PART FIVE OF THE ELLIPSE

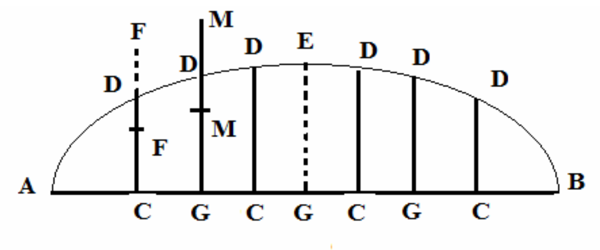
Showing the production of various kinds of ellipses.

PROPOSITION CXLVIII.

AB shall be a line divided in some manner at C, and from C some number of parallel lines CD shall be erected, so that the rectangle ACB shall become to the rectangle ACB thus as the square DC to the square DC.

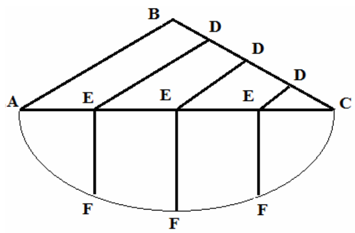
I say the points ADB to be the same for the ellipse or the circle.

Demonstration.



With AB bisected at G, GE shall be erected parallel to CD, so that the rectangle ACB shall become to the rectangle AGB, thus as the square CD to the square GE, thence it is understood the ellipses to be described the conjugate diameters of which shall be AG, GE : therefore if the ellipse does not pass through the point D, it may pass through the right line CD either above or below D at F so that AG, EG shall be conjugate diameters ; the lines DC will be put parallel to the ordinate EG for the diameter AB. Whereby since the ellipse may be said to pass through the point F, the square FC shall be to the square EG as the rectangle ACB to the rectangle AGB, that is, from the construction, to be as the square DC to the square EG: Which is absurd. Therefore there is no other point F either above or below D, which lies on the ellipse, so that the point D itself lies on the ellipse. Now some other point D may be taken D, for the sake of an example the following closer point ; again if the ellipse does not pass through D, it may pass through the right line CD at M, either above or below D. Because from the hypothesis, the square DE is to the square DE, thus as the rectangle ACB to the rectangle ACB, and from the construction, so that the square DE is to the square EG, thus the rectangle ACB to the rectangle ACB, from the equality it would arise that not only the square DE to the square EG, thus the rectangle ACB to the rectangle AGB: but also the square MC to the square EG, as the rectangle ACB to the rectangle AGB since the point M shall be placed on the ellipse. Therefore the squares DC, MC have the same ratio to the square EG. Which is absurd: Therefore no point M nor any other point besides D lies on the ellipse; we will be able to show by a similar discussion the remaining points D to lie on the ellipse. From which the truth of the theorem may be agreed on.

PROPOSITION CXLIX.



ABC shall be some triangle, and with the side BC divided in some manner at the points DD: from the points D the right lines DE may be drawn parallel to AB, and from A the lines EF may be dropped so that the squares EF shall be equal to the rectangles BDE.

I say the points A, F, C to lie on the same ellipse.

Demonstration.

As the rectangle BDE to the rectangle BDE, thus the rectangle BDC is to the rectangle BDC; that is, the rectangle AEC to the rectangle AEC: but as the rectangle BDE is to the rectangle BDE, thus the square EF is to the square EF; therefore as the rectangle AEC is to the rectangle AEC, thus the square EF is to the square EF. Therefore the points AFC lie on an ellipse.

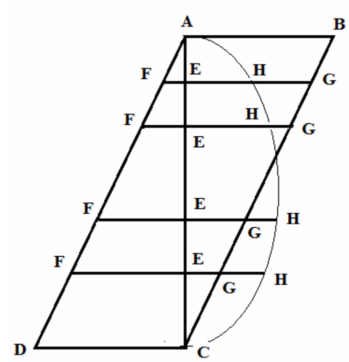
PROPOSITION CL

AC shall be the diameter of the parallelogram AB, CD; lines of some size FG may be drawn parallel to the side AB, cutting the line AC at E : then the [geometric] means EH between FE, EG may be put in place.

I say the points A, C, and all the points H to lie on the same ellipse, unless they shall lie on a circle.

Demonstration.

The ratio of the rectangle FEG to the rectangle FEG is composed from the ratio FE to FE, that is AE to AE; and from the ratio EG to EG, that is EC to EC: but from the same the ratio is composed of the rectangle AEC to the rectangle AEC, therefore as the rectangle FEG to the rectangle FEG; that is as the square EH to the square EH, thus the rectangle AEC to the rectangle AEC. Whereby the points AH, HC lie on an ellipse. Q.e.d.

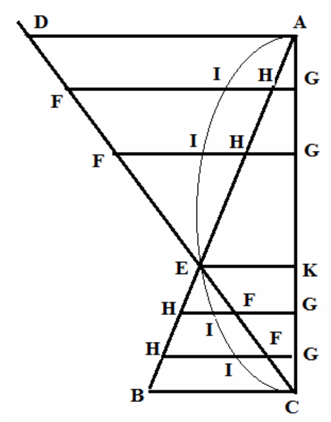


Corollary.

If the right line AB shall be normal to AC, and were equal to that, the points AHH lie on the same circle; for the rectangle AEC will be equal to the rectangle FEG, that is for the square EH; and thus the points HH lie on a circle.

PROPOSITION CLI.

Any two lines AB, CD shall cut each other at E which shall connect the two parallel lines AD, BC; likewise the points AC may be joined together : then the right lines FG may be drawn parallel to the lines AD, BC, meeting the line AB at HH, and the rectangles HGF may be made equal to the squares GI.
I say the points I, I, I to lie on an ellipse.



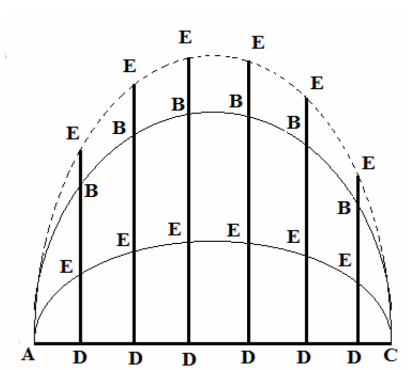
Demonstration.

The ratio of the rectangle HGF to the rectangle HGF is composed from the ratio HG ad HG, that is AG to AG, and from the ratio FG to FG, that is GC to GC : but the ratio of the rectangles AGC to AGC is composed from the same, therefore as the rectangle HGF is to the rectangle HGF, that is the square IG to the square IG, thus the rectangle AGC to the rectangle AGC: whereby the points I, I lie on an ellipse. Q.e.d.

Corollary.

The right line EK may be drawn parallel to the line AD from the point of intersection E : if AK, EK, KC will have been continued, and the line EK normal to the right line AC, I say the points I, I, to lie on a circle where indeed it shall be so that as the rectangle AKC to the rectangle AGC, thus the square EK to the square IG: (that indeed we have proved by the same discussion, where we have shown the rectangle AGK to be to the rectangle AGK as the square GI to the square GI); on interchanging will become: as the rectangle AKC to the square EK, thus the rectangle AGC to the square GI. And thus the square IG is equal to the rectangle AGC. Therefore I, I lie on a circle.

[It is of course a trivial exercise to prove this theorem for two points on the standard ellipse using coordinate geometry.]



PROPOSITION CLII.

AC shall be the diameter of the semicircle ABC, divided in some manner at D D, and from D the normals DE may be erected, and they shall be produced so that as BD to BD, there shall be ED to ED. I say the points E, E to lie on the same ellipse.

Demonstration.

