[124]

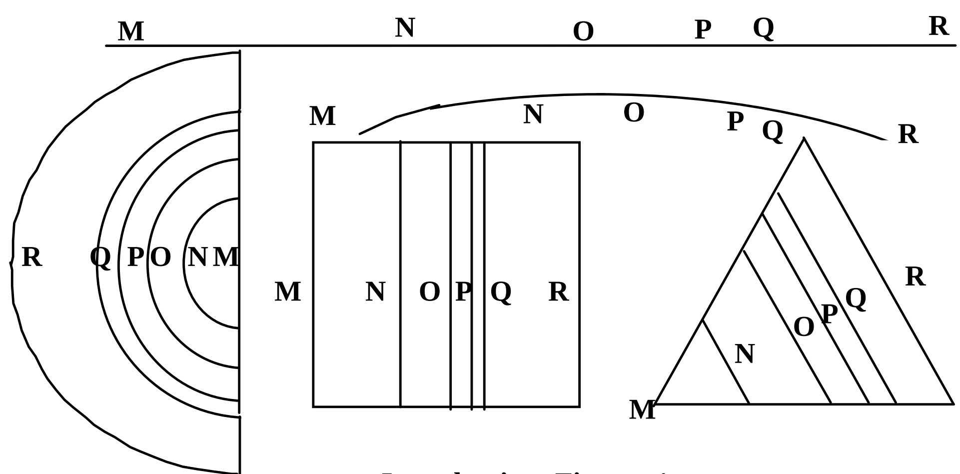
PROGRESSIONUM GEOMETRICARUM

PARS TERTIA

Progressiones terminatas planis applicat, praesertim similibus.

The Third Part is applied to terminations [i. e. limits] of progressions for plane figures, especially those which are similar.

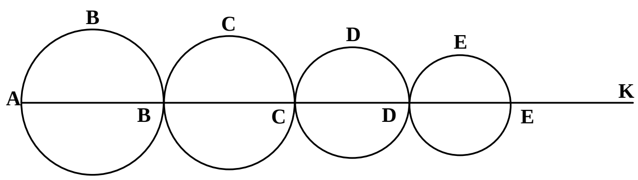
Qua de progressionibus Geometricis secunda parte hactenus demonstravimus, absque ullo discrimine lineis, superficiebus, corporibusque conveniunt: hac enim de causa nomen magnitudinis, non linea perpetuo assumpsimus, ut propositionum universalitatis indicaretur. quia tamen superficierum corporumque similium similiterque positorum progressiones, si extra invicem in directum constituantur, singulares habent proprietates non paucas, visum est operae pretium futurum illas hac tertia ac quarta partes explicare.



Introduction: Figures 1.

As far as the geometric progressions are concerned that we have already demonstrated in the second part, we may note that the methods introduced there can be applied directly to any lines, planes or areas, and solid bodies or volumes. The following development is more concerned then, with particular instances where such lines are parts of plane or solid figures, rather than with properties of the progressions of lines themselves. The propositions developed in the first part are assumed to have the same universal application, as the proposition presented about general use may indicate. Hence, for progressions of series derived from similarly placed similar surfaces and bodies, if these are set up in turn for shapes involving several lines, then the individual terms have more properties than those already considered for ratios along a single line: this is the interest in performing the present work, to be set out in this and the following part.

Nota, duplici modo planorum ac corporum progressiones infstitui posse. primo quidem ut termini progressionis simul sumpti, unam magnitudinem continuam, ac homogeneam componant: ut in figuris appositis exhibitur. secundus enim terminus NO cum primo MN, unam magnitudinem MO componit; & tertius OP, cum secundo ac primo, constituit unam magnitudinem MP ; omnes denique termini simul sumpti unam componunt magnitudinem MR, continuam ac homogeneam.

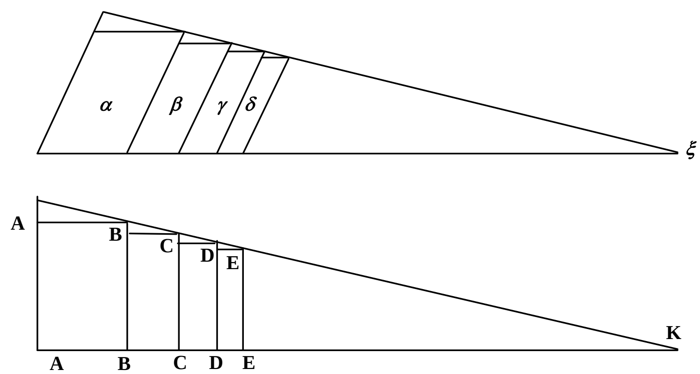


Introduction Fig. 2.

Secundus modus est quando termini progressionis similes inter se sunt, similiterque positi, neque iuxta positionem qua dantur, constituunt simul sumpti unam magnitudinem: huiusmodi progressiones (quas quidem in sequentibus prosequemur) exhibent figurae oppositae A, B, C, D, E, K : in quibus termini omnes similes sunt similiterque positi, ac in directum constituti; ita ut neque secundus terminus BC, cum primo AB, neque tertius CD, cum primo & secundo, neque caeteri subsequentes cum praecedentibus componant unam magnitudinem.

[125]

& figures sic positis terminum quidem ad quem bases illarum figurarum excurrent scilicet K, per praecedentia reperiemus; in heterogeneae vero illius seriei (ita enim lubet appellare) quam figurae similes similiterque; & extra se positae componunt, cognitionem non veniemus, nisi figuras has similes similiterque extra se positas, ad illas revocando, quarum prima cum secunda, & secunda cum tertia, & sic deinceps unam aliquam magnitudinem constituit. Ut si exempli gratia sint figurae AB, BC, CD, & similes similiterque & extra se invicem positae; harum termini in infinitum continuatarum sumpti magnitudinem non unam aliquam, sed aggregatum quoddam figurarum constituent: si igitur magnitudo huic seriei figurarum aequalis quaeratur, oportebit figuras similes AB, BC, CD, DE, &c. ad figuras MN, NO, OP, &c. revocare. quae figurae MN, NO, OP magnitudinem unam constituunt; & si proportio MN ad NO, & NO ad OP continuata terminetur in R : erit MR toti seriei progressionis figurarum AB, BC, &c. aequalis; quae omnia sequenti propositione demonstrata fient clariora.



Introduction Fig's. 3.

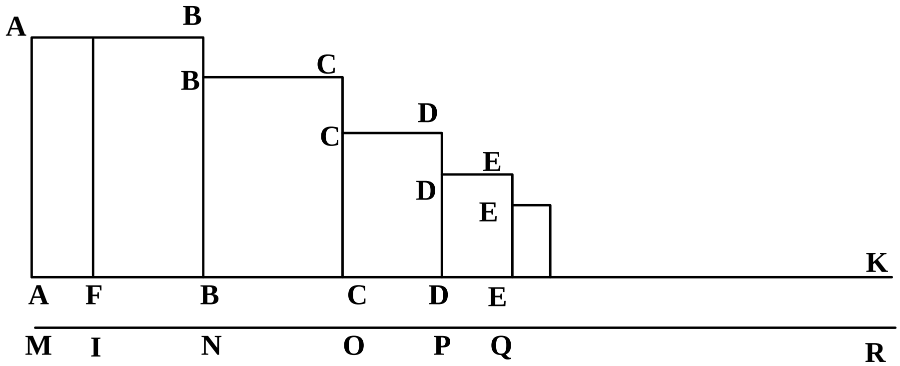
Note: There are two ways in which progressions can be established for planes and solid bodies. According to the first way, in order that the terms of a progression can be summed together, successive terms of the same kind of a single magnitude are placed together, as is show in the above figures (Introduction: Fig. 1): *for the second term NO taken with the first term MN gives a single magnitude MO; again, the third term OP summed with the second and the first, gives a single magnitude MP; all the terms* *summed in the same manner give a single magnitude MR of the same kind in the succession of terms.*

The second manner in which a progression can be formed is one in which the figures are similar to each other, and similarly placed in given positions just touching each other, but the sum of the terms does not give a single magnitude of the same kind. Progressions of this kind (which indeed we shall describe later in detail) are set out for the figures A, B, C, D, E, K opposite [Introduction:Fig.2]*: in which all the similar terms or corresponding figures are similarly put in place, and set together on a given line. However, in this case, neither the sum of the second BC with the first AB, nor the third with the second, nor those remaining in turn with the ones that have gone before, can be* [simply] *added together give a single similar figure of some magnitude.*

