{aae}{4bb}y^3\ \frac{aa}{4bb}xy^3$. And thus by multiplying the equation by $\frac{bb}{aa}$, there will be found $D = \pm\frac{1}{4}bb\text{A} - \frac{1}{4}ey^3 - \frac{1}{4}xy^3$.

Further in all these formulas A indicates the area ABC, so that, if the values of the letters found B, C, D may be substituted into the first two equations of the latter table, the unknown value of *t* will be had for every conic section disturbed in the plane and laterally both in air or a vacuum, as well as within any liquid ; if in a vacuum, there will be $g = \beta$, and if in some liquid, the specific gravity of which shall be to the specific gravity of the oscillating figure , as 1 to *n*, it will be as 1 to *n*, therefore $\beta = \frac{n}{n-1}g$.

From these found everything will be able to be deduced in an easy manner, which the celebrated James Bernoulli has included in three tables in the *Transactions Royal Academy Science*, Paris for 1st Dec.1703 on which I shall not tarry long, nor also is it necessary for me to show, how the values of the letters I, K, L must be found in conic sections, since this matter indeed, unless by beginners, shall be an operation able to be performed, since *y* in the magnitudes of these elements rises everywhere to two dimensions, thus so that the integral quantities will be resolved without further reduction.

APPENDIX.

Postquam præcedens tractatus Typographo jam missus esset, varia subinde in mentem venerunt, quæ tum ad illustrationem Operis, tum etiam ad qualemcumque doctrinæ profectum facere videntur. Eorum pauca in hac Appendice præcedentibus *libris* addere constitui, quando temporis angustia atque Bibliopolæ festinatio non concedunt mihi oportuniatem omnia ad umbilicum deducendi.

I. Circa notissimam naturæ legem, qua *cuilibet actioni æqualis & contraria esse dicitur reactio,* quam ex §. 11. derivatam *in* sequenti §. 12. retuli, egregii nonnulli viri hærere videntur, existimantes nunquam motum sequi debere actionem, si huic æqualis semper & contraria sit reactio; nam si, ut in exemplo a Celeb. Newtono inter alia adducto, equus lapidem funi alligatum trahens æqua vi retrahitur in lapidem, quomodo, inquiunt, progredi *equus lapidem* que movere potest, si vis agens ab æquali & contraria resistentia absorbetur tota atque retunditur ? Sed hæc objectio ab æquivocatione circa nomina *vis* & *actionis* nata esse videtur ; quæ tamen duo accurate debent distingui. Vis corporum non est actio ipsa; nam actio est applicatio vis cujusdam subjecto habili, seu cui applicari potest: illi vero soli corpori applicari censetur, quod resistit, quod renititur, quod reagit. In rebus materialibus nulla est actio proprie sic dicta, ubi nulla est reactio; equus enim, qui corpus *nihil* resistens post se trahit, *nihil* trahere vel *non* trahere, id est, nil aliud agere, quam simpliciter incedere; censetur. In omni ergo actione corporea est collisio inter vim agentem & renixum corporis patientis, applicatio vis agentis in corpore actionem suscipiente, hoc est, actio ipsa æqualis est & contraria renixui patientis qui est ejus reactio, quia hic renixus vel hæc vis inertiæ corporis patientis debet tolli ad id ut corpus moveri possit ab agente. Non tamen ideo sequitur *vim totalem* corporis agentis totam impendi superandæ rectioni patientis, sed ejus partem tantum, & hæc pars vis totalis corporis, quæ tollendæ resistentiæ corporis patientis insumitur, ea est vis, a qua actio proprie manat; *residuum* enim ejusdem vis totalis, cum nullam habeat resistentiam vel renixum absorbendum, in actionem minime influit. Idcirco cum actio quæcunque æqualis & contraria dicitur reactioni corporis patientis, hoc aliud non significat quam istud, *in omni actioni corporea tantum virium corpori agenti decedere, quantum corpus actionem suscipiens lucratum sit*.

II. Propositio IV. Lib. 1. §. 44. facilem suppeditat modum demonstrandi centrum gravitatis *dari* in unoquoque corpore, & hoc centrum *unicum* esse, si gravium directiones parallelæ fuerint. Nam, si fuerint corpuscula quotcunque quam minima A, B , C, D , &c. Fig.150. quomodocunque posita, & tanquam elementa corporis etiam cujuslibet spectata, a quibus demissæ sint ad planum P*c* positione datum perpendiculares Aa, B*b,* C*c,* D*d*, &c. & plano P*c* aliud parallelum ducatur GH, cujus distantia E*e* a plano P*c* ea sit, ut factum ex E*e* in aggregatum corporum $A+B+C+D+\&c.$ æquet

