PART FOUR : THE ELLIPSE

The poles [*or foci*] *of a section* [*of the cone*]*: that shall designate the shortest line from a given point on the axis to a point on the periphery,* [*and back again to the other pole or focus*.]

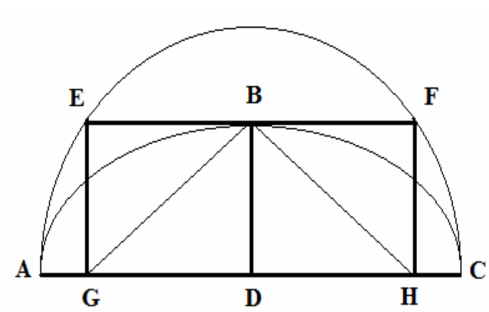
This part, which I am about to undertake, is concerned with the poles, to which we will advance several items to that investigation of the poles, demonstrated by Apollonios, in Propositions 41& 45 of Book Three ; *and indeed it has been necessary to be introduced by this work; since besides the logical development of these which Apollonius has brought together, so that everything is understood clearly, thus more items shall be introduced that I consider necessary. Therefore Apollonius shall show the poles on the axis of the ellipse, use is made of the construction from propositions 45 &46. It is understood to be equal to the fourth part of the figure prepared from each part*; *that is, the axis of the section* AC *may be cut thus by the two points* G & H, *so that both the rectangle* AGC *as well as the rectangle* CHA [*i.e*. AG.GC & CH.HA] *shall be equal to a fourth part of the figure: with which in place further it will be shown* G *and* H *to be the poles of the section: which points he calls forth from the construction made; clearly from the comparison of the rectangles under the segments from the fourth part of the figure. Again Apollonius here calls the figure of the ellipse rectangular since it shall be with a right side under the major axis with a right side under the major, and with the axis itself: and that with its fourth part may serve to be of exceptional use; clearly before the discovery of the individual poles etc., the figure with the remaining right angles he called by the ancient name rectangle: but the fourth part of this is equal to the square of the minor semi-axis: which Pergaeus demonstrated splendidly in Book 3, Prop. 42, and thus we will indicate by a single word.*

Lemma.

From §11, a rectangular figure as if under the major axis and with the right side of this, is equal to the square of the minor axis ; but the fourth part of the square of the minor axis is the square of half the minor axis; therefore the square of half the minor axis is equal to the fourth part of the figure. From which I will call this the fourth part to be used, by which I mean to be understood the square of the semi-minor axis.

In this part some propositions arise concerning focal points, the same as those which Apollonius demonstrated: which I have made here with further deliberations, lest the more studious reader may wish for more on this subject matter.

PROPOSITION CXX.



The axes of the ellipse ABC shall be AC, BD, and with the tangent EF acting at B: with centre D the circle AEFC is described with radius DA, which it may meet the circle touching at E and F. Then the

normals EG, FH may be dropped from E and F to the axes AC.
I say both the rectangle AGC as well as AHC to be equal to the fourth part of the figure.

Demonstration.

Since both the line EB shall be parallel to the line AD as well as EG parallel to BD, the lines EG, BD will be equal : but the rectangle AGC is equal to the square EG, since the right line EG shall be drawn normal to the diameter of the circle ; and therefore the square BD is equal to the rectangle AGC. In the same manner FH, that is the square BD is equal to the rectangle AHC, but the square BD is equal to the fourth part of the figure, therefore both the rectangle AGC as well as the rectangle AHC is equal to the fourth part of the figure. Q.e.d.

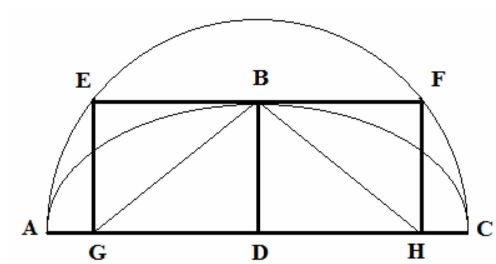
Corollary.

Hence it is apparent the lines AG, HC to be equal and GH to be bisected at D.

PROPOSITION CXXI.

With the same in place the right lines BG, BH may be drawn.
I say the lines BG, BH taken together to be equal to the axis AC.

Demonstration.

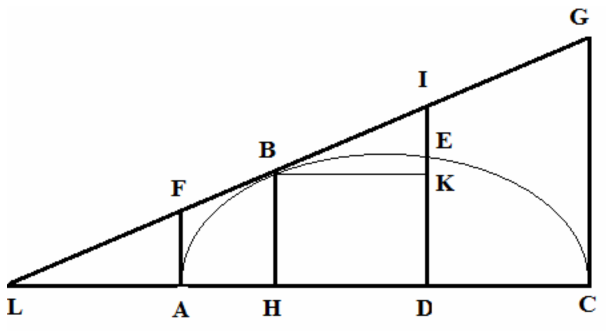


Because AG, HC are equal (§120.*Cor*.), the square AG is equal to the rectangle from AG and HC. Again since GD, DH are equal, the rectangle AGD taken twice will be equal to the rectangle AGH. Whereby since the square AD shall be equal to the squares DG, AG and with the rectangle AGD taken twice; the same square AD will be equal to the square DG, and with the rectangle from AG, HC together with the rectangle AGH. And the rectangles AG, HC, and AGH are equal to the rectangle AGC. Therefore the square AD is equal to the square DG together with the rectangle AGC; that is to the squares DG, GE, that is to the squares DG, DB. But from the same, the squares GB are equal; therefore the squares AD, GB are equal, and therefore the right lines AD, GB are equal. It will be shown in the same manner the right lines CD, HB to be equal. Therefore both GB, BH taken together are equal to the axis. Q.e.d.

Corollary.

Hence it follows: if the axes of the ellipse ABC were AC, BD, and from the vertex B of the minor axis the right lines BG, BH were dropped equal to the lines AD, DC, cutting the axes AC at G and H . So that both the rectangles AGC as well as AHC shall be equal to the fourth part of the figure, thus G and H are poles of the section.

PROPOSITIO CXXII.



The ellipse ABC, of which the conjugate diameters AC, DE have tangents at A and C, and at some other point B the three lines AF, CG, FG are tangents at A, B, and some other point B: and indeed FG shall cross the right lines AF, CG at F and G, and on being produced, ED will cross the line FG at I, and the right ordinate line BH dropped from B to the diameter AC:

I say the rectangle on the lines AF, CG to be equal to the rectangle on BHID.

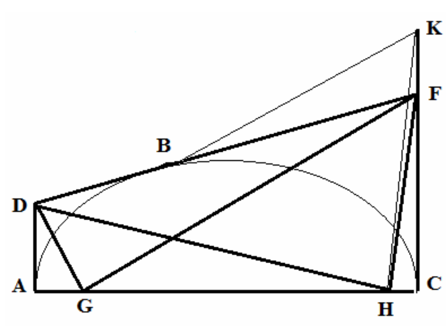
Demonstration.

The line FG may be produced until it reaches the axis at L. Since DH, DA, DL are lines in continued proportion [§32], there will become: LD to AD, that is as DC thus LA to AH, and on inverting and adding, so that as CL to DL, thus HL to AL, but as CL to DL, thus there is CG to DI, and as HL to AL, thus HB to AF, therefore as CG to DI, thus HB to FA: and thus the rectangle on the lines AF, CG to be equal to the rectangle BH, ID. Q.e.d.

Corollary.

Hence it follows the rectangle AF, CG or HB, ID, to be equal to the fourth part of the figure: for BK may be drawn parallel to the axis AC: the rectangle DK, DI shall be equal to the square ED; and therefore the rectangle AF, CG is equal to the square ED, that is to the fourth part of the figure.

PROPOSITION CXXIII.



The lines AD, CF, and DF shall be tangents to the ellipse ABC, of which the axis is AC, at A and C and at some other point B, indeed it may be agreed with the lines AD, GF at D and F, but with the line AC may be divided at G and H, so that AGC, AHC to be rectangles equal to the fourth part of the figure; and lines DG, GF, DH, HF shall be drawn.

I say the angle DGF, DHF to be right; and if they shall be right: I say DF to be a tangent line of the ellipse.

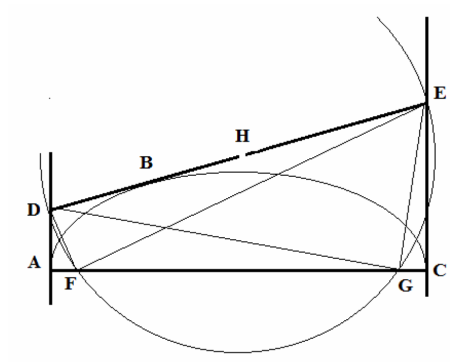
Demonstration.

From the preceding Cor., the rectangle DACF is equal to the fourth part of the figure, that is, to the rectangle AGC. Therefore as AG to AD thus FC to CG , but the angles DAG, FCG are right; therefore DAG, FCG are similar triangles : and the angle ADG is equal to the angle CGF, but the angle ADG, together with the angle AGD is equal to a right angle, since the angle DAG in triangle ADG shall be right : and therefore the angle CGF, together with the angle AGD are equal to one right angle: therefore the remaining angle DGF is right: it may be shown in the same manner the angle DHF is right. Q.e.d.

PROPOSITION CXXIV.

The lines AD, CE, DE shall be tangents to the ellipse ABC at A, C, B, the axis of which is AC, and indeed DE shall meet the lines AD , CE at the points D and E. Moreover they shall become the fourth part of the figure, equal to the rectangles AFC, AGC, and ED shall be bisected at H:

I say the circle described with centre H and with the radius DH and EH, to pass through F and G.



Demonstration.

The points DF, FE, DG, GE may be joined. Since both the angles DFE, as well as DGE, is right, and DE some line bisected at H, it is evident the circle described with centre H and with the radius HD to pass through F and G. Q.e.d.

Corollary.

Hence it follows the angles EDG, FDA to be equal to each other, for the angle ADF is shown in the preceding demonstration to be equal to the angle GFE: but the angle EDG is equal to the angle GFE since it stands on the same arc EG, therefore the angles EDG, FDA are equal to each other.

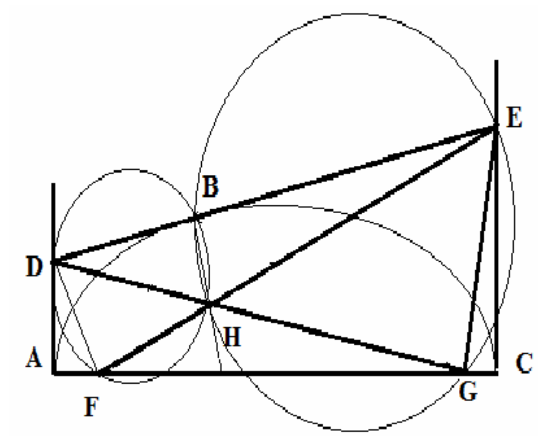
PROPOSITION CXXV.

The lines AD, CE, DE shall be tangents of the ellipse ABC, of which AC is the axis, at A, C, B : and DE indeed shall meet the lines AD, CE at D and E. Moreover AFG, AGC shall become rectangles equal to the fourth part of the figure: and with the lines FE, GD drawn which intersect each other at H, from the point H to the contact point B, the right line HB may be drawn.

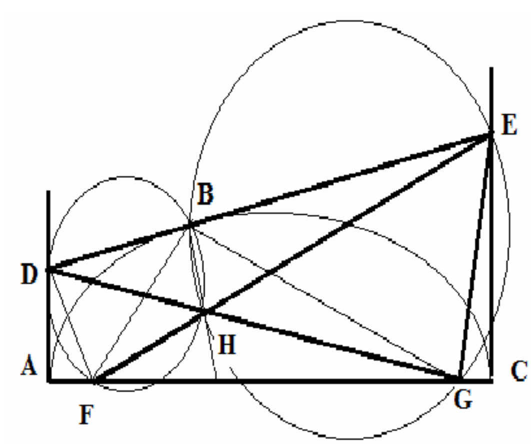
I say HB to be normal to the tangent DE.

Demonstration.

The right lines ED, CE are drawn. The angles DFE, EGD are right. Now with the lines HD, HE as the diameters , the circles DBH, EBH are described. Since the lines DH, HE are not parallel, it is evident the circles DBH, EBH in turn cut each other at some point B. Therefore with the points H and B joined; the right lines DB, EB may be drawn: the angles DBH, EBH are right, and thus the lines DB, EB are collinear, and HB shall be normal to the line DE. But as shown before in §123, the angles DFE, EGD are right; therefore the line DE is a tangent. Whereby the right line BH is normal to the tangent ED. Q.e.d.



PROPOSITION CXXVI.

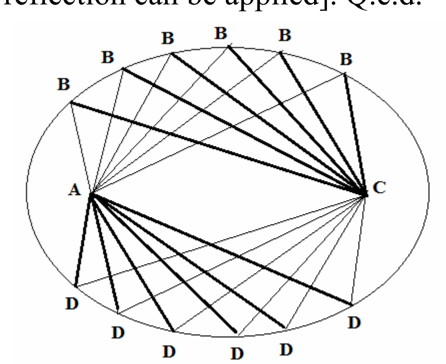


With the same figure remaining: FB, BG are drawn.