As the square DB is to the square DB, thus as the square ED is to the square ED: but as the square BD is to the square BD thus the rectangle ADC is to the rectangle ADC:

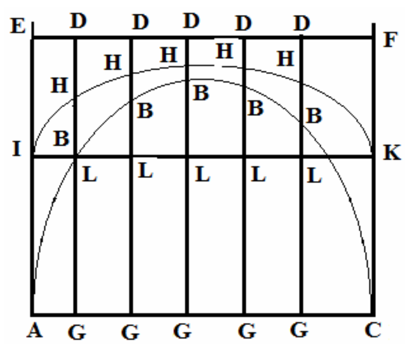
therefore as the square ED is to the square ED, thus the rectangle ADC is to the rectangle ADC.

Whereby the points E, E lie on an ellipse. Q.e.d.

PROPOSITIO CLIII.

A rectangle AF shall be described on the semicircle ABC with the diameter AC : and the lines DG may drawn parallel to the side AE which cross the circle at BB, some line IK shall be drawn parallel to the right line ED crossing the lines DG at LL; and there shall become, as AI to IE thus BH to HD.

I say the points HH lie on the same ellipse.



Demonstration.

As AE to AI, that is GD to LD, thus BD is to DH, therefore by interchanging and by dividing, and by interchanging again so that as GB to LH, thus BD to DH, and thus BD is to HD, as HD to HD, for both the ratios BD to HD, are of the same ratio AE to IE; whereby as GB to LH, thus GB to LH: and by interchanging as GB to GB, thus LH to LH: and as the square GB to the square GB, thus the square LH to the square LH: but as the square GB to the square GB thus the rectangle AGC to the rectangle AGC, that is, the rectangle IIK to the rectangle IIK; therefore as the square LH is to the square LH, thus the rectangle IIK is to the rectangle IIK : whereby the points H, H lie on an ellipse. Q.e.d.

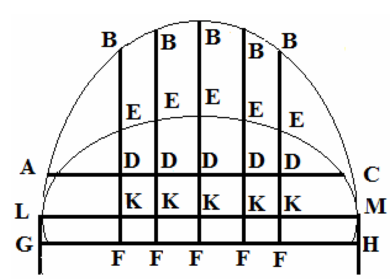
PROPOSITION CLIV.

ABC shall be a segment of some circle, subtended below from which from AC, divided in some manner at DD, the normals DB may be erected from D, and it shall become so that as BD to BD, thus ED to ED.

I say the points E, E to lie on an ellipse.

Demonstration.

With the semicircle ABC completed : the diameter GH of the circle GBH way be drawn parallel to the line AC, which lines BD produced may cut GH at FF: and it shall be made so that as BD to DF, thus ED ad DK: then from G and H the right lines GL, HM may be erected parallel to the lines BF cutting the line KK at L and M. Since as BD is to DF, thus DE is to DK, on interchanging there



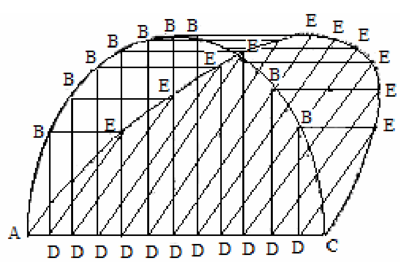
will become, as BD to ED, thus DF to DK. And thus BD is to DE, as BD to DE. Therefore as DF to DK, thus DF to DK : whereby the points K, K lie on the same line, and indeed are parallel to the line GH. Again since BD shall be to DK, therefore as ED to DK, on adding together and on interchanging, BF shall be to EK, as DF to DK. Therefore as BF to EK, and again as interchanging, as BF to BF, thus EK to EK, and as the square BF to the square BF, the square EK shall be to the square EK : but as the square BF shall be to the square BF, the rectangle HFG is to the rectangle HFG, that is the rectangle MKL to the rectangle MKL, therefore as the rectangle MKL shall be to the rectangle MKL, the square EK shall be to the square EK. Whereby the points E, E lie on an ellipse. Q.e.d.

PROPOSITIO CLV.

Let AD be the diameter of the semicircle ABC, which however many normals may cut BC at D, then with the right lines BD may become equal to BE parallel to the diameter AC.

I say the points E, E to belong to the ellipse, the diameter of which AC.

Demonstration.

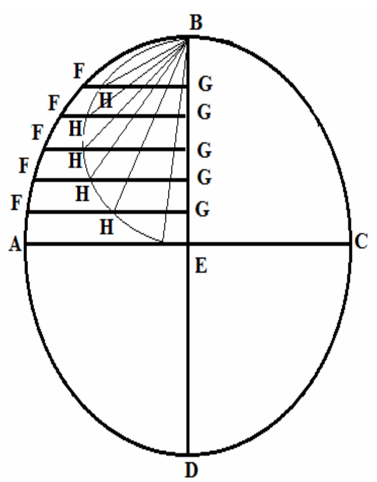


The right lines DE shall be drawn: because the angles BDC are right, and the lines BE are parallel, the angles DBE are right also; therefore the squares DE are equal to the squares BD, BE, that is, since BD, BE are equal, they are twice the squares BD. Therefore as the square BD to the square BD, that is, as the rectangle ADC to the rectangle ADC, thus the square DE to the square DE. Truly the right lines DE are parallel to each other : since indeed the angles DBE shall be right; and the sides BD, BE equal *;* BDE will be semi-right angles. Therefore since also BDC shall be right, the remaining angles EDC are semi-right, and thus equal: hence the lines DE are parallel. Therefore the points E, E lie on an ellipse. Moreover since AC shall be the diameter, it will be readily apparent by the same construction with a complete circle a complete ellipse for the other part as well shall be produced, then indeed all the parallel lines DE will be bisected by the right line AC.

PROPOSITION CLVI.

The diameters AC, BD shall cut the circle ABC at right angles, and with the right lines FG which shall be parallel to the diameter AC, the right lines may be dropped from B equal to the right lines BH cutting the lines FG in HH.

I say the points B, H, E to lie on the same ellipse.



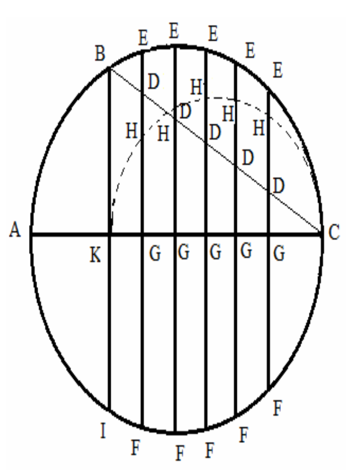
Demonstration.

Since the square FG is equal to the rectangle BGD, that [in turn] is equal to the rectangle BGE taken twice, together with the square BG; but the square HB is equal to the squares HG, BG: therefore with the common square BG taken away the square HG remains, equal to the rectangle BGE taken twice; similarly the remaining squares HG are equal to double of the remaining rectangles BGE; therefore so that as the square HG shall be to the square HG: the rectangle BGE is to the rectangle BGE; whereby the points B,E, and all the points H, lie on the same ellipse. Q.e.d.

PROPOSITIO CLVII.

Let AC be the diameter of the circle ABC and from C with some lines CB drawn crossing the perimeter of the circle at B, then with the right line BI dropped from B which will cut the diameter AC at right angles at K. Some number of lines EF may be drawn parallel to the line BI crossing the diameter AC at G, and the line BC in D: and the rectangles EDF shall be made equal to the squares GH.

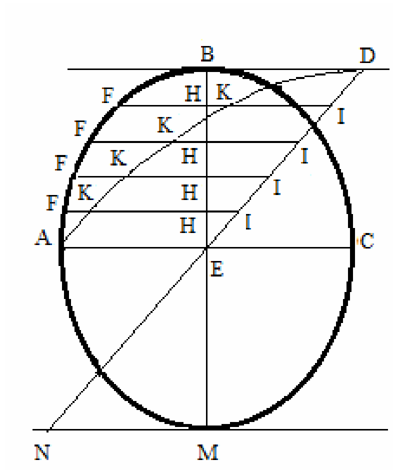
I say the points KHC to belong to the same ellipse.



Demonstration.

