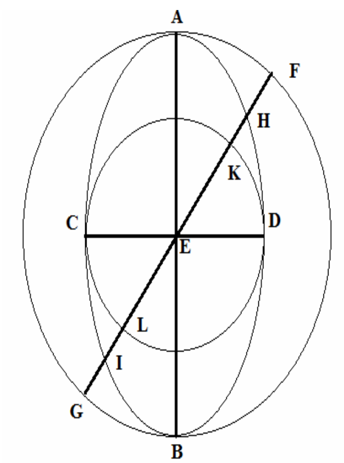
THE ELLIPSE: PART THREE

Part three considers both the similar and dissimilar properties of conjugate axes and diameters of the ellipse.

PROPOSITION LXXI.

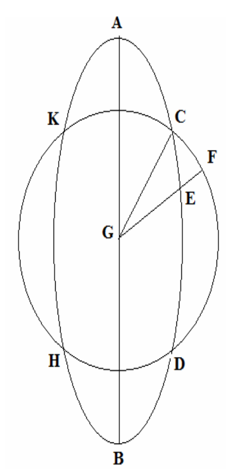
For an ellipse, the axes are the maximum and minimum of the diameters.

Demonstration.



The axes of the ellipse ABC shall be AB and CD ; and indeed AB shall be the major axis and CD truly the minor axis. I say AB to be the maximum of the diameters, CD truly the minimum, with the centre of the ellipse E: the circle AFG shall be described with the interval EA : it passes through B, and the whole of its remaining part will lie outside the ellipse: then some diameter FG may be drawn through E, intersecting the ellipse at H and I, but the circle at F and G.
Because the whole circle AFG falls outside the ellipse, the line FG will be greater than the line HI : and therefore AB is greater than HI. It is shown likewise for any other diameter; therefore the axis AB is the greatest of the diameters of the ellipse ABC. Which was the first part to be established.

Again, with the centre E and with the interval ED, the circle DKL is described intersecting the line FG at K and L: this circle passes through C and the whole of its remaining part will fall within the section; therefore the line HI is greater than the line KL, that is equal to CD: Whereby since the same may be shown for all the other lines which do not pass through C and D, CD will be the minimum diameter of all the lines which are able to be drawn through the ellipse ADB. Q.e.d.

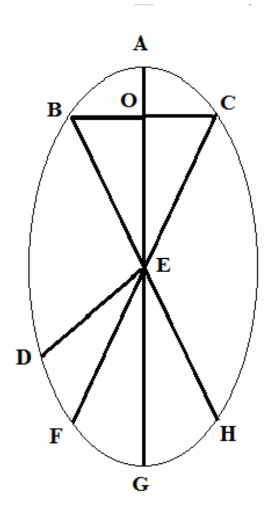


First Corollary.

AB shall be the major axis of the ellipse ABC, centre G; a diameter which is closer to that major axis is greater; and that which is more removed is smaller: and the diameter GC may be put closer to the axis than GE. I say GC to be greater compared with that GE, indeed with centre G and with the interval GC, a circle may be described which will cross that ellipse only at the four points CKHD, whereby GE does not extend to the periphery, since it is smaller than GC. Q.e.d.

Second Corollary.

Again, I show as follows alternately from the first corollary: a diameter is closer to the axis, which makes a smaller angle or sector with the axis. The major axis of the ellipse shall be AG and the diameter BE shall make the smaller sector BEA with the axis, than the diameter DE shall make the sector DEG with the axis.

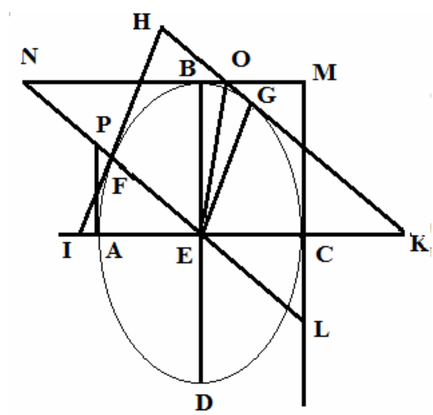


Therefore since the sector DEG is greater than the sector BEA, the sector FEG shall be equal to the sector BEA, and FE shall cross the ellipse at C, and the line BOC may be drawn; the sector BEA is equal to the sector FEG ; from the construction this is the sector to the vertex AEC. Therefore BC is bisected at O by the axis AG. Therefore the angles at O are right. Therefore it is evident the angle BEA to be equal to the angle AEC, that is, to the angle FEG, that is smaller than the angle DEG; therefore it is clear from the first diameter BE that the sector made to be smaller closer to the axis than DE, which shall be greater.

PROPOSITION LXXII.

The rectangle formed under half the axes is equal to the parallelogram formed by the conjugate radii.

Demonstration.



AC, BD shall be the axes of the ellipse ABC with centre E, and EF, EG shall be any radii taken together as it pleases. And with the tangents acting through F and G which shall meet at H, which shall meet the axis AC at I and K: also the tangent lines are drawn through C and B: which moreover shall meet at M: and which tangent lines shall cut [the other tangent line] HK, and EF in O, L, N: then with the points E O joined the tangent acts through A, cutting the line EN at P. Since both the lines NO, KE as well as the lines OK, NE are themselves mutually equidistant; NO, KE will be a parallelogram, with half the diameter OE: moreover the triangles EOB, EOC are equal, and therefore the remaining triangles EBN, EGK are equal to each other. Again since the lines AP, CL shall be equidistant, and the lines AE, CE shall be equal, the triangle ECL shall be equal to the triangle EAP, that is to the triangle EIF. Therefore since the triangle EGK is equal to the triangle EBN, and the triangle IFE to the triangle ECL, these four triangles will be in proportion; truly also since they shall be similar to each other, especially EGK to IFE, and EBN to ECL, therefore the right lines KE, EI, NE, EL, are proportionals, whereby since the above with the proportionalities requiring to be put in place in a straight line, the triangles IFE, EGK, IHK are similar to each other, and the triangles CLF, EBN, LMN also are similar between each other, so that the two triangles IFE, EGK are equal to the two triangles IFE, EGK thus as shown above, to the two triangles LCE, EBN. Therefore also the triangle IHK is equal to the triangle LMN, and hence with the equal parallelogram GEPH from the conjugate radii taken together, shall be equal to the rectangle BECM contained by the half axes. Q.e.d.

First Corollary.

Hence it follows, if in the ellipse any two of the conjugate diameters AE, EB, EF, EG the triangles upon EA, EB, EF, EG with the angles AEB, FEG, to be equal to each other: indeed are equal to half of the equal parallelograms.

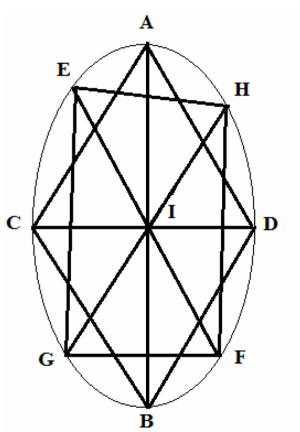
Second Corollary.

It follows in the second place, the parallelogram taken jointly within the whole diameters, to be equal to each other: since the squares of these have been shown to be equal by this proposition.

PROPOSITIO LXXIII.

The parallelogram, which arises from the outermost lines of the conjugate axes in an ellipse, is equal to the parallelogram contained by some of the conjugate diameters, on being compared.

Demonstration.



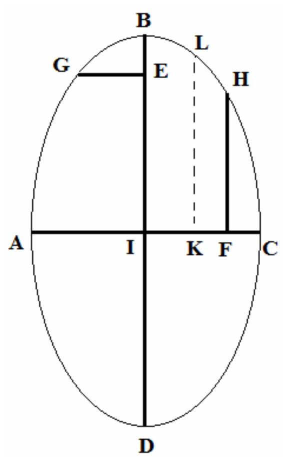
The axes of the ellipse ABC shall be AB, CD, and some other conjugate axes of the diameters shall be EF, GH. And both the extremes both of the axes, as well as of the diameters: I say the parallelogram CA,DB to be equal to the parallelogram EHFG. The rectangle formed from AB and DI is the double of the triangle ADB: and the rectangle from AB and CI, is the double of the triangle ACB. Therefore the rectangle formed from AB, CD, is the double of the parallelogram ACBD, or the parallelogram ACBD, is half of the rectangle on AB, CD. Similarly I shall show the parallelogram EGFH to be half of the parallelogram on EF, GH within the angle EIH: but the parallelogram on EF, GH is equal to the parallelogram on ABCD; and therefore the parallelogram ACBD is equal to the parallelogram EGFH. Q.F.D.

PROPOSITION LXXIV.

The axes or conjugate diameters of the ellipse ABC shall be AC, BD. And with BD divided in some manner at E, AC shall be divided in the same proportion at F, and the ordinate lines EG, FH may be drawn through E and F.

I say the rectangle AFC to be equal to the square GE, and the rectangle BED to be taken equal to the square HF, and if the square GE shall be equal to the rectangle AFC. I say BD, AC to be divided in the same proportion at E and F.

Demonstration.



Since from the hypothesis DE shall be to EB as AF to FC, there will become on interchanging, DE to AF, as BE to FC. And whereby the whole length DB, will be to the whole length AC, as DE to AF and as EB to FC. Therefore the ratios BE to FC, and DE to AF taken likewise, are the squares of the ratio DB to AC. And the ratio of the rectangle BED to the rectangle AFC, is composed from the ratios BE to FC, and DE to AF. Therefore the ratio of the rectangle BED to the rectangle AFC is the square of the ratio BD to AC, that is, of the ratio BI ad AI. Therefore the rectangle BED is to the rectangle AFC, as the square BI to the square AI; but likewise also the rectangle BED is to the square GE, as the rectangle BID; that is, the square BI to the square AI :

$$i.e.\left[\frac{DE}{EB}=\frac{AF}{FC};\frac{DE}{AF}=\frac{EB}{FC}\therefore\frac{DB}{EB}=\frac{AC}{FC}\,\&\,\frac{DB}{AC}=\frac{EB}{FC}=\frac{DE}{AF}.\right]$$
$$\&\left[\therefore\left(\frac{DB}{AC}\right)^2=\frac{EB}{FC}.\frac{DE}{AF}=\frac{BED}{AFC}=\frac{BI^2}{AI^2}.\right];$$

therefore the square GE is equal to the rectangle AFC. Similarly we may show the rectangle BED and the square HF to be equal.

Now the square GE and the rectangle AFC shall be taken equal: I say the sectors BD, AC to be in proportion: For if the ratio were not be as BE to ED, then neither thus will AF to FC be as BE to ED, nor thus AK to KC. Therefore the square GE will not be equal to the rectangle AKC by the first part of this above: Which cannot happen, since by the hypothesis the square GE shall be equal to the rectangle AFC. Therefore AC cannot be cut in proportion at any point K other than the point F. Q.E.D.

PROPOSITION LXXV.

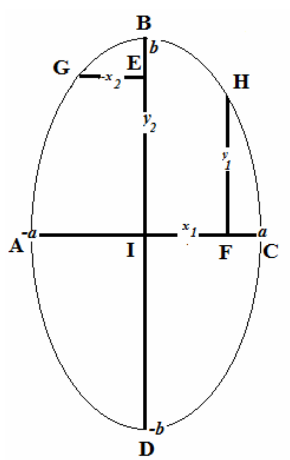
If the axes or conjugate diameters may be cut in proportion at E and F, and the ordinates EG, FH may be drawn.
I say the square FH to be to the square EG, as the square BD to the square AC.

Demonstration.

The square FH is equal to the rectangle BED, moreover the rectangle BED [§.74 above] is equal to the square EG as the rectangle BID, that is the square BI, is to the square IA. Therefore also the square FH is to the square EG, as the square BI to the square IA, that is, as the square BD is to the square AC. Q.e.d.

So that if the ordinates EG, FH shall be put in place for the conjugate diameters, and so that the square BD to the square AC, thus shall be as the square FH to the square EG : I say BD, AC to be proportional to the sections at E and F. Indeed if AF to FC may be negative, so that DE to EB, there shall become AK to KC, as DE to EB, and the ordinate shall be KL. Therefore so that as the square BD shall be to the square AC, the square KL shall be to the square EG: which cannot happen, since from the hypothesis the square FH shall be to the square EG, as the square BD to the square AC. Therefore so that there shall not be the ratio the square BD to the square AC, thus the square KL, or any other besides the square FH, to the square EG. Q.e.d.

****



Let us apply the above formulae to obtain the formula for the standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *for these conjugate points, taken arbitrarily.*

In the first place, we label the x and y intercept lengths $a,-a\,;b,-b$ *in the usual manner with the origin I, and consider at first the formula in §74:* $FH^2 = BED = BE.ED$, *applied to the points E* (x_2, y_2) *and F* (x_1, y_1) *as shown on the diagram, where* $BE = b - y_2$ *and* $ED = b + y_2$; *while* $FC = a - x_1$, *and* $AF = a + x_1$ *hence* $FH^2 = BED = BE.ED$ *becomes* $FH^2 = y_1^2 = BE.ED = (b - y_2)(b + y_2) = b^2 - y_2^2$, or $y_1^2 + y_2^2 = b^2$

Likewise, from the above formula $GE^2 = AF.FC$, *we have* :
$GE = x_2;\ AF = a + x_1;\ FC = a - x_1;$ *and* $GE^2 = AF.FC$ *becomes* :

$$GE^2 = x_2^2 = AF.FC = (a + x_1)(a - x_1) = a^2 - x_1^2, \text{ or } x_1^2 + x_2^2 = a^2 .$$

Hence, we may write:

$$\frac{x_1^2}{a^2}+\frac{x_2{}^2}{a^2}=1\,and\,\frac{y_1^2}{b^2}+\frac{y_2{}^2}{b^2}=1\;\therefore\frac{x_1^2}{a^2}=1-\frac{x_2{}^2}{a^2}\,and\,\frac{y_1^2}{b^2}=1-\frac{y_2{}^2}{b^2}$$

$$\therefore\;\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=1-\frac{x_2{}^2}{a^2}+1-\frac{y_2{}^2}{b^2}-1=1-\left(\frac{x_2{}^2}{a^2}+\frac{y_2{}^2}{b^2}\right).$$

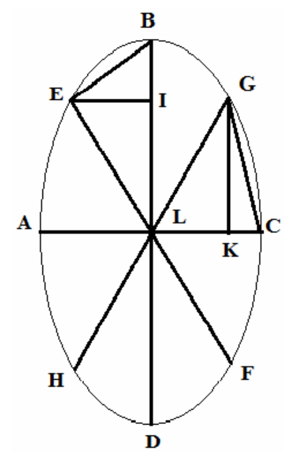
For this equation : $l.h.s.\leq 0; r.h.s.\geq 0$. *Hence both sides of the equation are equal to zero, and the standard formula for the ellipse is established for these points*.

****

PROPOSITION LXXVI.

In the ellipse ABC there shall be any two conjugate pairs of diameters AC, BD; EF, GH; and from E and G the ordinate lines EI, GK may be drawn to the diameters BD, AC.
I say the square EI to be equal to the rectangle AKC, and the rectangle BID shall be equal to the square GK.

Demonstration.

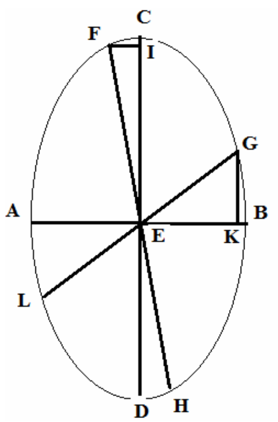


EB, GC may be put in place. Therefore since both the diameters AC, BD, as well as EF, GH are conjugate, the sector BLC will be equal to the sector ELG [*Cor*.§46], and with the common sector BLG removed, the sector ELB shall be equal to the sector CLG. And indeed the triangle LEB to be equal to the triangle LGC [§.45 *Cor*.]: and since BD is conjugate to AC itself, BD is parallel to KG, which is put to be the ordinate for AC. Therefore the angle GKL is equal to the angle BLA. Similarly since AC is conjugate to BD, AC is parallel to the ordinate EI put in place for BD; therefore the angle EIB is equal to the angle BLA, that is to the angle GKL; therefore since the triangles shall be equal, there will become (as shown below): as the base LB to the base LC, thus KG to EI, and thus as the square BL shall be to the square LC, that is, as the square BD to the square AC, thus the square KG to the square EI. From which the lines BD, AC are divided in proportion at I and K. Whereby the square EI is equal to the rectangle AKC, likewise the rectangle BID is equal to the square GK. Q.f.d.

PROPOSITION LXXVII.

The squares of the axes taken together are equal to the squares of any conjugation diameters taken together.

Demonstration.

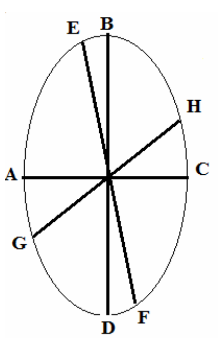


AB; CD shall be the axes of the ellipse ABC, and FH, GL some other conjugation of the diameters. I say the squares AB, CD taken together to equal the squares of FH, GL taken together. The ordinate lines FI, GK may be drawn, which will be perpendiculars since they are drawn to the axis; moreover E may be put to be the centre of the section. The square EC is equal to the square EI together with the rectangle CID, that is to the square GK: moreover, the square EB is equal to the square EK together with the rectangle AKB, that is to the square FI **:** from which the two squares EB, EC taken together are equal to the squares FI, IE, EK, GK taken together. But the squares EF, EG are equal to the same squares; therefore the squares EB, EC are equal to the squares SE, EG. Whereby with the squares AB, CD taken together, the quadruple of the squares shall be EB, EC and FH, GL the square of the quadruple of the squares EF, EG; it is apparent the squares AB, CD taken together to be equal to the squares FH, GL taken together. Q.e.d.

PROPOSITION LXXVIII.

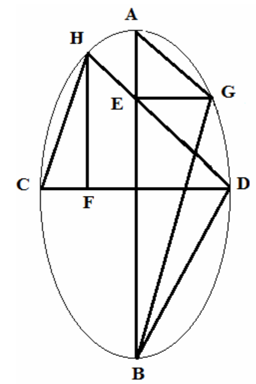
The axes of the ellipse taken together shall be the minimum of all the conjugate diameters taken together.

Demonstration.



AC, BD shall be the axes and EF, GH some other conjugate of diameters : I say the axes taken together to be the smaller of the conjugate diameters taken together. Because the squares AC, BD taken together are equal to the squares EF, GH taken together: moreover BD shall be the maximum of the diameters, and AC truly the minimum, AC, BD taken together will be smaller than the right lines EF, GH; therefore etc. Q.f.d.

PROPOSITION LXXIX.



AB, CD shall be the axes of the ellipse ABC, divided proportionally at E and F: and with the ordinate lines EG, FH drawn (which here are perpendiculars): AG, GB, CH, HD may be joined: I say the four squares AG, GB, CH, HD**,** taken together, to be equal to the squares of the two axes.

Demonstration.

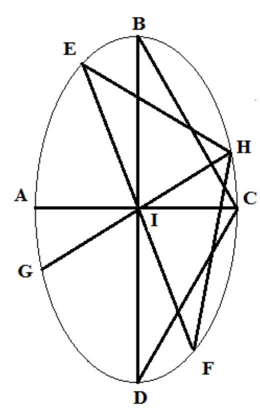
The square AB is equal to the squares of AE and EB together, with the rectangle AEB equal to the square HF [§76 *above*]; in the same manner, the square CD is equal to the square of CF and FD together, with the rectangle CFD equal to the square EG; [consequently on combining, and via Pythagoras,] from that the same squares are equal to the squares AG, GB, CH, HD. Therefore the squares of the axes taken together are equal to the AG, GB, CH, HD. Q.e.d.

[*i.e.* $$\mathrm{AG}^2+\mathrm{GB}^2+\mathrm{CH}^2+\mathrm{HD}^2=\left(\mathrm{AE}^2+\mathrm{EG}^2\right)+\left(\mathrm{EB}^2+\mathrm{EG}^2\right)+\left(\mathrm{CF}^2+\mathrm{FH}^2\right)$$
$$=\left(\mathrm{AE}^2+\mathrm{CF.FD}\right)+\left(\mathrm{EB}^2+\mathrm{CF.FD}\right)$$
$$+\left(\mathrm{CF}^2+\mathrm{AE.EB}\right)+\left(\mathrm{FD}^2+\mathrm{AE.EB}\right)$$
$$=2\mathrm{CF.FD}+\mathrm{CF}^2+\mathrm{FD}^2+2\mathrm{AE.EB}+\mathrm{AE}^2+\mathrm{EB}^2$$
$$=\left(\mathrm{CF}^2+\mathrm{FD}^2\right)+\left(\mathrm{AE}+\mathrm{EB}\right)^2=\mathrm{CD}^2+\mathrm{AB}^2.$$]

PROPOSITION LXXX.

The squares of the whole lines of the axes taken together are equal to the squares of the whole of each of the conjugates taken together.

Demonstration.