I say the angles DBF, EBG at the tangential point B to be equal.

Demonstration.

Since the angles DFH, EGA are right, the circles DFH, EGH will pass through F and G, but each also passes through B: since the angles EBH, DBH are right: therefore both the angles DBF, DHF as well as the angles EBG, EHG are equal to each other: but the angle DHF is equal to the angle EHG: and therefore DBF is equal to the angle EBG [*i.e.* the point B on the ellipse can be considered as an elemental mirror, to which the usual law of reflection can be applied]. Q.e.d.



Scholium

Since the point B *shall be assumed to lie on the periphery, it follows all the lines drawn from* F *to whatever point on the periphery are required to be reflected through* G. *Whereby the points* F *and* G *are to be called by no other name than the foci or poles: which are called from the comparison made with Apollonius, again these have extraordinary properties for ellipses :between those it has pleased* *to add the following here*.

A, C *shall be the foci of the ellipse, the distance between which shall be the separation of the eyes, and the left eye may be placed at A, and the right eye at* C. *I say that everything reflected by the whole mirror to appear to the right eye placed C: and in turn the right eye placed at C to see everything seen by the left eye placed at C: for the kinds of objects* A *reflected by the whole mirror, are reflected at C, and the kinds of objects seen at* C *by the whole mirror are reflected at* A. *Whereby an object placed at* A *reflected by the whole mirror to be used by the eye at* C, *and an object at* C, *by the eye at* A. *Hence it follows that a small visible object placed at C will appear large to the eye placed at A: since it will be reflected by the whole surface of the mirror.*

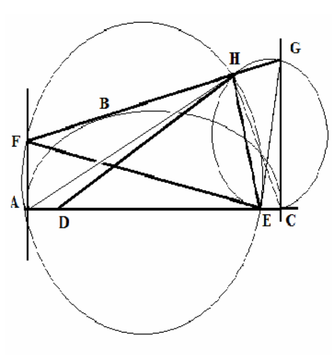
[There is a certain naivety in this description; for it is an inverted real image of the object at one focus that can be viewed at the other focus, subsequently the rays will return to the initial focal point on being inverted again; with the eye placed elsewhere conveniently; the eye itself should never placed at the focal point of a large mirror: if it were, damage to the retina could occur, and in any case, none of the light would be reflected, as it would be absorbed mostly by the eye. However, it is a pleasing demo., using sound waves to have two large elliptical or spherical mirrors set up confocally as parts of the ellipse considered above, and to have someone whisper at one focus, to be heard by someone with an ear placed at the other focus.]

PROPOSITION CXXVII.

The ellipse ABC, of which the axis is AC and poles D E, has the right line tangents AF, CG, FG at the points A, C, B ; and indeed FG shall meet the lines AF, CG at F and G. The line EH, erected from E, shall be put in place normal to the tangent FG, and the points AH, CH joined.

I say the angle AHC to be right.

Demonstration.

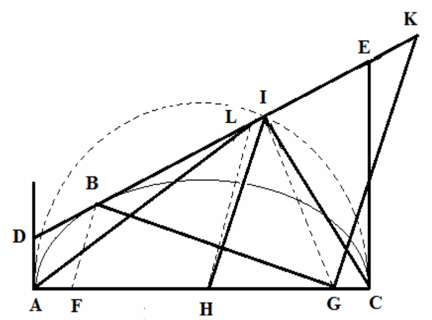


With the lines FE, EG drawn, the circles FHE, HGC are described with the diameters FE , EG : and indeed the circle FHE, with the right angles EHF, EAF, will pass through the points H, F, A ; truly the circle HGC: also shall be with the right angles DHE, ECG, shall pass through H, C. Therefore both the angles AHF, AEF as well as the angles EHC, EGC, shall be equal angles: but the angle GCE by the demonstration shown above in §121, is equal to the angle FEA, therefore the angle FHE is equal to the right angle AHC, and whereby to be right itself. Q.e.d.

PROPOSITION CXXVIII.

The lines AD, CE, DE shall be tangents to the ellipse ABC at A, C, B, the axis of which is AC, and indeed DE shall cross the lines AD, CE at D and E : moreover the poles shall be F, G, the centre H, and the right line FB drawn from F to the point of contact, and from H the line HI shall be drawn parallel to the line FB, crossing the line ED at I.
I say the line HI to be equal to the line HC, and if HI crosses the line ED, it shall be equal to HC. I say the line HI to be parallel to FB.

Demonstration.



BI shall be made equal to IK: and BG, GK may be joined, and the right lines AI, IC may be drawn: Since IB, IK are equal, BI will be to IK, as FH to HG: and thus the lines BF, KG are parallel, and the angle BKG equal to the angle DBF, that is IBG, from §115 : whereby the lines BG, GK are equal: moreover the two remaining sides BI, IG are equal to the sides KI, IG. Therefore the angle BIG, is equal to the angle KIG: and thus GI is normal to the tangent DE, and the angle DIG is right. Whereby the circle with centre H and described with radius HC will pass through I, and the line HI will be equal to the line HC. Which was to be shown first.

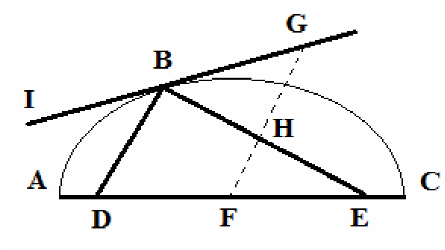
For the rest remaining, now HI shall be a line which crosses the tangential line ED at I, equal to the line HC. I say the right line HI to be parallel to the line BF: truly on the other hand, if a line HL may be drawn from H, parallel to the right line FB crossing the tangential line ED at L; hence the line HL would be equal to the line HC, that is, to HI. Whereby a circle with centre H described with the radius HC would pass through the points I and L. Which cannot happen ; therefore HL is not parallel to FB: nor any other line apart from the line HI. Q.e.d.

PROPOSITION CXXIX.

ABC shall be an ellipse with axis AC, moreover the poles shall be D, E ; from D and E the lines DB, EB meeting at some point B of the periphery will be reflected.
I say the lines DB, EB taken together shall be equal to the axis AC.

Demonstration.

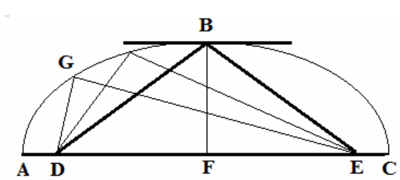
F shall be the centre of the ellipse; and with the tangent BG acting through: the right line FG shall be drawn parallel to the line DB cutting the line EB at H. Since BD, FG are parallel lines, therefore the angle FGB is equal to the angle DBI that is, to the angle EBG, and thus the lines HB, HG are equal : again, since DE shall be to FE, thus as BE to HE, and DE shall be twice as much as FE, and EB shall be twice as much as right line BH, that is HG: thus also BD is FH doubled, thus since DE shall be to FH, thus as DB to FH; therefore the lines EB, BD taken together are twice as much as FG, that is, FC, [from the previous prop.]: and whereby equal to the axis AC. Q.e.d.



PROPOSITION CXXX.

The maximum of the isoperimetric triangles is isosceles.

Demonstration.



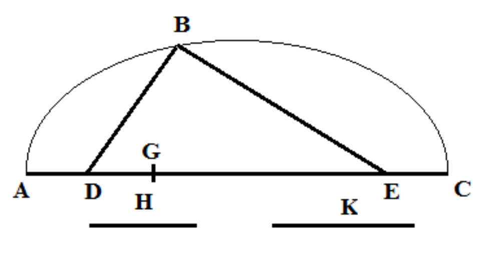
Some ellipse ABC may be described, of which the axes shall be AC, FB; the poles D, E; and the points D B, B E may be joined while above the base ED some triangles may be put in place and the points may be joined then some triangles DGE may be constituted upon the base of the triangle, the vertices of which G shall be on the periphery. Since both the lines DB, BE as well as DG, GE taken together are equal to the axis AC: it is apparent the triangles DBE, DGE to be isoperimetric: moreover I say the maximum triangle of these to be the triangle DBE acting through the tangent B: which since it shall occur only at the one point B of the ellipse, and the rest of that kind to fall outside the ellipse, it is clear the triangles DGE which are terminated on the ellipse to have a lesser height than the triangle DBE, and thus these to have a lesser perimeter: moreover the triangle DBE is isosceles, because the equal sides DF, FB are with the sides EF, FB, and with these to contain right angles; therefore of the triangles the isosceles triangle is of the maximum isoperimetric form. Q.e.d.

PROPOSITION CXXXI.

It shall be required to incline two lines from the foci D E of the ellipse to the same point of the perimeter which may be held in the given H to K.

Moreover, the ratio must be greater than the ratio AD to DC, yet smaller than the ratio AE to EC.

Construction & demonstration.



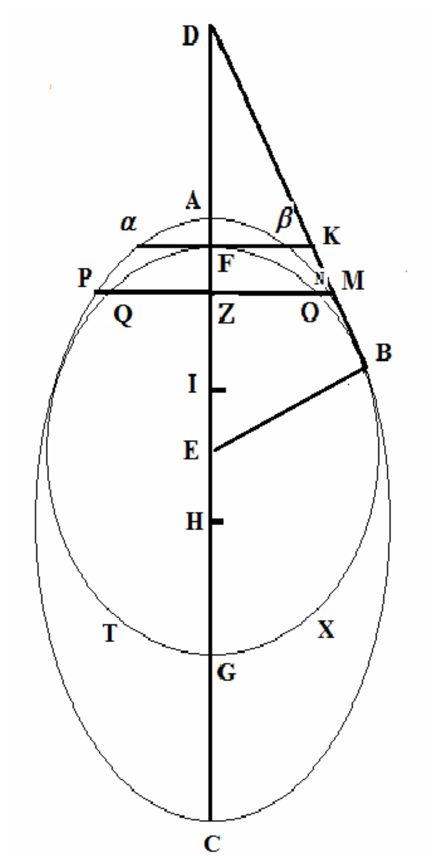
The axis AC shall be cut at G, following the given ratio H to K, which since it shall be put greater than the ratio AD to DC, and lesser than the ratio AE ad EC, evidently the line AG to be greater than the line AD: truly smaller than AE, and hence the point G to fall between the poles D, E and therefore the right line DB may be put in place from D to the perimeter equal to the right line AG. And the points B E shall be joined. I say what is required to be accomplished. For since the two lines DB, BE taken together shall be equal to the axis AC, moreover by the construction the line DB shall be equal to the line AG, BE will be equal to the remaining line GC: therefore DB is to BE, as AG to GC, that is as H to K. Therefore we have inclined the lines, etc. Q.e.f.

PROPOSITION CXXXII.

The line BD shall be a tangent to the ellipse ABC at B meeting the major axis CA at D; moreover, from the point of contact B, the normal BE to the tangent may be put in place, meeting the axis at E.

I say EB to be the shortest line of these which are able to be drawn from the point E to the periphery of the ellipse.

Demonstration.

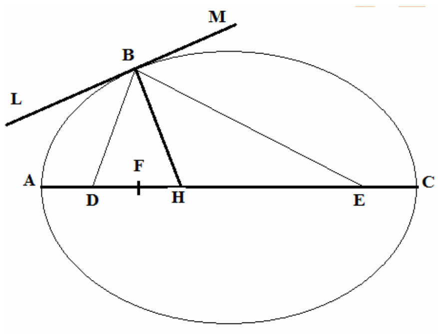


The circle FBG is described with centre E and radius EB crossing the axis at F and G; the centre of the ellipse shall be H. Since the ellipse tangent line DB meets the major axis at D, and the angle DBE is right: the line BE does not pass through the centre of the ellipse H: for if E were the centre, the right line EB will be placed normally to the conjugate axis AC, (since all the tangent lines DB shall be parallel and divided into two right lines) and thus DB would be parallel to the axis AC: therefore neither is E the centre of the ellipse, nor is EB the diameter : truly since DB meets the axis in the region of A, the line EB is smaller than the radius parallel to itself: and thus is smaller than the semi-axis HC, and much smaller than the right line EC, whereby the circle described with radius EB, meets the axis at G within the ellipse, therefore the point G is above C.