Indeed, from the preceding arguments for series of the first kind, we find that the bases of such figures, if the terms are thus put in place in a geometric progression, do extend to meet in some point K. The different forms of these series (as thus they may be called) consist of similar figures similarly placed, and the series are formed by putting these terms in place. It is recognised that if more of these similar figures likewise are placed beyond the diagram, then we recall that the sum of the first and second, and following with the third, and thus henceforth do not in this case constitute some similar single figure of some magnitude. As for example, if the figures AB, BC, CD, and like ones are placed similarly in turn beyond, as in Fig.2, then the sum of these continued indefinitely do not constitute a single figure, but rather an infinite aggregate of such particular figures. Therefore if a magnitude equal to the sum of this series of figures is sought, then it is necessary to recall figures MN, NO, OP, &c. similar to the figures AB, BC, CD, DE, &c., where the sum of the figures MN, NO, OP, &c. has a given magnitude. If the continued proportions MN to NO,NO to OP, &c. terminate in R, then MR is equal to the sum of the progression of the figures AB, BC, &c., [from which the sum of the other series of figures follows]. It is this point which all the following demonstrations of the proposition try to make clear.

PROPOSITIO CXXIII.

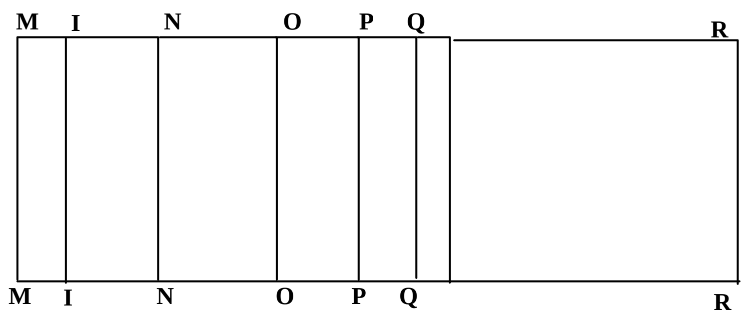
Data igitur sit secundi generis progressio AB, BC, CD, DE, &c. conflata ex similibus terminis similiterque; & in directum positis, sive planis, sive solidis, & quidem planis vel solidis cuiuscumque generis.



Prop.123. Fig. 1.

Dico omnes progressionum proprietates superiori parte demonstratas huiusmodi etiam progressionibus convenire, ac proinde propositiones, in quibus illae proprietates demonstrantur, prorsus universales esse: in hac igitur demonstratione, progressio posterior sive heterogenae, reducitur ad priorem sive homogeneam

[126]
Demonstratio.



Prop.123. Fig. 2.

Si enim primae magnitudini AB, aequalis quaecumque alia MN; & ut AB ad BC, ita sit MN ad aliam NO, quae cum MN, unam magnitudinem continuam & homogeneam componat: continueturque ratio MN ad NO, in infinitum per plures semper terminos OP, PQ, &c. qui perpetuo cum praecedibus terminis unam magnitudinem componant. Quoniam igitur aequales sunt AB, MN, erit AB ad NO, ut MN ad NO; sed MN est ad NO, ut AB ad BC; ergo AB est ad NO, ut AB ad BC: aequales ergo sunt BC, NO; ergo BC est ad OP, ut NO ad OP; sed NO est ad OP, ut MN ad NO, id est ut AB ad BC, id est ut BC ad CD; ergo BC est ad NO, ut BC est ad CD: aequales ergo sunt CD, OP. similiter ostendam singulos utriusque progressionis terminos inter se aequari in infiniti. Quare & series tota AB, &c. aequalis est toti seriei MN, &c. utpote constans aequalibus sive iisdem terminis: atqui quaecumque toto secundo libro demonstrata sunt de progressionum proprietatibus, conveniunt progressioni MN, NO, &c.: ergo etiam conveniunt progressioni AB, BC, &c. Quod erat demonstratum. Verum ut res clarius pateat, idipsum per aliquot consectaria seu corollaria explicabimus.

L2.§3. PROPOSITION 123.

A progression AB, BC, CD, DE, &c. of the second kind is therefore given, put together from similar terms, and similarly placed together on a line either in a plane or in space, and the progression consists indeed of some kind of plane or solid shapes.

I say that all the properties of the progressions shown above are also appropriate for progressions of this kind, and that the propositions by which these properties are demonstrated are hence of general use. Therefore, in this demonstration, a heterogenous progression of the second kind can be reduced to a homogenous progression of the first kind.

Demonstration.

For the first magnitudes, AB is set equal to some other MN; and subsequently as AB is to BC, so MN is to the other NO, which can be added to MN to give the single continuing and homogenous magnitude MO. Ratios of the form MN to NO can be continued indefinitely by adding more terms such as OP and PQ, etc., and these can in turn be added to the preceding terms to give single magnitudes.

Therefore, since AB and MN are equal, AB is to NO, as MN is to NO; but MN is to NO, as AB to BC; hence AB is to NO, as AB to BC: and hence BC and NO are equal. In like fashion, BC is to OP, as NO to OP; but NO is to OP as MN to NO, or as AB to BC, or as BC to CD; hence BC is to NO as BC is to CD, and hence CD and OP are equal.

Similarly, I can show that the individual successive terms of both progressions are equal to each other indefinitely. Whereby the sum of the series AB, etc. is equal to the sum of the series MN, etc., for they are in agreement by having all their consecutive terms equal to each other. But whatever sum is agreed upon, according to the properties of progressions set out in the second book, for the terms of the progression MN, NO, the sum of the terms of the progression AB, BC, &c., is also in agreement. Q.e.d.

However, in order that the idea becomes more apparent, we next set out some logical consequences or corollaries.

Corollarium primum.

Ex his igitur (iisdem positis) infero primo : progressionem universam magnitudinum AB, BC, &c. producere eandem determinatam magnitudinem sive quantitatem quam series MN, NO.

Demonstratio.

Series enim AB, BC, &c. aequalis est seriei MN, NO, &c. ergo eandem producit quantitatem: atqui series MN constituit [a] finitam & determinatam quantitatem (exempli causa MR) ergo & series AB producit quantitem finitam MR.

First Corollary.

In the first place, I can infer from these points, lines, etc. (in the same positions) that a general progression of the magnitudes AB, BC, etc. produces a sum of the same size as the series MN, NO, etc.

Demonstration.

For the series AB, BC, etc. is equal to the series MN, NO, etc., term by term; hence the same quantity or sum is produced by each; but the series MN sets up a finite bounded amount [a] (such as MR) , and hence the series AB also sets up the same finite quantity MR.

Corollarium secundum.

Iisdem positis infero secundo, series AK (id est omnes antecedentes) est ad seriem BK (id est omnes consequentes) ut AB ad BC unam consequentem.

Et series BK est ad seriem CK ut AB ad BC.
Et tres series AK, BK, CK, sunt in continua analogia, & similiter alia inferemus quae prop. octuagesima secunda habentur.

Demonstratio.

Sit MR aequale toti seriei MN, NO, &c. erit ergo & tota series, AB, BC, &c. etiam, ut ostensum antia, aequalis ipsi MR: cum igitur & AB aequalis sit MN, erit quoque series BC, CD, &c. aequalis ipsi NR. Igitur series AK est ad seriem BK ut MR ad NR: Atqui MR [b] est ad NR, ut MN ad NO. ergo series AK est ad seriem

[127]

BK ut MN ad NR, id est ut AB ad BC. quod erat primum. Similiter reliquas quoque demonstrabimus.
a 79 huius; b 82 huius.

Second Corollary.

With everything as above I can infer in the second place that the series AK (or the sum of the terms in the initial part of the ratio) is to the series BK (or the sum of the terms in the following part) as AB is to BC or the first term to the next term. Again, the sum of the series BK is to the sum of the series CK as AB is to BC. The three series AK, BK and CK are in continued proportion, and similarly we can infer the rest that we have found from Prop. 82.

Demonstration.

Let MR be equal to the whole series MN, NO, &c. Hence the whole series AB, BC, &c. also, as I have already shown, is equal to MR: and since AB is equal to MN, the series BC, CD, &c. too is equal to the series NR. Therefore the series AK is to the series BK as MR is to NR: But MR [b] is to NR as MN is to NO. Hence the series AK is to the series BK as MN is to NR, or as AB to BC, which shows the first part of the corollory. We can show the rest in a similar manner.
a 79 huius; b 82 huius.