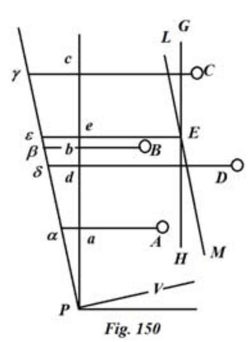


Fig. 150

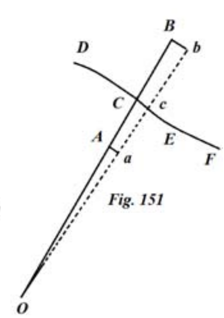


Fig. 151

summam factorum $\mathrm{A.A}a+\mathrm{B.B}b+\mathrm{C.C}c+\mathrm{D.D}d+\&c.$ Corporum A, B, C, D, *&c.* in suas respectivas a plano P*c* distantias A*a,* Bb, C*c,&c.* & in P*c* sumatur P*e*, talis ut $\left(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\&\mathrm{c}\right)$. P*e* sit $=\mathrm{A.P}a+\mathrm{B.P}b+\mathrm{C.P}c+\mathrm{D.P}d+\&\mathrm{c.}$ ductaque per *e* recta *e*E occurret GH in puncto E, quod est commune centrum gravitatis corpusculorum A, B, C , D , *&c.* vel corporis totalis, quod ex hisce corpusculis componitur. Hoc punctum E ideo dicitur corporis totalis centrum gravitatis, quia ejus partes seu corpuscula A, B, ex una parte lineæ, vel plani GH, æqualium sunt momentorum cum corpusculis C, D, &c. ex altera parte ejusdem plani vel lineæ GH, & quia hoc idem accidit respectu cujuslibet alius lineæ LM, quæ per hoc idem punctum E duci potest. Nam ducta per P linea $\mathrm{P}\gamma$ parallela LM, & productis A*a*, B*b*, *Cc,* D*d*, &c. & E*e*, in $\alpha,\beta,\gamma,\delta,\&\mathrm{c.}\ \&\ \varepsilon$ reperietur
$(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\&\mathrm{c}).E\varepsilon=\mathrm{A.A}\alpha+\mathrm{B.B}\beta+\mathrm{C.C}\gamma+\mathrm{D.D}\delta+\&\mathrm{c.}$
& $\left(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\&\mathrm{c}\right):\mathrm{P}\varepsilon=\mathrm{A.P}\alpha+\mathrm{B.P}\beta+\mathrm{C.P}\gamma+\mathrm{D.P}\delta+\&\mathrm{c}$. quandoquidem omnia triangula $\mathrm{P}a\alpha,\ \mathrm{P}b\beta,\mathrm{P}c\gamma,\&\mathrm{c.}$ similia sunt. Jam ex hisce duabus æqualitatibus ultro sequitur, omnia momenta corporum A, B, C, D, &c. respectu utriusque plani Px & PV ejus normalis simul æqualia esse respectivæ factis ex aggregato corporum in distantias centri E ab utroque plano vel linea $\mathrm{P}\gamma$,PV, quod cum ita fit, necessum etiam est ut corpuscula, quæ sunt ad unam partem rectæ LM æqualium sint momentorum cum pondusculis, quæ sunt in altera parte ejusdem LM. Cum itaque omnes lineæ, quæ per punctum E duæ possunt, sint axes æquilibrw ponduscolorum A, B, C, D, &c. liquet corporis ex hisce corpusculis tanquam suis elementis conflati centrum gravitatis *unicum* esse in E. Quod erat demonstrandum.

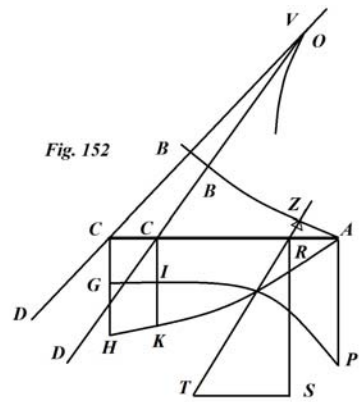
Hanc eandem propositionem Wallisius ex aliis principiis etiam elicuit.

III. Circa regulam Guldini, cujus demonstratio supra (§.47.) allata est, duo notanda sunt. 1°. Eam non solum in figuris aut *solidis* rotundis obtinere, sed generaliter in omnibus aliis, quæ generari possunt figuræ genitricis motu hac lege facto, ut figuræ planum semper perpendiculare maneat lineæ, quam centrum gravitatis figuræ genitricis motu ejus describit. Hoc enim casu *solidum figuræ motu genitum semper æquale est facto ex figura in viam centri gravitatis ejusdem.* Fig.151.

Nam si ponatur centrum gravitatis C figuræ cujuslibet *AB* incedens in curva DEF, cui figura ubique normalis sit, descripsisse arculum curvæ C*c*, atque ex situ BA venisse in situm *ba* priori indefinite vicinum, adeo ut rectæ CO & *cO,* quæ sint in planis figurarum AB, & *ab,* & in plano curvæ DEF, eritque O centrum circuli osculatoris curvæ datæ DF in ejus puncto C vel ejus proximo *c.* Adeoque solidum, quod figura AB describit cum ejus centrum C percurrit arculum *Cc,* per §.47. æquatur factio ex figura AB in arculum C*c*, ergo omnia ejusmodi solida vel solidum, aut *figura,* quæ producitur incessu figuræ AB per totam curvam DEF, æquabitur facto ex eadem AB in omnes arculos, qui in curva DF continentur, id est, facto ex figura AB in curvæ DEF longitudinem. Quod erat demonstrandum.

