QUADRATURE OF THE CIRCLE

BOOK IV : THE ELLIPSE: Part I.

ARGUMENT

It has been a pleasing task to divide the subject matter into six parts, and therein the distinctive properties of the ellipse and their natures are to be proposed methodically.

Section one: Indeed initially, the essential division of these parts has arisen from the section of the cone, and for the necessary remaining fundamental qualities to arise henceforth.

Section two: Here the ellipse is divided in some manner, and a comparison of the sectors and segments of these ellipses is undertaken.

Section three: The consideration is to be undertaken here of both equal, as well as unequal axes and diameters. And indeed in the first place the powers of these to be considered : these being the lines which connect the ends of the diameters together.

Section four: The poles and the understanding of these to be designated by the shortest distance from a given point on the axis to a line through the periphery.

Section five: various kinds of ellipse may arise, such as from lines, as well as from circles, or even from that contained by an ellipse itself.

Section six: The ellipse may be compared with the circle, in which an order may be maintained here also, as in the first place of the proportions and powers of lines, secondly segments & the sections of these, then the figures from each; to be brought together and inscribed amongst themselves.

Several of the remaining propositions of this book, and of the two following books, are those of Apollonius, but demonstrated in another longer way by me, with a few exceptions, which nevertheless may be seen to be similar to the others in this work, which the more studious reader of geometry may not desire pertaining to the teaching of conics. All the other propositions, and which in particular constitute by far the greater part of the work, have been found and demonstrated by us. Whereby if anyone indeed may find certain theorems in the more recent books of geometry which may agree with us, I shall wish it to be understood that the authors of these books shall have emerged into the light, and have now been published for many years before being discovered by me. With which few matters, I have wished to inform my reader, not that I may be detracted on account of anything found, but rather that I may remove the suspicion of plagiarism from myself.

THE ELLIPSE.

DEFINITIONS.

I.

A diameter of an ellipse is a right line drawn within the ellipse, which bisects all lines drawn parallel to a certain line within the same, and if indeed it may cut these at right angles, it shall be called an axis: moreover there shall be two axes in any ellipse, and indeed with these taken together (which are called the extreme diameters), i.e. bisecting each other mutually at right angles, which will become apparent from their position.

II.

Symmetry is said to apply to the diameter associated with each of the sets of parallel lines, and to the bisection of the same by division.

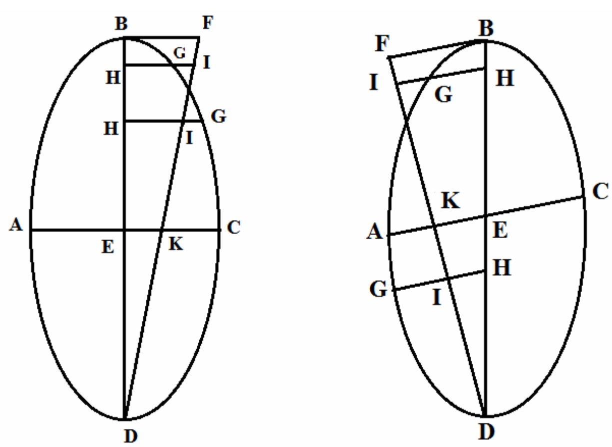
III.

The centre of an ellipse is the point which bisects a diameter.
Moreover, we will show in Prop. 7 of this book, that all lines drawn in the ellipse through the centre will be bisected.

IV.

Diameters which mutually bisect each other are said to be conjugate.

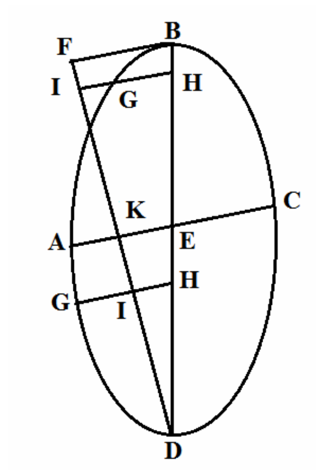
V.



I call the latus rectum such a line, that it is able to put the applied lines on the diameter in order ; or as if the latus rectum acts as a measure of the powers of the ordered lines put in place coupled together on the diameter. [The latus rectum becomes identified as the ordinate of a special point on a diameter, or along an axis ; the forerunner of *x* and *y* coordinates.]

The reasoning may be made clearer by an example: the diameter of the ellipse ABC shall be BD, of which FB will represent the latus rectum : and with FD joined, some point H may be assumed on the diameter, and HG shall be normal to the diameter BD, crossing FD at I : therefore the individual squares of the normal lines HG taken in order along the axis will be equal to the individual rectangles BHI, (as we will demonstrate in Prop. 2 of this section) which shall be smaller than the rectangle FBH, but similar to the rectangle FBD itself.

And then besides the latus rectum of Apollonius, at this point others are required to be set out to follow that special one. Truly for me it is seen to be necessary at least that the latus rectum may be applied at right angles to the diameter : [and for other angles to a diameter] and in place of the squares and rectangles, rhombi and rhomboidal figures to be compared with each other. And thus to the old kinds of latus rectum accepted, I add another new line of this kind which shall be required for the diameter of ellipses, whatever the ordering of the right lines GH put in place shall become: and indeed the latus rectum BF may be put parallel to the ordinates of the applied lines : the ordering of the individual rhombuses HG put in place at the angles IHB will be equal to the angles of the individual rhombuses IHB, and for a certain latus rectum FB, with the ordered applied lines put parallel: the order of the individual positions of the rhomboids with the same angles, which are diminished from the rhomboid FBH by the similar rhomboids FBD. See Prop. 12 of this book for the demonstration.



Again it has been found there from the deliberations by the ancients, that some of the certainty will have to be noted, by which the properties of the remaining sections may be understood properly and themselves observed may be able to be rendered more clearly: as in the individual conic sections these plainly are diverse, and thus different kinds of latus rectum will receive attention in the different cases ; and the rectangles with right sides and with the parts between the vertex of the same and the points by which the parts of the diameters between the vertex of the same and where the intercepts are cut held different for a long time in the individual cases, have a proportion to the ordered squares of the positions; indeed with the ellipse these squares become smaller to these similar figures which the right and transverse lines contain from the preceding rectangle ; in the parabola with these the same, are equal ; in the hyperbola truly they exceed with a similar figure to these, &c. From which the individual figures sort out their nomenclature.

Other powers of the ordinate positions for different diameters are different also, thus for the individual diameters, they are to be designated by a single and proper latus rectum: which in all respects, as with the right sides found, you will find demonstrated in its own place.

VI.

The figure is a rectangle because it is contained by a right side and by a transverse side (that is by the diameter, for that also is accustomed to be called transverse).

VII.

Poles or elliptic foci, are the points (which Apollonius thus calls from the comparison made) in which the axis shows division of the rectangle under the axis with segments held equal to the fourth part of the figure: by which they are enacted according to their location.

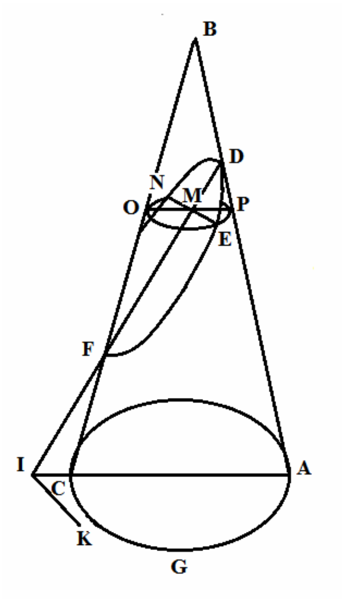
VIII.

The section below opposite is when the cone is cut by a plane through the axis with a triangle being producing , again otherwise it may be cut by a plane, which indeed may mark out a similar triangle (to the triangle produced), but thus put in place so that the angles which are in each triangle are equal, but the sides shall be different.

THE ELLIPSE

FIRST PART

The section to be drawn from a cone, and initially the essential properties of the same is shown.



PROPOSITION I.

The right cone AGCB shall be cut by a plane through the axis producing the triangle ABC. Then it shall be cut by another plane not parallel to the base of the cone AGC, meeting each side of the triangle at D and F: from which (the section produced) shall be the figure DEFN in the cone, but the common section of that cutting plane with triangle ABC shall be the line DFI; truly the common section of the same plane with the plane in which the base of the cone is AGC, shall be the right line IK, as it will be required to be perpendicular to the diameter AC of the base of the cone, or it may be put in place at right angles to the direction which is maintained by the diameter AC .

I say the figure DEFN not to be a circle.

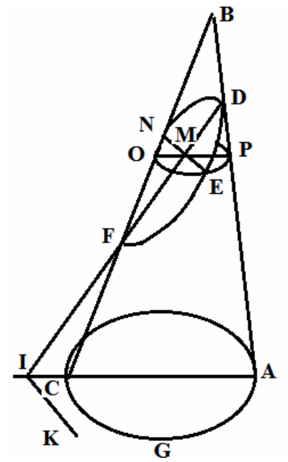
Demonstration.

Through some point M of the right line DF, NE is drawn parallel to IK, in plane of the figure DEFN: and through that same point M, the right line OP is drawn in the plane of the triangle ABC parallel to the right line ACI, but a plane is acting through the lines NE, OP. This will be parallel to the base AGC [§15.*book 11*], and thence the circle OEPN will be produced [§16.*prolog.*], of which the diameter will be OP.

Therefore since OP is parallel to AC, the triangles BOP , BCA are similar, but BCA is isosceles, and therefore BOP is isosceles. Therefore [§34.*lines*] the rectangle FMD is larger than the rectangle OMP ; but the rectangle OMP [§35.*book 3*] is equal to the rectangle NME. Therefore the rectangle FMD is greater than the rectangle NME; therefore it is apparent from §35.*book 3* the figure DEFN not to be a circle. Q.e.d.

PROPOSITION II.

Now a scalene cone ABC shall be given, and the cutting plane which produces the figure DEFN in the cone, and neither shall the figure be parallel to the base of the cone AGC, nor arranged below the axis NE in the opposite manner. Truly all the rest may be put in place and shall become the same as in the first proposition.



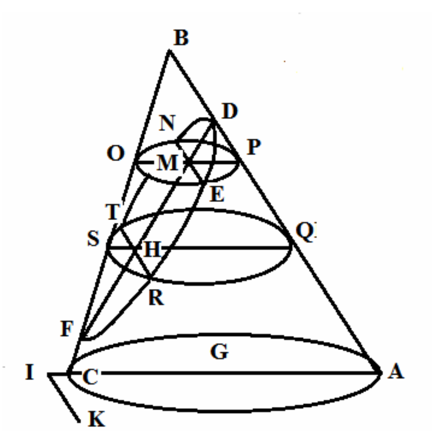
Again I say the figure DEFN not to be a circle.

Demonstration.

Since OP is parallel to AC and that shall cut FD in M, but not contrariwise below, that is the angle BFD not being put in place equal to the angle BAC, it is apparent from § 36 of our first book that the rectangle FMD to be unequal to the rectangle OMP; but the rectangle OMP is equal to the rectangle NME. Therefore the rectangle FMD also is unequal to the rectangle NME, therefore it is clear from §35 of our third book that the figure DEFN not to be a circle. Q.e.d.

PROPOSITION III.

Some cone shall be given, either right or scalene, and the remaining items may be put place and they will become the same as above:



I say the right line NE to be cut into equal parts by the right line DF at M.

Demonstration.

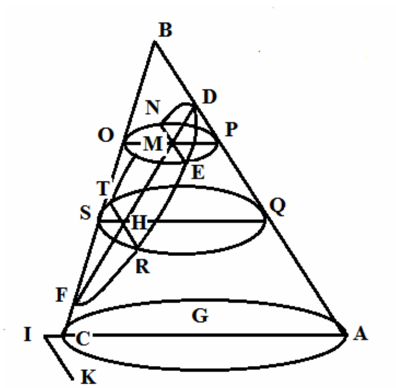
From the hypothesis the right line PM is parallel to the right line AC, and ME parallel to IK. Whereby PM and EM define equal angles [with AI and KI] ; and the angle AIK from the hypothesis is right, for the common section IK were placed perpendicular to ACI, therefore from the first proposition also PME is right. And thus since the section ONPE shall be a circle [§.26 *prolog*.], and its diameter OP, it is evident EMN [§.3], to which it is normal, to be bisected at M by the diameter of the circle OP. Moreover, from the hypothesis, the point M is common to the three right lines OP, NE, DF. Therefore NE to be bisected at M by DF. Q.e.d.

Corollary.

Hence it is apparent, if some number of right lines may be drawn parallel to IK or NE, all are to be bisected by DF: indeed the same is demonstrated in all cases. From which it shall be evident further of the section DEFN ; (which henceforth we will call an ellipse) the diameter to be the line DF [*Def*.1], the right line truly NE [*Def*.2], and the remaining lines parallel to this to be the ordered applied lines for the diameter DF.

PROPOSITION IV.

With the same in place, some line RT of the ellipse DEFN may be drawn, parallel to EN or IK, cutting the diameter DF at the point H.
I say, the rectangle DMF to be to the rectangle DHF, as the square EM to the square RH.

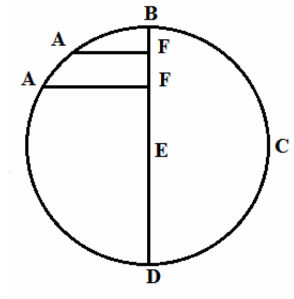


Demonstration.

The right line QHS may be drawn through the point H parallel to the right line OP meeting with the sides of the triangle ABC at Q and S. Then a plane is acting through the right lines QS, TR , this will be parallel to the base AGC and hence the section will produce the circle QRS; now truly the ratio of the rectangle DMF to the rectangle DHF is composed from the ratio DM to DH, (that is, since PM, QH are parallel from the construction from the ratio PM to QH) and from the ratio MF to HF (that is, since MO, HS are parallel from the construction, from the ratio MO to HS.) And the ratio of the rectangle PMO to the rectangle QHS is composed also from the ratios PM to QH, and MO to HS. Therefore the rectangle DMF is to the DHF as the rectangle PMO to the rectangle QHS; that is since the sections PEO, ORS are circles, so that the rectangle EMN to the rectangle DHF shall be as the rectangle to the rectangle RHT, that is, since EN, RT are bisected by the diameter DF in M and H, as the square EM to the square RH. Q.e.d.

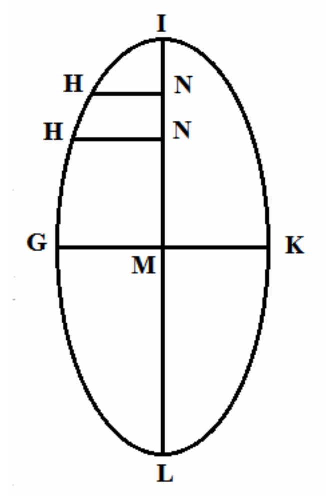
$$\left[\text{rect.DMF} : \text{rect.DHF} = \text{DM.MF} : \text{DH.HF} = (\text{DM} : \text{DH}) \times (\text{MF} : \text{HF}) = (\text{PM} : \text{QH}) \times (\text{MF} : \text{HF})\right.$$
$$= (\text{PM} : \text{QH}) \times (\text{MO} : \text{HS}); \text{ But rect.PMO} : \text{rect.QHS} = (\text{PM} : \text{QH}) \times (\text{MO} : \text{HS});$$
$$\left.\therefore \text{rect.DMF} : \text{rect.DHF} = \text{rect.PMO} : \text{rect.QHS} = \text{rect.EMN} : \text{rect.RHT} = \text{EM}^2 : \text{RH}^2 .\right]$$

Scholium.



We have shown by this proposition the proportion of the rectangles to the squares, which are established from the segments of the diameter of the ellipse, to be ordered according to the same diameter of the applied lines : which indeed is a primary essential property of the ellipse : truly thus since this belongs to the ellipse, so that also it may be found in its own way in the circle, I have thought it worthwhile for me, if briefly I may show the difference, between that and the ellipse in the nearby diagram.

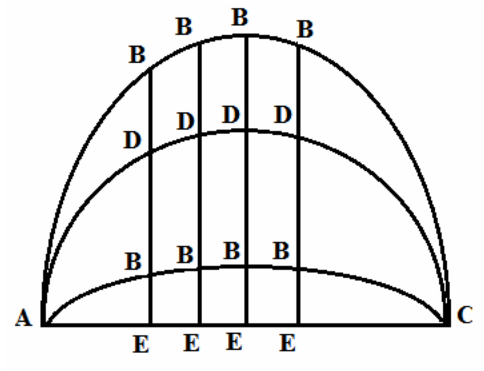
Let BD *be the diameter of the circle* ABC; *the centre* E*: and the normal* AF *to the diameter: moreover* IL *shall be some diameter of the ellipse* HIK *, that HN may cut in order: truly* M *shall be the centre of the section. Therefore since in the circle, the right lines* AF *are normal to the diameter* BD*, the rectangles* BFD *will be equal to the squares* AF*, and hence the square* AF *is to the square* AF *as the rectangle* BFD *to the rectangle* BFD*: in the same manner in the case of the ellipse since the right lines* HN *shall be placed in order to the diameter IL, the square HN will be to the square* HN*, as the rectangle* INL *to the rectangle* INL: *therefore that is agreed for each section; that there be a proportion between the squares put in order, which is of the rectangles, under the diameters of the segments to which they are put in order: truly they differ in this respect, since in the circle the proportions of the rectangles under the segments of the diameter shall be equal to the squares of the positions in order; truly of the inequalities of the ellipse (if the case of the equality of the diameters taken together may be excepted, concerning which several with their own position), which we make the first and second of this plane.*



From which it follows in the first place, in the axis of the ellipse, one axis is to be greater than the other, which can be shown thus : in the ellipse HIK there shall be some axis IL *which* HN *shall cut in order*: *the right line* GK *is acting through the centre* M *parallel to* HN *itself*: *I say these axes to be unequal: indeed so that as the rectangle* INL *to the rectangle* IML, *thus the square* HN *to the square* GM*, and on interchanging, as the rectangle* INL *to the square HN thus the rectangle* IML *to the square* GM; *but the rectangle* INL *is unequal to the square* HN ; *and therefore the rectangle* IML, (*that is the square* IM) *is unequal to the square* GM*: therefore the right line GM is unequal to the right line* IM*, therefore the whole* IL*, truly the axis, is unequal to the whole* GK, *that is, to the other axis. Which was proposed.*

Then if the axes shall be equal in a given ellipse, now the ellipse will not differ from a circle: therefore so that the rectangles under the axis of the segment shall be equal to the squares of the positions in order.

Secondly it follows, if above the axis AC of the ellipse ABC, the semicircle ADC may be described , and the lines BE may be drawn in order crossing the semicircle at D, so that BE shall be to BE, as DE is to DE; indeed it is the case both in the ellipse as well as in the semicircle, that the rectangle AEC to be to the rectangle AEC, thus as the square BE to the square BE, and the square DE to the square DE; and also as the square BE to the square BE thus the square DE to the square DE.