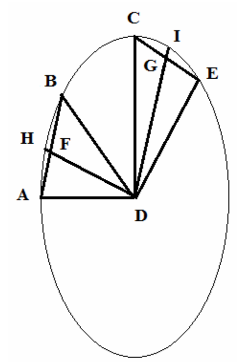


AC, BD shall be the axes of the ellipse ABC, and EF, GH shall be some other conjugate diameters. And BC, CD, EH, FH shall be joined. I say the squares BC,CD taken together to be equal to the squares EH, FH taken together. The squares BC, CD taken together are equal to the squares BI, IC taken together twice; but the squares EH, HF are equal to the squares EI, IH taken together twice; therefore the squares BC, CD taken together, are equal to the to the squares EH, HF taken together. Q.e.d.

PROPOSITION LXXXI.

Two conjugations of the diameters AD, DC; BD, DE of which the centre is D, cut the ellipse ABC, and with the points AB, CE joined, the lines AB, CE are bisected at F and G, and DF, DG may be drawn, which produced cross the ellipse at H and I.
I say HD, ID to be conjugate diameters.

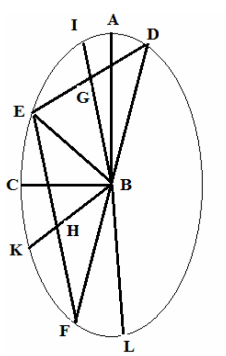
Demonstration.



Since AD, DC; BD, DE are conjugate diameters, ADC, BDE are equal sectors: therefore with the common sector BDC removed, the remaining sectors ADB, CDE are equal : again since the line AB shall be cut at F, the diameter DH bisecting into two equal parts, both the sectors ADH, BDH shall be equal, as well as the triangles AFD, BFD. In the same way the sector EDI is equal to the sector CDI; therefore the sectors ADH, EDI are equal to each other : Therefore with the addition of the common sector HDI, the sector ADI to equal the sector HDE, therefore DH, DI are conjugate.

PROPOSITION LXXXII.

The axes of the ellipse ABC shall be AB, BC; but the conjugate of some other diameter DF shall be EB; DE, FE may be joined; indeed DE which cuts the major axis, EF truly the minor : then DE, FE themselves are bisected by the diameters BGI, BHK.
I say IB to be the greater diameter than the diameter KB.



Demonstration.

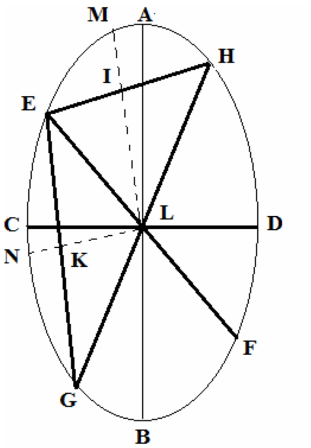
Because the diameters BI, BK bisect the right lines ED, EF both the sectors DBE, EBF are bisected. Whereby since they themselves shall be equal, also the half sectors of these IBD, KBF will be equal; therefore the sector IBA is smaller than the sector KBL. Therefore IB is closer to the axis than BK, and therefore is greater than BK. Q.e.d.

PROPOSITION LXXXIII.

AB, CD shall be the axes of the ellipse ABC, and EF, GH some other pair of the conjugate diameters, and EH, EG may be joined.

I say the line EH which cuts the major axis, to be smaller than the line EG, which cuts the minor axis.

Demonstration.

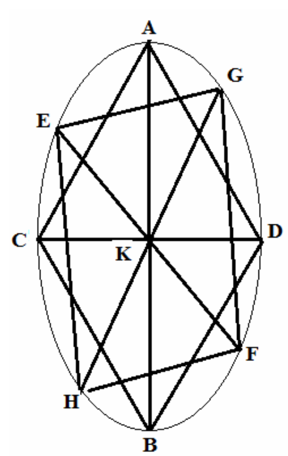


EH, EG are bisected by the diameters LM, LN, at I and K. Because GH, EF are conjugate; the sectors GLE, ELH are equal, and therefore the segments GNE, EMH are equal. Whereby LM, LN bisecting the chords EH, EG, are divided proportionally. Therefore MI shall be to IL as NK to KL; and on placing together and interchanging, so that as LM to LN, thus LI ad LK. But LM is greater than LN; therefore LI also is greater than LK. Now truly since LN, LM also shall be conjugates, and EG from the construction shall be put the ordinate to LN, LM will be parallel to EK. For the same reason, LN, EI will be parallel ; therefore EI, LK is a parallelogram, and thus LI, KE; LK, EI are equal. Therefore since LI shall be shown to be greater than LK, and KE greater than LK, that is than EI. Whereby the double of this, EG, is greater than the double of EH. Q.e.d.

PROPOSITIO LXXXIV.

The largest values of the axes taken together are the maxima of all the diameters of any conjugations taken together to the limit.

Demonstration.

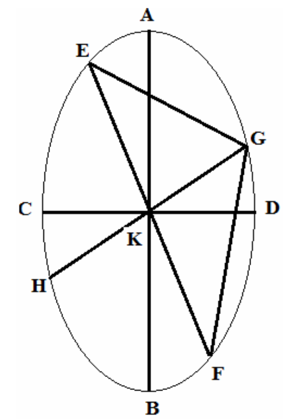


AB, CD shall be the axes and EF, GH shall be some other conjugation of the diameters : and both the greatest of the diameters as well as of the axes may be joined together. I say the greatest values of the lines CA, AD, DB, BC taken together, to be greater than EG, GF, FH, HE taken together. Because GK is equal to HK itself, truly with EK common, and the right line EH greater than EG [§.83], the angle EKH is greater than the angle EKG: and therefore the angle EKH is greater than the right angle AKC. But the triangles AKC, EKH are equal[§.73], whereby EH is greater than AC [*Elem.*]: in the same manner it is shown the line AC to be greater than the line EG. Therefore EG is the minimum, and EH the maximum of the lines EH, AC, AD, EG. Therefore since the squares EH, EG taken together shall be equal to the squares AC, AD taken together, the lines EH, EG taken together shall be less than the lines AC, CD taken together. In the same manner it will be shown the lines GF, FH to be less than the lines CB, BD. Therefore, etc. Q.e.d.

PROPOSITION LXXXV.

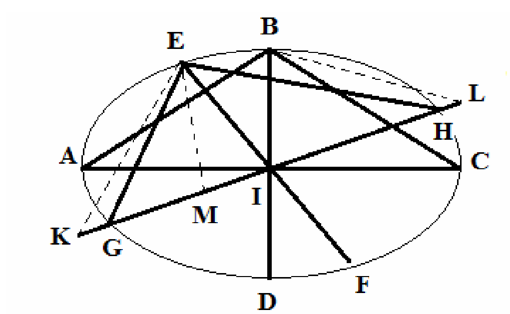
In no ellipse are conjugate diameters to be found which cut each other at right angles, besides the axes.

Demonstration.



AB shall be the major axis, CD the minor axis, and some other conjugate diameters EF, GH: moreover, K shall be the centre of the ellipse. I say neither of the angles EKG, GKF to be right: for the points EG, GF may be joined : Since EK, KG are equal to the two right lines FK, KG, and EG smaller than FG [§.83], therefore the angle EKG shall be smaller than the angle GKF [*Elem.*]: Whereby since the sum shall be equal to two right angles, neither of these is right: the same is shown for the rest. Therefore no ellipse is to be found, etc. Q.e.d.

PROPOSITION LXXXVI.



AC, BD shall be the axes of the ellipse ABC and EF, GH the conjugate pair of some other diameters: and with the points AB, BC joined, the right lines EH, EG are drawn.

I say the angle ABC which exists about the smaller axes, to be greater than the angle GEH, and hence to be the maximum of all the angles which will held by the greatest of the conjugate diameters connected together.

Demonstration.

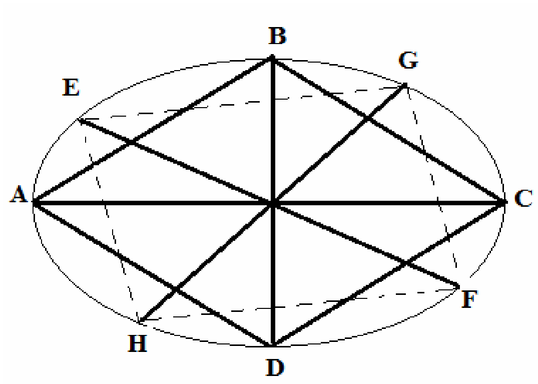
Since the line GH shall be smaller than the axes AC, it may be produced equally at each end to K and L so that LK shall be equal to the axis AC; and the points EK, EL may be joined, and the line EM may be sent from E normal to LK: therefore the triangle KEL shall be greater than the triangle GEH, that is to the triangle ABC ; whereby since the bases AC, LK of the unequal triangles shall be equal, the altitude EM of the triangle KEL, shall be greater than the height IB of the triangle ABC: and indeed the angle KEL shall be greater than the angle ABC: therefore the angle GEH is much smaller than the angle ABC. Q.e.d.

PROPOSITION LXXXVII.

AC, BD shall be the axes of the ellipse ABC : and the ends of these AB, BC, CD, DA are joined: moreover some other of the conjugate diameters shall be EF, GH, the ends of which also are connected.

I say the angles ABC, EGF, HFG, BCD to be in arithmetical proportion.

Demonstration.



Since both AC as well as EF is a parallelogram, both the angles ABC, BCD, as well as the angle EGF, GFH, are equal to two right angles: and whereby the angles ABC, BCD taken together are equal to the angles EGF, GFH taken together: but we have shown the angle ABC thus to be greater than the angle EGF, and therefore GFH is greater than the angle BCD: and because, as I have shown now, the angles B and C taken together are equal to the angles G and F taken together, by which excess the angle ABC exceeds the angle EGF, likewise it is necessary that the angle GFH will exceed the angle BCD. Q.e.d.

Corollary.

Hence it is evident BCD, which is the angle present about the major axes, to be the smallest of all the angles which are constructed from the diameters taken conjointly.

****

Let A, B, C, D & E, F, G, H *represent the angles of the parallelograms* ABCD & EFGH:

Since $B + C = G + F$; *we have shown* $B > C, G > F$ & $B > G \therefore F > C$.

Hence, for the equality to hold : $B > G > F > C$.

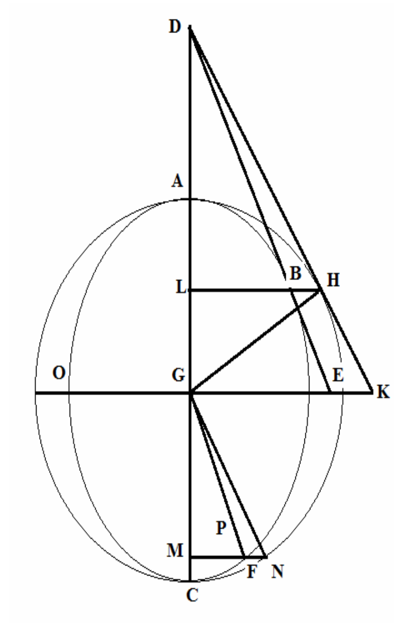
****

PROPOSITION LXXXVIII.

A certain line DE is a tangent to the ellipse ABC at B, of which the axes are AC, EG, meeting each axis at D & E: Truly a right line GF may be sent from the centre G parallel to the line DE.

I say the lines DB, GF, BE to be in continued proportion.

Demonstration.



The circle AHC is described with centre G and radius AG, : and from D the tangent DK is sent to the circle at H crossing the axis to the ellipse EG at K, and from H the line HL normal to the axis, which by the Cor. to §.33, also passes through B; another line FMN normal to the axis, passes through the point F on the ellipse, and the points NG, HG may be joined. Since the lines LH, MN are equidistant, the angles LDB, MGF will be equal: but the angles BLD, FMG are right by the construction; and therefore the remaining angle LBD is equal to the remaining angle MFG. Whereby the angles DBH, GFN are equal to each other. Moreover, since the triangles DLB, GMF are similar, so that there shall be LB to MF, thus as DB to GF. But from the demonstration in the Scholium of §.4 above, as LB to MF, thus BH to FN. Therefore DB to GF, as BH to FN. Whereby since the angle DBH, GFN now shall be shown to be equal, the triangles DBH, GFN will be equal. Therefore HD is to BD, that is HK is to BE, as GN to GF, and on interchanging, as HK to GN, thus BE to GF. Then since in triangle DGK, the angle at G shall be right and GH drawn from the centre to the point of contact, shall be normal to DK, HK will be to GH, as GH to HD. But GN, GH are equal, therefore, as KH to GN, that is just as shown before, to be as BE to GF, thus GN to DH. But on account of the similitude of the triangles as GN to DH, thus GF to DB. Therefore, as BE to GF, thus GF to DB. Q.e.d.

PROPOSITION LXXXIX.

With the same in place, if GF shall be the mean proportional between DB, BE. I say the point F to be found from the ellipse curve.

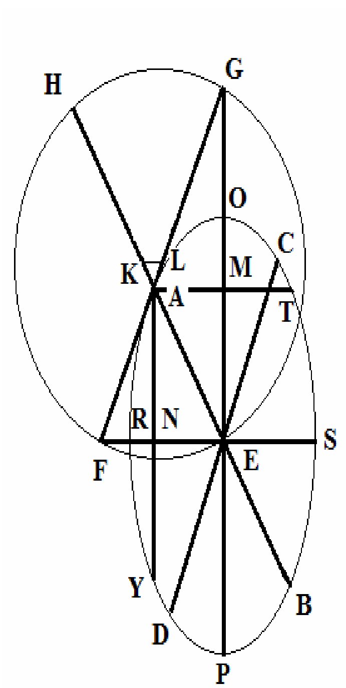
Demonstration.

If the point F did not pertain to the ellipse, then the right line GF will pass through the ellipse at P, either above or below F. Therefore from the preceding, DB, GP, BE are continued proportionals; which cannot happen, since DB, GF, BE shall be the continued proportionals in place. Therefore no other point of the right line GF lies on the ellipse, apart from F. Q.e.d.

PROPOSITION XC.

With the conjugate of some diameters given, to find the axes of the ellipse.

Construction & demonstration.



The conjugate diameters AB, CD shall be given, bisecting each other at E; and the line FG acting through A which shall be parallel to CD, so that as AE to ED, thus there become ED to AH; then EH shall be bisected at K, the normal KL erected from K crossing FG at L, then with the centre L and with the radius LE, the circle EFG is described, this will pass through H and will cut the line FG at some points F and G: finally the points EF, EG are joined. I say the lines EF and EG found to be satisfactory. Indeed since the right lines AE, ED, AH from the construction are in continued proportion, for the square ED will be equal to the rectangle EAH, that is to the rectangle FAG. And thus the lines GA, DE, FA also are in continued proportion, and hence the point D lies on the ellipse, the axes of which lie on the lines EF, EG. But A also lies on the same ellipse, and therefore the points B, C lie on the same ellipse.

Again the ends of the axes will be found thus: with AM drawn, with AN drawn normal to the line EG, the mean EO may be found between EM, EG, and the mean ER between inter EN, EF: and the ellipse may be described through the points D, A, C ; since CD is conjugate to AB itself, and thus for the same ordinate put in place, and the line FG parallel to CD itself, FG will be a tangent to the ellipse ABC; but AM, AN are normal to the lines in which the axes for the section are present; and both EM, EO, EG, as well as EN, ER, EF to be in continued proportion, therefore the ellipse ABC passes through the points R and O: whereby R and O are the ends of the axes, which it was required to show.

Scholium.

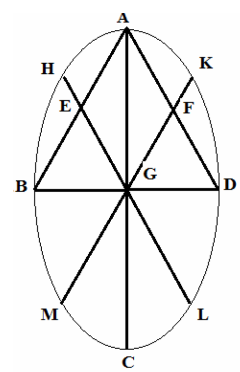
This proposition is taken from the problem of Pappus, Book 8, *Mathematical Collection, Prop.*14, *and indeed the truth of this construction evidently will follow both from this source, as well as from that which we now present, but have not yet demonstrated, Frederico Commandino had endeavored to supply a demonstration, thus writing:*
AM may be produced as far as T thus so that TM shall be equal to MA itself: also AN may be produced as far as to Y, so that YN shall be equal to NA : the points T and Y lie on the ellipse, from these matters which have been demonstrated by Apollonius in Prop. 47, Book 2 on Conic Sections. But RS is parallel to AT itself, indeed there is a right angle in a semicircle, and whereby OP itself will be parallel to AY. Therefore since the ordinate to AB is the applied line drawn through A parallel to DC, evidently FG will touch the section at the point A, and since FG touching the section shall cross the diameter at G and AM shall be the applied ordinate, by §.37 of the first book of Conics of Apollonius, the rectangle GEM is equal to the square from EO or EP. Also, by the same reasoning, the ordinate AN may be applied to the rectangle FEN equal to the square from ER or ES; therefore OP, RS will be the conjugate axes of the ellipse.

This explanation by which Commandino has shown the right lines OP *and* RS *to be the conjugate axes of the ellipse, which advances through the points* A T Y, *and is a tangent at the point by the line* FG*: truly this proposition cannot be true. For according to the conjugate axis requiring to be found there is no need for the circle* FEG *to be described, to make the rectangle* EAH *equal to the square* ED, *and to bisect* EH *at K and thence to erect the normal, which meeting* FG *at* L, *provides the centre* L *of the circle* FEG, *if indeed with the centre taken somewhere on the line* FG, *if the circle may be drawn around* E, *which will cut the line* FG*: now indeed the right angle will be contained not by the points* F *and* G *but by others which are erected from the right line through* E, *and not cut by* EF *and* EG, *and whereby if from* A *some normals may be drawn to these latter lines themselves, whereby there will become* AM & AN, *which may be doubled, so that* AT *and* AY *will provide the points which in turn allow the entrance of the points* T *and* Y *on the ellipse, and which are destined to become infinitely close, the tangent of the line* FG, *which passes through* A. *But it is observed this ellipse (which was required to be shown) to pass through the end points C & D, therefore so that the centre of the circle shall be assumed to be from some other point L, neither shall the square ED be equal to the rectangle under EA, nor be equal to any other line contained by AH. And thus so that it may be shown OP and RS to be the conjugate axes of the ellipse, which pass through the ends of the conjugate diameters AB and CD, another account is required to be entered into, which we have now proposed in our demonstration.*
This so far has been concerned with the Commandino demonstration.
Thus indeed , the text remaining of Pappus may be had, for which I do not know what unfortunate circumstances may have passed over time [for the indistinct parts lost in the text used by Gregorius]: But it is easy to find the axis related to these, for any conjugate diameters of the ellipse. Which indeed will be done by this method. *Words which have no proper sense do not find a place, since these are not related to the construction. But*

everything shall be geometric, so that it becomes apparent on being read: whereby I consider the omission of geometric words, and thus these being required to be read: But for an ellipse found with any conjugations of the diameters, its related axes to be found easily, which indeed will be done in this account geometrically. *Then these words are added into the same construction itself:* since DE shall be greater than EA: *for since the construction shall be general, DE shall be lesser or greater, or may be put equal to EA itself, as can be deduced from our demonstration, Pappus also is in no way certain about each side, frustrated he assumed DE greater than EA. Hence it is a wonder that Frederick Commandino did not turn his attention to this error, especially since with that assumption in his demonstration, as we have given above, he made no use: but he could have brought it into the general demonstration: and since from which the demonstration itself was abandoned, to which there is no doubt Pappus would have added. It is clear enough from the error of these matters that were touched upon, from the hand of whom this proposition arrives (which scarcely was considered from the whole, as I would judge), which now clearly changed and imperfect, have frustrated attempts to restore the original.*

PROPOSITION XCI.

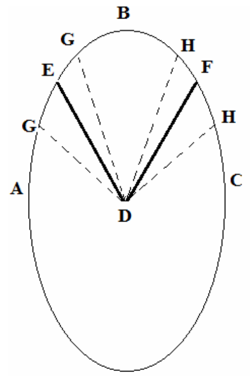
To show equal conjugate diameters from the given axes of the ellipse.



Construction and demonstration.

AC, BD shall be the axes of the ellipse ABC : moreover it shall be required to show the conjugate diameters to be equal to the joined points A B, A D; the right lines AB, AC shall be bisected at E and F; and from the centre G the right lines GH, GK shall be drawn through E and F, meeting the ellipse at the points H, K, L, M. I say these points satisfy the proposition. For since the two right lines EB, BG, are equal to the two right lines FD, DG (moreover are equal to each other, with equal sides and contained angles) , also the angles to the base EGB, FGD and thus the remaining angles AGE, AGF are equal to each other. Again since the angle AGB shall be right and the base AB bisected at E, if a circle with centre E and with the radius EA may be described, it will pass through B also, and thus EA, EG are equal lines. Whereby the angle EAG to be equal to the angle EGA, that is, AGF. Therefore AB, KM shall be parallel : in the same way the lines AD, HL are shown to be parallel: so that since the diameters HL, KM will bisect the mutually parallel lines, they will be conjugate. Truly since the angle HGA has been shown to be equal to the angle AGK, also the line HG is equal to the line GK, as is evident from §18 above; therefore the diameters HL, KM are conjugate and equal. Therefore we have shown, etc. Q.e.d.