Again since HC, that is AH, shall be greater than EB shown, that is EG, with the common term EH removed, AE will remain greater than HG, but on putting EI equal to EH, the right line FI is equal to HG, therefore AE also is greater than FI: therefore with the removal of the common term IE from FE and AE, IA will remain greater than IF: and from which the point F falls within the ellipse below A. Further the tangent FK may be put through F, to which PQNM will be parallel, therefore so that the square MB shall be to the square BK thus as the rectangle QMO to the square FK; but so that the square MB shall be to the square BK thus as the rectangle PMN to the rectangle αKβ ; therefore so that the rectangle QMO is to the square FK thus as the rectangle PMN is to the rectangle αKβ: and on interchanging and inverting, so that the square FK shall be to the rectangle αKβ, thus as the rectangle QMO is to the rectangle PMN: but the square FK is greater than the rectangle αKβ, and therefore the rectangle QMO is greater than the rectangle PMN : again the rectangle QMO together with the square ZO is equal to the square ZM, and the rectangle PMN together with the square ZN, is equal to the same square ZM ; therefore the rectangle QMO, together with the square ZO, is equal to the rectangle PMN, together with the square ZN; from which if unequally the rectangles QMO, PMN may be taken away, the unequal squares remain ZO and ZN: and since the rectangle QMO is greater than the rectangle PMN, the square ZO is smaller than the square ZN: and the square ZQ less than the square PZ; therefore the points O and Q are within the ellipse: the points X T may be shown similarly, and any other points of the perimeter of the circle FHG to be within the ellipse ; therefore the whole circle FBG falls within the ellipse: from which since all the right lines drawn from the centre of the circle E to the periphery of the ellipse, first shall meet the circle then the ellipse : and thus the radii of the same shall be greater than EB, the shortest of all of these is that which is terminated at the common point B of the ellipse and circle to be drawn from the point E to the periphery of the circle. Q.e.d.

PROPOSITION CXXXIII.

To draw the shortest line from the point (H) on the axes of the ellipse to the perimeter.



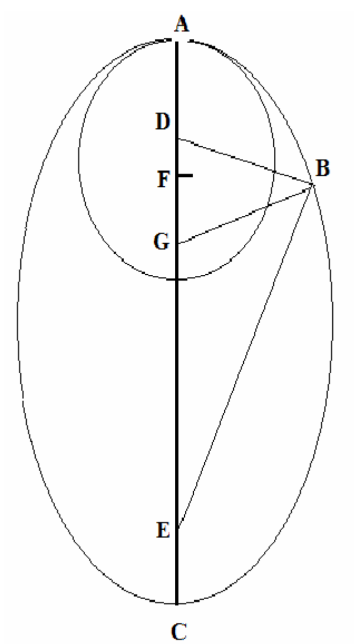
Construction & demonstration.

D and E shall be the foci of the ellipse. Cut the axis AC at F, thus so that AF shall be to FC, just as DH is to HE. Then from the pole to the perimeter the distance DB may be prepared equal to AF itself, and HB may be joined. I say HB to be the shortest distance.

For the right line EB may be drawn from the pole E to B, and LM the tangent to the ellipse at B; DB, and BE are equal in length to the axis. But DB is equal to AF. Therefore BE is equal to FC. Therefore DB is to BE, as AF to FC, that is , from the construction, as DH to HE. Therefore the angles DBH, EBH are equal, but also the angles are equal to the angles at the tangent DBL, EBM; therefore the sum of the angles HBL, HBM are equal ; therefore HB in normal to the tangent, therefore from the preceding the shortest of all the lines which can be drawn from the point H can be drawn to the perimeter. Therefore what was desired has been done.

If the point F may fall on the pole D, or between A and D; then the shortest distance from the given point H to the perimeter will be part of the axis, as is apparent from the first construction and demonstration being considered.

PROPOSITION CXXXIV.



In the given ellipse to describe the maximum circle of these which are tangential at the end of the axis and which are held within the ellipse.

Construction & demonstration.

The poles of the ellipse shall be D and E. The ratio shall become so that CD to DA; thus EF to FD. I say the circle described with centre F and with the radius AF to be that which is desired. Indeed since from the construction, there shall be CD to DA, thus as EF to FD. It is apparent from the preceding, FA to be the shortest of all the lines which can be drawn from the point F to the perimeter ; therefore the circle described with centre F is tangential to the ellipse at A, which was the first part:

Moreover I may show thus that it shall be the maximum of all the tangents within the ellipse. For with some other point

G taken for the centre of a greater circle, since therefore EG is to GD in a smaller ratio than EF to FD, that is, than CD ad DA, for the sake of an example there shall become EG to GD, thus CF ad FA: and by necessity FA shall be greater than DA: and thus the point F will fall beyond the pole D towards E: therefore if from the pole D to the perimeter DB may be put in place equal to GA, and GB may be joined, it will be clear from the preceding that GB to become the minimum of all the lines which may be drawn from G to the perimeter. Whereby GA is greater than GB, therefore the circle described through A with centre G falls outside the ellipse. Similarly we may show any other greater circle that that described before with the radius FA, to fall outside the ellipse : therefore that is the maximum of all the circles touching within the ellipse. Therefore in the given ellipse, etc. Q.e.f.

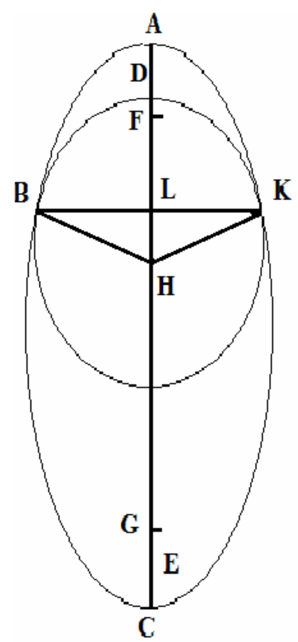
Corollary.

Clearly it is agreed from the discussion of this proposition, the circles described for all the radii, that the smallest to be tangential at the point A is for the radius FA. If the vertical axis, with the centre put in place between F and A, may pertain as far as A. These circles also will be tangential with that circle which is described with the radius FA, and these will be smaller, whereby also will be tangential within the ellipse, the axis of which is AC.

PROPOSITION CXXXV.

The major axis of the ellipse ABC shall be AC and in that the poles D, E, shall become as CD to DA, thus EF to FD, and DG to GE.
I say circles can be described from any point of the right line FG which touch the ellipse inside at two points: truly the centres of these to stand between the two excluding ends F and G.

Demonstration.



For some point H may be taken on the right line FS, and from H the line HB may be drawn, the shortest of these which will be able to be drawn from H to the periphery ; then from B the ordinate BLK to the axis may be drawn, and with HK, HB, joined, it is clear from the Elements that HK to be equal to HB, and thus the circle with centre H and with the element HB described to pass over through K and B: and since the lines HK, HB shall be the shortest by the construction, it is evident the whole circle BDK to fall within the ellipse, and on that account for the points B and K on that to be tangential. But since the centres of the circles tangential to the ellipse at two points, shall stand between the two points F and G: from that it is evident that the shortest lines FA, GC shall be those which are able to be drawn from F and G to the perimeter, and thus to be the circles with centres F or G, and for any radius which shall be described greater than FA or GC, the circle will cut the ellipse: truly the greatest will be described with the radius FA or GC, of these which lie inside the ellipse being tangential at only one point.

PROPOSITION CXXXVI.

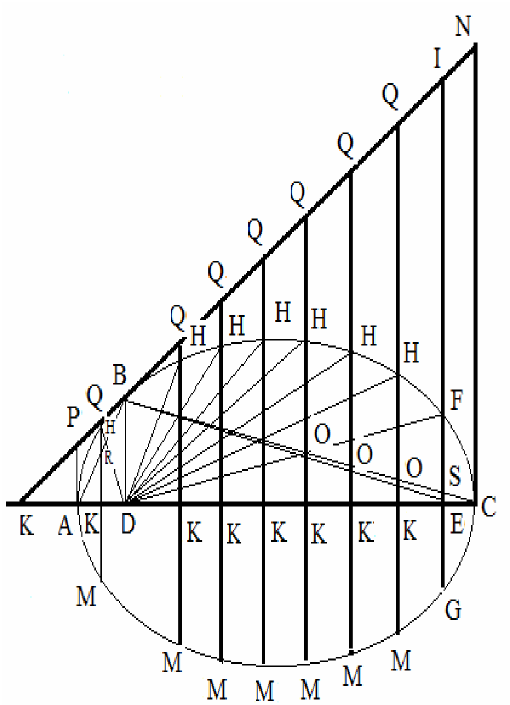
With the same figure remaining: it shall be proposed to designate a point on the axis from which centre a circle may be described, which shall be tangential at the given point.

Construction & demonstration.

The point B shall be given on the perimeter, through which if it is understood to be the touching point, the line BH shall be sent from B, normal to the tangent, crossing the axis at H. Clearly H is the point requiring to be satisfied; for since HB shall be the shortest of these which can be drawn from H to the periphery, the circle described with centre H and with radius HB shall be tangential to the ellipse at the point B; therefore, &c. Q.e.f.

PROPOSITION CXXXVII.

The axis of the ellipse shall be A, C, the poles D, E, from which the normals DB, EFG shall be drawn to the axis ; moreover KBI shall be a tangent to the ellipse at B, crossing the axis at K, truly the line GF at I, and DF may be joined.
I say the rectangle FIG to be equal to the square DE.



Demonstration.

Since DB, EB are drawn from the poles to the point of contact B, the angles KBD, IBE are equal, but, since DB, EI are parallel, the angle KBD is equal to the angle EIB, therefore the angles IBE, EIB, and therefore BE, IE are equal. Then since the rectangles CEA, CDA are equal, also the squares EF, DB are equal, and thus the right lines EF, DB are equal. Whereby since [the sums of:] DB & BC are equal to EF& FD, see §121, which also will be equal to BE & DF. And BE has been shown equal to EI. And therefore DF is equal to EI, and hence the squares DF, EI are equal. But the squares DE, EF are equal to the square DF; and since GF is bisected at E, and FI may be added to that, the rectangle GIF with the square EF is equal to the square EI. Therefore the squares DE, EF are equal to the rectangle GIF with the square EF.

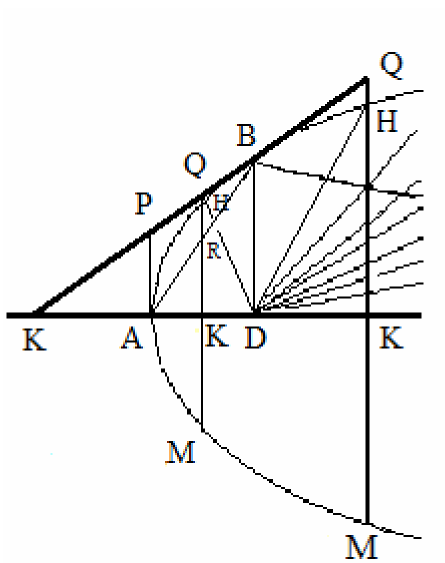
Therefore with the common square EF removed, the rectangle GIF and the square DE will remain equal. Q.e.d.

PROPOSITION CXXXVIII.

With the same in place some other Q, H, K, M, normal to the axis may be put in place. I say the rectangles HQM, to be equal to the squares DK.

Demonstration.

CN shall be a tangent to the ellipse at C, crossing the tangent line BN at N, and BC may be drawn cutting QM at O, and IG at S ; since NC, IG, QM are normal to the axis, they are parallel ; therefore by that which we have shown in §99, the rectangle FIG shall be equal to the square SI, and the rectangle HQN is equal to the square QO; whereby the square IS to the square QO, that is as the square SB to the square OB, that is as the square ED to the square KD, thus the rectangle FIG to the rectangle GQM; and on interchanging so that as the rectangle FIG to the square ED, thus the rectangle HQM to the square KD, but by the preceding, the rectangle FIG is equal to the square ED. Therefore also the rectangle HQM is equal to the square KD.



Similarly we will demonstrate for another position of the pole D, the rectangle HQM is equal to the square KD. Indeed AP is a tangent to the ellipse at A, crossing the other tangent line at P, and AB cutting QM at R. In the triangle BCN [in the larger diagram], some other QO shall be drawn, parallel to NC normal to the axes, thus itself having the ratio QO to QB, as QR is to QB, therefore on interchanging there will become thus, QB to BQ as KD ad DK, thus as QO to QR. Therefore so that the rectangle HQM to the square DK, thus the square QO to the square QR, that is the rectangle HQM to the square DK; thus so that on interchanging, as the rectangle HQM to the square KD, thus the rectangle HQM to the square DK. But it has been shown above, the rectangle HQM (evidently that which is towards C) to be equal to the square KD, therefore the rectangle HKM also is towards A, is equal to the square DK. Therefore all the rectangles HKM, &c. Q.e.d.

Corollary.

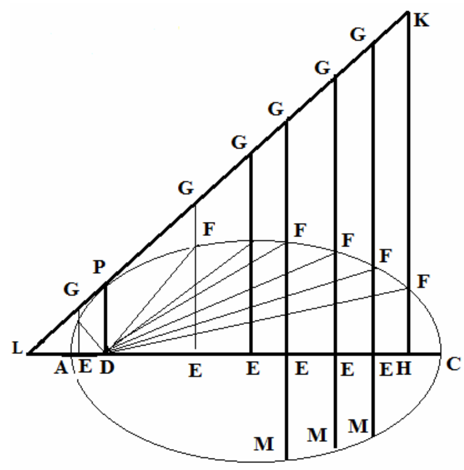
From the discussion of the demonstration now brought forth, it is allowed to deduce the squares of the tangents CN, AP, to be equal to the squares CD, DA.

PROPOSITION CXXXIX.