Corollarium tertium.

Iisdem positis infero tertio; AF differentia primi & secundi termini, AB primus terminus, tota series AK, sunt in continua analogia.
Et AF differentia, est ad AB primum terminum, ut BC secundus terminus ad totam seriem dempto primo termino nempe ad seriem BK : & AF differentia , BC secundus terminus, tota series demptis duobus primis terminis; (series nempe CK) sunt in continua analogia.

Demonstratio.

Sit MN differentia primi & secundi termini, in progressione MR: quoniam igitur AB, MN, BC, NO aequantur, excessus quoque AF, MI, aequales erunt : & quia iam AF ipsi MI, & AB ipsi MN; aequalis est, ipsis AF, AB eadem magnitudo erit tertia proportionalis, quae ipsis MI, MN, ut patet ex elementis: atqui ipsis [a] MI, MN, tertia proportionalis MR: est tota series rationis MN ad NO; ergo ipsis etiam AF, AB eadem tota series MR, tertia proportionalis erit: atqui series AB, BC, per corollorium primum eandem producit magnitudinem MR, quam series MN, NO, ergo etiam productum seriei AB, BC, &c. sive tota series AK, erit tertia proportionis ipsis AF, AB: sunt itaque AF differentia, AB primus terminus, tota series AK in continua analogia. Quod erat primum. Similiter reliquas corollarii partes demonstrabimus.
a 83 huius.

Third Corollary.

For the same situation, I can infer in the third place that the difference AF between the first and second terms, the first term AB, and the whole series AK, are in continued proportion.
And the difference AF is to the first term AB, as the second term BC is to the whole series with the first term removed, surely the series BK. The difference AF, the second term BC, and the whole series with the first two terms removed (that is, the series CK) are in continued proportion.

Demonstration.

Let MI be the difference of the first and second terms in the progression MR: hence, since AB is equal to MN, and BC is equal to NO, then the differences AF and MI are equal too : and now, since AF is equal to MI, and AB is equal to MN; it follows that AF, AB has the same magnitude for the third of the proportions as MI, MN, as is apparent from first principles: but MR is the third of the proportionals [a] MI and MN: and MR is the sum of the series of ratios MN to NO; hence also the sum of the series MR is the same third proportion of AF, AB : but the series AB, BC, by the first corollary gives the same magnitude MR, as the series MN, NO; hence the sum of the series AB, BC, &c. that is AK, is also the third of the proportions of AF, AB: hence the difference AF, the first term AB, and the sum of the series AK are in continued proportion. Which demonstrates the first part of the corollary. Similarly, the remaining parts of the corollary can be demonstrated. *a 83 huius.*

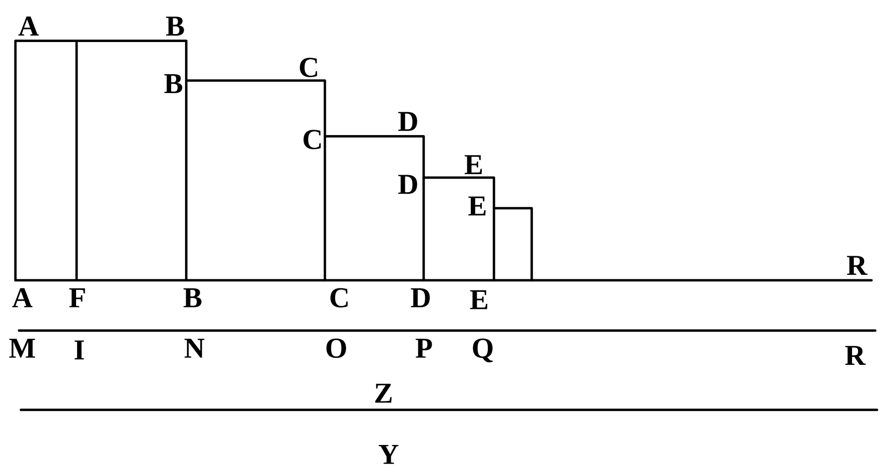
Corollarium quartum.

Iisdem positis infero quatro, hic etiam valere universalem illam ac triplicem constructionem qua propositione datae seriei magnitudinem aequalem invenimus.
Fiat enim ut AF differentia primoram terminoram ad AB, sic AB ad aliam magnitudinem Z. Dico, Z aequalem esse toti similium magnitudinum seriei AK.

Demonstratio.

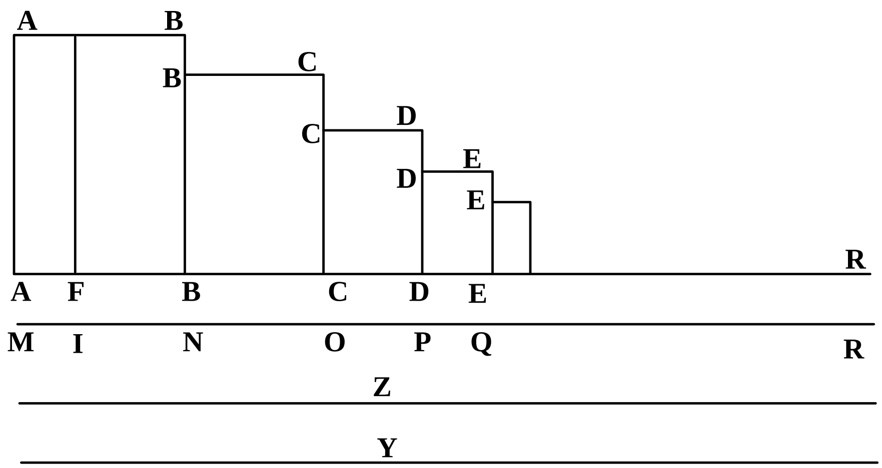
Si non est Z aequalis toti seriei, ergo alia magnitudo maior vel minor quam Z ipsi aequalis erit; (aliqua enim magnitudo per Corollarium primum toti seriei AK aequalis est) sit illa Y. ergo per corollarium praecedens AF, AB, Y, sunt continuae. atqui etiam ex constructione AF, AB, Z sunt continuae, ergo AB est ad Z ut AF est ad AB; & AB est ad Y, ut AF est ad AB. eandem igitur AB ad Z & Y rationem habet: aequales igitur sunt Z & Y contra hypothesim: ponebatur enim Y maior aut minor quam Z. non erit ergo alia minor maioruc quam Z, aequalis seriei AK. ergo Z aequalis erit. similiter duas alias propositionis octavagesimae huius constructioni demonstrabimus.

[128]



Prop.123. Fig. 3.

Fourth Corollary.



Prop.123. Fig. 3.

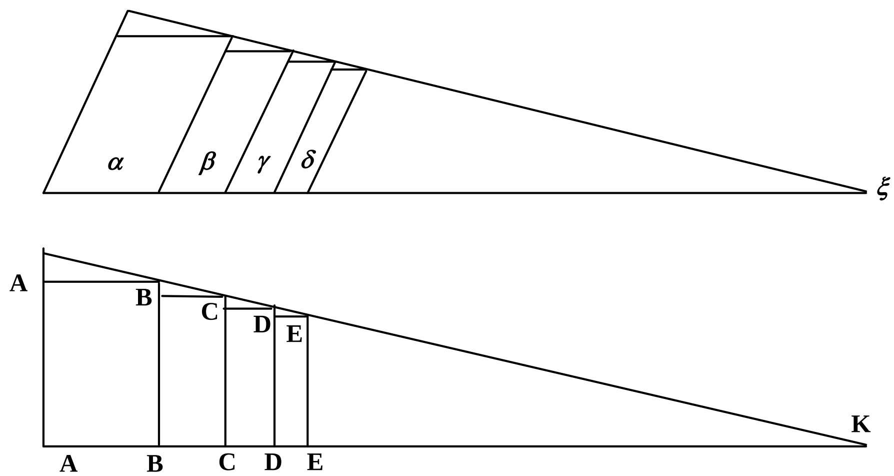
For the same situation, in the fourth place, I can infer that the general theorem prevails here also, and we find three constructions for which the magnitude of the series for the given proposition is equal.
For it happens that as AF the difference of the first terms AB and BC, is to AB, thus AB is to a different magnitude Z. I say that Z is equal to the sum AK of the series of similar magnitudes.

