THE ELLIPSE: Part II.

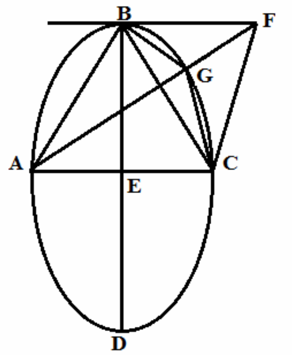
Concerning the Sectors and Segments of the Ellipse.

PROPOSITION XLII.

BD shall be the diameter of the ellipse ABC, for which the ordinate AEC may be put in place; and ABC may be joined.

I say ABC to be the maximum triangle of these which are able to be inscribed in the segment ABC.

Demonstration.

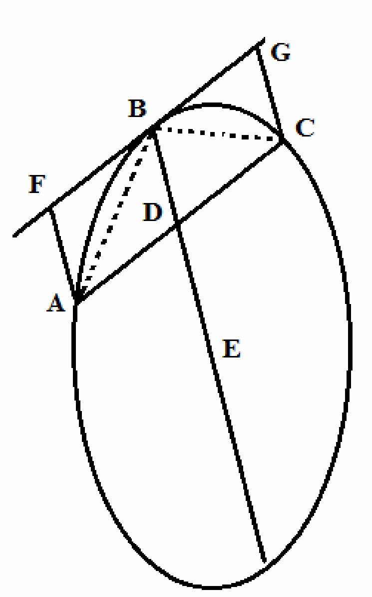


With the tangent BF acting through B, from A some right line AF may be drawn crossing the ellipse at G, the tangent at F and the lines GC, FC may be joined : Since the tangent FG falls above G, therefore the triangle AFC is greater than the triangle AGC; but the triangle AFC is equal to the triangle ABC on account of AC, BF being parallel lines; and therefore ABC is greater than the triangle AGC: from which since the same may be shown from all the other triangles ; to be apparent the triangle ABC, to be the maximum of those which are able to be described within the segment ABC. Q.e.d.

Corollary.

Hence the practise is readily deduced for inscribing the maximum triangle for any segment: without doubt by raising to the diameter BD, and by joining the points AB, BC. The demonstration apparent from the previous.

PROPOSITION XLIII.



The maximum triangle inscribed in any segment cannot to be greater than half of the inscribed ellipse, but to be greater than half of this same segment.

Demonstration.

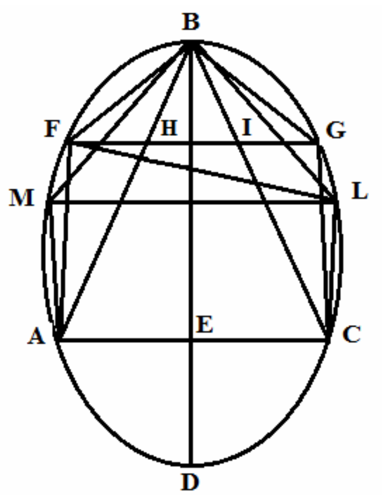
Let the segment of the ellipse be ABC, the maximum triangle ABC inscribed not to exceed half of the ellipse. I say that to be the greater than half of the segment ABC: indeed with the diameter BE drawn, which divides the chord AC equally at D, the lines AF, CG may be erected from A and C parallel to the diameter BE, which meet the tangent line FG drawn through B, at F and G ; and AB, CB may be joined: the triangle ABC [§.41, Ch.1] is half of the parallelogram AG. And the parallelogram AG is greater than the segment of the ellipse ABC, since the lines AF, CG, FG fall outside the ellipse; and therefore the triangle ABC is greater than half of this same segment. Q.f.d.

PROPOSITION XLIV.

The diameter BD shall cut the ellipse ABC, to which the ordinate AEC may be put in place thus: with AB, CB joined, the maximum triangle AFB may be inscribed for the segment AFB, and from F, FG may be placed parallel to AC, and BG, GC may be joined.

I say the maximum triangle BGC to belong to these which can be inscribed for the segment ABC, and if the triangles were the largest, I say FG to be parallel to AC.

Demonstration.



Because FH, GI are equal lines [§.26], the triangles FBH, GBI having the same altitude, will be equal, similarly the triangles FAH, GIC set up between the parallels FG, AC, will be equal and hence the whole triangles BFA, BGC will be equal. Therefore, if BGC were not be the maximum, BGC may be replaced by another greater triangle BLC, and from L, LM may be drawn parallel to AC; therefore so that the first triangle BLC will be equal to the triangle AMB, that is AFB to be greater than the triangle BGC, which is contrary to the supposition, since BFA shall be put to be the maximum. Therefore BGC is the maximum triangle of these which can be described by the segment BGC, which establishes the first part.

Since the triangles BFA, BGC shall be the greatest, we will demonstrate AC to be parallel to the joined line FG: for if it were not parallel, there shall be another line either above or below FG itself parallel to AC; without doubt to be the right line FL; and BL, CL may be joined. Therefore, by the first part of this Prop., the triangle BLC would be the greatest, which cannot occur, since from the hypothesis, FGC shall be the maximum. Therefore neither FL, nor any other line besides FG, is parallel to AC. Q.e.d.

First Corollary.

Hence it follows, that if the triangles BFA, BGC shall have been the greatest of these which shall be described by the segments, then they are to be equal. For by the second part of this Prop., FG is parallel to AC. From which FH, GI are equal, and also the triangles FBH, BIG, and FAK, GCI are equal; and thus the whole triangles BFA, BGC are equal.

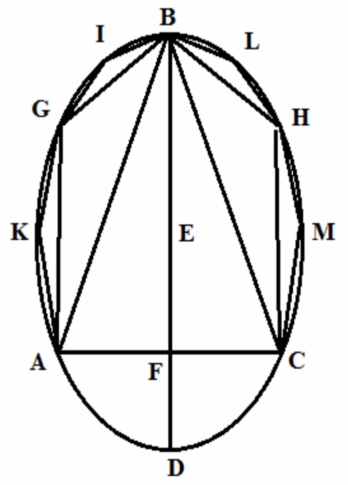
Second Corollary.

So that if the two lines AC, FG were put in place to be the ordinate lines for the diameter, the greatest triangles are inscribed by the segments AF, CG, also to be equal to each other, plainly we will demonstrate by the same discussion, which we have used in the proposition and in the first corollary, with nothing else changed, just as in place of Prop. 26 here it will be required to assume Prop. 27.

PROPOSITION XLV.

BD shall be some diameter of the ellipse ABC for which an ordinate line AFC may be put in place. I say the segment AGBF to be equal to the segment CHBF.

Demonstration.

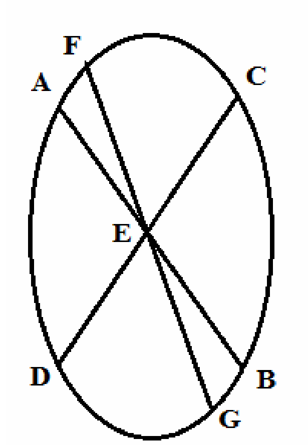


With the remaining segments joined AB, CB, the greatest triangles AGB, CHB will be inscribed [§.42, Ch.1] equal to each other, by the first Cor. of the preceding Proposition, and [§.43, Ch.1] from the halves of the greater segments ABG, CBH: then the greatest triangles AKG, GIB, CMH, BLH will be inscribed as before, and the greatest triangles will be inscribed on each side of the segment, by the remaining triangles AKG, GIB, will be as part of the first triangle by the first cor., part by the following equal triangles CMH, BLH, and with half of the greater segments therefore since that inscribed always shall be able to be continued in each segment AGB, BHC, and the parts removed on both sides shall be equal to each other, and with the greater halves of the segments from which they are removed, it is agreed [§.216, Progress.] the segment AGB to be equal to the segment CHB. Whereby with the equal triangles ABF, CBF added, the whole segments AGBF, CHBF will be equal. Q.e.d.

Corollary.

Hence it follows the ellipse to be cut into two equal parts by any : for BD shall be any diameter, and the ordinate AFC may be drawn through any point of this, by the proposition now demonstrated, the segment ABF is equal to the segment BCF, again from the same proposition the segment ADF shall be equal to the segment CDF; therefore the segments ABF, ADF, that is the whole segment DAB, are equal to the segments BCF, CDF, that is, to the whole segment DCB; therefore the ellipse is divided into two equal parts by the diameter BD.

PROPOSITION XVI.



Two conjugate diameters divide the ellipse into four parts, and the four diameters dividing the ellipse, are conjugate to each other.

Demonstration.

AB, CD shall be two conjugate diameters in the ellipse ABC ; I say these to divide the ellipse into four parts; and if AB, CD shall divide the ellipse into four parts, I say these to be conjugate. Since AB, CD are conjugate diameters ; AB will be the ordinate put in place for the diameter DC, so that now AEC, CEB as well as AED, BED shall be equal parts ; moreover AEC, AED are equal sectors on account of the same ratio; therefore the four sectors are equal to each other AEC, CEB, BED, DEA, are equal to each other, and the lines AB, CD divide the ellipse into four parts: Which was the first part.

Now if the ellipse shall be divided into four parts, I say AB, CD to be conjugate diameters; truly if CD itself may not be drawn, conjugate to FG, therefore FEC shall be the fourth sector of the ellipse through its first part. And the sector AEC from the hypothesis also is the fourth part of the ellipse, therefore the sectors

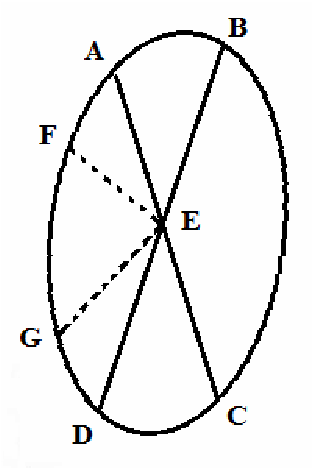
FEC, AEC are equal, both the part and the whole, which cannot happen; therefore the diameter FG is not the conjugate of CD itself, nor any other besides AB. Q.e.d.

Corollary.

Hence to be apparent, the sectors of whatever conjugations, to be equal with the individual sectors of whatever the kind of the other singular conjugation, and if the sectors shall be equal and the sides of one shall be conjugate, also the sides of the other to be conjugate.

PROPOSITION XLVII.

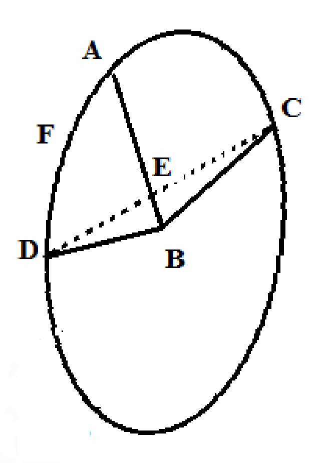
Sectors to the opposite vertex are equal to each other.



Demonstration.