The ellipse shall be given the axis of which shall be AC, the poles D, H; from the pole D to the perimeter DP shall be drawn normal to the axis, and the line GPG shall be a tangent to the ellipse at P. Now some normals GFE may be drawn to the axis, and DF, DF may be joined.

I say all the lines DF, to be equal to all the lines GE.

Demonstration.



One of the right lines GE may be produced to M: from the preceding, the rectangle FGM is equal to the square DE; therefore by adding the common square EF, the squares DE, EF, are equal to the square DF, which is equal to the rectangle FGM with the square EF, that is, to the [next] square GE. Therefore since the square DF is equal to the square GE, also the right line DF is equal to the right line GE. By the same discussion all the remaining DF, are equal to the remaining GE. Q.e.d.

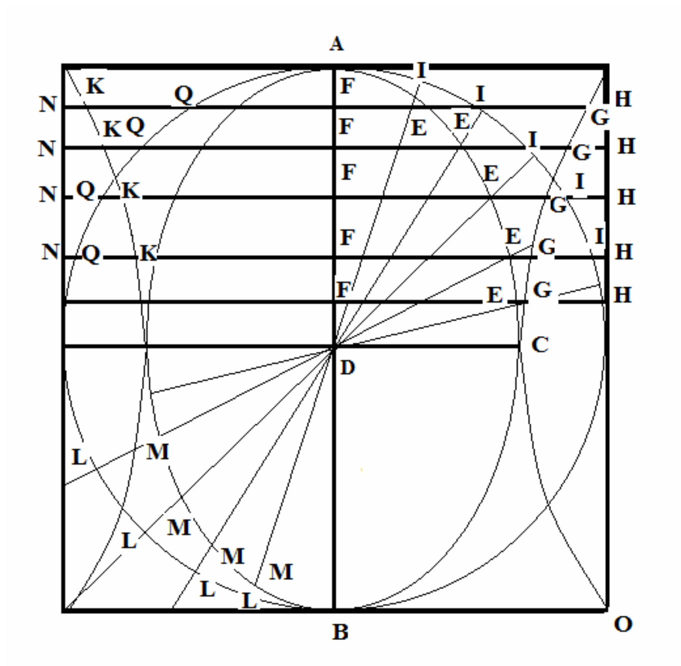
The three following theorems shall be allowed to be demonstrated in the book on the hyperbola, which shall depend on the properties of hyperbolas, yet on account of the favorable disposition of the properties of ellipses related to hyperbolas, this is not seen to be the place to propose these other properties.

PROPOSITION CXL.

With the same figure remaining, if the lines EFG normal to the axis AC may be equal to the right line DF drawn from the pole. I say the line drawn through the point G shall be a tangent to the ellipse at the point P.

Evidently the demonstration is from the preceding proposition, with the tangent at C. See the demonstration in the book on the hyperbola.

PROPOSITION CXLI.

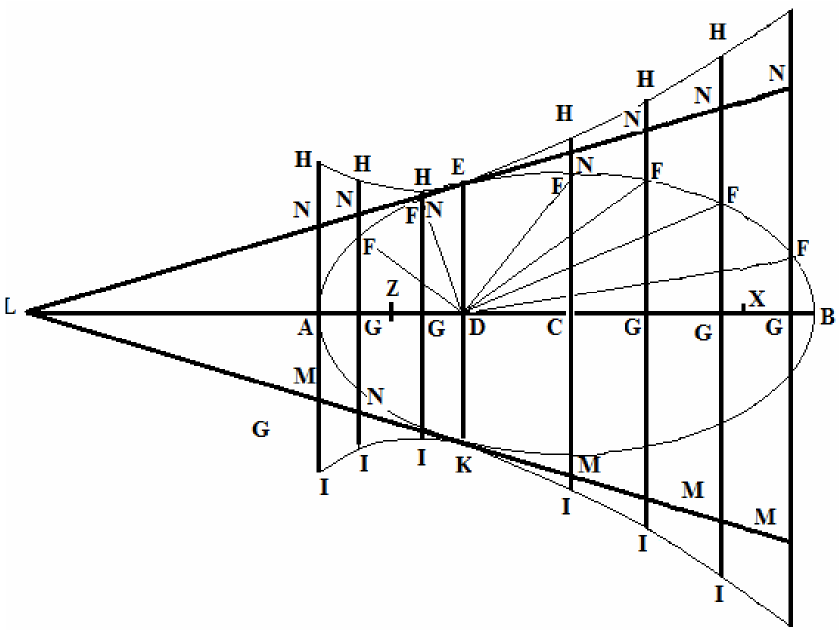


The ellipse shall be given having the axis AB, the centre point D may be taken on the axis which in the first place shall be the centre of the ellipse, from that DC is drawn to the perimeter normal to the axis, and then some other lines DE, DE : from which the lines FEG shall be made normal to the axis.

I say the line described through the points G to be a hyperbola which shall have the same centre D as the ellipse, and shall be a tangent to the same at C.

See the demonstration in the book on the hyperbola.

PROPOSITION CXLII.



An ellipse shall be given having the axis AB, centre C, the poles X, Z ; take some other point D on the axis between the centre C and the pole X, from which DE is drawn to the periphery normal to the axis, and then some others DF, DF are drawn; from which equal normals GFH may be drawn to the axis.

I say the line described through the points H, H to be a hyperbola, which shall be a tangent at F.

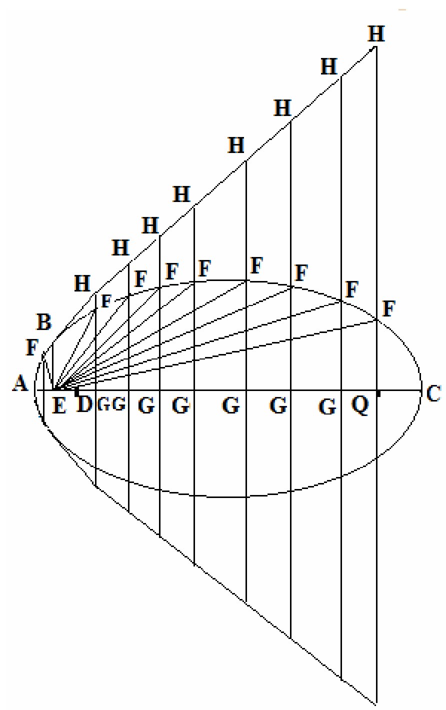
This will be demonstrated in the book concerning the hyperbola.

PROPOSITION CXLIII.

Again the ellipse shall be given, having the axis AC, the poles D, Q, the point E may be taken on the axis between the pole D and the vertex A, from which the normal EB may be drawn from the axis to the perimeter : and some other normals EF, by which GFH become normal to the axis.

I say the line described by the point H on the hyperbola may be embraced by the ellipse and shall be a tangent at the point B.

We will give the demonstration in the book on the hyperbola.

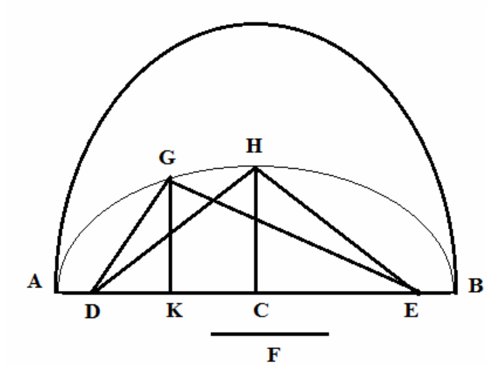


PROPOSITION CXLIV.

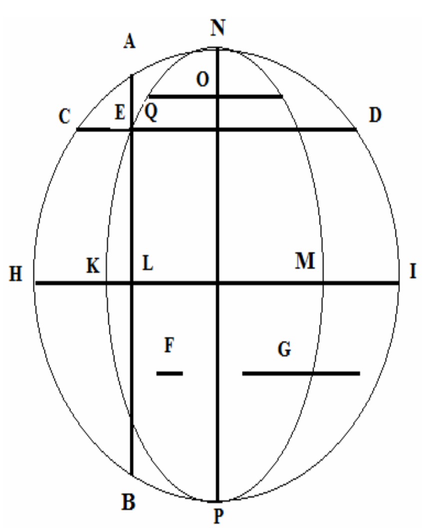
To show the triangle for a given sum of the sides, and for a given height and base.

Construction & demonstration.

AB may be put equal to the given sum of the sides of the triangle, DE shall be made equal to the base of the triangle which shall be bisected at C, thus so that at each end the equal lengths AD, BE may be left off, moreover F shall be put equal to the height. Triangle DHE may be composed from the sides AC, CB, DE (for AC, CB, taken together are greater than DE,) and therefore DHE will be isosceles. Thence it shall happen that the square HC shall be to the square F, thus as the rectangle ACB to the rectangle AKB, and KG may be erected equal to F and parallel to HC, and DG, GE may be joined. I say DGE to be the triangle sought, since the rectangle ACB is to the rectangle AKB, as the square HC to the square F, that is to the square GK, therefore A,G,H,B will be points on the same ellipse, of which AB is the axis: and since AD shall be equal to EB itself, and likewise DH together with HE, shall be equal to AB itself, DE will be the points made from the comparison, or from the foci of the ellipse, whereby DGE are equal to the sides of the axis AB, that is the sum of the sides is from the given base DE, and from the height F that is GK. Therefore we have shown the triangle which was sought.



PROPOSITION CXLV.



The right line AB, subtended by some arc of the circle ABC, to cut another CD, the same at right angles, so that CE to ED, may maintain the ratio F to G.

Construction & demonstration.

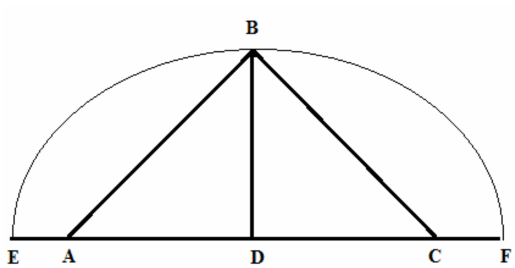
We have proposed this problem in the book concerned with the properties of the circle : but since its demonstration depends on a property of the ellipse, therefore we have delayed that demonstration to this place: truly the construction is as follows.

The diameter HI shall be drawn normal to AB cutting AB at L. And there shall become as F to G, thus HK to KI: then with IM taken equal to HK itself, it may divide the diameter HP, at the point O so that KM shall be divided at L: then the rectangle NOP may become equal to the square LE; finally the right line CED may be drawn through E. I say CED to be divided at E, following the ratio F to G. Since the lines NP, KM are bisected at right angles ; therefore the ellipse NKP may described around these described put as the axis of the ellipse, and thus OQ, LE will be put in place in order for the individual axes; and because the axes are similarly divided at O and L, the rectangle NOP is equal to the square LE, it is evident the point E to belong to the ellipse described through the points N, K, P, M. Therefore as HK is to KI, that is, as F to G, there shall be CE to ED, it is apparent the applied right line at D to be normally to the circle for CD, so that CE to ED may obtain F to G in the given ratio. Q.f.d.

PROPOSITION CXLVI.

With the right line AC and the height BD given, to describe the ellipse the poles of which shall be A and C.

Construction & demonstration.



The isosceles triangle ABC shall be established on the line AC with the height BD , then the line AC may be produced equally in each side to E and F: so that the total length EF shall be equal to twice AB, BC, since the ellipse may be described through the points E, B, F. I say that to be what is desired. Since the line EF is bisected at D and not to be bisected at A: the rectangle EAF together with the square AD, to be equal to the square ED, that is equal to the square AB by the construction ; but also the squares AD, BD are equal to the square AB; therefore with the common square AD removed, the rectangle EAF remains equal to the square BD, that is, to the fourth part of the figure. It is shown in the same way for the square BD to be equal to the rectangle FCE : whereby A and C, are the foci of the ellipse EBF described. Therefore, given the line and the height, etc. Q.e.f.

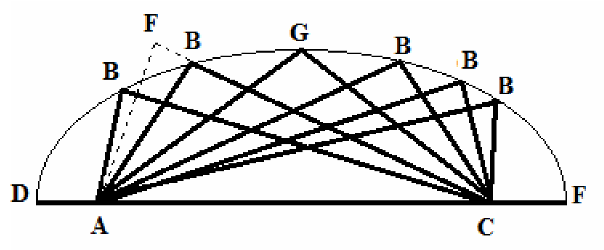
Corollary.

Hence it follows for some given isosceles triangle ABC with some angle held to the vertex, an ellipse can be described of which the foci shall be the ends of the given base of the triangle ABC. The demonstration to be apparent from the proposition.

PROPOSITION CXLVII.

Some isoperimetric triangle ABC, AGC shall be described on the line AC.
I say the points G, B, G to be on the same ellipse, the poles of which shall be A and C.

Demonstration.