Demonstration.

If Z is not equal to the sum of the whole series, then the sum is equal to a different magnitude which is larger or smaller ; (for by the first Corollary the sum of the series AK is equal to some magnitude) let that quantity be Y. Hence by the preceding corollary AF, AB, and Y are in continued proportion. But also from the construction AF, AB, and Z are in continued proportion, hence AB is to Z as AF is to AB; and AB is to Y as AF is to AB. Therefore AB to Y and Z have the same ratio: therefore Z and Y are equal contrary to the hypothesis whereby Y is made greater or less than Z. Hence there is not a different quantity less or greater than Z that is equal to the series AK, and thus the series is equal to Z. Similarly, we can demonstrate the two other arguments of proposition eighty for this construction.

Corollarium quintum.

Iisdem positis infero quinto: si fuerit progressio AB, &c. similitum magnitudinum itemque alia progressio similium inter se magnitudinem α, β, γ, δ, &c. sive similes illae sint terminis alterius sive dissimiles; sit autem progressio utraque eiusdem proportionis: infero inquam totas series AK, α, ξ eam habere rationem inter se, quam primi termini A, B, &c.



Prop. 123: Fig. 4.

Demonstratio.

Per corollarium secundum series AK, est ad seriem BK, ut AB ad BC : sed ex hypothesi AB est ad BC, ut α ad β; ergo series AK est ad seriem BK, ut α ad β, hoc est per idem corollarium ut series αξ ad seriem βξ. Igitur per constructionem rationis series AK est ad AB, ut series αξ, ad α : & permutando series AK est ad seriem αξ ut AB ad α. Quod erat demonstrandum.

Corollary five.

With the same points and lines in position I can infer in the fifth place that if there is a progression AB, &c. of similar magnitudes and likewise another progression of magnitudes similar between themselves α, β, γ, δ, &c. , and these terms are either similar or dissimilar with the terms of the other progression; but each progression is of the same proportions: then I can indeed infer that the sum of the series AK, α, and ξ have the same ratio between themselves as the first series A, B, etc.

Demonstration.

By the second corollary, the series AK is to the series BK as AB is to BC : but by hypothesis AB is to BC, as α is to β; hence the series AK is to the series BK, as α is to β, or by the same corollary as the series αξ is to the series βξ. Therefore from the construction of the ratios, the series AK is to AB as the series αξ is to α : and on interchanging, the series AK is to the series αξ as AB is to α. Q. e. d.

Corollarium sextum.

Et quamquam hactenus solum assumpserimus progressionem planorem, corporumve similium similiterque positorum, non est tamen quod existimet lector, quae hactenus demonstrata sunt non subsistere, si planorum aut corporum non similium statuatur progressio, eadem quippe utrobique, ut cuilibet rem expedenti manifestum est, & veritas est & veritatis demonstratio, idcirco autem figuras similes assumere placuit, quod & usus earum frequentior, & magis sint ad demonstrandum accommodatae.

Ex his hunc in modum demonstratis manifestum est progressionum proprietates, secunda parte explicatas progressionibus magnitudinum in directum positarum, quas deinceps prosequimur, non minus quam aliis convenire : ac proinde propositiones superioris partis in quibus illae tractantur, prorsus universales esse. Quare has deinceps uti revera tales in sequentium theorematum demonstrationibus citabimus

Corollary six.

And though up to this point, we have only assumed progressions of similar plane or solid figures placed in some regular arrangement, the reader should not think that this is the end of the matter. For until now we have not stopped to demonstrate the case of a progression of plane or solid figures set in place which is not directly similar to a progression along a line, though obviously there is agreement between the terms in both places, in order that the desired outcome can be shown and there is a demonstration of the true. Hence more general series are to be considered which will be used more often, and these are arranged here for further explanation.

From these in this manner the characteristics of progressions are to be made clear, as the following section explains the magnitudes of progressions placed in given directions, which we can then examine in detail to establish that they are in agreement no less than for the others: and hence the propositions set out in the above sections are shown in short to be quite general. Whereby we set in motion these progressions that are actually to be used as such in the demonstrations of the theorems to follow.

PROPOSITIO CXXIV.

Data sint proportionales continuae AB, BC, CD, &c. & super iis constructa plana similia.

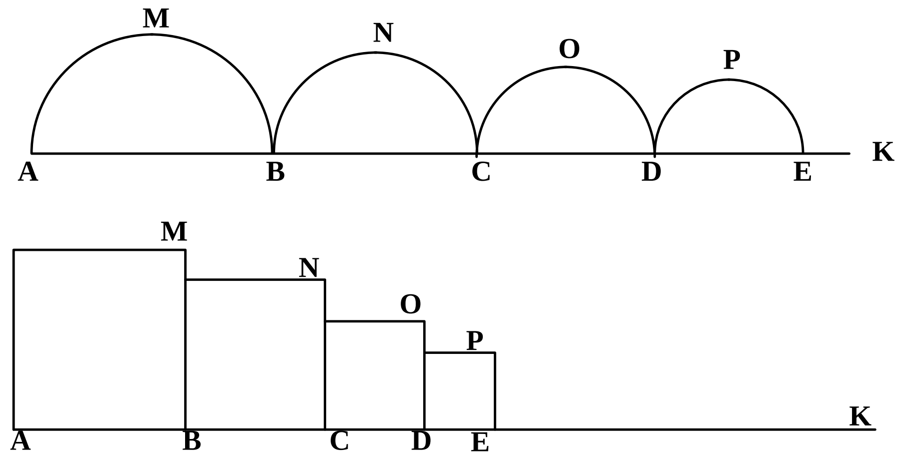
Dico plana esse in continua analogia : & si plana dentur continue proportionalia:

Dico etiam bases fore continue proportionales.

[129]

Demonstratio.

Planum AM est ad planum BN, in duplicata ratione AB ad BC; & planum BN est ad planum CO, in duplicata ratione BC ad CD, id est ex datis AB ad BC : similiter planum CO est ad planum DO, in duplicata ratione CD ad DE, id est rursus AB ad BC: similiter ostendam omnia reliqua inter se esse in duplicata ratione AB ad BC : manifestum est igitur omnia esse in continua analogia : Quod erat primum. secunda pars simili plane discursu ostendatur; patet igitur veritas propositionis.



Prop.124. Fig. 1.

L2.§3. *PROPOSITION 124.*

The continued proportionals AB, BC,CD, etc. are given, and above these similar plane figures are constructed.

I say that the plane figures are in continued proportion; and if the planes figures are given on continued proportion, then I say that the bases are also in continued proportion.

Demonstration.

The plane figure AM is to the plane figure BN in the square ratio AB to BC; and the plane figure BN is to the plane figure CO in the square ratio BC to CD, that is from the given ratio AB to BC. Similarly the plane figure CO is to the plane figure DO, in the square ratio CD to DE, or again as AB to BC. Similarly I can show that all the remaining terms are in the square ratio AB to BC to each other: which demonstrates the first part, that the plane figures are in proportion; the second part can be shown by a similar argument, and so the truth of the proposition is apparent.
Q.e.d.

PROPOSITIO CXXV.

Eadem posita figura data sint duo plana similis, basibus homologis in directum positis, AM maius, BN, minus. Petitur inveniri terminus longitudinis, ad quem proportio dictorum planorum sine statu continuata excurret.

Constructio & Demonstratio.

Per octavagesima huius inveniatur progressionis basium AB, BC terminus; sitque; K.

Dico etiam K terminum esse longitudinis ad quem series planorum excurret : plana enim similia quae fient super terminis progressionis basium, per praecedentium erunt continuae proportionalia, ac proinde linearum planorumque; in infinitum progressio, pari passu procedent: quare utriusque terminus erit K. Quod erat demonstrandum.

Corollarium.

Idem igitur punctum, terminus est progressionis basium, & terminus longitudinis, quem habet series figurarum similium : sive, quod idem est, linea quae aequalis est seriei basium, est longitudo seriei figurarum similium, super basibus descriptarum.

L2.§3. *PROPOSITION 125.*

With the same given figure position, two sets of similar plane figures are given, with bases of the same kind placed along a line, with AM larger than BN, etc. It is required to find the terminus of the length to which the proportions of the said plane figures extends on being continued indefinitely.

Construction & Demonstration.