The diameters cut the ellipse ABC in some manner AC, BD: I say the sectors opposite the vertex to be equal to each other. Indeed two diameters EF, EG may be drawn from the centre E, and EF the conjugate of EB itself: EG truly the conjugate of AE itself. Therefore since the sectors BEF, AEG, [§.46, Ch.1] are equal, with the common sector AEF removed, the sector AEB will be equal to the sector FEG: again since the sectors SED, GEC shall be equal, with the common sector DEG removed, the sector DEC will be equal to the sector FEG, that is AEB to be equal to the opposite vertex. In the same manner it will be shown the sectors AED, BEC to be equal. Therefore, &c. Q.f.d.

PROPOSITION XLVIII.



ABC shall be some sector in the ellipse ADC, it is required to draw a right line from B to the periphery, which may constitute with the line AB a sector equal to the given ABC.

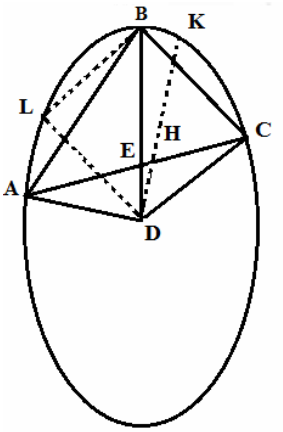
Construction & demonstration.

The ordinate may be drawn from C to the diameter AB, with the right line CED, and BD may be joined; I say what is required to be done, indeed the segment AED [§.45, Ch.1] is equal to the segment AEC, and since CE, ED shall be equal ; the triangle DEB to be equal to the triangle CEB, and therefore the sector ABD to be equal to the sector ABC. Therefore we have shown, etc. Q.f.f.

PROPOSITION XLIX.

Equal sectors ADB, CDB may have a common line BD, and AB, CB may be joined.
I say the segments of the lines AB, CB removed to be equal to each other, and if the segments were equal, I say the sectors also to be equal.

Demonstration.

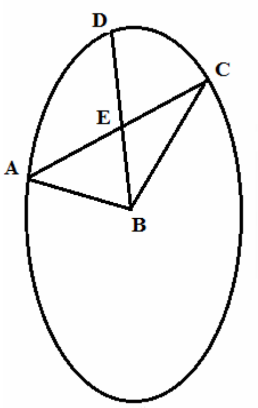


A,C may be joined and AC crosses the line for the diameter BD at E, then if AC may not be bisected at E: it shall be bisected at H, and the diameter HK acts through H. Because AC has been drawn the ordinate line to the diameter DK, the segments AHK, CHK will be equal; moreover AHD and CHD are equal triangles, therefore the sectors ADK, CDK are equal to each other, therefore the sector CDK is equal to a part of the whole sector CDB. Which is absurd; whereby the line AC is not bisected at H ; nor in another point except at E: and therefore thus AC, is placed to be the ordinate for the diameter BD. From which the segment AEB is equal to the segment CEB; but AEB, CEB are equal triangles, therefore the remaining segment AB, is equal to the segment CB; which was the first to be shown.

Now AB, CB shall be equal segments, and from the points A, B, C there may be placed the diameters AD, BD, CD, I say the sectors ADB, CDB, to be equal to each other; but if truly the sector CDB shall be equal to the sector BDL, and the points LB may be joined, therefore the segment LB is equal to the segment CB, that is to AB by the hypothesis, and thus a part is equal to the whole.

Since this cannot happen; therefore the sector BDL [§.48 above] is not equal to the sector CDB: nor otherwise to any sector besides the sector ADB. Q.e.d.

PROPOSITION L.



ABC shall be some sector.
I say a line drawn from the centre, which bisects the chord AC, also cuts the sector into two equal parts.

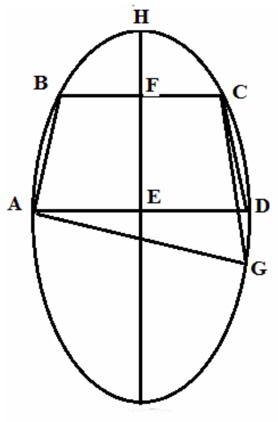
Demonstration.

The diameter BD may be drawn from the centre B; bisecting the line AC at E: I say the sectors ABD, CBD to be equal: Indeed since AC shall be bisected at E, ABE, CBE shall be equal triangles, but, since the ordinate AC has been put in place for the diameter BD, the segments AED, CED also are equal [§.45 above], therefore the whole sector ABD, is equal to the sector CBD. Q.e.d.

PROPOSITION LI.

Any two parallel lines AD, BC may cut the ellipse ABC ; and AB, CD may be joined.

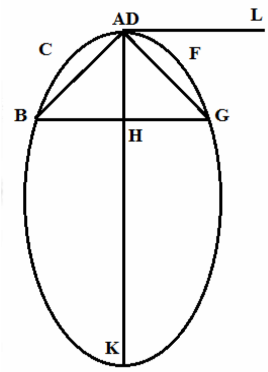
I say the segments AB, CD to be equal and if the segments were equal, I say the lines BC, AD to be parallel.



Demonstration.

AD, BC to be bisected at F and E by the line FE acting through F and E, crossing the ellipse at H, that line will be the diameter, whereby the segments BFH, CFH are equal; again because AE, DE are equal, and the segments AHE, DHE will be equal : therefore with the equal segments BFH, CFH removed the segments AB, FE, DC, FE will remain equal. Then because AE, ED are equal, and the heights of the common parallel lines BC, AD, will be AE, FB, equal to the trapezium DEFC, therefore from the equalities, the remaining segments AB, CD remain equal to each other. Which was the first part.

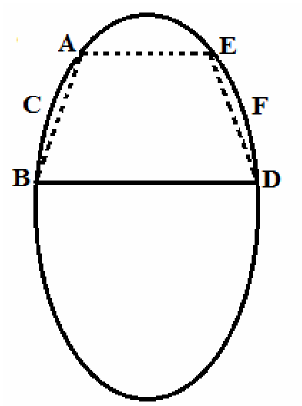
Now the segments AB, CD shall be equal, and BC, AD may be joined ; I say the lines AD, BC to be parallel to each other: truly if not, AG may be drawn parallel to BC itself, and CG may be joined: therefore by the first part of this Prop., the segment AB will be equal to the segment CG: however from the hypothesis the segment CD is equal to the segment AB; therefore the segments CG, CD are equal, which cannot happen, since the point G either falls above or below D, and thus the segment CG shall be greater or less than the segment CD: therefore the line AG shall not be parallel to BC, but to AD only. Q.e.d.



But if the given point D the same as the point A, the tangent AL to the ellipse at the point A, AL shall be drawn tangent to the ellipse at the point A or D, (indeed now from the hypothesis A and D are one and the same point) and with BG drawn parallel to AL, join DG.

I say by this contraction the segment DFG to be equal to the segment ACB.

From the point of contact the diameter AK may be drawn; therefore since BG shall be parallel to the tangent, it is the ordinate put in place for the diameter AK, and thus bisected at H, therefore the triangles BAH, GAH are equal; and truly the segments BCAH, GFAH shall be equal ; therefore the remaining segments BCA, GFA or GFD are equal. Therefore what was sought has been accomplished.



PROPOSITION LII.

Some right line AB may cut the ellipse : taking away the segment ACB, and some point D may be given on the periphery D, it will be required to draw the right line DE from D, which shall remove the segment DEF, equal to the segment ABC.

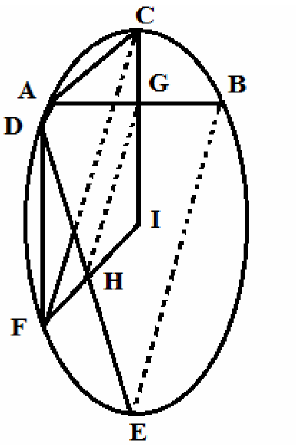
Construction and demonstration.

BD shall be joined, and from A there may be put AE parallel to BD, and ED may be joined; it is evident from the preceding that the segment DEF to be equal to the segment ABC. Therefore from the given point, &c. Q.e.f.

PROPOSITION LIII.

Some two chords AB, DE may cut the ellipse ABC, removing equal segments : moreover with the right lines AB, DE bisected at G and H , the diameters IGC, IHF shall be drawn through G and H.
I say these to be divided proportionally at G and H. And if the diameters shall be divided proportionally : I say the segments to be equal.

Demonstration.

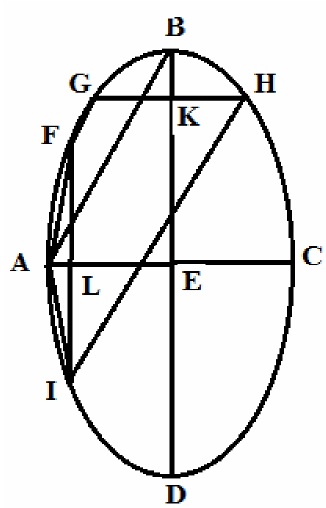


The lines AD, BE, AC, DF, GH, CB, FE, CF may be joined. Because the segments ACB, DFE may be placed equal, the lines AD, EB are parallel: moreover as AG to GB, thus DH to HE, since the lines AB, DE [§.51. above] shall be bisected at G and H, and therefore the line GH to be parallel to AD, BE. Now truly since AB shall be the ordinate put in place to the diameter IC, the segments AGC, BGC [§.45. above] shall be equal, and thus the segment AGC shall be half the segment ACB; from the same reason the segment DFH is half the segment DFE. Whereby since the whole segments ACB, DFE may be put equal, there will be also the segments AGC, DFH [§.42. above], the halves of which shall be equal to each other. Then the triangles ACB, DFE shall be the greatest of these triangles able to be inscribed, and since AD, BE have been shown to be parallel, also are equal to each other [§.44. Cor.2], and the halves of these, the triangles AGC, DFH: which, if taken from the equal segments AGC, DFH, then the segments AC, DC will remain equal, therefore the line CF [§.52. above] will be parallel to the right line AD, that is GH: whereby as CG to GI, thus FH to HI. Q.e.d. Hence now the truth of the converse shall be evident.

PROPOSITION LIV.

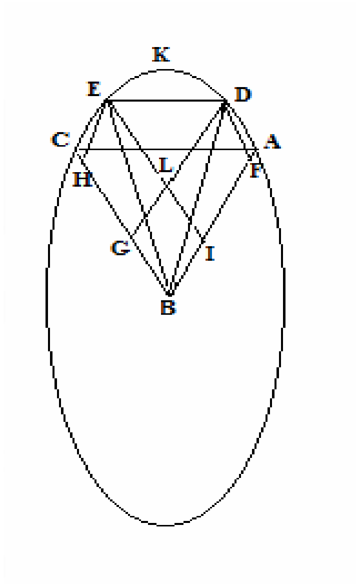
AC, BD shall be some of the diameters conjugate in the ellipse ABC and AB may be joined, some line FG may be drawn parallel to AB; and from F and G, the right lines GH, FI may be placed as ordinates to the diameters BD, AC.
I say the diameters AC, BD to be divided proportionally at K and L.

Demonstration.



The lines GB, BH, FA, AI, HI may be drawn. Because the lines AB, GF are parallel, the segments GB, FA will be equal [§.52. above]. Moreover the segment GB is equal to the segment HB, (for the segment of the line GKB is equal to the segment HKB, and the triangle GBK to the triangle HBK) and, by the same reason, the segment FA is equal to the segment AI. Therefore the segment HB is equal to the segment AI, and HI is parallel to the line AB [§.45. above], that is, FG. Whereby the whole segment GBH is equal to the whole segment FAI: and thus by the preceding, the diameters AC, BD are divided similarly at K and L. Q.e.d.