AC shall be produced equally on each side to D and F, so that the whole length DF shall be equal to double the lengths AB, BC, then the ellipse may be described through the points D, B, F. I say that truly it be allowed to pass through the remaining points B; it may pass either above or below B, and first above to pass through the point F, CB produced until it may cross the periphery at F, AF may be joined. Therefore since the points G and F are points on the ellipse, of which the poles are A and C, AGC, AFC shall be isoperimetric triangles ; but the triangle AGC by the construction is isoperimetric to the triangle ABC; therefore AFC, ABC are isoperimetric triangles, which cannot happen; whereby the ellipse DGF does not pass through the above point B, but it can be shown in the same way, neither may it fall below B. Therefore the ellipse passes through the points B, B; therefore the points G B B belong to the ellipse of which the poles are A, C. Q.e.d.

[There are numerous errors in labeling the last diagram especially, in the original text.]

ELLIPSEOS PARS QUARTA

Sectionis polos :& lineam a puncto in axe dato ad peripheriam, brevissimam designat .

Partem hanc, *quae de poles est*, *aggressuri*, *paucis praemittemus est quae ad inventionem polorum ab Apollonio libro tertia propositione 41& 45 demonstrata sunt* ; *& quidem hoc necessarium esse duxi; tum quod ad illorum intestigentiam quae Apollonius in rem hanc contulit, nec omnium captui ita patent, plurimum conducant; tum quod ad rem nostram plane iudicem necessaria. Apollonius igitur ut in axe ellipseos polos exhibeat,haec utitur constructione propositione 45 & 6.* Quartae, *inquit*, parti figurae aequale rectangulum comparetur ex utraque parte; *id est, sectonis axis* AC *ita secetur in duobus punctis* G *&* H, *ut tam* AGC *quam* CHA *rectangulum aequale sit quartae partifigurae: quo posito ulterius ostendit* G *&* H *polos esse sectionis: quos punctae vocat ex comparatione facta; videlicet ex comparatione rectangulorum sub segmentis axeos, cum quarta parte figurae. Figuram porro hic vocat Apollonius rectangulum quod sit sub latere recto axeos maioris & ipso axe: atque illud cum quarta sui parte ad usus seruiret eximios; videlicet inventionem polorum &c. singulari prae reliquis rectangulis appellatione figuram appellavit antiquitas: huius autem quartae parti aequale est quadratum semiaxeos minoris: quod Pergaeus lib.tertio, propos.*42. *praeclare demonstravit, & nos verbo uno sic ostendimus.*

Lemma.

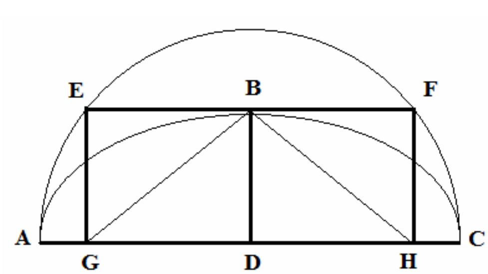
Per undecimam huius, figura sive rectangulo sub axe maiore & latere illius recto aequale est quadratum axeos minoris; sed quadrati minoris axis quarta pars est quadratum dimidii axis minoris; igitur quadratum dimidii axeos minoris aequale est quartae parti figurae. Unde cum voce illa in hac parte utar, quarta pars figurae, intelligi volo quadratum semiaxeos minoris.

Occurrent in hac parte propositiones aliquot eadem cum illis quas Apollonius de focis, demonstravit: quod eo consilio feci, ne quid in hac materia studiosus lector desideraret.

PROPOSITIO CXX.

Sint ABC ellipsis axes AC, BD, actaque per B tangente EF: centro BD intervallo DA circulus describatur AEFC, qui tangenti occurrat in E & F. dein ex E & F, normales demittantur EG, FH ad axem AC.

Dico tam AGC quam AHC rectangulum aequale esse quartae parti figurae



Demonstratio.

Quoniam tam EB linea aequidistat rectae AD quam EG ipsi BD, erunt EG, BD lineae aequales : est autem AGC rectangulum aequale quadrato EG, quod recta EG ducta sit ad diametrum circuli normalis; igitur & quadrato BD aequale est rectangulum AGC. eodem modo est FH quadrato, hoc est quadrato BD aequale rectangulum AHC, sed BD quadratum est aequale quartae parti figurae, igitur tam AGC quam AHC rectangulum est aequale quartae parti figurae. Quod erat demonstrandum.

Corollarium.

Hinc patet AG, HC lineas aequales & GH bifariam esse divisam in D.

PROPOSITIO CXXI.

Iisdem positis ducantur rectae BG, BH.
Dico BG, BH lineas simul sumptas axi AC esse aequales.

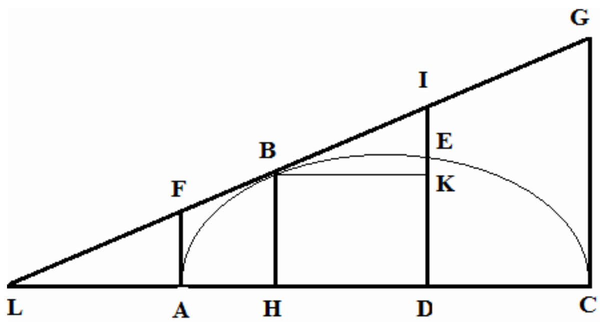
Demonstratio.

Quoniam aequales sunt (§120.*Cor*.) AG, HC,quadratum AG aequatur rectangulo ex AG & HC. lterum quia aequales sunt GD, DH, aequabitur rectangulum AGD bis sumptum rectangulo AGH. Quare cum quadratum AD aequale sit quadratis DG, AG & rectangulo AGD bis, idem quadratum AD aequabitur quadrato DG & rectangulo ex AG , HC una cum rectangulo AGH. Atqui rectangula AG, HC, & AGH, aequantur rectangulo AGC. Ergo quadratum AD aequale est quadrato DG una cum rectangulo AGC; hoc est quadratis DG, GE, hoc est quadratis DG, DB. sed iisdem aequatur quadratum GB, aequantur igitur quadrata AD, GB, ac proinde rectae AD, G B aequales sunt. Eodem modo demonstrabitur rectas CD, HB aequales esse. Ambae igitur GB, BH simul sumptae axi sunt aequales. Quod erat demonstrandum.

Corollarium.

Hinc sequitur: si ABC ellipsis axes fuerint AC, BD, & ex B vertice minoris axis rectae dimittantur BG, BH, aequales lineis AD, DC, secantes axem AC in G & H. Quod tam AGC quam AHC rectangulum, aequale sit quartae parti figurae, adeoque G & H sectionis poli sunt.

PROPOSITIO CXXII.



Ellipsim ABC, cuius diametri coniugatae AC, DE contingant in A & C, & alio puncto quovis B tres lineae AF, CG, FG: & FG quidem occurrat AF, CG rectis in F & G. productaque ED occurrat FG lineae in I, & ex B, recta demittatur BH ordinatim ad AC diametrum:

Dico rectangulum super AF, CG lineis aequari rectangulo super BHID.

Demonstratio.

Producatur FG linea donec cum axe conveniat in L. Quoniam DH, DA, DL lineae sunt in continua ratione, erit ut LD ad AD, hoc est ad DC sic LA ad AH, & invertendo componendo ut CL ad DL, sic HL ad AL, sed est ut CL ad DL, sic CG ad DI, & ut HL ad AL, sic HB ad AF, igitur ut CG ad DI, sic HB ad FA: adeoque rectangulum super AF, CG lineis aequale rectangulo BH, ID. Quod erat demonstrandum.

Corollarium.

Hinc sequitur rectangulum AF, CG vel HB, ID, aequale esse quartae parti figurae: ducatur enim BK parallela axi AC: erit rectangulum DK, DI aequale quadrato ED; igitur & AF, CG rectangulum est aequale quadrato ED hoc est quartae parti figurae.

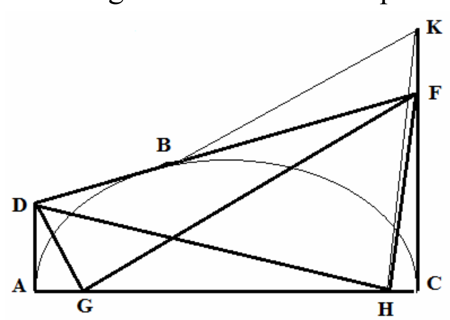
PROPOSITIO CXXIII.

Ellipsim ABC cuius axis AC, contingant in A & C, & alio quovis puncta B, lineae AD, CF, DF: & D F quidem conveniat cum AD, GF lineis in D & F dividatur autem linea AC in G & H , ut AGC, AHC rectangula sint aequalia quartae parti figurae ducanturque; lineae DG, GF,DH, HF.

Dico angulos DGF, DHF esse rectos; & si sint recti: dico DF, lineam tangere ellipsim.

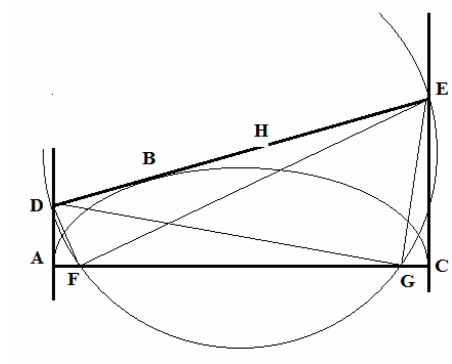
Demonstratio.

Rectangulum DACF est aequale quartae parti figurae, hoc est rectangulo AGC. Ergo ut AG ad AD sic FC ad CG , sunt autem anguli DAG, FCG recti; igitur DAG, FCG triangula similia : & angulus ADG aequalis angulo CGF, est autem angulus ADG, una cum angulo AGD aequalis uni recto, cum DAG angulus in triangulo ADG sit rectus: igitur & angulus CGF, una cum angulo AGD uni recto sunt aequales: ergo reliquus DGF est rectus: eodemmodo ostenditur angulus DHF rectus. Quod erat demonstrandum.



PROPOSITIO CXXIV.

Ellipsim ABC cuius axis AC contingant in A, C, B, punctis AD, CE, DE: & DE quidem occurrat rectis AD:CE in D & E. Fiant autem quartae parti figurae, aequalia rectangula AFC, AGC, seceturque ED bifariam in H:
Dico circulum centro H intervallo D& E, descriptum transire per F & G.



Demonstratio.

Iungantur puncta DF, FE, DG, GE. Quoniam tam angulus DFE, quam GE est rectus, & DE linea utcumque subtendens divisa bifariam in H, patet circulumm centro H intervallo HD descriptum transire per F& G. Quod erat demonstrandum.

Corollarium.

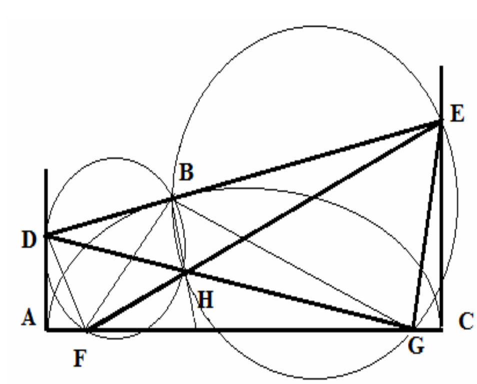
Hinc sequitur angulos EDG, FDA esse inter se aequales , est enim angulus ADF in demonstrationc praecedentis aequalis ostensus angulo GFE: sed angulo GFE aequatur angulus EDG cum eidem arcui EG insistat, ergo anguli EDG, FDA sunt inter se aequales.

PROPOSITIO CXXV.

Ellipsim ABC cuius axis AC contingant in A, C, B, lineae AD, CE, DE: & DE quidem conveniat cum AD, CE lineis in D & E. Fiant autem AFG, AGC rectangula aequalia quartae parti figurae: ductisque lineis FE, GD quae se intersecent in H, ex puncto H ad contactum B, ducatur recta HB.
Dico HB normalem esse ad tangentem DE.

Demonstratio.

Ducatur recta ED, CE. Anguli DFE, ECD recti erunt. Iam super H D, HE lineis ut diametris circuli describantur DBH, EBH, Quoniam DH, HE lineae non sunt in directum, patet DBH, EBH circulos se invice secare in puncto aliquo B. Iunctis igitur punctis HB; ducantur rectae DB, EB: erunt anguli DBH, EBH recti, adeoque DB,EB linea in directum, & HB lineae normalis rectae DE. sunt autem ut ante ostendi DFE, EGD anguli recti ; igitur DE linea est tangens. Quare recta BH est normalis ad ED tangentem. Quod

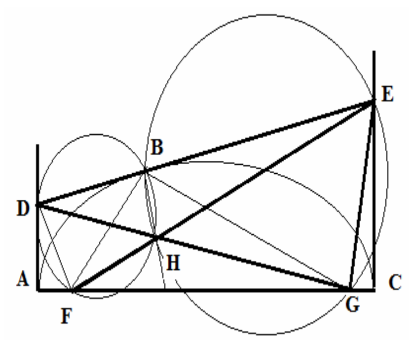


erat demonstrandum.

PROPOSITIO CXXVI.