PROPOSITION XCII.



In an individual ellipse only two conjugate equal diameters are to be found.

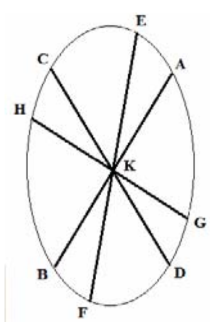
Demonstration.

D shall be the centre of the ellipse ABC and in that the equal conjugate diameters ED, FD : I say no other equal and conjugate diameters can be shown in that: for if it can be done, besides the diameters ED, FD: if there shall be the other equal and conjugate diameters GD, HD: therefore the sector GDH will be equal to the sector EDF. Which cannot happen, for if the diameters GD, HD shall be equal since it is by necessity these must be greater or smaller with the diameters ED; FD, and thus, both taken together to fall either greater or smaller than the diameters ED, FD: Therefore, besides the equal conjugate diameters ED, FD, no others can be shown equal in the ellipse. Q.e.d.

PROPOSITION XCIII.

In an ellipse, the equal conjugate diameters taken together, are the greatest of all the conjugate diameters taken together.

Demonstration.

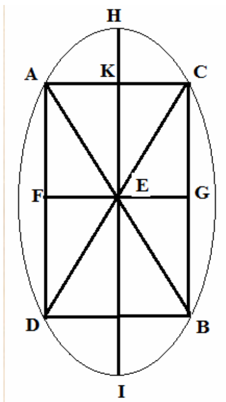


AB, CD shall be the equal conjugate diameters, but if there shall be some other of the diameters EF, GH conjugate : I say the diameters AB, CD likewise taken to be greater than the diameters EF, GH taken together: for since the sectors AKC, GKE shall be equal to each other ; it is necessary one of the unequal (it shall be EF) to be closer to the axis than whichever of the equal axis AB, CD*:* truly than the other HG to be more distant, so that from the four diameters EF is the maximum and GH the minimum. But the squares EF, GH taken together are equal to the squares AB, CD taken together ; therefore the lines AB, CD taken together are greater than the lines EF, GH also taken together: Q.e.d.

PROPOSITION XCIV.

The lines which are conjugate to the greatest of the conjugate diameters, are bisected by the axes.

Demonstration.

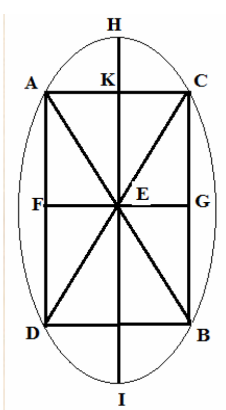


AB, CD shall be the equal conjugate diameters, and the ends of these joined together AD, AC, CB, DB. I say these to be bisected by the axes, with AD bisected at F, FEG acting through the centre E crossing the right line CB at G. On account of which therefore AB, CD are put equal, the halves of these AE, DE, also are equal. Moreover from the construction , similarly AF, DF are equal. And thus in the triangles AEF, DEF since FE shall be common, all the sides themselves are equal in turn. Therefore the angles at F are equal, and thus also FEA, FED are equal with right angles. Whereby also the angles GEC, GEB vertically opposite to the first are equal. Truly again the sides CE, EB are equal and with EG common to each of the triangles GEC, GEB. Therefore CG, BG are equal, and thus the angles at G to be right. Therefore the right lines AD, CB (which by §19 are parallel) are bisected at right angles by the axis FG. Therefore with the right lines AD, CB bisected by the axis , the ends of the equal conjugates AB and CD are connected. We may show in the same manner the two remaining lines AC, BD to be bisected by the axis HI. Therefore the truth of the proposition is agreed on.

PROPOSITION XCV.

If the lines which join the ends of the conjugates may be bisected by the axes:
I say these diameters to be equal to each other.

Demonstration.



The same figure as before may be put in place, and the lines AD, CB, AC, the extremities of the conjugates bisected at F, G, and K and divided at right angles by the axes Hl, FG. I say the conjugate diameters AB, CD to be equal to each other: indeed since AD, CB per §19 shall be parallel and from the hypothesis shall be bisected by the axis at F and G, the angles at F are right, and the two sides AF, FE are equal to the two sides DF, FE; therefore the remaining sides AE, ED also are equal to each other. Similarly I may show CE, EB to be equal. From which the whole diameters AB, CD are equal. Q.f.d.

Corollary.

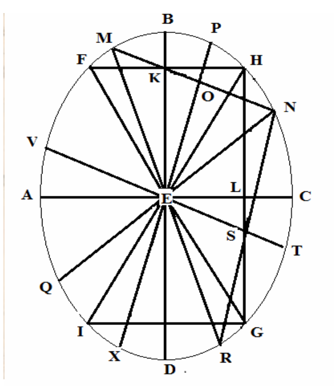
Hence it is evident the lines, which connect the ends of unequal conjugates diameters, at no time to be bisected by the axes or by any other diameter, or to be cut at right angles.

PROPOSITION XCVI.

Of the lines which join the ends of any conjugate diameters, that one is the greatest which joins equal conjugates which shall cut the minor axis; the smallest the one which shall cut the major axis.

Demonstration.

AC, BD shall be the axes of the ellipse ABC, truly the equal conjugates FG, HI ; and the join FH meets the major axis at K: and HG the minor axes at L. I say HG to be the maximum line of these which join the ends of any conjugations whatsoever, and FH the minimum. Indeed some other conjugations of the axes may become MR, NQ. The ends of which M N, N R may be joined, the lines XOP, VST may be drawn by which these conjugate lines are bisected at O and S through their centres. Therefore since FG, HI are conjugates, the sectors EFBH, EHCG are equal, (§46). Therefore the segments FBH, HCG are equal, (§60). Whereby, since the axes BD, AC also bisect the right lines FH, HG, (§94), which join equal conjugates, the axes themselves to be cut proportionally at K and L (§53). Therefore the rectangle BKD is equal to the square LH (§74). Clearly by a similar discourse we will show the rectangle POX to be equal to the square NS. Then (§46, *Cor*.) since the sectors MEN, FEH are equal (§60), and thus the segments MPN, FBH, and both FH, MN bisected at K and O, DB, XP (§53) are cut in proportion at K and O; but the axis DB is greater than the axis XP: therefore the rectangle BKD, that is, as shown before, the square HL, is greater than the rectangle POX, that is to the square NS. Therefore the right line HL is greater than the right line NS, but HG is the double of HL, as shown before, and NR (§53) from the construction is the double of NS. Therefore HG is greater than NR: but since NR is greater than MN, HG also will be greater than MN. We show in the same manner HG to be greater than the connected ends of any other conjugates. Therefore HG is the greatest of all. Which was the first part of the proposition.

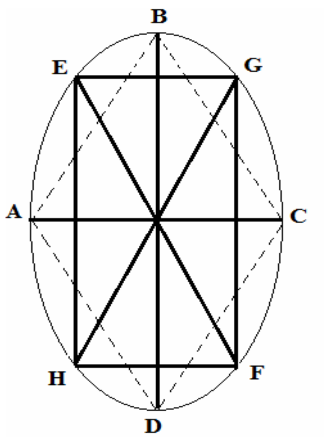


Clearly we will demonstrate by the opposite means that FH may be the smallest of all the connectors. For the construction may be repeated for some other conjugation of the diameters MR, NQ taken, than has been used above. We may show in the same way the square FK to be smaller than the square MO, and the line FK to be smaller than the line MO, and hence FH, to be smaller than MN; but RN is greater than MN. Therefore FH also is smaller than NR. And thus we will show FH to be smaller than any of the ends of the conjugate diameters joined together. Therefore it is the smallest of all. Which was required to be shown in the second place.

PROPOSITION XCVII.

The ends of the equal conjugate diameters joined together are the smallest of all the conjugate diameters of any kind joined together.

Demonstration.

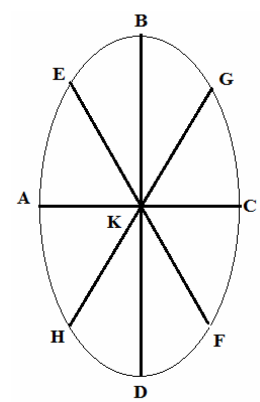


EF, GH shall be the equal conjugate diameters in the ellipse ABC. Moreover some other conjugate of the diameters EF, GH may be put in place. I say the lines which join the ends of these equal conjugate diameters taken together to be lesser than the lines which connect the ends of the other conjugate diameters (§80). Indeed the squares EG, GF are equal to the squares AB, BC, and in addition EG is the shortest line of the connection, and FG the greatest by the preceding; therefore the lines EG, GF are shorter than the lines AB, BC; in the same manner the lines EH, HF are shown to be lesser than the lines AD, DC : Therefore the lines , etc. Q.f.d.

PROPOSITION XCVIII.

AC, BD shall be the axes of the ellipse ABC and EF one of the equal conjugate diameters.
I say the squares AK, BK taken together to be the double of the square EK.

Demonstration.

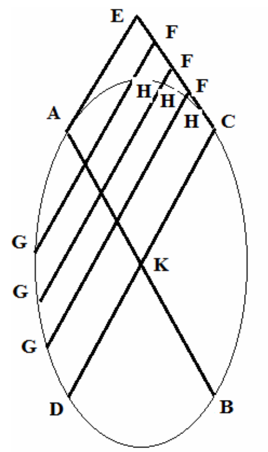


GH shall be drawn, the other of the equal conjugate diameters. Since the squares AC, BD taken together are equal to the squares EF, GH taken together (§77), and the squares under half the axes will be AK, BK, equal to the squares EK, GK under half the equal diameters; but the squares EK, GK are equal to each other, therefore the squares AK, BK taken together are equal to the square EK. Q.e.d.

PROPOSITION XCIX.

*Apollonius has a proposition of this kind in book.*3.*Conics.Prop.16 : if the tangents to an ellipse meeting at E shall be* AE, CE, *and on taking some point G in the section,* GHF *may be drawn parallel to one of the tangents AE, the rectangle* GFH *shall be to the square* BC, *as the square* AE *to the square* CF.
Truly not only the same ratio shall be found, but also the areas of equal conjugate diameters shall be found.

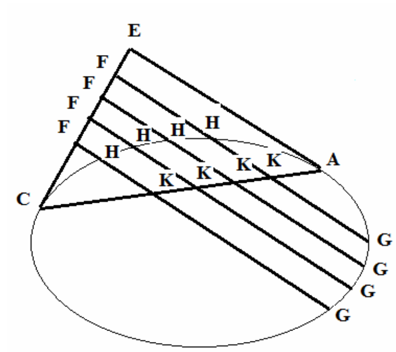
The equal conjugate diameters of the ellipse shall be AB, CD, at the ends A, C of which the two tangent right lines shall meet at E, if several more may be drawn parallel to GH, the rectangles GFH will be equal to the squares FC.



Demonstration.

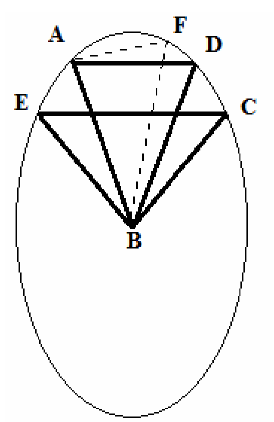
Since CD is the conjugate diameter of the diameter AB, for that the ordinate itself will be put in place: therefore CD is parallel to the tangent AE. In the same way AK is parallel to the tangent EC. Therefore the figure KAEC is a parallelogram. Whereby since AK, KC shall be equal from the hypothesis, also AE, CE are equal: therefore the squares AE, EC are equal. And as the square AE to the square EC, thus the rectangle GFH to the square FC, therefore the rectangle GFH is equal to the square FC. Q.e.d.

And since we now have that theorem of Apollonius at hand, I add three things to this that likewise Apollonius does not seem to have observed: without doubt if with the tangents drawn AE, CE, the points of contact A, C may be joined together, with the rectangles GFH to be equal to the squares KF.



Since FK, AE are parallel, the triangles AEC, KFC are similar. Therefore AE to EC, thus as KF to FC. There as the square AE to the square EC, thus the square KF to the square FC. But also as the square AC to the square EC, thus the rectangle GFH to the square KF. Therefore the square KF and the rectangle GFH have the same ration to the square EC; therefore the equations are equal. Q.e.d.

PROPOSITIO C.

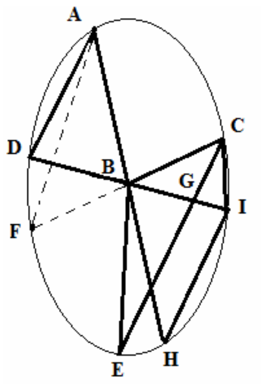


AB, BC shall be unequal conjugate diameters, and from A some right line AD may be drawn cutting the ellipse at D; to which the a line CE may be drawn parallel from C, and EB, DB may be joined.

I say EB, DB to be the opposite conjugate diameters.

Demonstration.

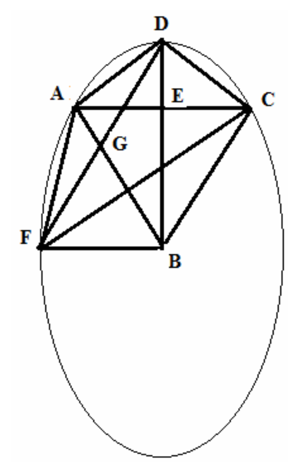
In the first case they lie parallel on the same part of the ellipse. The right lines EA, DC are drawn. Since the lines AD, EC are mutually equidistant from each other, the segments EA, DE are equal to each other (§51), and thus the sectors ABE, DBC are equal (§59). Therefore with the common angle ABD added, the sectors EBD, ABC will be equal to each other. Whereby the diameters BA, BC shall be conjugate with the sides of one sector, also the other sides EB, BD are conjugate.



In the second case the lines AD, CE lie parallel on opposite parts of the ellipse : the radii AB, DB may be produced to H and I, and the points H and I may be joined. Since AB, BC are conjugate the sector ABC shall be the fourth part of the ellipse, but the diameter AH bisects the ellipse, and thus the portion ACH is half of the ellipse. Therefore the sector ABC is half of the semi-ellipse ACH. And hence half of the sector CBH. But since IH by §19 is parallel to DA, to which from the hypothesis CE also is parallel, IH and CE shall be parallel to each other. Therefore the segments CI, EH and thus the sectors CBI, HBE are equal ; therefore with the common sector IBH added, the sector IBE is equal to the sector CBH, that is, as now shown before, equal to the sector ABC. Whereby since the sector ABC shall be a quarter of the ellipse, or half of the half ellipse, also the sector IBE will be half of the half ellipse, that is, of the portion IED, which is apparent to be half of the semi ellipse from the Coroll. to §45. Therefore the sector IBE, that is the sector ABC, is equal to the sector EBD. Whereby since AB, BC shall be conjugate, also DB, EB will be conjugate.

Now AB, CB, likewise EB, DB shall be conjugate diameters and AD, EC may be joined together. I say the lines AD, EC to be parallel. Truly if not, AF may be drawn parallel to EC itself from A , and FB joined: therefore FB will be the conjugate diameter to EB itself through the second part of this: but DB by the construction is the conjugate to the diameter EB, therefore several diameters are conjugate to that same EB. Which cannot be the case. Therefore AF cannot be parallel to EC. The same is shown for any other. Therefore AD alone is parallel to the right line BC. Q.e.d.

PROPOSITION CI.



AB, BC shall be some conjugate of the diameters of the ellipse ADC, the centre of which shall be B: and with the points A C joined, the diameter BD may cut the line AC at some point E, of which the conjugate BF may be drawn, and with the line FD drawn, it shall cut the diameter AB at some point G.

I say the lines AC, FD, as well as the lines BD, AB to be divided in proportion at E and G.

Demonstration.

Since both the diameters AB, BC as well as the diameters DB, FB are conjugate, the sectors ABC, FBD will be equal (§46.*Cor*.): therefore with the common sector ABD removed, the equal sectors DBC, ABF remain. So that the lines BD, AB (§61), likewise the lines AC, FD are divided proportionally at E and G.

PROPOSITION CII.

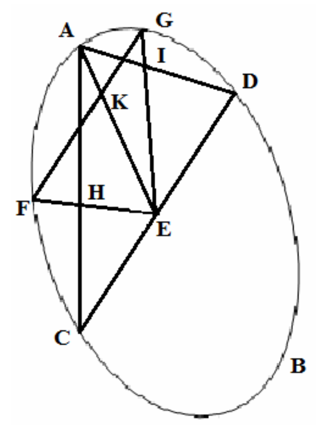
With the same in place:
I say the joined lines AD, FC to be parallel.

Demonstration.

By the preceding the sectors DBC, ABF have been shown to be equal; therefore the segments DE, AF also are equal to each other (§60). Therefore the lines AD, FC are parallel (§61). Q.e.d.

PROPOSITION CIII.

The conjugate of some diameter AE shall cut the ellipse ABC at CD: and moreover there shall be the conjugate of other diameters, FE, GE which cut the joined lines AC, AD at H and I:
I say that as AH to HC, thus there shall be DI to IA.



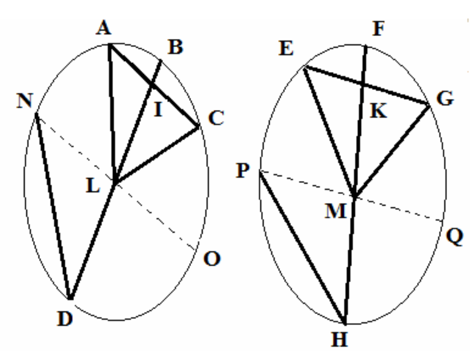
Demonstration.

FG shall be drawn which will cut AE at K so that AH to HC, thus as FK to KG: but as FK is to KG, thus DI is to AI (§101), therefore as AH to HC thus DI to AI. Q.e.d.

PROPOSITION CIV.

Some diameter BD shall cut the ellipse ABC: moreover EFG shall be an ellipse similar and equal to ABC : which some other diameter FH may cut : henceforth with the diameters BD, FH divided in proportion at I and K, the ordinate lines AC, EG shall act through the points I and K : the radii of the lines with the same ends AL, CL, EM, GM are drawn.
I say ALC, EMG to be equal triangles.

Demonstration.



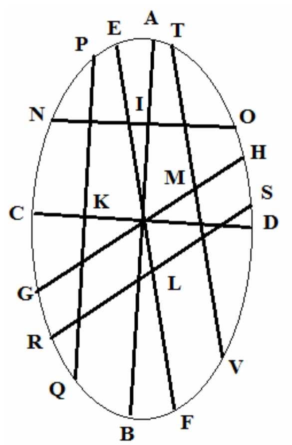
The conjugate NO is drawn to the diameter BD, and PQ to the diameter FH: ND and PH shall be drawn.
Since the diameters BD, FH are divided proportionally both by IK as well as LM, so that the rectangle BID will be to the rectangle BLD, thus as the rectangle FKH to the rectangle FMH: whereby, as the square AI to the square NL and thus as the square EK to the square PM : and as

the line AI to the line NL, thus EK to PM: but also there is by the construction, as IL to BL, that is LD, thus KM to FM, that is, MH; therefore so that as the triangle NLD to the triangle AIL, thus the triangle PMH to the triangle EKM (since the ratio is composed from the same sides of these:) and by interchanging, as the triangle NLD to the triangle PMH, thus the triangle AIL to the triangle EKM. But the triangles NLD, PMH are equal, therefore the triangles AIL, EKM also, and thus the whole triangles ACL, EGM are equal. Q.e.d.

PROPOSITION CV.

Two conjugations of diameters AB, CD, EF, GH shall cut the ellipse ABC and all four diameters shall be divided proportionally at the points I, K, M, L, through which the ordinate lines are drawn NO, PQ, RS, TV.

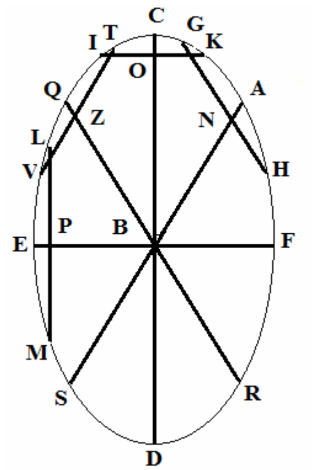
I say the squares NO, PQ taken together to be equal to the squares RS,TV taken together.



Demonstration.

Since both AB, EF, as well as CD, HG are divided proportionally, so that the square AB shall be to the rectangle AIB, thus as the square EF to the rectangle ELF, and the square CD to the rectangle CKD, and the square HG to the rectangle HMG. Indeed from these same ratios, the proportions of the individual squares to the rectangles are composed. Therefore so that the squares AB, CD taken together with the rectangles AIB, CKD taken together, shall be the squares EF, GH taken together, are as the rectangles ELF, HMG ; that is the squares PK, NI to the squares SL, TM (§74) : but the squares AB, CD taken together are equal to the squares EF, GH taken together (§77); and therefore for the squares SL, TM to be equal to the squares NI, PK : therefore the squares NO, PQ taken together are equal to the squares SR, TV taken together. Q.e.d.