PROPOSITION LV.



In the given ellipse some of the conjugate diameters AB, CB shall be given, and AC shall be joined, to which some ED shall be given parallel, moreover from the points D and E the ordinate for the diameters DF, EH, DG, El, may be drawn : therefore the figures DFBG, EIBH will be parallelograms. I say these parallelograms FG, HI to be equal.

Demonstration.

Since by the preceding BA, BC are divided proportionally, AB will be to BF, as CB to BH: and on interchanging, as AB to CB, thus BF to BH; similarly by the preceding AB is to BI as CB to BG, and on interchanging, as AB to CB, thus BI to BG; therefore BF is to BH, as BI to BG. Therefore [§.14. ch.6], the parallelograms AG, HI are equal. Q.e.d.

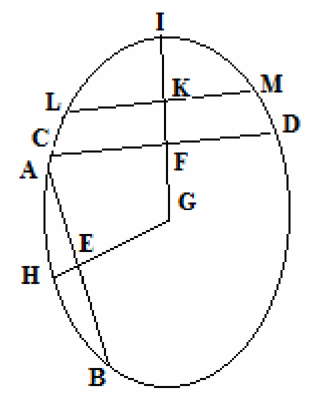
PROPOSITION LVI.

The diameters AB, BC shall be joined; with the points joined AC, ED may be drawn parallel to the right line AC, then from D and E the ordinate right lines EI, DF, DG, EH may be put in place for the diameters AB, CB, and EB, DB may be joined.
I say the sector EBD to be equal to the figure EIFDKE.

Demonstration.

From the halves of the preceding equal parallelograms FG, HI, the triangles DFB, EIB are equal to each other; therefore with the removal of the common triangle LIB, the triangle ELB will be equal to the trapezium DFIL; whereby with the addition of the common figure ELDKE, the sector EBD is equal to the figure EIFDKE. Q.e.d.

PROPOSITION LVII.



Any two lines AB, CD cut the ellipse ABC; it is required to draw some line parallel to the line CD, which will remove a segment equal to the segment AHB.

Construction & demonstration.

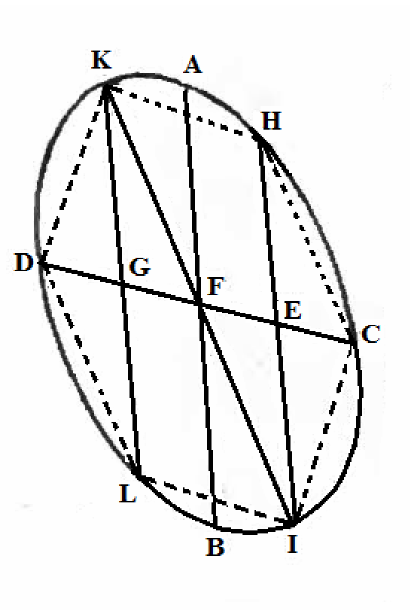
Bisect AB, CD at E and F, the diameters GH, GI shall act from the centre G through E and F; then GI shall be divided at K, just as HG has been divided at E, and the ordinate LM may be placed through K , it is apparent from §53 that LIM, AHB to be equal segments ; but LM is parallel to the given CD, therefore with the segment of the ellipse given, &c. Q.e.f.

PROPOSITION LVIII.

Two conjugate diameters AB, CD shall cut the ellipse ABC ; and of these the other AB divides CD at E and G fourfold, with the right lines HI, KL acting through E and G parallel to the diameter AB, and the points D, K, H, C, I, L, D may be joined.

I say the lines LD, DK, KH, HC, CI, IL to bear equal segments.

Demonstration.



Because KL, HI are parallel to the right line AB, which is the diameter conjugate to DC itself, KL, Hl will be the ordinates put in place for DC Therefore so that the rectangle DGC shall be to the rectangle DEC, thus as the square KG to the square HE, and thus since the rectangles shall be equal, also the squares will be equal, so that the right lines KG, HE also are equal.

Truly GF, EF are equal, and the angles KGF, FEI (since Hl, KL shall be parallel) are equal; therefore the triangle KGF is equal to the triangle FEI, and the angle IFE is equal to the angle GFK. Since GFE is a right line, the angles IFE, KFG are going to constitute equal angles with the vertex: from which the points of KFI lie on a straight line, and thus the sectors KFD, CFI are to be made equal by the vertex.

Moreover, since I have shown the triangle KFG to be equal to the triangle FEI, and by a similar discussion it may be shown also, that the triangle DKG to be equal to the triangle ICE, the whole triangle DKF to be equal to the whole triangle ICF.

In addition, the sectors DFK, IFC are equal with regard to the vertex. Therefore also the segments remaining DK, IC will be equal to each other. In the same manner, the segments DL, HC are shown to be equal to each other.

Again, since the two sides KG, GF shall be equal to the two sides GL, DG, and the angles contained by equal sides to be equal to the angles through the vertex; the triangles GKF, DGL to be equal to each other, and the angle GKF is equal to the other angle GLD.

Thus KFI, DL to be parallel lines; whereby the segments DK, LI are equal to each other [§.51. above]. Now truly it has been shown the segments CI, DK also to be equal to each other, therefore the three segments DK, LI, CI are equal. Further since DC, LI taken together are parallel, and for which GL, EI, themselves also are parallel, from which again the segments CI, DL are to be equal, for the four segments DL, CI, LI, DK to be equal. Again since KH, LI, taken together with KL, HI are equal and themselves parallel : therefore the segments KH, Ll are equal. Therefore the five segments KH, LI, KD, DL, CI are equal; and also the equality of the segments HC, DL has been shown; therefore all six segments are equal. Q.f.d.

Moreover the figure DKHCIL inscribed in the ellipse may be called a regular polygon.

PROPOSITION LIX.

With the same figure proposed remaining, to inscribe a regular hexagon figure in the ellipse.

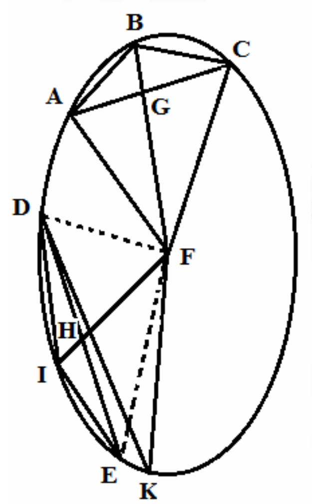
Construction and demonstration.

Any two conjugate diameters AB, CD may be assumed and with CD divided fourfold at E and G, the lines HI, KL shall act through the points E and G parallel to AB: and DK, KH, HC, CI, IL, LD may be drawn: it will be evident from the preceding: these right segments to be removed equally, and thus the hexagon figure DK, HC, ILD to be regular; therefore we have inscribed the figure of a regular hexagon. Q.e.f.

PROPOSITION LX.

Any two lines may cut the ellipse ABC bearing the equal segments AC, DE, and the ends of the radii FA, FC, FD, FE may be drawn to these.
I say the sectors AFC, DFE to be equal, and if the sectors were equal, I say the segments to be equal.

Demonstration.



With AC, DE bisected at G and H, the diameters FB, FI shall act through G and H, and AB, BC, DI, EI may be joined; since the segments AC, DE are equal from the hypothesis, therefore the right lines, which may join the points C and E, A and D, may become parallel; therefore the greatest triangles from the inscribed segments AC, DE shall be equal, but ABC, DIE, are the greatest of the prescribed, therefore the triangles ABC, DIE shall be equal : but since the diameters FB, FI at G and H are divided proportionally: therefore as the triangle ABC is to the triangle AFC, thus triangle DIE is to triangle DFE, and on interchanging so that triangle ABC is to triangle DIE, thus triangle AFC is to triangle DFE. Therefore since the triangles ABC, DIE shall be equal, also the triangles AFC, DFE will be equal. Whereby with the equal segments AC, DE added , they will be equal to the sectors FAC, FDE.
Now the sectors AFC, DFE shall be equal, and AC, DE may be joined. I say the segments ABC, DIE also to be equal; truly if not : either may be considered (for example) DIE lesser than the other, and the line DK may be drawn from D, the segment removed is equal to the segment ABC, the sector DFK is equal to the sector AFC, that is equal to the sector DFE, which cannot happen; therefore the segments AC, DE are not unequal, but equal. Q.e.d.

Corollary.

From this, and from §.53 above, it follows in the first place, that if AFC, DFE were some equal sectors, the chords of which were bisected at G and H, then the diameters drawn through G and H, may be divided proportionally by the same points.
In the second place if the two sectors AFC, DFE were equal, and right angles AC, DE were subtended by these : so that the triangles AFC, DER shall be equal.
In the third place such problems hence are solved, with some sector AFB given, it will be required to draw radii from F which shall constitute a sector equal to the sector AFB: for the construction join AB, some

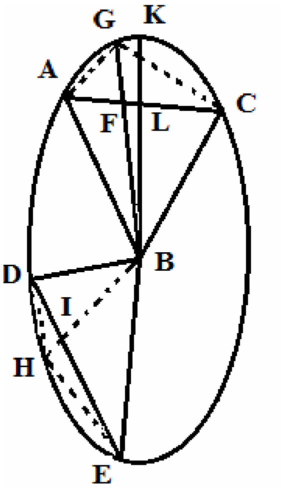
radius FD shall be drawn from F: then from D the right line DI may be drawn taking away a segment DI equal to the segment AB and IF may be joined, it is evident the sector IFD to be equal to the sector AFB.

PROPOSITION LXI.

ABC, DBE shall be equal sectors: AC, DE may be joined and some line BE shall be drawn from B as if cutting the line AC at F: then the sector DBH shall be made equal to the sector ABG, and the line HB may cut the right line ED at I.

I say both the lines AC, DE as well as BG, HB to be divided proportionally at F and I.

Demonstration.



AG, GC, DH, HE shall be joined. Since from the hypothesis both the sectors ABG, DBH, as well as ABC, DBE are equal, and the remaining GBC, HBE shall be equal; whereby the triangle AGB is equal to the triangle DHB, and the triangle CGB is equal to the triangle HEB, whereby in order that the triangle BAG shall be to the triangle GCB, thus as the triangle BDH triangle shall be to the triangle BHE, but (which has been shown easily from ext.6.) the ratios of the right lines AF, FC, and DI, IE, are the same as from the ratios of the triangles BAG, GCB, and BDH, BHE.

Therefore also, AF is to FC, as DI is to IE. Then since the trapeziums GABC, BDHE shall be equal (for indeed the triangle BAG is equal to the triangle BDH, and the triangle BGC is equal to the triangle BHE) moreover the triangle ACB shall be equal to the triangle DEB, the remaining triangle AGC shall be equal to the remaining triangle DHE: therefore as the triangle AGC to the triangle ACE, that is as GF to FB, thus the triangle DHE to the triangle DEB, that is, HI to IB. Q.e.d.