As the rectangle EDF is to the rectangle EDF, thus the rectangle BDC is to the rectangle BDC, that is, the rectangle KGC to the rectangle KGC: but (just as becomes apparent from the hypothesis in turn), as the rectangle EDF to the rectangle EDF, thus the square HG is to the square HG: therefore as the rectangle KGC is to the rectangle KGC, thus the square HG to the square HG; whereby the points KHC pertain to an ellipse. Q.e.d.

PROPOSITION CLVIII.



The diameters AC, BE shall cut the circle ABC orthogonally, and with the tangent BD acting at the point B some right line ED shall be drawn through the centre E meeting the tangent BD at some point D. Then the right lines FHI may be drawn, parallel to the tangent BD, meeting the diameter EB at HH, and the line ED at I, I : and there becomes FH to FH, thus as IK to IK.

I say the points AKD to be for the same ellipse.

Demonstration.

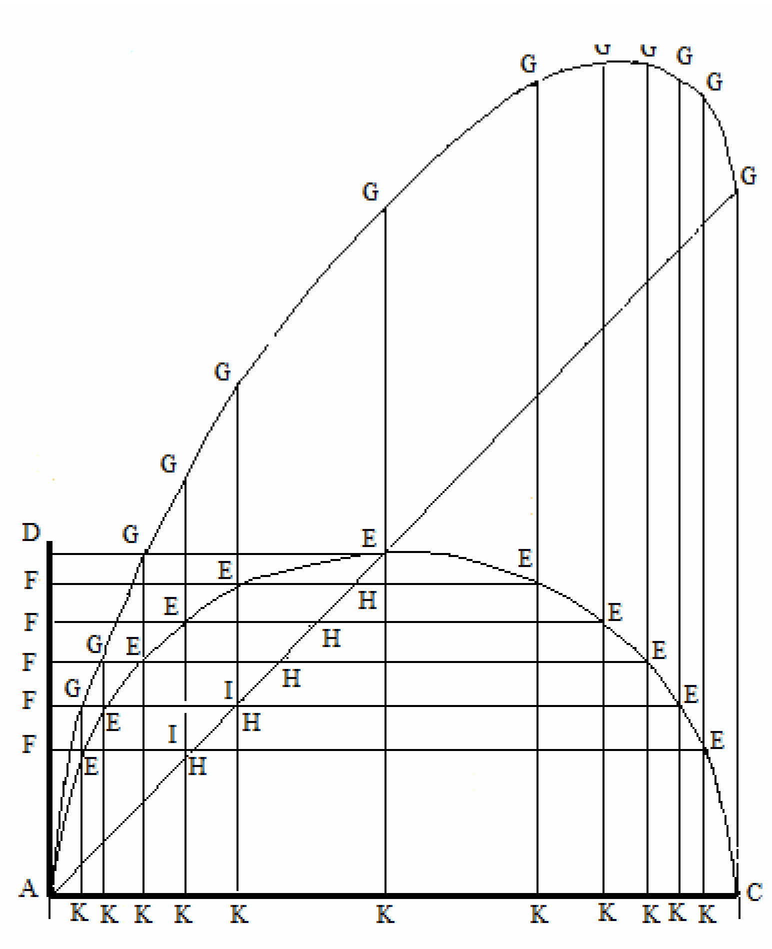
The diameter BE shall be produced to M; and the line DE shall be produced until it may meet the tangent acting through M at N. Since BD, NM, HI shall be parallel lines, so that the rectangle BHM shall be to the rectangle BHM, thus as the rectangle DIN shall be to the rectangle DIN ; but as the rectangle BHM shall be to the rectangle BHM, thus the square FH shall be to the square FH, that is the square IK to the square IK: therefore as the rectangle DIN shall be to the rectangle DIN, thus the square IK shall be to the square IK. Whereby the points AKD are for an ellipse. Q.e.d.

PROPOSITION CLIX.

The two lines AD, BD intersecting each other at right angles at D, shall be tangents to the circle ABC, the diameter of which is AC: and with the points AB joined, and with the tangent CL acting through C: and with the points AB joined, the tangent CL shall be acting through C, crossing the line AB at L; then some lines FE may be drawn parallel to the line DB crossing the line AB at HH, and the circle at EE; then the right lines GK shall be drawn through E parallel to the line AD crossing the diameter AC at K and the line AL at I I. And all the lines FE equal to EG.

I say the points AGL to be for an ellipse.

Demonstration.

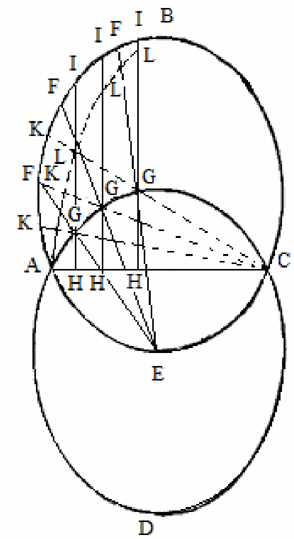


As AD is to DB, thus AF is to FH: but AD, DB are equal lines, and therefore AF, FH shall be equal lines ; and whereby EK, FH shall be equal lines. Again, since AF shall be to FH, thus as EI to EH, the lines EI, EH are equal to each other: moreover from the construction the lines FE are equal to the lines EG: therefore the whole length IG, is equal to the whole length FH, that is FA is equal to EK; whereby as the square EK to the square EK, thus the square IG is to the square IG: but as the square EK is to the square EK, thus the rectangle ACK is to the rectangle ACK, that is the rectangle AIL to the rectangle AIL; therefore as the square IG is to the square IG, thus the rectangle AIL is to the rectangle AIL. Whereby the points AGL are for an ellipse. Q.e.d.

So that if the same construction may be continued to the other part, this part of the ellipse also will have been completed, which will fall within the circle. Whereby it is noteworthy to be observed from the hypothesis, that the circle and the ellipse themselves in turn may be allowed to be cut by the same lines GK, still the same right line DA to be a tangent at the same mutual point A for the circle and the same also shall be the tangent for the ellipse, thence it becomes evident that all the points of the perimeter of the ellipse shall be on the lines GK which are drawn parallel to DA between the points C and A.

PROPOSITION CLX.

The two circles ABC, ADC shall cut each other so that the former ABC shall pass through the centre E of the latter ABD, and with these joined together by the points AC, some line EF shall be drawn from the point E meeting the circle ABC at the points F and with the circle ADC at the points G: then through G the right lines HI are acting normal to the line AC, meeting the line AC at H H and the circle ABC in I I: and the right lines GF shall be made equal to the lines GL.
I say the points ALL to lie on an ellipse.

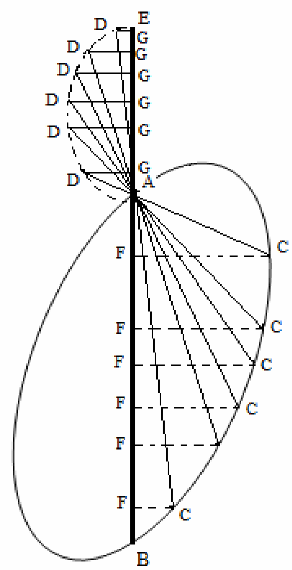


Demonstration.

The lines CGK shall be drawn from C through G, so that as the rectangle KGC shall be to the rectangle KGC, thus the rectangle FGE shall be to the rectangle FGE; but as the rectangle KGC shall be to the rectangle KGC, the line GH shall be to the line GH (§84 Cor., *Circle Book* III), and as the rectangle FGE to the rectangle FGE, thus the line FG to the line FG, therefore so that as GH to GH, thus FG to FG, that is, LG to LG, and on adding together and interchanging as LH to LH, thus GH to GH. Whereby the points ALL or an ellipse.

PROPOSITION CLXI.

AB shall be a diameter of some ellipse ABC, and with the line CD acting through the point A, which crosses over the ellipse at CC, there shall become as AC to AC, thus AD to AD, and as AC to AD, thus AB to AE.
I say the points A, D, E to lie on the same ellipse.