By Prop. 80 of this book, the terminus of the progression of the bases is found to be K.
I say that the terminus K is also the length to which the series of plane figures extends: for the plane figures constructed on the bases of the progression are in continued proportion, by the preceding theorem, and hence both the lines and the plane figures are in an infinite progression, proceeding with like steps; whereby the terminus of both series is K. Q.e.d.

Corollary.

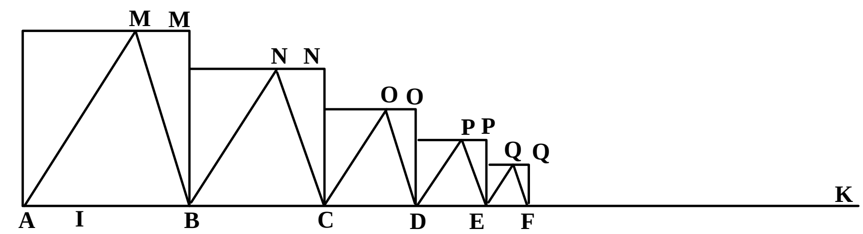
Therefore the terminus of the progression of basis is the same point as the terminus of the length of the series of similar figures; or, what amounts to the same thing, the line equal to the sum of the infinite series of bases is also the length of the line of the sum of the series of similar figures described on the bases.

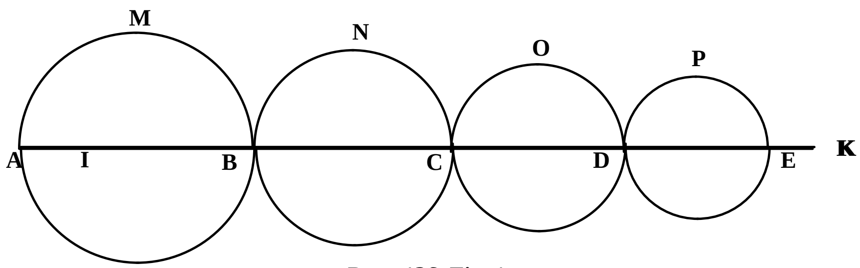
PROPOSITIO CXXVI.

Data sit planorum similium continue proportionalium series, homologis basibus AB, BC, CD, &c. in directum positis, habens terminum longitudinis punctum K;

Dico, totam planorum seriem MK esse ad primum terminum AM, ut est tota series basium imparium AB, CD, EF, &c. ad primam AB.

[130]





Prop.126. Fig. 1.

Demonstratio.

Per octuagesimam secundam huius, tota planorum series MK est ad seriem NK, ut planum AM ad planum BN : atqui cum plana sint similia, duplicatam habent ratione basium AB, BC; ergo series MK ad seriem NK duplicatem habet rationem AB ad BC : quia autem per corollarium proposit. praecedentis, progressionis basium AB, BC, &c. terminus est K, erunt igitur AK, BK, [a] CK tres continuae proportionales. unde ratio AK ad CK, duplicata est rationis AK ad BK, id est rationis AB ad BC : ergo series MK est ad seriem NK, ut AK ad CK: quare per conversionem rationis series MK, est ad planum AM, ut AK ad CA : deinde quia series AB, BC, CD, DE, &c. id est linea AK, est ad seriem AB, CD, &c. ut CA ad BA, erit alternando AK ad CA, ut series AB, CD &c. ad AB : sed series planorum MK, est ad planum AM, ut AK ad CA, ergo series MK est ad planum AM, ut series AB, CD, &c. ad AB. Quod erat demonstrandum.

a 82 huius; b 1 huius; c 103 huius.

Manifestum.

Ex demonstrationis discursu patet totam planorum seriem esse ad primum planum, ut KA ad CA. Quod quia postea usui veniet, sigillatim notare placuit.

L2.§3. *PROPOSITION 126.*

A series is given of similar continued proportions, with bases of the same kind placed on a line, having the terminus of the length [taken up by the series] at the point K.

I say that the sum of the series of plane figures MK is to the first term AM as the sum of the odd bases AB, CD, EF, &c. is to the first term AB.

Demonstration.

By Prop. 80 of this book, the sum of the plane figures MK is to the series NK, as the plane figure AM is to the plane figure BN : but as the plane figures are similar, they are in the ratio of the squares of the bases AB, BC; hence the ratio of the series MK to the series NK is as the square of AB to BC : however, by a corollary of the preceding proposition, the terminus of the progression of the bases AB, BC, &c. is K. Hence [a] AK, BK, and CK are three lines in continued proportion, from which it follows that the ratio AK to CK is the square of the ratio AK to BK, or AB to BC. Hence the series of similar plane figures MK is to the series of figures NK as AK is to CK; whereby on rearranging the ratio by subtraction, the series of plane figures MK is to the plane figure AM, as AK is to CA. Hence, since the sum of the series AB, BC, CD, DE, &c., or line AK, is to the series AB, CD, &c, as CA is to BA, then on alternating the ratio, AK is to CA as the series AB, CD, &c. is to AB. But the series of plane figures MK, is to the plane figure AM, as AK is to CA; hence the series of plane figures MK is to the plane figure AM as the series AB, CD, &c. is to AB. Q.e.d.

[Initially we have: (Sum of plane figures MK)/(Sum of plane figures NK) = area AM/area BN = AB^2/BC^2; however, AK/BK = BK/CK = ...; and on subtraction, AB/BK = BC/CK; and AK/BK = AB/BC; hence AK/CK = $AK^2/BK^2 = AB^2/BC^2$;
hence (Sum of plane figures MK)/(Sum of plane figures NK) = AK/CK, from which it follows that (area of figure AM)/(Sum of plane figures MK) = AC/AK; (changing from NK and CK to MK and AK). Again: (sum of AB, BC, CD, ...)/(sum even terms AB, CD, EF, ...) = CA/BA, on application of Prop. 103,
then AK/CA = (sum of even terms AB, CD, EF, etc.)/BA = (Sum of plane figures MK) /(area of figure AM), as required.
In modern terms, $S(r)/S(r^2) = a/(1 - r) \times (1 - r^2) /a = 1 + r = a(1 + r)/a$ = CA/BA; whence $S(r)/a(1 + r) = aS(r^2)/a^2$ = S(MK)/area AM]

Conclusion.

From the discussion of the demonstration, it is apparent that the ratio of the sum of the series of plane figures to the first plane figure is in the same ratio as KA to CA. Since later use will come about for these propositions, it has pleased us to note them individually.

PROPOSITIO CXXVII.

Iisdem positis sit AI differentia primae AB, & tertia CD.

Dico totam similium planorum seriem esse ad primum planum AM, ut AB prima basis, ad AI primae & tertiae differentiam.

Demonstratio.

Fiat lineis AI, AB tertia ST continue proportionalis. Igitur ST [d] aequalis est tota seriei basium imparium AB, CD, EF, &c. ergo per praecedentem tota series planorum MK est ad planum AM ut ST ad AB : Atqui ex constructione AB est ad AI, ut ST ad AB; ergo tota series est ad planum AM, ut AB ad AI. Quod erat demonstrandum. *d 79 huius.*

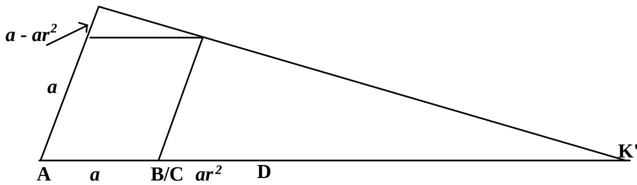
L2.§3. PROPOSITION 127.

With the same points and figures in place, AI is the difference of the first and third terms AB and CD.

I say that the sum of the series of similar plane figures MK is to the first plane figure AM, as the first base AB is to AI, the difference of the first and third.

Demonstration.

ST is made the third proportion with the lines AI and AB in continued proportion. Therefore ST [d] is equal to the sum of the series of the odd [*i. e.* in the sense the first, the third, etc. terms, though these actually correspond to even powers of the common ratio, which is not of course in use here] bases AB, CD, EF, &c. Hence by the preceding proposition, the sum of the series of plane figures MK is to the plane figure AM as ST is to AB : But from the construction AB is to AI, as ST is to AB; hence the sum of the series is to the plane figure AM as AB is to AI. Q.e.d.



Prop.127. Fig. 2.