PROPOSITION LXII.

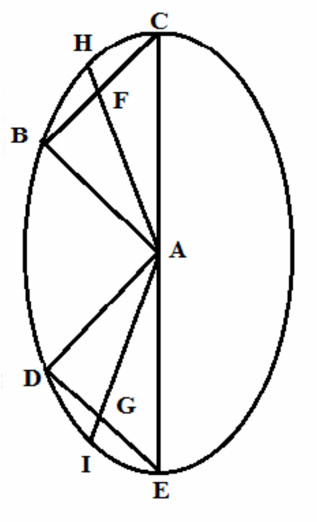
Now the lines AC, DE shall be divided proportionally at F and I : and the radii BG, BH acting through F and I.

I say the sectors ABG, DBH to be equal.

Demonstration.

But if truly one such as ABG shall be smaller than the other : ABK shall be made the sector equal to the sector DBH, and BK shall cut the right line AC at L. Since the sectors ABK, DBH are equal: by the first part of this Prop. there will become AL to LC, as DI to IE. And also AF is to FC, as DI to IE, therefore as AF to FC, thus AL to LC, which cannot happen : since the point L falls either further or nearer than F; whereby the sector ABK is not equal to the sector DBH, nor to any other besides ABG. Q.e.d.

PROPOSITION LXIII.

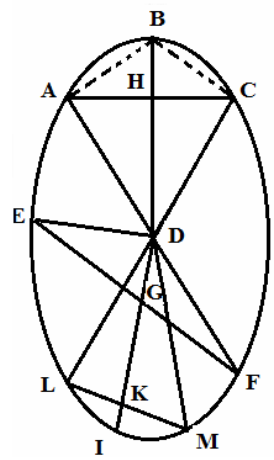


AB, AC shall be any two diameters and with the ends of these joined, some other two diameters AD, AE may be drawn such that with DE joined, ABC, ADE shall be equal triangles : moreover with BC, DE bisected at F and G, the diameters AH, AI acting through F and G shall be divided proportionally at F and G.
I say the sectors BAC, DAE to be equal.

Demonstration.

Because the diameters AH, AI are divided proportionally at F and G, the segments BHC, DAE will be equal: but from the hypothesis the triangles ABC, ADE are equal; therefore the sectors BAC, DAE are equal to each other. Q.e.d.

PROPOSITION LXIV.

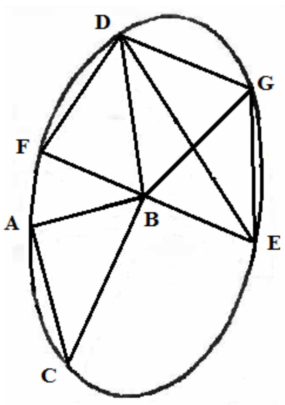


There shall be two sectors DABC, DEIF, and with the right lines drawn AC, LM, and bisected at H and G, the diameters DHB, DGI may be drawn. Moreover the ratio DG to DI shall be smaller than the ratio DH to DB.
I say the sector DEIF to be greater than the sector DABC.

Construction and demonstration.

Because DG to DI is in a smaller proportion than DH to DB. There may become DK to DI, as DH to DB, therefore DK will be greater than DG, and the point K falls between G and I, the ordinate LKM shall be drawn through K ; and DL, DM may be joined; there the segments LIM, ABC are equal. Whereby the sectors DLIM, DABC are equal also, and therefore the sector DEIF is greater than the sector DAVC. Q.e.d.

PROPOSITION LXV.

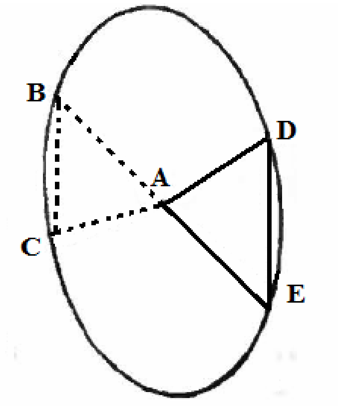


ABC, DBE shall be two unequal sectors, thus so that ABC, DBE shall be equal triangles:
I say the sectors ABC, DBE taken together to equal a semi-ellipse.

Demonstration.

Since, from the hypothesis, the sectors shall be unequal, BDGE shall be the greater sector, as, according to the hypothesis, the triangle DBE is equal to the triangle BAC, therefore the segment DGE to be greater than the segment AC, and thus a segment EG equal to the segment AC itself may be removed from EG, and BG, GD may be joined and EB produced to F, FD shall be drawn : therefore since the segments GE, AC are equal, the sectors also are equal, and indeed the triangle GEB is equal to the triangle ACB, that is, equal to the triangle BDE ; whereby BE, DG are parallel and thus the segment GE, that is the segment AC is equal to the segment DF; and thus the sectors BAC, BFD are equal. And the sectors BFD, BDGE, constitute a semi-ellipse. Q.e.d.

PROPOSITION LXVI.

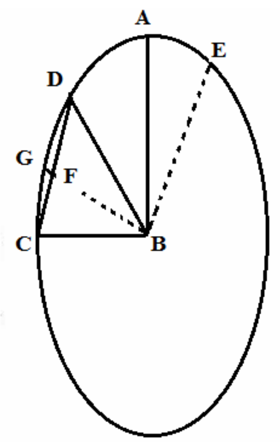


The two sectors BAC, DAE may be taken, thus so that the triangles ABC, ADE shall be equal: if the sectors of these likewise may be taken, they would be greater or smaller than a semi-ellipse:
I say these sectors to be equal to each other.

Demonstration.

Indeed if they were not equal, let BAC be smaller than the sector DAE: therefore since the triangles ABC, DAE are equal, the sectors BAC, DAE taken together shall be equal to a semi-ellipse : Which is contrary to the hypothesis.
Therefore the sectors BAC, DAE are not unequal, but equal. Q.e.e.

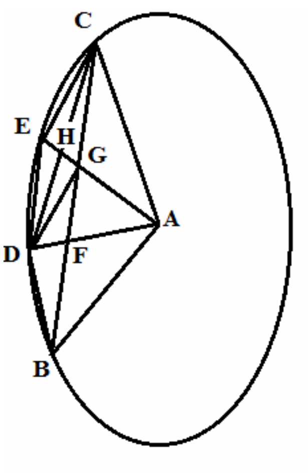
PROPOSITION LXVII.



AB, BC shall be conjugate diameters, and some right line CD may be drawn from C, from B the line BE may be drawn parallel to the line CD; and DB shall be joined.
I say the sector DBC to be the double of the sector ABE.

Demonstration.

CD shall be bisected at F, the diameter BG may be drawn, and since BE shall be parallel to the ordinate CD put in place for the diameter BG, so that also is an ordinate indeed through the centre. Therefore BG, BE are conjugate diameters. Moreover, from the hypothesis, CB, BA also are conjugate. Therefore the sector CBA is equal to the sector GBE, and with the common sector GBA removed, the sectors CBG, ABE are equal. And the sector CBD is the double of the sector CBG. Therefore also CBD is the double of the sector ABE. Q.e.d.



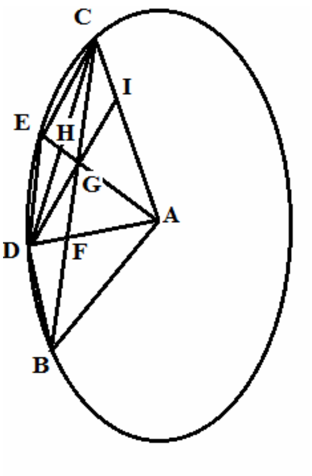
PROPOSITION XVIII

If the sectors ABD, ADE, AEC shall be equal, and the right lines DE and BC may be drawn, cutting the right lines AD, AE in F & G.
I say BF, DE, EC to be equal.

Demonstration.

The right lines BD ,EC, DC, DG may be drawn. Since the sectors ADE, AEC are equal, also the triangles CHA, DHA are equal; therefore the right lines DH, HC are equal. Then since the sectors ADB, ACE are equal, the segments BD, EC also are equal. Therefore CB, ED are parallel. Therefore the angles GCH, EDH, are equal : truly the angles GHC, DHE are equal also, and now the right lines DH, HC are shown to be equal; therefore CG, ED shall be equal. Similarly we may show BF, DE to be equal. Therefore the truth of the proposition may be agreed on.

PROPOSITION LXIX.



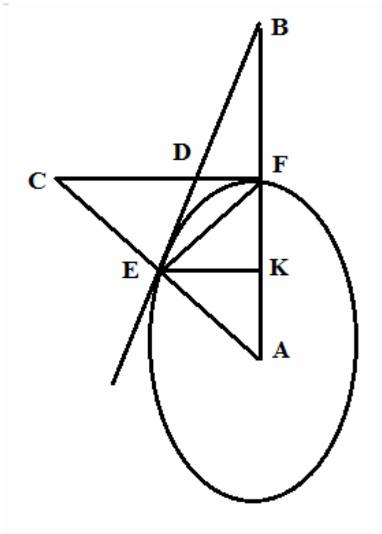
With the same in place DG may be produced, then it shall cross the line AC at I. I say AGI, AGC, AEC to be triangles in continued progression.

Demonstration.

From the above demonstration, it is evident everything is equal in triangles CHG, EHD, and thus also the sides EH, HG, then only the triangles DHG, BHC may be considered to be equal. In which since the two sides DH, HE, shall be equal to the two sides CH, HE, and the angle DHG equal to the angle CHE, for the base also the angles HGD, CEH will be equal; hence DI, CE shall be parallel and thus as AI to AC, thus AG to AE. But as AI is to AC, thus triangle AGI to triangle AGC: and as AG to AE, thus triangle AGC to triangle AEC. Therefore as triangle AGI is to triangle AGC; thus triangle AGC is to triangle AEC. Q.e.d.

PROPOSITION LXX.

Two diameters AE, AF shall cut the ellipse at the points E and F, from which the tangents to the ellipse FC, EB may be drawn, crossing the diameters at C and B, and EF may be joined.
I say the triangle ACF to be equal to the triangle ABE.



Demonstration.

From the point E, draw the ordinate EK to AF, this will be parallel to the tangent at F, and thus the triangles AEK, ACF are similar, and therefore have the two fold ratio of the ratio AK ad AF : that is because AK, AF, AB are three ratios in continued proportions, they have the ratio as AK ad AB, but it is also as AK ad AB , thus as the triangle AEK to the triangle AEB: therefore the triangle AEK has the same ratio to triangle ACF, as it has to triangle AEB; therefore the triangles ACF, AEB are equal. Q.e.d.

ELLIPSIS

PARS SECUNDA

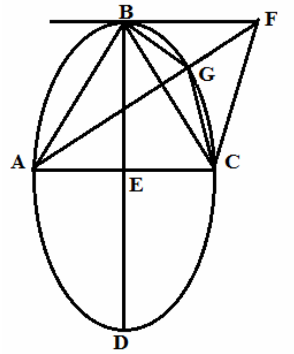
De sectoribus & segmentis Ellipseos.

PROPOSITION XLII.

Sit ABC ellipseos diameter BD, ad quam ordinatim ponatur AEC; iunganturque ABC.