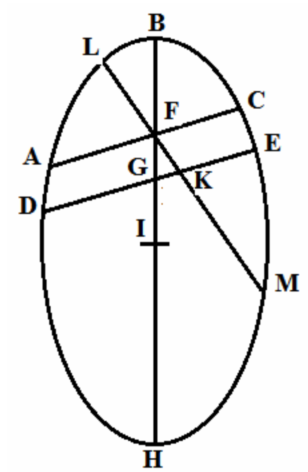


PROPOSITION V.

To find the diameter of a given ellipse.

Construction & demonstration.

The parallel lines AC, DE are drawn within the ellipse, which are bisected at the points F, G by the right line BH ; and the right line BH is drawn through F and G.
I say this to be the diameter.



The demonstration is clear, if indeed BH were not the diameter, then it shall become LFM, cutting DE in K. Therefore since LM is put to be the diameter, and from the parallel lines it shall bisect the one AC, at F, and it shall bisect the other DE in K. Which cannot happen since from the construction, DE shall be bisected at G . Therefore neither is LM nor any other line drawn through F to be the diameter besides that, which also passes through G, that is besides BH itself. Therefore we have found the diameter in the given ellipse. Q.e.f.

PROPOSITION VI.

To find the centre of a given ellipse.

Construction & demonstration.

To seek the diameter of the ellipse BH by the preceding, as bisected in L. From the third definition it is apparent I to be the centre.

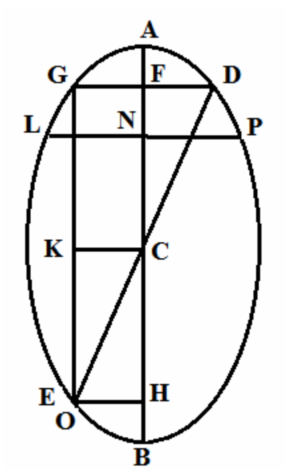
Corollary.

It is apparent from this proposition every diameter to pass through the centre. And from which you deduce the other easily, without doubt all the lines passing through the centre to be diameters.

PROPOSITION VII.

The ellipse ADB shall be given, of which the diameter shall be AB, truly the right line LP, shall be one of these, which we have demonstrated in proposition three of this section to be bisected by the diameter: C shall be the centre of the ellipse C.
I say all the lines drawn through the centre to be bisected in the centre.

Demonstration.



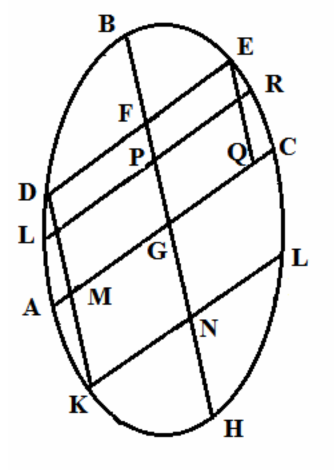
Indeed some right line DO may be drawn through the centre C; DFG shall be drawn from D parallel to LP, GE parallel to AB, & EH, CK parallel to GD, or to LP. Therefore since FGEH is a parallelogram, GF, EH will be equal : and hence the squares GF, EH are equal. And as the square GF is to the square EH, thus the rectangle AFB is to the rectangle AHB, therefore the rectangles AFB, AHB are equal, therefore as AF to AH, thus so BH to BF : therefore on dividing, thus AF to FH and BH to HF are equal ; therefore AF, BH are equal. Whereby since also the whole diameter AB shall be bisected at C, as is apparent from the definition of the centre, also the remainder FH is bisected at C. Therefore since KC is parallel to GD, EH, themselves to be parallel to LP itself, also the right line GE shall be bisected at K, and truly DG bisected at F, as itself being parallel to LP. Therefore there shall be as DG to GF, that is as DG to CK, thus GE to KE. Therefore the points DCE are on a line ; but also the points DCO are on a line, since from the hypothesis DCO shall be a right line. Therefore DCE & DCO are one and the same right line. And DCE is bisected at C, (since indeed from the construction: GE, FC shall be parallel, therefore as DF to FG, thus DC to CE.) Therefore also DCO is bisected at C. Q.e.d.

PROPOSITION VIII.

The ellipse ABCH shall be given, of which the diameter shall be BH; truly in order for the applied diameter to be LPR: the centre of the ellipse to become G. Moreover the right line AGC will be drawn through the centre G, parallel to the applied right lines in order.
I say BH, AC to be conjugate diameters.

Demonstration.

Some point M may be taken on AG, through which KD shall be drawn parallel to the diameter BH, meeting the ellipse at the points D and K ; from which there may be drawn DFE, KNL parallel to LR itself. Therefore since DK, NF is a parallelogram, thus the right lines DF, KN are equal, and the squares DF, KN also are equal. Whereby since the rectangle BFH shall be to the rectangle BNH as the square DF to the square KN, the rectangles BFH, BNH also are equal, and hence, as we have shown in the preceding, BF and NH are equal. Truly BG and HG are equal. And therefore the remaining FG, NG are equal, or FN is bisected at G. Therefore KD parallel to the diameter BH shall be bisected at M for AC. Similarly I may show any other parallel diameter to AC to be bisected by BH. Whereby since also BH bisects DE, LR and all the remaining which are put in order to BH, and from the hypothesis parallel to AC itself; it is clear from the definition BH, AC to be conjugate diameters .



Corollary.

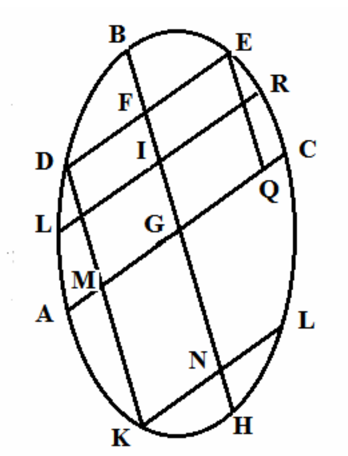
An axis given through the centre of the ellipse drawn perpendicular to a given axis is said to be the conjugate to this axis.
From the discussion now brought forwards it will be easy for the reader himself to elicit a demonstration of this.

PROPOSITION IX.

An ellipse shall be given and its diameter BH.
It shall be required to show the diameter conjugate to BH.

Construction and demonstration.

Some other right line KD may be drawn parallel to BH; DK, BH each bisected at M and G, and AC may be drawn through M and G .



I say AC, BH to be conjugate diameters.

And indeed in the first place it is apparent from Prop. 5 of this section that the right line AC to be a diameter, and BH to be a diameter from the hypothesis; therefore both are diameters. But which I shall now show to be conjugate. Because AC is a diameter and it bisects KD, KD to be applied as the ordinate to AC , that is, these parallel lines are going to be bisected by the diameter AC. Therefore the rest to be parallel to KD itself. But DK, from the construction with the lines parallel to it, is parallel to the diameter BH. Therefore the diameter AC shall bisect the lines parallel to the diameter BH. Thereupon the line DE may be drawn, parallel to the diameter AC and EQ, parallel to the right line DK; therefore MDEQ is a parallelogram, in which since FG is parallel to DM itself, EQ is drawn parallel, therefore DF to FE, as MG to GQ; also the sides of the parallelogram DM, EQ to be equal, and the squares DM, EQ will be equal. Now truly since DM, EQ, are placed in order to the diameter AC, the ratio between the rectangles AMC, AQC will be the same as between the squares DM; EQ, that is, of equality : and hence as is apparent from the demonstration in Prop. 7 of this section, AM, QC will be equal. Whereby since all the AG,CG shall be equal (indeed since G is the centre of the ellipse ; since it bisects the diameter BH) also the remaining MG, QG, are equal; therefore since there is MG to GQ, thus DF to FE, also DF, FE are equal, that is DE is bisected at F. Therefore from that it is defined, DE is put in order to the diameter BH, and therefore the remaining are themselves placed parallel to BH in order. And from the construction DE, with the other diameter AC parallel to itself. Therefore the diameter BH bisects the parallel diameter AC. Whereby since also I shall have shown earlier AC to bisect the parallels to BH, thus BH, AC to be conjugate diameters. Therefore what was demanded has been done.

First Corollary.

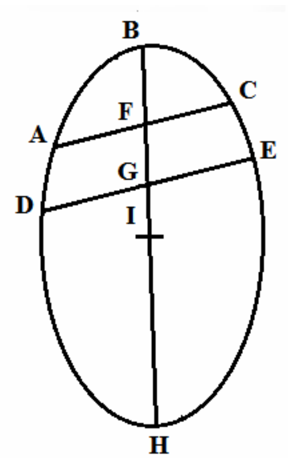
For a given axis of the ellipse, you will find the conjugate axis, if you will draw a right line through the centre of the ellipse perpendicular to the given axis, the matter to be evident from the corollary of Prop. eight.

Second Corollary.

From this problem it shall be evident how, from a given point D on the ellipse, a right line must be adjoined in order to the diameter BH. Indeed the diameter AC may be found conjugate to the diameter BH; and from the given point D, DFE may be drawn parallel to AC itself.

I say DFE may be placed in order to the diameter BH. The demonstration to be evident from the proposition.

PROPOSITION X.



Of the diameters placed in order (AC, DE, &c.) that is greater which is closer to the centre (I).

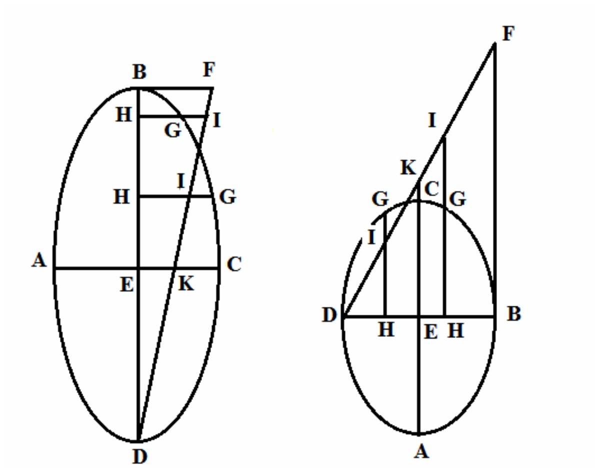
Demonstration.

The rectangle BGH is greater than the rectangle BFH as is apparent from the second part of Prop. five. And the square EG is to the square CF as the rectangle BGH to the rectangle BFH. Therefore the square EG is greater than the square CF. And therefore the ordinate for the position EG is greater than for the position CF. Q.e.d

PROPOSITION XI.

BD shall be the axis of the ellipse ABC ; it is required to show the latus rectum of this ellipse.

Construction & demonstration.



AC is connected to the axis BD through the centre E , and the continued proportions BD, AC, BF arise [coroll.9 of this section]*:* I say that FB to be the latus rectum ; indeed BF to be parallel to AC itself: and the ordinate FD of the line GH may be drawn which cross each other at I, truly FD will cut the line AC at K. Since EC, GH are put in place in order on the axis BD, as the square GH will be to the square EC, thus as the rectangle BHD shall be to the rectangle BED: but as the rectangle BHD to the rectangular BED, the rectangle IHB is to the rectangle KEB (because evidently they have been composed from the same ratio BH to BE, and from HD to ED, that is, HI to EK.) therefore as the square GH to the square CE, thus as the rectangle IHB is to the rectangle KEB, and by permutation and inverted so that as the rectangle KEB to the square CE, thus the rectangle IHB is to the square GH: but since the square AC shall be equal to the rectangle FBBD (since from the construction: BD, AC, BF are three lengths in continued proportion) EC will be a square equal to the rectangle KEB, (without doubt the fourth part of the square AC, is indeed the square of AC bisected at E) equal to the fourth part of the rectangle KEB on FBBD. And therefore the square HG is equal to the rectangle IHB: therefore HG can be the distance which may be added to the length FB having HB, being deficient by the rectangle FBH, similar to the figure from the rectangle BFD : whereby we have shown the latus rectum FB [*def*.6] is …etc. Q.e.f.

Corollary.

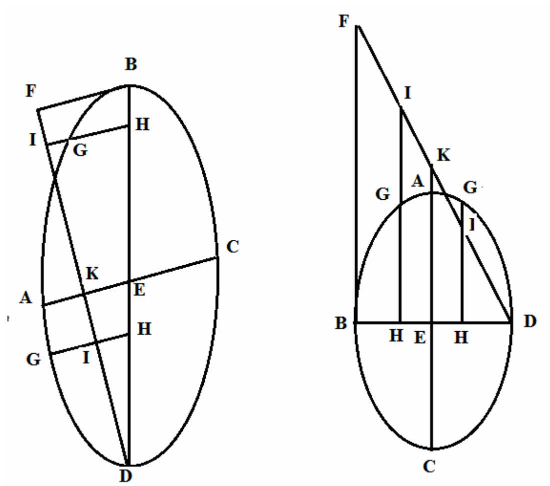
Hence it follows the first four lines, without doubt the latus rectum of the minor axis, the major axis, the minor axis, and the latus rectum of the major axis to be in continued proportion.

In the second place it follows, with the latus rectum both of the minor and major axis given, whereby an ellipse will be show, so that between the two lengths given, the two axes lengths will be found in the middle.

PROPOSITION XII.

Let BD be the diameter of some ellipse ABC, it will be required to show its latus rectum.

Construction & demonstration.

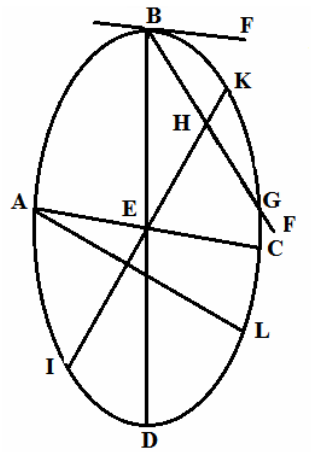


The diameter AC may be drawn through E conjugate to BD itself [§.9], and the continued proportions BD, AC, BF, and BF shall arise parallel to the diameter AC : and the line FD may be drawn, which will cut the right line AC at K, and the ordered lines GH may be drawn, which cross the line FD at I. Since BD, AC , FB are lines in continued proportion, the square AC shall be equal to the rectangle on FB, BD; and therefore the square AE (without doubt equal to the fourth part of the square AC, indeed AC bisected at E) equal to the rectangle on KE $[=\frac{1}{2}\cdot \mathrm{AE}]$, EB$[=\frac{1}{2}\cdot \mathrm{BD}]$, the fourth part of the rectangle FBD: whereby, as we have shown in the preceding, thus also we have shown the square HG to be equal to the rectangle IHHB, and for the rhombus IH at the angle IHB is equal to the rhomboidal IHB at the same angle IHB , as the square IH to the rectangle IHB, (for the ratios of the rhombus to the rhomboid and of the square to the rectangle are composed from the same ratios, truly from IH to IH and IH to HB,) therefore since the square IH shall be equal to the rectangle IHB, also the rhombus IH will be equal to the rhomboid IHB ; therefore the right line IH can be the rhomboid at the angle IHB of the applied ordinate, which (since it is shown easily) applied to the rhomboid FBH at the same angle, is smaller by a similar rhomboid to that which shall be at the same angle for the diameter DB and the right line BF. Therefore FB is the latus rectum. Which was sought.

PROPOSITION XIII.

Every right line (BF,) which is drawn through the end of the diameter (BD) parallel to the ordinate (AC) of the applied line, is a tangent to the ellipse.
And that which is drawn parallel to the tangent, is the ordinate [or the ordered line] for the applied diameter.

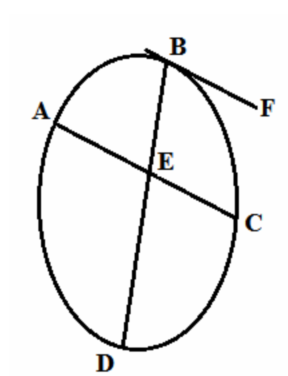
Demonstration.



Indeed if the right line BF may not be a tangent of the ellipse, it may cut that at G; and with BG bisected at H, KI passes through H and E, crossing both sides of the perimeter at I and K. Since, from the construction, BG is parallel to AC, moreover each will bisected by the right line IK, and IK will be a diameter and AC, BG the ordinates put in place for that; but, from the construction, the right line AC also cuts the diameter BD, therefore one and the same right line ordinate AC shall cut the two diameters BD, IK. Which cannot happen; otherwise indeed each diameter BD, KI also must be drawn parallel to AC itself, and hence may be bisected at two points, therefore it is clear the line FB, to be tangential to the cut: as it was in the first place. So that if some line AC may be drawn parallel to the tangent BF crossing the diameter at E, it will be an ordinate put in place to the diameter. If the ordinate AL may not be drawn from A : AL will be parallel to the tangent BF; and whereby it shall be parallel to AC itself, which cannot happen, since the same will be cut at A *:* therefore AL cannot be an ordinate put in place nor any other besides AC, which was the other. Therefore the truth of the proposition is evident.

PROPOSITION XIV.

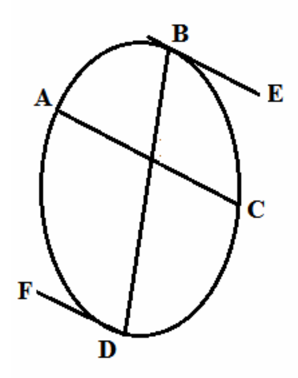
To draw a tangent through a given point on the periphery.



Construction & demonstration.

ABC shall be an ellipse and B a given point on the periphery, it is required to draw a right line through B which section shall be a tangent at B; to find the centre and through this draw the diameter BD, to which there may be put some ordinate line AC, to which there may act the parallel line BF through B, it is evident therefore BF to be the tangent; therefore through a given point on the periphery….etc. Q.e.f.

PROPOSITION XV.

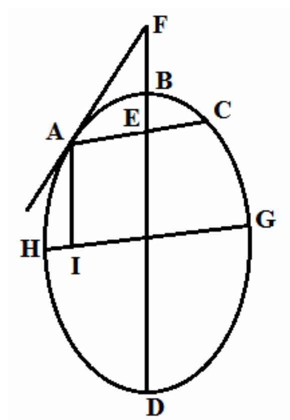


Draw lines through the ends of a diameter, parallel to each other, being tangents to the ellipse.

Demonstration.

Indeed it is evident from §.13 in this section, some ordinate AC drawn to the diameter BD, both the tangents BE as well as DF to be parallel to that ordinate, and thus to each other. Q.f.d.

PROPOSITION XVI.

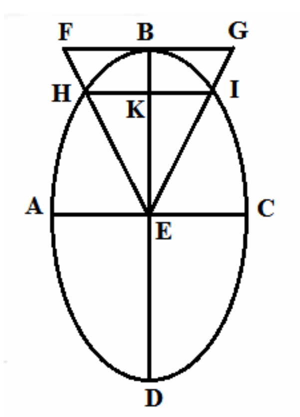


A tangent drawn through an ordinate end meets the diameter beyond the section.