Demonstration.

Since DG, FG are parallel, and hence the triangles FCA, DGA are similar, so that the individual lines CA will be to the individual lines AD one to one in turn, thus so that the individual FC will be to the individual GD. And thus the individual CA to the individual AD are as BA to AE.

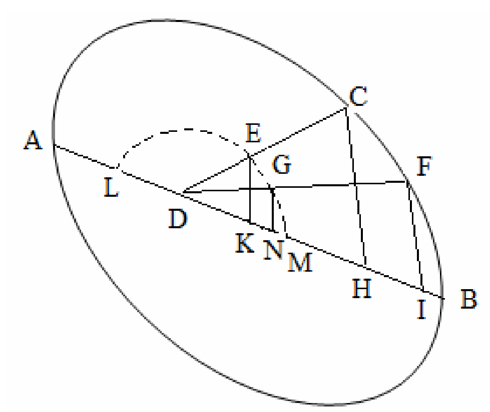
Therefore the individual FC are to the individual DG, as BA to AE. Whereby since the ratio has one particular FC to one particular DG, the remaining individual FG have the same ratio to the individual remaining DG. Therefore on interchanging so that as the FC are to the FC, thus the GD are to the GD, and thus as the squares FC are to the squares FC, thus the squares GD are to the squares GD; similarly we will demonstrate, as the AF are to the AF, there shall be AG to AG. From which as the remaining FB are to the remaining FB, thus the remaining GE are to the remaining GE. Whereby since the rectangles AFB shall have the same ratio between themselves, in turn to be composed from the ratios AF to AF, and FB to FB, which are shown to be in the same ratios AG to AG, and GE to GE, from which the ratio of the rectangles AGE is composed ; so that the rectangles AFB to the rectangle AFB, shall be as the rectangles AGE to the rectangles AGE. And the rectangles AFB are to the rectangles AFB, as the square FC to the square FC; this has been shown above, to be as the square GD to the square GD; therefore the rectangles AGE are to the rectangles AGE, as the squares GD to the squares GD. Therefore the points D A, A D E belong to an ellipse. Q.E.D.

PROPOSITION CLXII.

The diameter of some ellipse ABC shall be AB, divided in some manner at D and from D the right lines DC, DF shall be drawn to the periphery, and which lines shall be divided in some manner at E and G: then AD shall be divided at L, and DB at M, so that DC, DF are divided at E and G.

I say the points L E G M to belong to an ellipse.

Demonstration.

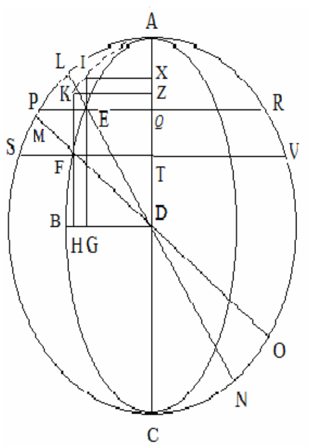


The ordinate lines CH, FI may be drawn from C and F to the diameter AB, from which, from E and G the parallel lines EK, GN may be drawn so that as DC to DE, thus DF to GD, therefore so that as CH to EK, thus FI to GN, and EK to GN, as CH to FI: and therefore the square EK to the square GN, as the square CH to the square FI. Then, since DI is to DN, as DF to DG, and DA to DL, as DF to DG, as there will become DI to DN, there shall be DA to DL, therefore as the preceding one DI to the following one DN, (*i.e.* , as DF to DG, *i.e.* as DC to DE, i.e. as DH to DK) thus both the antecedents, *i.e.* the whole AI to both the following, *i.e.* to the whole LN, similarly we may infer AH to be to IK, as DH to DK. From which AI is to LN, as AH to DK, and on interchanging AI is to AH, as LN to IK. Besides, since as DF is to DG (*i.e.* as the whole DB is to the whole DM) there will be with DH removed to DN removed, the remainder IB to the remainder NM, as the whole DB to the whole DM. Similarly we may deduce HB to KM, to be as DB to DM. Therefore on interchanging IB to NM, and HB as to KM, and by inverting, HB to IB to be as KM ad NM. Therefore since we shall have shown the ratios AH to AI, and AK to AN, likewise the ratios HB to IB, and KM to NM to be the same, also the ratios of the rectangle AHB to the rectangle, and of the rectangle AIB to the rectangle AKB, composed from these equal ratios, will be the same; but the rectangle AHB is to the rectangle AIB, as the square CH to the square FI; that is by the above demonstration as the square EK to the square GN. Therefore the rectangle IKM is to the rectangle LNM, as the square EK to the square GN. Therefore the points L, E, G, M lie on an ellipse. Q.e.d.

PROPOSITION CLXIII.

The axes of the ellipse ABC shall be AC, BD: and with several radii DE, DF drawn from D, the lines IG, KH may act through E and F equal to ED, FD themselves, truly parallel to the axis AC, crossing the axis BD at G and H.
I say the points AIK to be for the ellipse of this axis.

Demonstration.



The circle ALC shall be described on AC as diameter, and the lines DE, DF may be produced in each way until the cross the perimeter of the circle at L, M, N, O: and with the lines PER, SFV acting through E and F, which cross the circle at R, V and cross the diameter AC at Q and T, and will be parallel to the axis BD, the ordinate lines IX, KZ may be drawn for the axis AC. Since the lines PER, SFV are divided in proportion at E and F, the ratio of the rectangle PER to the rectangle SFV is twice the ratio of PE to SF, and thus the rectangle PER to the rectangle SFV shall be as the square PE to the square SF, that is, as the square EQ to the square FT, that is as the square IX to the square KZ. Whereby since the rectangles LEN, PFO shall be equal to the rectangles PER, SFV, also the rectangle LEN is to the rectangle MFO, as the square IX to the square KZ: then since IG, that is, XD is equal to ED, and DC is equal to DN, XC will be equal to EN; and truly the whole AC is equal to the whole LN. Therefore the remainder AX is equal to the remainder LE : and thus the rectangle AXC equal to the rectangle LEN*:* it may be shown in the same manner that the rectangle AZC [§148] to be equal to the rectangle MFO, therefore there will be as the rectangle AXC to the rectangle AZC, thus the square IX to the square KZ. Whereby AIKC shall be points on an ellipse. Q.e.d.

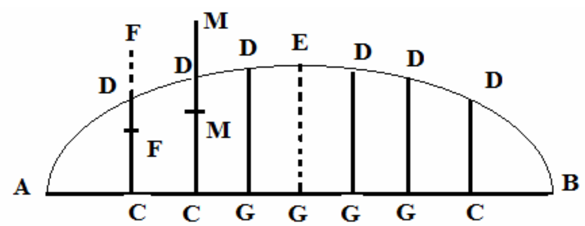
ELLIPSEOS PARS QUINTA

Varias exhibet ellipsis geneses.

PROPOSITIO CXLVIII.

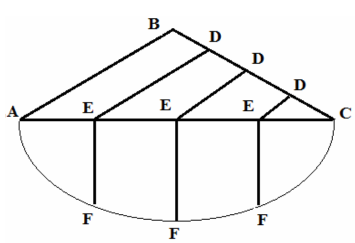
Sit AB linea utcunque divisa in C, & ex C quotcunque erigantur parallelae CD, fiatque ut ACB rectangulum ad rectangulum ACB sic DC quadratum ad quadratum DC.
Dico ADB puncta esse ad eandem ellipsim vel circulum.

Demonstratio.