Dico ABC triangulum maximum esse illorum quae segmento ABC inscribi possunt.

Demonstratio.

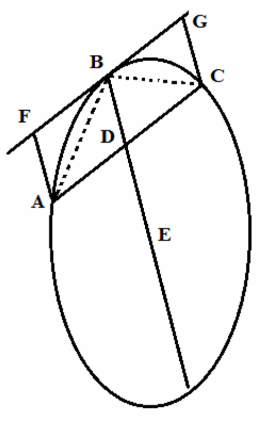


Acta per B contingence BF, ex A recta ducatur quaevis AF, occurrens ellipsi in G & contingenti in F iunganturque GC, FC : Quoniam FG contingens cadit supra G, igitur triangulum AFC maius est triangulo AGC; sed AFC triangulo aequale est triangulum ABC ob AC, BF aequidistantes; igitur & ABC triangulum maius est triangulo AGC: unde cum idem de aliis omnibus triangulis ostendatur; patet ABC triangulum, maximum eorum esse quae segmento ABC inscribi possunt. Quod erat demonstrandum.

Corollary.

Hinc facilis praxis elicitur ad inscribendum cuivis segmento triangulum maximum: erigendo nimirum diametrum BD, iungendoque AB, BC puncta. Demonstratio patet ex priori.

PROPOSITION XLIII.



Triangulum maximum segmento cuivis non maiori semiellipsi inscriptum, maius est dimidio eiusdem segmenti.

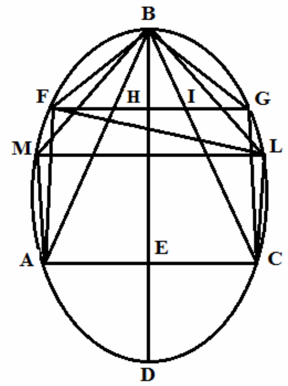
Demonstratio.

Esto ABC segmento, non maiori semiellipisi inscriptum triangulum maximum ABC. Dico illud maius esse dimidio segmenti ABC: ducta enim diametro BE, quae AC subtensam diuidat bifariam in D, erigature ex A & C lineae AF, CG parallela diametro BE quae FG contingenti per B actae occurrant in F; iunganturque AB, CB: triangulum ABC [§.41, Ch.1] dimidium est parallelogrammi AG. Atqui AG parallelogrammum maius est segmento ABC, cum AF, CG, FG lineae cadant extra ellipsim; igitur & triangulum ABC maius est dimidio cuiisdem segmenti. Quod fuit demomstramdum.

PROPOSITION XLIV.

Ellipsim ABC secet diameter BD, ad quam ordinatim posita sit AEC: iunctis AB, CB inscribatur segmento AFB triangulum maximum AFB, & ex F ponatur FG parallela AC, iunganturque BG, GC.
Dico BGC triangulum maximum esse eorum quae segmento ABC inscribi possunt, & si triangula fuerint maxima, dico FG esse parallelam ad AC.

Demonstratio.



Quoniam FH, GI lineae sunt aequales, triangula FBH, GBL eandem habentia altitudinem, aequalia erunt, similiter triangula FAH, GIC inter parallelas EG, AC, constituta erunt aequalia ac proinde aequabuntur tota triangula BFA, BGC : si igitur BGC non sit maximum, ponatur aliud BLC, maius triangulo BGC, & ex L ducatur LM aequidistans AC; erit igitur ut prius triangulum BLC, aequale triangulo AMB, adeoque & AMB triangulum, maius triangulo BGC id est AFB, quod est contra suppositum, cum BFA maximum ponatur ; igitur BGC triangulum maximum est eorum quae segmento BGC inscribi possunt, quod erat primum; sint deinde triangula BFA, BGC maxima, demonstrabimus iunctam FG parallelam esse AC, si enim non est parallela , sit alia supra vel infra ipsam FG parallela ad AC, nimirum recta FL; iunganturque BL, CL; ergo per primam partem huius triangulum BLC erit maximum, quod fieri non potest cum FGC ex hypothesi sit maximum. Non igitur FL, aut alia ulla praeter FG est parallela ad AC. Quod erat demonstrandum.

Corollary primum.

Hinc sequitur si triangula BFA, BGC maxima sint eorum quae segmentis inscribi possunt, esse aequalia. Nam per secundam partem huius FG est parallela ad AC. Unde FH, GI aequales sunt, ac proinde triangula FBH, BIG, & FAK, GCI ; adeoque & tota BFA, BGC aequalia sunt.

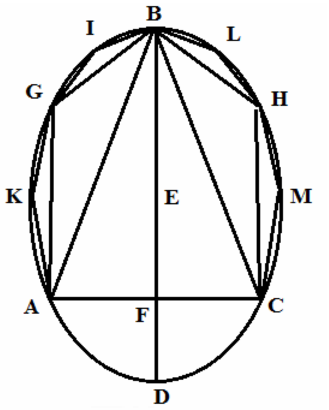
Corollary secundum.

Quod si fuerint binae AC, FG ad diametrum ordinatim positae, iunganturque FA, CG, triangula quae segmentis AF, CG inscribuntur maxima, inter se quoque aequalia esse, eodem plane discursu demonstrabimus, quo usi sumus in propositione & corollario primo, nullo alio immutato, quam quod loco 26 huius assumenda sit vigesima septima.

PROPOSITION XLV.

Sit ABC ellipsis diameter quaecunque BD ad quam ordinatim ponatur AFC.
Dico AG BF segmentum aequari segmento CHBF.

Demonstratio.



Iunctis AB, CB,inscribantur [§.42, Ch.1] segmentis reliquis triangula maxima AGB, CHB erunt illa per primum Coroll. praecedentis propositionis inter se aequalia & [§.43, Ch.1] maiora dimidio segmentorum ABG, CBH: dein & residuis utrimque segmentis triangula inscribantur maxima AKG, GIB, CMH, BLH erunt ut prius triangula AKG, GIB partim per corollar. primum, partim per secundum aequalia triangulis CMH, BLH, & maiora dimidiis segmentorum: igitur cum ea inscriptio semper possit continuari in utroque segmento AGB, BHC, & utrimque partes ablatae sint inter se aequales, & maiores dimidio, segmentorum a quibus auferuntur, constat [§.216, Progress.] AGB segmentum aequale esse segmento CHB. Quare additis aequalibus triangulis ABF, CBF erunt tota segmenta AGBF, CHBF aequalia. Quod erat demonstrandum.

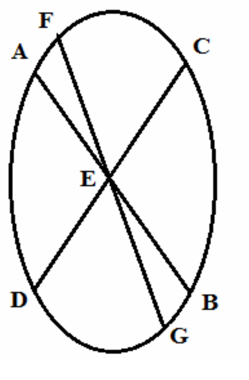
Corollary.

Hinc sequitur a quavis diametro ellipsim bifariam secari: sit enim diameter quaevis BD, & ducatur per quodvis illius punctum ordinatim AFC, per propositionem iam demonstratam segmentum ABF, segmento BCF, aequatur rursum per eandem proposition e segmentum ADF, segmento CDF aequale est; ergo segmenta ABF, ADF, hoc est totum segmentum DAB, aequantur segmentis BCF, CDF, hoc est toti segmento DCB; bifariam igitur divisa est ellipsis a diametro BD.

PROPOSITION XVI.

Diametri duae coniugatae ellipsim quadrifariam dividunt, & diametri ellipsim quadrifariam diuidences, sunt inter se coniugatae.

Demonstratio.



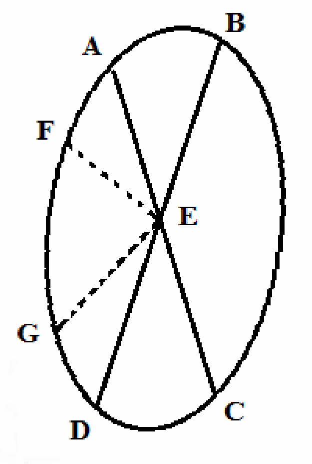
Sint in ABC ellipsi diametri duae coniugatae AB, CD; dico illas ellipsim quadrifariam dividere; & si AB, CD diametri ellipsim quadrifariam dividant, dico illas esse conugatas. Quoniam AB, CD diametri sunt coniugatae ; erit AB ordinatim posita ad diametrum DC, unde iam AEC, CEB quam AED, BED sectores sunt aequales ; sunt autem & AEC, AED sectores ob eandem rationem aequales; sectores igitur quatuor AEC, CEB, BED, DEA, sunt inter se aequales, & AB, CD lineae quadrifariam dividunt ellipsim: Quod erat primum.
Sit iam ellipsis quadrifariam divisa, dico AB, CD diametreos esse coniugatas; sin vero ducatur ipsi CD, coniugata FG, igitur FEC, sectorquadrans ellipseos est per priorem partem huius. Atque sector AEC ex hypothesi etiam quarta ellipseos pars est, ergo sectores FEC, AEC aeqales sunt, pars & totum, quod fieri nequit ; igitur FG diameter non est conjugata ipsius CD, nec quaevis alia praeter AB. Quod erat demonstrandum.

Corollary.

Hinc patet sectores quarumcumque coniugatarum, aequales esse sectoribus cuiuscunque alterius coniugationis singulas singulis : singul enim quadrantes sunt ellipseos, & si sectores sint aequales ac latera unius sint coniugatae, alterius etiam latera esse coniugatas.

PROPOSITION XLVII.

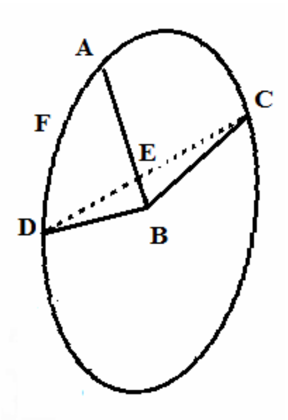
Sectores ad verticem oppositi sunt inter se aequales.



Demonstratio.

Secent ABC ellipsim diametri quaecunque AC, BD. dico sectores ad verticem oppositos esse inter se aequales. Ducantur enim ex E centro diametri duae EF, EG & EF quidem coniugata ipsi: EG vero coniugata ipsi AE. Quoniam igitur sectores BEF, AEG, [§.46, Ch.1] aequales sunt, dempto communi AEF, erit sector AEB, aequalis sectori FEG: rursum cum sectores SED, GEC sint aequales, dempto communi DEG, erit sector DEC aequalis sectori FEG, id est AEB ad verticem opposito. Eodem modo ostenduntur AED, BEC sectores aequales. Igitur, &c. Quod fuit demonstrandum.

PROPOSITION XLVIII.



Sit in ADC ellipsi sector quicunque ABC, oportet ex B rectam ad peripheriam ducere, quae cum AB linea sectorem constituat dato ABC aequalem.

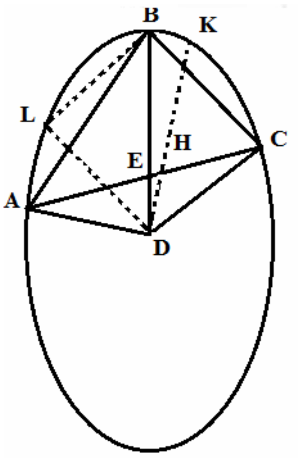
Constructio & demonstratio.