Demonstration

BD shall be the diameter of the ellipse ABC, and the ordinate AEC put in place, and the tangent AF is acting through A; I say that tangent meets the diameter at F. For the diameter HG is found in conjunction with its diameter BD, from A the line AI may be sent parallel to BD, therefore since AI, BD are parallel, and AF shall meet the right line AI, it is evident DB produced also to meet with BD. Q.f.d.

PROPOSITION XVII.



And on taking the ellipse ABC, the axis of which shall be BD, the line FG shall be a tangent at B, with the equal parts FB, BG tangents together; I shall make the two diameters FE, GE to depart from F and G crossing the ellipse at H and I. I say the connected line HI to be parallel to FG.

Demonstration.

HK may be put parallel to FG, which produced crosses EG at I; and thus HK will be equal to KI. Therefore, since the rectangle BKD shall have the same ratio to BED, as the square HK to the square AE [§.4], there will be also the rectangle BKD to the rectangle BED, as the square IK to the square EC: from which the point I taken to be a point on the ellipse, and HI to intersect with the same point I of the perimeter BIC, and the right line EG intersects at the same point I. Q.f.d.

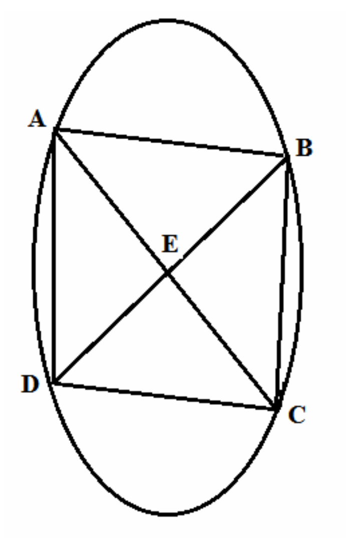
PROPOSITION XVIII.

With the same figure remaining, the axes of the ellipse ABC shall be AC, BD, and HE some diameter, it will be required to deduce the diameter from E towards C, to be equal to HE itself.

Construction & demonstration.

The angle BEH shall be made equal to the angle BEI, I say the right line be required to be satisfied, indeed the lines HE, EI produced cross the line of the tangent acting through B at F and G. The points H, I may be joined; since the angles BEH, BEI may be put equal, moreover the angles EBF, EBG to be right, and BE the common line, it is clear the triangles FBE, GBE, and thus the sides FB, BG to be equal to each other, and HI shall be parallel to FG, and thus as FE to GE, there shall be HE to IE, whereby HE, IE to be equal lines; therefore we may deduce the diameter from E….etc. Q.e.f.

PROPOSITION XIX.



The lines connecting the ends of any diameters in an ellipse are equal and parallel to each other.

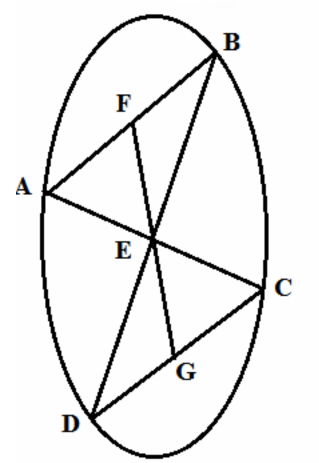
Demonstration.

Any two diameters AC, BD shall cut the ellipse ABC ; I say the joined lines AB, CD, likewise AD, BC, to be equal and parallel to each other: since DB, AC shall be bisected at E, as DE to EB, thus there shall be CE to AE , and on permutation there are, as DE to CE, thus BE to AE, and truly the angles E are equal, therefore the triangles DEC, AER are similar ; therefore as DE to EB, there shall be DC ad AB; whereby since DE: EB are equal, also AB, DC are equal. Similarly, we may show AD, BC to be equal. Q.e.d.

PROPOSITION XX.

Lines which are put parallel to the ends of a diameter, also are equal to each other.

Demonstration.



Some diameter BD shall cut the ellipse ABC, and the parallel lines AB, CD shall be drawn from B and D within the section. I say these are to be equal to each other. With the centre E found, and AB bisected at F, join FE, and produce to G, and since the diameter EF shall bisect AB, also it shall bisect DC, parallel to AB itself. Then since the triangles FEB, DEG are similar; DE will be to DG, as EB to BF*:* and by interchanging, as DE to EB there shall be DG to BF; but DE, EB are equal, and therefore are equal BF, DC, so that now shown to be half of these AB, DC. Therefore the whole lengths AB, DC are equal, Q.e.d.

Corollary.

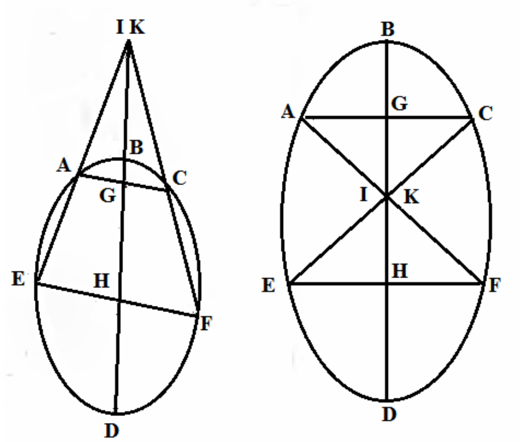
Hence if follows the joined lines AE, EC to be collinear: indeed since the sides AF, FE shall be equal to the two sides CG, GE, and the angles contained by equal sides are equal, it is clear AFE, CGE to be triangles equal to each other, and the angle AEF to be equal to the angle CEG, and thus AE, EG to be collinear.

PROPOSITION XXI.

The lines through the ends of two unequal parallel lines drawn in the ellipse, meet in the same point with the diameter, to which the ordinates put in place are parallel.

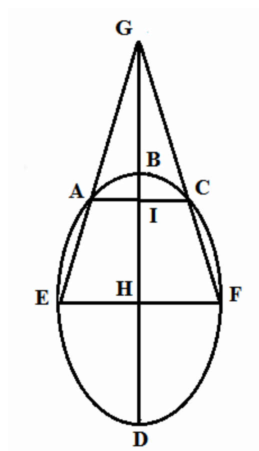
Demonstration.

Any two unequal parallel ordinate lines AC, EF placed in order for the diameter DB, cut the ellipse ECD. I say EA, FC come together and cut the diameter BD at the same ordinate point. Since the lines AC, EF are ordinates for the diameter BD, both ordinate lines are bisected at G and H, from which AG to GC shall be as EH to HF, and on interchanging as AG ad EH, thus GC to HR, now EA concurs with the diameter at Ii truly the other FC at K, therefore there will be as IG to IH, thus IA to IE, that is, as before, as AG to EH, that is as IG to IH, therefore the points I and K are the same; therefore the point I is common to the intersection of the lines EI, FI, HI. Q.e.d.



PROPOSITION XXII.

BD shall be the diameter of the ellipse ABC to which the ordinate EF shall be put in place, and from E and F lines may be drawn crossing with the diameter at the point G, truly with the ellipse at A and C.
I say the joined line AC, to be parallel to EF.

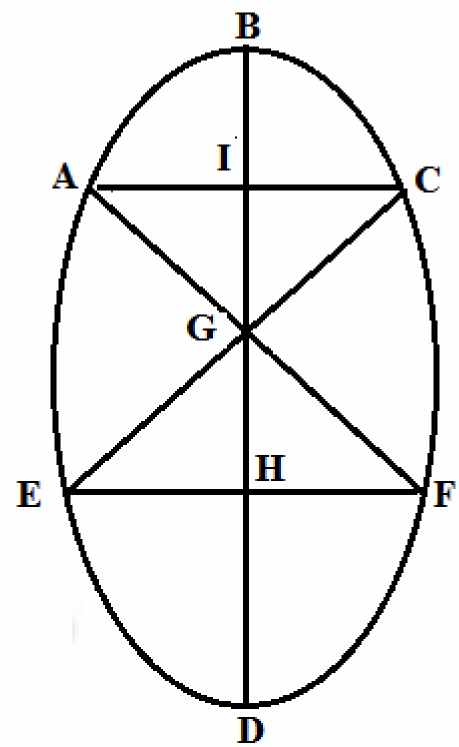


Demonstration.

AI may be drawn parallel to EF and produced shall cross the line FG at C, therefore since EH is equal to HF, and AI itself will be equal to IC; but since AI shall be parallel to EF, the rectangle BID will be to the rectangle BHD, as the square AI to the square EH. Also therefore, as the rectangle BID to the rectangle BHD, thus the square IC to the square HF; from which the point C is the common intersection of the right lines FG, AI with the perimeter BCF of the ellipse; and therefore AC joins the points A, C, parallel to EF. Q.e.d.

PROPOSITION XXIII.

The parallel lines AC, EF shall be drawn in the ellipse, through the ends of which EC, AF may be drawn through joined together at G and GIH may be drawn through G shall bisect AC at I.
I say the other line also to be bisected.

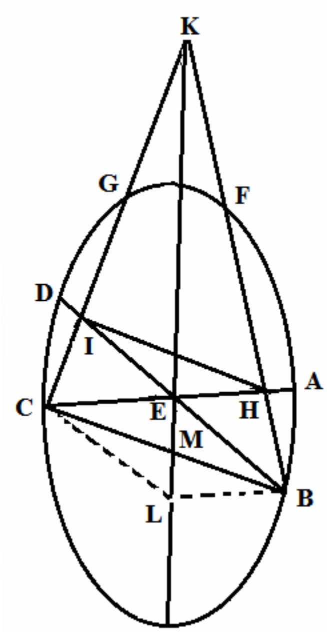


Demonstration.

As HG to IG, thus EH to AI, and as HG to IG, thus FH to CI; therefore EH to AI, as HF to IC; therefore on interchanging, EH to HF as AI to IC; but AI, IC are equal, and therefore EH, HF are equal, and thus both EF as well as AC are bisected; therefore GIH is a diameter est. Q.e.d.

PROPOSITION XXIV.

The two diameter AC, BD cut the ellipse ABC, and BC may be joined, the diameter KL shall act through the centre E, bisecting BC in M, and from B and C the right lines BF, CG, are drawn to the same point K of the diameter cutting the lines AC, BD at H and I.
I say the rectangle AHC to be to the rectangle DIB as the square AC to the square DB.



Demonstration.

The line CL may be put from C parallel to BD, crossing the diameter KL at L, and with BL joined, since CL shall be parallel to DE, EMB, CML will be triangles similar to each other : truly since CM, MB are equal, also the triangle CML, EMB will be equal and the side LM equal to the side EM ; therefore in the triangles BML, CME, the two sides CM, ME are equal to the two sides BM, ML, and moreover the angles BML, EMC contained by these are equal. Therefore the bases of the angles LBM, ECB are equal; therefore BL, CEA are parallel. Therefore BH to HK shall be as LE to EK, that is, (since from the construction

BI, CL are parallel), as CI to IK. Therefore IH shall be parallel to CB, and so that HE is to EC, thus as IE to ER, and on putting together and interchanging, as EC to EB, thus HC to BI, but as CE to BE, thus AC is to BD, since each shall be bisected in the centre; therefore as AC to BD, thus HC to BI. Therefore also as AC to DB, thus AH to dL. Whereby since the rectangle AHC shall have the ratio to the rectangle DIB composed from the sides by the ratios AH to DI, and HC to BI, which both are shown to be the same with the ratio AC to BD, the ratio of the rectangles will be double the ratio AC to BD, that is the same as the squares of AC, BD. Q.e.d.

PROPOSITION XXV.

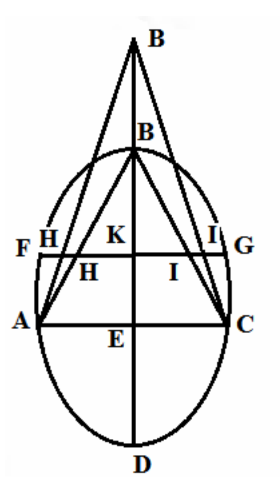
The two lines CG, BF drawn within the ellipse cross the diameter of the ellipse MK at the same point K. Then two other diameters BD, CA which thus will be cut by the right lines CG, BF so that the rectangles BID, CHA shall be proportional to the squares BD, AC .

I say the joined lines IH, CB to be parallel.

Demonstration.

Since it is, as the square BD to the square CA, that is, as the square ED to the square EA, thus the rectangle BID to the rectangle CHA ; there will become on interchanging, so that the square ED, (that is the rectangle BID with the square EI to the rectangle BID is to the square EI) to the rectangle BID, as the square EA, (that is the rectangle CHA with the square EH) to the rectangle CHA to the square EH. Therefore on interchanging, the rectangle BID is to the rectangle CHA as the square EI to the square EH, but also the rectangle BID is to the rectangle CHA as the square BD to the square CA; that is as the square ED to the square EA. And thus the square EI is to the square EH as the square EH is to the square EA: and thus the right line EI to the right line ED, that is EB, as the right line EH to the right line EA, that is EC, therefore IH, CB are parallel. Q.e.d.

PROPOSITIO XXVI.



Let BD be the diameter of the ellipse ABC, for which the position of the ordinate shall be the right line AC : and with the right lines drawn from A and C, which shall cut the diameter at the same point B, FG shall be drawn parallel to AC, crossing AB, CB at H and I, truly with the diameter BD at K.

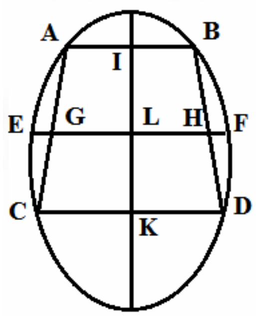
I say FH, GI to be equal lines.

Demonstration.

Since FG shall be an ordinate placed parallel to AC, and also the ordinate put in place for the diameter BD, and thus bisected at K ; but also HI is bisected at K , just as AC in E, therefore with the equalities HK, IK, removed, the remaining FH, IG are equal. Q.e.d.

PROPOSITION XXVII.

Any two parallels AB, CD drawn may cut the ellipse ABC, and with AC, BD joined, EF may be drawn parallel AB, cutting the lines AC, BD at G and H.
I say the right lines EG, FG to be equal.

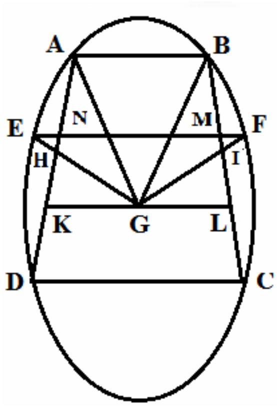


Demonstration.

The line IK is acting through the points I and K with AB, CD bisected at I and K; IK will be that diameter, and the line EF parallel to the line AB parallel will be bisected at L; moreover HG is bisected at L as is CD at K; or AB at I, therefore with the equal lines GL, LH removed, the remaining lines EG, FH remain equal. Q.e.d.

PROPOSITION XXVIII.

Any two parallel lines AB, CD may cut the ellipse ABC, and with AD, BC joined, ENMF may be drawn parallel to AB, and from E and F, the semi-diameters EG, FG may be put in place which shall bisect the lines AD, BC at H and I.
I say EG, FG to be divided proportionally at H and I.



Demonstration.

KL may be drawn through G, parallel to AB , crossing AD, BC, at K and L. Since EF, KL are parallel, there will be: as EN to KG, thus EH to HG; and as FI to IG, thus FM to LG; but as EN to KG, thus FM is to LG, (since EN, FM likewise KG, LG, shall be equal,) therefore as EH to HG, there shall be FI to IG. Q.f.d.

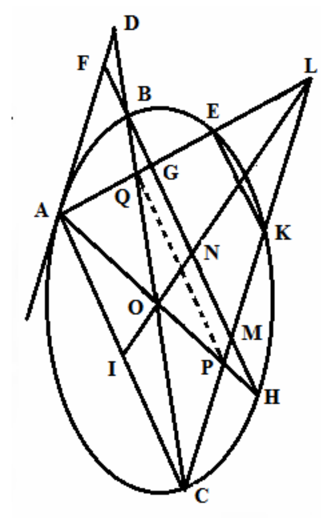
Corollary.

Hence it is clear the join HI to be equidistant from DC, and thus the lines AD, BC to be divided proportionally at H and I.

PROPOSITION XXIX.

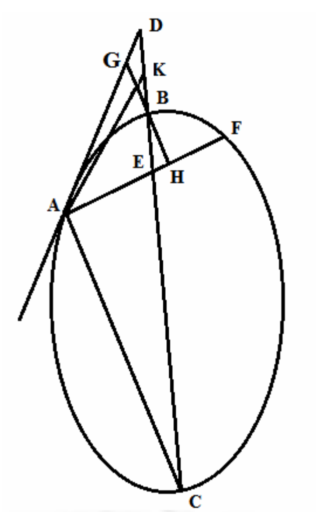
AD shall be a tangent to the ellipse, of which a diameter is BC and the centre O, meeting the diameter at D, and with the ordinate AQE drawn from the point A, and AC joined, the right line FBG may be drawn through B parallel to the right line AC.
I say FB, BG to be equal.

Demonstration.



FG shall meet the ellipse at H, and HO, AO which are lie on the same line may be joined [§.20] Then with AC bisected at I, the diameter IOL may be drawn through I meeting the right line AE at L and the points L, C may be joined by the right line LC, crossing the ellipse at K, and the right line FG in M; truly the right line HO at P.
Since from the construction the ordinate AC has been put in place for the diameter IL, and the two right lines through A and C meet the diameter at the same point L [§.25] , EK will be parallel to AC. And truly BH parallel to AC from the hypothesis: and the radii OB, OH shall cut AE, CK at Q and P, therefore QP shall be parallel to BH [§.28.*cor*.] ; therefore the three right lines are parallel: AC, QP, EK. Whereby since from the hypothesis AE shall be bisected at Q, and CK bisected in P, and hence placed to be an ordinate to the diameter AH. And thus CK shall be parallel to the tangent AD. Moreover FM is parallel to AC by the hypothesis, therefore FM is equal to AC, but also BH is equal to AC. Therefore FM, BH are equal; whereby with the common part BM removed FB, HM are equal. Also GB, HM are equal, and thus FB, GB are equal. Q.e.d.

PROPOSITION XXX



The right line AD shall be a tangent to the ellipse ABC at A meeting the diameter at D : and the ordinate line AF shall be drawn from A to the diameter BD.
I say the right line DC to be divided in the extreme and mean ratio of the proportions at B and E, that is, as CD is to BD, thus as CH is to HB: if it were divided at B and E in the extreme and mean ratio of the proportions AF to BC, and if the ordinate line were acted on in the mean ratio of the proportions through E : I say the join AD to be a tangent to the section.

Demonstration.