PROPOSITIO CVI.



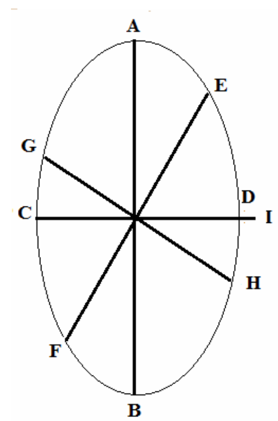
Some conjugates may cut the diameters CD, EF of the ellipse ABC, by which the diameters are divided proportionally at O and P : together with the conjugate drawn from one of the two equal conjugate diameter AS, which may be divided at N, in the same proportions as CD is divided at O. The ordinates GH, IK, LM are drawn through N, O, and P. I say the squares IK, LM taken together to be the double of the square GH.

Demonstration.

The other conjugate QR of the two equal conjugates may be drawn, which is to be divided similarly at Z, as SA is divided at N, and CD, EF, at O and P, and the ordinate VT may be placed through the point Z : since QR, AS therefore are cut similarly, the rectangle QZR is equal to the rectangle ANS (§74). But the rectangles QZR, ANS are equal to the squares GN, TZ. Therefore the squares GN, TZ and the squares GH, TV thus are equal. But the squares ML, IK are equal to the squares VT, GH. Therefore the squares ML, IK (§105) together are twice the square GH. Q.e.d.

PROPOSITION CVII.

Two pairs of conjugate diameters AB, CD; EF, GH shall cut the ellipse ABC : and AB shall be the largest in magnitude and EF the second largest.
I say the ratio AB to EF to be smaller than the ratio GH to CD.



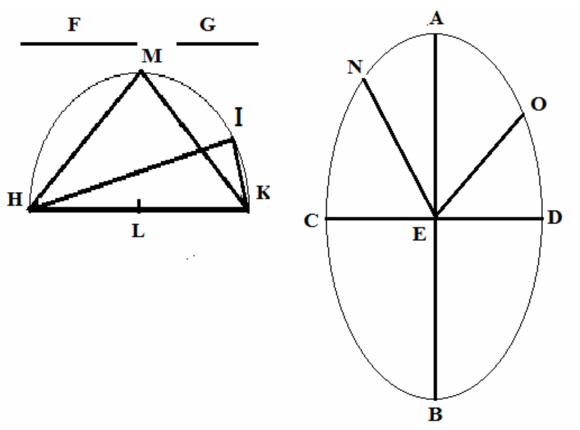
Demonstration.

Since the squares AB, CD are taken equal to the squares EF, GH taken together: the ratios cannot be as AB to EF, thus GH to CD, for then AB, CD shall be the maxima of the squares and the minima of the squares EF, GH. Therefore there shall become, as AB to EF, thus GH ad CI: and AB, CI shall be the greater square of the squares EF, GH; that is of the squares AB, CD. Whereby the line CI is greater than the line CD, and the ratio GH to CD; that is from the construction, the ratio AB to EF is smaller than the ratio GH to CD. Q.e.d.

PROPOSITION CVIII.

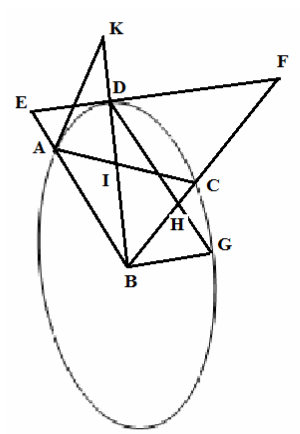
In a given ellipse, to show the conjugate diameters to be in a given ratio.

Construction and demonstration.



AB, CD shall be the axes of the ellipse ABC with centre E : truly as the given ratio of the greater to the lesser axis shall be as F to G, will be required for that not to be greater than the ratio of the axes AE, EC. If the ratio given is equal to the ratio of the axes, the situation is clear: if the ratio is less, the lines HI, IK may be taken, equal to the right lines AE, EC, and HIK may be put equal to a right angle: and with HK drawn, and bisected at L, with centre L the circle HIK is described with radius LH, which will pass through the points I and K, then in the segment HIK (§49, *Circle*) the two lines HIK shall be inclined in the given ratio F to G. Therefore since HM is to MK as F to G, and the ratio F to G is less than the ratio of the axes, that is, from the construction in the ratio HI to IK, it is clear the point M to fall between I and H, and thus the right line HM to be smaller than HI, that is, smaller than EA; truly the line MK to be greater than IK, that is than EC. Therefore with centre E and with radius HM, the described circle shall cut the ellipse at some point N between A and C, therefore the diameter NE may be drawn to that; with EO joined. I say NE, OE (§77) are required to satisfy the diameters. Since these squares NE, OE are to be equal to the squares AE, EC ; that is HI, IK to the squares HM, MK: but the line NE is equal to the right line HM. Therefore the line OE is equal to the line MK. Therefore as HM to MK, that is as F to G, thus NE to OE. Therefore we have shown, etc. Q.e.f.

PROPOSITION CIX.

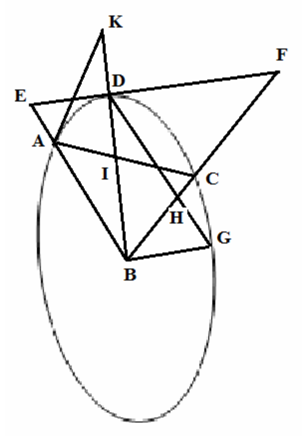


Some conjugate of the diameters AB, BC shall cut the ellipse ACD, and with some point D assumed on the periphery between A and C, the tangent EF is acting through D, which since it shall cross the lines AB, BC at E and F; the line BG is drawn parallel to that tangent through the centre.
I say the lines DF, BG, DE to be in continued proportion.

Demonstration.

Since from the hypothesis DF, BG are parallel, the triangles DFH, BHG will be similar, as evident from the *Elements*, and hence DF is to BG, as DH to HG; that is, as the triangle DHB is similar to the triangle

BHG. Thence, since BG drawn through the centre is parallel to the tangent DF, it is clear the conjugate to be the diameter DB itself: truly from the hypothesis also AB, BF to be



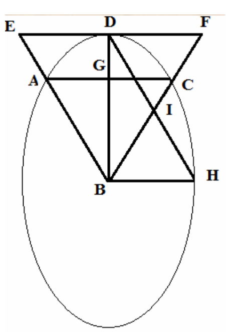
conjugates. Therefore the sectors BDCG, BADC (§46,*Cor*.), and thus also the triangles BDG, BAC are equal: of which the bases DG, AC, since they shall be divided proportionally by H and I (§60, *Cor*.), so that DH shall be to HG, thus as CI is to IA, it may be agreed from the *Elements* the triangles BHD, BIC and BHG, BIA to be equal. Therefore since before I shall have shown DF to be to BC as the triangle BHD to the triangle BHG, there will be also DF to BG, as [the area of] triangle BHD to triangle BHG. Also there will become DF to BG, as triangle BIC to triangle BIA. In addition since triangle BHG shall be to triangle BHD, as GH to HD, that is, as AI to IC, that is as KI to IB (since indeed CB shall be the conjugate of AB, and AK, the tangent from the construction, it is apparent AK, CB to be parallel) that is as triangle AIK to triangle AIB : therefore by adding triangle BDG to triangle BHD, to become as triangle AKB to triangle AIB; and by interchanging, triangle BDG to triangle AKB, as triangle BHD, just as shown before, as triangle BIC to triangle AIB. But triangle AKB is to triangle EDB, that is triangle BDG is to triangle EDB, that is, since ED, BG are parallel, BG is to ED, as triangle BIC to triangle AIB, that is just as shown above, as DF to BG. Therefore they are in the continued ratio DF, BG, ED. Q.e.d.

PROPOSITION CX.

AB, BC shall be some conjugate diameters, and on assuming some point D on the periphery between A and C, the tangent shall be acting through D, crossing the diameters AB, BC at E and F, and with AC joined it shall cross the diameter DB at G.

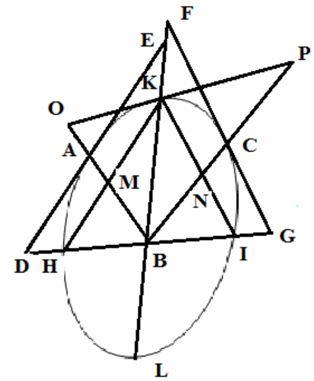
I say the right line DF to DE, to be in the square ratio of that which CG has to GA.

Demonstration.



The line BH may be drawn from B parallel to EF, and from D the right line DH crossing FB in I. Therefore, from the preceding, the lines FD, BH, ED will be in continued proportion. And therefore the ratio FD to ED, will be in the square ratio of FD to BH, that is DI to IH, since DF, BH will be parallel, from the construction: again since the right line HB shall be parallel to the tangent DF, the diameters DB, BH will be conjugate; moreover from the construction AB, BC also shall be conjugate; therefore as DI to IH, thus CG ad GA (§101): whereby the ratio FD to DE, is the square of the ratio CG ad GA. Q.e.d.

PROPOSITION CXI.



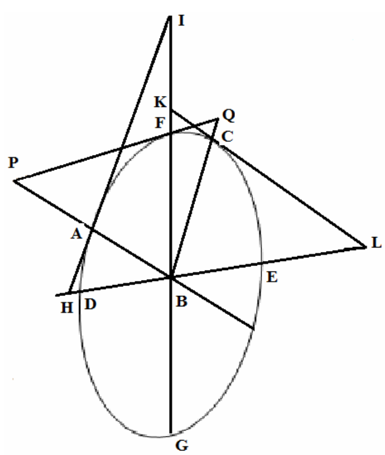
AB, BC shall be conjugate diameters and the tangents DE, FG may be drawn through A and C. Moreover there shall be some other conjugates of the diameters, HI, KI, which produced will intersect the tangents DE, FG at E, F, D, and G.
I say the lines DE, FG to be divided proportionally at A and C, without doubt to be EA to AD, as GC to CF.

Demonstration.

The lines HK, KI may be drawn which will cut the right lines AB, BC at M and N. The ratio EA to AD is the square of the ratio KM ad MH (§110), and GC to CF is the square of the ratio IN to NK. And the ratio KM to MH is equal to the ratio IN to NK. Therefore the ratios EA to AD and GC to CF of the squares of equal ratios, are equal; therefore DE, GF are cut proportionally at the points A, C. Q.e.d..

Corollary.

So that if a third tangent may be drawn through K, agreeing with the conjunctions BA, BC at O and P, I say there becomes OK to KP, as EA to AD. Which will be shown with the right line AC drawn in the same manner as we have used.
And thus the three tangents DE, OP, FG thus are divided similarly so that EA shall be to AD, just as OK to KP, and GC to CF.



PROPOSITION CXII.

AB, BC, FG, DE shall be two sets of conjugate diameters, with the tangents HI, KL acting through A and C, which intersect the diameters FG, DE at the points H, I, K, L.
I say BC shall be to BA thus as HI to KL.

Demonstration.

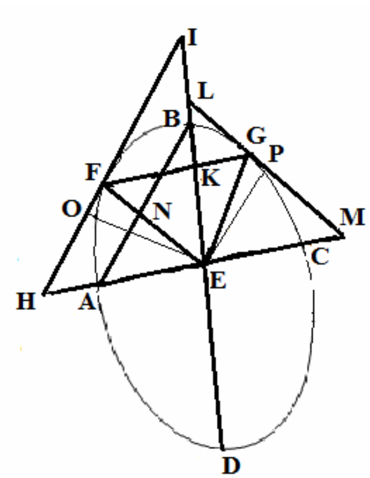
Since the right line BC shall be parallel to the right line HI, and KL parallel to AB, both HA, BC, AI as well as KC, AB, CL are lines in continued proportion. Therefore since just as for HA to AI, first to third, thus also there will be KC to CL, first to third; there will be also HA to BC, first to second, as well as KC to AB; I have shown before HA to be to AI, as KC to CL, and thus on inverting the compounded ratios, and on interchanging, there shall be as HA to KC, thus HI to KL, so that there shall be BC to BA, thus as HI to KL.

$$\left[\begin{array}{l} \text{HA, BC, AI \& KC, AB, CL in continued proportion provide} \\ BC^2 = HA \cdot AI \ \& \ AB^2 = KC \cdot CL \\ \dfrac{BC^2}{AB^2} = \dfrac{HA \cdot AI}{KC \cdot CL} = \dfrac{HI^2}{KL^2}. \end{array}\right]$$

Corollary.

It is shown in the same manner, if a tangent may be acting through F which crosses AB, BC at P and Q , that BC shall be to BD, thus as HI to PQ.

PROPOSITION CXIII.



AC, BD, EF, EG shall be the two pairs of conjugate diameters: and with the tangents acting through F and G which cross the diameters AC, BD at H, I, L, M, the right line FG may be drawn cutting the diameter BE at K.

I say the lines LK, KE, KI to be in continued proportion.

Demonstration.

Since FE, as the conjugate of EG, shall be parallel to the tangent LG , so that LK to KE, thus GK will be to KF; Similarly since EG, as the conjugate of EF, shall be parallel to the tangent FI; therefore as LK to KE, thus KE is to KI . Q.e.d.

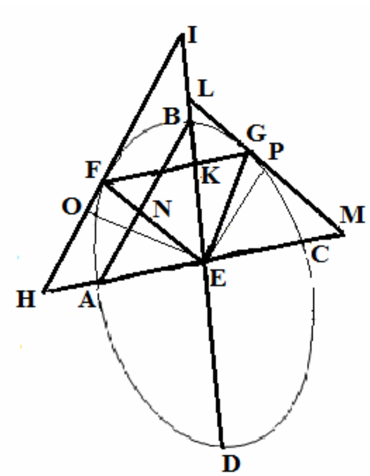
PROPOSITION CXIV.

With the same in place:
I say the triangles IHE, LME to be similar.

Demonstration.

The right line AB shall be drawn which shall cut the line FE at N : and from E the right lines EO, EP shall be drawn normal to the lines HI, LM. Since the line LM shall be parallel to FE (for LM is the tangent, and FE conjugate to EG), the angle FEG shall be equal to the angle EGP. In the same manner the angle OFE will be equal to the angle FEG, where the angles OFE, EGP are equal to each other: moreover the angles EPG, EOF are right; Therefore the triangles EGP, EFO are similar. Whereby as EG to EF, thus EP to EO. But as EG is to EF, thus HI to LM. Therefore as EP to OE, thus inversely HI to LM. Therefore the triangles IHE, LME are similar. Q.e.d.

PROPOSITION CXV.

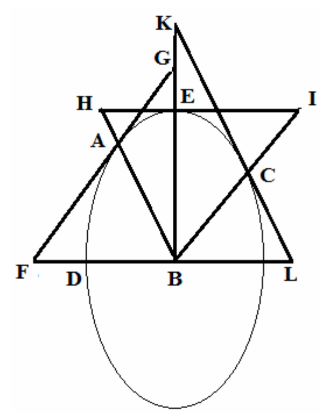


With the same triangle EGM in place equal to the triangle EFI, then triangle EGL is equal to triangle EFH.

Demonstration.

Indeed as HF is to FI, thus LG to GM, and on adding together, as Hl to FI, thus LM to GM: but as HI to FI, thus HIE triangle is to triangle FIE, and as LM ad GM, thus triangle ELM to triangle EGM, therefore as triangle HIE to triangle FIE, thus triangle ELM is to triangle EGM. Whereby the triangles FIE, EGM are equal. In the same manner the remaining triangles EGL, HFE are shown to be equal.

PROPOSITION CXVI.



If the ellipse ADC may cut the two sets of conjugate diameters AB, BC; EB, BD in such a way that by acting through A, E, C, the tangents FG, HI, KL shall indeed cross the diameters EB, BD at G, K, F, L. Truly with the diameters AB, BC, at H and I.

I say the triangles FGB, HBI, KLB to be equal to each other.

Demonstration.

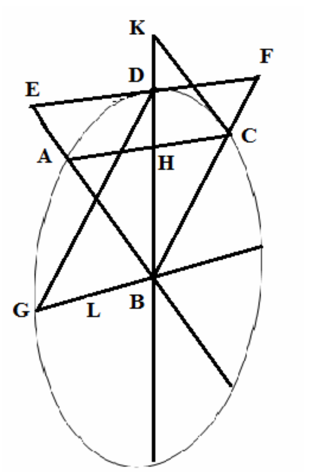
For triangle BIE is equal to triangle BKC (§.70), that is, by §.115, with triangle BFA and triangle BHE, shall be equal to triangle GAB, therefore the whole triangle BIH is equal to the whole triangle BFG. Whereby since also FGB, KBL shall be equal; it is evident the three triangles to be equal.

PROPOSITION CXVII.

AB, BC shall be two conjugate diameters of the ellipse ABC, and from some point assumed between A and C, evidently D, the tangent DE may be drawn crossing over the conjugates AB, BC produced at E and F: then BG may be drawn from the centre B, parallel to EF itself, and GB shall be the mean between ED, DF.

I say the point G to be in the periphery of the ellipse, of which the conjugate diameters are AB, BC, and the tangent ED.

Demonstration.



BD is drawn from the centre to the tangent point, and since by the hypothesis GB is parallel to the tangent ED, GB will be the ordinate in place for BD, and indeed for the centre. From which thus BD, GB are conjugate diameters. Then AHC and DG may be joined, and from C, CK may be drawn parallel to AB, crossing BD produced at K : which is a tangent to the section at C, and there will become as AH to HC, thus BH to HK: but as AH to HC, thus the triangle ABH is to the triangle HBC; and thus as BH ad HK, thus triangle HBC is to triangle HCK: therefore as triangle ABH to triangle HBC itself, thus triangle HBC is to triangle HCK: therefore on adding, and interchanging triangle ABC with triangle BCK , thus there is triangle BHC to triangle HCK, that is now as shown, so that triangle ABH is to triangle HBC, that is, as the line AH to the line HC: but the triangles BCK, BDF are equal (§.70); therefore also the triangle ABC to the triangle BDF, shall be as AH to HC. Further since DF, GB are parallel, there will be triangle GDB to triangle DBF, as GB to DF: but, since from the hypothesis ED, GB, DF are in continued proportion , the ratio GB to DF, is half of the ratio ED to DF [*i.e.* the geometric mean of the ratio]. Therefore the ratio of triangle GDB to triangle DBF is half of the ratio ED to DF. And the ratio AH to HC, that is as shown above, the ratio of triangle ABC to triangle DBF, also is half of the ratio ED ad DP. Therefore triangle GDB is to triangle DBF, as triangle ABC to the same triangle DBF. Therefore the triangles GDB, ABC are equal. Therefore the parallelogram held by the radii, as shown above, with the conjugates GB, BD in the angle GBD is equal to the parallelogram held under the conjugate radii AB, BC. Therefore the point G lies on the ellipse (§.72). Q.e.d.

PROPOSITION CXVIII.

Again there shall be the two conjugate diameters BA, BC. And with the point D taken on the perimeter of the ellipse between A and C, it shall be the tangent EF of the ellipse at D, crossing with the diameters E and F. Then from the centre B, BG is drawn to the perimeter parallel to the tangent.

I say ED, GB, DF to be in continued proportion.

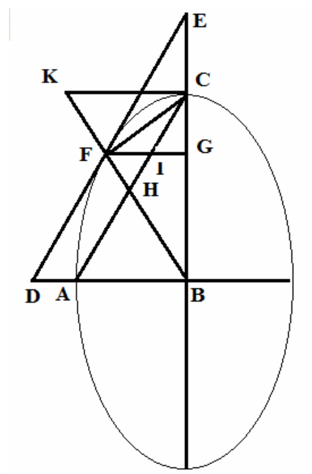
Demonstration.

Thus if not, there shall be some other mean value LB between ED, DF greater or smaller than GB. Therefore by the preceding the point L lies on the ellipse, which cannot happen, since from the hypothesis the point G shall be present on the ellipse. Therefore no other besides GB, is the mean between ED, DF. Therefore ED, GB, DF are continued proportionals. Q.e.d.

PROPOSITION CXIX.

AB, BC shall be conjugate diameters for the ellipse and the ends of these may be joined ; the line DE parallel to this line AC shall be a tangent at F, and crossing the conjugate diameters extended at D and E.
I say the triangle EDB to be the double of triangle CAB.

Demonstration.