Ducatur ex C ordinatim ad diametrum AB, recta CED, iungaturque BD; dico factum esse quod petitur est enim segmentum AED [§.45, Ch.1] aequale segmento AEC, & cum aequales sint CE, ED; triangulum DE aequale triangulo CEB, igitur & sector ABD aequalis sectori ABC. Eduximus igitur, &c. Quod fuit faciendum.

PROPOSITION XLIX.

Habeant ADB, CDB sectores aequales commune latus BD, iunganturque AB, CB.
Dico segmenta lineis AB, CB ablata esse inter se aequalia, & si segmenta fuerint aequalia, dico & sectores aequari.

Demonstratio.



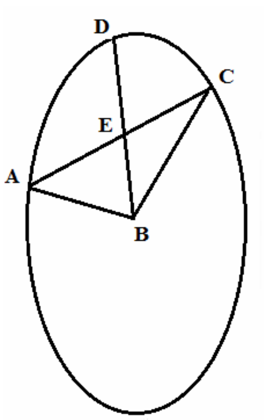
Iungantur A,C occurratque AC, linea diametro BD in E, tum si AC non sit diuisa bifariam in E: dividatur bifariam in H, agaturque per H diameter DK. Quoniam AC, linea ordinatim ducta est ad diametrum DK, erunt AHK, CHK segmenta aequalia ; sunt autem & AHD, CHD triangula aequalia, sectores igitur ADK, CDK inter se aequales sunt, igitur sector CDK aequalis est sectori CDB pars toti. Quod absurdum; quare AH linea non dividitur in H bifariam; nec in alio puncto qua in E: adeoque igitur AC, lineae posita est ordinatim ad diametrum BD. Unde AEB segmentum est aequale segmento CEB; sunt autem AEB, CEB triangula aequalia, ergo reliquum segmentum AB, aequale est segmento CB; quod erat primum.

Sint iam AB, CB segmenta aequalia & ex A, B, C punctis diametri ponatur AD, BD, CD, dico sectores ADB, CDB, esse inter se aequales; sin vero: fiat CDB sectori aequalis sector BDL, iunganturque puncta LB, erit igitur LB segmentum aequale segmento CB, hoc est AB per hypothesin, adeoque pars aequalis toti. Quod fieri non potest; igitur sector BDL [§.48 above] non est aequalis sectori CDB: nec alius quisquam preater ADB sectorem. Quod erat demonstrandum.

PROPOSITION L.

Sit ABC sector quicunque.
Dico lineam ex centro ductam, quae AC subtensam dividit bifariam, sectorem quoque bifariam secare.

Demonstratio.

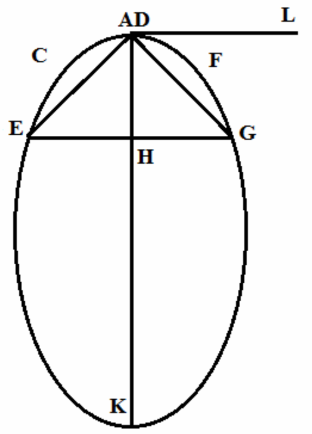


Ducatur ex B centro diameter BD; secans bifariam AC lineam in E: dico ABD, CBD sectores esse aequales: Cum enim AC in E divisa sit bifraiam, erunt ABE, CBE triangula aequalia, sed, quia AC est ordinatim posita ad diametrum BD, etiam segmenta AED, CED sunt aequalia [§.45 above] , igitur totus sector ABD, sectori CBD aequale est. Quod erat demonstrandum.

PROPOSITION LI.

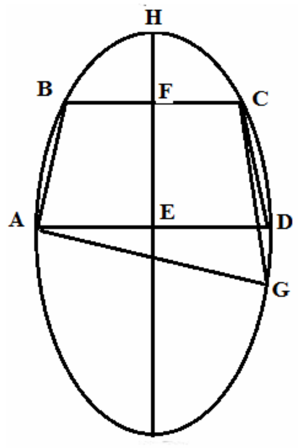
Ellipsim ABC secent duae quaevis parallelae AD, BC; iunganturque ; AB, CD.
Dico AB, CG segmenta esse aequalia & si segmenta fuerint aequalia, dico BC, AD lineas aequidistare.

Demonstratio.



Divisis AD, BC bifariam in F & E agatur per F & E linea FE, occurrens ellipsi in H, erit illa diameter, quare BFH, CFH segmentia sunt aequalia; rursum quoniam AE, DE aequales sunt, & segmenta AHE, DHE aequalia erunt : ablatis igitur aequalibus segmentis BFH, CFH remanent segmenta AB, FE, DC, FE aequalia. Deinde quoniam AE, ED sunt aequales, & altitudo communis parallelarum BC, AD, erunt AE, FB, DEFC trapezia aequalis, igitur ab aequalibus, manent AB, CD reliqua segmenta inter se aequalia. Quod erat primum.

Sint iam AB, CD segmenta aequalia, iunganturque BC, AD; dico AD, BC lineas aequidistare: sin vero, ducatur ipsi BC parallela AG, iunganturque CG : erit igitur per primam partem huius segmentum AB aequale segmento CG: sed & CD segmentum ex hypothesi aequale est segmento AB; segmenta igitur CD, CG sunt aequalia, quod fieri non potest, cum punctum G cadat supra vel infra D, adeoque CG segmentum maius vel minus sit segmento CD: igitur AG linea non aequidistat ipsi BC, sed sola AD. Quod erat demonstrandum.

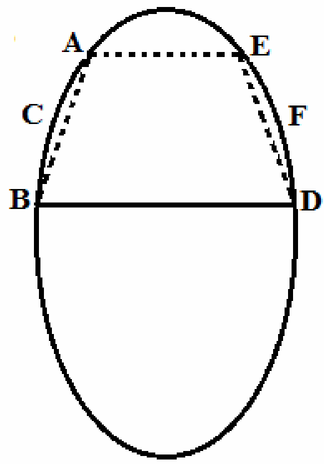


Quod si punctum datum D idem sit cum puncto A, ducatur AL tangens ellipsim in puncto A, ducatur AL tangens ellipsim in puncto A sive D, (sunt enim A & D iam ex hypothesi unum idemque punctum) ductaque BG parallela ad AL iunge DG.

Dico hanc abscindere segmentum DFG aequale segmento ACB.

Ex contactu ducatur diameter AK; igitur BG quia tangenti aequidistat, est ordinatim posita ad diametrum AK, adeoque bisecta in H, triangula igitur BAH, GAH aequantur; aequantur vero & segmenta BCAH, GFAH; ergo reliqua etiam segmenta BCA, GFA siveGFG aequantur. Factum igitur est quod petebatur.

PROPOSITION LII.



Secet ellipsim recta quaevis AB : auferens ACB segmentum, & detur in peripheria punctum quodvis D, oportet ex D rectam ducere DE, quae auferat segmentum DEF, aequale segmento ABC.

Constructio & demonstratio.

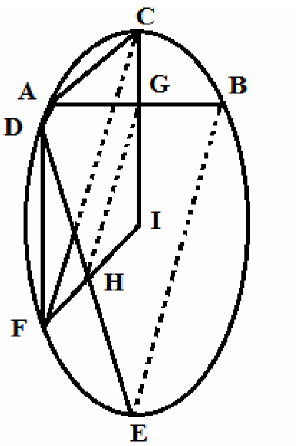
Iungantur BD, & ex A ponatur AE aequidistans BD, iunganturque ED; patet per praecedentem DEF segmentum aequale esse segmento ABC. Igitur ex puncto dato, &c. Quod erat faciendum.

PROPOSITION LIII.

Secent ABC ellipsim duae quaevis lineae AB, DE segmentas auferentes aequalia: divisis autem AB, DE segmenta auferentes aequalia: diuisis autem AB , DE rectis bifariam in G & H , ducantur per G & H diametri IGC, IHF.

Dico illas in G & H proportionaliter esse divisas. Et si diametri sint proportionaliter divisae: dico segmenta esse aequalia.

Demonstratio.

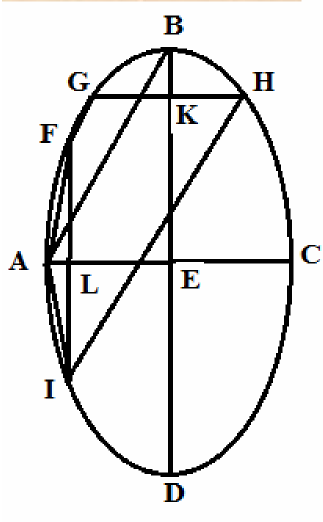


Iungantur AD, BE, AC, DF, GH, CB, FE, CF. Quoniam ACB, DFE segmenta ponuntur aequalia, AD, EB lineae parallelae sunt: est autem ut AG ad GB, sic DH ad HE, cum AB, DE [§.51. above] lineae in G & H diuisae sint bifariam, igitur & GH linea aequidistat AD, BE. Iam vero cum AB, sit ad diametrum IC ordinatim posita, erunt segmenta AGC, BGC [§.45. above] aequalia, adeoque segmentum AGC dimidium segmenti ACB; simili de causa segmentum DFH dimidium est segmenti DFE. Quare cum tota segmenta ACB, DFE ponantur aequalia, erunt etiam [§.42. above] segmenta AGC, DFH, eorum dimidia inter se aequalia. Deinde triangula ACB, DFE maxima sunt eorumquae segmentis inscribi possunt, & quoniam AD, BE ostensae sunt parallelae, etiam inter se aequalia erunt [§.44. Cor.2] ; aequabuntur igitur & eorum dimidia triangula AGC, DFH: quae si auferas a segmentis aequalibus AGC, DFH, remanent segmenta aequalia AC, DC, ergo CF lineae [§.52. above] aequidistat rectae AD hoc est GH: quare ut CG ad GI, sic FH ad HI. Quod erat demonstrandum. Hinc iam veritas conversae fit manifesta.

PROPOSITION LIV.

Sit in ABC ellipsi quaevis diametrorum coniugatio AC, BD, iunctaque AB, ducatur quaeuis FG parallela AB; & ex F & G, rectae ponantur GH, FI ordinatim ad diametros BD, AC.

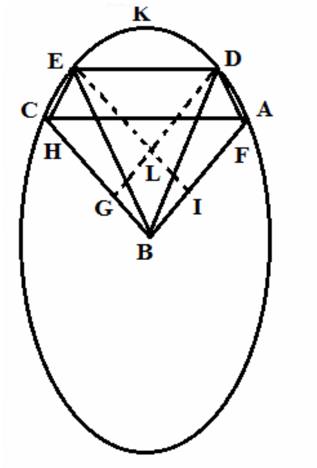
Dico AC, BD diametros in K & L proportionaliter esse divisas.



Demonstratio.

Iungantur GB, BH, FA, AI, HI. Quoniam AB, GF lineae aequidistant, erunt GB, [§.52. above] FA segmenta aequalia; sed GB segmento est aequale segmentum HB. (nam segmentum lineae GKB aequatur segmento HKB, & triangulum GBK triangulo HBK) & FA segmento ob eandem causam aequatur segmento AI, igitur & HB segmentum est aequale segmento AI, & HI aequidistat linea AB [§.45. above], hoc est FG; quare totum segmentum GBH, aequale est toti segmento FAI: adeoque per praecedentem AC, BD diametri in K & L similiter sunt divisas. Quod erat demonstrandum.