Join AC, the line GH is acting through B parallel to the line AC crossing the line AF at H and the tangent AD at G. Since AC, BH shall be parallel lines, there shall be as AC to BH, thus CE to EH, and thus CE to EB: but as AC to BH, thus there is AC to GB, (since GB, BH are equal [§.29]) therefore as AC to GB, thus there is CE to BE: moreover as AC to GB, thus CD to DB (since GB, AC are parallel) therefore as CD to DB, thus CE is to BE. Since in the first place there were now as CD to BD, thus CH to HB, if the ordinate AF were acting through E: I say the nearby line AD to be a tangent to the section at A. For if AD were not a tangent, suppose a tangent may be put through A which shall cross the diameter BD at K, therefore there will be, as CE to EB, thus CK to KB, but as CE is to EB, thus CD to DB, and therefore CK to KB, which cannot happen, since the point K shall fall either above or below D. Therefore AK is not a tangent nor any other besides AD. Q.f.d.

Corollary.

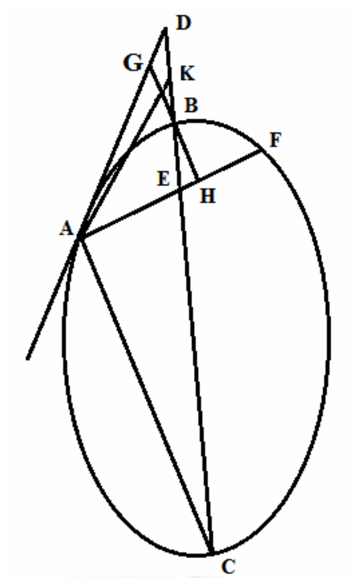
Propositions 29 and 30 also are true for the circle, although moreover there may be more cases where there may be a tangent for the circle than we have shown in this book of the ellipse, yet I make no mention of the circle except that it must be assumed for the following demonstrations.

PROPOSITION XXXI

With the same figure proposed remaining, to deduce the tangent for a given point D beyond the section.

Construction and Demonstration.

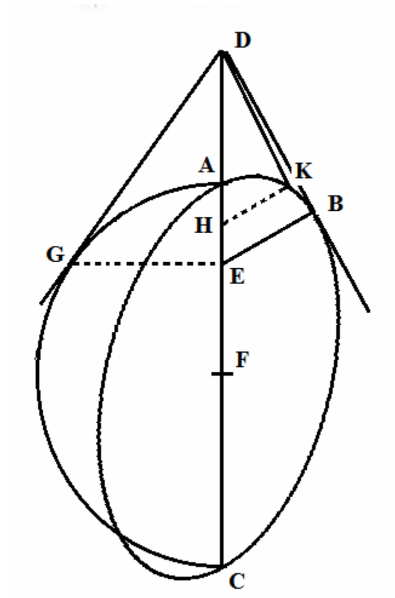
The diameter DBC may be drawn from the point D beyond the section, and there shall become as CD to DB, thus CB to EB, and through E to BC, the ordinate AF may be put in place, and AD may be joined, it is apparent from the preceding the line AD to be a tangent to the section at A; therefore for a given point beyond the ellipse, etc. Q.e.d.



PROPOSITION XXXII

The right line BD shall be a tangent to the ellipse ABC, of which the diameter shall be AC, meeting the diameter at D: and from B the ordinate BE may be drawn to the diameter AC: moreover the centre of the section shall be F.

I say FE, FA, FD to be lines in continued proportion and if FE, FA, FD were continued in proportional, and the right line EB were acting through the ordinate E, I say the section to contain the joined line BD. A proposition of Apollonius.

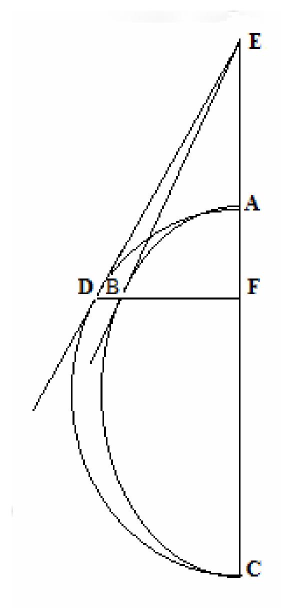


Demonstration.

The circle AGC is described with centre F and with the radius FA , then from the point E the normal may be drawn to the diameter AC crossing the circle at G: and the right line GD may be drawn, since the right line ordinate EB is put in place for the diameter AC and the tangent acting through B it meets the same diameter at D, there will be as CD to DA [§.30] thus CE to EA ; moreover in the circle, with the right line EG normal to the diameter AC [§.30 cor.], and therefore the right line GD is a tangent to the circle at G; [§.30 cor.] whereby in the circle the lines FE, FA, FD will be in continued proportional : but the same lines are common to the ellipse, and therefore in the ellipse FE, FA, FD will be in continued proportion. Because if FE, FA, FD shall be in continued proportion, and through E the ordinate EB may be drawn, I say the junction BD to be a tangent of the ellipse at B; truly if the right line DK may be drawn from D touching the ellipse at K ; and from K the ordinate KH may be put in place ; therefore by the first part of this section, FH to FA, shall be as FA to FD; but also by the hypothesis : FE is to FA, as FA to FD. Therefore FE is to FA, as FH is to FA, which cannot be done, since FG shall be greater or less than FE. According to which DK is not going to be a tangent, but rather DB. Q.f.d.

PROPOSITION XXXIII

Let AC be the axis of the ellipse ABC, so that upon which with the diameter AC the semicircle ADC, and with the assumed point F found on the axis which shall not be the centre, from FD crosses the ellipse at B.
I say the tangents acting through B and D, occur at one and the same point on the axis AC.



Demonstration.

The tangent BE is acting through B, meeting the axis at E, and ED may be joined; since the ordinate FB has been put in place to the axis and the tangent BE to the section drawn, there will become CF to FA, as CE to EA*:* and from which with ED the tangent to the circle; therefore the tangents acting through B and D, are agreed to meet the axis in one and the same point. Q.f.d.

Corollary.

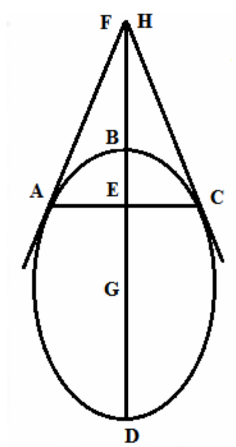
Hence we will show easily, if the two tangents meet the normal FD at the same point in the diameter FD, which if it may become a single point of contact D, to go through the other tangent as well.

PROPOSITION XXXIV

Let BD be the diameter of the ellipse ABC, for which AC may be put for the ordinate; the tangents through A and C.
I say this diameter to be present at one and the same point.

Demonstration.

By §16. above, it is evident the individual tangents drawn through A and C intersect on the diameter ; therefore if they may not meet at the same point, the tangent AF shall meet the diameter at F, and CH at H: Since the tangent AF concurs with the diameter at F, to that there will become DE to EB, thus as DF to FB; thus as DH to BH, and on dividing as DB to BF, thus DB to BH, which cannot happen; whereby the tangents do not cross the diameter at different points: therefore at the same point. Q.e.d.

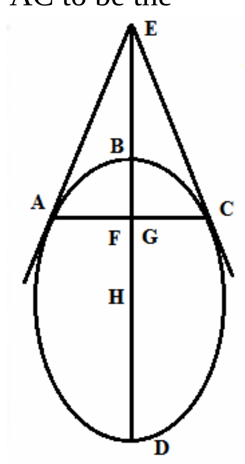


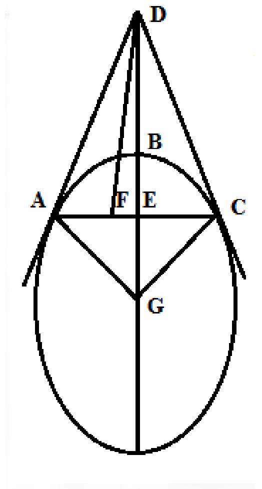
PROPOSITION XXXV

Let ABC be an ellipse, the diameter BD produced to some point E, and from E the lines EA, EC shall be tangents to the section at A & C. I say the joined line AC to be the ordinate put in place for the diameter BD.

Demonstration.

AF shall be put as an ordinate for BD and H shall be the centre of the ellipse ; therefore [§.32 above] the line EH will be divided at B and F into three continued proportionals, some ordinate CG will be dropped down to BD and again EH shall be divided at B and G into the three lines in continued proportion; therefore F and G are the same points ; whereby the right line AFC is the ordinate put in place for the diameter BD. Q.f.d.





Corollary.

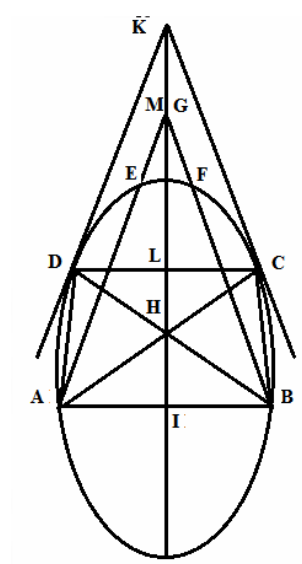
Hence it follows if the ellipse ABC shall have tangents at A and C, the two right lines AE, CE meeting at D; and AC may be joined, which will be bisected at E; the right line DE to pass through the centre or the joined line DE to be the diameter of the section; indeed if ED shall not be the diameter, the diameter DF may be drawn from D; meeting the line AC at F; therefore by the preceding the line AC is bisected at F, and thus the point F, to be the same as the point E, from which the right line DF to be the same as the line DE: which is contrary to the supposition; whereby DE is the diameter of the section. Q.e.d.

PROPOSITION XXXVI.

If the two right lines meeting at D may be tangents to the ellipse, and from the centre there may be drawn GA, GC, GD.
I say the triangles GCD, GAD to be equal.

Demonstration.

The right line AC shall join the points of contact, since AC is bisected at E, the triangles GAE, GEC, likewise DEC, DEA will be equal : and thus the two triangles DEC, DEA, that is the whole triangle DCG, will be equal to the two triangles DEA, EAG, that is, to the whole triangle GAD. Q.e.d.



PROPOSITION XXXVII.

AC and DB shall cut the ellipse : whatever tangents are acting through the tangents C and D, which by §.34 above meet the diameter HK at the same point.

I say the lines AG, BG drawn from A and B, to be parallel with DK, CK themselves, to intersect the diameter HK at the same single point.

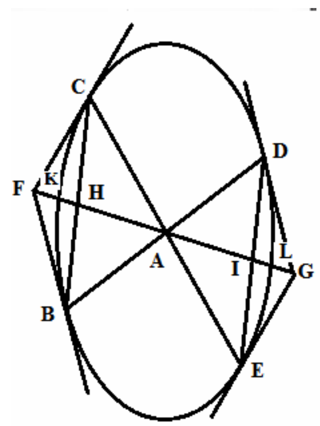
Demonstration.

AE may be put to cross the diameter at G and BF at M; then DC, AB, DA, CB may be joined. Since DC connects the tangents DK, CK, it shall be bisected by the diameter HK at L [§.35], truly there is AB parallel to DC[§.19] ; and therefore this shall be bisected by the diameter at I; whereby since the whole lengths DC, AB are equal, therefore since also DL, AI, will be parallel, which are joined by these parallel lines DA, IL, therefore this figure AGDK is a parallelogram; therefore the figure AGKD is a parallelogram, and on that account DA is equal to GK; in a similar manner we show that BC is equal to MK. Whereby since DA, BC, shall be equal, also KG, KM will be equal, therefore G and M are one and the same point by which, with the lines DC, CK drawn parallel to the tangents from the points A, B, cross the diameter. Q.f.d.

PROPOSITION XXXVIII.

Any two diameters BD, CE shall cut the ellipse of which the centre is A and with BC, DE joined, the diameter FG may be put in place, which shall bisect BC at H, and moreover ED shall be bisected at I, then the tangents are acting through C and B, likewise the tangents through D and E, which will cross the diameter FG at the same points F and G.

I say the following triangles to be equal to each other; in the first place the triangles ACF, ABF, secondly the triangles ACF, ADG, thirdly the triangles CBF, DEG.



Demonstration.

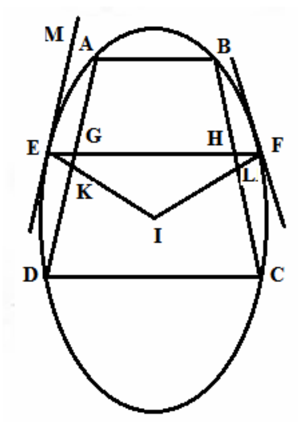
The diameter FG shall meet the ellipse at K and L. Therefore since from the hypothesis, BC shall be bisected at H, there will be both the triangle ACH equal to the triangle AHB, as well as the triangle HCF equal to the triangle HFB, from which the whole triangle ACF is equal to the whole triangle AFB, which was the first to be shown. Again since BC, DE shall be parallel [§.19] and BC shall be bisected at H by the diameter FG, and likewise ED shall be bisected at I; whereby since CB, DE shall be equal, the halves of these HC, DI are equal, whereby also the right lines CD, HI, the right lines of which may be joined, also are parallel; therefore the triangles ACH, ADI are between the same parallel lines. Moreover the bases AH, AI are equal, indeed AK is equal to AL, and the bases KH, AI are equal, (indeed AK is equal to IL): therefore the triangle ACH shall be equal to the triangle ADI. Now truly AH, AK, AF, and likewise AI, AL, AG, are in continued proportion ; whereby the ratio AH to AF, is the twofold ratio of AH to AK; and the ratio AI to AG is the twofold [*i.e.* the square] of the ratio AI to AL. Therefore since the ratios AH to AK, AI to AL, shall be the same (for AH is equal to AI and AK is equal to AL), and the ratios AH to AF, AI to AG shall be the same, and the duplicate ratios of the same ratios will be the same between themselves : and hence also the triangle AHC is to the triangle AEC, as the triangle AID to the triangle AGD. Whereby since the triangles ACH, ADI may be shown to be equal, also AFC, AGD will be equal, which was the other equality of the triangles. From which it is now apparent, if you may add FCH, GDI, for which to be equal if you may add FBH, GEI, which we have shown clearly to be equal by the same discussion, thus the whole triangles CBF, DEG, will be equal. Which was required to be demonstrated in the third place.

Corollary.

Hence it is apparent the quadrilaterals CFBA, DGEA, to be equal, indeed by the same discourse we will prove the triangles ABF, AEG to be equal in the same manner as we have proved the triangles ACF, ADG to be equal.

PROPOSITION XXXIX.

Any two parallel lines AB, CD may cut the ellipse and with AD, CB joined, the right line EM parallel to AD itself, may be a tangent to the section at E, and from E there may be drawn EF, parallel to AB, cutting the lines AD, CB at G and H.
I say the tangent line drawn through F to be parallel to BC itself BC.

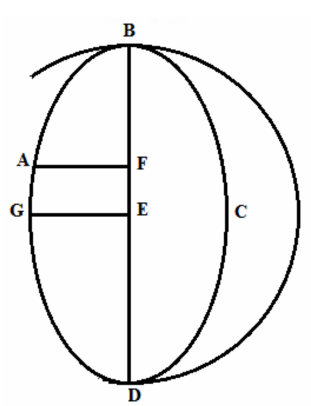


Demonstration.

The lines IE, IF may be drawn from the centre I crossing AD, CB at K and L : therefore since the tangent EM parallel to AD, and the diameter IE drawn to the tangent will cut the line AD at K, AD will be bisected at K [§.13]; moreover the right line BC is divided at L just as AD at K, indeed the right line KL joining the points K, L is parallel to AB, DC. Therefore BC is bisected at L by the diameter IF; and thus is parallel to the tangent drawn through F. Q.f.d.

PROPOSITION XL.

The circle described on the major axes as diameter exterior to the ellipse will meet the ellipse in two points only.



Demonstration.

BD shall be the major axis of the ellipse ABC, and from the centre E of this ellipse, and with the radius EB a circle is described, I say to cross that ellipse only at the two points B and D. For if it were possible to happen in addition at the point A, and the ordinate AF to the axis shall be acting through A, and the minor axis GE may be drawn, therefore there will become : as the rectangle BFD to the square FA thus the rectangle BED to the square EG: but the rectangle BFD in the circle is equal to the square FA; and therefore the rectangle BED, that is the square BE is equal to the square GE, which cannot happen since BE shall be a greater line GE, therefore the circle cannot meet the ellipse at A: and not in any other point, besides B and D. Q.f.d.

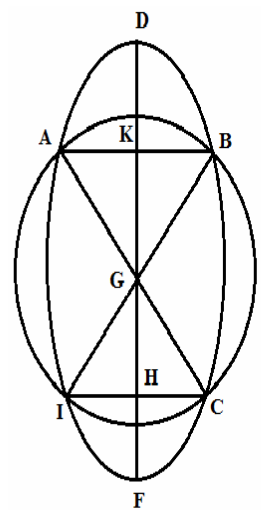
Corollary.

We will demonstrate by a similar discussion a circle described about the minor axis of the ellipse only to meet at the two furthest points of the ellipse, and the whole circle to lie inside the ellipse.

PROPOSITION XLI.

A circle is described from the centre of the ellipse, if it may cut the ellipse, then it will be cut at four points.

Demonstration.



There shall be a circle described from the centre G of the ellipse, and indeed you may cut the ellipse at B, the axis FD may be drawn, and the right line BGI; then the ordinate BKA shall be applied crossing the ellipse at A, likewise the right lines AGC, IC may be drawn; in the triangles BKG, AKG, BK, AK shall be equal, and KG is common, and the angle at K to be right; therefore GB, GA are equal ; whereby since the point B shall be on the circle, and moreover the same point A also is a point on the ellipse, therefore the circle shall cut the ellipse at A. Then AB, IC are parallel, and thus since the angle AKH shall be right, also IHK will be a right angle IHK; and hence the ordinate IC is placed on the axis at DF, and is bisected at H, moreover the whole lengths AB, IC are equal, therefore AK, IH the halves of these also are equal. Therefore in the triangles GKA, GHI, AK is equal to IH, and KG itself is equal to HG, truly the angles AKG, IHG also are equal; therefore GA, GI are equal; whereby since the point A and also the point I shall be on the circle, and also the point I is on the ellipse. Therefore the circle shall cut the ellipse at I. Similarly we may show the circle to meet the ellipse at C. Therefore the circle cuts the ellipse at the four points. Q.e.d.

Corollary.

But so that the circle may not cut the ellipse in more than four points, may be easily deduced from the demonstration now put in place.

QUADRATURAE CIRCULI

LIBER QUARTUS : DE ELLIPSI.

ARGUMENTUM

Ellipsis proprietates illiusque naturam methodicè propositu ri, rem totam in sex partes dividire placuit. Ac prima quidem è cono sectionem educit, affectionesque illius essentiales, dein accidentales reliquis necessarius & fundamentales.

Secunda ellipsim dividit illiusque sectores & segmenta comparat.

Tertia, axium ac diametrorum coniugatarum tam aequalium quam inaequalium ampliorem continet considerationem. Ac illarum primo quidem contemplatur potentiam:inde lineas, quae extrema diametrorum coniungunt.

Quarta sectionis polos eorumque passiones ac lineam breuissmam à puncto in axe dato ad peripheriam designat.

Quinta varias ellipsis geneses quae tum ex lineis, tum è circula, tum ex ipsa ellipsi oriuntur continet.

