[155]

PROGRESSIONUM GEOMETRICARUM

PARS QUARTA

Part Four of Geometrical Progressions.

Doctrinam praecedenti parte in planis demonstratam, corporibus, solidisque, applicat.

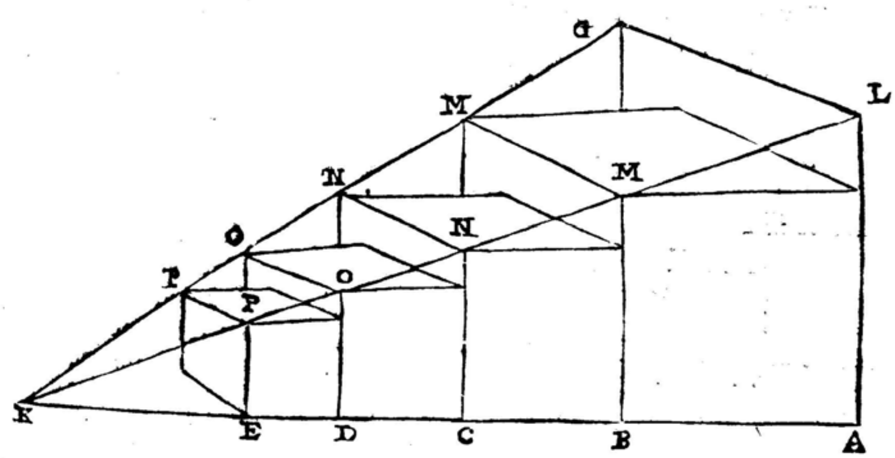
Having demonstrated the principles in the preceding part for plane figures, it is now applied to bodies and solid figures.

PROPOSITIO CLXIII.

Data sit quadratorum continue proportionalium series, basibus in directum positis, cuius longitudinis terminus sit K; super singulis autem quadratis cubi construantur.

Dico constitui seriem cuborum continue proportionalium, quae eumdem quoque habeant terminum longitudinis K.

Demonstratio.



Prop. 163. Fig. 1.

Ratio cubi AM ad cubum BN, triplicata est rationis [a] AB ad BC, item ratio cubi BN ad cubum CO, triplicata est rationis [b] BC ad CD; id est rationis AB ad BC; (ponuntur enim quadrata AM, BN, CO &c. in continua analogia) quare cum rationes utraeque, cubi AM ad cubum BN, & cubi BN ad cubum CO, triplicatae sint rationis AB ad BC, eadem erunt. Sunt igitur cubi AM, BN, CO in continua analogia; eodem modo erunt & reliqui omnes continue proportionales. Quod erat primum, ex quo patet etiam secundum : Cum enim quadratorum & cuborum series, pariter semper procedant, idem utriusque terminus sit longitudinis necesse est. Quod erant demonstranda. *a 33 undicimi; b 20 sexti ?*

L2.§ 4. PROPOSITION 163.

A series of squares is given in continued proportion, with the bases ordered along a line, and the terminus of the series is at a length K ; moreover, cubes are constructed on the individual squares.

I say that a series of cubes in continued proportion has been constructed, which have the same terminus of length K.

Demonstration.

The ratio of the cube [in the volume sense] AM to the cube BN, is the cube [in the power sense: the text calls this the triplicate ratio, which is confusing for us as we thing of the triplicate rather as × 3] of the ratio [a] AB to BC, likewise the ratio of the cube BN to the cube CO, is the cube of the ratio [b] BC to CD; or of the ratio AB to BC; (for the squares AM, BN, CO, etc. are placed in an analogous continued proportion) since each of the ratios of the cube AM to the cube BN, and of the cube BN to the cube CO, are the cubes of the same ratio AB to BC. Therefore the cubes AM, BN, CO are in continued analogous proportion [in the sense that the common ratio is derived from another simpler ratio]; and in the same manner the rest of the cubes are in continued proportion. Which proves the first part of the proposition, from which the second part is also apparent : For indeed the series of squares and cubes always proceed equally, and likewise the terminus of each by necessity is of the same length. Q.e.d. *a 33 undicimi; b 20 sexti ?*

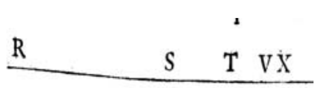
PROPOSITIO CLXIV.

Iisdem positis; primae AB, & quartae DE, aequales fiant RS, ST : continueturque ratio RS ad ST, per plures semper terminos TV, VX, &c.

Dico cubum primum AM, esse ad quemlibet cubum seriei propositis, verbi gratia ad quartum DP, ut est linea RS ad quartam VX.

[p. 156]

Demonstratio.



Prop. 164. Fig. 1

Cubus AM ad cubum DP, est in triplicata [a] rationis AB ad DE, hoc est per constructionem rationis RS ad ST. Atque etiam RS ad quartam VX, est in triplicata ratione eius, quam habet RS ad ST : ergo ut RS ad VX, sic cubus AM ad cubum DP. Simili ratiocinatione ostendemus cubum primum, ad quemvis seriei cubum, eandem habere rationem, quam habet RS ad lineam quae aeque distabit a prima RS, atque cubus a cubo primo AM. Quod erat demonstrandum. *a 33 undecimi*.

Corollarium.

Duo haec theoremata eadem servata demonstratione, ad omnia similium corporum genere licebit extendere.

L2.§4. ***PROPOSITION 164.***

With the same figure in place ; RS and ST are made equal to the first and fourth lengths AB and DE: and the ratio RS to ST is continued through many terms TV, VX, etc., in the same manner.

I say that the first cube AM, is to any cube of the proposed series, for argument's sake to the fourth DP, as the line RS is to the line VX.

[p. 156]

Demonstration.

The cube AM to the cube DP, is in the ratio a of the cube of AB to DE, that is by the construction as RS to ST. And indeed RS to the fourth term VX, is in the cubic ratio that RS has to ST : hence as RS to VX, thus the cube AM to the cube DP. By similar reasoning, we can show that the first cube, to any cube of the series you wish, has the same ratio as RS has to the line which is at the same distance from the first RS, and the cube from the first cube AM. Q.e.d. *a 33 undecimi.*

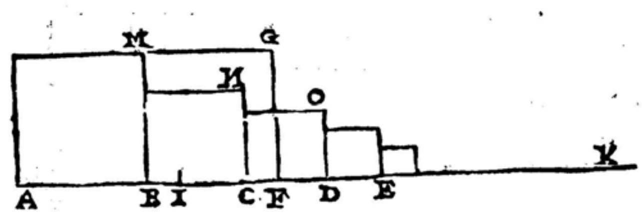
Corollarium.

These two theorems established by the same demonstration can be extended to kinds of similar bodies.

PROPOSITIO CLXV.

Data sit quadratorum series, habens bases in directum, & terminum longitudinis K. super quadratis autem singulis, extructi sint cubi : Petitur seriei cubicae aequale parallelipipedum exhiberi.

Constructio &Demonstratio.



Prop. 165. Fig. 1

Seriei rationis primae baseos AB, ad quartam DE, fac [b] aequalem AF; & super AF, in altitudinem AB, rectangulum AG : deinde super rectangulo AG, in altitudine AB, construe parallelepipedum rectangulum. Dico hoc seriei cubiae aequari. Vel super quadrato AB in altitudine lineae, aequalis seriei rationis AB ad DE, fac parallelepipedum. Dico hoc esse quaesitum. Fiat enim BI aequalis DE. Quoniam igitur ex constructione seriei rationis AB ad BI, hoc est ut AB ad DE, aequalis est AF, erit [c] AF ad BF, ut AB ad BI, hoc est ut AB ad DE : quia autem quadrata ex hypothesi sunt continua, erunt [d] lineae AB, BC, CE, DE, &c. in continua analogia : unde & cubus [e] AM ad cubum BN, ut AB ad DE, hoc est (sicut ostendi) ut AF ad BF : Quare cum parallelepipedum AG, sit ad parallelepipedum BG, [f] ut basis AG ad basim BG, hoc est ut [g] AF ad BF, erit cubus AM ad cubum BN, ut parallelepipedum AG, ad parallelepipedum BG : Atqui tota [h] series cubica MK, est ad seriem cubicam, NK, ut cubus AM ad cubum BN, ergo parallelepipedum AG, est ad parallelepipedum BG, ut series cubica MK, ad seriem cubicam NK : ergo dividendo cubus AM, est ad parallelepipedum BG, ut cubus idem AM, ad seriem cubicam NK; (cum enim parallelepipeda AG, BG constructa sint supra bases AG, BG, in communi altitudine AB, patet cubum AM esse excessum parallelepipedum BG super parallelepipedum BG.) Itaque series cubica NK & parallelepipedum, erunt aequalia; communique addito cubo AM, tota series cubica, & parallelepipedum AG aequalia erunt. Factum igitur est quod petebatur. *b 80 huius; c 82 huius; d 22 sexti; e 33 undecimi; f 25 undecimi; g 1 sexti; h 82 huius.*

[p.157]

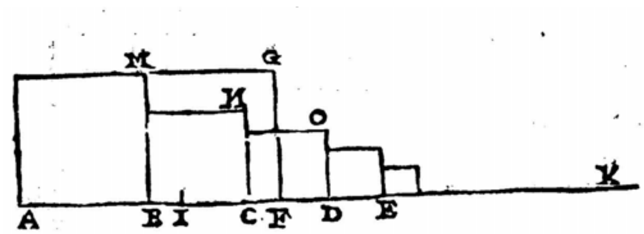
Corollarium.