The diameter BHF shall be drawn from the centre B through the point of contact F, to which it shall meet at K the right line tangent to the ellipse at C, then from the point of contact F the ordinate FIG may be put in place. Since AC is parallel to the tangent DE from the hypothesis, it will be the ordinate to the diameter BF (§.13). Therefore KB, FB, HB are continued proportionals (§.32). Therefore as KB is to FB, thus KF is to FH. But KB is to FB as the triangle KCB is to the triangle FCB, that is, since the triangles KCB, EFB are equal (§.70), so that as triangle EFB to the same triangle CFB, that is, as EB to CB. Therefore as EB to CB, thus KF to FH. Then since from the construction KC to be a tangent, and FG is put the ordinate to BC, the right lines EB, CB, GB are in continued proportion. Therefore as EB to CB, thus EC to CG. But also as I have now shown, thus as EB to CB, thus KF to FH. Therefore as KF to FH, thus EC to CG. Therefore as triangle KCF to triangle FCH: thus triangle EFC to triangle CFG. And since the whole triangles KCB, EFB shall be equal, with the common triangle FBC removed, KCF, EFC are equal. Therefore FCH and CFG are equal; therefore with the common triangle FIC removed, FIH, CIG remain equal, from which if with the common area BHIG added, FGB will be equal to CHB. Now truly since the ordinate AC to BF put in place as shown above, is AC bisected at H, and thus the triangle CAB is twice the triangle CHB, and DE parallel to AC also shall be bisected at F: truly is FG, just as with the ordinate drawn to CB, parallel to CB, the conjugate diameter to CB. Therefore BE shall be bisected at G. And thence EFB is twice GFB. And CHB, FGB have been shown to be equal. Therefore the double of these CAB, EFB are equal. But triangle DEB doubled is triangle EFB, is indeed DE bisected at F. Therefore triangle DEB doubled is also the double of triangle CAB. Q.e.d.

ELLIPSIS

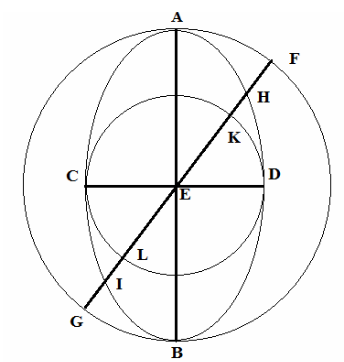
PARS TERTIA

Considerat axium ac diametrorum coniugatarum tam aequalium quam inaequalium proprietates.

PROPOSITIO LXXI.

In ellipsi diametrorum maxima & minima sunt axes.

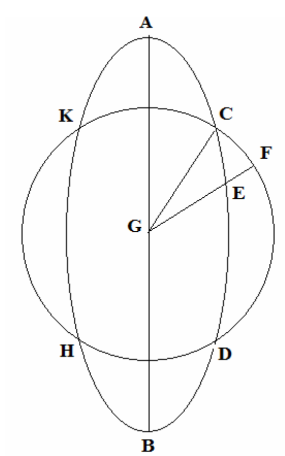
Demonstratio.



Sint ABC ellipseos axes AB, CD, & AB quidem maior, D vero minor. Dico AB, diametrorum esse maximam, CD vero minimam, centro ellipsis E: intervallo EA circulus describatur AFG*:* transilit per B, et reliquo sui totus extra ellipsim cadet: ducatur dein per E diameter quaecunque FG occurrens ellipsin H & I circulo autem in F & G. Quoniam circulus AFG totus cadit extra ellipsim, erit FG linea maior recta HI : igitur & AB maior est quam HI. Idem ostenditur de quavis alia diametro; igitur AB axis maximus est diametrorum ellipsis ABC. Quod erat primum.

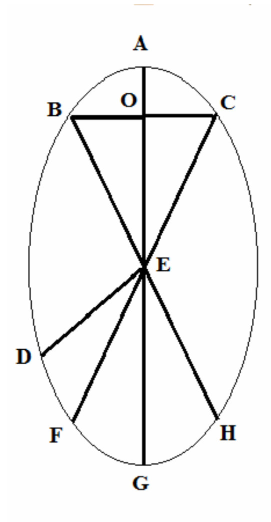
Rursum centro E intervallo ED circulus describatur DKL occurrens FG lineae in K & L: transilit is per C & reliqua sui parte totus intra sectionem cadet, igitur HI linea maior est quam KL hoc est CD: Quare cum idem de omni alia linea quae per C & D non transit ostendatur, erit CD diameter omnium minima quae in ellipse ADB duci possunt. Quod erat demonstrandum.

Corollorium primum.



Diameter quae maiori axi est propior, illa maior est; & quae remotior, minor sit ABC ellipseos axis AB, centrum G: ponaturque GC diameter propior axi quam GE. Dico GC maiorem esse ipsa GE centro enim G intervallo GC circulus describatur occurret is ellipsi in quatuor tantum punctis CKHD, quare GE ad peripheriam non pertingit. Unde minor est quam GC. Quod erat demonstrandum.

Corollarium secundum.



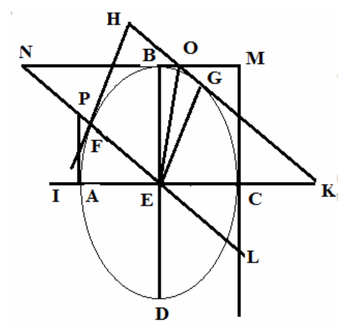
Porro diameter axi vicinior est, quae cum axe minorem vel angulum facit vel sectorem, primum patet : alterum e primo sic ostendo. Ellipseos axis maior sit AG faciatque diameter BE cum axe sectorem BEA. minorem sectore DEG, quem cum axe facit diameter DE.

Quoniam igitur sector DEG maior est sectore BEA, fiat sector FEG aequalis sectori BEA, & FE occurrat ellipsi in C, ducaturque BOC, sector BEA aequatur sectori FEG ex constructione hoc est sector ad verticem AEC. Ergo BC bisecta est in O ab axe AG. Anguli ergoad O recti sunt. Patet ergo angulum BEA aequari angulo AEC, hoc est angulo FEG, hoc est minorem esse angulo DEG; liquet igitur ex primo BE quae sectorem facit cum axe minorem, axi propiorem esse quam DE, quae maiorem.

PROPOSITIO LXXII.

Rectangulum sub dimidiis axibus aequale est parallelogammo sub semidiametris coniugatis.

Demonstratio.



Sint ABC ellipsis axes AC, BD: centrum E, & quaevis semidiametri coniugatae EF, EG. Actisque per F & G tangentibus quae conveniant in H, & axi AC occurrant in I & K: ducantur etiam per C & B lineae quae ellipsim contingant in C & B: conveniant autem in M: & HK, EF secent in O, L, N: tum iunctis punctis EO agatur per A tangens, secans EN lineam in P. Quoniam tam NO, KE linea quam OK, NE sibi mutuo aequidistant; erit NO, KE parallelogrammum, diametro OE divisum bifariam: sunt autem triangula a EOB, EOC aequalia, igitur & reliqua triangula EBN, EGK interse aequantur. Rursum cum AP, CL lineae aequidistent, & AE, CE lineae sint aequales, erit ECL triangulo aequale triangulum EAP hoc est EIF. Quoniam igitur triangulum EGK aequatur triangulo EBN, & triangulum IFE triangulo ECL, proportionalia erunt quatuor illa triangula; sunt vero etiam similia inter se, nimirum EGK ipsi IFE, & EBN ipsi ECL, ergo rectae KE, EI, NE, EL, proportionales sunt, quare cum super proportionalibus in directum positis constitura sunt triangula IFE, EGK, IHK inter se similia & triangula CLF, EBN, LMN inter se quoque similia, ut sunt duo triangula IFE, EGK ad duo triangula IFE,EGK aequantur, ut ostendi supra, duobus LCE, EBN. Ergo etiam triangulum IHK triangulo LMN aequale est, ac proinde demptis aequalibus parallelogrammum GEPH sub semidiametris coniugatis, aequatur rectangulo BECH sub dimidiis axibus contento. Quod erat demonstrandum.

Corollorium primum.

Hinc sequitur si in ellipsi duae quaevis sint diametrorum coniugationes AE, EB, EF, EG triangula super EA, EB, EF, EG in angulis AEB, FEG, esse inter se aequalia: sunt enim dimidia parallelogrammorum aequalium.

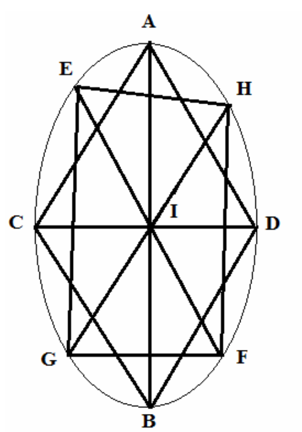
Corollorium secundum.

Sequitur secundo parallelogramma sub totis diametris coniugatis, inter se esse aequalia : cum sint quadrupla eorumque hac propositione ostensa sunt aequalia.

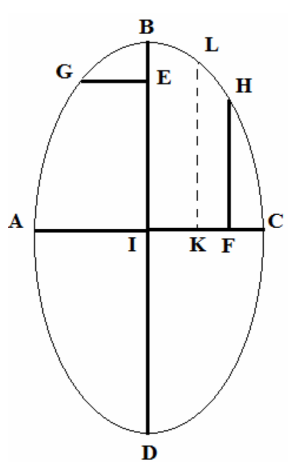
PROPOSITIO LXXIII.

IN ellipsi parallelogrammum quod fit a lineis extrema axium coniungentibus aequale est parallelogrammo contento lineis extrema quarumvis diametrorum coniugatarum coniungentibus.

Demonstratio.



Sint ABC ellipseos axes ABCD, & alia quaevis diametrorum coniugatio EFGH. Iunganturque tam axium, quam diametrorum extremae: dico CADB parallelogrammum aequari parallelogrammo EHFG. Rectangulum ex AB & DI duplum est trianguli ADB: & rectangulum ex AB & CI, duplum est trianguli ACB. Ergo rectangulum ex AB, CD, duplum est parallelogrammi ACBD, sive ACBD parallelogramum, dimidium est rectanguli super AB, CD. Similiter ostendam parallelogrammum EGFH dimidium esse parallelogrammi super EF, GH in angulo EIH: sed parallelogrammum super EF, GH aequale est parallelogrammo super ABCD; igitur & ACBD parallelogrammum aequale est parallogramo EGFH. Quod fuit demonstrandum.



PROPOSITIO LXXIV.

Sint ABC ellipseos axes vel diametri coniugatae, AC, BD. Divisaque BD utcunqae in E, dividatur AC in F proportionaliter, & per E & F ordinatim ducantur lineae EG, FH:

Dico rectangulum AFC aequari quadrato GE, & BED rectangulum quadrato HF, & si quadratum GE sit aequale rectangulo AFC. dico BD, AC proportionaliter esse divisas in E & F.

Demonstratio.

Cum ex hypothesi DE sit ad EB ut AF ad FC, erit permutando DE ad AF, ut BE ad FC. Quare & tota DB, ad totam AC, ut DE ad AF & EB ad FC. igitur rationes BE ad FC, & DE ad AF, simul sumptae duplicatae sunt rationis DB ad AC. Atqui ratio rectanguli BED ad rectangulum AFC, componitur ex rationibus BE ad FC, & DE ad AF. Ergo ratio rectanguli BED ad rectangulum AFC duplicata est rationis BD ad AC, hoc est rationis BI ad AI. Ergo rectangulum BED est ad rectangulum AFC, ut quadratum BI ad quadratum AI, sed idem quoque rectangulum BED est ad quadratum GE, ut rectangulum BID. Hoc est, quadratum BI, ad quadratum AI aequantur igitur quadratum GE & rectangulum AFC. Similiter ostendemus rectangulum BED & quadratum HF aequalia esse.

Sint iam aequalia quadratum GE & rectangulum AFC: dico BD, AC proportionaliter esse sectas: Nam si non est ut BE ad ED, sic AF ad FC, sit ut BE, ad ED, sic AK ad KC. Erit ergo quadratum GE aequale rectangulo AKC per primam partem huius: Quod fieri non potest, cum quadratum GE ex hypothesi sit aequale rectangulo AFC. Non igitur AC in K aut alibi quam in F erit secta proportionaliter ad BD. Quod erat demonstrandum.

PROPOSITIO LXXV.

Si axes aut diametri coniugatae sint proportionaliter sectae in E & F, & ordinatim ducantur EG, FH.

Dico quadratum FH esse ad quadratum EG, ut quadratum BD, ad quadratum AC.

Demonstratio.

Quadratum FH aequatur rectangulo BED, sed rectangulum BED est ad quadratum EG ut rectangulum BID, hoc est quadratum BI, ad quadratum IA. Ergo etiam quadratum FH est ad quadratum EG, ut quadratum BI ad quadratum IA, hoc est, ut quadratum BD ad quadratum AC. Quod erat demonstrandum.

Quod si ad diametros coniugatas ordinatim positae sint EG, FH, & sit ut quadratum BD ad quadratum AC, ita quadratum FH ad quadratum EG : Dico BD, AC proportionaliter esse sectas in E & F. Si enim negas esse AF ad FC, ut DE ad EB, fiat AK ad KC, ut DE ad EB, & sit ordinarim KL. Ergo ut quadratum BD ad quadratum AC, sit quadratum KL ad quadratum EG: quod fieri non potest, cum ex hypothesi quadratum FH sit ad quadratum EG, ut quadratum BD ad quadratum AC. Non igitur est ut quadratum BD ad quadratum AC, ita quadram KL, aut quodvis aliud praeter quadratum FH, ad quadratum EG. Quod erat demonstrandum.

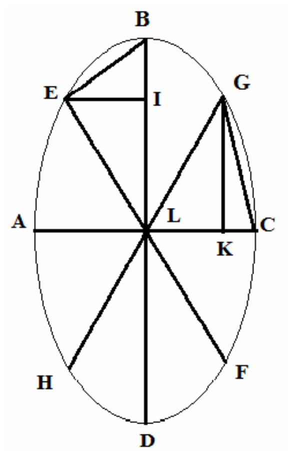
PROPOSITIO LXXVI.

Sint in ABC ellipsi duae quaevis diametrorum coniugationes AC, BD, EF, GH; ducanturque ex E & G lineae EI, GK ordinatim ad diametros BD, AC.

Dico EI quadratun, aequari rectangulo AKC, & BID, rectangulum aequale esse quadrato GK.

Demonstratio.

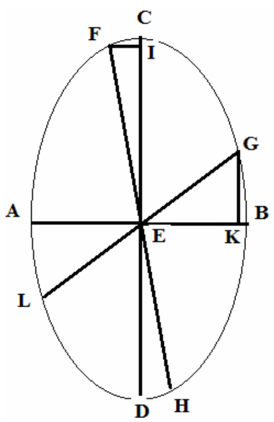
Ponantur EB, GC. Quoniam igitur tam AC, BD, quam EF, GH diametri sunt coniugatae, erit sector BLC aequalis sectori ELG, & dempto communi BLG, sector ELB aequalis sectori CLG. Adeoque & LEB triangulum aequale triangulo LGC: & quia BD est coniugata ipsi AC, erit BD parallela ad KG quae est ordinatim posita ad AC. Ergo angulus GKL aequalis angulo BLA. Similiter quia AC est coniugata ipsi BD, erit AC parallela ipsi EI ordinatim positae ad BD; angulus ergo EIB aequalis est angulo BLA, hoc est angulo GKL; igitur cum triangula sint aequalia, erit (ut infra ostendam) ut basis LB ad basim LC, ita KG ad EI, adeoque ut quadratum BL ad quadratum LC, hoc est ut quadratum BD, ad quadratum AC, sic quadratum KG ad quadratum EI. Unde BD, AC lineae in I & K proportionaliter sunt divisae. Quare EI quadratum aequale rectangulo AKC, item BID rectangulum aequale quadrato GK. Quod fuit demonstrandum.



PROPOSITIO LXXVII.

Axium quadrata simul sumpta aequalia sunt quadratis cuiuscunque coniugationis simul sumptis,

Demonstratio.

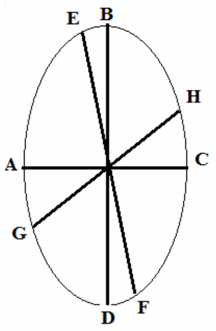


Sint ABC ellipseos axes AB; CD, & alia quaevis diametrorum coniugatio FH, GL. Dico AB, CD quadrata simul sumptis aequari quadratis FH, GL simul sumptis. Ducantur ordinatim lineae, FI, GK quae, quia ad axes ducantur, perpendiculares erunt; centrum autem sectionis ponatur E. Quadratum EC aequale est quadrato EI una cum CID rectangulo id est quadrato GK: quadratum autem EB aequale est quadrato EK una cum rectangulo AKB id est quadrato FI **:** unde quadrata duo EB, EC simul sumpta aequalia sunt quadratis FI, IE, EK, GK simul sumptis. Sed iisdem quadratis aequalia sunt quadrata SE, EG, quadratis igitur EF, EG aequalia sunt quadrata EB, EC. Quare cum AB, CD quadrata simul sumpta quadrupla sint quadratorum EB, EC & FH, GL quadrata Quadrupla quadratorum EF, EG; patet AB, CD quadrata simul sumpta aequari quadratis FH, GL simul sumptis. Quod erat demonstrandum.

PROPOSITIO LXXVIII.

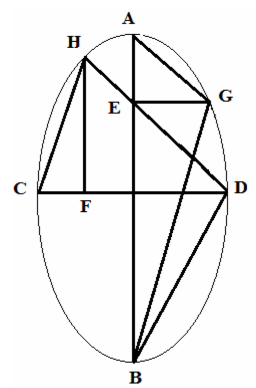
Axes ellipseos simul sumpti minimae sint omnium diametrorum coniugatarum simul sumptarum.

Demonstratio.



Sint axes AC, BD & quaevis diametrorum coniugatio, EF, GH: dico axes simul sumptos minores esse diametris coniugatis simul sumptis. Quoniam AC, BD quadrata simul sumpta, aequalia sunt quadratis EF, GH simul sumptis: sit autem & BD maxima diametorum, AC vero minima, erunt AC, BD simul sumptae minores rectis EF, GH igitur, &c. Quod fuit demonstrandum.

PROPOSITIO LXXIX.



Sint ABC ellipsis axes AB, CD, devisi proportionaliter in E & F: ductisque ordinatim (quae hic sunt perpendiculares) lineis EG, FH: iungantur AG, GB, CH, HD: Dico quatuor quadrata AG, GB, CH, HD**,** simul sumpta, aequari duobus axium quadratis.

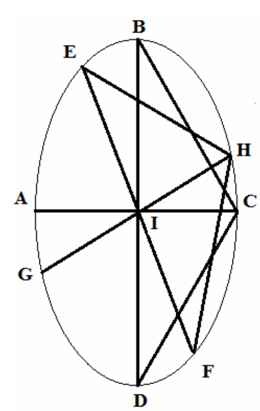
Demonstratio.

Quadratum AB aequale est quadratis AE, EB una cum rectangulo AEB id est quadrato HF bis sumpto quadratum vero CD aequalc est quadratis CF, FD, & CFD rectangulo id est quadrato EG bis sumto, sed iisdem quadratis aequalia sunt quadrata AG, GB, CH, HD, igitur axium quadrata simul sumpta aequalia sunt quadratis AG, GB, CH, HD. Quod erat demonstrandum.

PROPOSITIO LXXX.

Quadrata linearum extrema axium coniungentium aequalia sunt quadratis linearum quae extrema cuiusuis coniugationis coniungut.

Demonstratio.

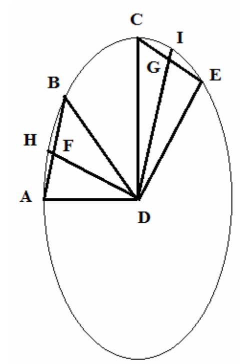


Sint ABC ellipseos axes AC, BD & alia quaevis diametrorum coniugatio EF, GH. iunganturque BC,CD, EH, FH. Dico quadrata BC,CD simul sumpta aequari quadratis EH, FH simul sumptis quadrata BC, CD simul sumpta aequalia sunt quadratis BI, IC bis sumptis quadrata autem EH, HF aequantur quadratis EI, IH bis sumptis: sed quadrata EI, IH simul sumpta sunt aequalia quadratis BI, IC simul sumptis; igitur quadrata BC, CD simul sumpta, aequalia sunt quadratis EH, HF simul sumptis. Quod erat demonstrandum.

PROPOSITIO LXXXI.

Secent ABC ellipsim, cuius centrum D duae diametrorum coniugationes AD, DC, BD, DE, iunctisque punctis AB, CD dividantur AB, CD linae bifariam in F & G, ducanturque DF, DG quae productae occurrant ellipsi in H & I.
Dico HD, ID diametros esse coniugatas.

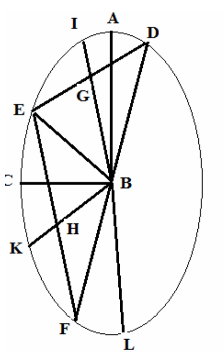
Demonstratio.



Quoniam AD, DC, BD, DE, diametri sunt coniugatae, erunt ADC, BDE sectores aequales: ablato igitur communi BDC, erunt ADB, CDE reliqui aequales : rursum cum AB lineam secet in F, bifariam diameter DH, erunt tam ADH, BDH sectores, quam AFD, BFD triangula aequalia. Eadem modo ostenditur EDI sector aequalis sectori CDI; sectores igitur ADH, EDI sunt aequales inter se: Addico igitur communi HDI, erit sector ADI aequali sectori HDE, coniugatae ergo sunt DH, DI.