Sexta ellipsim cum circulo comparat, in qua hic etiam ordo tenetur, ut primo linearum proportiones ac potentiae, secundo segmenta & ipsae sectiones, dein figurae utriq; inscriptae inter se conferantur.

Caeterùm propositiones nonnullae huius libri ac sequentium duorum sunt Apollonii, sed via longè aliâ à me demonstratae, paucis exceptis, quas nihilominus caeteris apponere visum fuit, ne quid hoc in opere quod ad conicam doctrinam pertineat, studiosus Geometriae lector desideraret. Caetera omnia, quae longe maximam atque praecipuam operis partem constituunt, a nobis & inventa sunt & demonstrata. Quare si quis in recentium quorundam Geometratum libris theoremata quaedam reperiat quae cum nostris conveniant, is velim intelligat, ea ab annis iam plurimis ac multo ante fuisse a me reperta, quam authorum illorum libri in lucem prodierint. Quae paucis lectorem meum docere volui, non ut cuiusquam inventis detraham, sed ut plagii suspicionem a me remoneam.

ELLIPISIS.

DEFINITIONES.

I.

Diameter ellipseos est, recta linea intra ellipsim ducta, quae omnes lineas, rectae cuidam aequidistantes bifariam dividit & si quidem ad rectos illas secet angulos, axis dicetur: in quaevis autem ellipsi binos esse axes, & quidem coniugatos (qui extremae dicuntur diametri) hoc est qui mutuas parallelas bisecent ad angulos rectos, suo loco patebit.

II.

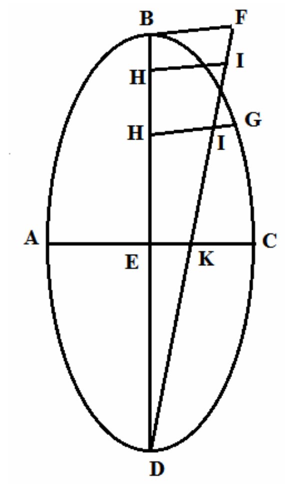
Ordinatim ad diametrum applicari dicitur unaquaeque linearum aequidistantium, ac bifariam divisarum.

III.

Centrum ellipseos est punctum quod diametrum bifariam dividit. Quod autem lineae in ellipsi per centrum ductae bifariam secentur, propos. septima huius libri demonstrabimus.

IV.

Diametri coniugatae dicuntur quae mutuas parallelas bifariam secant.

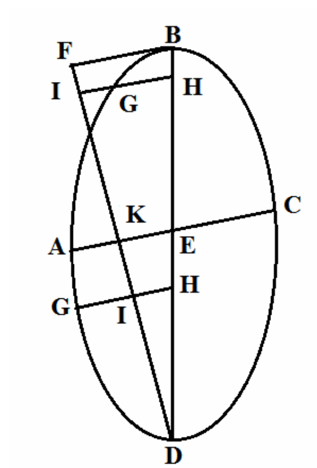


V.

Latus rectum voco lineam, iuxta quam possunt ordinatim ad diametrum applicatae sive, latus rectum est mensura iuxta quam comparantur potentiae linearum ordinatim ad diametrum positarum.

Res in exemplo erit clarior: sit ABC ellipseos diameter BD, illiusque latus rectum repraesentet FB: iunctisque FD, sumantur in diametro puncta quaevis H, ponanturque HG normales diametro BD, occurrences FD in I : singula igitur quadrata ordinatim positarum aequalia erunt singulis rectangulis BHI, (ut propositione undecimi huius demonstrabimus) quae deficiant a rectangulis FBH, rectangulo simili, ipsi FBD.

Atque quidem latus rectum tum Apollonius, tum caeteri illum hactenus secuti exposuere. Verum mihi minime videtur necessarium ut latus rectum diametro ad rectos applicetur angulos : & quadratorum ac rectangulorum loco possunt Rhombi ac Rhomboides inter se compariri. Itaque ad veterem lateris recti acceptionem, novam aliam adiicio eiusmodi ad ellipseos diametrum sint ordinatim positae quotvis rectae GH : & quaedam BF latus rectum, aequidistans ponatur ordinatim applicatis : singuli ordinatim positarum Rhombi HG in angulis IHB aequales erunt singulis IHB, Rhomboidibus in iisdem angulis, qui deficiunt a Rhomboidibus FBH per Rhomboides similes Rhomboidi FBD: demonstratiomen huius vide propos.12 huius libri.



Porro latus rectus eo ab antiquis consilio inventum est, ut certi aliquid & noti haberent, per quod reliquas sectionem proprietates intelligere ac notas sibi reddere facilius possent: ut in singulis coni sectionibus illae plane diversae sunt, ita & latera recta diversas in singulis obtinent passiones ; & rectangula lateribus rectis ac diametrorum partibus inter verticem earundem & puncta quibus ab ordinatim positis secantur interceptis contenta longe diversam in singulis, ad quadrata ordinatim positarum habent proportionem; in ellipsi quidem quadrata illa deficiunt figura simili illi quae latere recto & transverso continetur a rectangulis praedictis ; in parabola iisdem aequantur; in hyperbola vero excedunt figura simili illi, &c. Unde & nomenclaturam singulae suam sortitae sunt.

Caeterum uti potentiae ordinatim positarum ad diversas diametros , diversae quoque sunt, ita & diametris singulis, proprium & unicum latus rectum assignatur: quae omnia, uti & lateris recti inventionem, suis locis demonstrata invenies

VI.

Figura est rectangulum quod latere recto & transverso (id est diametro, nam illa quoque transversa vocari solet) continetur.

VII.

Poli seu foci ellipseos, puncta sunt (quae ex comparatione facta vocat Apollonius) in quibus axis divisus rectangulum exhibet sub segmentis contentum aequale quartae parti_ figurae: de quo suo loco agendum.

VIII.

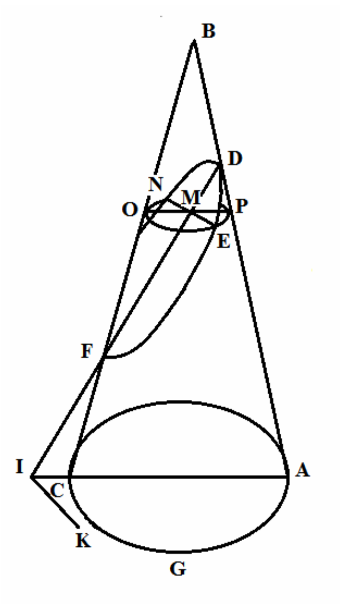
Sectio subcontraria est quando conus plano per axem sectus triangulum producente, alio rursum secatur plano, quod abscindat (triangulo producto) triangulum simile quidem, sed ita positum ut anguli qui in utroque triangulo sunt aequales ad diversa sint latera.

ELLIPSIS

PARS PRIMA

Sectionem e cono educit, primasque essentiales eiusdem exhibet proprietates.

PROPOSITIO PRIMA.



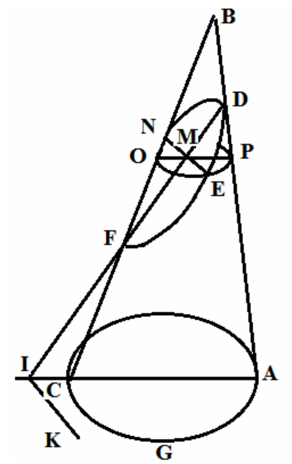
Conus rectus AGCB sectus sit plano per axem faciente triangulum ABC. Secetur alio deinde plano basi coni AGC non parallelo, cum utroque trianguli latere conveniente in D & F: ex qua (sectione producta) sit in cono figura DEFN , communis autem sectio illius plani secantis cum triangulo ABC sit DFI; eiusdem vero sectio communis cum plano in quo est coni basis AGC, sit recta IK, quam perpendicularem esse oportet ad AC diametrum basis coni, vel ad rectam quae diametro AC in directum constituitur.

Dico figuram DEFN circulum non esse.

Demonstratio.

Per punctum aliquod M rectae DF ducatur NE parallela ad IK, in plano figurae DEFN: & per idem illud punctum M ducatur in plano trianguli ABC recta OP parallela ad ACI, per lineas autem NE, OP agatur planum. Erit hoc a parallelum basi AGC [15.*undecimi*], ac proinde producet circulum OEPN [16.*prolegone*], cuius diameter erit OP.

Quoniam igitur OP est parallela ad AC, triangula BOP , BCA similia sunt, sed BCA isosceles est, ergo & BOP isosceles est. Ergo [34.*de lineis*] rectangulum FMD maius est rectangulo OMP ; sed rectangulum OMP [35.*tertii*] aequale est rectangulo NME. Ergo rectangulum FMD maius est rectangulo NME patet igitur ex 35.tertii figuram DEFN circulum non esse. Quod erat demonstrandum.



PROPOSITIO II.

Datus iam sit conus scalenus ABC, & planum secans quod producit in cono figuram DEFN, neque sit parallelum basi coni AGC, neque subcontrarie positum. Caetera vero omnia ponantur & fiant eadem quae propositione prima.

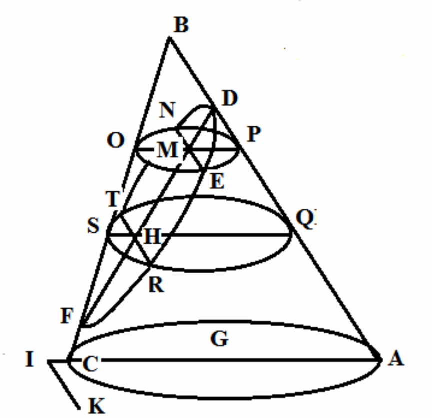
Dico rursum figuram DEFN circulum non esse.

Demonstratio.

Quoniam OP est parallela ad AC eamque secat FD in M non subcontrarie, hoc est angulum BFD non constituens aequalem angulo BAC, patet ex 36. libri nostri primi rectangulum FMD inaequale esse rectangulo OMP; sed rectangulum OMP aequatur rectangulo NME. Ergo rectangulum FMD rectangulo etiam NME inaequale est, liquet igitur ex 35.tertii figuram DEFN non esse circulum. Quod erat demonstrandum.

PROPOSITIO III.

Datus sit conus quicunque sive rectus sive scalenus, & caetera ponantur & fiant eadem quae supra:



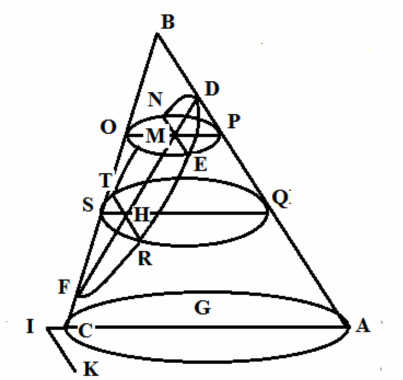
Dico rectam NE a recta DF secari bifariam in M.

Demonstratio.

Recta PM ex hypothesi est parallela ad rectam AC, & ME parallela ad IK. Quare PM, EM angulos comprehendunt aequales; atqui angulus AIK ex hypothesi rectus est communis enim sectio IK posita fuit perpendicularis ad ACI, propositione primae ergo etiam PME rectus est. Itaque cum sectioONPE sit circulus, eiusque diameter OP, manifestum est EMN, a diametro circuli OP, ad quam normalis est , bisecati in M. Sed ex hypothesi punctum M tribus rectis OP, NE, DF communis est. Ergo NE ad DF, bisecatur in M. Quod erat demonstrandum.

Corollarium.

Hinc patet, si ducantur quotcunque rectae ad IK sive NE parallelae, omnes a DF bifariam dividi eadem enim est in omnibus demonstratio. Ex quo ulterius sit manifestum sectionis DEFN ; (quam ellipsim deiceps nominabimus) diametrum esse lineam DF [*Def*.1], rectam vero NE [*Def*.2] caeterasque huic parallelas ordinatum esse ad diametrum DF applicatas.



PROPOSITIO IV.

Iisdem positis, ducatur ellipsi DEFN linea quaevis RT, parallela ad EN sive IK, secans diametrum DF in puncto H.

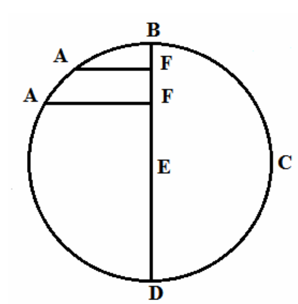
Dico rectangulum DMF esse ad rectangulum DHF ut quadratum EM ad quadratum RH.

Demonstratio.

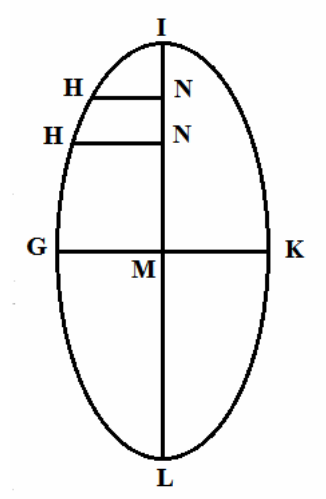
Per punctum H ducatur recta QHS parallela recta OP occurrens lateribus triangluli ABC in Q & S. Tum per rectas QS, TR agatur planum, erit hoc parallelum basi AGC ac proinde sectionem producet circulum QRS, iam vero ratio rectanguli. DMF ad rectangulum DHF componitur ex ratio DM ad DH, (hoc est, quia PM, QH sunt parallelae ex construct. ex ratione PM ad QH) & ex ratione MF ad HF (hoc est, quia MO, HS ex const. sunt parallelae, ex ratio MO ad HS.) Atqui ratio rectanguli PMO ad rectangulum QHS componitur etiam ex rationibus PM ad QH, & MO ad HS. ergo rectangulum DMF est ad rectangulum DHF ut rectangulum PMO ad rectangulum QHS; hoc est quoniam sectiones PEO, ORS circuli sunt, ut rectangulum EMN ad rectangulum DHF ut rectangulum RHT, hoc est, quia EN, RT bisectae sunt a diametro DF in M & H, ut quadratum EM ad quadratum RH. Quod erat demonstrandum.

Scholion.

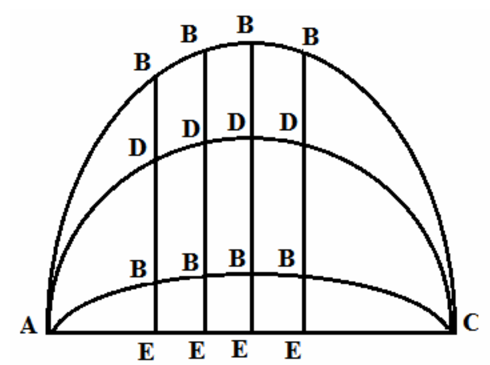
Exhibuimus propositione hac proportionem rectangulorum quae a segmentis diametri ellipseos constituuntur ad quadrata, ordinatim ad eandem diametrum applicatarum: quae quidem proprietas ellipseos est primaria, & essentialis: verum quia haec ita ellipsi inest, ut etiam in circulo suo modo reperiatur, operae pretium me existamavi, si differentiam, illum inter & ellipsim, breviter in scemate apposito ostendam.



Esto circuli ABC *diameter* BD *centrum* E*: & normales to diametrum* AF*: sit autem & ellipsis* HIK *diameter quaecunque* IL*, quam ordinatim secent HN: centrum vero sectionis* M. *Quoniam igitur in circulo,* AF*, rectae sunt normales ad diametrum* BD*, erunt rectangula* BFD *aequalia quadratis* AF*, proindeque,* AF *quadratum, est ad quadratum* AF *ut* BFD *rectangulum : eodem modo cum* HN *rectae in ellipsi ordimatim* *positae sint ad diametrum IL, erit HN quadratum ad quadratum* HN, *ut* INL *rectangulum ad rectangulum* INL: *illud igitur utrique sectioni convenit; eam esse proportionem inter quadrata ordinatim positarum, quae rectangulorum est, sub segmentis diametri ad quam ordinatim sunt positae: in hoc vero differunt, quod in circulo proportio rectangulorum sub segmentis diametri, ad quadrata ordinatim positarum sit aequalitatis; in ellipsi vero (si casum diametrorum coniugatarum aequalium excipias, de quo plura suo loco) inaequalitatis. quod prima & secunda huius planum fecimus.*



Ex quo sequitur primo; in ellipsi axem unum altero maiorem esse, *quod sic ostendo: sin in HIK ellipsi axis aliquis* IL *quem ordinatim secent* HN: *agatur per* M *centrum recta* GK *aquidistans ipsi* HN: *dico illos axes esse inaequales: est enim ut* INL rectangulum *ad rectangulum* IML, *sic quadratum* HN *ad quadratum* GM, *&permutando ut* INL *rectangulum ad quadratum HN sic* IML *rectangulum ad quadratum* GM; *sed rectangulum* INL *quadrato* HN, *est inequale* ; *ergo & rectangulum* IML, (*hoc est quadratum* IM) *quadrato* GM *inaequale est: ergo recto* IM *recta* GM *est inaequalis, ergo totas* IL, nempe *axis, toti* GK *hoc est axi alteri, inaequalis est. Quod erat propositum.*



Deinde si axes in ellipsi aequales essent, iam non differret ellipsis a circulo: eo quod rectangula sub segmentis axeos aequalis essent quadratis ordinatim positarum.

Sequitur secundo, si super axe AC ellipseos ABC, describatur semicirculus ADC, ducanturque ordinatim lineae BE occurrentes semicirculo in D , quod BE sit ad BE, ut DE est ad DE; est enim tam in ellipse quam in semicirculo, ut AEC rectangulum ad rectangulum AEC, sic BC quadratum ad quadratum BE, & DE quadratum ad quadratum DE; unde quoque est quadratum BE ad quadratum BE, sit DE quadratum ad quadratum DE.

PROPOSITIO V.

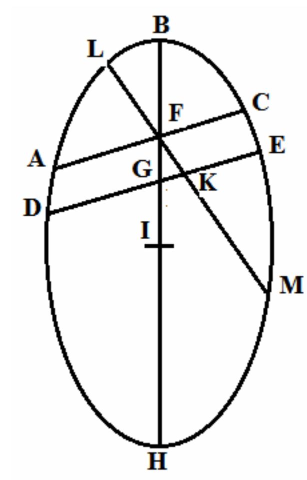
In data ellipsi diametrum invenire.

Constructio & demonstratio.

Intra ellipsim ducantur parallelae AC, DE, quas bifariam seca in punctis F, G; & per F ac G, ducatur recta BH.

Dico hanc esse diametrum.

Demonstratio est manifesta, si enim BH non est diameter, sit LFM, secans DE in K. Quoniam igitur LM ponitur esse diameter, & bisecat e parallelis unam AC, in F, bisecat alteram quoque DE



in K. Quod fieri non potest cum ex construct, DE bisecta sit G.Non igitur LM aut alia quaevis ducta per F est diameter praeter eam , quae etiam transit per G, hoc est praeter ipsam BH. In data igitur ellipsi invenimus diametrum. Quod erat faciendum.