2°. Notandum, hæc tantum ita se habere, si figura genetrix gravitatis sit uniformis, & fallere Guldini regulam, figura genitrice existente difformiter gravi.

IV. Fig. 152. Propositio VIII. Lib. I. (§.59.) etiam deduci potest vel saltem ipsi æquivalens ex proprietate universali vectis recti in singulis punctis a potentiis obliquis impulsi, quam hoc loco explicare placet. Sit linea recta inflexilis AC, quæ in singulis punctis C, C &c. urgeatur potentiis CD, CD uniformi lege ad ipsam inclinatis, & circa eam tanquam axem constitutæ intelligantur figuræ ACGIP, & ACHKA, ea conditione, ut ordinata quælibet CG (CI) sit ad homologam potentiain CD in duplicata proportione, *sinus* anguli respectivi BCA ad sinum totum; ac ordinata CH (CK) alterius figuræ ACH se habeat ad ordinatam CG (CI) primæ ACGP, ut sinus complementi anguli BCA ad ejusdem sinum rectum, & sic ubique, positis hisce, per centrum gravitatis areæ ACGP agatur RS axi AC recta huic lineæ AC occurrens in R, eritque linea RT cum RS angulum TRS continens, cujus sinus est ad sinum complementi, ut area AHC ad aream ACGP, media directio omnium potentiarum CD, CD lineæ AC applicatarum, & onus quo hypomochlion in R urgetur, erit ad omnes potentias lineæ AC rectas ex obliquis CD, CD derivatas, ut sinus totus ad sinum complementi anguli TRS. Demonstratio hujus Lectoris industriæ relinquitur brevitatis amore. Hoc tamen minime reticendum est, in iisdem directionibus BD, BD potentias, curvæ cuilibet AB applicatas, quæ sint ad homologas CD ad lineam AC pertinentes, ut respectivæ CV ad BV (supponendo directiones BD, BD tangere curvam quandam CV) eandem habere mediam directionem ZT cum potentiis CD, CD, &c. rectæ AC applicatis, ut demonstratu id facile est. Unde si curvæ cuicunque AB potentiæ quæcunque in directionibus BC, BC, &c. applicatæ sint, facili negotio invenietur earum media directio, ducendo per curvæ verticem A tangentem AC, vel quamlibet aliam rectam, & disquirendo quænam potentiæ CD in directionibus BC huic rectæ AC applicandæ sint ad id, ut harum potentiarum media directio RT etiam præbeat mediam directionem potentiarum in directionibus BC, BC, &c. datæ curvæ ABB applicatarum; ad id facienda ubique est CD ad homologam potentiam curvæ AB applicatam, sicut BV ad CV.



V. Usum universalem propositionis XII. Lib. I. In subjunctis ejus corollariis & scholio jam utcunque ostendimus. Ad majorem tamen propositionis & applicationis ejusdem illustrationem, pauca hæc, quæ sequuntur, addidisse forte juvabit. Præter symbola §. 101. jam explicata, firmitatem cujusque curvæ elementi B*b* vel Bβ nominabimus *t*, quæ illic insigniebatur litera T; potentias juxta BH & BG,*dq* & *dp* respective, tangentem & secantem anguli GBH, *h* & *k*, ac denique radium osculi seu curvitatis in B, *r;* item per *a* sinum totum designabimus . Fig. 29. Hisce positis, erit (num.1. §. 93.) GH =*dt,* & tria elementa *dp*, *dq,* & *dt,* ipsis *a* ,*k* & *h* proportionalia erunt, sic etiam $\mathrm{AP}:\mathrm{PQ}=r:ds$, atqui (§.103.) est $\mathrm{BG}(dp):\mathrm{T}(t)=\mathrm{PQ}:\mathrm{AP}$, ergo $dp:t=ds:r$, item $dp:dt=a:h$, ergo $dt:t=hds:ar$, & $log.t=\int(hds:ar)$, adeoque ex hisce analogiis quatuor habentur æquationes, quales in hoc adjecto laterculo notantur. Hinc ex quantitatibus *dp, dq, dt,* vel *t* & *h* data qualibet in indeterminatis curvæ *x, y dx, dy*, &c, & quantitatibus constantibus, invenientur reliquæ; si non algebraice, saltem transcendenter.

1. $dt=hdp:a=hdq:k.$
2. $t=rdp:ds.$
3. $log.\,t=\int hds:ar.$
4. $h=ardt:tds.$

Exempl. Sit curva data ABZ hyperbola æquilatera, cujus æquatio $yy = xx - aa$, in qua abscissæ *x* sumantur a centro, semilatere transverso existente $= a$, sintque tenacitates aut firmitates *curvæ* proportionales distantiis punctorum curvæ a centro , hoc est $t = \sqrt{(xx + yy)}$. Et §.161. præbebit $r = t^3 : aa, \& \ h\left(= ardt : tds\right) = \frac{2xy}{a}$, nec non $k = tt : a$; hinc æquatio prima $dt = hdq : k$ præbebit $dq = ttdt : 2xy = 2txdx : 2xy = tdx : y$; atqui $tdx = yds$, ergo $dq (= tdx : y) = yds : y = ds$. Idcirco potentiæ BH, seu *dq,* sunt ut elementa curvæ *Bb,* earumque directiones in centro hyperbolæ coeunt, quoniam tangens anguli DBH inventa est $= 2xy : a$, & hæc expressio etiam tangentem anguli, quem normalis hyperbolæ BD cum recta ex puncto B ad centrum hyperbolæ ducta continet. Propterea est hyperbola æquilatera *catenaria,* si gravium directiones in centro hyperbolæ concurrere supponuntur, ut supra circa finem §.105. sine demonstratione dictum est. Pariter si posuissem $dq = ds$ & $h = 2xy : a$ in hyperbola eadem, invenissemus $t = \sqrt{(xx + yy)}$.