[From the last proposition: AK/CA = (sum of even terms AB, CD, EF, etc.)/BA = (Sum of plane figures MK) /(area of figure AM) = ST/BA . Now, AI, AB - CD, AB, and AK' = ST are in continued proportion, (from similar triangles, see Fig. 2, not in original text), hence AB /(AB - CD) = ST/AB; from which it follows that AB/AI = ST/AB = (Sum of plane figures MK) /(area of figure AM), as required.]

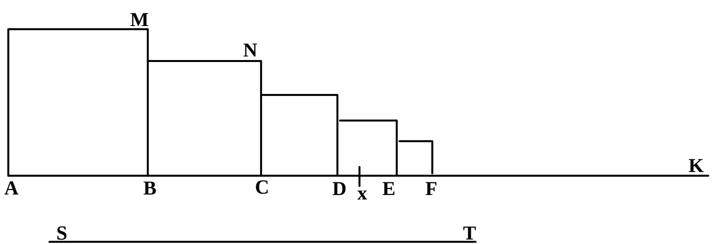
PROPOSITIO CXXVIII.

Eadem manente figura, data sit quadratorum series basibus in directum positis, terminum habens longitudinis, punctum K : fiat autem per octuagesima huius ST, aequalis seriei basium imparium AB, CD, &c.

Dico rectangulum super ST in altitudine AB aequali toti seriei quadratorum MK.

[131]

Demonstratio.



Prop.128. Fig. 1.

Per propositionem centesimam vegesimam sextam hiuis, tota series quadratorum MK, est ad primum quadratum AM, ut ST ad AB; atqui rectangulum super ST in altitudine AB, est ad quadratum AM, ut [a] ST ad AB; ergo series quadratorum MK est ad quadratum AM, ut rectangulum ST AB ad quadratum AM : aequalia sunt igitur rectangulum, & tota series. Quod erat demonstrandum. *a 1 sexti ?.*

Corollarium.

Hinc sequitur quadratum ST, totam seriem MK, & quadratum AM in continua esse analogia; nam quadratum ST ad rectangulum super ST & AB, est ut ST linea ad AB lineam: sed rectangulum idem, hoc est tota series MK, est ad AM quadratum in eadem ratione; ergo, &c.

L2.§3. PROPOSITION 128.

Continuing with the same figure, a series of squares is given with bases placed on the line having the terminus at the point K along the line : while by proposition eighty of this work, ST is set equal to the series of the odd numbered bases AB, CD, &c.

I say that the rectangle erected on ST with altitude AB is equal to the sum of the series of squares MK.

Demonstration.

By proposition 126 of this book, the sum of the series of squares MK is to the first square AM, as ST is to AB; but the rectangle upon ST with altitude AB is to the square AM, as [a] ST is to AB; hence the series of squares MK is to the square AM, as the rectangle ST AB is to the square AM: therefore the rectangle and the sum of the series are equal. Q.e.d. *a 1 sexti ?.*

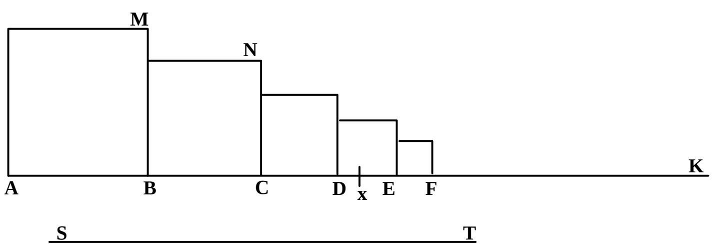
[From the last proposition, AK/CA = (sum of even terms AB, CD, EF, etc.)/BA
= (Sum of plane figures MK) /(area of figure AM) = ST/BA .
Now, Sum of plane figures MK = (area of figure AM) × ST/BA = AB^2 × ST/BA = AB × ST, as required.]

PROPOSITIO CXXIX.

Iisdem positis ut AB ad BC, sic fiat AX ad XK.

Dico rectangulum XAB toti quadratorum seriei aequale esse.

Demonstratio.



Prop.129. Fig. 1.

Quia AX est ad XK, ut AB ad BC, erit invertendo ac componendo, KA ad XA, ut CA ad BA: Atqui etiam KA b est ad seriem linearum AB, CD, ut CA ad BA; ergo KA eandem habet rationem ad XA, & ad seriem AB, CD, aequales sunt igitur series AB, CD & linea XA. unde per praecedentem rectangulum XAB, toti quadratorum seriei est aequale. Quod erat demonstrandum. *b 103huius.*

Corollarium.

Ex duabus propositionibus colligere modum licet, quo quadratorum datae seriei repertiri possit quadratum unum aequale. nimirum si inter AB & ST, vel inter AB, &c. AX, media fiat proportionalibus; erit haec latus quadrati, toti seriei ut patet ex duobus iam demonstratis theorematibus : Verum luculentius & universalius hoc Theorema sequenti propositione construemus.

L2.§3. *PROPOSITION 129.*

With the same points in position, as AB is to BC, thus AX is made to XK.
I say that the rectangle XAB is equal to the sum of the series of squares.

Demonstration.

Since AX is to XK as AB is to BC, on inverting and adding, KA is to XA as CA is to BA: But also KA [b] is to the series of the lines AB, CD, as CA is to BA; hence KA has the same ratio to XA and to the series AB, CD: therefore the series AB, CD and the line XA are equal. Hence by the preceding, the rectangle XAB is equal to the sum of the whole series of squares. Q.e.d. *b 103huius.*

[AX/XB = AB/BC = 1/r, hence KA/XA = CA/BA = 1 + r ;
but CA/BA = KA/XA = (Sum of plane figures MK) /(sum of square terms AB, CD, EF, etc.); hence XA is equal to sum of square terms AB, CD, EF, etc.]

Corollary.

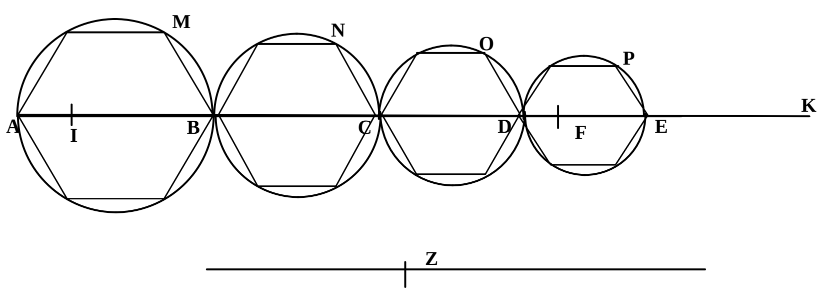
From the two propositions taken together the rule follows that a single square can be found equal to the sum of the given squares of the series: for without doubt between AB and ST, or between AB, &c. and AX, the mean of the proportionals can be found; this will be the side of the square of the sum of the whole series, as is apparent from the two theorems now demonstrated : The following proposition that we construct will show the truth and universal nature of this theorem most clearly.

PROPOSITIO CXXX.

Data sit series planorum quorumcumque basibus in directum AB, BC, CD, &c. positis, ac terminum habens longitudinis punctum K. petitur planis seriei universae planum aequale ac simile exhiberi.

[132]

Constructio ac demonstratio prima.



Prop.130. Fig. 1.

Fiat ut AC ad AK, sic primum seriei planum AM, ad aliud simile cuius diameter vel basis sit Z. Dico hoc toti seriei aequale esse.

Per manifestum propositionis 126 huius, tota planorum series MK, est ad planum AM, ut AK ad AC, id est ex constructione ut Z, ad AM planum : ergo planum Z est ad planum AM, ut tota series MK, ad idem planum AM. aequantur igitur inter se planum Z, & tota series MK. Cum itaque etiam simile sit ex constructione planum Z, planis seriei datae MK, perfecimus quod in problemate petebatur.

Constructio & demonstratio secunda.

Sumatur AI primae AB, ac tertiae CD, basium differentia: fiatque ut AI ad AB, sic primum seriei planum AM, ad aliud sibi simile Z:

Dico Z planum satisfacere problemati. Nam tota series MK est ad planum AM ut [a] AB ad AI: Atqui ex constructione etiam planum Z est ad idem planum AM, ut AB ad AI; ergo planum Z, & tota series aequalia sunt. Invenimus igitur datae planorum similium seriei, planum aequale ac simile. Quod erat demonstratum.

a 127 huius.

Constructio & demonstratio tertia.