PROPOSITIO VI.

Datae ellipseos centrum reperire.

Constructio & demonstratio.

Per praecedentem quaere diametrum ellipseos BH, quam seca bifariam in L. Ex definitione tertia patet ellipseos centrum esse I.

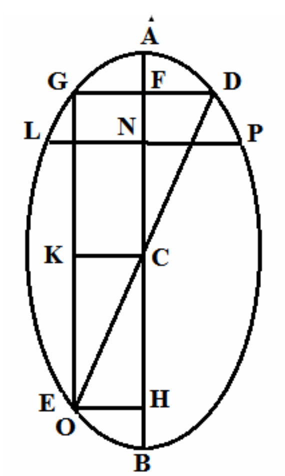
Corollarium.

Patet ex hac propositione omnem diametrum transire per centrum. Ex quo & conversam facile deduces, omnes nimirum lineas per centrum transeuntes esse diametros.

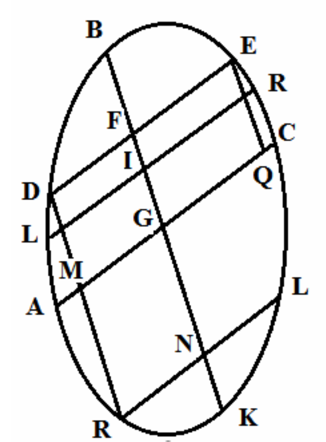
PROPOSITIO VII.

Data sit ellipsis ADB, cuius diameter AB, recta vero LP, una sit earum, quas propositione tertia huius demonstravimus a diametro secari bifariam: centrum ellipseos sit C.
Dico omnes lineas per centrum ductas in centro dividi bifariam.

Demonstratio.



Ducta sit enim quaecunque recta DO, per centrum C; ex D ducantur DFG parallela ad LP, GE parallela ad AB, & EH, CK parallelae ad GD, sive LP. Quoniam igitur FGEH parallelogramum est, erunt GF, EH aequales : unde & quadrata GF, EH aequales sunt. Atqui ut quadratum GF est ad quadratum EH, ita rectangulum AFB est ad rectangulum AHB, aequantur igitur rectangula AFB, AHB, ergo ut AF ad AH , sic BH ad BF : ergo dividendo ut AF ad FH, sic BH ad HF aequantur ; igitur AF, BH. Quare cum tota quoque diameter AB bisecta sit in C, ut paret ex definitione centri, reliqua etiam FH, bisecta est in C. Quoniam igitur KC ipsis GD, EH est parallela, recta quoque GE bisecatur in K, est vero & DG bisecta in F, utpote ipsi LP parallela. Ergo est ut DG ad GF, hoc est ut DG ad CK, sic GE ad KE. Ergo puncta DCE sunt in directum ; sed etiam puncta DCO sunt in directum, cum ex hypothesi DCO sit linea recta. Una igitur eademque recta sunt DCE, & DCO. Atqui DCE bisecta est in C, (cum enim ex constr: GE, FC sint parallelae, erit ut DF ad FG, sic DC ad CE.) Ergo etiam DCO bisecta est in C. Quod erat demonstrandum.



PROPOSITIO VIII.

Data sit ellipsis ABCH, cuius diameter sit BH; ordinatim vero; ad diametrum applicata LPR: centrum ellipsos G. Ducta autem sit per centrum G, recta AGC ordinatim applicatae parallela.

Dico BH, AC diametros esse coniugatas.

Demonstratio.

Sumatur in AG quodvis punctum M, per quod ducatur KD diametro BH parallela, occurrens ellipsi in punctis D & K ; ex quibus ducantur DFE, KNL; ipsi LR parallel. Quoniam igitur DK, NF parallelogramum est, rectae DF, KN, adeoque & quadrata DF, KN aequantur. Quare cum rectangulum BFH sit ad rectangulum BNH ut quadratum DF ad quadratum KN, rectangula BFH, BNH etiam sunt aequalia, ac proinde, ut ostensum in praecedenti, BF & NH aequantur. Sunt vero & BG, HG aequales. Ergo & reliquae FG, N G aequales sunt, sive FN bisecta est in G. Ergo & KD parallela diametro BH bisecatur in M ab AC. Similiter ostendam quasvis alias diametro BH parallelas bisecati ab AC. Quare cum etiam AB bisecet DE, LR ceterasque omnes quae sunt ordinatim positae ad BH, & parallelae ex hypothesi ipsi AC; patet ex definitione
BH, AC diametros esse coniugatas.

Corollarium.

Quae per centrum ad ellipseos axem datum perpendicularis ducitur, est axis dato axi coniugatus.

Ex discursu iam allato facile sibi lector demonstration huius rei eliciet.

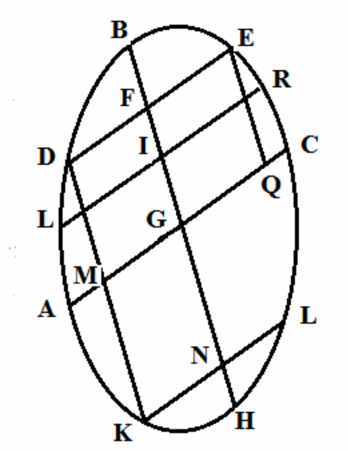
PROPOSITIO IX.

Data sit ellipsis eiusque diameter BH.
Oporteat diametro BH coniugitam diametrum exhibere.

Constructio et demonstratio.

Ducatur recta aliqua KD parallela ad BH, & utraque D K, BH divisa bifariam M & G, per M & G, ducatur AC.

Dico AC, BH coniugatas esse diametros.



Ac primo quidem rectam AC esse diametrum patet ex 5.huius & BH diameter est ex hypothesi; ambae igitur sunt diametri. Quod autem sint coniugatae sic ostendo. Quoniam AC diameter est & bisecat KD, erit KD ad AC ordinatim applicatae; ergo & reliquae ipsi KD parallelae, erunt ad AC, ordinatim applicatae, hoc est a diametro AC bifariam secabuntur. Sed DK ex constructione cum sibi parallelis, parallela est ad diametrum BH. Ergo diameter AC bisecat diametro BH parallelas. Ducantur deinde DE, parallela diametro AC & EQ, paralla rectae DK parallelogrammum igitur est MDEQ, in quo quia FG ipsis DM, EQ ducta est parallela, erit DF ad FE, ut MG ad GQ; latera quoque parallelogrammi DM, EQ adeoque & quadrata DM, EQ aequalia erunt. Iam vero quia DM, EQ, sunt ad diametrum AC, ordinatim positae, ratio inter rectangula AMC, AQC erit eadem quae inter quadrata DM; EQ, hoc est aequalitatis : ac proinde ut patet ex demostratis in septima huius, aequales erunt AM, QC. Quare cum & totae AG,CG aequales sint (est enim B centrum ellipseos ; quia bisecat diametrum BH) etiam reliquae MG, QC, aequales sunt; quoniam igitur est ut MG ad GQ, sic DF ad FE, etiam DF, FE aequantur, hoc est DE bisecta est in F. Ergo ex definit, DE est ad diametrum BH ordinatim posita, ergo & reliquae ipsi parallelae sunt ad BH ordinatim positae, hoc est bisecantur a BH. Atqui DE ex constructione, cum sibi parallelis, est alteri diametro AC parallela. Ergo diameter BH bisecat parallelas diametro AC. Quare cum etiam prius ostenderim AC bisecare parallelas ad BH, erunt BH, AC diametri coniugatae. Factum igitur est quod petebatur.

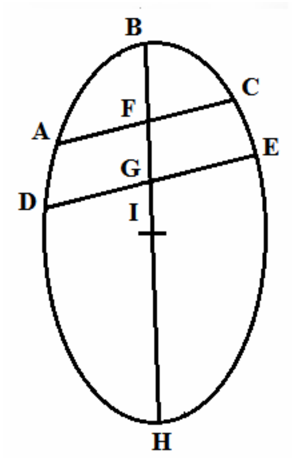
Corollarium primum.

Dato ellipseos axi; axem coniugatum invenies, si per centrum ellipseos duxeris rectam lineam dato axi perpendicularem, res patet ex corollario octavae.

Corollarium secundum.

Ex hoc problemate sit manifestum qua ratione ex dato in ellipsi puncto D, ad diametram BH, recta linea ordinatim debeat applicari. lnveniatur enim AC diameter coniugata diametro BH; & ex dato puncto D ducatur DFE ipsi AC parallela.

Dico DFE ordinatim esse positam ad diametrum BH. Demonstratio patet ex propositione.



PROPOSITIO X.

Ordinatim positarum (AC, DE, &c.) illa maior est quae centro (I) vicinior.

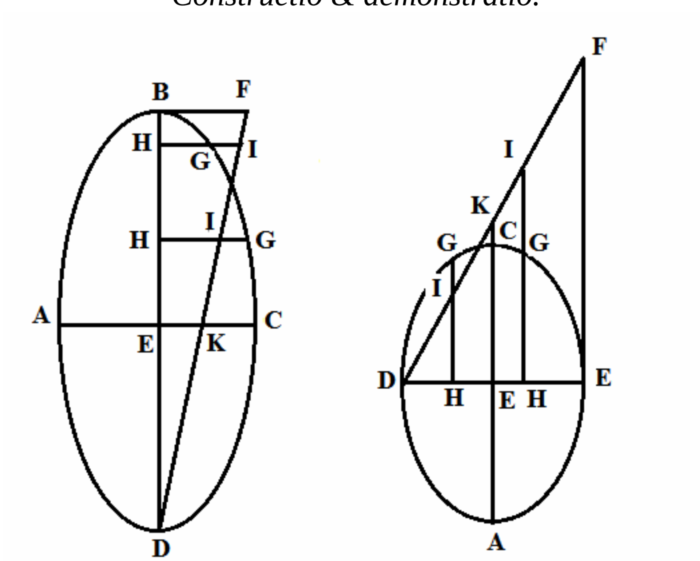
Demonstratio.

Rectangulum BGH maius est rectangulo BFH ut patet ex quinta secundi. Atqui quadratum EG est ad quadratum CF ut rectangulum BGH ad rectangulum BFH. Ergo quadratum EG maius est quadrato CF. Ergo & ordinatim posita EG maior est ordinatim posita CF. Quod erat demonstrandum.

PROPOSITIO XI.

Esto ABC ellipsis axis BD; oportet illius latus rectum exhibere.

Constructio & demonstratio.



Axi BD per E centrum ducatur coniugatus AC, fiantq; continuae BD, AC, BF *:* dico FB esse latus rectum aequidistet enim BF ipsi AC: ducanturque ordinatim lineae GH quae iunctae FD occurrant in I ipsa vero FD secet AC lineam in K. Quoniam EC, GH ordinatim positae sunt ad axem BD, erit ut quadratum GH ad quadratum EC, sit BHD rectangulum ad rectangulum BED: sed ut BHD rectangulum ad rectangulum BED, sit IHB rectangulum est ad rectangulum KEB (quia ex iisdem rationem habent compositam scilicet ex BH ad BE, & ex HD ad ED, hoc est HI ad EK.) igitur ut quadratum GH ad quadratum CE, sic IHB rectangulum est ad rectangulum KEB, & permutando invertendo ut KEB rectangulum ad quadratum CE, sit IHB rectangulum est ad quadratum GH: sed cum AC quadratum sit aequale rectangulo super FBBD (cum ex construct: BD, AC,BF sint trcs continuae) erit EC quadratum, (nimirum quarta pars quadrati AC, est enim F, AC bisecta in E) aequale rectangulo KEB quartae parti rectanguli sit per FBBD. Igitur & quadratum HG aequale est rectangulo IHB: ergo HG potest spatium quod adiacet ipsi FB latitudinem habens HB, deficiens ab FBH rectangulo, similis figurae rectangulo, BFD quare FB latus rectum est; exhibuimus ergo, &c. Quod erat faciendum.

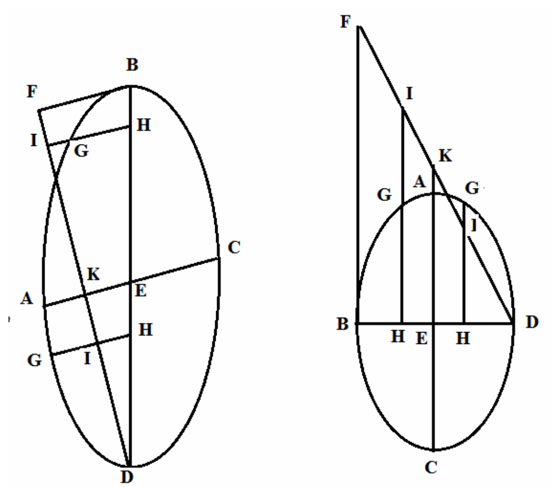
Corollarium.

Hinc sequitur primo quatuor lineas, nimirum latus rectum axis minoris, axem maiorem, axem minorem , & latus rectum axis maioris in continua esse analogia.
Sequitur secundo qui datis lateribus rectis axium, ellipsiu exhibuerit, quod inter binas datas, duas medias invenerit.

PROPOSITIO XII.

Esto ABC ellipsis diameter quaecunque BD , oportet illius latus rectum exhibere.

Constructio & demonstratio.



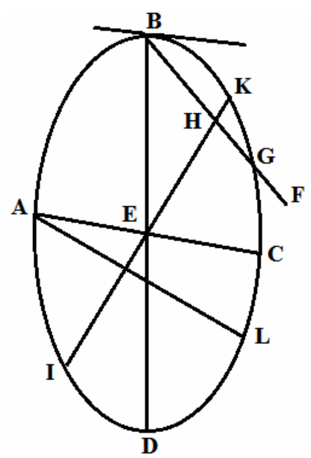
Ducatur per E centrum diameter AC coniugata ipsi BD, fiantque continuae BD, AC, BF, ac BF quidem aequidistet diametro AC : ducaturque linea FD, quae AC rectum secet in K, ducanturque ordinatim lineae GH, quae FD lineae occurrant in E Quoniam BD, AC , FB lineae sunt continuae, erit AC quadrato aequale rectangulum super FB, BD; igitur & quadrato AE (nimirum quartae parti quadrati AC , est enim AC bisecta in E) aequale rectangulum super KE, EB, quarta pars rectanguli FBD: quare, ut in praecedenti ostendimus, ita etiam ostendemus HG quadrato aequale esse rectangulum IHHB, atqui Rhombus IH in angulo IHB est ad Rhombo idem IHB in eodem angulo, ut quadratum IH ad rectangulum IHB, (rationes enim Rhombi ad Rhomboidem & quadrati ad rectangulum ex iisdem rationibus componuntur nempe ex IH, ad IH & IH ad HB,) ergo cum quadratum IH aequale sit rectangulo IHB, etiam Rhombus IH, Rhomboidi IHB aequalis erit; recta igitur IH potest Rhomboidem in angulo ordinatim applicatae IHB, qui (quod demonstratum est facile) a Rhomboide FBH in eodem angulo, deficit Rhomboide simili ei qui in eodem angulo sit a diametro DB & recta BF. Igitur FB est latus rectum. Quod petebatur.

PROPOSITIO XIII.

Omnis recta (BF,) quae per terminum diametri (BD) ducitur ordinatim applicatae (AC) aequidistans, ellipsim contingit.

Et quae tangenti ducitur parallela, est ordinatim ad diametrum applicata.

Demonstratio.

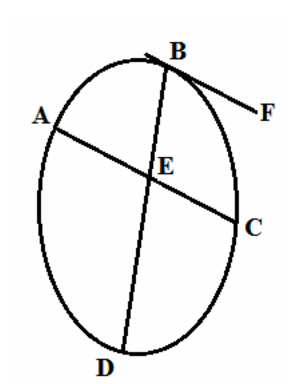


Si enim recta BF non contingat ellipsim, secet illam in G; divisaque BG bifariam in H, agatur per H & E, KI occurrens utrimque peripheriae in I & K. Quoniam recta BG per constructionem aequidistat AC, utramque autem bifariam secet recta IK erit IK diameter & AC, BG lineae ordinatim ad illam positae; secat autem per constructionem recta AC ordinatim quoque diametrum BD, igitur una eademque recta AC ordinatim secat duas diametros BD, IK. Quod fieri non potest; alias enim quae ipsi AC duceretur parallela, etiam utraque diametro BD, KI ac proinde in duobus punctis bisecaretur, igitur patet FB lineam , sectionem contingere : quod erat primum. Quod si tangenti BF parallela ducatur quaevis AC occurrens diametro in E, erit ordinatim ad diametrum posita. Si non ducatur ex A ordinatim AL: erit AL parallela contingenti BF; quare & ipsi AC aequidistat, quod fieri non potest, cum eandem secet in A *:* igitur AL non est ordinatim posita nec quaevis alia praetcr AC, quod erat alterum. Patet igitur veritas propositionis.

PROPOSITIO XIV.

Per datum in peripheria punctum contingentem ducere.

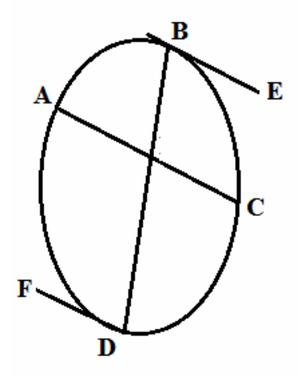
Constructio & demonstratio.



Esto ABC ellipsis & punctum in peripheria datum B, oportet per B rectam ducere quae sectionem contingat in B inveni centrum, & per hoc ex dato puncto B duc diametrum BD, ad quam ponatur ordinatim quaevis linea AC, cui per B agatur parallela BF, manifestum igitur est BF esse tangentem; igitur per datum in peripheria punctum, &c. Quod erat faciendum.

PROPOSITIO XV.

Lineae quae per extremitates diametri ductae, ellipsim contingunt, inter se aequidistant.

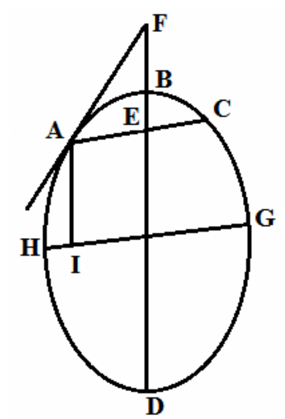


Demonstratio.

Ducatur enim quaevis AC ordinatim ad diametrum manifestum est ex 13.huius tam BC quam DF lineas illa aequidistare, adeoque & inter se. Quod fuit demonstrandum.

PROPOSITIO XVI.

Contingentes ductae per extremitates ordinatim positae, conveniunt cum diametro
extra sectionem.