Generalius, si directiones gravium convergunt in punctum, abscissæque sumantur in axe ab hoc centro gravium, erit $h = axdx + aydy : xdy - ydx$, & hujus ac alterius $dt = hdq : k$ ope, in suppositione $dq = ds$ invenissemus $t = \sqrt{(xx + yy)}$ & $yy = xx - aa$.

Si directones gravium axi AC parallelæ sunt, erit $h = adx : dy$, ac substituendo loco *dx,* & *dy* earum proportionales *n* & *m,* fiet $h\left(= adx : dy\right) = an : m$. Item in æquatione quarta, surrogando *m* & *dn* pro *r* & *ds,* quibus proportionales sunt, fiet $an : m = amdt : tdn,$ & $dt : t = ndn : mm$ (aut, quia $mm + nn = aa$, & $ndn = -mdm) = -mdm : mm = -dm : m$, ergo $lt = l\left(aa : m\right)$, id est, $t = aa : m = ads : dy$, ut habet equatio nona. Substituendo igitur valorem inventum ipsius *t* in æquatione secunda invenietur æquatio septima ; & ex hac ope analogiæ $dp : dq = a : k, = dy : ds$, elicietur æquatio octiva. Hinc,ex hisce tribus *dp, dy* & *t* una data in indeterminatis curvæ & constantibus utlibet invicem permixtis, facile invenientur reliquæ duæ, si non algebraice, saltem transcendenter.

5. $h = adx : dy$
6. $k = ads : dy$
7. $dp = ads^2 : rdy$
8. $dq = ads^3 : rdy^2$
9. $t = ads : dy$

Si directiones potentiarum BH sunt curvæ normales, erit $h = 0,$ & $k = a$, & sola æquatio secunda sufficit; hunc casum particularem Celeb. Joh. Bernoullius solutum dedit jam pridem in Commentariis Academiæ Reg. Paris Scientiarum 1706, & nuper adhuc Libro cui titulum fecit *Essay d'une Nouvelle Theorie de la Manoeuvre des Vaisseaix.*

Fig.153; VI. *De sublimitate gravitatis difformis, qua projectilia sectiones conicas in vacuo describunt.* Quoniam mobile perimetrum alicujus sectionis conicæ AQ*a* describens citatur solicitationibus, quæ quadratis distantiarum mobilis a centro sunt reciproce proportionales, *scala* harum solicitationum GB erit hyperbola quadratica, cujus applicatæ HG, AB, quæ solicitationem gravitatis in H & A exponunt, quadratis abscissarum DH, DA sunt reciproce proportionales, centro hyperbolæ existente in D. Quæritur jam quantam assumere oporteat *sublimitatem,* hoc est, lineam HA talem, ut in ea descendens grave motum a quiete in H incipiens, & solicitationibus, quas ordinatæ hyperbolæ in quadrilineo HGBA exponunt, celeritatem acquirat in A, cum qua propulsum mobile, secundum lineam Aα alteri BA in directum positam, datam sectionem conicam AQ*a* in vacuo describat.

Ponitur punctum A pro alterutra apside, ac propterea DA erit sectioni perpendicularis in A, atque adeo hoc punctum A alteruter vertex erit sectionis conicæ AQ*a*, & radius curvitatis in A æquabitur $\frac{1}{2}\mathrm{L}$, id est, semilateri recto sectionis conicæ, & solicitatio curvæ in *A* normalis ex centrali AB derivata eadem erit cum hac centrali; idcirco indicando celeritatem mobilis in puncto A, per V, vi §154. habebitur $\mathrm{V}^2=\frac{1}{2}\mathrm{L.AB}$. Atqui eadem velocitas V, sed (secundum hypothesin) descensu in HA acquisita, exponitur etiam (§. 136.) latere quadrato ex duplo areæ HGBA, ergo $\mathrm{V}^2=2.\mathrm{HGBA}$, atqui $\mathrm{AHGB}=\mathrm{AD.AB}-\mathrm{HD.HG}$, ergo etiam $\frac{1}{2}\mathrm{L.AB}=2.\mathrm{AD.AB}-2.\mathrm{HD.HG}$, vel, propter hyperbolam, subrogando loco AB & HG earum proportionalia quadrata $\mathrm{HD}^2, \mathrm{AD}^2$,