Fiat ut AB ad seriem basium imparium AB, CD, &c. sic primum planum ad aliud simile Z.
Dico hoc seriei planorum datae aequari. vel (quod idem est) Fiat ut AB ad BC, sic AF ad FK, utque AB est ad AF, sic planum primum fiat ad aliud simile.
Dico etiam hoc conficere problema: demonstratio eadem est quae primae ac secundae constructionis, ea tantum differentia, quod propositio 126. huius, sit adhibenda. Dixi autem secundum huius tertiae constructionis modum coincidere cum primo, eiusdem constructionis tertiae, quod ex praecedenti manifestum sit, FA aequalem esse seriei basium imparium.

L2.§3. *PROPOSITION 130.*

A series of some kind of plane figures is given with bases AB, BC, CD, etc., placed along a line and having the terminus point at a distance K.

It is required that a similar plane figure is produced that is equal in area to the sum of the whole series.

First construction and demonstration.

As AC is to AK, thus the first plane figure of the series AM is made in the same ratio to a different like figure, the base or diameter of which is Z. I say that this figure is equal in area to the whole series.
As proposition 126 of this book has shown, the sum of the series of the plane figures MK is to the plane figure AM, as AK is to AC, or by construction as Z to the plane figure AM : hence the plane figure Z is to the plane figure AM, as the sum of the series MK is likewise to the plane figure AM. Therefore the figure Z is equal to the sum of the whole series of figures MK. Since the plane figure Z is also similar by construction to the plane figures in the given series MK, we have accomplished what was demanded in the problem.

[(Sum of plane figures MK) /(first plane figure AM.) = AK/AC = Z /(first plane figure AM.), etc.
For, as we have already shown above, the sum Z of the series of plane figures is proportional to the sum of the squares of the diameters of the figures, AB, CD, etc. ;
hence algebraically, Z = (area AM). $(a/(1 - r))/a(1 + r)$ = (area AM)$/(1 - r^2)$ = (area AM)$(1 + r^2 + r^4 + ...)$]

Construction & second demonstration.

The difference AI is taken of the bases of the first and third terms AB and CD : and as AI is to AB, thus the first plane figure of the series AM is made to in the same ratio to some other similar plane figure Z:

I say that the plane figure Z provides a satisfactory answer to the problem. For the sum of the series of plane figures MK is to the plane figure AM as [a] AB is to AI: But from the construction the plane figure Z also is to the same plane figure AM, as AB is to AI; hence Z is equal to the sum of the series. Therefore we have found a plane figure equal to the sum of the similar given plane figures. Q. e. d.
[In this case, AI = AB - CD, and AI/AB = area AM/area Z; but from Prop. 127:
AB/AI = ST/AB = (Sum of plane figures MK) /(area of figure AM), from which the result follows at once.]

Third construction & demonstration.

As the ratio of AB is made to the sum of the bases of the series of odd terms [i. e. even powers] AB, CD, etc., thus the first plane figure is made in the same ratio to some similar plane figure Z.
I say that this figure Z is equal to the given series of plane figures, in the sense that it has the same area. As AB is to BC, thus AF is to FK, and as AB is to AF, thus the first plane figure is made to some other similar figure.
I also say that in solving this problem: the demonstration of the first and second constructions is the same, however the difference, according to Prop. 126 of this book, is to be applied. In addition I say that the method of this subsequent third construction coincides with the first, for from the preceding, FA can be shown to be equal to the sum of the bases of the odd series.

[In this case, AB/sum of even powers = $1/(1 + r^2 + r^4 + ..) = 1 - r^2$ = area AM/Z, as previously; however, in this case, AB/BC = AF/FK, from which AC/BC = AK/FK on addition;
hence, FK = AK.BC/AC = $a/(1 - r) \,.\, r/(1 + r) = ar/(1 - r^2)$ and AF = $a/(1 - r) - ar/(1 - r^2) = a/(1 - r^2)$.]

Scholium.

Adverte constructionem illam triplicem propositionis octuagesimae huius, cum universalis sit, huic etiam series convenire : verum quia in progressionibus huius generis, faciliores subinde ac magis expedita constructiones suppetunt, visum est opera pretium, illas tum hoc loco, iam aliis etiam deinceps in medium proferre.

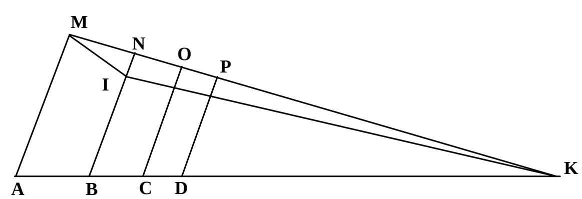
[133]

Scholium.

By directing one's attention to this triple construction instead of making use of the eightieth proposition directly, which is general, we note that the series is also in agreement with the proposition : however, for progressions of this kind, it is usually easier and more expedient to make use of some construction at hand, which is seen to be of value, not only for the present case, but also for other cases henceforth as they come to pass.

Lemma.

Esto linearum AB, BC, CD progressio terminata in K, & ex punctis A, B, C, &c. erigantur parallelae AM, BN, CO, &c. quae proportionales sint ipsis AB, BC, CD, &c. , ducaturque; ex puncto M ad terminum progressionis K, linea MK. Dico hanc per omnium parallelarum extrematates, N, O, P, &c. transire.



Prop.130. Fig. 2.

Demonstratio.

Consideremus primo lineam BN; si ergo MK non transit per N, secabit lineam BN supra aut infra N, in I. Erit ergo MIK una recta. Et quoniam progressionis AB, BC, &c. terminus est K per 82. huius, erit ut AK ad BK sic AB ad BC, hoc est, ex datis AM ad BN: atqui etiam ut AK est ad BK, sic AM ad BI; ergo AM est ad BN, ut AM ad BI maiorem aut minorem quam BN, quod est impossibile. Non ergo secabit MK ipsam BN supra aut infra N, ergo in N; similiter ostendemus rectam NK (hoc est rectam MNK, ostendimus enim modo puncto MNK esse in una recta) transire per O. & sic de ceteris in infinitum: Patet igitur veritas lemmatis.

Lemma.

Let the progression of the lines AB, BC, CD be terminated in K, and from the points A, B, C, &c. the parallel lines AM, BN, CO, etc. are erected, which are themselves proportional to AB, BC, CD, etc.; and a line MK is drawn from the point M to the terminus of the progression K. I say that this line passes through the ends of all the parallel lines N, O, P, etc.

Demonstration.

We consider the first line BN; for if MK does not pass through N, then it cuts the line BN either above or below N at the point I. Therefore MIK is a single line. And since the terminus of the progression AB, BC, etc. is K by Prop. 82, AK is to BK thus as AB is to BC, or from what is given, as AM is to BN: but also as AK is to BK, thus AM is to BI; hence AM is to BN as AM is to BI, which is larger or smaller than BN, which is impossible. Hence MK does not cut BN itself above or below N, but instead passes through N; similarly we can show that the line NK (or the line MNK, as we have shown in this way that the point lies on a single line MNK) passes through O, and thus for the other points indefinitely: The truth of the lemma is thus established.

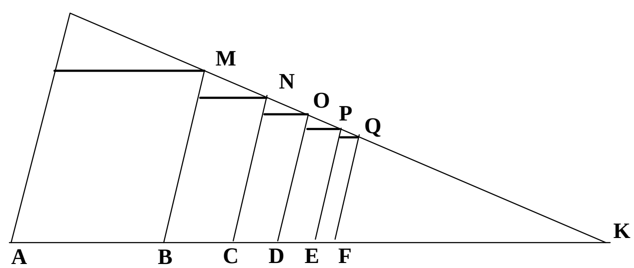
PROPOSITIO CXXXI.

Esto planorum rectilineorum similium similiterque positorum series MK, basibus in directum collatis, terminum habens longitudinis punctum K. Ex vertice autem M, ad K ducatur recta MK.

Dico hanc per omnes omnium angulorum totius seriei vertices N, O, P, Q; &c. transire : sive totam planorum seriem angulo AKM inscriptam esse.