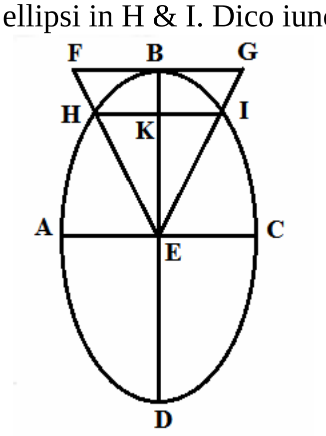


Demonstratio

Sit ABC ellipsis diametcr BD, & ordinatim posita AEC, agaturque per tangens AF; dico illam cum diametro convenire in F. Inventa enim HG diametro coniugata ipsius BD, demittatur ex A linea AI aequidistans BD, quoniam igitur AI, BD aequidistant, & AF occurrat rectae AI, patet productam quoque convenire cum BD. Quod fuit demonstrandum.

PROPOSITIO XVII.

Ellipsim ABC cuius axis BD, contingat in B linea FG, sumptisque; in contingente aequalibus partibus FB, BG, dem ittantur ex F & G diametri duae FE, GE occurrentes ellipsi in H & I. Dico iunctam HI aequidistare ipsi FG



Demonstratio.

Ponatur HK parallela FG , quae producta occurrat EG in I; erit itaque HK aequalis KI. Ergo cum rectangulum BKD ad BED, eam habeat rationem quam HK quadratum ad quadratum AE, erit quoque BKD, ut IK quadratum ad quadratum EC: unde punctum I est ad ellipsim, & HI linea perimetrum BIC, & EG rectam in eodem puncto intersecat. Quod fuit demonstrandum.

PROPOSITIO XVIII.

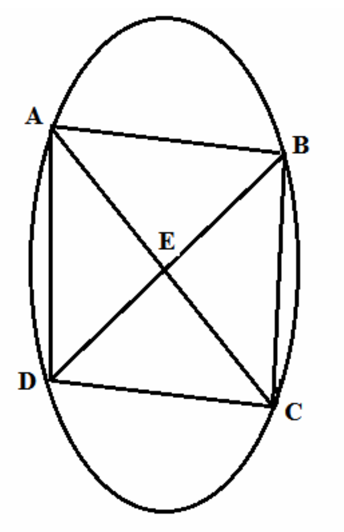
Eadem manente figura, sint ABC ellipseos axes AC, BD, & HE diameter quaecunque, oportet ex E versus C diametrum educere, aequalem ipsi HE.

Constructio & demonstratio.

Fiat angulo BEH aequalis angulus BEI, dico rectam EI satisfacere petitioni; productae enim lineae HE, EI, occurrant actae per B contingenti in F & G. Iungantur puncta H, I; quoniam anguli BEH, BEI ponuntur aequales, sunt autem & EBF, EBG anguli recti, & BE linea communis, patet FBE, GBE triangula, adeoque & latera FB, BG inter se aequaliae unde & HI aequididistat FG, estque Ut FE ad GE, sit HE ad le, quare HE, IE lineae aequales; igitur ex E diametrum eduximus, &c. Quod erat faciendum.

PROPOSITIO XIX.

Lineae in ellipsi coniungentes extrema quarumcunq; diametrorum inter se sunt aequales & parallelae.



Demonstratio.

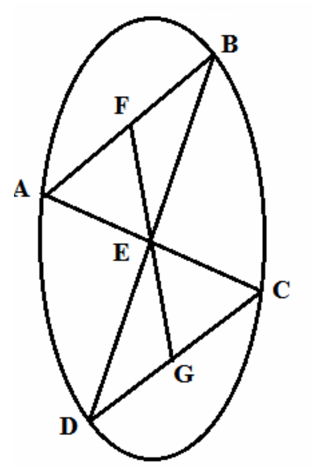
Secent ABC ellipsim diametri duae quaevis AC, BD; dico iunctas AB, CD, item AD, BC, esse interse aequales & parallelas: cum DB, AC bisectae sint in E, erit ut DE ad EB , sic CE ad AE , & permutando ut DE ad CE, sic BE ad AE sunt vero & anguli E aequales, similia igitur sunt triangula DEC, AER; ergo ut DE ad EB, sit DC ad AB; quare cum DE: EB aequentur, etiam AB, DC aequales erunt. Similiter ostentemus AD, BC aequales esse. Quod erat demonstrandum.

PROPOSITIO XX.

Lineae quae ad extremitates diametri, intra sectionem aequidistances ponuntur, aequales quoque erunt inter se,

Demonstratio.

Secet ABC ellipsim diameter quaecunque BD, ducanturque ex B & D, intra sectionem parallelae AB, CD. Dico illas inter se esse aequales. Invento centro E, & AB bisecta in F, iunge FE, & produc in G, & quoniam EF diameter bisecat AB, biseceat etiam DC, ipsi AB parallelam. Deinde quia similia sunt triangula FEB, DEG; erit DE ad DG, ut EB ad BF*:* & permutando ut DE ad EB sit DG ad BF; sed DE, EB aequantur, ergo & BF, DC, qui sunt, ut iam ostendi, ipsarum AB, DC dimidiae. Ergo & totae AB, DC aequales sunt, Quod erat demonstrandum.

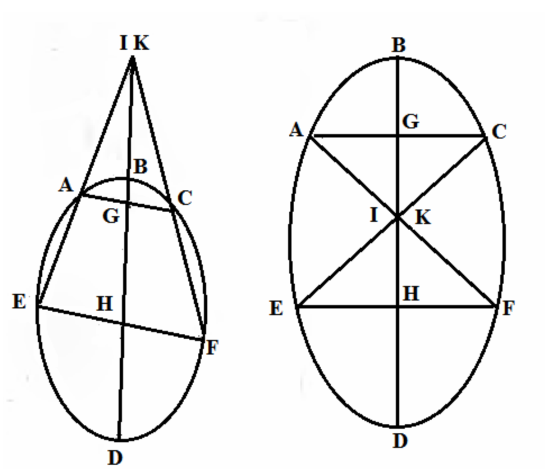


Corollarium.

Hinc sequitur iunctas AE, EC esse in directum: cum enim latera AF, FE aequalia sint duobus lateribus CG, GE & anguli aequalibus lateribus contenti, aequales, patet AFE, CGE triangula esse inter se aequalia, & angulum AEF aequalem angulo CEG, adeoque AE:EG lineas in directum.

PROPOSITIO XXI.

Lineae per extremitates duarum parallelarum inaequalium in ellipsi ductae, conveniunt in eodem puncto cum diametro, ad quam ordinatim positae sunt paralleleae.



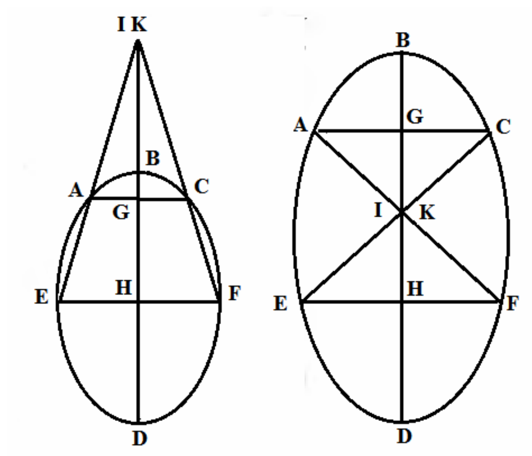
Demonstratio.

Secent ECD ellipsin duae quaevis parallelae inaequales AC, EF, ordinatim positam ad diametrum DB. Dico iunctas EA, FC cum BD diametro quam secant ordinatim in eodem puncto convenite. Quoniam ordinatim ponuntur lineae AC, EF ad diametrum BD, ambae bisecantur in G & H, unde AG ad GC ut EH ad HF, & permutando ut AG ad EH, sic GC ad HR, concurrat iam EA cum diametro in Ii alter vero FC in K, erit ergo ut IG ad IH, sic IA ad IE, sed est, ut ante ostendi, ut AG ad EH, hoc est ut IG ad IH, ergo puncta I & K eadem sunt; ergo punctum I communis est intersectio rectarum EI, FI, HI. Quod erat demonstrandum.

PROPOSITIO XXII.

Sit ABC ellipseos diameter BD ad quam ordinatim posita sit EF, ducanturque ex E & F linea occurrentes diametro in puncto G, ellipsi vero in A &C.
Dico iunctam AC, aequidistare EF.

Demonstratio.



Ponatur AI parallela EF & producta occurrat FG lineae in C, quoniam igitur EH aequalis est HF, erit & AI ipsi IC aequalis; sed quia AI aequidistat EF, erit BID rectangulum ad rectangulum BHD , ut AI quadratum ad quadratum EH. Ergo etiam, ut rectangulum BID ad rectangulum BHD, ita quadratum IC ad quadratum HF; unde punctum C est ad ellipsim & communis intersectio rectarum FG, AI cum perimetro BCF; ac proinde AC iungens puncta A, C, aequidistat EF. Quod erat demonstrandum.

PROPOSITIO XXIII.

In ellipsi ductae sint parallelae AC, EF, per quarum terminos ducantur EA, FC coeuntes in G & per G ducta GIH bisecet parallelam AC.
Dico etiam alteram bisecati.

Demonstratio.

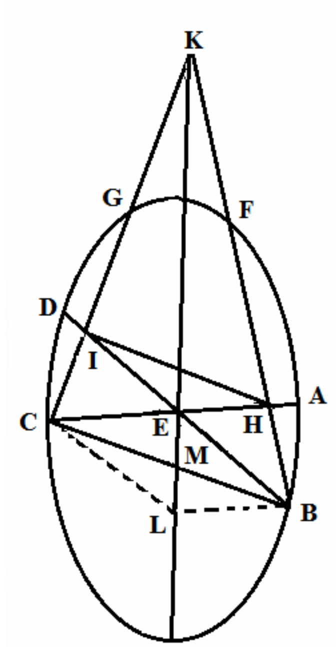
Ut HG ad IC, sic EH ad AI, & ut HG ad IG, sic FH ad CI; ergo EH ad AI, ut HF ad IC; ergo permutando EH ad HF, ut AI ad IC; sed AI, IC aequantur, ergo & EH, HF aequantur, adeoque tam EF quam AC sunt bisectae; ergo GIH diameter est. Quod erat demonstrandum.

PROPOSITIO XXIV.

Secent ABC ellipsim diametri duae AC, BD, iunctaque BC, agatur per E centrum diameter KL, secans BC bifariam K secantes secans BC bifariam in M, & ex B & C rectae ducantur BF, CG, ad idem diametri punctum K secantes AC, BD lineas in H & I.

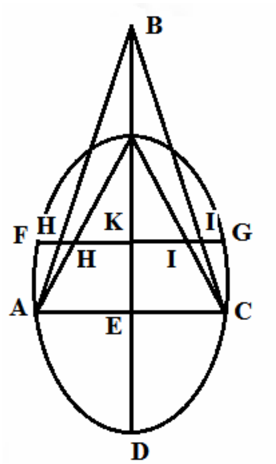
Dico rectangulum AHC esse ad rectangulum DIB ut quadratum AC ad quadratum DB.

Demonstratio.



Ponatur ex C linea CL parallela BD , occurrens diametro KL in L, & iunge BL, quoniam CL aequidistat DE, erunt EMB, CML triangula inter se similia : quia vero CM, MB aequales sunt, aequalia quoque erunt triangula CML, EMB & lateri EM, aequale latus LM; igitur in triangulis BML, CME, duae latera CM, ME,aequalia sunt duobus lateribus BM, ML, sed & anguli iis contenti BML, EMC aequantur. Ergo ad bases anguli LBM, ECB aequantur ; ergo BL, CEA sunt parallelae. Ergo BH ad HK , ut IE ad EK, hoc est (quoniam ex constructione BI, CL sunt parallelae ut CI ad IK. Ergo IH aequidistat CB, & est ut HE ad EC, sic IE ad ER, & componendo ac permutando ut EC ad EB, sic HC ad BI, sed ut CE ad BE, sic AC est ad BD, cum utraque in centro divisa sit bifariam; igitur ut AC ad BD, sic HC ad BI. Ergo etiam ut AC ad DB, sic AH ad dL. Quare cum rectangulum AHC ad rectangulum DIB rationem habeat compositam ex laterum rationibus AH ad DI, & HC ad BI, quae ambae ostensae sunt eadem esse cum ratione AC ad BD, erit rectangulorum ratio duplicata ratione AC ad BD, hoc est eadem quae quadratorum AC, BD. Quod erat demonstrandum.

PROPOSITIO XXV.



Duae lineae CG,BF intra ellipsim ductae occurrant diametro ellipseos MK in eodem puncto K. Ductae sint deinde binae aliae diametri BD, CA quae ita secentur a rectis CG, BF ut rectangula BID, CHA quadratis BD, AC proportionalis sint.
Dico iunctas IH, CB esse parallelas.

Demonstratio.

Quoniam est ut quadratum BD ad quadratum CA, hoc est ut quadratum ED ad quadratic EA, sic rectangulum BID ad rectangulum CHA ; erit permutando ut quadratum ED, (hoc est rectangulum BID cum quadrato EI ad rectangulum BID est ad quadratum EI) ad rectangulum BID, ut quadratum EA, (hoc est rectangulum CHA cum quadrato EH) ad rectangulum CHA ad quadratum EH. Permutando igitur rectangulum BID est ad rectangulum CHA ut quadratum EI ad quadratum EH sed etiam est rectangulum BID ad retangulum CHA ut quadratum BD ad quadratum CA; hoc est ut quadratum ED ad quadratum EA. Itaque quadratum EI est ad quadratum EH ut quadratum ED ad quadratum EA: adeoque recta EI ad rectam ED, hoc est EB, ut recta EH ad rectam EA, hoc est EC, parallelae sunt igitur IH, CB. Quod erat demonstrandum.

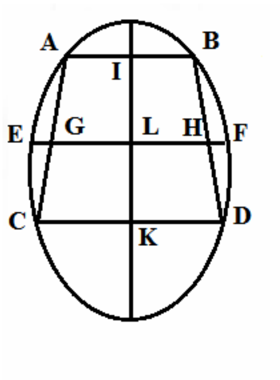
PROPOSITIO XXVI.

Esto ABC ellipsos diameter BD, ad quam ordinati, posita sit recta AC : ductisque ex A & C lineis quae diametrum in eodem puncto B secent, ducatur FG parallela AG, occurrens AB, CB in H & I, diametro vera BD in K.
Dico FH, GI lineas esse aequales.

Demonstratio.

Quoniam FG aequidistat AC ordinatim posite ad BD, erit & FG, quoque ordinatim posita ad diametrum BD, adeoque in K bifariam divisa ; sed & HI in K divisa est bifariam, uti AC in E, demptis igitur aequalibus HK, IK, reliquae FH, IG aequales sunt. Quod erat demonstrandum.

PROPOSITIO XXVII.



Secent ABC ellipsim ducae quaevos parallelae AB, CD, iunctisque AC, BD, ducatur EF parallela AB, secans AC, BD lineas in G & H.
Dico EG, FG rectas esse aequales.

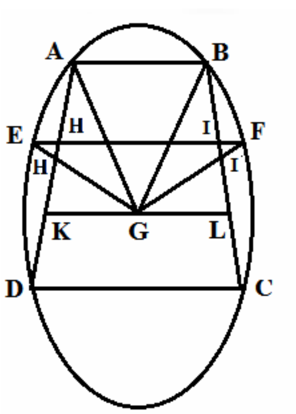
Demonstratio.

Divisis AB, CD bifariam in I & K, agatur per I & K, linea IK; erit illa diameter, & EF lineam, rectae AB parallelam secabit bifariam in L; sed & HG in L secta est bifariam ut CD in K; vel AB in I , ablatis igitur aequalibus GL, LH, manent EG, FH, reliqua aequales.Quod erat demonstrandum.

PROPOSITIO XXVIII.

Secent ABC ellipsim duae quaevis parallelae AB, CD, iunctisque AD, BC ducatur ENMF parallela AB, & ex E & F, semidiametri ponantur EG, FG, quae AD, BC lineas secent in H & I.
Dico EG, FG in H & I proportionaliter esse divisas.

Demonstratio.



Ducatur per G, KL aequidistans AB , occurrens AD, BC, in K & L. Quoniam EF, KL aequidistant, erit ut EN ad KG, sic EH ad HG; & FI ad IG, ut FM ad LG; sed ut EN ad KG, sic FM est ad LG, (cum EN, FM item KG, LG, aequales sint,) igitur ut EH ad HG, sit FI ad IG. Quod fuit demonstrandum.

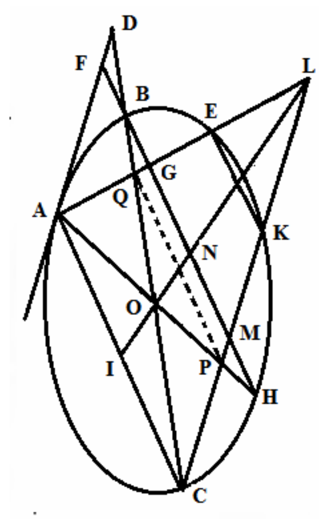
Corollarium.

Hinc patet iunctam HI aequidistate DC, adeoqae lineas AD, BC, in H & I proportionaliter esse divisas.

PROPOSITIO XXIX.

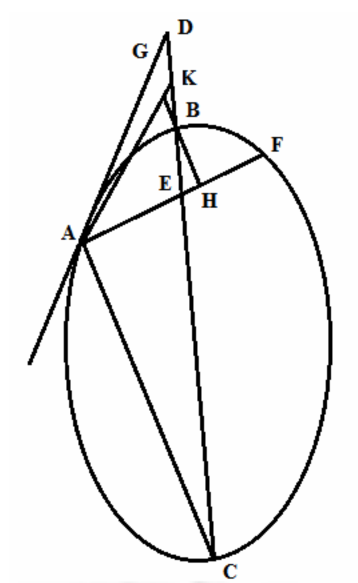
Ellipsim, cuius diameter BC centrum O, contingat AD occurrens diametro in D, ductaque ex puncto A ordinatim AQE, & iuncta AC, per B ponatur recta FBG parallela rectae AC.
Dico FB, BG aequales esse.

Demonstratio.



FG occurrat ellipsi in H, iunganturque HO, AO quae erunt in directum. Tum AC bisecta in I, ducatur per I diameter IOL occurrens rectae AE in L & iungantur puncta LC, per rectam LC, occurrentem ellipsi in K, & rectae FG in M; rectae vero HO in P.
Quoniam AC ex constructione ordinatim posita est ad diametrum IL, rectaeque per A & C duplae occurrunt diametro in eodem puncto L, erit EK parallela AC. Est vero & BH parallela ipsi AC ex hypothesi : & semidiametra OB, OH secant AE, CK in Q & P, ergo QP aequidistat rectae BH, tres igitur AC, QP, EK sunt parallelae. Quare cum ex hypothesi AE bisecta sit in Q, erit & CK bisecta in P, ac proinde ordinatim posita ad diametrum AH. Itaque CK aequidistat tangenti AD. Est autem & FM ex hypothesi parallela ad AC, ergo FM aequalis est AC , sed etiam BH aequalis est AC. Igitur FM, BH aequales sunt; quare communi dempta BM aequantur FB, HM. Atqui etiam GB, HM aequales sunt. itaque FB, GB aequales sunt. Quod erat demonstrandum.