Itaque si fuerint propositae binae, vel plures cuborum progressiones, etiam rationum dissimilium, cognoscetur earum proportio inter se, si per hanc propositionem singulis cuborum progressionibus aequalia parallelepipeda constituantur.

L2.§4. *PROPOSITION 165.*

A series of squares is given, having the bases ordered along a line, and having the terminus at a length K. Moreover, upon the individual squares, cubes are to be set up : It is required to exhibit a parallelepiped equal in volume to the series of cubes.

Construction &Demonstration.



Prop. 165. Fig. 1

Make AF equal to the sum of the series of ratios of the first base AB to the fourth base DE b; and on AF with height AB, make the rectangle AG : then on the rectangle AG, with height AB, construct a rectangular parallelepiped [ppd.]. I say that this ppd. has a volume equal to the series of cubes. Or alternatively, on the square AB, construct a parallelepiped with a height equal to the sum of the series of ratios AB to DE. I say that this is the volume sought. For if BI is made equal to DE, then from the construction of a series of ratios AB to BI, that is in the same ratio as AB to DE, for which the sum is equal to AF, then c AF is to BF as AB to BI, or as AB to DE : moreover since the squares are in a continued progression by hypothesis, the lines d AB, BC, CE, DE, &c. are in an analogous continuous progression: from which the cube e AM is to the cube BN, as [the cube of] AB is to [the cube of] DE, or (as was shown) as AF to BF : Whereby, since the parallelepiped AG is to the parallelepiped BG, f as the base AG is to the base BG, or as g AF to BF, the cube AM is to the cube BN, as the ppd. AG is to the ppd. BG : But h the sum of the whole series of cubes MK is to the series of cubes NK, as the cube AM is to the cube BN, hence the ppd. AG is to the ppd. BG, as the sum of the series of cubes MK is to the series of cubes NK : hence by division, the cube AM is to ppd. BG, as likewise the cube AM is to the series of cubes NK; (since indeed the ppd's AG and BG are constructed on the bases AG and BG, with the common altitude AB, it is apparent that the cube AM is the difference of the ppd's AG and BG.) Thus the series of cubes NK and the given ppd BG are equal; and by adding the common cube AM, the sum of the series of cubes and the ppd. AG are equal to each other. Therefore what was sought has been done. *b 80 huius; c 82 huius; d 22 sexti; e 33 undecimi; f 25 undecimi; g 1 sexti; h 82 huius.*

[For BI = DE, then AB/BI = AB/ DE, and the sum of the cubes is AF, then AF/BF = AB to BI, or AB/ DE : also AB, BC, CE, DE, &c. are in an analogous continuous progression: and cubeAM/cube BN = cube AB^3/DE^3, or AF/BF : Whereby, as ppd.AG/ppd. BG = rect.AG/rect.BG = AF/BF,
AM/ BN = ppd. AG /ppd.BG : But the sum MK/sum NK = AM/BN, hence ppd.AG/ppd.BG = sum MK/NK : (ppd.AG/ppd.BG – 1) = (sum MK/sumNK – 1); hence cube AM/ppd. BG = cube AM/sum NK; Thus the series of cubes NK = ppd BG ; In some respects this proof resembles a modern inductive proof, and up to this stage it shows the self – consistency of the argument, rather than an actual formula for the sum.

In modern terms, the sum of the series of cubes is $\frac{a^3}{1-r^3}$; and the sum of the series of lengths of the sides of the cubes, AF in the text, is $\frac{a}{1-r^3}$, while the series BF, with the first term missing, is $\frac{ar^3}{1-r^3}$; hence the

ratio AF/BF = $1/r^3$. This is the method used to evaluate the infinite sums of linear, square, or cubic terms, where the sum is assumed for the whole series from the first term, and likewise from the second term, and the ratio taken, which is then set equal to either $1/r$, $1/r^2$, or $1/r^3$. Thus, in this case,
AF/BF = $1/r^3$ = AB / DE. Hence, AF can be determined as above to be $\frac{a}{1-r^3}$, from which the sum for the series of cubes $\frac{a^3}{1-r^3}$ follows.]

[p.157]

Corollarium.

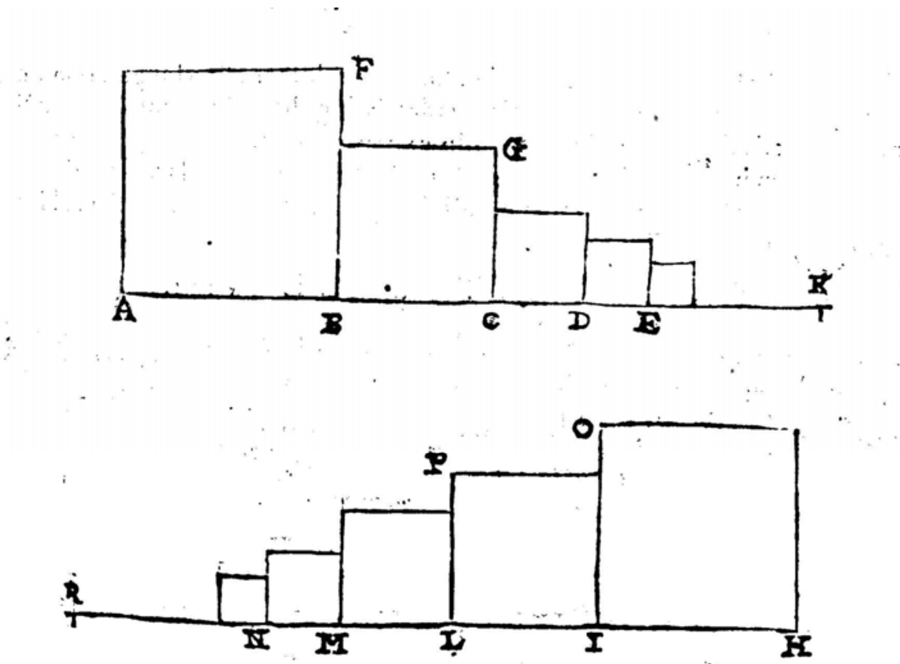
Thus if two or more progressions of cubes are proposed, also having different ratios, the proportion is known between these, if the parallelepipeds are constructed for the individual progressions of the sums of the cubes.

PROPOSITIO CLXVI.

Dentur binae, sed aequales cuborum progressiones rationum dissimilium. Dico seriem cubicam AK, ad seriem cubicam HR, rationem habere compositam, ex ratione primi quadrati AF, ad primum quadratum HO, & ratione seriei rationis AB primae, ad quartam DE, ad seriem rationis HI, primae, ad quartam MN.

Demonstratio.

Parallelepipedum factum super quadrato AB, in altitudine lineae seriei rationis AB ad DE, per praecedentem seriei cubicae AK, erit aequale : similiter parallelepipedum super quadrato HI, in altitudine lineae rationis HI ad MN, seriei cubicae HR aequale est. Quare cum series cubicae ponantur aequales, dicta quoque parallelepipida aequalia erunt; ergo reciprocam habent basium & [a] altitudinem rationem, hoc est habent rationem compositam ex rationibus basium & altitudinum rationem : Quare & series cubicae AK & HR illis aequales, rationem habent compositam ex ratione dictarum altitudinem, hoc est ex ratione seriei AB, DE, &c. ad seriem HI, MN, &c. & ex ratione basium, hoc est ex ratione quadrati AF ad quadratum HO. Quod erat demonstrandum. *a 34 undecimi.*



Prop.166. Fig. 1.

L2.§4. PROPOSITION 166.

Two series of cubes in progressions with dissimilar ratios but with the same sum are given.

I say that the series of cubes AK, to the series of cubes HR, has a ratio composed from the ratio of the first square AF to the first square HO, and from the ratio [composed from two ratios, the first of which is the sum] of the series with a ratio of the first term AB to fourth term DE, etc., [to the second which is] the sum of the series with the ratio of first term HI to fourth MN, etc.

Demonstration.

A parallelepiped is constructed on the square AB, with the height in the ratio of AB to DE, which is equal to the sum of the cubes AK by the preceding proposition : similarly a parallelepiped constructed on the square HI, with a height in the ratio HI to MN, is equal to the series of cubes HR. Whereby since the series of cubes are put equal, then the said parallelepipeds are also equal ; hence they have a reciprocal ratio of bases and heights [a], that is they have a ratio composed from the ratios of the bases and the ratio of the heights : Whereby the series of cubes AK and HR from these are equal, they have a ratio composed from the ratio of the given heights, that is from the ratio of the series AB and DE, &c. to the series HI and MN, &c. and from the ratio of the bases, that is from the ratio of the square AF to the square HO. Q.e.d.
a 34 undecimi.

[This proposition follows directly from the previous proposition.]

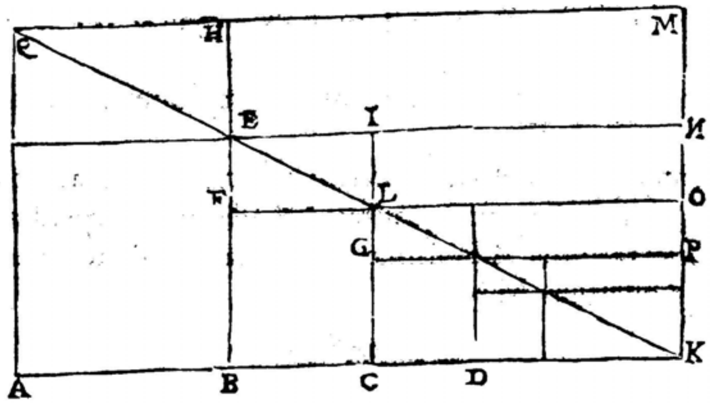
PROPOSITIO CLXVII.