AB bisecta in G, erigatur GE parallela ipsis CD, fiatque ut rectangulum ACB ad rectangulum ACB, sic quadratum CD ad quadratum GE, intelligatur deinde descripta esse ellipsis cuius diametri coniugatae sint AG, GE si ergo ellipsis non transit per punctum D, occurrat rectae CD supra vel infra D in F quia AG, EG sunt diametri coniugatae; erunt DC ipsi EG parallelae ordinatim positae ad diametrum AB. Quare cum ellipsis dicatur transire per F, erit quadratum FC ad quadratum EG ut rectangulum ACB ad rectangulum AGB, hoc est ex constr. ut quadratum DC ad quadratum EG: Quod est absurdum. Non igitur punctum ullum F supra aut infra D, est ad ellipsim, sed ipsum D punctum ad ellipsim est. Sumatur iam aliud quodlibet punctum D, exempli gratia punctum propius sequens; si rursum ellipsis non transit per D, occurrat rectae CD supra vel infra D in M. Quoniam est ex hypothesi ut quadratum DE ad quadratum DE, sic rectangulum ACB ad rectangulum ACB, & ex constr. ut quadratum DE ad quadratum EG, ita rectangulum ACB ad rectangulum ACB, erit ex aequali ut quadratum DE ad quadratum EG, ita ACB rectangulum ad rectangulum AGB: sed etiam est quadratum MC ad quadratum EG, ut rectangulum ACB ad rectangulum AGB cum punctum M ponatur esse ad ellipsim. Ergo quadrata DC, MC ad quadratum EG, eandem habent rationem. Quod est absurdum: Non igitur punctum M aut aliud ullum praeter D ad ellipsim est; simili discursu reliqua puncta D ad ellipsim esse demonstrabimus. Ex quibus constat veritas Theorematis.

PROPOSITIO CXLIX.



Sit ABC triangulum quodcunque, divisoque latere BC utcunque in punctis DD: ducantur ex D rectae DE parallelae AB, & ex A demittantur lineae EF sit ut quadrata EF,sint BDE rectangulis aequalia.
Dico puncta A, F, C esse ad eandem ellipsim.

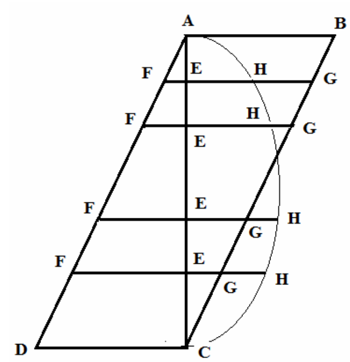
Demonstratio.

Ut BDE rectangulum ad rectangulum BDE, sic BDC, rectangulum est ad rectangulum BDC, hoc est rectangulum AEC ad rectangulum AEC: sed ut BDE rectangulum est ad rectangulum BDE, sic EF quadratum est ad quadratum EF, igitur ut rectangulum AEC est ad rectangulum AEC, sic EF quaratum est ad quadratum EF. Ergo AFC, puncta sunt ad ellipsim.

PROPOSITIO CL

Sit AB, CD parallelogrammi diameter AC, ducanturque; AB lateri quotcumque parallelae FG, secantes AC lineam in E : dein fiant inter FE, EG mediae E H.
Dico puncta A, C, & omnia puncta H esse ad eandem ellipsim, nisi sint ad circulum.

Demonstratio.

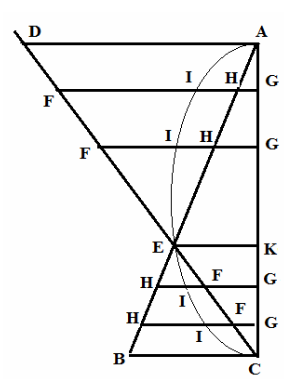


Ratio FEG rectanguli ad rectangulum FEG est composita ex ratione FE ad FE, id est AE ad AE; & ex EG ad EG, id est EC ad EC: sed ex iisdem componitur ratio rectanguli AEC ad AEC rectangulum, igitur ut FEG rectangulum ad rectangulum FEG; hoc est quadratum EH ad quadratum EH, sic AEC rectangulum est ad rectangulum AEC. Quare AH, HC puncta sunt ad ellipsim. Quod erat demonstrandum.

Corollarium.

Si AB recta sit normalis ad AC; & illi fuerit aequalis, erunt AHH puncta ad eundem circulum; erit enim AEC rectangulum aequale rectangulo FEG, hoc est quadrato EH. adeoque puncta HH ad circulum.

PROPOSITIO CLI.



Secent se in E duae quaevis lineae AB, CD quas coniungant duae parallelae AD, BC; iungantur item puncta AC: tum rectae ducantur FG parallelae lineis AD, BC, occurrentes AB lineae in HH, fiantque; HGF, rectangulis equalia quadrata GI.
Dico puncta I, I, I esse ad ellipsim.

Demonstratio.

Ratio HGF rectanguli ad rectangulum HGF est composita ex ratione HG ad HG, id est

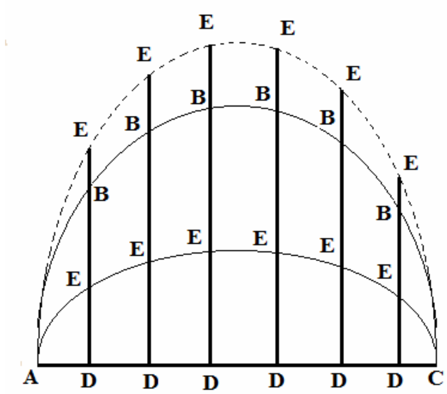
AG ad AG, &ex ratione FG ad FG, id est GC ad GC : sed ex iisdem est composita ratio rectanguli AGC ad AGC, igitur ut HGF rectangulum est ad rectangulum HGF, hoc est quadratum IG ad quadratum IG, sic AGC rectangulum ad rectangulum AGC: quare I, I puncta sunt ad ellipsim. Quod erat demonstrandum.

Corollarium.

Ducatur ex E puncto intersectionis recta EK parallela lineae AD: si AK, EK, KC fuerint continuae, & EK linea normalis ad rectam AC, dico I, I, puncta esse ad circulam cum enim sit ut AKC rectangulum ad AGC rectangulum, sic EK quadratum ad quadratum IG: (id enim eodem discursu probabimus, quo rectangula AGK ostendimus esse ad rectangula AGK ut quadrata GI ad quadrata GI) erit permutando ut AKC rectangulum ad quadratum EK sic AGC rectangulum ad quadratum GI. Adeoque quadratum IG aequale rectangulo AGC. Igitur I,I sunt ad circulum.

PROPOSITIO CLII.

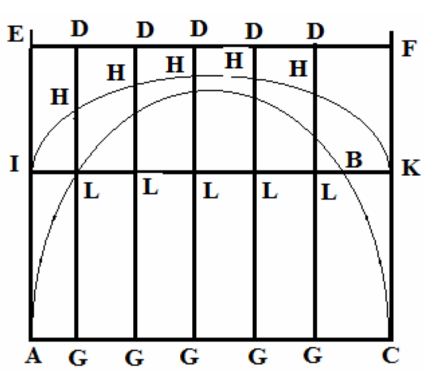
Sit ABC semicirculi diameter AC, divisa utcunque in DD, & ex D normales erigantur DE, fiantque ut BD ad BD, sit ED ad ED.
Dico puncta E, E esse ad eandem ellipsim.



Demonstratio.

Ut quadratum DB ad quadratum DB, sic ED quadratum est ad quadratum ED: sed ut quadratum BD ad quadratum BD sic ADC rectangulum est ad rectangulum ADC:
igitur ut quadratum ED ad quadratum ED sic ADC rectangulum est ad rectangulum ADC.
Quare E,E puncta sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLIII.



Super ABC semicirculi diametro AC rectangulum describatur AF: ductisque lineis DG parallelis lateri AE quae circulo occurrant in BB, ducatur qaaevis IK parallela rectae ED occurrens DG lineis in LL; fiatque ut AI ad IE sic BH ad HD.
Dico puncta HH esse ad eandem ellipsim.

Demonstratio.