PROPOSITION LV.



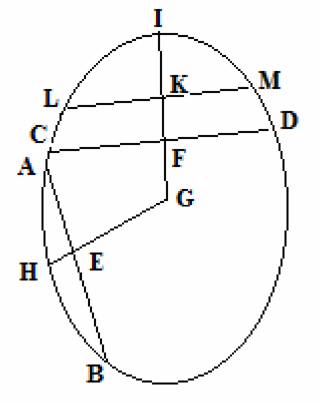
In ellipsi data sit quaecunque diametrorum coniugatio AB, CB , & iungatur AC, cui parallela sit quaevis ED, ex punctis autem D, & E ducantur ordinatim ad diametros DF, EH, DG, El, erunt igitur figurae DFBG, EIBH parallelogramma. Dico parallelogramma illa aequalia esse.

Demonstratio.

Quia per praecedentem BA, BC proportionaliter sunt divisae, erit AB ad BF, ut CB ad BH: & permutando ut AB ad CB, ut BF ad BH; similiter AB per praecedentem est ad BI ut CB ad BG, & permutando ut AB ad CB, sic BI ad BG; ergo BF est ad BH, ut BI ad BG. Ergo [§.14. ch.6], parallelogramma AG, HI aequalia sunt. Quod erat demonstrandum.

PROPOSITION LVI.

Sint AB, BC diametri coniugate; iunctis punctis AC, ducatur ED parallela rectae AC, tum ex D & E rectae ponantur EI, DF, DG, EH ordinatim ad diametros AB, CB, iunganturque EB, DB. Dico EBD sectorem, aequari figurae EIFDKE.



Demonstratio.

Per praecedentem parallelogramma FG, HI: & illorum dimidia, triangula DFB, EIB inter se sunt aequalia; ablato igitur communi LIB, erit ELB triangulum aequale trapezio DFIL; quare addica figurae communi ELDKE, sector EBD aequalis est figurae EIFDKE. Quod erat demonstrandum.

PROPOSITION LVII.

Secant ABC ellipsim duae quaevis lineae AB, CD, oportet CD lineae parallelam ducere, quae segmentum auferat aequale segmemto AHB.

Constructio & demonstratio.

Divisis AB, CD bifariam in E & F, agantur ex G centro per E & F diametri GH, GI; tum GI dividatur in K, sicut HG divisa est in E, ponaturque per K ordinatim LM, patet per 53.huius LIM, AHB segmenta esse aequalia ; est autem LM, parallela datae CD, igitur dato in ellipsi segmento, &c. Quod erat faciendum.

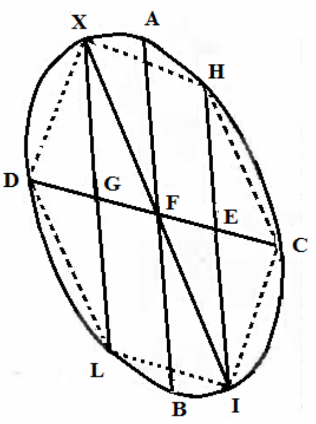
PROPOSITION LVIII.

Secent ABC ellipsim coniugatae duae diametri AB, CD; divisaque CD illarum altera quadrifariam in EG, agantur per E & G rectae HI, KL aequidistantes diametri AB, iunganturque puncta D, K, H, C, I, L, D.

Dico LD, DK, KH, HC, CI, IL lineas segmenta auferre aequalia.

Demonstratio.

Quoniam KL, HI, sunt parallelae rectae AB, quae est diameter coniugata ipsi DC, erunt KL, Hl ordinatim positae ad DC, ergo ut rectangulum DGC, ad rectangulum DEC, ita quadratum KG, ad quadratum HE, adeoque cum rectangula sint aequalia, etiam quadrata erunt aequalia, unde & rectae KG, HE aequales sunt ; sunt vero & GF, EF aequales & anguli KGF, FEI (quod Hl, KL sint parallelae) aequales sunt; igitur KGF triangulum aequale est triangulo FEI, angulusque IFE aequalis angulo GFK, & quia GFE linea recta



est, anguli IFE, KFG ad verticem constituri sunt aequales: unde KFI, puncta sunt in directum, adeoque; sectores KFD, CFI, sunt ad verticem constuti: quia autem ostendi triangulum KFG, aequari triangulo IFE, & simili discursu ostendi possit triangulum quoque ; DKG aequari triangulo ICE, erit triangulum totum DKF, aequale toti triangulo ICF. Atqui & sectores DFK, IFC ad verticem positi sunt aequales. Igitur reliqua etiam segmenta DK, IC inter se aequalia erunt ; eodem modo ostenduntur DL, HC segmenta aequalia. Iterum cum duo latera KG, GF sint duobus lateribus DG, GL aequalia, & anguli lateribus aequalibus contenti ad verticae aequalibus contenti ad vertice aequales ; erunt triangula GKF, DGLinter se aequalia, & angulus GKF aequalis angulo altero GLD: adeoque KFI, DLlineae parallelae; quare DK , LI segmenta sunt inter se aequalia; est vero iam ostensum segmenta quoque CI, DK aequalia esse, aequantur igitur tria segmenta DK, LI, CI. ulterius quia DC, LI iungunt aequales & parallelas GL, FEI, ipsae etiam erunt parallelae. Unde rursum segmenta CI, DL aequalia sunt aequantur igitur quatuor segmenta DL, CI, LI, DK. Rursum quia KH, LI, iungunt KL, HI aequales & parallelas, sunt ipsae etiam parallelae: erunt ergo aequalia etiam segmenta KH,Ll. aequantur igitur quinqae segmenta KH, LI, KD, DL, CI. Atqui etiam ostensum est aequalia esse segmentas HC,DL; aequantur igitur omnia sex segmenta. Quod fuerat demonstrandum.

Vocetur autem figura DKHCIL, ellipsi inscripta, polygonum regulare.

PROPOSITION LIX.

Eadem manente figura propositum sit ellipsi hexagonum regulare inscribere.

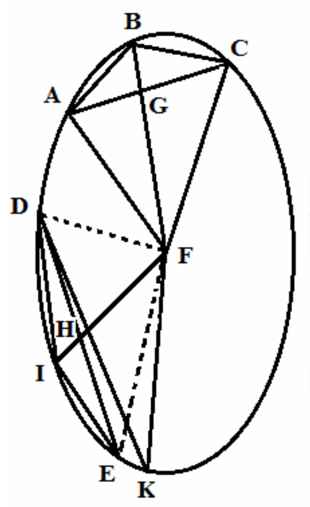
Constructio & demonstratio.

Sumantur duae quaevis diametri coniugatae AB, CD, divisaque CD quadrifatiam in E & G, agantur per E & G lineae HI, KL aequidistances A B: ducanturque DK, KH, HC, CI, IL, LD: patet per praecedentem: rectas illas segmenta auferre aequalia, adeoque figuram hexagonam DK, HC, ILD, esse regularem; igitur ellipsi hexagonum inscripsimus regulare. Quod erat faciendum.

PROPOSITION LX.

Secent ABC ellipsim duae quaevis lineas AC, DE auferentes segmenta aequalia, ducanturque ad illarum extremitates semidiametri FA, FC, FD, FE.
Dico sectores AFC, DFE, esse aequales, & si sectores fuerint aequales, dico segmentae esse aequalia.

Demonstratio.



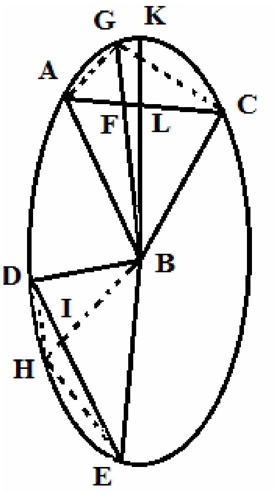
Dividis AC, DE bifariam in G & H agantur per G & H, diametri FB, FI, iunganturque AB, BC, DI, EI; quoniam segmenta AC, DE ex hypothesi sunt aequalia, ergo rectae, quae puncta C & E, A & D iungerent , forent parallelae; ergo triangula segmentis AC, DE inscriptorum maxima, sunt aequalia, sed ABC, DIE, sunt inscriptorum maxia, erunt igitur triangula ABC, DIE, aequalia : quia autem FB, FI diametri in G & H, sunt proportionalite divisae : igitur ut ABC triangulum est ad triangulum AFC, sic DIE triangulum est ad triangulum DFE, & permutando ut ABC triangulum est ad triangulum DIE, sic AFC triangulum est ad triangulum DFE. Cum igitur triangula ABC, DIE aequalis sint, etiam triangula AFC, DFE aequalia erunt. Quare additis aequalibus segmentis AC, DE , erunt sectores FAC, FDE aequales.
Sint iam sectores AFC, DFE aequales, iunganturque AC, DE. Dico ABC, DIE segmenta quoque esse aequalia; sin vero : sit alterutrum (puta) DIE minus altero, ducaturque ex D linea DK, segmentum auferens aequale est segmento ABC, sector DFK, aequalis est sectori AFC, id est sectori DFE quod fieri non potest; segmenta ergo AC, DE non inaequalia sunt, sed aequalia. Quod erat demonstrandum.

Corollary.

Ex hac & ex 53 huius sequitur primo, si AFC, DFE sectores quivis fuerint aequales, eorumque subtensae in G & H divisae bifariam, quod diametri per G & H ductae, iiisdem punctis proportionalite dividantur.
Secundo si sectores duo AFC, DFE fuerint aequales, subtendanturque ipsorum anguli rectis AC, DE: quod triangula AFC , DER sint aequalia.
Tertio hinc tale problema solvitur , dato AFB sectore quocunque, oportet ex F duas educere semidiametros quae sectorem constituant aequalem sectori AFB: pro constructione iuncta AB, ducatur ex F semidiameter quae cunque FD: tum ex D recta ducatur DI segmentum auferens aequale segmento AB iungaturque IF, patet IFD sectorem aequari sectori AFB.

PROPOSITlO LXI.

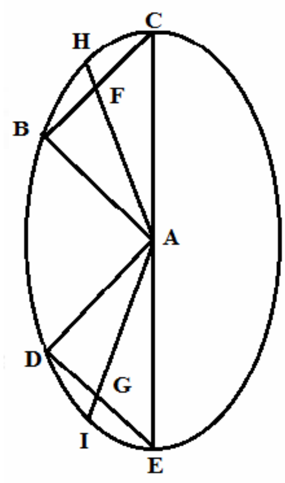
Sint ABC, DBE sectores aequales : iunctisque AC, DE ducatur ex B quaevis linea BG fecans AC lineam in F: dein sectori ABG fiat aequalis sector DBH secetque; HB linea, rectam ED in I
Dico tam AC, DE lineas quam BG, HB in F & I proportionaliter esse divisas.



Demonstratio.