Data sit quadratorum progressio, basibus in directum positis, quae terminum longitudinis habeat K, & iuxta 131, huius inscripta sit triangulo AQK; completo autem rectangulo AM, producantur latera quadratorum in N, O, P, &c. & in H, I, L, &c. [p. 158]

Dico seriem parallelepipedorum, super rectangulis EM, FN, GO, &c, in altitudine linearum BE, CF, DG, &c. aequalem esse seriei cubicae, super quadratis exstructae.

Demonstratio.

Dicta enim EM, FN, &c. rectangula, sunt complementa rectangulorum, quae sunt circa diametrum, ergo a singula quadratis singulis ordine sunt aequalia. Quare Parallelepipeda super complementis illis exstructa, cum easdem quoque cum cubis quadratorum habeant altitudines BE, CF, &c. patet singula b parallelepipeda singulis cubis aequalea esse : ergo tota parallelepipidorum series, toti cubicae aequatur : quod erat demonstrandum. *a 45 primi; b 7 duodecimi.*



Prop. 167, Fig. 1.

L2.§4. ***PROPOSITION 167.***

A series of squares is given, with bases arranged in order along a line, and which has a terminus of length K, and just as in Prop. 131 of this book, the series is inscribed in the triangle AQK; moreover with the rectangle AM completed, the sides of the squares are extended to N, O, P, &c. & to H, I, L, &c. [p. 158]

I say that the sum of the series of parallelepipeds, erected on the rectangles EM, FN, GO, &c, with the heights of the vertical lines BE, CF, DG, &c., is equal to the sum of the of the cubes built up on the squares.

Demonstration.

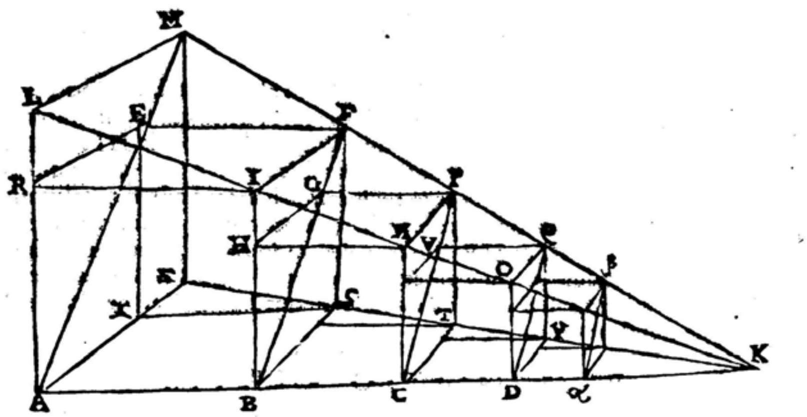
For the said rectangles EM, LN, &c., are the complements of the rectangles which lie around the diagonal, hence [a] the individual rectangles are equal to the squares in order. [Thus, the rectangle EM has the same area as the square AB, rect. LN = sq. BL, next rect. ?O = sq. on CD, etc;] Whereby the Parallelepipeds constructed on these complements, with the same heights also with the cubes of the squares BE, CL, etc. it is apparent that the individual [b] parallelepipeds are equal to the individual cubes : Hence the whole series of parallelepipeds is equal to the whole series of cubes: q.e.d. *a 45 primi; b 7 duodecimi.*

[See note to Prop. 155 of section 3.]

PROPOSITIO CLXVIII.

Data sit progressio AB, BC, CD, &c. terminata in K, & super lineis quadrata, super quadratis autem cubis. Petitur cuborum series inscribi pyramidi, quadratam basim habenti.

Constructio &Demonstratio.



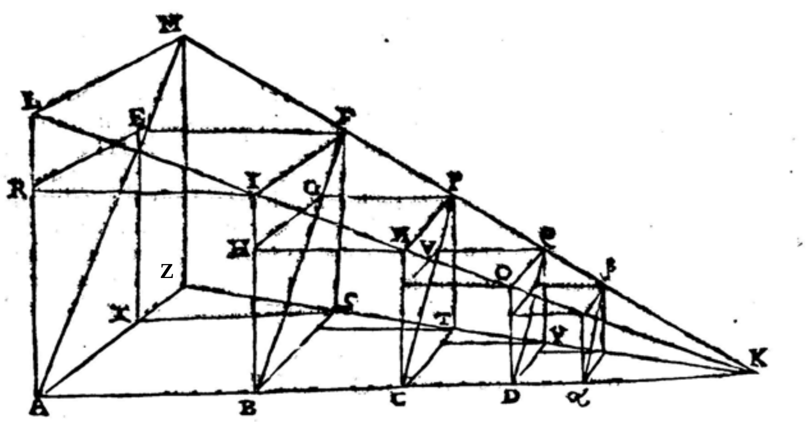
Prop. 168, Fig. 1.

Ex puncto K, per I, S, F, ducantur recte KI, KS, KF, quarum duae primae KI, KS, occurrant lineis AR, AX productis in L & Z : producto deinde plane AE, occurrat linea KF in M, iunganturque LM, ZM. Dico factum quod petabatur. Ducantur enim in adversus cuborum planis diametri AE, BF, CP, DQ, &c. primo igitur ex hypothesi manifestum est utriusque; seriei quadrata AI, BN, CO, &c. AS, BT, &c. [p.159]esse in eodem plano. quare lineae [a]: KIL, KSZ transeunt per omnia puncta N, O, &c. T, Y, &c. hoc est tangunt totam cuborum seriem. Superest ergo ut demonstremus lineam KFM transire etiam per omnia puncta P, Q. &c. quod sic praestabimus. IF est ad HG, ut IB ad HB, id est ut IK ad NK, hoc est ut BK ad CK; Atqui cum seriei AB, BC, CD terminus sit K, [b] AK, BK, CK sunt continuae proportionales; ergo BK est ad CK, ut AB ad BC : & IF ad HG ut AB ad BC : quare cum AB, IF aequales sint, etiam BC & intercepta HG, aequales erunt; ergo BF transit per verticem anguli G, quadrati SBHG, cum intercipiat parallelam HG aequalem lateri dicti quadrati; unde B, G, F sunt in directum. Itaque cum ex elementis constet diametros adversas CP, BG esse parallelas, etiam CP, BF erunt parallelae. Quia igitur linea BF est in [c] plano BFK, etiam CP in eodem [d] plano BFK erit. Similiter ostendemus lineas DQ, CP, esse parallelas, & proinde cum CP sit in plano BFK, etiam DQ esse in plano BFK. eodem discursu demontrabimus omnes BF, CP, DQ, Xβ, &c. esse in eodem plano BFK, sive AMK : deinde BF, CP, &c. cum sint in oppositis planis parallelis, productae nunquam convenient. quare cum sint omnes in plano BFC, erunt omnes inter [e] se parallelae. Praeterea ex elementis & ex datis patet diametros BF, CP, &c. esse lateribus IB, NC, &c. hoc est AB, BC, CE, &c. proportionales: quare [f] KFM transit per omnia puncta P, Q, , &c. Quod autem etiam basis ZM quadrata sit, sic ostendo: ZA est ad XA, ut ZK ad SK, hoc est ut AK ad BK, hoc est ut LK ad IK, hoc est denique ut LA ad RA: Quia ergo XA, RA aequales sunt, etiam ZA, LA [g] aequales erunt. Praeterae LM est ad IF ut LK ad hoc est ut AK ad BK, hoc est ut AZ ad BS, atqui IF, BS aequales sunt, ergo etiam LM, AZ aequales erunt. Deinde cum LK sit ad IK, ut [h] MK ad FK, erit LM parallela ad IF, quae cum ad AXZ parallela sit, etiam LM ad AXZ parallela erit. Quia igitur MZ & AL aequales & parallelas LM, AZ connectunt, ipse quoque [i] aequales & parallelae erunt; est autem angulus LAZ rectus, ac proinde etiam angulus MZA, & consequenter anguli illis oppositi sunt recti, basis igitur ZL est quadrata : Factum ergo est quod petebatur. quod erat demonstrandum. *a 131huius; b 82 huius;c 2 undecimi; d Defin. 34 primi;e ibid;f Lemms ad 131 huius; g 14 quinti; h 17 undecimi; i 33 primi .*

L2.§4. PROPOSITION 168.

A progression is given AB, BC, CD, &c. with the terminus in K, and upon the lines squares, and again upon the squares, cubes are constructed. It is required to inscribe the series of cubes in a pyramid having a square base.

Construction &Demonstration.



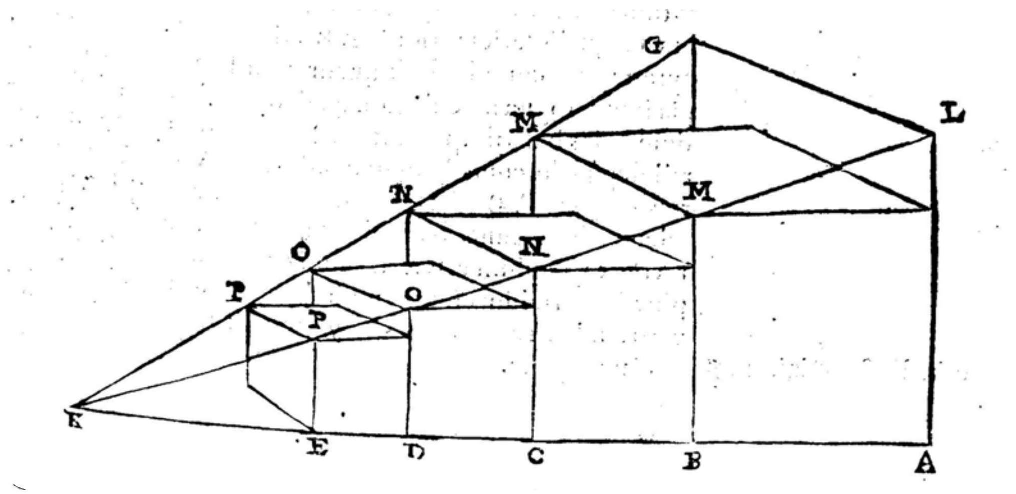
Prop. 169, Fig. 1.