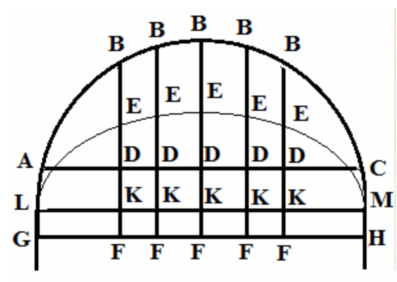
Ut AE ad AI, hoc est GD ad LD, sic BD est ad DH, igitur permutado dividendo iterumque permutando ut GB ad LH, sic BD ad DH, atqui BD est ad HD, ut HD ad HD, sunt enim ambae rationes BD ad HD, eadem rationi AE ad IE; quare ut GB ad LH, sic GB ad LH: & permutando ut GB ad GB, sic LH ad LH: & ut quadratum GB ad quadratum GB , sic LH quadratum ad quadratum LH: est autem ut quadratum GB ad quadratum GB sic AGC rectangulum ad rectangulum AGC, id est IIK, rectangulum ad rectangulum IIK, igitur ut LH, quadratum est ad quadratum LH, sic IIK rectangulum est ad rectangulum IIK : quare H, H puncta sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLIV.

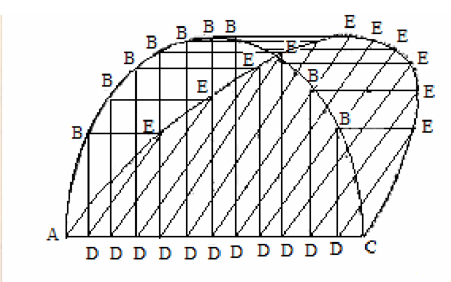
Sit ABC segmentum circuli quodcunque, cuius AC subtensa, divisa utcunque in DD, erigantur ex D normales DB, fiatque ut BD ad BD sic ED ad ED.
Dico puncta E,E esse ad ellipsim.

Demonstratio.

Perfecto semicirculo ABC: ducatur GH diameter circuli GBH parallela lineae AC, quae BD, lineas productas secet in FF: fiatque ut BD ad DF, sic ED ad DK: tum ex G & H rectae erigantur GL, HM parallelae lineis BF secantes KK lineam in L & M, Quoniam est ut BD ad DF, sic DE ad DK, erit permutando, ut BD ad ED, sic DF ad DK. Atqui BD est ad DE, ut BD ad DE. igitur ut DF ad DK, sic DF ad DK : quare puncta K, K ad eandem lineam, & quidem parallelam lineae GH. Rursum cum sit ut BD ad DK, igitur ut ED ad DK, erit componendo & permutando BF ad EK, ut DF ad DK. igitur ut BF ad



ad EK, & rursum permutando, ut BF ad BF, sic EK ad EK, & ut quadratum BF ad quadratum BF ad quadratum BF, sit EK, quadratum ad quadratum EK : sed ut BF quadratum ad quadratum BF, sit HFG rectangulum est ad rectangulum HFG, id est MKL rectangulum ad rectangulum MKL, igitur ut MKL rectangulum ad rectangulum MKL sit quadratum EK ad quadratum EK. Quare E, E puncta sunt ad ellipsim. Quod erat demonstrandum.



PROPOSITIO CLV.

Esto ABC semicirculi diameter AD, quam in D secent quotcunque normales BC dein rectis BD fiant aequales BE parallelae diametro AC.
Dico puncta E, E esse ad ellipsim cuius diameter est AC.

Demonstratio.

Ducantur rectae DE: quoniam anguli BDC recti sunt, & BE parallelae, anguli quoque DBE erunt rectiquadrata igitur DE, aequantur quadratis BD, BE, hoc est, quia BD, BE sunt aequales, dupla sunt quadratarum BD. Ergo ut quadratum BD ad quadratum BD, hoc est: ut rectangulam ADC ad rectangulum ADC, ita quadratum DE ad quadratum DE. Sunt vero & DE rectae inter se parallelae: cum enim anguli DBE recti sint; & latera BD, BE aequalia *;* erunt BDE semirecti. cum ergo etiam BDC rectus sit, reliqui EDC sunt semirecti, adeoque aequales: unde DE parallelae. Puncta igitur E, E sunt ad ellipsim. Quod autem AC sit diameter, facile apparebit si perfecto circulo ellipsis eadem constructione ad partem alteram tam producatur, tunc enim parallelae omnes DE a recta AC bifariam dividentur.

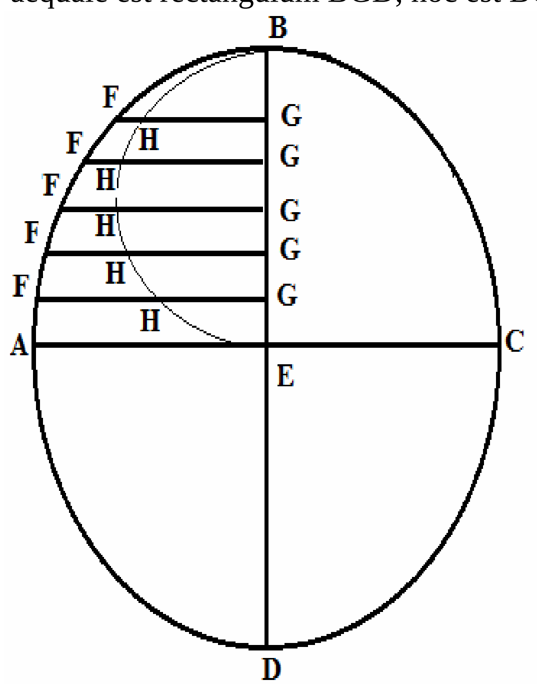
PROPOSITIO CLVI.

Circulum ABC secent ad angulos rectos diametri AC, BD ductisque rectis FG quae AC, diametro aequidistent, demittantur ex B lineae BH aequales rectis FG secantes FG lineas in HH.

Dico puncta B, H, E esse ad eandem ellipsim.

Demonstratio.

Quoniam FG quadrato aequale est rectangulum BGD, hoc est BGE, a rectangulum bis

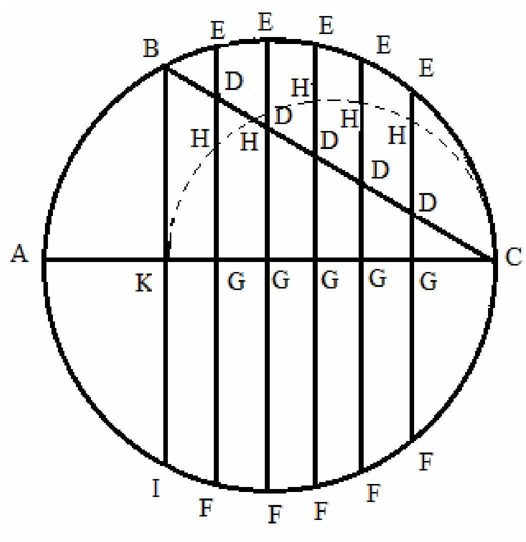


sumptum una cum quadrato BG, erit & quadratum HB aequale rectangulo BGE bis sumpto una cum quadrato BC: sed HB quadratum est aequale quadratis HG, BG: ablato igitur communi quodrato BG manet HG, quadratum aequale rectangulo BGE bis sumpto; similiter reliqua quadrata HG dupla sunt rectangulorum BGE; igitur ut quadratum HG ad quadratum HG: sit BGE rectangulum est ad rectangulum BGE quare puncta B,E, & omnia puncta H, ad eandem sunt ellipsim. Quod erat demonstrandum.

PROPOSITIO CLVII.

Esto ABC circuli diameter AC & ex C recta quaevis ducta CB occurrant circuli perimetro in B, dein ex B demissa recta BI quae AC, diametrum ad rectos angulos secet in K. Ducantur quotcunque lineae EF parallelae rectae BI occurrentes AC diametro in G, & lineae BC in D: fiantque EDF rectangula aequalia quadratis GH.

Dico KHC puncta esse ad eandem ellipsim.