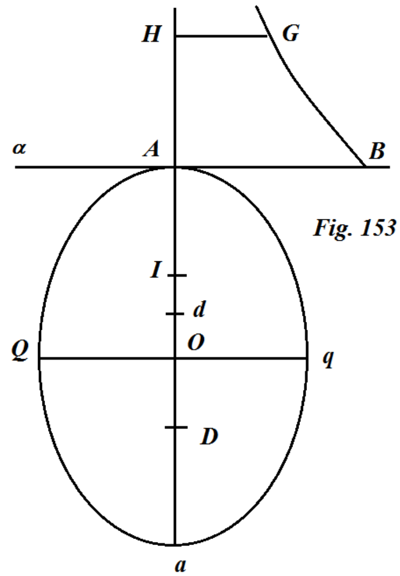


Fig. 153

fiet $\frac{1}{2}\mathrm{L.HD}^2=2\mathrm{AD.\,HD}^2-2\mathrm{HD.AD}^2=2\mathrm{HD.AD.HA}$, hinc etiam $\mathrm{L.HD}=4.\mathrm{HA.AD}$, vel $\mathrm{L.A}a.\mathrm{HD}=4\mathrm{A}a.\mathrm{HA.AD}$. Verum ex conicis est $4\mathrm{QO}^2=\mathrm{L.A}a=\mathrm{AD.A}d$, ergo $4\mathrm{A}a.\mathrm{HA.AD}=4\mathrm{AD.A}d.\mathrm{HD}$, id est, $\mathrm{A}a.\mathrm{HA}=\mathrm{A}d.\mathrm{HD}$, & $\mathrm{HD}:\mathrm{HA}=\mathrm{A}a:\mathrm{A}d$. & dividendo, $\mathrm{AD}:\mathrm{HA}=\mathrm{AD}$ (vel ad): $\mathrm{A}d$, propterea est $\mathrm{HA}=\mathrm{A}d$, atque adeo $\mathrm{HD}=\mathrm{A}n$. *Est* ergo *sublimitas in sectionibus conicis æqualis distantiæ apsidis remotioris a centro solicitationum, a foco ejus propiore.* Hinc varia deduci possunt, quorum non nulla sequuntur.

1°. Uni eidemque sublimitati HA convenire quidem possunt infinitæ sectiones conicæ AQ*a;* cum eadem manente A*d* distantia focorum *dD* possit esse magnitudinis cujuscunque; sed non ideo una eademque omnibus competet celeritas jactus secundum A*a*. Nam, quia $\mathrm{V}^2=\frac{1}{2}\mathrm{L.AB}=\frac{1}{2}\mathrm{L.AD}^2.\,\mathrm{AB}:\mathrm{AD}^2$ (seu, quia $\mathrm{AD}^2.\mathrm{AB}$ est solidum constans) est $\frac{1}{2}\mathrm{L}:\mathrm{AD}^2$, erit V ut $\sqrt{\frac{1}{2}\mathrm{L}}:\mathrm{AD}$. Id est, in diversis sectionibus conicis. velocitates mobilis in alterutra apside erunt in composita ratione ex subduplicata directa ratione laterum rectorum sectionum, & reciproca ratione distantiarum apsidis a centro solicitationum.

2°. Si umbilicorum distantia *Dd* fuerit infinita, sectio conica A*qa* st parabola focum habens in *d*, ac proinde sublimitas HA:A*d* sit æqualis quartæ parti lateris recti L, ut Galilæus, aliique post ipsum plures ex aliis fundamentis demonstrarunt. In hoc ergo consectario continetur universa doctrina motus projectorum in hypotheti gravitatis parallelarum.

3°. Si mobile eadem velocitate V in *A* acquisita post descensum in HA circulum ex centro I, radiumque $\mathrm{AI}=\frac{1}{2}\mathrm{L}$ habentem describat, conatus centrifugus mobilis in circulo eadem erit cum solicitatione centrali AB in apside A sectionis *conicæ* AQ*a*.

4°. Gravitas uniformis, qualis apud nos est, se habet ad conatum centrifugum in dicto circulo, seu ad solicitationem centralem AB in A ut $\frac{1}{2}\mathrm{AI}$ seu $\frac{1}{4}\mathrm{L}$ ad HA seu A*d*. Adeoque in parabola ex foco *d* & vertice A descripta missilis gravitas in vertice ejus A, æquabitur conatui centrifugo in circulo prædicto.

VII. *De tempore Periodico in Sectionibus Conicis.* Vocentur tempus unius circuitus in sectione AQ*a,* T, diameter A*a,* D; ejus conjugatus *Qq, d;* latus rectum *l,* distantia seu altitudo AD, A, solicitatio centralis AB in apside A, G. Eruntque $d=\sqrt{lD}$,& area totius ellipseos, hoc est, $2.\mathrm{AQ}a=\frac{1}{2}p\mathrm{D}d$, ubi *p* est exponens rationis circumferentiæ circuli ad radium. Est vero (§.157.) $\mathrm{T}=2.\mathrm{AQ}a\mathrm{A}:\frac{1}{2}\mathrm{AD.V}$, ergo substituendo valores analyticos loco areæ & linearum *AD, V;* habebitur $\mathrm{T}=p\mathrm{D}d:\mathrm{A}u$, seu $\mathrm{T}^2=pp\mathrm{D}^2d^2:\mathrm{A}^2uu$, est vero $dd=l\mathrm{D},\&\ uu=\frac{1}{2}l\mathrm{G}$, ergo $\mathrm{T}^2\left(=pp\mathrm{D}^3l:\frac{1}{2}l\mathrm{A}^2\mathrm{G}\right)=2pp\mathrm{D}^3:\mathrm{A}^2.\mathrm{G}$. Hinc *quadratum ex tempore periodico est in composita ratione ex directa triplicata ratione lateris transversi, & reciproca composita ratione duplicatæ distantiæ apsidis a foco, seu centro solicitationis, & simplæ solicationis* G *in apside.* Unde, quia solicitationes gravitatis sunt in reciproca duplicata ratione distantiarum mobilis a centro, rationes compositæ ex duplicata distantiæ apsidum a centro & simpla gravitatis in apsidibus sunt rationes æqualitatis, propterea sunt quadrata temporum periodicorum, ut cubi diametrorum principalium in sectionibus conicis. Et hæc est demonstratio Celebris Kepleri canonis.