From the point K, the lines KI, KS, and KF are drawn through I, S, F, of which the first two lines cross with the lines AR and AX produced in L and Z : then by producing the plane AE, this crosses the line KF in M, and LM, ZM are joined. I say that what was desired has been done. For the transverse diagonals AE, BF, CP, DQ, &c are drawn in the planes of the cubes. Therefore in the first place by hypothesis, it is clear that each series AI, BN, CO, &c. and AS, BT, &c. lie in the same planes. [p.159] Whereby the lines [a]: KIL, KSZ pass through all the points N, O, &c. T, Y, &c. that is they are tangents to all the series of the cubes. Therefore it remains that we must show the line KFM also cuts through all the points P, Q. &c. , which thus we will establish. IF is to HG, as IB to HB, that is as IK to NK, that is as BK ad CK; But since the terminus of the series AB, BC, CD is K, then [b] AK, BK, CK are in continued proportion; hence BK is to CK, as AB to BC : & IF to HG is as AB to BC : whereby since AB and IF are equal, also BC and the intercept HG are equal ; hence BF passes through the vertex of the angle G of the square SBHG, since it intercepts the parallel line HG of the said square; hence B, G, F are on a line. Thus since from the Elements it is agreed that the transverse diagonals CP and BG are parallel, also CP and BF are parallel. Therefore since the line BF is in [c] the plane BFK, also CP is in the same plane [d]BFK. Similarly we can show that the lines DQ and CP are parallel, and hence since CP is in plane BFK, also DQ is in the plane BFK. By the same argument we can demonstrate that all of BF, CP, DQ, Xβ, &c. are in the same plane BFK, or AMK : then it follows that BF, CP, &c. , since they are in opposite parallel planes, their productions never meet. Whereby since they are all in the plane BFC, they are all parallel amongst themselves [e]. In addition, from the Elements and from what is given, it is apparent that the diagonals BF, CP, &c. are with the sides IB, NC, &c. ; that is AB, BC, CE, &c. are in proportion: whereby [f] KFM crosses all the points P, Q, β, etc. Moreover, since also the base is the square of ZM, I show thus : ZA is to XA, as ZK to SK, that is as AK to BK, or as LK to IK, and that is hence as LA to RA: Therefore, since XA and RA are equal, also ZA and LA [g] are equal. In addition LM is to IF as LK to IK, that is as AK to BK, that is as AZ ad BS, but IF and BS are equal, hence LM and AZ are equal. Then since LK is to IK, as [h] MK to FK, LM is parallel to IF, which since it is parallel to AXZ, also LM is parallel to AXZ. Therefore since MZ and AL are equal and parallel; LM, AZ taken together, are themselves equal and parallel also; moreover the angle LAZ is right, and hence also the angle MZA, and consequently the opposite angles are right, therefore the base ZL is a square : What was asked has been done. Q.e.d. *a 131huius; b 82 huius;c 2 undecimi; d Defin. 34 primi;e ibid;f Lemms ad 131 huius; g 14 quinti; h 17 undecimi; i 33 primi .*

PROPOSITIO CLXIX.

Series seu pyramis cubica inscripta sit pyramidi ALGK. Oporteat pyramidis includentis, & inculsae differentiam exhibere.

Constructio &Demonstratio.



Prop. 169, Fig. 1.

Super quadrato AB in altitudine lineae aequalis seriei rationis AB ad DE fac parallelepidedum, hoc est seriei [k] cubicae aequale erit. Deinde fiat ut quadratum AB ad quadratum ALG, ita pyramidis GLK altitudo AK ad aliquam Q : denique super quadrato AB in altitudine lineae, quae contineat unam tertiam rectae Q fiat parallelipedum, erit hoc pyramidi LK aequale; nam parallelipedum super quadratio [p. 160]

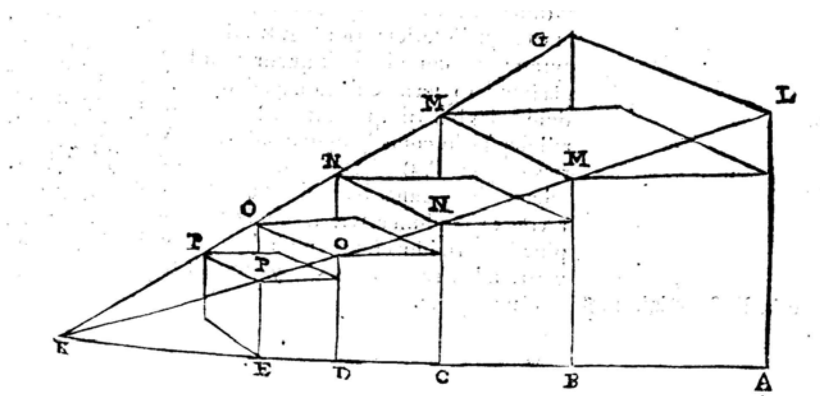
AB in altitudine lineae Q, aequale est [a] parallelipedo super quadrato AG in altitudine AK, cum habeant bases ex constr. & altitudines reciprocas. Quare cum pyramis [b] LK sit una tertia parallelipedi super AG in altitudine AK (est enim AK altitudo, quia AK ut ex construct. praecenentis propositionis patet, est normalis ad AL) itemque parallelipedum super quadrato AG in altitudine tertiae partis rectae Q, sit eiusdem parallelipedui tertia pars, erunt pyramis & dictum parallelipedum aequalia. Quare cum pyramis maior sit inscripta cubica pyramide, etiam dictum parallelipedum nempe super quadrato AB in altitudine tertiae partis rectae Q, erit maius parallelipedo quod pyramidi cubicae aequale feceramus. Eadem igitur erit pyramidis includentis & inclusa quae horum parallelipedorum differentia. exhibuimus ergo, &c. quod petebatur.

k 165 huius; *a 34 undecimi; b 7 undicemi.*

L2.§4. PROPOSITION 169.

A progression is given AB, BC, CD, &c. with the terminus in K, and upon the lines squares, and again upon the squares, cubes are constructed. It is required to inscribe the series of cubes in a pyramid having a square base.

Construction &Demonstration.



Make a parallelepiped [ppd. AB^2.LG] on the square AB with a height LG, equal to the sum of the cubes in the series, in the ratio AB to DE, that is, the ppd. is equal [in volume] to the sum of the series of cubes [k]. Then make the ratio of the square AB to the square ALG to be the same as the height AK of the pyramid GLK to some other length Q : and then upon the square AB [p. 160] with a line which has a length equal to one third of the line Q make a ppd.: this is equal to the pyramid LK; for the ppd. upon the square AB with the height of the line Q, is equal to the ppd. [a] on the square AG with height AK [these extra shapes are not shown on the diagram], since from the construction, they have their bases and heights in reciprocal proportions. Whereby since the pyramid [b] LK is one third of the ppd. on the square AG with height AK (for the altitude is AK, since AK is normal to AL, as is apparent from the construction of the preceding proposition), and likewise the ppd. on the square AG with height one third of Q is a third part of the same ppd., then the [greatest] pyramid and the said ppd. are equal in volume. Whereby, since the largest pyramid is greater than the inscribed series of cubes, and also truly equal to the said ppd. erected on the square AB with height equal to one third of the line Q, then the ppd. is greater than the series of cubes we have constructed. Therefore the series of included cubes is also less, and includes these for which the ppd's. are different; hence we have shown what was required.

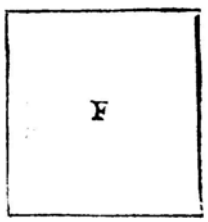
k 165 huius; *a 34 undecimi; b 7 undicemi.*

[Thus, the volume of the series of cubes is $AB^3(1 + ED/AB + \text{..........})$, or in modern terms, $V = a^3(1 + r^3 + \text{...........}) = \frac{a^3}{1-r^3}$, where as before a is the length AB, and r = CB/AB, etc; Initially, construct a ppd. on the square AB with height GL, equal to the sum of the cubes V; then $GL = a/(1 - r^3)$. Subsequently, put the ratio of the end squares $AB^2/GL^2 = KA/Q$, where AK is the height of the largest pyramid, and Q is some other length; then $AB^2 \times Q/3$ is the volume of the ppd. on the square AM of height Q/3 is equal to volume of the largest pyramid $GL^2 \times AK/3$, since $GL^2 \times KA = AB^2 \times Q$. Consequently, since the volume of the pyramid LK is greater than V, also the volume of the ppd. on the square AM is greater than V; thus the series of cubes is enclosed.]

PROPOSITIO CLXX.

Datae seriei sive pyramidi cubicae, pyramidem super F quadrato dato, aequalem exhibere.

Constructio &Demonstratio.



Prop. 170. Fig. 1.