PROPOSITIO LXXXII.

Sint ABC ellipseos axes AB, CD, sit autem & alia quaevis diametrorum coniugatio, DF, EB; quas iungant DE, FE; DE quidem secans axem maiorem, EF vero minorem : ipsas deinde DE, FE bifariarm secent diametri BGI, BHK.
Dico IB diametrum maiorem esse diametro KB.



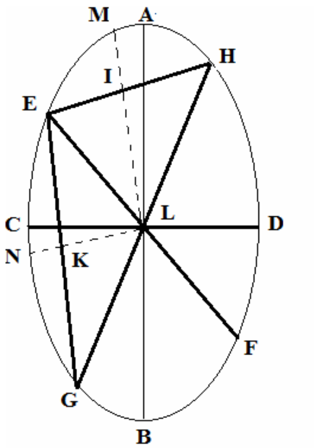
Demonstratio.

Quoniam diametri BI, BK bisecant rectas ED, EF sectores ambo DBE, EBF bisecantur. Quare cum ipsi aequales sint, etiam ipsorum sectores dimidii IBD, KBF aequales erunt. Sector igitur IBD minor est sectore KBL; ergo sector IBA multo minor sectore KBL. Ergo IB propior axi est quam BK, ac proinde maior quam BK. Quod erat demonstrandum.

PROPOSITIO LXXXIII.

Sint ABC ellipseos axes AB, CD, & alia quaevis diametrorum coniugatio EF, GH, iunganturque EH, EG.
Dico lineam EH quae axem maiorem secat, minorem esse lineam EG, quae minorem secat.

Demonstratio.



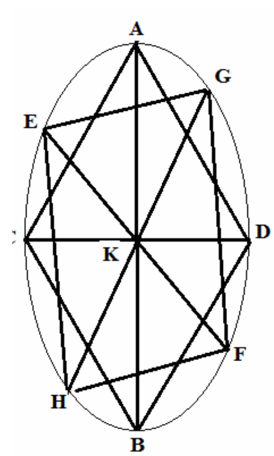
EH, EG dividantur bifariam per diametros LM, LN, in I & K. Quoniam GH, EF sunt contiugatae; sectores GLE, ELH aequales erunt, ac proinde segmenta GNE, EMH aequalia sunt. Quare LM, LN bisecantes subtensas EH, EG, proportionaliter sunt divisae Ergo MI ad IL ut NK ad KL; & componendo ac permutando ut LM ad LN, sic LI ad LK. Sed LM maior est quam LN; ergo LI etiam maior quam LK. Iam vero cum LN, LM etiam sint coniugatae, & EG sit ordinatim ex constr. posita ad LN, erit LM parallela ad EK. ob similem causam LN, EI parallelae erunt ; parallelogrammum igitur est EI, LK, adeoque LI, KE, LK, EI aequantur. Cum ergo LI ostensa sit maior esse quam LK, erit & KE maior quam LK, hoc
quam EI. Quare dupla eius EG, maior dupla EH. Quod erat demonstrandum.

PROPOSITIO LXXXIV.

Axium extrema coniungentes simul sumptae maximae sunt omnium quae quarumvis diametrorum coniugatarum conuertunt extrema.

Demonstratio.

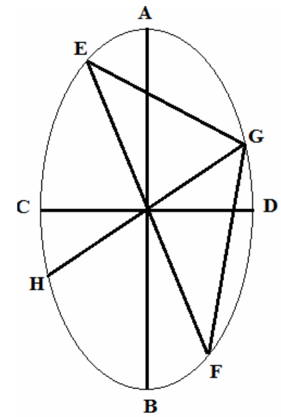
Sint axes AB, CD & alia quaevis diametrorum coniugatio EF, GH : iunganturque extrema tam axium, quam aliarum diametrorum. Dico lineas CA, AD, DB, si C simul sumptas maiores esse lineis FG, GF, FH, HE, BC simul sumptas. Quoniam CK ipsi HK aequalis est, EK vero communis, & EH recta maior quam EG, erit angulus EKH, maior angulo EKG: igitur & angulus EKH maior est recto AKC. Sunt autem AKC, EKH triangula aequalia, quare & EH est maior quam AC: eadem modo ostenditur AC linea maior est recta EG. Ergo EG minima est, & EH, maxima linearum EH, AC, AD, EG. igitur cum EH, EG quadrata simul sumptis sint aequalia quadratis AC, AD simul sumptis, erunt EH, EG lineae fimul sumptae minores lineis AC, CD simul sumptis. Eodem modo ostenduntur lineae GF, FH minores lineis CB, BD, ergo, & c. Quod erat demonstrandum,



PROPOSITIO LXXXV.

ln nulla ellipsi est invenire diametros coniugatas quae sese ad rectos secent, praeter axes.

Demonstratio.



Sint axes, AB maior, CD minor & alia quaevis diametrorum coniugatio EF, GH: centrum autem ellipsis sit K. Dico neutrum angulorum EKG, GKF rectum esse: iungantur enim puncta EG, GF: Quonian EK, KG duabus rectis FK, KG aequales sunt, & EG minor quam FG. erit angulus, EKG minor angulo GKF; Quare cum erum summa sit duobus rectis aequalis, neuter illorum rectus est: idem de aliis omnibus ostenditur: igitur in nulla ellipsi est invenire, &c. Quod erat demonstrandum.

PROPOSITIO LXXXVI.

Sint ABC ellipsis axes AC, BD & alia quaevis diametrorum coniugatio EF, GH : iunctisque punctis AB, BC, rectae ducantur EH, EG.

Dico angulum ABC qui circa minorem axem existit, maiorem esse angulo GEH, ac proinde maximum esse omnium angulorum qui continentur a lineis extrema diametrorum coniugatarum coniungentibus.

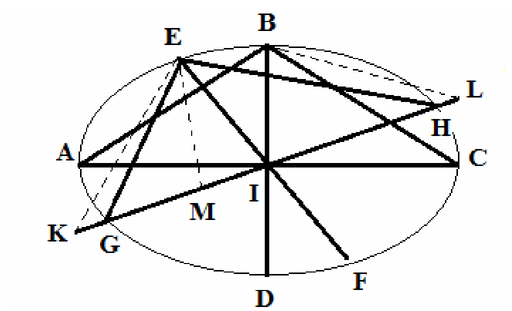
Demonstratio.

Cum GH linea minor sit axe AC, producatur utrimque aequaliter in K & L ut LK, sit aequalis axi AC, iunganturque puncta EK, EL, & ex E demittatur EM normalis ad LK: erit igitur KEL triangulurom maius triangulo GEH id est triangulo ABC ; quare cum inaequalium triangulorum bases AC, LK sint aequales, erit EM altitudo trianguli KEL, maior IB altitudine trianguli ABC: adeoque angulus KEL maior angulo ABC: angulus igitur GEH multo minor est angula ABC. Quod erat demonstrandum.

PROPOSITIO LXXXVII.

Sint ABC ellipseos axes AC, BD : iunganturque illorum extrema AB, BC, CD, DA : sit autem & alia quaecunque diametrorum coniugatio EF, GH, quarum extrema quoque coniungantur.
Dico angulos ABC, EGF, HFG, BCD arithmetice esse proportionales.

Demonstratio.



Quoniam tam AC quam EF parallelogrammum est, erunt tam ABC est, erunt tam ABC, BCD anguli, quam EGF, GFH duobus rectis aequales: quare & anguli ABC, BCD simul sumpti aequantur angulis EGF, GFH simul sumptis: est autem angulus ABC, ostensus igitur maior angulo EGF, igitur & GFH maior est angulo BCD: & quia, ut iam ostendi, anguli B, & C simul sumpti aequantur angulis G, & F, simul sumptis quo excessu ABC angulus superat angulum EGF, eodem necesse est ut angulus GFH superet angulum BCD. Quod erat demonstrandum.

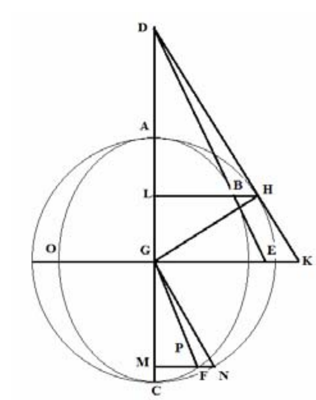
Corollarium.

Hinc patet angulum BCD qui circa axem maiorem existit, minimum esse omnium angulorum qui sunt a lineis diametrorum coniugatarum extrema coniungentibus.

PROPOSITIO LXXXVIII.

Ellipsum ABC cuius axes AC, EG contingat in B recta quaedam DE conveniens cum utroque axe in D & E: ex centro vero G recta demittatur GF parallela lineae DE.
Dico DB, GF, BE lineas esse in continua proportione.

Demonstratio.



Centro G,intervallo AG circulus describatur AHC: & ex D demittatur DK contingens circulum in A occurrens EG axi ellipseos in K, ductaque ex H linea HL normali ad axem quae per Coroll.33. huius transit etiam per B, agatur per F normalis alia FMN, iunganturque puncta NG, HG. Quoniam LH, MN lineae aequidistant, erunt anguli LDB, MGF aequales: sunt autem & anguli BLD, FMG recti per constructionem; igitur & reliquus LBD, reliquo MFG aequalis est. Quare anguli DBH, GFN inter se aequantur. Quia autem triangula DLB, GMF sunt similia, erit ut LB ad M F, sic DB ad G F. sed ex demonstratis in scholio quartae huius, ut LB ad MF, sic BH ad FN. Ergo DB ad GF, ut BH ad FN. Quare cum anguli DBH, GFN iam ostensi sint aequales, similia erunt triangula DBH, GFN. Ergo HD est ad BD, hoc est HK est ad BE, ut GN ad GF, & permutando ut HK ad GN, sic BE ad GF. Deinde cum in triangulo DGK angulus ad G rectus sit & GH ex centro ad contactum ducta, normalis ad DK, erit HK ad GH, ut GH ad HD. Sed GN, GH aequantur, ergo, ut KH ad GN, hoc est sicut ante ostendi, ut BE ad G F, sic GN ad DH. Sed ob similitudinem triangulorum ut GN ad DH, sic GF ad DB. ergo, ut BE ad GF, sic GF ad DB. Quod erat demonstrandum.

PROPOSITIO LXXXIX.

Iisdem positis si GF sit proportionalis media inter DB, BE.
Dico punctum F esse ad ellipsim.

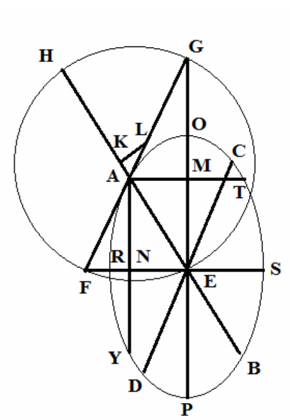
Demonstratio.

Si punctum F non est ad ellipsim, occurrat ergo ellipsis rectae GF in P, supra vel infra F. Ergo per praecedentem DB, GP, BE sunt continue proportionales; quod fieri non potest, cum DB, GF, BE ponantur continue. Non igitur aliud punctum rectae GF ad ellipsim est, quam F. Quod erat demonstrandum.

PROPOSITIO XC.

Data quavis diametrorum coniugatione, ellipseos axes reperire.

Constructio & demonstratio.



Sint datae diametri coniugatae AB, CD secantes se bifariam in E; actaque per A linea FG quae aequidistat CD, fiat ut AE ad ED, sic ED ad AH; tum EH divisa bifariam in K, erigatur ex K normalis KL occurrens FG in L, dein centro L intervallo LE circulus describatur EFG, transibit hic per H secabitque FG lineam in punctis quibusdam F & G: iungantur demum puncta FF, EG. Dico lineas EF, EG satisfacere petitioni. Quoniam enim rectae AE, ED, AH sunt ex constructione in continua analogia, erit quadrato ED aequale rectangulum EAH, hoc est FAG rectangulum. Itaque GA, DE, FA rectae in continua etiam sunt analogia, ac proinde, erit punctum D ad ellipsim, cuius axes sunt in lineis EF, EG. Sed & A in eadem est ellipsi, igitur & B, C puncta, in eadem erunt ellipsi.

Porro termini axium ita invenientur: ductis AM, AN normalibus ad EG lineam, inveniatur inter EM, EG media EO, & inter EN, EF media ER : describaturque,per D, A, C puncta ellipsis; quoniam CD ipsi AB est coniugata, adeoque ad ipsam ordinatim posita & FG linea aequidistat ipsi CD, erit FG tangens ellipsim ABC; sunt autem AM, AN normales ad lineas in quibus axes sectionis existunt; & tam EM, EO, EG, quam EN, ER, EF continuae, igitur ellipsis ABC transit per puncto RO: quare R & O termini sunt axium, quos oportuit exhibere.

Scholion.

*Propositum est hoc problema a Pappo, lib.*8.*Mathem.Collect.prop.*14.*ac veram quidem eius constructionem eam nempe quam ex illo nos iam attulimus, sed non demonstrat.Fredericus Commandinus demonstrationem supplere conatus est, ita scribens:*

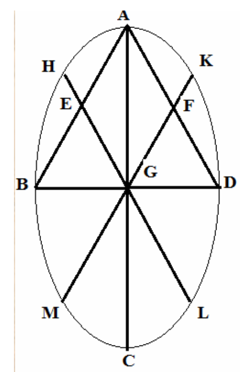
Producatur AM usque ad T ita ut TM ipsi MA, sit aequalis: producatur etiam AN usque ad Y ut YN sit aequalis NA *:* erunt puncta TY in ellipsi, ex iis quae demonstrata sunt ab Apollonio in propop:47 a lib. Conic. Sed RS parallela est ipsi AT, est enim angulus in semicirculo rectus, quare & OP ipsi AY parallela erit. Quoniam igitur CD ad AB ordinatim est applicata quae per A ipsi DC parallela ducitur, videlicet FG sectionem in puncto A continget, & cum FG sectionem contingens diametro occurrat in G & AM ordinatim applicetur, erit per 37. prim.Coni.Apollon, rectangulum GEM aquale quadrato ex EO vel EP. Eadem quoque ratione cum AN ordinatim applicetur rectangulum FEN quadrato ex ER vel ES aequale est ergo OP, RS ellipsis coniugati axes erunt.

Haec Commandinus quibus recta ostendit OP *&* RS *conugatos axes esse ellipsis, quae per puncta* ATY *incedit, & a linea* FG *in puncto tangitur: verum hoc propositum non fuit. Nam ad inveniendum eius modi coniugatos axes non opus erat ad describendum circulum* FEG*, facere rectangulum* EAH *quadrato* ED *aequale, & secare* EH *bifariam in* K *indeque normalem excitare, quae congrediens cum* FG *in* L*, centrum praeberet* L *circuli* FEG*, sumpto si quidem in linea* FG *centro quocumque, si per* E *circulus circumducatur, qui secet lineam* FG*: non iam quidem in punctis* F *&* G *sed in aliis quae ex* E *recta emittentur angulum rectum continebunt, non secus ac* EF *&* EG*, quare si ab* A *ad has ipsas postremo ducts lineas, normales ducanturque, quales erant* AM *&* AN*, duplicentur , ut* AT *&* AY*, erunt puncta quae vices punctorum* T *&* Y *subibunt in ellipsi, quae per* A *incedit, tangiturque lineae* FG *infinitae destinantur. At perspicuum est hanc ellipsin (quod fuerat demonstrandum) per puncta C & D minima transire, propterea quod circuli centrum aliud ab L assumptum sit, nec sit quadratum ED rectangulo sub EA, & alia linea quam AH contentum aequale. Itaque ut ostendatur OP & RS coniugatos axes esse ellipsis, quae per terminos diametrorum coniugatarum AB, & CD incedit, alia ratio est ineunda, quam in demonstratrione nostra iam proposuimus.*

Haec hactenus super Commandini demonstratio.

Caeterum ipse Pappi textu temporum iniuriam sescio quid infortunii passus videtur, ita enim habet : Facile autem est inventis quibuscumque coniugationibus diametrorum ellipsis, axis eius organice invenire. Quod quidem hac ratione fiet. *Quae verba legitimum sensum non habent, cum ea, quam adfert constructio non organica. Sed omnino Geometrica sit, uti eam legenti satis patet: quare puto omissum verbum Geometrice, sicque legendum:* Facile autem est inventis quibuscumque coniugationibus diametrorum ellipsis, axes eius organice invenire, quod quidem Geometrice hac ratione fiet. *Deinde addita sunt in ipsa constructione illa verba:* cum sit DE maior quam EA: *cum enim constructio universalis sit, sive DE minor sive maior, sive ipsi EA aequalis ponatur, ut ex nostra demonstratione colligi potest, quodque Pappum etiam latere nullo modo potuisse certum est , frustra assumitur DE maior ipsa EA. Mirum proinde est hunc errorem Fredericum Commandium non advertisse, praesertim cum illo assumpto in demonstratione sua, quam superius dedimus, usus non fuerit: sed universalem attulerit* *demonstrationem: unde cum & ipsa desit demonstratio, quam quom Pappus addiderit , dubium non est. Satis manifestum est eorum errore id contigisse, in quorum manus venit haec propositio (quae pene tota, ut existimo, interciderat:) qui eam plane iam mutilam & imperfectam, frustra restituere conati sunt.*

PROPOSITIO XCI.



Datis axibus in ellipsi, aequales diametros coniugatas exhibere.

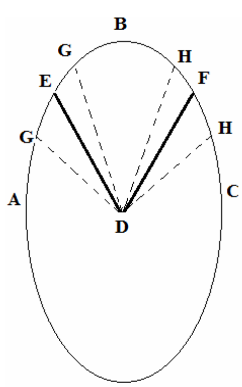
Constructio et demonstratio.

Sint ABC ellipsis axes AC, BD : oporteat autem exhibere diametros coniugatas aequales iunctis AB, AD; dividantur rectae AB, AC bifariam in E & F; & per E & F ex G centro rectae ducantur GH, GK, occurrentes ellipsi in H, K, L, M punctis. Dico illas satisfacere propositioni. Quoniam enim rectae duae EB, BG, aequales sunt duabus lineis FD, DG (sunt autem & anguli, aequalibus lateribus contenti inter se aequales) erunt etiam anguli ad basin, EGB, FGD adeoque & reliqui AGE, AGF aequales. Rursum cum angulus AGB sit rectus & basis AB in E divisa bifariam, si centro E intervallo EA describatur circulus, transibit etiam per B, adeoque EA, EG lineae erunt aequales. Quare & angulus EAG, aequalis angulo EGA, hoc erit AGF. Ergo AB, KM lineae parallelae: eodem modo ostenduntur rectae AD, HL parallelae: unde cum diametri HL, KM mutuas parallelas bisecent, erunt coniugatae. Quia vero angulus HGA est angulo AGK ostensus aequalis, erit quoque HG linea aequalis GK, ut patet ex 18.huius ergo HL, KM diametri sunt coniugatae & aequales exhibuimus ergo, &c. Quod erat demonstrandum.

PROPOSITIO XCII.

In una ellipsi duas tantum est reperire diametros coniugatas aequales.

Demonstratio.

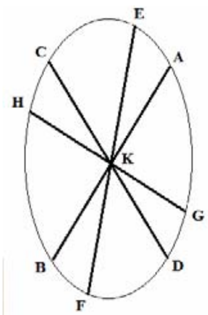


Sit ABC ellipsis centrum D & in ea aequales diametri coniugatae. ED, FD : dico alias diametros coniugatas & aequales in ea exhiberi non posse: sint enim, si potest fieri, praeter ED, FD diametros : aliae aequales & coniugatae GD, HD: erit igitur EDF sectori aequalis sector GDH. Quod fieri non potest , nam GD, HD diametri cum sint aequales necesse est maiores vel minores illas esse diametris ED; FD, adeoque, ambas simul cadere supra vel infra diametros ED, FD: Igitur praeter ED, FD diametros coniugatas aequales, nullas alias aequales in ellipst est exhibere. Quod erat demonstrandum.

PROPOSITIO XCIII.

In ellipsi aequales diametri coniugatae simul sumptae, maximae sunt omnium diametrorum coniugatorum simul sumptarum.

Demonstratio.

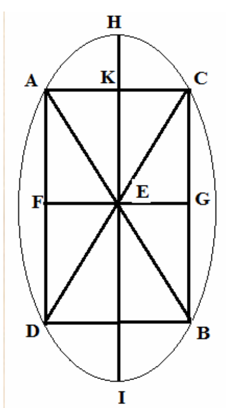


Sint AB, CD diametri coniugatae aequales, sit autem & alia quaevis diametrorum coniugatio EF, GH: dico diametros AB, CD simul sumptas maiores esse diametris EF, GH simul sumtis: cum enim sectores AKG, GKE sint inter se aequales ; necesse est unam coniugatarum inaequalium (sit EF) axi viciniorem esse utravis aequalium AB, CD*:* alteram vero HG; remotiorem unde ex quatuor diametris EF maxima, & GH minima est. Sunt autem EF, GH quadrata simul sumpta aequalia quadratis AB, CD simul sumptis ; igitur AB, CD lineae simul sumptae maiores quoque sunt lineis EF, GH simul sumptis: Quod erat demonstrandum.