VIII. *De Centro Oscillationis.* In theoria, centri oscillationis Cap. V. Lib. I. exposita brevitatis tantum gratia posui, non vero ex necessitate, centrum oscillationis reperiri, in linea centri, hoc est, in recta ex puncto suspensionis per centrum gravitatis figuræ oscillantis ducta. Methodus enim nostra perindæ valet quæcunque linea accipiatur per punctum suspensionis, in qua centrum oscillationis existat, invenientur semper hanc lineam per centrum gravitatus figuræ oscillantis transire debere, prorsus ut idem etiam Celeb. Joh. Bernoullius in eleganti suo Schediasmate Act. Lips. 1714. mens. Junii §.28 ostendit : Nam si in fig. 46. pendulum simplex CN composito isochronum per aliud punctum *m* quam per centrum gravitatis M figuræ oscillantis transire ponatur, quod tamen comparatum, seu positum, esse debet, ut linea ex hoc puncto *m* (in si figura quidem non signato sed calamo aut mente facile supplendo) per centrum gravitatis M ducta horizontali CY ad angulos rectos semper occurrat, invenietur, semper, ut in §.206, Fig 46. $\mathrm{NC}=(\mathrm{P.PC}^2+\mathrm{Q.QC}^2+\&\mathrm{c.}).\mathrm{G}:\mathrm{M}.m\mathrm{C},$ ubi M, ut in citato loco, significat $\mathrm{E.P}+\mathrm{F.Q}+\&\mathrm{c.}$ seu aggregatum ponderum omnium figuræ oscillantis partium. Jam vero, mutato utlibet angulo YCN, mutabitur simul magit, itudo ipsius *m*C; ac propterea quantitas $\mathrm{NC}=\left(\mathrm{P.\ PC}^2+\mathrm{Q.QC}^2+\&\mathrm{c}\ ..\ \right).\mathrm{G}:\mathrm{M}.m\mathrm{C}$ constans & invariata fieri nequit; nisi coincidente *m*C, cum MC, atque adeo puncta *m* & M ubique confundantur, adeo ut hinc appareat centrum oscillationis necessario in linea centri existere. Quod erat demonstrandum.

IX. Applicatio hujus novæ theoriæ centri oscillationis nunc paulo uberius declarandæ est, quamque in §.208, ubi formula tantum exhibetur pro figuris uniformis gravitatis. Designando particulas P, Q, &c. ut ibi, per *dp,* earumque pondus per *βdp,* ita ut β gradum gravitatis denotet, &, si g significet gravitatem, qua simplex pendulum agitatur, & z distantiam particulæ oscillantis ab axe oscillationis, canon §. 206,

$\mathrm{CN}=\left(\mathrm{P.PC}^2+\&\mathrm{c.}\right).\mathrm{G}:\mathrm{M.MC}$ præbebit $t=g\int zzdp:\int \beta xdp$. Hinc, si

$\beta=gzz:ax,$ fiet $t=ag\int zzdp:g\int zzdp=a.$

Oscilletur Jam figura BAD in planum; seu ita ut axis oscillationis Fig.154 semper maneat in plano figuræ oscillantis, atque adeo ordinatæ figuræ BD axi oscillationis QQ constanter parallelæ sint; erunt hoc casu ipsæ z, seu distantiæ punctorum ordinatæ,

$\mathrm{BD}=\mathrm{QC}=x$, atquæ adeo $\int zzdp=\int xxdp$, &, quia $dp=\mathrm{BC.C}c=ydx$, *erit*

$\int zzdp=\int xxydx$, & $t(=g\int zzdp:\int \beta xdp)=\int gxxydx:\int \beta xdp$, vel, si omnes partes

ordinatæ BD sint uniformis gravitatis, $=g\int xxydx:\int \beta xydp$. Si eadem figura movetur in latus, id est, si planum figuræ oscillationis axi oscillationis rectum est; sit quælibet in ordinata BC ejus portio $\mathrm{CI}=u$, & $dp=dudx$, erit $z=\sqrt{\left(\mathrm{QC}^2+\mathrm{CI}^2\right)}=\sqrt{xx+uu}$,