Sit series cuborum AM, BN, CO, &c. & datum quadratum sit F, & fiat ut quadratum F ad quadratum AB, ita linea aequalis seriei rationis AB ad DE ad lineam Q. Dico pyramidem cuius basis sit quadratum F, altitudino autem tripla ipsius Q seriei cubicae aequalem esse. Nam parallelepipedum cuius basis sit quadratum F altitudo Q aequatur [c] parallelepipedo, cuius basis sit quadratum AB, altitudo vero series rationis AB ad DE, hoc est [d] seriei cubicad. Ergo parallelepipedum cuius basis sit quadratum F & altitudo tripla rectae Q triplum erit seriei cubicae. atqui [e] idem parallelepipedum triplum est pyramidis habentis basim F, altitudinem vero triplicata rectae Q, ergo pyramis illi seriei cubicae aequalis erit. Factum igitur est quod petebatur. *c 34 undecimi;d 165 huius;e 7 duodecimi.*

[p. 161]

L2.§4. *PROPOSITION 170.*

A progression is given AB, BC, CD, &c. with the terminus in K, and upon the lines squared are formed, and again upon the squares, cubes are formed. It is required to inscribe the series of cubes in a pyramid having a certain square base.

Construction &Demonstration.

Prop. 170. Fig. 1.

Let AM, BN, CO, etc. be a series of cubes, and F is a given square, and as the square F to the square AB, thus a line equal to the sum of the series of ratios AB to DE is to some line Q. I say that the pyramid the base of which is the square F, and moreover with a height equal to three times Q, is equal to the sum of the series of cubes. For the ppd. the base of which is the square F and with height Q is equal to the volume of the ppd. [c], the base of which is the square AB, with the height truly equal to the sum of series of ratio AB to DE, that is, to the sum of the cubes [d]. Hence the ppd. with the base equal to the square F, and with height equal to the triple of the length of the line Q is three times the sum of series of cubes. But [e] the same ppd. is three times the volume of the pyramid having the base F, and with height three times the length Q, and hence the pyramid is equal to that series of cubes. Therefore what was required has been done. *c 34 undecimi;d 165 huius;e 7 duodecimi.*

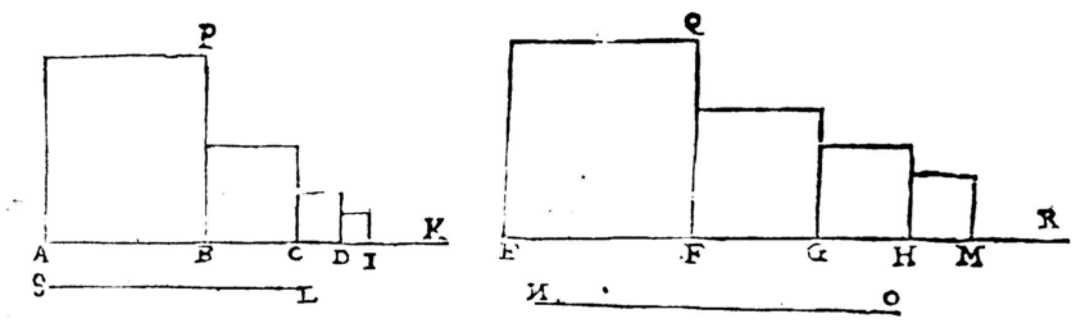
[Let f^2 be the size of the given square F, and a the length of the first side AB of the series in geometric proportion with a common ratio r. Let S be the sum of the series $a(1 + r^3 + r^6 + ...)$, $i.e. S = \frac{a}{1-r^3}$; then $f^2 / a^2 = S / Q$. According to the proposition, the sum of the required cubes is equal to the volume of the pyramid with base f^2 and height $3Q$: For, the volume of such a pyramid is $\frac{1}{3} f^2.3Q = f^2 \times a^2 S / f^2 = \frac{a^3}{1-r^3}$ as required. The demonstration considers the ppd. with base F and height 3Q, which is equal to the length 3S of the series, and the volume of the pyramid is 1/3 of this amount as required.]

[p. 161]

PROPOSITIO CLXXI.

Binae cuborum series quarumvis rationum ab aequalibus cubis AD, EQ incipiant.
Dico cubicas series eamdem habere ad invicem proportionem, quam habent lineae SL, NO aequales seriebus rationum AB ad DI, & EF ad HM.

Constructio &Demonstratio.



Prop. 171, Fig. 1.

Parallelepipedum super quadrato AB in altitudine SL, aequatur [a] seriei cubicae PK. Item parallelepipedum super quadratio EF in altitudine NO, aequatur seriei cubicae QR : cum autem cubi AP, EQ ponatur aequales, etiam quadrata AB, EF aequalia erunt. Quare dicta parallelepipeda easdem bases habebunt; itaque dicta parallelepipeda, hoc est series cubicae eamdem habebunt rationem, quam altitudines SL, NO, hoc est quam habent series rationis AB ad DI, & rationis EF ad HM : quod erat demonstrandum.
a 165 huius.

L2.§4. PROPOSITION 171.

Two series of cubes of any ratios start from the equal cubes AD and EQ.
I say that the sums of the series of cubes are in the inverse proportion of the lines SL and NO, equal to the ratios AB to DI and EF to HM of the series.

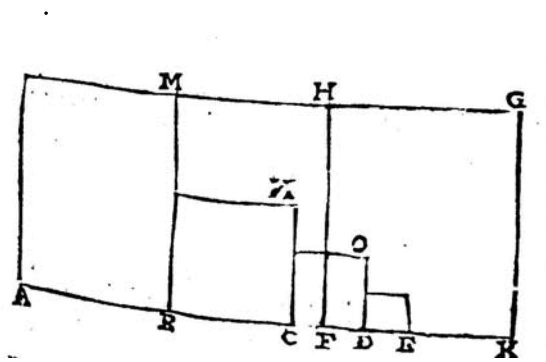
Construction &Demonstration.

The ppd. on the square AB with height SL, is equal to [a] the series of cubes PK. Likewise the ppd. on the square EF with height NO, is equal to the series of cubes QR : moreover since the cubes AP and EQ are placed equal, also the squares AB and EF are equal. Whereby the said ppd's have the same bases; thus the said ppds, that is the series of cubes have the same ratio, as the heights SL and NO, that is as the series of ratio AB to DI, and of the ratio EF to HM : q.e.d. *a 165 huius.*

PROPOSITIO CLXXII.

Data sit quadratorum progressio solita, superque illis exstructa series cuborum, & inscripta parallelepipedo AG, cuius basis sit quadratum AB, altitudo AK, eadem nempe quae longitudino seriei cubicae.
Dico parallelepipedi ad seriem cubicam, eamdem esse rationem, quae est DA trium primorum laterum, ad latus primum AB.

Demonstratio.



Prop. 172, Fig. 1.

Linea AF aequalis fiat seriei rationis AB ad DE; erit parallelepipedum super quadrato AB in altitudine AF (quod vocemus parallelepipedum AH) aequale [b] seriei cubicae, sed parallelepipedum AG est ad parallelepipedum AH, (cum eadem sit basis utriusque) ut AK ad AF, hoc est [c] DA ad BA. Ergo parallelepipedum AG etiam erit ad seriem cubicam ut DA ad BA. Quod erat demonstrandum.

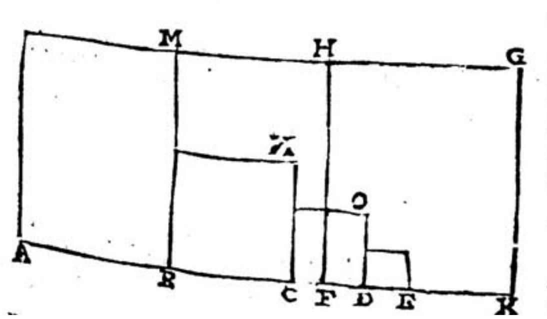
b ibid; c 103 huius.

L2.§4. PROPOSITION 172.

A single progression of squares is given, and upon these a series of cubes are formed, and inscribed in a ppd. AG, the base of which is the square AB, with height AK which is the same as the length of the series of cubes.

I say that the ratio of the ppd. to the sum of the series of cubes is the same as that which the length of the first three sides DA has to the first side AB.

Construction &Demonstration.



Prop. 172, Fig. 1.

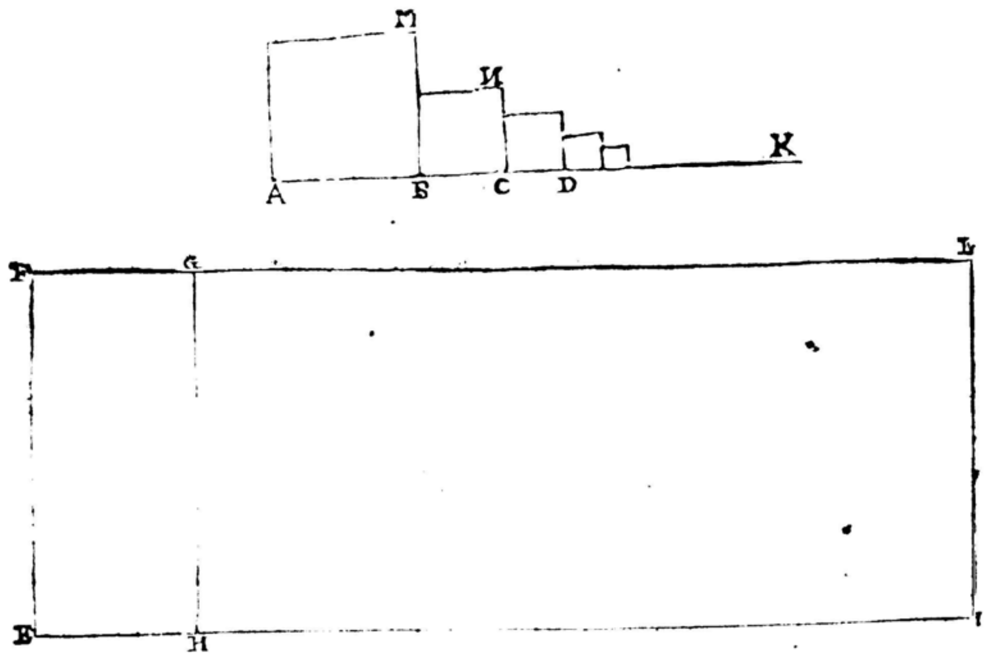
The line AF is made equal to the sum of the series of ratios AB to DE; the ppd. on the square AB with height AF (that we call the ppd. AH) is equal [b] to the series of cubes, but the ppd. AG is to the ppd. AH, (since both have the same base) as AK to AF, that is [c] DA to BA. Hence the ppd. AG also is to the series of cubes as DA to BA. Q.e.d.

b ibid; c 103 huius.