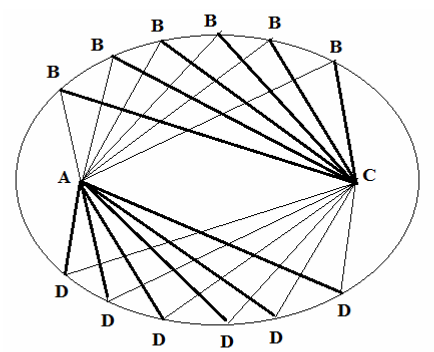
Eadem manente figura: ducantur FB, BG.
Dico angulos DBF, EBG ad contingentem esse aequales.

Demonstratio.



Quoniam anguli DFH, EGA recti sunt, transibunt per F & G. circuli DFH, EGH transit autem uterque etiam per B: quia anguli EBH, DBH sunt recti: igitur tam anguli DBF, DHF quam EBG, EHG anguli sunt inter se aequales: sed angulus DHF aequalis est angulo EHG: ergo & DBF aequatur angulo EBG. Quod erat demonstrandum.

Scholion

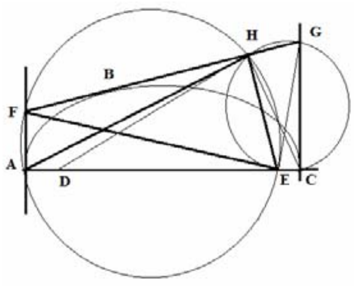


Cum punctum B *in peripheria assumptum, sit quodcunque sequitur lineas omnes ex* F *in peripheriam ellipsis ductas, reflectendas in G. Quare & puncta FG poli seu foci a nonnullis vocantur: quae ab Apollonio puncta ex comparatione facta dicuntur, porro haec in ellipticis mirabiles habent proprietates : inter reliquas placuit sequentiem hic adiungere. Sint* A, C , *foci ellipsios, quorum distantia par sit intervallo oculorum, ponaturque in A oculus sinister, & dexter in* C. *Dico illum per totum speculum videri ab oculo sinistro in* A ; *demonstratio patet: species enim obiecti* A, *per totum speculum diffusa, reflectuntur in C, & species obiecti* C *per totum diffusae reflectuntur in* A. *Quare obiectum* A *per totum apparebit speculum, oculo* C, *uti & obiectum* C, *oculo* A. *Hunc sequitur quod minimum & visibile positum in C, maximum apparebit oculo in A posito: quia apparebit per totam speculis superficiem diffusum.*

PROPOSITIO CXXVII.

Ellipsim ABC, cuius axis AC & poli DE contingant in punctis A, C, B rectae AF, CG, FG; & FG quidem conveniat cum AF, CG lineis in F& G. erigatur ex E, linea EH normalis ad tangentem FG, iunganturque puncta AH, CH.
Dico angulum AHC rectum esse.

Demonstratio.



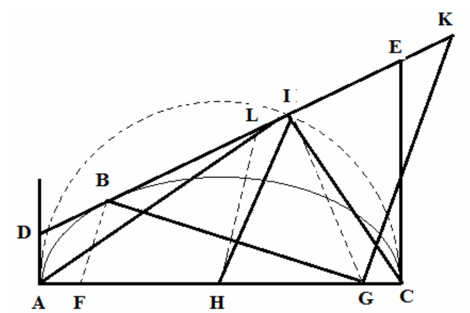
Ductis lineis FE, EG describantur super FE , EG diametris circuli FHE, HGC: ac circulus quidem FHE,cum anguli EHF, EAF sint recti, transibit per H, F, A, puncta ; circulus vero HGC: cum DHE, ECG anguli quoque recti sint transibit per H, C. Erunt igitur tam anguli AHF, AEF quam EHC, EGC, anguli aequales : sed angulus GCE per demonstrata in 121. huius aequalis est angulo FEA, igitur & angelo FHE recto aequalis angulus AHC, quare & ipse rectus. Quod erat demonstrandum.

PROPOSITIO CXXVIII.

Ellipsim ABC, cuius axis AC contingant in A, C, B, lineae AD, CE, DE, ac DE quidem occurrat AD, CE lineis in D & E : sint autem poli F, G, centrum H ductaque ex F recta FB ad punctum contactus ducatur ex H linea HI parallela rectae FB occurrens ED lineae in I.

Dico HI lineam aequalem lineae HC, & si HI occurrens ED rectae, sit aequalis HC. Dico HI lineam aequidistare FB.

Demonstratio.



Fiat BI aequalis IK: iunganturque BG, GK, & rectae ducantur AI,IC: Quoniam IB, IK sunt aequales, erit BI ad IK, ut FH ad HG: adeoque BF, KG lineae parallelae, & angulus BKG aequalis angulo DBF, hoc est IBG: quare BG,GK lineae aequales: sunt autem & duo reliqua latera BI,IG aequalia duobus lateribus KI, IG. Angulus ergo BIG, aequalis angulo KIG: adeoque GI linea normalis tangenti DE, & angulus AIC rectus. Quare circulus centro H intervallo HC descriptus transibit per I, eritque HI linea aequalis lineae HC. Quod erat primum.

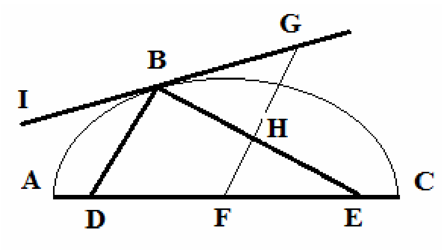
Reliquis manentibus, sit iam HI linea quae occurrat tangenti ED in I, aequalis lineae HC. Dico HI rectam aequidistare lineae BF: sin vero; ducatur ex H linea HL, parallela rectae FB occurrens ED tangenti in L; erit igitur HL linea aequalis lineae HC, hoc est HI. Quare circulus centro H intervallo HC descriptus transilit per I & L puncta. Quod impossibile igitur HL non est parallela ipsi FB: nec quaevis aliae praeter HI lineam. Quod erat demonstrandum.

PROPOSITIO CXXIX.

Sit ABC ellipseos axis AC, poli autem D, E ex D & E rectae inflectantur DB, EB convenientes in puncto quodam peripheriae B.
Dico DB, EB lineas simul sumptas aequari axi AC.

Demonstratio.

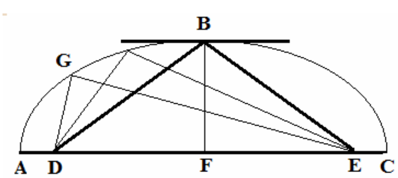
Sit F centrum ellipseos; actaque per B tangente BG: ducatur recta FG parallela lineae DB secans EB lineam in H. Quoniam BD, FG lineae parallelae, erit angulus FGB aequalis angulo DBI hoc est EBG, adeoque HB, HG lineae aequales: rursum cum sit ut DE ad FE, sic BE ad HE, sitque DE dupla FE, erit & EB, dupla rectae BH id est HG: sed etiam BD dupla est FH, cum sic ut DE ad FH, sic DB ad FH; igitur EB, BD lineae simul sumptae duplae sunt rectae FG hoc est FC*:* quare & aequales axi AC. Quod erat demonstrandum.



PROPOSITIO CXXX.

Triangulorum isoperimetrorum maximum est isoscelium.

Demonstratio.



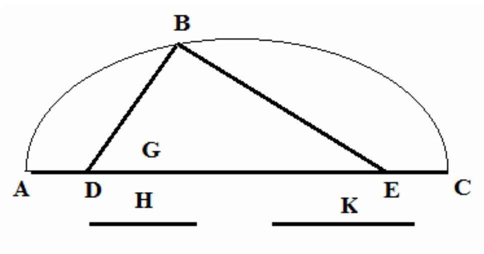
Describatur ellipsis quaecunque ABC cuius axes AC, FB poli D, E, iunganturque puncta DB, BE tum super ED bali triangula constituantur quaecunque DGE, quorum vertices G sint in peripheria. Quoniam tam DB, BE lineae quam DG, GE simul sumptae sunt aequales axi AC: patet DBE, DGE triangula esse isoperimetra: dico autem illorum esse maximum triangulum DBE agatur enim per B tangens: quae cum in uno tantum puncto B ellipsi occurrat & reliqua sui parte tota cadat extra, patet DGE triangula quae terminantur in ellipsi minorem habere altitudinem triangulo DBE, adeoque illo esse minora : est autem DBE triangulum isosceles, quia DF, FB latera aequalia sunt lateribus EF, FB & anguli illis contenti recti; igitur triangulorum isoperimetrorum maximum est isosceles. Quod erat demonstrandum.

PROPOSITIO CXXXI.

Oporteat e focis ellipseos DE duas inclinare ad idem punctum perimetri quae datam contineant rationem H ad K.

Debet autem data ratio maior esse ratione AD ad DC, minor vero ratione AE ad EC.

Constructio & demonstratio.



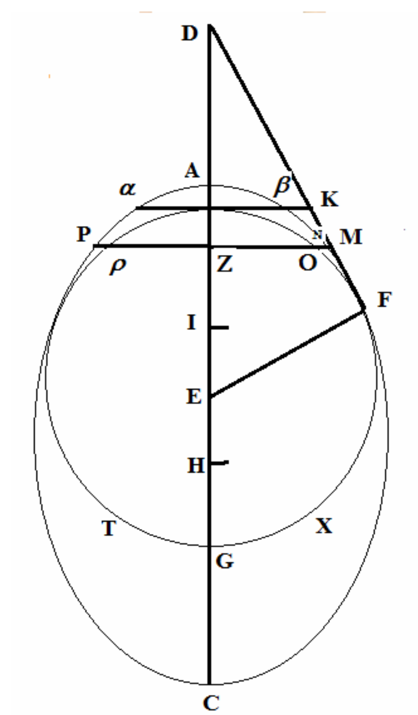
Secetur axis AC in G, secundum datam rationem H ad K, quae cum ponatur maior ratione AD ad DC, & minor ratione AE ad EC, manifestum est AG lineam maiorem esse recta AD: minorem vero AE, ac proinde punctum G cadere inter polos D, E erigatur igitur ex D ad peripheriam linea DB aequalis rectae AG. Iunganturque puncta BE. Dico factum esse quod petitur. Cum enim rectae duae DB, BE simul sumptae sint aequales axi AC, sit autem per constructionem DB linea aequalis lineae AG, erit BE reliqua aequa aequalis reliquae GC: igitur DB est ad BE, ut AG ad GC, id est ut H ad K. Inclavimus igitur, &c. Quod erat faciendum.

PROPOSITIO CXXXII.

Ellipsim ABC contingat in B linea BD conveniens cum axe maiore CA, in D; ex B autem contactu, normalis ad contingentem ponatur BE, occurrens axi in E.

Dico EB lineam brevissimam esse illarum quae ex E puncto ad peripheriam ellipseos duci possunt.

Demonstratio.

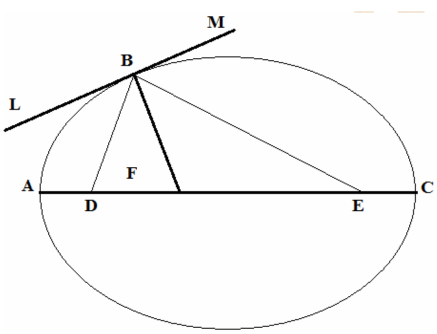


Centro E intervallo EB circulus describatur FBG occurrens axi in F & G centrum ellipseos sit H. Quoniam DB ellipsim contingens cui axe maiori convenit in D, & angulus DBE rectus est BE linea non transit Per H centrum ellipseos : si enim E centrum est, recta EB, normaliter ad contingentem posita axis erit coniugatus axi AC, (cum aequidistances omnes contingenti DB bifariam & ad rectos divideret) adeoque DB aequidistaret axi AC: non igitur E centrum est ellipseos, nec EB diameter : quia vero DB cum axe convenit ad partes A, EB linea minor est semidiametro, sibi parallela : adeoque & minor est semiaxe HC,& multo minor recta EC, quare circulus radio EB descriptus, occurrit axi in G intra ellipsim, punctum igitur G, supra C est.

Rursum cum HC id est AH, maior sit ostensa quam EB id est EG, ablato communi EH, manet AE maior quam HG, posita autem EI aequali EH, recta FI aequatur HG, igitur AE quoque maior est FI: ablato ergo ex FE & AE, communi IE, manet IA maior quam IF: unde & F punctum cadit intra ellipsim infra A. Ulterius ponatur per F, contingens FK, cui aequidistet PQNM, erit igitur ut MB quadratum ad quadratum BK sic QMO rectangulum ad quadratum FK; sed ut MB quadratum ad quadratum BK sic PMN rectangulum ad rectangulum αKβ ; igitur ut QMO rectangulum ad quadratum FK sic PMN rectangulum est ad rectangulum αKβ: & permutando, invertendo ut FK quadratum ad rectangulum αKβ, sic QMO, rectangulum est ad rectangulum PMN: est autem FK quadratum maius rectangulo αKβ, igitur & QMO rectangulum maius est rectangulo PMN : iterum QMO rectangulum una cum quadrato ZO aequale est quadrato ZM, & PMN rectangulam una cum quadrato ZN, eidem quadrato Z M aequale est; aequale igitur est rectangulum QMO & una cum quadrato ZO, rectangulo PMN, una cum quadrato ZN; a quibus si inaequalia auferantur rectangula QMO, PMN , inaequalia remanent quadrata ZO, ZN: & quia QMO rectangulum maius est rectangulo PMN, quadratum ZO minus est quadrato ZN: & ZQ minus quadrato PZ; puncta igitur O & Q intra ellipsim sunt: similiter ostendentur puncta XT, & quaevis alia perimetri circuli FHG esse intra ellipsin ; circulus igitur FBG totus intra ellipsim cadit: unde cum rectae omnes ex E centro circuli ad ellipsis peripheriam duci; prius circulo occurrant quam ellipsi: adeoque semidiametris eiusdem maiores sint igitur EB, quae in B puncto communi ellipsi & circulo terminatur omnium illarum brevisssima est quae ex E puncto ad ellipsis peripheriam duci possunt. Quod erat demonstrandum.