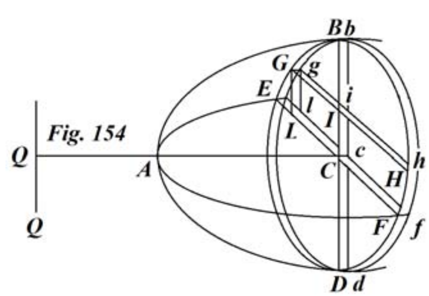
hinc $zzdp=xxdudx+uududx$, & integrando, positis x & dx constantibus, invenietur

$\int zzdp=uxxdx+\frac{1}{3}u^3dx$, vel (facta $x=y$) $=xxydx+\frac{1}{3}y^3dx$, sed hoc est tantum

elementum respectu totius figuræ ABD. Et $\int \beta xdp=\int \beta xydx$, adeoque

$t(=g\int zzdp:\int \beta xdp)=\left(\int xxydx+\int \frac{1}{3}y^3dx\right).g:\int \beta xydx.$

Pro solidis rotundis sic est indaganda formula : sit BEAD ejusmodi solidum, cujus basis sit circulus BEDH, cujus diameter EF axi oscillationis QQ parallelus supponitur. Ad hanc diametrum EF demissis perpendiculis GL, gl, fiant $\mathrm{CL}=t$, & ut antea LG seu $\mathrm{CI}=u$ requiruntur primiun omnia $zzdp$, quæ in rectangulo GLl continentur. Quia omnia $zzdp$, quæ in ordinata GL reperta sunt $xxu+\frac{1}{3}u^3$, omnia $zzdp$, quæ in rectangulo GLl sunt, erunt $=xxudt+\frac{1}{3}u^3dt$.



Ponantur jam $udt=d\alpha,$ & $u^3t=\alpha$, atque adeo $d\alpha=u^3dt+3uutdu$. Sed circulus EBF præbet $tt+uu=yy$: vel ducendo hanc in udt, fiet $ttudt+u^3dt$, seu (ob $tdt=-udu$) $=-uutdu+u^3dt=yyudt=yyd\alpha$, vel etiam $3u^3dt-3uutdu=3yyd\alpha$, & addatur hæc æquatio ad alteram $u^3dt+3uutdu=d\alpha$, fietque

$4u^3dt=3yyd\alpha+d\alpha$ vel $u^3dt=\frac{3}{4}yyd\alpha+\frac{1}{4}d\alpha$, qui valor substituatur in $xxudt+\frac{1}{3}u^3dt$,

atque $d\alpha$ pro udt, fietque $zzdp(=xxudt+\frac{1}{3}u^3dt)=xxd\alpha+\frac{1}{4}yyd\alpha+\frac{1}{12}d\alpha$.

Adeoque omnia *zzdp*, quæ in segmento circulari CBGL, seu α, continentur sunt $= xx\alpha + \frac{1}{4}yy\alpha + \frac{1}{12}\alpha$, nam ipsæ QC, CB invariatæ, seu constantes, manent, utlibet variatis ipsis CL & GL, seu *t* ac *u*, adeo ut istarum respectu indeterminatæ *x*, *y* constantes sint. Jam, quia α significat segmentum CBGL, & *u* factum $\mathrm{CL.GL}^3$, erunt omnia *zzdp* quæ in quadrante CBE continentur $= (xx + \frac{1}{4}yy).\mathrm{CBEC}$, cum evanescente GL in E, etiam & factum $\mathrm{CL.GL}^3$ seu *u* evanescat. Hinc, si π designet exponentem rationis periphriæ cujusque circuli ad radium, reperietur $\mathrm{CBEC} = \frac{1}{2}\pi yy$, ac solidum $\mathrm{CBEC.C}c = \frac{1}{2}\pi yydx$, adeoque $\int zzdp$ in hoc solido, $= \left(\frac{1}{2}xxyydx + \frac{1}{12}y^4dx\right).\pi g$. Propterea erit $t = \left(\frac{1}{2}\int xxyydx + \frac{1}{2}\int y^4dx\right).\pi g : \int \beta xdp$. Atque ex hisce jam omnibus emergent canones generalissimi, quales sequens tabella exhibet :

si ponantur

$$\mathrm{A} = \int ydx, \qquad \mathrm{I} = \int xyydx.$$

$$\mathrm{B} = \int xydx, \qquad \mathrm{K} = \int xxyydx.$$

$$\mathrm{C} = \int xxydx \qquad \mathrm{L} = \int y^4dx.$$

$$\mathrm{D} = \int y^3dx. \qquad \mathrm{N} = y^3x.$$

$$\text{In figuris agitatis in} \begin{cases} \text{Planum } t = gC : \int \beta xdp. \\ \text{Latus } t = \left(gC + \frac{1}{2}gD\right) : \int \beta xdp. \end{cases}$$

In solidis rotundis $t = \left(\frac{1}{3}\pi g\mathrm{K} + \frac{1}{4}\pi g\mathrm{L}\right) : \int \beta xdp.$

Hi canones funt generales, quia deserviunt in omni casu gravitatis uniformis aut pro libitu variabilis, nam, ut jam supra dictum, βdp generaliter indicat pondus absolutum cujusque elementi *dp* figuræ oscillantis, unde cum β infinitis modis variare possit, exinde satis in propatulo est, hos canones infinities infinitos diversos casus omnes complecti.