PROPOSITIO XXX



Ellipsim ABC cuius diameter BC contingat in A recta AD conveniens cum diametro in D : ductaque ex A sit linea AF ordinatim ad diametrum BD.
Dico rectam DC in B & E divisam esse extrema & media ratione proportionali, hoc est, ut CD est ad BD, sic CH est ad HB: &: si divisa fuerit in B & E extrema & media ratione proportionali agaturque per E ordinatim linea AF ad BC: dico iunctam AD, sectionem contingere.

Demonstratio.

Iuncta AC, agatur per B linea GH parallela recta AC ocurrens AF lineae in H & AD tangenti in G. Quoniam AC, BH lineae aequidistant, erit ut AC ad BH, sic CE ad EH, sic CE ad EB: sed ut AC ad BH, sic AC est ad GB, (quia GB, EH sunt aequales) igitur ut AC ad GB, sic CE est ad BE: est autem ut AC ad GB, sic CD ad DB (quia

GB, AC aequidistant) igitur ut CD ad DB, sic CE est ad BE. Quod erat primum iam ut CD ad BD, sic CH ad HB, si per E ordinatim agatur AF: dico iunctam AD sectionem contingere in A. si enim AD non tangit, ponatur per A tangens quae BD diametro occurrat in K, erit igitur ut CE ad EB, sic CK ad KB, sed est ut CE ad EB, sic CD ad DB, igitur CK ad KB, quod fieri non potest, cum punctum K supra vel infra D cadat. Igitur AK non est tangens nec quaevis alia praeter AD. Quod fuit demonstrandum,

Corollarium.

Propositiones 29 & 30 etiam in circulo sunt verae, quamvis autem saepius contingat ut quae hoc libro de ellipsi demonstramus locum etiam habeant in circlo, circuli tamen mentionem non facio nisi ad sequences demonstrationes assumi debeat.

PROPOSITIO XXXI

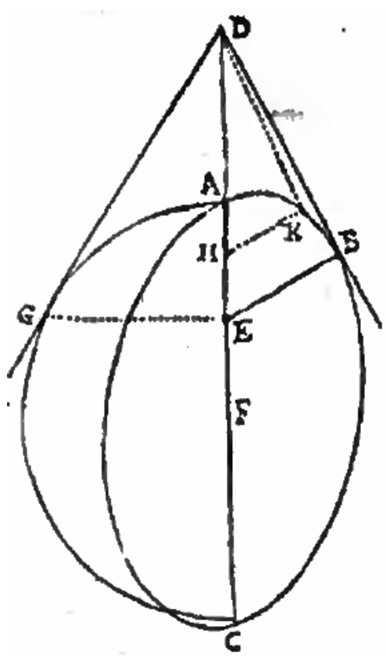
Eadem manente figura propositum sit a dato extra sectionem puncto D, tangentem ducere.

Constructio & Demonstratio.

Ducatur ex D diameter DBC, fiatque ut CD ad DB, sic CB ad EB, & per E ad BC, ordinatim ponatur AF, iungturque AD , patet per praecedentem AD lineam sectionem in A contingere; igitur a, dato extra ellipsim puncto; &c. Quod erat faciendum,

PROPOSITIO XXXII

Ellipsim ABC cuius diameter AC contingat recta BD in B, conveniens cum diametro in D : & ex B ducatur BE ordinatim ad diametrum AC: centrum autem sectionis sit F.

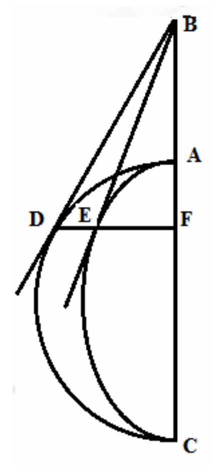


Dico FE, FA, FD lineas esse in continua ratione & si FE, FA, FD fuerint continuae proportionaLEs, & per E ordinatim recta agatur EB, dico iunctam BD sectionem contingere, Est Apollonii.

Demonstratio.

Centro F intervallo FA circulus describatur AGC, tum ex E puncto normalis educatur ad diametrum AC occurrens circulo in G: ducaturque recta GD, quoniam EB recta ponitur ordinatim ad diametrum AC & per B acta tangens convenit cum eadem diametro in D, erit ut CD ad DA sic CE ad EA : est autem in circulo, recto EG normalis ad diametrum EG, igitur &. recta GD circulum contingit in G; quare in circulo erunt FE, FA, FD lineae continuae proportionaLEs : sunt autem eaedem lineae communes ellipsi, igitur & in ellipsi erunt FE, FA, FD in continua analogia. Quod si FE, FA, FD continuae proportionaLEs sint, & per E ducatur ordinatim EB, dico iunctam BD ellipsim contingere in B; sin vero: ducatur ex D recta DK contingens ellipsim in K, &: ex K ordinatim ponatur KH; igitur per primum partem huius FH ad FA, ut FA ad FD; sed etiam ex hypothesi est, FE est ad FA, ut FA ad FD. Ergo FE est ad FA, ut FH est ad FA, quod fleri non potest, cum FG sit maior aut minor quam FE. unde DK non est contingeris, sed DB. Quod fuit demonstrandum.

PROPOSITIO XXXIII



Esto ABC ellipseos axis AC, super quo ut diametro semicirculus describatur ADC, assumptoque in axe puncto F quod non sit centrum, erigatur ex F orchogona FD occurrens ellipsi in B.
Dico contingentes per B & D actas, axi AC in uno eodemque puncto occurrere.

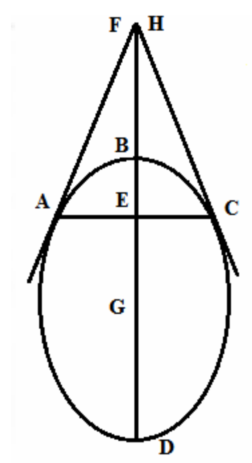
Demonstratio.

Agatur per B contingens BE, conveniens cum axe in E, iunganturque ED; quoniam FB ordinatim posita est ad axem & BE sectionem contingit, erit CF ad FA, ut CE ad EA*:* unde & iuncta ED circulum contingit; igitur contingentes per B & D actae, conveniunt cum axe in uno eodemque puncto. Quod fuit demonstrandum.

Corollarium.

Hinc facile etiam demonstrabimus si duae tangentes in eodem puncto diametro occurrant normalem FD, quae si per unum contactum D transeat, transire etiam per alterum.

PROPOSITIO XXXIV



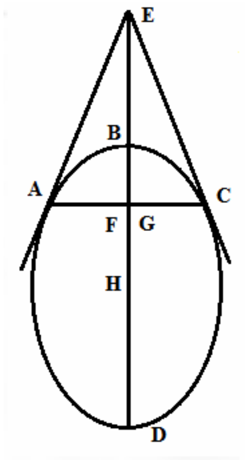
Esto ABC ellipsis diameter BD, ad quam ordinatim ponatur AC aganturque; per A & C contingences.
Dico illas diametro in uno eodemque puncto occurrere.

Demonstratio.

Per 16.huius patet singulas contingences per A & C ductas cum diametro convenire si igitur non conveniant in eodem puncto, occurrat AF contingens diametro in F, & CH in H: Quoniam tangens AF concurrit cum diametro in F, erit ut DE ad EB ; sic DF ad FB; sic DH ad BH, quod fieri not potest; quare tangentes non occurrunt diametro in diversis punctis: ergo in eodem. Quod erat demonstrandum.

PROPOSITIO XXXV

Esto ABC ellipsis, diameter BD producta utcunque in E, & ex E demissae EA, EC sectionem contingant in A & C .
Dico iunctam AC, ordinatim esse positam ad diametrum BD.

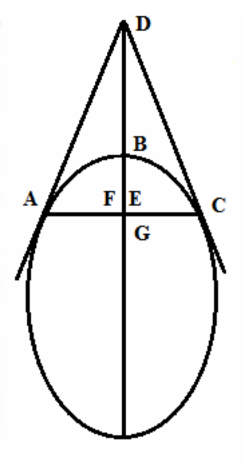


Demonstratio.

Ponatur AF ordinatim ad BD sitque H centrum ellipseos; erit igitur linea EH divisa
in B & F in tres continuae proportionales, demittatur quoque CG ordinatim ad BI erit
denuo EH divisa in B, & G, in tres lineas in analogia continuae igitur F & G, puncta sunt
eadem; quare recta AFC, est ordinatim posita ad diametum BI. Quod fuit demonstrandum.

Corollarium.

Hinc sequitur si ellipsim ABC contingant in A & C, rectae duae AD, CD convenientes in D; iunctaque AC, bifariam secetur in E; rectam DE transire per centrum sive iunctam DE esse diametrum sectionis, si enim ED non sit diameter, ducatur ex D diameter DF; occurrens AC lineae in F; erit igitur per praecedentem AC linea in F; divisa bifariam, adeoque punctum F, idem cum E, unde DF recta eadem cum linea DE: quod est contra suppositum; quare DE sectionis est diameter. Quod erat demonstrandum.



PROPOSITIO XXXVI.

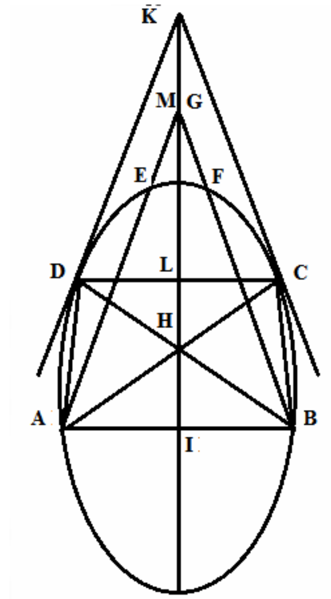
Si ellipsim tangant binae rectae coeuntes in D, & ex centro ducantur GA, GC, GD.
Dico triangula GCD, GAD esse aequalia.

Demonstratio.

Puncta contactuum iungantur recta AC, quoniam AC bisecta est in E, triangula
GAE, GEC, item DEC, DEA aequalia erunt *:* duo itaque triangula DEC, DEA, hoc est totum DCG, aequabuntur duobus triangulis DEA, EAG, hoc est toti GAD. Quod erat demonstrandum.

PROPOSITIO XXXVII.

Ellipsim ABC secent AC, DB: diametri quaevis agantur per C & D tangentes, quae per 34. huius conveniunt cum diametro HK in eodem puncto.



Dico lineas AG, BG ex A & B ductas, ipsis DK, CK aequidistantes, diametrum HK, in uno eodemque puncto intersecare.

Demonstratio.

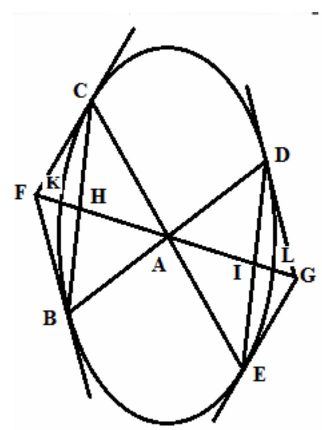
Ponatur AE occurrere diametro in G & BF in M; iunganturque DC, AB, DA, CB. Quoniam DC iungit tangentes DK, CK, bisecatur a diametro HK in L, est vero AB parallela ad DC; ergo & haec a diametro bisecatur in I; quare cum totae DC, AB aequales ; cum igitur etiam DL, AI, sint parallelae, quae eas iungunt DA, IL parallelae, ergo figura AGDK parallelogrammum; ergo figure AGKD parallelogrammum est, proindeque DA aequales est GK; simili modo ostendimus BC aequalem esse MK. Quare cum DA, BC,sint aequales, etiam KG, KM aequales erunt, unum igitur idemque punctum sunt G & M in quo parallelae tangentibus DC, CK, ductae e punctis A, B, occurrunt diametro. Quad fuit demonstrandum.

PROPOSITIO XXXVIII.

Ellipsim cuius centrum A secent quaevis duae diametri BD, CE, iunctisque BC, DE ponatur FG diameter, quae BC bifariam dividat in H, ipsi autem ED occurrat in I, tum per C & B, item D & E, contingentes agantur,que FG diametro occurrent in ijsdem punctis F & G.

Dico aequalia esse inter se; primo triangula ACF, ABF, secundo triangula ACF, ADG, tertio triangula CBF, DEG.

Demonstratio.



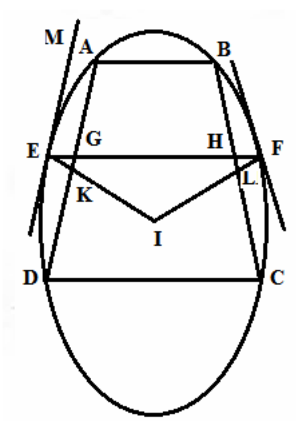
Occurrat FG diameter ellipsi in K & L. Quoniam igitur BC ex hypothesi in H divisa est bifariam, erit tam ACH triangulum aequale triangulo AHB, quam HCF aequale triangulo HFB, unde totum triangulum ACF, aequale est toti triangulo AFB, quod erat primum. Rursum cum BC, DE aequidistent & BC in H divisa sit bifariam a diametro FG, erit & ED in I, bifariam divisa. quare cum totae CB, DE sint aequales, erunt & harum dimidiae HC, DI aequales, quare cui, etiam sint parallelae rectae, CD, HI,quae illas iungut, sunt parallelae ; triangula igitur ACH, ADI sunt inter easdem parallelas. Sunt autem & bases AH, AI aequales, est enim AK aequalis, AL, & bases KH, AI aequales , (est enim AK aequalis IL) ergo triangulam ACH aequatur triangulo ADI. Iam vero AH, AK, AF, idemque AI, AL, AG sunt continue ; quare ratio AH ad AF, duplicata est rationis AH ad AK; & ratio AI ad AG duplicata est rationis AI ad AL Cum igitur rationes AH ad AK, AI ad AL, eadem sint (AH enim ipsi AI & AK, ipsi AL aequalis est) erunt & rationes AH ad AF, AI ad AG earundem rationum duplicatae; eaedem inter se : ac proinde triangulum quoque; AHC est ad triangulum AEC, ut triangulum AID ad triangulum AGD. Quare cum triangula ACH, ADI ostensa sint aequalia, etiam AFC, AGD aequalia erunt, quod erat alternum. Ex quo iam patet quibus si addas FCH, GDI, aequalia esse, quibus si addas FBH,GEI, quae eodem plane discursu ostendimus aequalia, erunt tota triangula CBF, DEG, aequalia. Quod erat tertio loco demonstrandum.

Corollarium.

Hinc patet quadrilatera CFBA, DGEA, aequalia esse, eodem enim discursu probabimus aequalia esse triangula ABF, AEG, quo probavimus aequari ACF, ADG.

PROPOSITIO XXXIX.

Secent ABC ellipsim duae quaevis parallelae AB, CD, iunctisque AD, CB, recta EM parallela ipsi AD, contingat sectionem in E, & ex E ducatur EF, aequidistans AB, secans AD, CB lineas in G & H.
Dico contingentem per F ductam aequidistare ipsi BC.



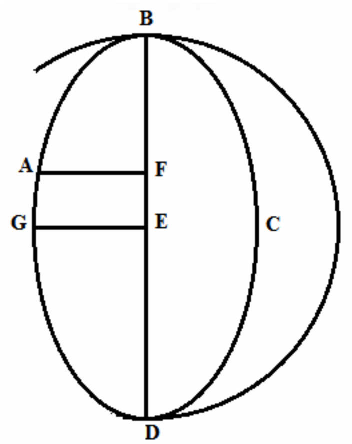
Demonstratio.

Ducantur ex I centro lineae IE, IF occurrentes AD, CB in K & L : quoniam igitur EM contingens, aequidistat AD & IF diameter ad contingentem ducta, secet AD lineam in K, erit AD in K divisa bifariam; est autem recta BC in L divisa sicut AD in K, recta enim KL iungens puncta K, L parallela ellipsis AB, DC. Igitur & BC in L secta est bifariam a diametro IF; unde & tangenti per F ductae aequidistat. Quod fuit demonstrandum.

PROPOSITIO XL.

Circulus super axe maiore ut diametro descriptus ellipsi exterius in duobus tantum punctis occurrit.

Demonstratio.



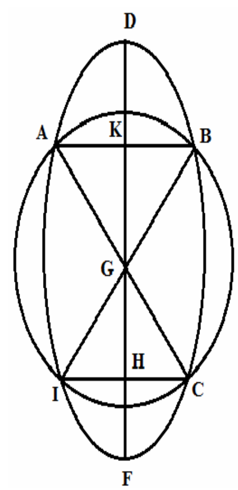
Sit ABC ellipseos axis maior BD, centroque illius E intervallo EB circulus describatur,
dico illum ellipsi in duobus tantum punctis B & D occurrere. Occurrat enim si fieri possit insuper in puncto A & per A ordinatim ad axem agatur AF, ducaturque axis minor GE, erit igitur ut BFD rectangulum ad quadratum FA sic BED rectangulum ad quadratum EG: sed BFD rectangulum in circulo est aequale quadrato FA; igitur & rectangulum BED, id est quadratum BE aequale est quadrato GE, quod fieri non potest cum BE linea maior sit quam GE, igitur circulus ellipsi non occurrit in A: nec in alio quavis puncto, praeter B & D. Quod fuit demonstrandum.

Corollarium.

Simili discursu demonstrabimus circulum circa minorem ellipseos axem descriptum in duobus tantum punctis extremis axeos ellipsi occurrere, & totum intra ellipim existere.

PROPOSITIO XLI.

Circulus centro ellipseos descriptus, si ellipsi in secat, in quatuor punctis secabit.



Demonstratio.

Sit enim ellipseos centro G descriptus circulus secas ellipsim in B, ducatur axis FD, & recta BGI; tum ordinarim applicetur BKA occurrens ellipsim A, ducantur item recta AGC, IC; in triangulis BKG, AKG, BK, AK aequantur, & KG est communis, angulique ad K recti; ergo GB, GA aequales; quare cum punctum B sit ad circulum, erit & punctum A est autem idem punctum etiam ad ellipsim, ergo circulus ellipsim secat in A. Deinde AB, IC sunt parallelae, adeoque cum angulus AKH rectus sit, erit etiam rectus IHK; ac proinde IC ordinatim est posita ad axe in DF, ac bisecta in H sunt autem totea AB, IC aequales, ergo AK, IH earum dimidiae etiam sunt aequales. In triangulis igitur GKA, GHI, AK ipsi IH, & KG ipsi HG est aequalis, anguli vero AKG, IHG etiam aequales sunt; ergo GA, GI aequantur; quare cum punctum A sit ad circulum, erit & punctum I, atqui etiam punctum I est ad ellipsim. Ergo circulus ellipsim secat in I. similiter oftendemus circulum ellipsi occurrere in C. In quatuor igitur punctis secat. Quod erat demonstrandum.

Corollarium.

Quod autem non secet ellipsim circulus in pluribus punctis quam quatuor, facile colligetur ex demonstrationc iam posita.