PROPOSITIO CLXVIII.

Data sit vt supra cuborum series. Oportet exhibere superficiem omnibus superficiebus omnium cuborum progressione datae aequale.

Constructio &Demonstratio.



Prop. 173, Fig. 1.

Flat [a] rectangulum EHGF aequale progressioni quadratorum AM, BN, &c. tum E I sextupla fiat lineae E H. Dico rectangulum F I esse id quod quaeritur. Cum superficies singulorum cuborum constet sex quadratis aequalibus, manifestum est omnes feriei cubicae superficies consitui ex sex seriebus quadratorum AM, BN, &c. atqui rectangulum H F ex constructione seriei quadratorum AM, BN. est aequale. Ergo sex rectangula F H, hoc eft ex construct. rectangulam FI constituet omnes seriei cubicae datae fuperficies. Fecimus ergo quod petebatur.

L2.§4. PROPOSITION 173.

A series of squares is given as above. It is necessary to show a surface equal to the sum of the progression of the surfaces of all the given cubes.

Construction &Demonstration.

The rectangle [a] EHGF is made equal to the sum of the progression of the squares AM, BN, &c. then E I is made six times the length of the line E H. I say that the rectangle FI is that which is requires.

Since it is agreed that the surfaces of the individual cubes consist of six equal squares, it is seen that the series of the surfaces of all the cubes are constituted from six series of squares AM, BN, etc, but the rectangle H F is equal to the construction of the series of squares AM, BN. Hence six rectangles F H, that is FI from the construction, make up all the given surfaces of the given cubes. Therefore we have done what was required.

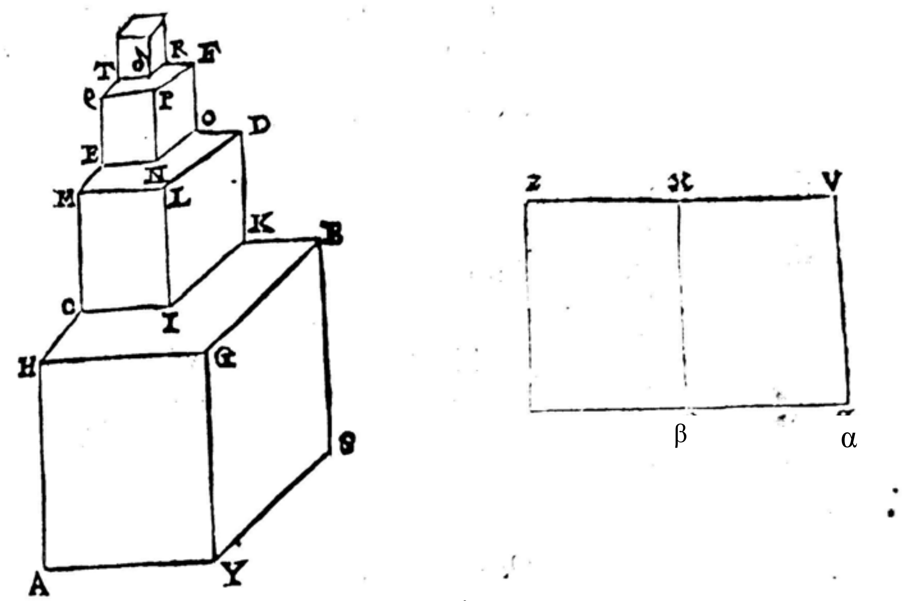
PROPOSITIO C L X X I V.

Data sit cuborum progressio sibi mutuo insistentium, constituens pyramidem cubicam.

Dico residuas basium superficies, nempc BKICHG , DONEML & reliquas omnes in infinitum simul sumptas, quadrato primi cubi aequales esse.

Demonstratio.

Fiat enim seriei rationis primae A H ad tertiam EQ aequalis VZ; super quavis altitudine AH fiat rectangulum Zα, sumptaque V X aequali, A H ducutur ad Vα parallela Xβ: quae abscindat quadratum Xα aequale quadrato AG seu HB. rectangulum αZ per 79 huius aequatur seriei quadratorum A G, C L, EP, &c. hoc est: (quonium cuborum plana omnia sunt quadrata aequalia) seriei quadratorum HB, MD, QF.&c.ergo cum αX ex constr. quadrato AB aequale sit, erit reliquum β Z rcliquae quadratorum seriei MD, QF. &c. aequa1e. Atqui series quadratorum [p. 163]



Prop. 174. Fig. 1.

MD, QF, etc. eadem est cum serie quadratorum KC, OE, &c. rectangulum igitur βZ seriei quadratorum CK, OE, &c. aequatur. Quare cum rectangulum αZ & series quadratorum HB, MD, QF, &c. itemque rectangulum βZ & series quadratorum CK, EO, &c., aequalia sint, etiam excessus rectanguli αZ super βZ, & excessus seriei HB, MD, *&c.* super seriem CK, EO, &c. aequales erunt. Atqui excessus αZ super βZ est αX, id est ex construct. quadratum HB: excessus vcro seriei quadratorum BH, MD, &c. super seriem quadratorum CK, EO, &c. sunt figurae BKICHG, DONEML,FRδTQP, &c. ergo figurae illae omnes simul sumptae aequantur quadrato HB. Quod erat demonstrandum.

L2.§4. ***PROPOSITION 174***.

A progression of stacked cubes is given as shown, making a cubic pyramid.

I say that the remaining surfaces of the bases, truly BKICHG, DONEML, etc., and the rest of the like surfaces of the bases of all the cubes summed together to infinity, is equal to the square of the base of the first cube.

Demonstration.

For VZ is made equal to the sum of the series with the ratio of the third term AH of the given series to the first term EQ; upon which line the rectangle Zα is constructed with height AH; and with VX taken equal to AH, V*α* is drawn parallel to X*β*: which cuts off the square Xα equal to the square AG or HB. By prop. 79 of this book, the rectangle αZ is equal to the sum of the series of squares AG, CL, EP, &c., that is: (since the squares in all the planes of the cubes are equal) to the series of squares HB, MD, QF, etc. Hence, since by construction αX is equal to the square AB, the remainder *β* Z is equal to the rest of the series of cubes [p. 163] MD, QF, etc. But the series of squares MD, QF, etc., is the same with the series of squares KC, OE, etc. Therefore the rectangle *β*Z is equal to the series of squares CK, OE, etc. Whereby since the rectangle *α*Z is equal to the series of squares HB, MD, QF, etc. and likewise the rectangle *β*Z is equal to the series of squares C K, EO, etc., then also the differences of the rectangle *α*Z over *β*Z, and the series H B, M D, etc. over the series C K, EO, etc. are equal. But the difference of *α*Z over *β*Z is *α*X, that is the square H B from the construction: truly the figures BKICHG, DONEML, FRδTQP, etc are the difference of the series of squares BH, MD, etc. over the series of squares CK, EO, etc. are. Hence the sum of all these figures is equal to the square HB. Q.e.d.

[We can set AH = a; EQ = ar^2 ; then VZ is the sum of the series $a(1+r^2+r^4+....)=\frac{a}{1-r^2}$. Thus, the rect. Zα $=a^2(1+r^2+r^4+....)=\frac{a^2}{1-r^2}$; consequently *β*Z = $a^2(r^2+r^4+....)=\frac{a^2r^2}{1-r^2}$. Hence, αZ - βZ = a^2; but this difference can be expressed as the differences of the squares BKICHG = $a^2(1-r^2)$; DONEML = $a^2r^2(1-r^2)$; etc.; the infinite sum of which is a^2.]

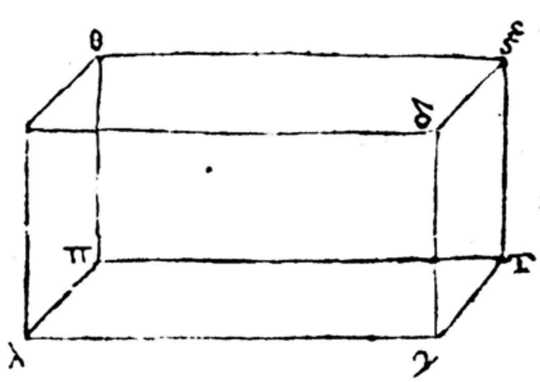
PROPOSITIO CLXXV.

Iisdem positis,

Dico supcrficiem cubicae pyramidis, aequalem esse fuperficiei parallelepipedi γθ, cuius basis *γξ,* sit primi cubi quadratum, altitudo vero, γλqua aequalis seriei rationis primae AH ad tertiam EQ. Per supeficiem autem pyramidis cubicae intelligo hic superficies omnium cuborum, exceptis quadratis CK, EO, T R, &c.

Demonstratio.