PROPOSITIO CXXXIII.

A puncto (H) in axe ellipseos assignato lineam ad perimetrum brevissimam ducere.



Constructio & demonstratio.

Sint D, & E foci ellipseos. Axem AC seca in F, ita ut AF sit ad FC, sicut DH est ad HE. Tum ex polo ad perimetrum aptetur DB aequalis ipsi AF, iunganturque HB. Dico HB esse brevissimam.

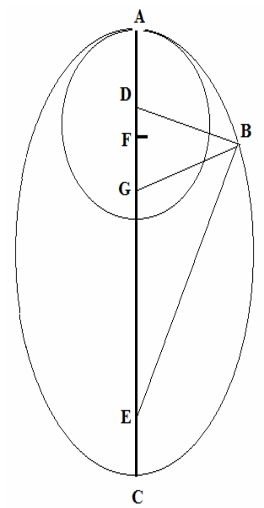
Ducatur enim ex polo E ad B recta EB, & LM tangens ellipsim in B, DB, BE aequantur axi. sed DB aequalis est AF. Ergo BE aequalis est FC. Ergo DB est ad BE, ut AF ad FC, hoc est ex const. ut DH ad HE. Ergo anguli DBH, EBH aequantur, aequantur autem & anguli ad contingentem DBL, EBM, toti igitur anguli HBL, HBM aequales sunt;normalis igitur est HB ad tangentem, ergo per praecedemem brevissima omnium quae ex puncto H ad perimetrum duci possunt. Factum igitur est quod petebatur.

Si punctum F incidat in polum D, aut inter A, & D; tunc brevissima ex dato puncto H ad perimetrum erit pars axis , ut patet constructionem ac demonstrationem priorem consideranti.

PROPOSITIO CXXXIV.

In data ellipsi circulum describere maximum eorum qui ellipsim in termino axis contingunt & ab ellipsi comprehenduntur.

Constructio & demonstratio.



Poli ellipseos sint D & E. Fiat ut CD ad DA; sic EF ad FD. Dico circulum centro F intervallo A descriptum eum esse qui petitur. Cum enim ex const. sit CD, ad DA, ut EF ad FD. Patet ex praeccd. FA esse brevissimam omnium, quae a puncto F ad perimetrum duci possunt; circulus igitur centro F per A descriptus tangit ellipsim, quod erat primum: quod autem tangentium intra ellipsim maximus sit,sic ostendo. Sume ulterius punctum aliquod G pro centro maioris circuli, quoniam igitur EG est ad GD, in minori ratione quam EF ad FD, hoc est quam CD ad DA, sit exemp.grat. ut EG ad GD, sic CF ad FA: eritque FA necessario maior quam DA: adeoque punctum F cadet ultra polum D versus E: si igitur ex polo D ad perimetrum aptetur DB aequalis GA, iungaturque GB, patet ex praeced. GB fore minimam omnium quae ex G ad perimetrum ducuntur. Quare GA maior est quam GB, circulus ergo centro G per A descriptus extra ellipsim cadit. Similiter ostendemus quemlibet circulum alium maiorem circulo qui intervallo FA ante descriptus est, cadere extra ellipsim: ergo ille omnium intra ellipsim tangentium, maximus est. In data igitur ellipsi, &c. Quod erat faciendum.

Corollarium.

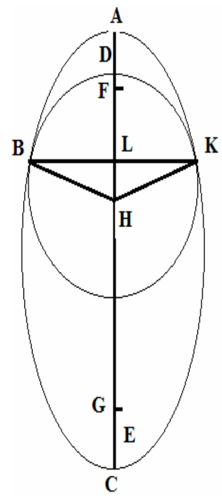
Ex huius propositionis discursu clare constat circulos omni intervallo descriptos quod minus est intervallo FA ellipsim intra contingere in puncto A. Si centro inter F & A constituto pertingant usque ad A, verticem axeos. Illi etenim circuli contingent eum qui radio FA descriptus est, eoque minores erunt; quare etiam ellipsim intra contingent cuius axis AC.

PROPOSITIO CXXXV.

Sit ABC ellipseos axis maior *A*C & in illo poli D, E, fiatque ut CD ad DA sic EF ad FD, & DG ad GE.

Dico ex quovis puncto rectae FG circulos posse describi qui ellipsim intus in duobus punctis contingant: centra vero illorum consistere inter F & G exclusis terminis.

Demonstratio.



Sumatur enim quodvis in FS recta punctum H, & ex linea ducatur HB, brevissima illarum a quae ex H ad peripheriam duci poterunt ; dein ex B ordinatim ad axem ponatur B L K, & iunge HK, HB, patet per elementa HK iuncta aequari HB, adeoque circulum centro H intervallo HB descriptum transire per K & B: & cum HK, HB lineae sint brevissimae per constructionem, patet circulum BDK , totum cadere intra ellipsim ac proinde eam in B & K, punctis contingere. Quod autem centra circulorum ellipsim in duobus punctis contingentium consistant inter F & G exclusis terminis, ex eo patet quod FA, GC lineae brevissimae sint illarum quae ex F & G, ad peripheriam duci poterunt, adeoque circuli centro F vel G, & intervallo quovis maiore quam sit FA vel GC descripti ellipsim secent: radiis vero FA vel GC descripti maximi sint, illorum qui ellipsim intus in uno tantum puncto contingunt.

PROPOSITIO CXXXVI.

Eadem manente figura: propositum sit in axe punctum designare quo centro circulus describatur, qui ellipsim in dato puncto intus contingat.

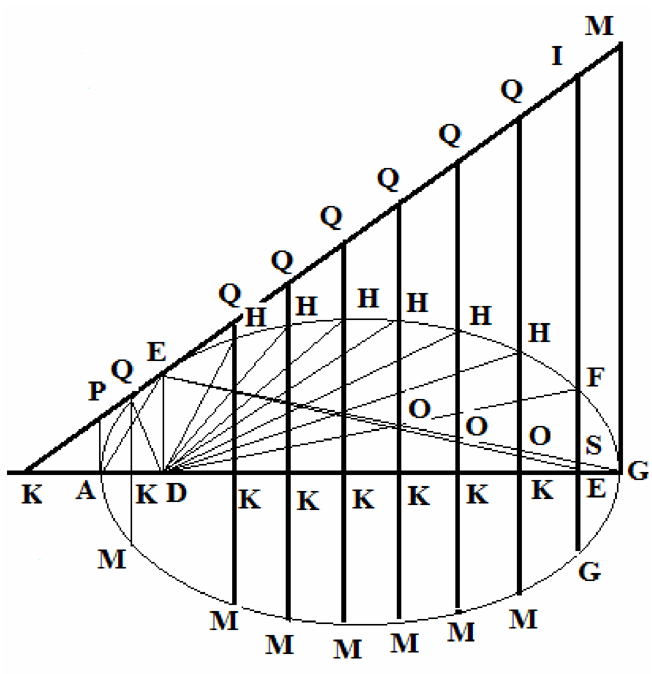
Constructio & demonstratio.

Sit datum in peripheria punctum B, per quod si acta intelligatur contingens, demittatur ex B linea B H, normalis ad tangentem, occurrens axi in H. Manifestum est H punctum satisfacere petitioni; nam cum HB linea brevissima sit earum quae ex H ad peripheriam duci possunt, continget circulus centro H intervallo HB descriptus ellipsimin puncto B*;* igitur, &c. Quod erat faciendum.

PROPOSITIO CXXXVII.

Ellipseos axis sit A, C, poli D, E, ex quibus ductae sint DB, EFG normales axi; ellipsim autem tangat KBI in B, occurrens axi in K, rectae vero GF in I, iungaturque DF.
Dico rectangulum FIG quadrato DE aequale esse.

Demonstratio.



Quoniam DB , EB ductae sunt e polis ad contactum, anguli KBD, IBE aequales sunt , sed, quia DB, EI parallelae , angulus KBD angulo EIB aqualis est, aequantur igitur anguli IBE, EIB, ergo BE, IE aequales sunt. Deinde quia rectangula CEA, CDA aequalia sunt, etiam quadrata EF. DB sunt aequalia, adeoque & rectae EF,DB aequales. Quare cum DB, BE aequentur EF,FE etiam BE, DF aequales erunt. Atqui BE ostensa est aequalis EI. Ergo & DF est aequalis EI, & quadrata proinde DF, EI aequalia sunt. Sed quadrata DE, EF aequantur quadrato DF; &, quia GF bisecta est in E, eique adiecta FI, rectangulum GIF cum quadrato EF aequatur quadrato EI. Quadrata ergo DE, EF aequantur rectangulo GIF cum quadrato EF. Dempto igitur communi quadrato EF, remanent aequalia rectangulum GIF & quadratum DE. Quod erat demonstrandum.

PROPOSITIO CXXXVIII.

Iisdem positis ducantur quotcunque aliae Q, H, K, M, normales axi.
Dico rectangula H, Q, M, quadratis DK esse aequalia.

Demonstratio.

Ellipsim tangat CN in C, occurrens tangenti BN in N, ducaturque BC secans QM in O, & IG in S quoniam NC, IG, QM sunt normales axi, aequidistant ; ergo per ea quae propos 99.huius demonstravimus, rectangulum FIG, aequatur quadrato SI, & rectangulum HQN quadrato QO aequale est quare ut quadratum IS ad quadratum QO, hoc est ut quadratum SB ad quadratum OB, hoc est ut quadratum ED ad quadratum KD, ita rectangulum FIG ad rectangulum GQM; & permutando ut rectangulum FIG ad quadratum ED, ita rectangulum HQM ad quadratum KD, sed rectangulum FIG per praeced. aequatur quadrato ED. Ergo rectangulum quoque HQM aequatur quadrato KD.

Similiter demonstrabimus ad alteram partem poli D, rectangula HQM aequatur quadratis KD esse aequalia tangat enim ellipsim AP, in A occurrens tangenti in P, & tactus iungat AB secans QM in R, in triangulo BCN. Ducatur aliqua QO, parallela NC normali ad axem, ita se habens ad QB, ut QR est ad QB, erit igitur permutando ut QB ad BQ hoc est ut KD ad DK, ita QO ad QR. Ergo ut quadratum KD ad quadratum DK, ita quadratum QO ad quadratum QR, hoc est rectangulum HQM ad rectangulum HQM;

permutando igitur ut rectangulum HQM ad quadratum KD, ita rectangulum HQM ad quadratum DK.
Atqui supra demonstrarum est rectangulum HQM (illud nempc quod est versus C) aequari quadrato KD, ergo rectangulum quoque HKM quod est versus A, aequatur quadrato DK. Omnia igitur rectangula HKM, &c. Quod erat demonstrandum.

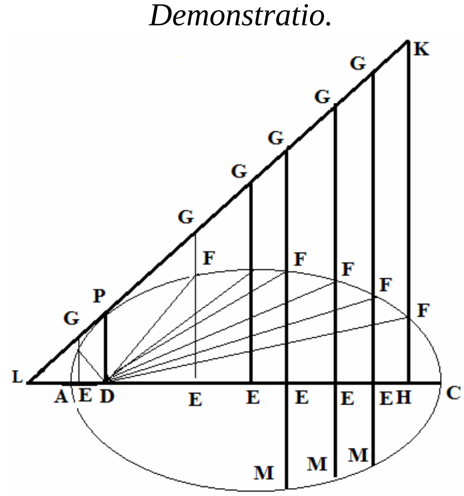
Corollarium.

Ex discursu demonstrationis iam allatae licet colligere quadrata tangentium CN, AP, quadratis CD, DA esse aequalia.

PROPOSITIO CXXXIX.

Data sit ellipsis cuius axis AC, poli D, H, ex polo D ducta sit ad perimetrum DP normalis axi, & in P ellipsim tangat linea GPG. Ducantur autem quotcunque normales axi GFE, iunganturq; DF, DF.
Dico lineas omnes DF, lineis omnibus GE aequales esse.

Demonstratio.