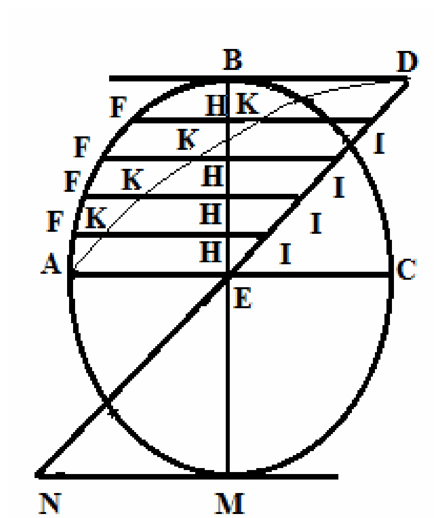
Demonstratio.

Ut EDF rectangulum ad rectangulum EDF, sic BDC rectangulum est ad rectangulum BDE, id est rectangulum KGC ad rectangulum KGC: sed (quemadmodum alternando patet ex hypothesi) ut EDF rectangulum ad rectangulum EDF, sit HG quadratum est ad quadratum HG: igitur ut KGC rectangulum est ad rectangulum KGC, sit H G quadratum est ad quadratum HG. quare HGC puncta sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLVIII.

Secent ABC circulum ortogonaliter diametri AC, BE actaq; per B tangente BD ducatur per E centrum recta quovis ED, occurrens tangenti BD in puncto quovis D. Dein rectae ducantur FHI, parallelae tangenti BD, occurrentes EB diametro in HH, & ED lineae in I, I :fiatque ut FH ad FH, sit IK ad IK.

Dico AKD puncta esse ad eandem ellipsim.



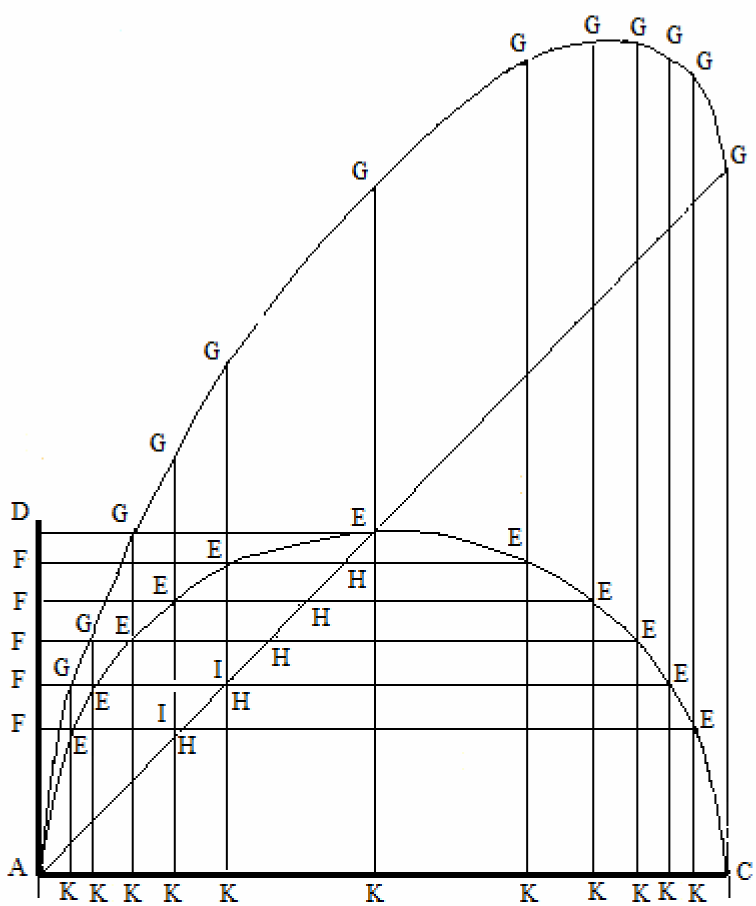
Demonstratio.

Producta BE diametro in M; producatur & DE linea donec actae per M tangenti occurrat in N. Quoniam BD, NM, HI lineae aequidistant, erit ut rectangulum BHM ad rectangulum BHM, sic DIN rectangulum ad rectangulum DIN ;sed est ut BHM rectangulum ad rectangulum BHM , sit FH quadratum ad quadratum FH, id est quadratum IK ad quadratum IK: igitur ut DIN rectangulum ad rectangulum DIN, sic est quadratum IK ad quadratum I K. Quare AKD puncta sunt ad ellipsin. Quod erat demonstrandum.

PROPOSITIO CLIX.

Circulum ABC cuius diameter AC contingant duae lineae AD, BD secantes sese ortogonaliter in D: iunctisque punctis AB, agatur per C tangens sese ortogonaliter in D: iunctisque punctis AB, agatur per C tangens CL, occurrens AB lineae in L; dein rectae ducantur quotqunque FE parallelae lineae DB occurrentes AB lineae in HH, & circulo in EE; tum per E rectae ducantur GK parallelae lineae AD occurrentes AC diametro in K & AL lineae in II. Fiantque FE lineis aequales EG.

Dico puncta ALL esse ad ellipsim.



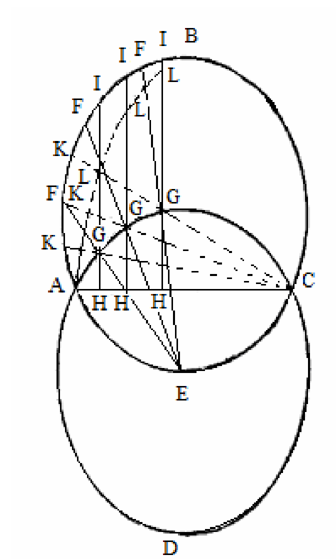
Demonstratio.

Ut AD est ad DB, sic AF est ad FH: sed AD, DB lineae sunt aequales, igitur & AF, FH lineae aequantur; quare & EK, FH lineae aequantur ; Rursum cum sit ut AF ad FH, sic EI ad EI ad EH, erunt EI, EH lineae inter se aequales: est autem ex constructione FE lineae aequalis lineae EG: igitur tota IG, est aequalis tota FH, hoc est FA id est EK; quare ut quadratum EK ad quadratum EK, sic IG est ad quadratum IG: sed ut EK quadratum est ad quadratum EK, sic ACK rectangulum est ad rectangulum ACK, id est AIL rectangulum ad rectangulum AIL, igitur ut quadratum IG est ad quadratum IG, sic AIL rectangulum est ad rectangulum AIL. Quare ALL puncta sunt ad ellipsim. Quod erat demonstrandum.

Quod si eadem constructio ad alteram partem continuetur, perficietur ellipsis altera fui parte, quae intra circulum cadet. Ubi hoc notatu dignum occurrit, quod licet circulus & ellipsis sese invicem secent, eandem tamen rectam DA patet ex hypothesi: quod eandem contingat etiam ellipsis, inde fit manifestum quod omnia perimetri elliptici puncta sint in lineis GK quae inter puncta C & A, ipsi DA ducuntur parallelae.

PROPOSITIO CLX.

Secent se duo circuli ABC, ADC ut illorum alter ABC transeat per E centrum circuli AB, iuncitisque ; punctis AC ducantur ex E lineae quaecunque EF occurrentes circulo ABC in punctis F & ADC circulo ABC in punctis F & ADC circulo in punctis G: tum per G rectae agantur HI normales ad lineam AC, occurrentes AC lineae in HH, & circulo ABC in II: fiantque rectis GF aequales lineae GL.
Dico puncta ALL esse ad ellipsim.



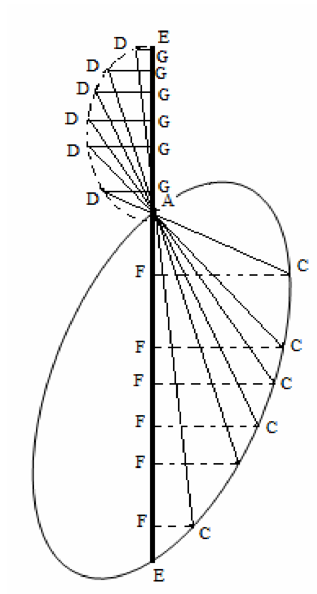
Demonstratio.