PROPOSITIO XCIV.

Lineae quae extrema diametrorum coniugatatum aequalium coniungunt, ab axibus bifariam secantur.

Demonstratio.



Sint AB, CD diametri coniugatea aequales, iunganturque illarum extrema AD, AC, CB, DB. Dico illas ab axibus bifariam secari, divisa enim AD bifariam in F, agatur per E centrum FEG occurrens CB rectae in G. Quandoquidem ergo AB, CD ponantur aequales, harum dimidiae AE, DE, etiam sunt aequales. Aequantur autem ex const: similiter AF, DF. Itaque in trianguli AEF, DEF cum FE sit commune, omnia latera sibi invicem aequantur. Ergo anguli ad F aequales, adeoque rectis & anguli quoque FEA, FED, aequales. Quare anguli etiam GEC, GEB prioribus ad verticem oppositi aequantur. Sunt vero latera rursum CE, EB aequaliter & EG commune utrique triangulo GEC, GEB. Igitur CG, BG aequales, & anguli ad G aequales adeoque recti. Cum ergo FG rectas AD, CE, (quae per 19.huius sunt parallelae) bifariam & ad angulos rectos secet, axis est. Secantur igitur ab axe bifariam recte AD, CE extrema coniugatarum aequaluum connectentes.Eodem modo ostendemus reliquas duas AC, BD ab axe HI bisecari. Constat ergo veritas propositionis.

PROPOSITIO XCV.

Si lineae quae extrema coniugatarum connectunt, ab axibus secentur bifariam: Dico diametros illas esse inter se aequales.

Demonstratio.

Ponatur eadem figura quae prius, sintque AD, CE, AC lineae, extrema coniugatarun connectentes in F, G, & K bifariam & ad rectos divisae axibus Hl, FG. Dico AB, CD, diametros coniugatas esse inter se aequales: cum enim AD, CE per huius sint parallelae & ex hypothesi ab axe in F & G bisecentur, anguli ad F, recti sunt, & latera duo AF, FE aequalia sunt lateribus DF, FE; reliqua igitur latera AE, ED quoque inter se aequalia. Similiter ostendam CE, EB aequales. Unde & totem diametri AB, CD aequales. Quod fuit demonstrandum.

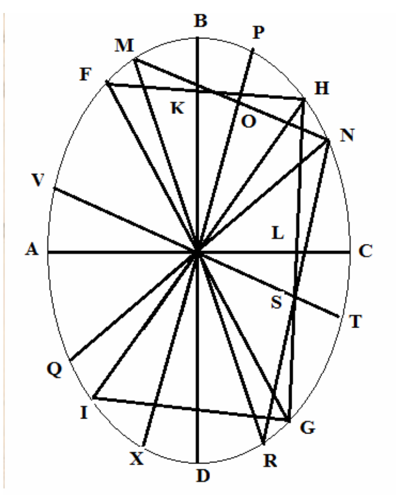
Corollarium.

Hinc patet lineas, quae inaequalium coniugatarum extrema coniungunt, nunquam ab axibus aut alia quavis diametro bifariam & ad rectos secari.

PROPOSITIO XCVI.

Linearum quae extrema coniugatarum quarumvis coniungunt, illa maxima est quae coniugatas aequales connectens, axem minor secat ; minima, quae maiorem.

Demonstratio.



Sint AC, BD axes ellipseos ABC, coniugatae vero aequales FG, HI iunctaque FH maiori axi occurrat in K: & HG minori in L. Dico HG lineam maximam esse illarum quae cuiuscunque coniugationis extrema coniungunt, & FH minimam. Fiat enim quaevis alia diametrorum coniugatio MR, NQ. Quarum extrema iungant MN, NR, quibus in O & S bisectus ducantur per centrum XOP, VST. Quoniam ergo FG, HI sunt coniugatae, sectores EFBH, EHCG aequantur sunt. Ergo segmenta FBH, HCG aequalia sunt. Quare, cum axes BD, AC etiam bisecent rectas FH, HG, quae aequales coniugatas iungunt, erunt axes ipsi in K & L, proportionaliter secti. Ergo rectangulum BKD aequale est quadrato LH. Simili plane discursu ostendemus rectangulum POX aequari quadrato NS. Deinde quia sectores MEN, FEH aequantur, adeoque segmenta MPN, FBH, suntque ambae FH, MN bisectae in K & O, erunt DB, XP proportionaliter sectae in K & O; sed DB axis maior est quam XP. Ergo rectangulum BKD, hoc est, ut ante ostendi, quadratum HL,maius est rectangulo POX, hoc est quadrato

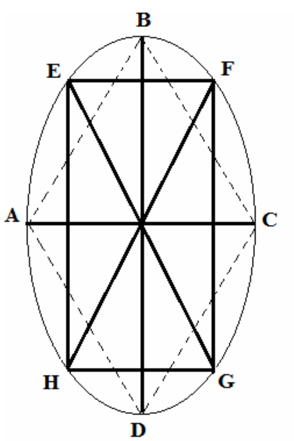
NS. Ergo recta HL maior recta NS. sed HG dupla est ipsius HL, ut ante ostendi, & NR ex const. ipsius NS dupla est. Ergo HG maior quam NR: quia autem NR maior est quam MN; erit HG etiam maior quam MN. Eodem modo ostendemus HG maiorem esse quarumvis aliarum coniugatarum extrema connectentibus. Ergo HG est omnium maxima. Quod erat primum.

Quod autem FH sit omnium minima, discursu plane disimili demonstrabimus. Sumatur enim quaevis alia diametrorum coniugatio MR, NQ, & eadem, quae supra est adhibitis, repetatur constructio. Eodem modo ostendemus quadratum FK minus esse quadrato MO, & rectam FK minorem recta MO, ac proinde FH, minorem quam MN. Est autem RN maior quam MN. Ergo FH etiam minor est quam NR. Atque ita demonstrabimus FH minorem esse quarumvis coniugatarum extrema connectentibus. Omnium igitur minima est. Quod erat secundo loco ostendendum.

PROPOSITIO XCVII.

Coniugatarum aequalium extrema coniungentes simul sumptae minimae sunt omnium quae quascunque diametros coniugatas coniungunt.

Demonstratio.

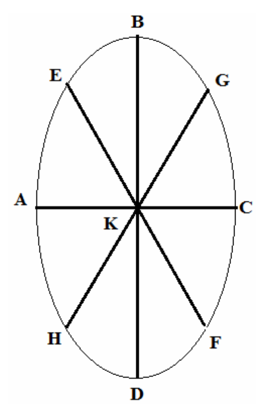


Sint in ABC ellipti coniugatae aequales EF, GH. Ponatur autem & alia quaevis diametrorum coniugato, EF, GH. Dico lineas quae extrema coniugatarum aequalium coniungunt, simul sumptas minores esse lineis quae extrema alterius coniugationis connectunt. Sunt enim EG, GF quadrata aequalia quadratis AB, BC, insuper & EG linea connectentium minima, & FG maxima per praecedentem; igitur EG, GF lineae minores sunt lineis AB, BC; eodem modo ostenduntur EH, HF lineae minorem lineis AD, DC : igitur lineae, &c. Quod fuit demonstrandum.

PROPOSITIO XCVlll.

Sint ABC ellipseos axes AC, BD & EF una exdiametris coniugatis aequalibus.
Dico quadrata AK, BK simul sumpta esse dupla quadrati EK.

Demonstratio.

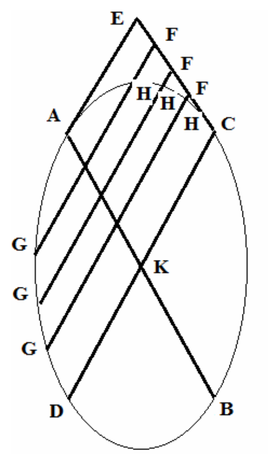


Ducatur GH altera diametrorum coniugatarum aequalium. Quoniam AC, BD quadrata simul sumptae qualia sunt quadratis EF, GH simul sumptis, erunt & quadrata AK, BK sub dimidiis axibus, aequalia quadratis EK, GK sub dimidiis diametris aequalibus; sunt autem EK, GK quadrata inter se aequalia, igitur quadrata AK, BK simul sumpta dupla sunt quadrati EK.
Quod erat demonstrandum.

PROPOSITIO XCIX.

Apollonius l. 3· *Conic. prop.16.huiusmodi habet theorema: si ellipsim tangant* AE, CE, *conuenientes in* E, *& sumpto in sectione puncto* G *ducatur* GHF *tangentium uni parallela* GHF, *erit rectanguluum* GFH *ad quadratum* BC, *ut quadratum* BC, *ut quadratum* AE *ad quadratum* CE.

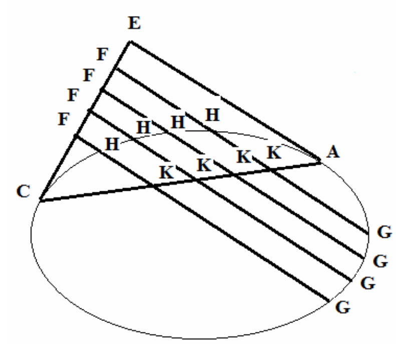
Verum non similitudo tantum rationum sed spatiorum etiam aequalitas reperietur si tangentes a diametrorum coniugatarum aequalium ductae fuerint.



Ellipseos diametri coniugatae aequales sint AB, CD, in quarum terminis A, C, ellipsim tangant duae rectae conuenientes in E, si alterutri ducantur quotius parallelae GH erunt rectangula GFH quadratis FC aequalia.

Demonstratio.

Quoniam CD est diameter coniugata diametri AB, erit ad ipsam ordinatim posita: ergo tangenti AE parallela est. Eodem modo AK tangenti EC parallels est. Figura igitur KAEC est parallelogrammum.Quare cum AK, KC ex hypothesi sint aequales, etiam AE, CE aequales sunt: aequantur igitur quadrata AE, EC. Atque est ut quadratum AE ad quadratum EC, ita rectangulum GFH ad quadratum FC, ergo rectangulum GFH quadrate FC aequale est. Quod erat demonstrandum.



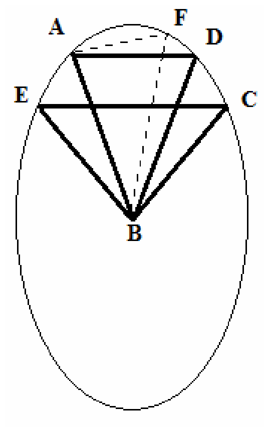
Et quoniam Theorema illud Apollonii iam habemus in manibus, triam hoc addo quod similiter Apollonius non videtur observasse: nimirum si ductis tangentibus AE, CE, iungantur puncta contactuum A, C, rectangula GFH, quadratis KF aequalia esse.

Quoniam FK, AE sunt parallelae, triangula AEC, KFC similia sunt. Ergo AE ad EC, sic KF ad FC. Ergo ut quadratum AE ad quadratum EC, sic quadratum KF ad quadratum FC. Sed etiam ut quadratum AC ad quadratum EC sic rectangula GFH ad quadratum KF. Ergo quadratum KF & rectangulum GFH ad quadratum EC, eandem habent rationem; aequantur igitur. Quod erat demonstrandum.

PROPOSITIO C.

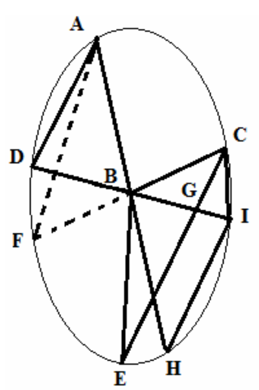
Sint AB, BC diametri coniugatae inaequales, & ex A recta quaevis ducatur AD secans ellipsim in D; cui ex C parallela ducatur C, iunganturque EB, DB.
Dico EB, DB diametros esse coniugatas & contra.

Demonstratio.



Primo cadant parallelae ad eandem partem ellipseos. Ducantur rectae lineae EA, DC. Quoiam AD, EC lineae sibi mutuo aequidistant, erunt segmenta EA, DE inter se aequalia adeoque ABE, DBC sectores aequales addito igitur communi ABD, erunt EBD, ABC sectores inter se aequales. Quare cum unius sectoris latera BA, BC sint diametri coniugatae, etiam alterius latera EB, BD sunt coniugatae.

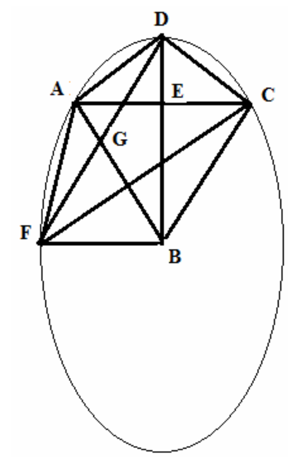
Secundo cadant AD, CE parallelae ad partes ellipsis oppositas: producantur semidiametri AB, DB in H & I, iunganturque puncta HI. Quoniam AB, BC sunt conjugatae erit sector ABC quarta pars ellipseos, sed AH diametro bisecat ellipsim, adeoque portio ACH, dimidium est ellipseos. Ergo ABC sector dimidius est semi-ellipseos ACH. Ac proinde aequalis sectori CBH. Quia autem IH per 19.huius est parallela ad DA, cui ex hypothese etiam CE est parallela , erunt IH, CE inter se parallelae. Ergo segmenta CI, EH adeoque & sectores CBI, HBE aequantur; addito igitur commumi IBH, ante ostendi, sectori ABC. Quare cum sector ABC sit quarta pars ellipseos, sive dimidium semiellipseos, etiam sector IBE erit dimidium semiellipseos, hoc est portionis IED, quam est semiellipsim patet ex Coroll. 45. huius . Ergo sector IBE hoc est sector ABC aequalis est sectori EBD. Quare cum AB, BC sint coniugatae, etiam DB, EB erunt coniugatae.



Sint iam AB, CB,item, EB, DB diametri coniugatae iunganturque AD, EC. Dico AD, EC lineas esse parallelas. Sin vero; ducatur ex A ipsi EC parallela AF iunganturque FB: erit igitur FB diameter coniugata ipse EB per secundam partem huius: sed DB per constructionem coniugata est diametro EB, ergo eidem EB plures diametri sunt coniugatae. Quod lieu non potest. Igitur AF non aequidist ipsi EC. Idem ostenditur de quavis alia. Ergo AD sola parallela est rectae BC. Quod erat demonstrandum.

PROPOSITIO CI.

Sit in ADC ellipsi cuius centrum B quaevis diametrorum coniugatio AB, BC: iunctisque punctis AC, secet AC lineam in E diametrum utcunque BD, cui coniugata ducatur BF, ductaque linea, FD secet AB diametrum utcunque in G.
Dico AC, FD lineas, uti & BD, AB in E & G proportionaliter esse divisas.



Demonstratio.

Quoniam tam AB, BC diametri quam DB, FB coniugatae sunt, sectores ABC, FBD aequales erunt : ablato igitur communi ABD, aequales manent DBC, ABF sectores. Unde BD, AB lineae, item AC, FD in E & G proponionaliter sunt divisae.

PROPOSITIO CII.

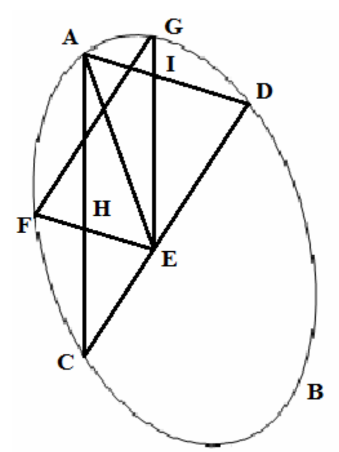
Iidem positis:
Dico iunctas AD, FC aequidistare.

Demonstratio.

Per praecedentem sectores DBC, ABF ostensi sunt aequales; segmenta igitur DE, AF quoque inter se aequantur: Ergo AD, FC lineae aequidistant. Quod erat demonstrandum.

PROPOSITIO CIII.

Secet ABC ellipsim quaevis diametrorum coniugatio AE, CD: sit autem & alia diametrorum coniugatio, FE, GE quae iunctas AD, AC secet in H & I:
Dico esse ut AH ad HC, sic DI ad IA.



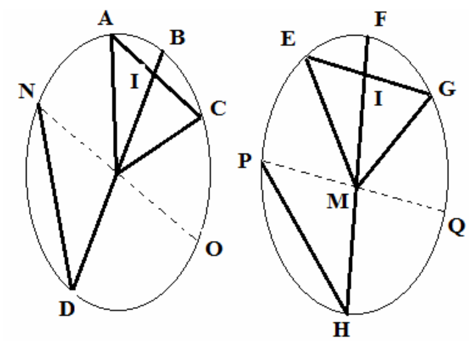
Demonstratio.

Ducatur FG quae AE secet in K ut AH ad HC, sic FK ad KG: sed ut FK ad KG, sic DI est ad AI, igitur ut AH ad HC sic DI ad AI. Quod erat demonstrandum.

PROPOSITIO CIV.

Secet ABC est ellipsim diameter quaecunque BD : sit autem & EFG, ellipsis similis & aequalis est ipsi ABC: quam secet quaevis alia diameter FH : dein BD, FH diametris proportonaliter divisis in I & K, agantur per I & K, ordinatim lineae AC, EG: aequarum extremitatibus ducantur semidiametri AL, CL, EM, GM.
Dico ALC, EMG triangula esse aequalia.

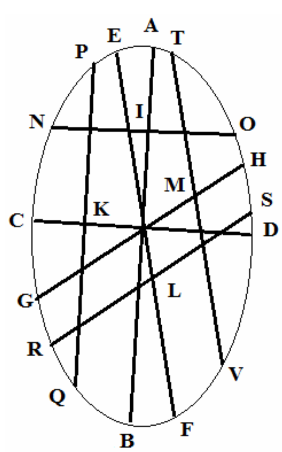
Demonstratio.



Diametro BD coniugata ducatur NO, & FH diametro PQ: iunganturque ND, PH. Quoniam diametri BD, FH tam in IK quam LM proportionaliter sunt divisae, erit ut rectangulum BID ad rectangulum BLD sic FKH, rectangulum ad rectangulum FMH: quare ut quadratum AI ad quadratum NL sic quadratum EK ad quadrum PM : & ut AI linea ad lineam NL sic EK ad PM: sed etiam est per constructionem ut IL ad BL, id est LD, sic KM ad FM, id est MH; igitur ut triangulum NLD ad triangulum AIL, sic PMH triangulum ad triangulum EKM (quia ex iisdem illorum ratio componitur:) & permutando ut NLD triangulum ad triangulum PMH, sic AIL triangulum ad truangulum EKM. Sed NLD, PMH triangula sunt aequalia, igitur AIL, EKM triangula, adeoque tota ACL, EGM aequantur, Quod erat demonstrandum.

PROPOSITIO CV.

Secent ABC ellipsim duae diametrorum coniugationes AB, CD, EF, GH, & omnes quatuor diametri proportionaliter sint divisae in I, K, M, L punctis, per quae ordinatim ducantur lineae NO, PQ, RS, TV.
Dico quadrata NO, PQ simul sumpta aequari quadratis RS,TV simul sumptis.



Demonstratio.

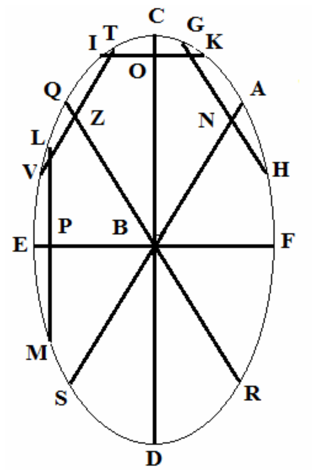
Quoniam tam AB, EF, quam CD, HG proportionaliter sunt divisae, erit ut AB quadratum ad rectangulum AIB, sic EF quadratum ad rectangulum ELF, & quadratum CD ad rectangulum CKD , & HG quadratum ad rectangulum HMG. Ex iisdem enim rationibus singulae quadratorum ad rectangula proportiones componuntur. Igitur ut AB, CD quadrata simul sumpta rectangula AIB, CKD simul sumpta, sit quadrata EF, GH simul sumpta, sunt ad rectangula ELF, HMG ; hoc est quadrata PK, NI ad quadrata SL, TM: sed AB, CD quadratis simul sumptis aequalia sunt quadrata EF, GH simul sumpta;

igitur & quadratis NI, PK aequalia sunt quadrata SL, TM: ergo NO, PQ quadrata simul sumpta aequalia sunt quadratis SR, TV. Quod erat demonstrandum.

PROPOSITIO CVI.

Secet ABC ellipsim quaevis diametrorum coniugatio CD, EF, quaevis proportionaliter divisis in O & P : ducatur una ex diametris coniugatis aequalibus AS queae dividatur in N, ut CD est divisa in O. Per NOP rectae ducantur ordinatim GH, IK, LM.
Dico IK, LM quadrata simul sumpta esse dupla quadrati GH.