Si β est constans, prædentes canones mutabuntur in eos, quos hæc altera tabella repræsentat.

$t = gC : \beta\mathrm{B}$ . . . . Planum agitatis

$t = (g\mathrm{C} + \frac{1}{2}g\mathrm{D}).\beta\mathrm{B}$ pro figuris in Latus agitatis.

$t = \left(g\mathrm{K} + \frac{1}{4}g\mathrm{L}\right) : \beta\mathrm{I}.$ Pro solidis rotundis.

Ut saltem usum horum posteriorum canonum ostendam, esto figura BAD circa axem QQ oscillans sectio conica, cujus æquatio generalis sit $\pm aa \mp ee \pm 2ex \mp xx = \frac{aa}{bb}yy$, in qua *a* designat semilatus transversus, *b* semiaxem conjugatum, *e* distantiam puncti suspensionis Q a centro sectionis, ac denique *x*, *y* coordinatas QC, CB. Si æquatio curvæ differentiatis $\pm edx \mp xdx = \frac{aa}{bb}yy$, ducatur in *y*, habebitur $\pm edx \mp xdx$ in

$y = \frac{aa}{bb}yydy = \pm eydx \mp xydx = \frac{aa}{bb}yydy = \pm ed\mathrm{A} \mp d\mathrm{B}$, &

per antithesin . . . . $\pm d\mathrm{B} = \mp ed\mathrm{A} - \frac{aa}{bb}yy$. Et

integrando . . . . $\pm \mathrm{B} = \mp e\mathrm{A} - \frac{aa}{3bb}y^3$.

Porro, quia (secundum hypothesin) $N = y^3x$, cujus differentialis ducta in $\frac{aa}{bb}$ præbet

in $\frac{aa}{bb}$ praebet . . . . $\frac{aa}{bb}d\mathrm{N} = \frac{aa}{bb}y^3dx + \frac{3aa}{bb}xyy$.

Ducatur aequatio curvae in ydx, & fiet $\frac{aa}{bb}y^3dx = \left(\pm aa \mp ee\right).d\mathrm{A} - \frac{2aae}{bb}yydy \mp d\mathrm{C}$
Aqu. vero differentialis in $3xy$ ducta praebet $\frac{3aae}{bb}xyydy = \pm 33eed\mathrm{A} - \frac{3aae}{bb}yydy \mp 3d\mathrm{C}$ } Add.

Eritque summa $\frac{aa}{bb}y^3dx + \frac{3aa}{bb}xyydy = \frac{aa}{bb}d\mathrm{N} = \left(\pm aa \mp 4ee\right).d\mathrm{A} - \frac{5aae}{bb}yydy \mp 4d\mathrm{C}$. Hujus integralis divisa per 4 præbebit $C = \left(aa + \frac{1}{4}ee\right).\mathrm{A} \mp \frac{5aa}{12bb}y^3 \mp \frac{aa}{4bb}xy^3$ (vel N).

Atqui B supra inventa est, scilicet $\mathrm{B} = e\mathrm{A} \mp \frac{aa}{3bb}y^3$.

Supra vero erat $\frac{aa}{bb}d\mathrm{D}(= \frac{aa}{bb}y^3dx) = \left(\pm aa \mp ee\right).d\mathrm{A} - \frac{2aae}{bb}yydy \mp d\mathrm{C}$.

Et integrando reperietur $\frac{aa}{bb}\mathrm{D} = \left(\pm aa \mp ee\right).\mathrm{A} - \frac{2aae}{3bb}y^3 \mp \mathrm{C}$ (vel substituendo hujus C valorem inventum) $= \pm\frac{1}{4}aa\mathrm{A} - \frac{aae}{4bb}xy^3$. Atque adeo multiplicando æquationem per $\frac{bb}{aa}$, invenietur $D = \pm\frac{1}{4}bb\mathrm{A} - \frac{1}{4}ey^3 - \frac{1}{4}xy^3$.

In hisce formulis omnibus A significat aream ABC, unde, si valores reperti literarum ; B, C, D substituantur in duabus primis æquationibus posterioris tabellæ, habebitur valor incognitæ *t* pro omni sectione conica in planum & in latus agitata tam in ære vel in vacuo, quam intra quemlibet liquorem; si in vacuo, erit $g = \beta$, & si in aliquo liquore, cuius gravitas specifica sit ad gravitatem specificam figuræ oscillantis, ad 1 ad *n*, erit ut 1 ad *n,* erit $\beta = \frac{n}{n-1}g$.

Ex hisce repertis facili negotio deduci possunt omnia, quæ Celeb. Jac. Bernouli tribus tabellis complexus est in Actis Acad. Reg. Paris. Scient. 1703 ad 1. Dec. cui propterea non diutius immorabor, nec etiam ostendere necessum duco, quomodo in sectionibus conicis valores literarum I, K, L, inveniri debeant, cum hæc res ne tyronibus quidem negotium facessere possit, quandoquidem *y* in harum quantitatum elementis ubique ad duas dimensiones ascendit, adeo ut quantitates inde resultent, quæ absque ulla alia reductione integrabiles existant.