Ducantur ex C per G lineae CGK, ut KGC rectangulum est ad rectangulum KGC, sic FGE rectangulum est ad rectangulum FGE sed est & ut KGC rectangulum ad rectangulum KGC, sit GH linea ad lineam GH, & ut FGE rectangulum ad rectangulum FGE, sic FG linea ad lineam FG, igitur ut GH ad GH, sic FG ad FG, id est LG ad LG, & componendo permutando LH ad LH, ut GH ad GH. Quare puncta ALL sunt ad ellipsim.

PROPOSITIO CLXI.

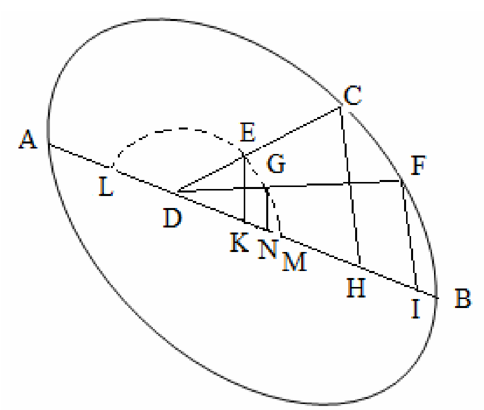
Sit ABC ellipsis diameter quaecunque AB, actisque per A lineis CD, quae ellipsi occurrant in CC, fiat ut AC ad AC, sic AD ad AD, & ut AC ad AD, sic AB ad AE. Dico puncta A, D, E ad eandem ellipsim esse.

Demonstratio.



Quoniam DG, FG sunt parallelae, triangulaque proinde FCA, DGA similia, erunt ut lineae CA ad lineas AD singulae ad singulas, ita singulae FC ad singulas GD. Atqui singulae CA sunt ad singulas AD ut BA ad AE. Ergo singulae FC sunt ad singulas DG, ut BA ad AE. Quare quam rationem habet una FC ad unam DG, eandem habent singulae reliquae FG ad singulas reliquas DG. Igitur permutando ut sunt FC ad FC, ita GD sunt ad GD, adeoque ut sunt quadrata FC ad quadrata FC, ita quadrata GD sunt ad quadrata GD; similiter demonstrabimus, ut AF funt ad AF, sit esse AG ad AG. Unde ut reliquae FB sunt ad reliquas FB, ita reliquae GE sunt ad reliquas GE. Quare cum rectangula AFB rationem habeant ad sese invicem compositam ex rationibus AF ad AF, & FB ad FB, quae ostensae sunt eadem esse rationibus AG ad AG, & GE ad GE , ex quibus componitur ratio rectangulorum AGE; erunt ut rectangula AFB ad rectangula AFB, sit rectangula AGE ad rectangula AGE. Atqui rectangula AFB sunt ad rectangula AFB, ut quadrata FC ad quadrata FC; hoc est per superius demonstrata, ut quadrata GD ad quadrata GD; ergo rectangula AGE sunt ad rectangula AGE, ut quadratae GD ad quadrata GD. Puncta igitur DA, ADB sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLXII.



Esto ABC ellipsis diameter quaevis AB, divisa utcunque in D & ex D ad peripheriam rectae ducantur DC, DF quae proportion aliter dividantur in E & G: dein AD dividatur in L , & DB in M, ut DC, DF divisae sunt in E & G. Dico L EGM puncta esse ad ellipsim.

Demonstratio.

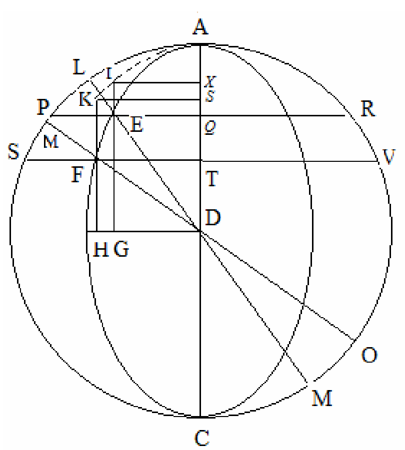
Ducantur ex C & F ordinatim lineae CH, FI ad AB, diametrum quibus ex E & G parallelae ducantur EK, GN ut DE ad DE, sic DF ad GD, igitur ut CH ad EK, sic FI ad GN, & EK ad GN, ut CH ad FI: igitur & quadratum EK ad quadratum GN ,ut CH quadratum ad quadratum FI. Deinde, quoniam DI est ad DN, ut DF ad DG, & DA ad DL,ut DF ad DG, erit ut DI ad DN, sit DA ad DL, ergo ut una anticedens

DI ad unum consequens DN, (hoc est: ut DF ad DG, hoc est ut DC ad DE, hoc est ut DH ad DK) ita ambae antecedentes, hoc est: tota AI ad ambas consequentes, hoc est totam LN, similiter inferemus AH esse ad IK, ut DH ad DK. Unde AI est ad LN, ut AH ad DK, & permutando AI est ad AH, ut LN ad IK. Praeterea, quoniam est ut DF ad DG (hoc est: ut tota DB ad totam DM) sit ablata DH ad ablatam DN, erit & reliqua IB ad reliquam NM, ut tota DB ad totam DM. Similiter inferemus HB esse ad KM, ut DB ad DM. Ergo IB ad NM, ut HB ad KM, permutando igitur ac inuertendo HB ad IB, ut KM ad NM. Cum igitur ostenderimus rationes AH ad AI, & AK ad AN, item rationes HB ad IB, & KM ad NM easdem esse, rationes quoque rectanguli AHB ad rectangulum AIB, & rectanguli AKB ad rectangulum ANB, ex rationibus illis aequalibus compositae, eaedem erunt; sed rectangulum AHB est ad rectangulo AIB, ut quadratum CH ad quadratum FI; hoc est per superius demonstrata ut quadratum EK ad quadratum GN. Ergo rectangulum IKM est ad rectangulum LNM, ut quadratum EK ad quadratum GN. Ergo puncta L, E, G. M sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLXIII.

Sint ABC ellipsis axes AC, BD: ductisque ex D semidiametris quibusvis DE, DF, agantur per E & F lineae IG, KH aequales ipsis ED, FD, parallelae vero axi AC, occurrentes axi BD in G & H.
Dico puncta AIK esse ad ellipsim cuius axis.

Demonstratio.



Super AC ut diametro describatur circulus ALC, & DE, DF lineae utrimque producantur donec circuli perimetro occurrant in L, M, N, O: actisque per E & F, lineis PER, SFV quae circulo occurrant in R, V & AC diametro in Q & T, & aequidistent axi BD, ducantur ordinatim ad axem AC lineae IX,KZ. Quoniam PER, SFV lineae in E & F, proportionaliter sunt divisae, ratio rectanguli PER ad rectangulum SFV duplicata est rationis PE ad SF, adeoque erit PER rectangulum ad rectangulum SFV ut quadratum PE ad quadratum SF, id est ut quadratum EQ ad quadratum FT, id est ut quadratum IX ad quadratum KZ. Quare cum rectangula LEN, PFO aequalia sint rectangulis PER, SFV, etiam LEN rectangulum est ad rectangulum MFO, ut quadratum IX ad quadratum KZ: deinde cum IG hoc est XD aequalis ED. & DC aequalis DN, erit XC aequalis EN; est vero & tota AC aequalis toti LN. Ergo reliqua AX reliquae LE aequalis est : adeoque AXC rectangulum aequale rectangulo LEN*:* eodem modo ostenditur rectangulum AZCa aequari rectangulo MFO, erit igitur ut AXC rectangulum ad rectangulum AZC, sic quadratum IX ad quadratum KZ. Quare AIKC, puncta ad ellipsim. Quod erat demonstrandum.