Demonstratio.



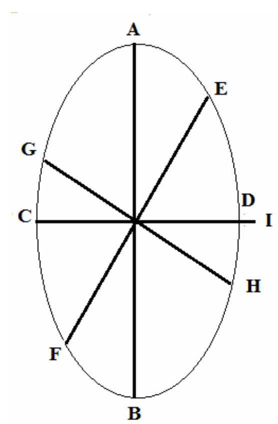
Ducatur altera coniugatarum aequalium QR, quam similiter divisi in Z, ut SA est in N, & CD, EF, in O & P, per punctum Z ponatur ordinatim VT quia igitur QR, AS sunt aequales & similiter sectae, rectangulum QZR aequatur rectangulo ANS. sed rectangula QZR, ANS aequantur quadratis GN, TZ. Ergo quadrata GN, TZ adeoque & quadrata GH, TV aequalia sunt. Sed quadrata ML, IK aequantur quadratis VT, GH. Ergo quadrata M L, IK dupla sunt quadratri. Quod erat demonstrandum.

PROPOSITIO CVII.

Secent ABC ellipsim duae diametrorum coniugationes AB, CD, EF, GH : sitque AB maxima & EF magnitudine secunda.
Dico rationem AB ad EF minorem esse ratione GH ad CD.

Demonstratio.

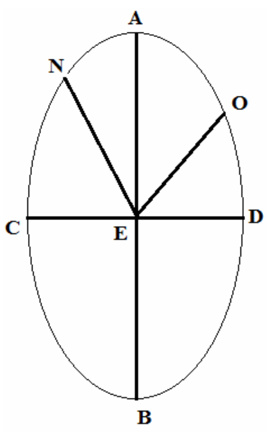
Quoniam AB, CD quadrata simul sumpta aequalia sunt quadratis EF, GH simul sumptis: non est ut AB ad EF, sic GH ad CD, nam tunc quadrata AB, CD maximae & minimae maiora essent quadratis EF, GH. Fiat igitur ut AB ad EF, sic GH ad CI: eruntque AB, CI quadrata maiora quadratis EF, G H; hoc est quadratis AB, CD. Quare CI linea est maior recta CD, & ratio GH ad CD; id est ex constr. ratio AB ad EF minor est ratione GH ad CD. Quod erat demonstrandum.



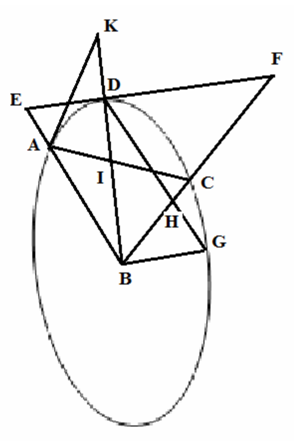
PROPOSITIO CVIII.

In data ellipse diametris exhibere coniugatas in data ratione.

Constructio & demonstratio.



Sint ABC ellipsis axes AB, CD: centrum E : ratio vero data maioris ad minus F ad G, quam non maiorem esse oportet ratione axium AE, EC. Si ratio data aequalis est axium rationi ; res patet: si minor, sumantur HI, IK lineae, rectis AE; EC aequales, ponanturque ad angulum rectum HIK: ductaque HK, & divisa bifariam in L, centro L intervallo LH circulus describatur HIK, qui transibit per puncta I & K, in segmento dein HIK inclinentur duae lineae HIK habentes datam rationem F ad G. Quoniam igitur HM est ad MK ut F ad G, estque ratio F ad G, minor ratione axium, hoc est: ex const. ratione HI ad IK, patet punctum M cadere inter I & H, adeoque rectam HM minorem quam HI, hoc est quam EA, rectam vero MK maiorem quam IK hoc est quam EC. Centro igitur E intervallo HM, descriptus circulus secat ellipsim in puncto aliquo N inter A & C, ducatur igitur diameter NE, eique; coniugata EO. dico NE, OE diametros satisfacere petitioni. Quoniam quadrata NE, OE sunt aequalia quadratis & AE, EC hoc est HI, IK hoc est quadratis HM, MK: sit autem & NE linea aequalis rectae HM. Erit & OE linea aequalis lineae MK. igitur ut HM ad MK, id est ut F ad G, sic NE ad OE. Exhibuimus ergo, &c. Quod erat faciendum.



PROPOSITIO CIX.

Secet ACD ellipsum quaevis diametrorum coniugatio AB, BC assumptoque in peripheria puncto quovis D inter A & C, agatur per D contingens EF, quae cum AB, BC lineis concurrat in E& F; cui per centrum parallela ducatur BG.
Dico DF, BG, DE lineas esse in continua analogia.

Demonstratio.

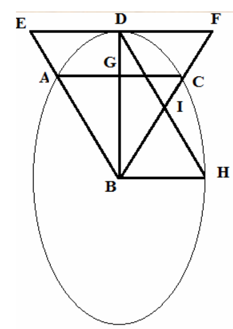
Quoniam ex hypothest DF, BG sunt parallelae, triangula DFH, BHG, ut ex elementis patet, erunt similia, ac proinde DF est ad BG, ut DH ad HG; hoc est ut triangulum DHB ad triangulum BHG. Deinde, quoniam BG per centrum est tangenti DF parallela, liquet eam coniugatam esse diametrum ipse DB: sunt vero ex hypothesi etiam AB, BF coniugatae. Ergo sectores BDCG, BADC, adeoque: & triangula BDG, BAC aequantur: Quorum bases DG, AC, cum proportionaliter sint divisae H & I, ut DH sit ad HG, sicut CI est ad IA , constat ex elementis triangula BHD, BIC & BHG, BIA esse aequalia. Ergo cum prius ostenderim DF esse ad BC ut triangulum BHD ad triangulum BHG, erit quoque DF ad BG, ut triangulum BHD ad triangulum BHG. Erit quoque DF ad BG, ut triangulum BIC ad triangulum BIA. Ulterius cum triangulum BHG sit ad triangulum BHD, ut GH ad HD, hoc est, ut AI ad IC, hoc est ut KI ad IB (cum enim CB sit coniugata ipsi AB , & AK, ex constructione tangens, patet AK, CB esse parallelas) hoc est ut triangulum AIK ad triangulum AIB : erit componendo triangulum BDG ad triangulum BHD, ut triangulum AKB ad triangulum AIB; et permutando triangulum BDG ad triangulum AKB, ut triangulum BHD, est sicut ante ostendi, ut triangulum BIC ad triangulum AIB. Sed triangulum AKB est triangulum EDB, hoc est triangulum BDG est ad triangulum EDB, hoc est,quoniam ED, BG sunt parallelae, BG est ad ED, ut triangulum BIC ad triangulum AIB, hoc est sicut ostendi supra, ut DF ad BG. Sunt igitur in ratio continua DF, BG, ED. Quod erat demonstrandum.

PROPOSITIO CX.

Sint AB, BC diametri quaecunque coniugatae, assumptoque; in peripheria, inter A & C puncto quouis D, agatur per D contingens, occurrens AB, BC diametris in E & F, iunctaque AC occurrat diametro DB in G.

Dico rectam DF ad DE, rationem habere duplicatam, eius quam habet CG ad GA.

Demonstratio.

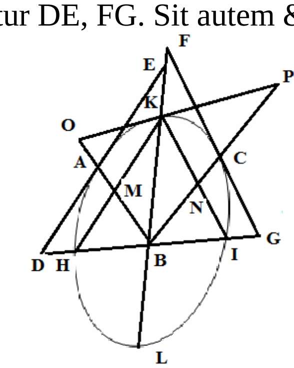


Ducatur ex B linea BH parallela ipsi EF, & ex DE recta DH occurrens FB in I. Erunt igitur per praecedentem, continuae FD, BH, ED. Adeoque ratio FD ad ED, duplicata rationis FD ad BH, id est DI ad IH, quia DF, BH per constructionnem aequidistant: rursum cum HB recta aequidistet tangenti DF, erunt DB, BH diametri coniugatae; sunt autem ex constructione etiam AB, BC coniugatae; igitur ut DI ad IH, sic CG ad GA: quare & ratio FD ad DE, duplicata est rationis CG ad GA. Quod erat demonstrandum.

PROPOSITIO CXI.

Sint AB, BC diametri coniugatae & per A & C tangentes ducantur DE, FG. Sit autem & alia quaevis diametrorum coniugatio HI, KL, quae producta occurrat tangenibus DE, FG in E, F, D, & G.

Dico lineas DE, FG in A & C proportionaliter esse divisas, nimirum esse EA ad AD, ut GC ad CF.



Demonstratio.

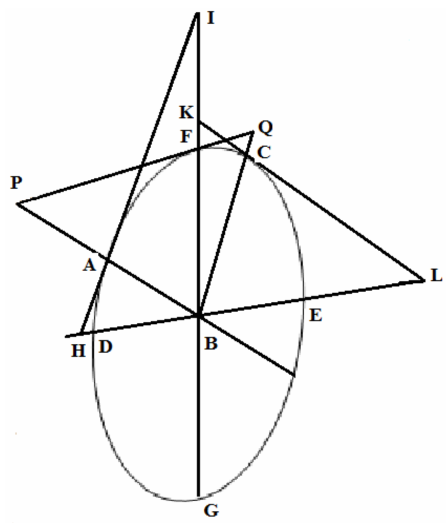
Ducantur lineae HK, KI quae rectas AB, BC secent in M & N. Ratio EA ad AD duplicata est rationis KM ad MH, & GC ad CF, duplicata est rationis IN ad NK. Atqui ratio KM ad MH aequalis est rationi IN ad NK. Ergo rationes EA ad AD, & GC ad CF aequalium rationum duplicatae, sunt aequales, proportionaliter ergo sectae sunt DE, GF in punctis C, A. Quod erat demonstrandum.

Corollarium.

Quod si per K ducatur tertia tangens, conveniens cum BA, BC coniugatis in O & P, dico fore OK ad KP, ut EA ad AD. Quod ducta recta AC eodem modo quo usi sumus demonstrabitur.
Itaque tres tangentes DE, OP, FG similiter sunt divisae sic ut EA sit ad AD, sicut OK ad KP, & GC ad CF.

PROPOSITIO CXII.

Sint duae diametrorum coniugationes A B, BC, FG, DE, aganturque per A & C tangentes HI, KL, quae FG, DE diametris occurrant in H, I, K, L punctis.
Dico esse ut BC ad BA sic HI ad KL.



Demonstratio.

Quoniam recta BC aequidistat rectae HI & KL linea ipsi AB , erunt tam HA, BC, AI quam KC, AB, CL lineae in continua analogia. Cum ergo sic ut HA ad AI , prima ad tertiam, sic KC ad CL, prima ad tertiam: erit etiam HA ad BC, prima ad secundam ut KA est ad BC, prima ad secundam : quare permutando est HA ad KC, ut BC ad AB, cum igitur, ante ostenderim HA esse ad AI, ut KC ad CL, adeoque invertendo componendo, ac permutando sit ut HA ad KC, sic HI ad KL, erit ut BC ad BA, ita HI ad KL.

Corollarium.

Eodem modo ostenditur, si per F agatur tangens quae cum AB, BC convenient in P & Q esse ut BC ad BD, sic HI ad PQ.

PROPOSITIO CXIII.

Sint duae diametrorum coniugationes AC: BD, EF, EG: actisque per F & G tangentibus quae diametris AC, BD occurrant in H, I, L, M ducatur recta FG secans BE diametrum in K.
Dico LK, KE, KI lineas esse continuas.

Demonstratio.

Quoniam FE, utpote coniugata ipsi EG, aequidistat tangenti LG , erit LK ad KE ; ut GK ad KF. Similiter quoniam EG, utpote coniugata ipsi EF, parallela sit tangenti FI, est ut GK ad KF, sic KE ad KL; igitur ut LK ad KE, sic KE est ad KI. Quod erat demonstrandum.

PROPOSITIO CXIV.

Iisdem positis:
Dico IHE, LME triangula esse aequalia.

Demonstratio.

Ducatur recta A B quae FE lineam secet in N : & ex E rectae ducantur EO, EP, normales ad lineas HI, LM. Quoniam LM linea aequidistat ipsi FE (est enim LM tangens, & FE coniugata ipsi EG,) erit angulo FEG aequalis angulus EGP. Eodem modo erit angulus OFE aequalis angulo FEG. quare anguli OFE, EGP sunt inter se aequales: sunt autem EPG, EOF anguli recti; igitur triangula EGP, EFO similia. Quare ut EG ad EF, sic EP ad EO. Sed est ut EG ad EF, sic HI ad LM. Igitur ut EP ad OE, sic reciprocem HI ad LM. Ergo IHE, LHM triangula sunt aequalia. Quod erat demonstrandum.

PROPOSITIO CXV.

Iisdem positis triangulum EGM, triangulo EFI, & EGL triangulum, triangulo EFH aequale est.

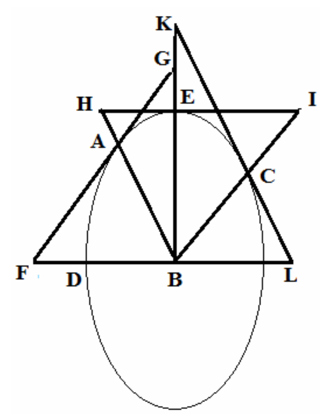
Demonstratio.

Est enim ut HF ad FI, sic LG ad GM, & componendo ut Hl ad FI, sic LM ad GM: sed est ut HI ad FI, sic HIE triangulum ad triangulum FIE, & ut LM ad GM, sic ELM triangulum ad triangulum EGM, igitur ut HIE triangulum ad triangulum FIE, sic ELM triangulum est ad triangulum EGM. Quare FIE, EGM triangula sunt aequalia. Eodem modo ostenduntur reliqua EGL, HFE aequalia.

PROPOSITIO CXVI.

Si ellipsim ADC duae secent diametrorum coniugationes AB, BC, EB, BD: aganturque per A, E, C. Contingentes FG, HI, KL quae diametris quidem EB, BD occurrat in G, K, F, L. Diametris vera AB, BC, in H & I.

Dico triangula FGB, HBI, KLB esse inter se aequalia.



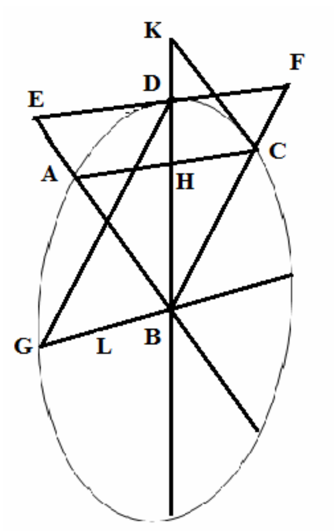
Demonstratio.

Nam triangulum BIE aequatur triangulo BKC, hoc est per 115, huius triangulo BFA : & triangulum BHE, aequatur triangulo GAB, ergo triangulum totum BIH aequatur toti BFG. Quare cum etiam FGB, KBL aequalia sint; liquet tria triangula esse aequalia.

PROPOSITIO CXVll.

Sint ellipseos duae diametri coniugatae AB, BC, &. ex puncto aliquo inter A & C assumpto, scilicet D ducatur tangens DE coniugatis AB, BC productis occurrens in E & F: deinde ex centro B ducatur BG, ipsi EF aequidistans, sitque GB media inter ED, DF.
Dico punctum G esse in peripheria ellipseos, cuius diametri coniugatae AB, BC, & tangens ED.

Demonstratio.



Ex centro ad contactum ducatur BD, & quia GB hypothesi est parallela tangenti ED, erit GB ordinatim posita ad BD, & quidem ad centrum. Unde BD, GB sunt diametri coniugatae. Iungantur deinde AHC & D G, ex C ducatur CK parallela ipsi AB,
occurrens BD protractae in K : quae sectionem C contingit, eritque ut AH ad HC, ita BH ad HK: sed ut AH ad HC, ita est ABH triangulum ad triangulum HBC, & ut BH ad HK, ita est triangulum HBC ad triangulum HCK: ergo ut triangulum ABH ad ipsum HBC ita est triangulum HBC ad triangulum HCK: ergo componendo, ac permutando triangulum gulum ABC, ad triangulum BCK est ut triangulum BHC ad triangulum HCK, id est ut iam ostensum, ut triangulum ABH ad triangulum HBC, id est ut linea AH ad lineam HC: rectae: sed aequalia sunt triangula BCK, BDF,ergo etiam erit triangulum ABC ad triangulum BDF, ut AH ad HC. Ulterius quoniam DF, GB sunt parallelae, erit triangulum GDB, ad DBF triangulum, ut G B ad DF: sed, quoniam ex hypothesi ED, GB, DF sunt continuae, ratio GB ad DF, est dimidiata rationis ED ad DF. Ergo ratio trianguli GDB ad triangulum DBF dimidiata est rationis ED ad DF. Atque ratio AH ad HC, hoc est ut ostensum supra, ratio trianguli ABC ad triangulum DBF, dimidiata quoque est rationis ED ad DP. Ergo triangulum GDB est ad triangulum DBF, ut triangulum ABC ad idem triangulum DBF. Aequantur igitur triangula GDB, ABC. Ergo & parallelogrammum contentum semidiametris, ut supra ostendi, coniugatis, GB, BD in angulo GBD aequatur parallelogrammo contento sub semidiametris coniugatis AB, BC. Ergo punctum G est ad ellipsim. Quod erat demonstraandum.

PROPOSITIO CXVIII.

Sint rursum binae diametri coniugatae BA, BC. Et sumpto in perimetro ellipsis puncto D inter A, ac C, tangat ellipsim EF in D, occurrens diametris in E & F. Deinde ex centro B ducatur ad perimetrum BG parallela tangenti.
Dico ED, GB, DF esse in continua analogia.

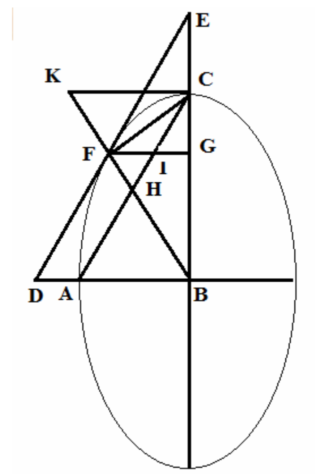
Demonstratio.

Si non sit aliqua LB minor vel maior quam CB media inter ED, DF. Ergo per praecedentem punctum L est ad ellipsim, quod fieri non potest, cum ex hypothesi punctum G ad ellipsim existat. Nulla igitur praeter GB, media est inter ED, DF. Ergo ED, GB, DF sunt continuae proportionales. Quod erat demonstrandum.

PROPOSITIO CXIX.

Sint in ellipsi diametri coniugatae AB, BC iungaturque; AC cui parallela fiat linea DE tangens est ipsum F, & occurrens diametris coniugatis protractis in D & E.
Dico triangulum EDB trianguli CAB duplum esse.

Demonstratio.



Ex centro B per tactum F ducatur diameter BHF, cui occurrat in K recta CK ellipsim tangens in C, deinde ex tactu F ponatur ordinatim FIG. Quoniam AC parallela est ex hypothesi tangenti DE, erit ad diametrum BF ordinatim posita. Ergo KB, FB, HB sunt continuae proportionales. Ergo est ut KB ad FB, sic KF ad FH. Sed KB est ad FB, & triangulum ECB ad triangulum FCB, hoc est, quia triangula KCB, EFB aequantur, ut triangulum EFB ad idem triangulum CFB, hoc est EB ad CB. Igitur ut EB ad CB, sic KF ad FH. Deinde quia ex constr. KC tangit, & FG est posita ordinatim ad BC, rectae EB, CB, GB sunt continuae. Ergo ut EB ad CB, sic EC ad CG. Sed etiam est ut iam ostendi, sic ut EB ad CB, ita KFad FH. Ergo ut KF ad FH, sic EC ad CG. Ergo ut triangulum KCF ad triangulum FCH: sic triangulum EFC ad triangulum CFG. Atqui cum tota KCB, EFB aequalia sint, ablato communi FBC reliqua KCF, EFC aequalia sunt. Ergo & FCH, CFG aequalia sunt; ablato igitur communi FIC, aequalia remanent FIH, CIG, quibus si commune addis BHIG, FGB aequabitur CHB. Iam vero quia AC ordinatim posita est, ut supra ostendi, ad BF, bisecta est AC in H, adeoque & triangulum CAB duplum est: trianguli CHB, & DE parallela ad AC etiam bisecatur in F: est vero FG, utpote ducta ordinarim ad CB, parallela ad CB, diametrum coniugata ipsi C B. Ergo & BE bisecatur in G. Proindeque EFB duplum est GFB. Atqui CHB, FGB ostensa sunt aequalia. Ergo & eorum dupla CAB, EFB aequalia sunt. Sed triangulum DEB duplum est trianguli EFB, est enim, DE bisecta in F. Ergo triangulum DEB duplum quoque est trianguli CAB. Quod erat demonstradum.