Iungantur AG, GC, DH, HE. Quoniam ex hypothesi sectores tam ABG, DBH, quam ABC, DBE aequales, erunt & reliqui GBC, HBE aequales; quare AGB triangulum aequale triangulo DHB, & triangulum CGB aequale triangulo HEB, quare ut triangulum BAG ad triangulum GCB, sic BDH triangulum ad triangulum BHE, sed (quod facile ext.6. est demonstratu) rationes rectarum AF, FC, & DI, IE, eadem sunt cum rationibus triangulorum BAG, GCB, & BDH, BHE. Ergo etiam, AF est ad FC, ut DI ad IE. Deinde cum trapezia GABC, BDHE aequalia sint (est enim triangulum BAG, triangulo BDH, & triangalum BGC, triangulo BHE, aequale) sit autem & ACB aequale triangulo DEB, erit reliquum triangulum AGC aequale reliquo DHE: igitur ut AGC triangulum ad triangulum ACE, id est ut GF ad FB sic DHE triangulum ad triangulum DEB, id est HI ad IB. Quod erat demonstrandum.

PROPOSITION LXII.

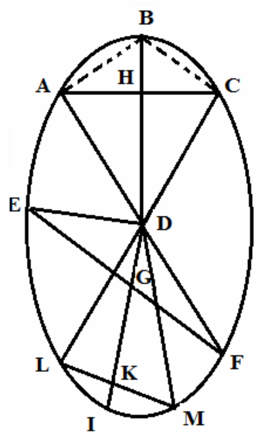


Sint iam AC, DE lineae in F & I proportionaliter divisae: aganturque per F & I semidiametri BG, BH.
Dico ABG, DBH sectores esse aequales.

Demonstratio.

Sin vero: sit alteruter ut ABG minor altero : fiat ABK sector aequalis sectori DBH, secetque BK linea rectam AC in L. quoniam ABK, DBH sectores sunt aequales: erit per primam partem huius AL ad LC, ut DI ad IE. Atqui etiam AF est ad FC, ut DI ad IE, igitur ut AF ad FC, sic AL ad LC, quod fieri non potest : quia punctum L cadit ultra aut citra F; quare sector ABK non est aequalis sectori DBH nec alius quisquam praetcr ABG. Quod erat demonstrandum.

PROPOSITION LXIII.



Sint duae quaevis diametri AB, AC, iunctisque illarum extremitatibus, ducantur duae quaevis aliae diametri AD, AE sic ut iuncta DE, sint ABC, ADE triangula aequalia : diuisis autem BC, DE bifariam in F & G, agantur per F & G, diametri AH, AI: quae si in F & G, proportionaliter sint divisae.
Dico BAC, DAE sectores esse aequales.

Demonstratio.

Quoniam AH, AI diametri in F & G proportionaliter sunt divisae, erunt BHC, DAE segmenta aequalia: sunt autem ex hypothesi triangula ABC, ADE aequala; sectores igitur BAC, DAE sunt inter se aequales.

Quod erat demonstrandum.

PROPOSITION LXIV.

Sint duo sectores DABC, DEIF, ductisque rectis AC, LM, & bisectis in H & G, ducantur diametri DHB, DGI. Sit auten ratio DG ad DI minor ratione DH ad DB.
Dico sectorem DEIF maiorem esse sectore DABC.

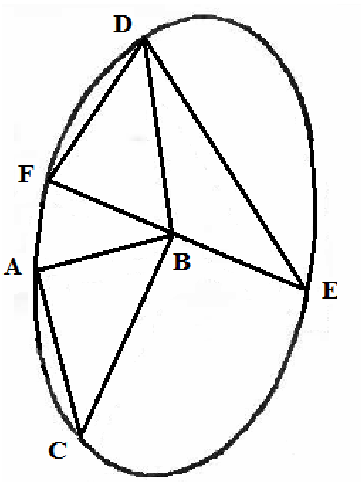
Constructio & demonstratio.

Quoniam DG est ad DI in minori proportione quam DH ad DB. Fiat DK ad DI, ut DH ad DB, maior igitur erit DK quam DG, & punctum K cadet inter G ac I, per K ducatur ordinatim LKM; iunganturque DL, DM; segmenta igitur LIM, ABC aequalia sunt. Quare sectores etiam DLIM, DABC sunt aequales, et proinde sector DEIF maior est sectore DAVC. Quod erat demonstrandum.

PROPOSITION LXV.

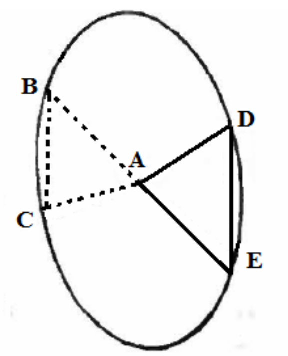
Sint ABC, DBE sectores duo inaequales, sic ut ABC, DBE triangula sint aequalia:
Dico ABC, DBE sectores simul sumptos aequari semiellipsi.

Demonstratio.



Cum sectores sint ex hypothesi inaequales, sit maior BDGE, quia ergo triangulum DBE, ex hypothesi aequatur triangulo BAC, erit segmentum DGE maius segmento AC, abscindatur itaque EG segmentum ipsi segmentum AC aequalc, iunganturque BG, GD, & EB producta in F ducatur FD: quia igitur segmenta GE, AC aequalia sunt, etiam sectores aequales sunt , adeoque & triangulum GEB, triangula ACB, hoc est triangulo BDE aequale erit ;quare BE, DG sunt parallelae adeoque segmentum GE, hoc est segmentum AC aequale est segmento DF; sectores itaque BAC, BFD aequantur. Atqui sectores BFD, BDGE, constituunte semiellipsim, ergo & sectores BCA, BDGE, semiellipsim constituunt. Quod erat demonstrandum.

PROPOSITION LXVI.



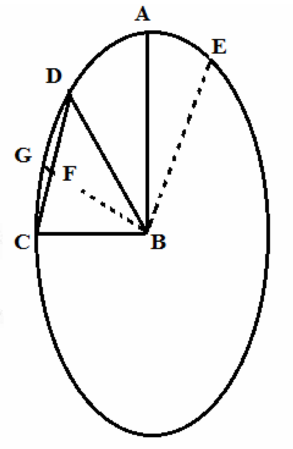
Sumantur sectores duo BAC, DAE, sic ut triangula ABC, ADE sint aequalia: si sectores illi simul sumpti, maiores fuerint vel minores semiellipsi:
Dico illos inter se aequales esse.

Demonstratio.

Si enim non sint aequales, sit BAC minor sectore DAE: cum igitur triangula ABC, DAE sunt aequalia, erunt BAC, DAE sectores simul sumpti aequales semiellipsi : Quod est contra hypothesim.
Igitur sectores BAC, DAE non sunt inaequales sed aequales. Quod erat demonstrandum.

PROPOSITION LXVII.

Sint AB, BC diametri coniugatae, ductaque ex C recta quavis CD , ducatur ex B linea BE parallela rectae CD; iungaturque DB.

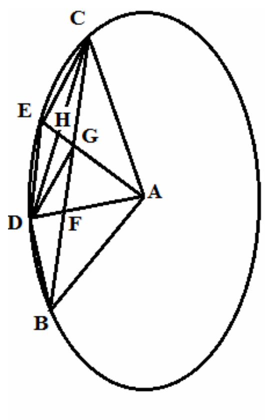


Dico DBC sectorem duplum esse sectoris ABE.

Demonstratio.

Divisa CD bifariam in F, ducatur diameter BG, & quoniam BE aequidistat ordinatim positae CD ad diametrum BG, ipsa etiam est posita ordinatim & quidem per centrum, sunt igitur coniugatae diametri BG, BE. sunt autem ex hypothesi CB, BA etiam coniugatae. Ergo sector CBA par est sectori GBE, demptoque communi GBA, sectores CBG, ABE aequales erunt. Atqui sector CBD duplus est sectoris CBG. Ergo CBD duplus quoque est sectoris ABE. Quod erat demonstrandum.

PROPOSITION XVIII



Si sectores ABD, ADE, AEC sint aequales, ducanturque rectae DE & BC, secans rectas AD, AE in F & G.
Dico BF, DE, EC esse aequales.

Demonstratio.

Ducantur rectae BD , EC, DC, DG. Quia sectores ADE , AEC sunt aequales, etiam triangula CHA, DHA aequalia sunt; ergo aequales sunt rectae DH, HC. Deinde quia sectores ADB, ACE aequales sunt,etiam segmenta BD, EC sunt aequalia. Ergo CB, ED parallelae. Anguli igitur GCH, EDH, aequantur : sunt vero anguli quoque GHC, DHE aequales, & iam ostendi aequales etiam esse rectas DH, HC; igitur CG, ED. aequales sint. Similiter ostendemus BF, DE aequales esse. Constat ergo veritas propositonis.

PROPOSITION LXIX.

Iisdem positis producatur DG, donec AG lineae occurat in I.
Dico AGI, AGC, AEC triangula in continua esse analogia.

Demonstratio.

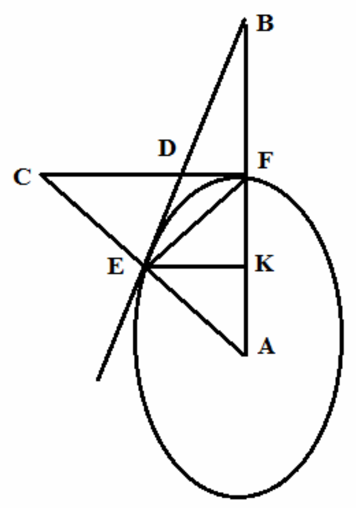
Ex superiori demonstratione, in triangulis CHG, EHD, patet omnia esse aequalia, adeoque latera etiam EH, HG tunc aequalia considerentur modo triangula DHG, BHC, in quibus cum duo latera DH, HE, duobus lateribus CH, HE aequalia sint, angulusque DHG aequalis angulo CHE, ad basim etiam aequales erunt anguli HGD, CEH, ac proinde DI, CE sunt parallelae adeoque ut AI ad AC, sic AG ad AE; sed est ut AI ad AC, sic AGI triangulum ad triangulum AGC: & ut AG ad AE sic AGC triangulum ad triangulum AEC: igitur ut AGI triangulum ad triangulum AGC: sic AGC triangulum ad triangulum AEC.

Quod erat demonstrandum.

PROPOSITION LXX.

Ellipsim secent binae diametri AE, AF in punctis E & F, ex quibus ducantur FC, EB tangentes ellipsim, occurrentes diametris in C & B, iunganturque; EF.

Dico triangulum ACF triangulo ABE aequale essse.



Demonstratio.

Ex puncto E ad AF duc ordinatim EK, erit haec tangenti F parallela, ideoque triangula AEK, ACF similia sunt, ac proinde duplicatam habent rationem rationis AK ad AF :hoc est quia AK, AF, AB sunt tres continuae proportionales, rationem habent quam AK ad AB, sed etiam est ut AK ad AB , sic triangulum AEK ad triangulum AEB: ergo triangulum AEK ad triangula ACF, AEB eandem habet rationem; aequantur igitur triangula ACF, AEB: Quod erat demonstrandum.