Rectangulum λδ continetur linea γλ, aequali seriei rationis AH ad EQ, & altitudine γδ, quae aequalis est AH; est enim quadratum γξ aequale quadrato AG : igitur [a] rectangulum λδ omnibus quadratis AG, CL, &c. aequale est : reliquae igitur hederae τθ, δθ, γπ aequales sunt seriei quadratorum oppositae seriei AG,CL, &c. & series quadratorum BY, DI, &c. nec non illi quae infra huius opposita est : est autem & quadratum γξ basis parallelepipedi aequalis quadrato [p.164] AS basi pyramidis cubicae, & per praecedentem; quadratum HB, id est λθ aequale est omnium basium residuis. Ergo tota superficies parallelepipedi, toti pyramidis cubicae superficiei aequalis est. Quod erat demonstrandum. *a 128 huius;*



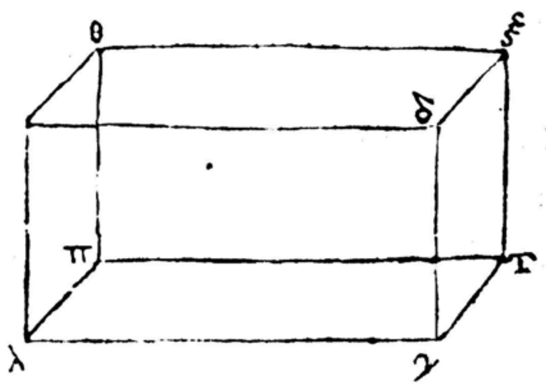
Prop. 175. Fig.1.

L2.§4. ***PROPOSITION 175.***

With the same figures in place,

I say that the surface of the pyramid of cubes is equal to the surface of the parallelepiped γθ, the base of which is *γξ,* the square of the first cube, with height γλ, which is equal to the sum of the series with the ratio of the third term EQ to the first term AH. Moreover I understand that the surface of the cubic pyramid is the surface of all the cubes of the series with the exception of the squares CK, EO, TR, &c.

Demonstration.



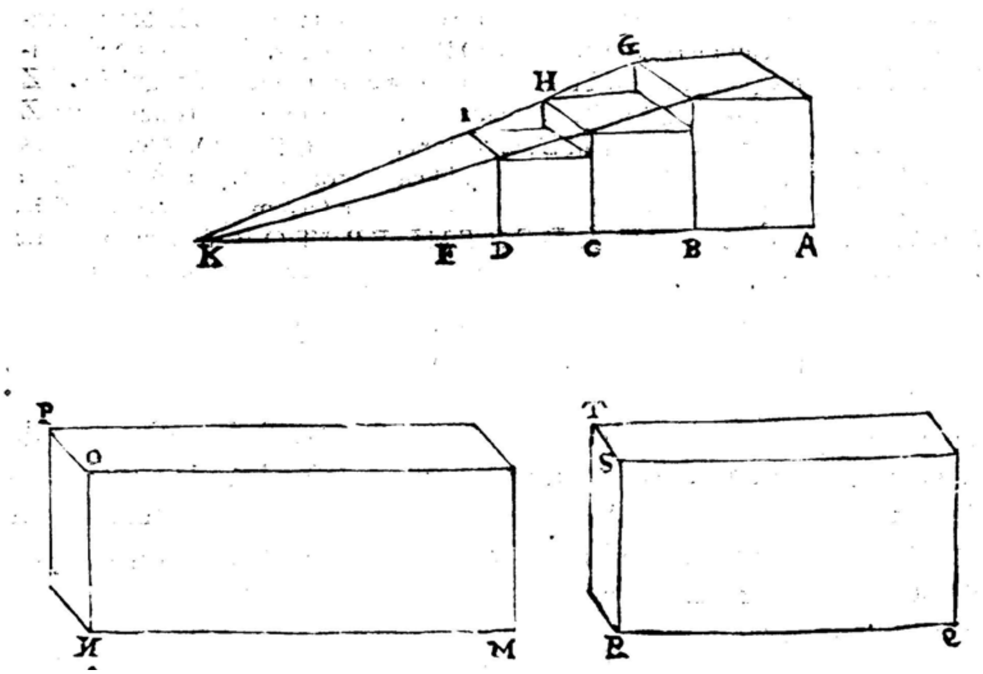
Prop. 175. Fig.1.

The rectangle λδ consists of the line γλ, equal to the sum of the series with the ratio EQ to AH, and with the height γδ, which is equal to AH; for indeed the square γξ is equal to the square AG : hence [a] the rectangle λδ is equal to the sum of all the squares AG, CL, etc.: therefore the rest of the rectangles on the sides τθ, δθ, γπ are equal to the series of squares opposite the series AG,CL, &c. & the series of squares BY, DI, &c. all these which are on opposite sides : moreover the square γξ is equal to the base of the ppd. which is equal to the base of the cubic pyramid [p.164] AS; and by the preceding theorem, the square HB, that is λθ, is equal to the sum of all the remaining bases. Therefore the whole surface of the ppd. is equal to the total surface of the cubic pyramid. Q.e.d. *a 128 huius.*

PROPOSITIO CLXXVI.

Data sit quadratorum progressio cui terminus Iongitudinis sit K; fuper quadratis autem exstructa fit cuborum series. Deinde per 165 huius factum fit parallelepipedum MP, aequale seriei cubicae. Dico superficiem huius parallelepipedi , ad superficiem pyramidis cubicae (sumendo hic superficiem pyramidis cubicae , vt in propositione prescedenti sumpsimus) eam habere rationem, quam linea aequalis seriei rationis A B prima: ad DE quartam, vna cum dimidia ipsius AB, habet ad aequalem seriei rationis primae A B ad CD tertiam, vna cum dimidia AB.

Demonstratio.



Prop. 176, Fig. 1.

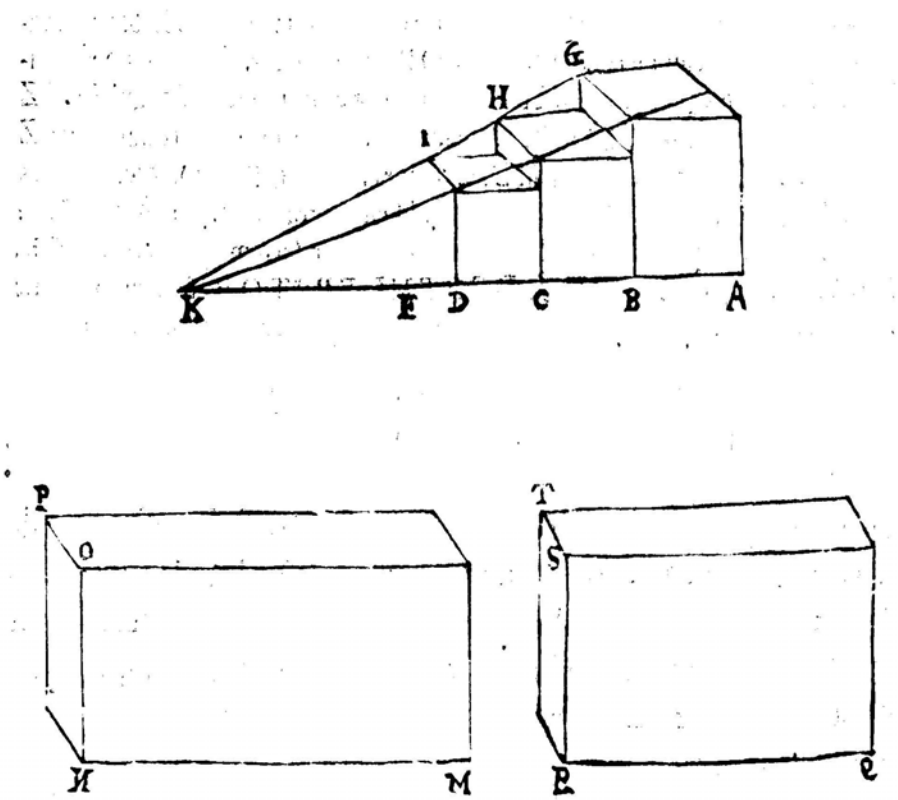
Ergo per praecedentem superficies parallelepipedi QT [p.165] aequalis erit superficiei cubicae pyramidis. Deinde superficies parallelepipedi M P aequalis est rectangulo [b] quod linea composita ex quadrupla M N & dupla NO,& altitudine NO sive OP continetur. Similiter supcrficies parallelepipedi Q T aequalis eft rectangulo cuius basis sit composita ex quadrupla QR & dupla RS ; altitudo vero RS sive ST (sunt enim R S, ST aequa1es) quia latera aumt quadrati R T. Quare cum dicta rectangula sint vt bases (altitudines enim NO, R S aequalcs habent eidem AB, ideoque aequales inter se) etiam erit parallelepipedi MP superficies, ad superficiem paralle1epipedi QT ut basis ad basim ; nempe ut composita ex quadrupla MN & dup1a NO, ad compositam ex quadrupla QR & dupla R S. Atqui ut quadrupla MN cum dupla NO, ad quadruplam Q R cum dupla R S, sic M N cum dimidia NO ad QR cum dimidia R S; ergo superficies parallelepipedi MP, est ad superficiem parallelepipedem QT, hoc est ad superficiem pyramidis cubicae, ut MN cum dimidia NO, hoc est ut series rationis A B ad D E cum dimidia AB, ad QR cum dimidia RS , hoc eft ad seriem rationis A B ad CD cum dimidia A B: quod erat demonstrandum. *a 165 huius; b 8 sexti;*

L2.§4. ***PROPOSITION 176.***

A progression of squares is given the terminus of which is at a distance K; moreover on the squares a series of cubes is set up. Then by Prop. 165 of this book, a parallelepiped MP is made equal to the sum of this series of cubes.