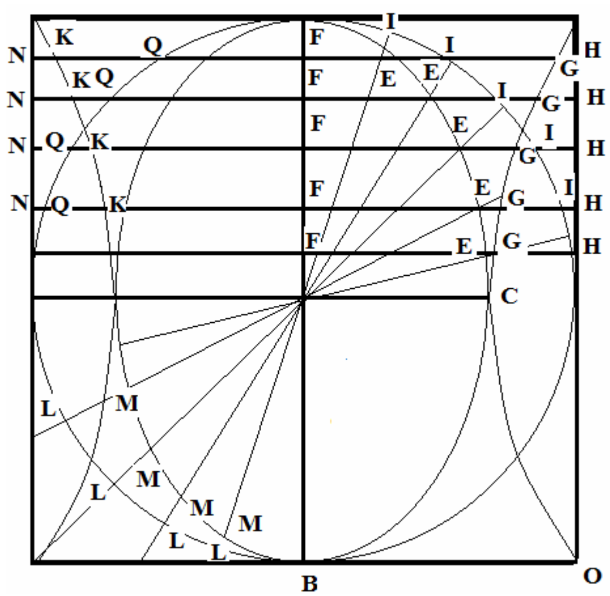
Producatur una rectarum GE in M, per praeced. rectangulum FGM aequatur quadrato DE; addito igitur communi quadrato EF, aequantur quadrata DE, EF, hoc est quadratum DE, rectangulo FGM cum quadrato EF, hoc est, quadrato GE. Quia igitur quadratum DF aequatur quadrato GE, etiam recta DF rectae GE aequalis est. Eodem discursu reliquae omnes DF, reliquis omnibus GE aequales sunt. Quod erat demonstrandum.
In libro de hyperbola, tria sequentia theoremata licet sint demonstranda quod ab hyperbolae proprietatibus dependeant, ob miram tamen cum ellipticis affectionibus connectionem visum est non alienum hoc loco proponere.

PROPOSITIO CXL.

Eadem manente figura, si rectis DF e polo ductis aequentur lineae EFG normales ad axem AC.

Dico lineam per puncta G ductam esse rectam quae ellipsim contingat in P.

Demonstratio manifesta est ex propositione praecedenti, contingit in C. Demonstrationem vide in lib. de hyperbola.



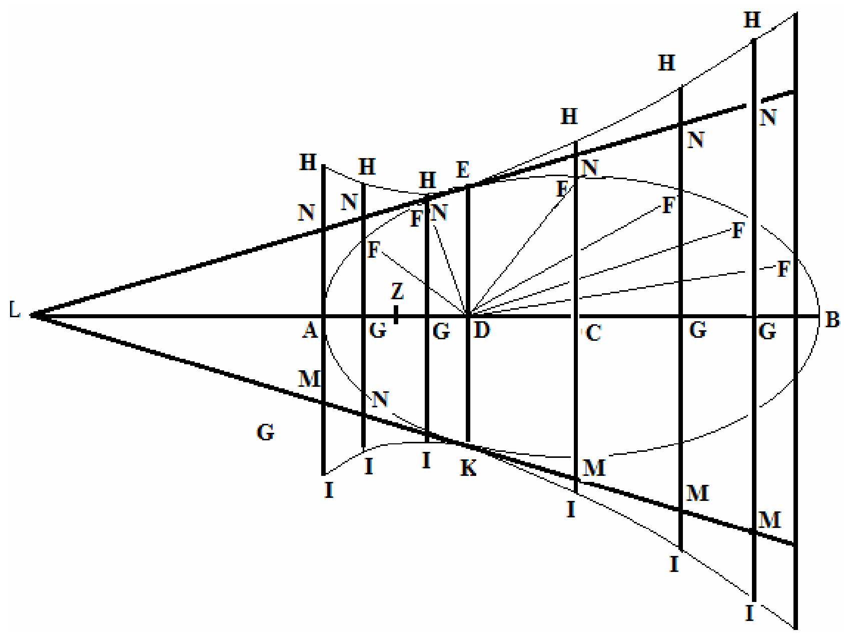
PROPOSITIO CXLI.

Data sit ellipsis axem habens AB, centrum D sumatur in axe punctum quod prima sit centrum ellipseos, ex eo ducatur ad perimetrum normalis axi DC ac deinde quotcunque aliae DE, DE : quibus aequales fiant lineae FEG axi normales.

Dico lineam per puncta G descriptam esse hyperbolam quae idem habeat cum ellipsi centrum D, eamque contingit in C.

Demonstrationem vide in lib. de hyperbola.

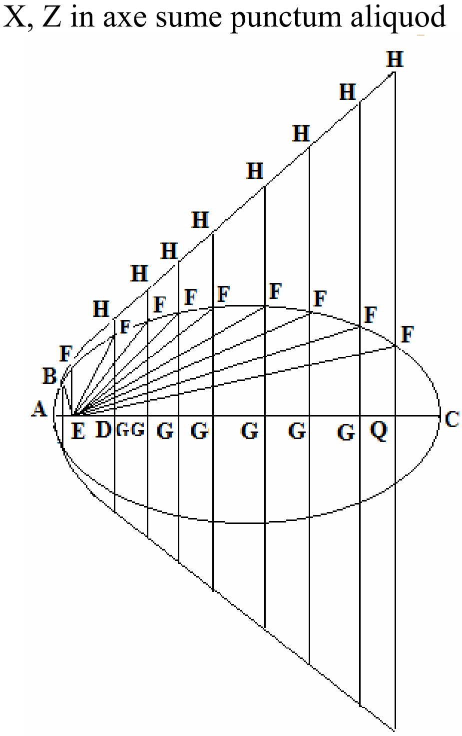
PROPOSITIO CXLII.



Data sit ellipsis axem habens AB, centrum C, polos X, Z in axe sume punctum aliquod D inter centrum C & polum D, ex quo ducatur ad perimetrum DE normalis axi, ac deinde quaevis aliae DF, DF; quibus aequales fiant GFH axi normales.

Dico lineam per puncta H, H descriptam esse hyperbolam, quae ellipsim tangat in F.

Demonstrabitur in libro de hyperbola.



PROPOSITIO CXLIII.

Data rursum sit ellipsis axem habens AC, polos D, Q, in axe sumatur punctum E inter polum D & verticem A, ex quo ducatur ad perimetrum normalis axi EB : & quotvis aliae EF, quibus aequales fiant GFH axi normales.

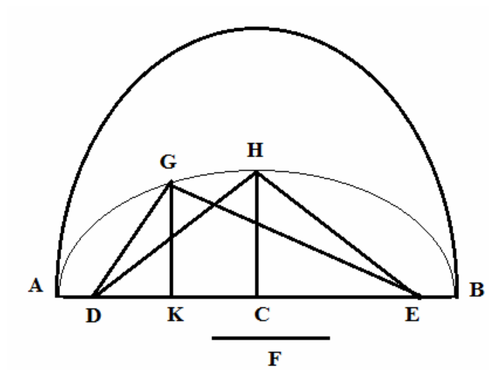
Dico lineam quae per puncta H describitur hyperbolam esse quae ellipsim ambiat & tangat in puncto B.

Demonstrationem dabimus in libro de hyperbola.

PROPOSITIO CXLIV.

Data basi aggregato laterum & altitudine triangulum exhibere.

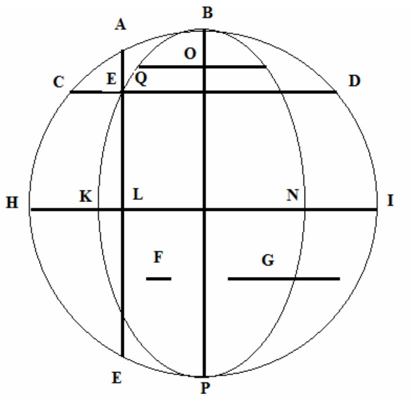
Constructio & demonstratio.



Dato aggregato laterum ponatur AB, aequalis, qua bifariam divisa in C fiat DE aequalis basi trianguli bifarium divisa in C sic ut utrimque relinquantur aequales AD, BE, altitudini autem sit aequalis F, ex lateribus AC, CB, DE fiat triangulum DHE (nam AC, CB, simul sumptae maiores sunt DE,) erit DHE isosceles. Deinde fiat ut quadratum HC ad F quadratum, ita rectangulum ACB ad AKB, & erigatur KG aequalis F parallela HC, & iungantur DG,GE. Dico DGE, esse triangulum quaesitum, quoniam ACB, rectangulum est ad AB rectangulum, ut quadratum HC ad quadratum F, hoc est, quadratum, erunt puncta A,G,H,B ad eandem ellipsim cuius AB, sit axis: & quia AD, ipsi EB, itemque DH, HE, aequales sunt ipsi AB, erunt DE, puncta ex comparison facta sive foci ellipseos, quare DGE, latera aequalia sunt axi AB, hoc est aggregatio laterum estque basis data DE, & altitudo F hoc est GK. Igitur exhibuimus triangulum quod quaerebatur.

PROPOSITIO CXLV.

Rectam AB, subtensam cuiusvis arcus circuli ABC, altera secare CD, eidem ad angulos rectos ut CE ad ED, datam obtineat rationem F ad G.



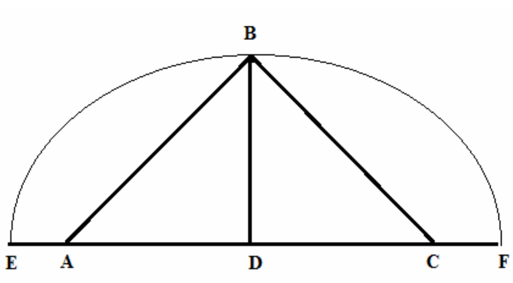
Constructio & demonstratio.

Libro de circulorum proprietatibus proposuimus hoc problema : *sed quoniam eius demonstratio ab elliptica proprietate dependet, id circo in hunc locum eam distulimus , constructio vero est.* Ducatur diameter HI normalis ad AB secans AB in L. Fiatque ut F ad G, sic HK ad KI: sumpa deinde IM aequali ipsi HK dividatur diameter HP, in O puncto ut dividitur KM in L: deinde rectangulo NOP, fiat aequale quadratum LE; tandem ducatur per E, recta normalis CED. Dico CED, divisam in E, secundum rationem F ad G. Quoniam sunt lineae NP, KM ad angulos rectos bifariam divisae; igitur descripta ponatur circa illas tamquam axes ellipsis NKP, erunt itaque OQ, LE, ordinatim positae ad singulos axes:& quia axes similiter divisae sunt in O & L, estque rectangulum NOP aequale quadrato LE, patet a punctum E esse ad ellipsim per puncta N, K, P, M descriptam. Ergo est ut HK ad KI, hoc est ut F ad G, sit CE ad ED, patet rectam
D applicatam esse in circulo normaliter ad CD, ut CE ad ED datam rationem obtineat F ad G. Quod fuit demonstrandum.

PROPOSITIO CXLVI.

Data recta AC & altitudine BD, ellipsim describere cuius poli sint A, & C.

Constructio & demonstratio.



Fiat super AC linea in altitudine BD triangulum isosceles ABC dein AC linea utrimque aequaliter producatur in E & F: ut tota EF sit aequalis duabus AB, BC, cum per E, B, F, puncta ellipsis describatur. Dico illam esse quae petitur. Quoniam EF linea divisa est bifariam in D & non bifariam in A: erit EAF rectangulum una cum quadrato AD, aequale quadrato ED hoc est per constructionem quadrato AB; sed etiam quadrato AB aequalia sunt quadrata AD, BD; dempto igitur communi quadrate AD, manet EAF rectangulum aequale quadrato BD id est quartae parti figurae. Eodem modo ostenditur quadrato BD aequari rectangulum FCE : quare A & C, foci sunt descriptae ellipseos EBF. data igitur linea & altitudine, &c. Quod erat faciendum.

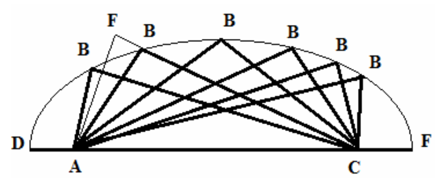
Corollarium.

Hinc sequitur dato quovis triangulo isosceli ABC continente ad verticem, angulum quemcunque, describi posse ellipsim cuius foci sint extrema basis trianguli dati ABC. Demonstratio patet ex propositione.

PROPOSITIO CXLVII.

Super AC linea descripti sint quotcunque triangula isoperimetra ABC, AGC. Dico puncta G, B, B esse ad eandem ellipsim cuius poli sint A & C.

Demonstratio.



Producatur AC utrimque aequaliter in D & E, ut tota DE sit aequalis duabus AB, BC, tum per puncta D, E, G ellipsis describatur. Dico illam transire per reliqua puncta B, B sin vero; transeat supra vel infra B, ac primum supra per punctum F, producta CB donec peripheriae occurrat in F, iungantur AF. Quoniam igitur GF puncta ad ellipsim sunt, cuius poli A & C, erunt AGC, AFC triangula isoperimetrae est autem AGC triangulum per constructionem isoperimetrum triangulo ABC; igitur AFC, ABC triangula sunt isoperimetra, quod fieri non potest. quare DGE, ellipsis non transit supra B, sed nec infra B, cadere eodem modo demonstrabitur. Ergo per B, B puncta; igitur GBB sunt ad ellipsim cuius poli sunt A, C. Quod erat demonstrandum.