I say that the surface of this parallelepiped is in a ratio to the surface of the pyramid of cubes (by taking this surface of a cubic pyramid, as we assumed in the previous proposition) equal to the ratio a line equal in length to the series in the ratio of the fourth term DE to the first term A B, together with half of AB, has to an equal series taken in the ratio of the third term CD to the first term AB, together with half of AB.

Demonstration.

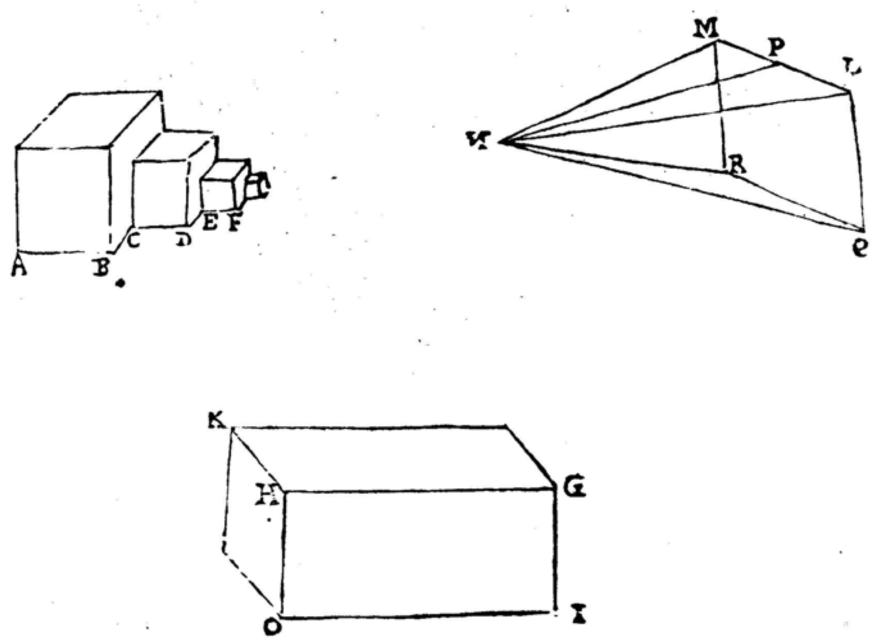


The parallelepiped M P is made equal to the sum of the series of cubes, therefore the side MN is equal to a series in the ratio DE to AB [a], and NO and OP are equal to the particular term AB. Now on the square RT that is equal to the square AB, a paral1elepiped QT is made, with height QR equal to the series in the ratio CD to AB. Hence by the preceding Proposition, the surface of the parallelepiped QT [p.165] is equal to the surface of the cubic pyramid. Then the surface of the parallelepiped M P is equal to a rectangle [b] composed from a line four times the length of M N and twice NO, and with height NO or OP. Similarly the surface of the ppd. Q T is equal to a rectangle of which the base is four times QR and twice the height RS, and with height RS or ST (for R S and ST are equal) since the sides are the squares R T. Whereby since the said rectangles are as the bases (for the heights NO and R S are equal to the same length AB, and so are equal to each other) also the surface of the ppd. MP, to the surface of the ppd. QT are as base to base ; truly as they are composed from the quadruple of MN and double NO, to that composed from the quadruple of QR and the double of R S. But as four times MN with twice NO, to four times QR with twice R S, thus MN with half of NO to QR with half of R S; hence the surface of the ppd. MP is to the surface of the ppd. QT, that is, the surface of the pyramid of cubes, as MN with half NO, that is as the sum of the series of ratios D E to AB as half AB, to QR with half RS , that is as the sum of the series of ratios CD to A B with half A B: Q.e.d. *a 165 huius; b 8 sexti;*

PROPOSITIO CLXXVII.

Proportionem exhibere quam superficies pyramidis habet ad inscriptae sibi pyramidis cubicae superficiem : eo modo intelligendo
superficiem seriei cubicae, quo in pracedenti propositione.

Constructio &Demonstratio.



Prop. 177. Fig. 1.

Assumemus hoc loco pyramidem isosceliam, facilitatis gratia. fit ergo pyramis QLMRN isoscelis, cuius basis sit quadratum QM, cui pyramis cubica AB, CD, &c. inscripta intelligatur, factoque quadrato OK aequali quadrato AB reperiatur linea GH aequalis seriei rationis AB primae ad tertium EF; & super quadrato in altitudine GH, fac parallelepipedum, cuius superficies aequebitur [a] pyramidis cubicae. Dico ut rectangulum super dupla LN & L M tamquam una recta, [p.166] in altitudine LM, est ad rectangulum super quadrupla GH & dupla AO tamquam una recta, in altitudine HO, sic pyramidis includentis superficies, ad superficiem inclusae pyramidis cubicae, Ducatur enim ex vertice pyramidis N ad LM normalis NP, quae ut ex datis facile colliges, bisecat L M in P: rectangulum igitur NLP duplum est trianguli LPN, ut patet ex elementis; ergo rectangulum NLP aequale est triangulo LMN. & rectangulum NLP, aequale eft triangulo LMN. & rectangulum NLM duplum est triangulum LMN. ergo rectangulum super dupla LN, in altitudine

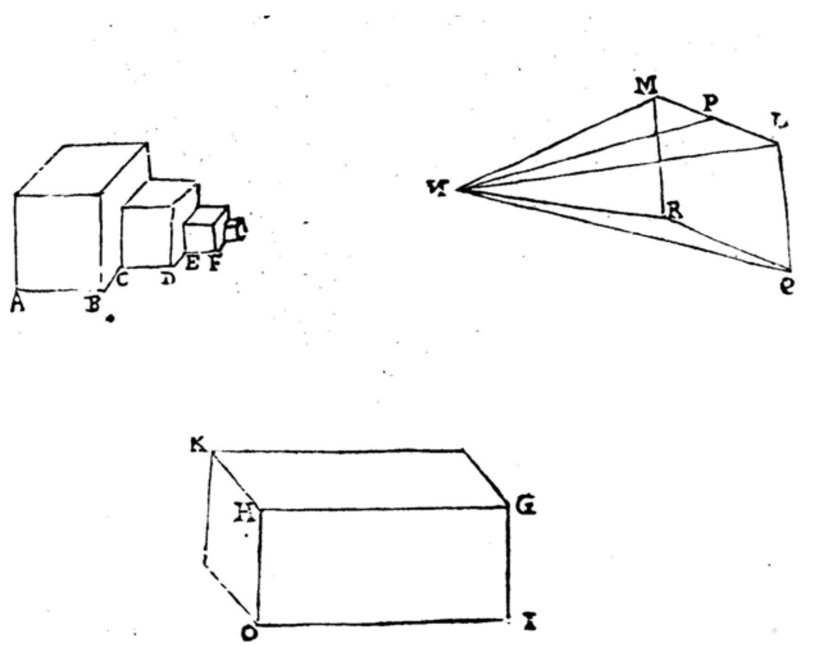
LM, est quadruplum triangulum LMN, hoc est aequatur toti superficiei pyramidis praeter basim; quare rectangulum super dupla L N & LJ tamquam una recta in altitudine L M, aequatur toti [a] superficies pyramidis. simili discursu demonftrabimus rectangulum super quadrupla GH & dupla HO tamquam una recta, in altitudine HO aequari superficiei parallelepipedi , ergo superficies pyramidis N, est ad superficiem parallelepipedi, hoc eft ex conftruct. ad
superficiem pyramidis cubicae , vt funt dicta rectangula inter se, Exhibuimus ergo,&c. quod petebatur
a 1. secundi.

Libri secundi finis.

L2.§4. PROPOSITION 177.

To show the proportion that the surface of the pyramid has to the surface of the cubes inscribed in the pyramid: with the surface of the series of cubes understood to be that according to the previous proposition.

Construction &Demonstration.



Prop. 177. Fig. 1.

Here we will assume an isosceles pyramid in place, for the sake of simplicity. This is therefore the isosceles pyramid QLMRN, the base of which is the square QM, and the cubes AB, CD, etc. are understood to be inscribed, and with the square OK made equal to the square AB the line GH is found to be equal to the sum of the series of ratios of the first AB to the third EF [the actual ratio in the progression being EF to AB of course]; and upon the square with height GH, construct a ppd., the surface of which is equal to the pyramid of cubes [a]. I say that as the rectangle on twice LN and LM as one line, [p.166] with height LM, is to the rectangle upon the quadruple of GH and with twice AO as one line, with height HO, thus the surface of the enclosing pyramid to the surface of the enclosed pyramid of cubes. For from the vertex of the pyramid N to LM a normal NP is drawn, which is easily understood from what is given, bisects LM in P: therefore the rectangle NLP is twice the triangle LPN, as is apparent from the Elements; hence the rectangle NLP is equal to the triangle LMN; and the rectangle NLP, is equal to the triangle LMN; and the rectangle NLM is twice the triangle LMN. Hence the rectangle of twice LN, with height LM, is four times the triangle LMN, that is equal to the whole surface of the pyramid except the base; whereby the rectangle on twice LN and LJ taken as one line with height L M, is equal to the total surface of the pyramid[a]. By a similar argument we can show that the rectangle on the quadruple of GH and twice HO considered as one line, with height HO is equal to the surface of the ppd. Hence the surface of the pyramid N, is to the surface of the ppd., that is from construction to the surface of the pyramid of cubes, as the said rectangles are to each other. Therefore we have shown what was required. *a 1. secundi.*

End of the Second Book.