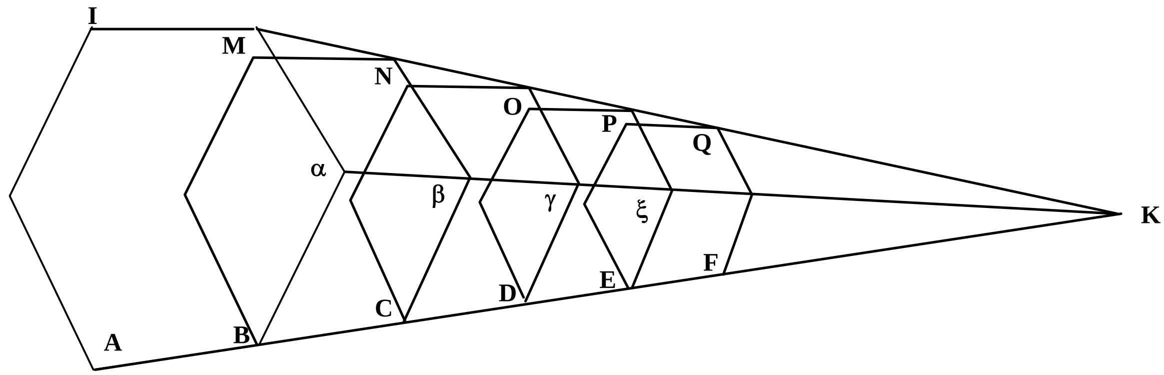
Demonstratio.

Data sit primum series figurarum trium quatuorque laterum. Cum igitur ex hypothesisi planorum series terminetur in K, basium quoque progressonis terminus erit K per 124. huius: & quoniam plana AM, BN, &c. similia sunt, similiterque; posita, ex elementis patet latera BM, CN, DO, &c. inter se parallela esse, ipsisque AB, BC, &c. proportionalia. ergo per lemma praecedens MK transit per omnes vertices N, O, P, Q, &c. Quod erat demonstrandum.



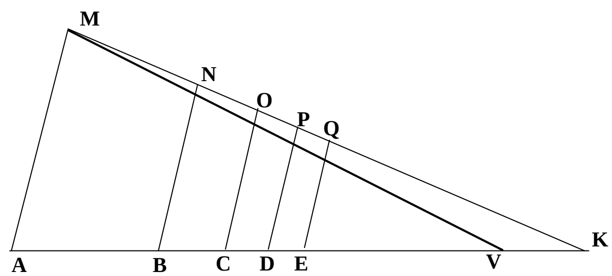
Prop.131. Fig. 1.

[134]



Prop.131. Fig. 2.

Sint iam plurium laterum figurae quarum series terminetur in K. Quoniam igitur plana AM, BN, &c. similia sunt similiterque posita, erunt Bα, Cβ, Dγ, Eξ, parallelae inter se, ipsisque AB, BC proportionales. quare per lemma precedens puncta α, β, γ, ξ &c. cum puncto K sunt in una eademque linea αK, quae cum similiter divisa sit, ac linea BK, manifestum est progressionem linearum αβ, βγ, &c. terminari quoque in K. Sunt autem ex punctis α, β, γ, &c. erecta parallela latera αM, βN, γO, &c. quae lateribus BC, CD, DE, hoc est ipsis αβ, βγ, γξ, sunt proportionalia (quae omnia patent ex eo quod AM, BN, CO, &c. similia plana sint similiterque posita) ergo per lemma, linea MK transit per omnia puncta M, N, O, P, Q, &c. Quod erat demonstrandum.



Prop.131. Fig. 3.

Lemma.

Esto linearum AB, BC, CD, &c. progressio terminata in K. & ex punctis A, B, C, D, &c. erigantur AM, BN, CO, DP, &c. inter se parallelae; & proportionales ipsis AB, BC, CD, &c. assumptisque duabus lineis AM, CO, per M & O ducatur recta MO.

Dico hanc productam incidere in K ac per omnium reliquarum extremitates transire.

Demonstratio.

Si enim ita non sit; igitur MO producta, cis vel ultra K in V concurret cum linea AK; & quoniam progressionis AB, BC, CD terminus est K, erunt AK, BK, [a] CK, DK, EK proportionales continuae. quare etiam AK, CK, EK ex aequo sunt continuae. unde [b] ut AK ad CK, sic AC ad CE. Atque [c] AC est ad CE, in duplicata rationis AB ad BC, id est ex hypothesi rationis AM ad BN; ergo AK est ad CK, in duplicata rationis AM ad BN : sed & ratio AM ad CO (ut ex datis colligitur) duplicata est rationis AM ad BN, ergo AM est ad CO, hoc est MOV ad OV, ut AK ad CK; quod est impossibile: non igitur occurret MO ipsi AK cis vel ultra K. Quod erat primum, hoc autem sic demonstrato, patet secunda pars ex lemmate propositionis preacedentis. Quae erant demonstranda. *a 82 huius; b 1. huius; c 106. huius.*

L2.§3. *PROPOSITION 131.*

Let MK be a series of similar rectilinear plane figures similar placed in position in a like manner with their bases arranged along a line, having the terminal point at a distance K. The line MK is drawn from the vertex M to K.

I say that this line passes through the vertices N, O, P, Q, etc. of the whole series of angles of the complete series : or the whole plane figure is inscribed within the angle AKM.

Demonstration.

A first series of figures of three and four sides is given. Since by hypothesis, the series of plane figures is terminated in K, the terminus of the bases of the progression is K too, by Prop. 124 of this work : and since the plane figures AM, BN, &c. are similar to each other, and placed in a similar manner, then it is apparent from elementary considerations that the sides BM, CN, DO, &c. are parallel to each other, and proportional to AB, BC, etc. Hence by the preceding lemma, MK passes through all the vertices N, O, P, Q, &c. Q. e. d.

Now there are figures of many sided for which the series terminates in K. Therefore, since the plane figures AM, BN, etc. are similar and similarly placed, then the lines Bα, Cβ, Dγ, Eξ are parallel to each other and proportional to AB and BC. Whereby by the preceding lemma the points α, β, γ, ξ etc. with the point K lie on the same line αK, and the line BK which as it is likewise divided, shows a progression of lines αβ, βγ, etc. to be terminated too in K. However, from the points α, β, γ, etc., there are erected parallel lines αM, βN, γO, etc. which are in proportion with the sides BC, CD, DE, or αβ, βγ, γξ (which are all apparent from this, since AM, BN, CO, etc. are similar plane figures similarly positioned) hence by the lemma, the line MK is cut by all the points M, N, O, P, Q, etc. Q. e. d.

Lemma.

The progression of lines AB, BC, CD, etc. terminates in K. From the points A, B, C, D, etc. the lines AM, BN, CO, DP, etc. are erected parallel to each other; and from the proportions AB, BC, CD, etc. and from the two lines taken AM, CO, the line MO is drawn through M and O.

I say that this line produced is to be incident in K, and to pass through all of the other extremities.

Demonstration.

For if this is not the case, then MO produced meets the line AK either on the near or far side of K in V; and since K is the terminus of the progression AB, BC, CD, then AK, BK, [a] CK, DK, EK are in continued proportion. Whereby AK, CK, EK are also in continued proportion from equality. Hence [b] as AK is to CK, thus AC is to CE. But [c] AC is to CE in the square of the ratio AB to BC, that is from the hypothesis for the ratios AM to BN; hence AK is to CK in the square of the ratio AM to BN : but the ratio AM to CO (as combined from what is given) is the square of the ratio AM to BN, hence AM is to CO, or MOV to OV, as AK is to CK; which is not possible : hence MO does not cut AK on the near or far side of K. Which establishes the first part of the proof, with this explained thus, the second part is apparent from the lemma of the preceding proposition. Q. e. d.

PROPOSITIO CXXXII.

Esto planorum rectilineorum similium similiterque positorum series MK, uti prius; & terminus sit K, per quorumlibet autem duorum planorum AM, CO vertices ducatur recta MO.

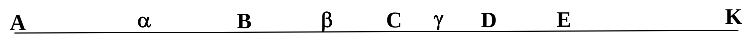
[135]

Dico hanc productam cadere in terminum K ac per omnium reliquorum vertices N, P, Q, &c. transire; sive totam planorum seriem angulo AKM inscriptam esse.

Demonstratio.

Discursus demonstrationis plane idem erit qui propositionis praecedentis. Nam quemadmodum illic per lemma illi propositioni appositum demonstravimus propositum, ita hic per lemmatis proximi applicationem propositionis veritatem concludemus.

Lemma.



Prop.132. Fig. 1.

Sint AK, BK, CK, DK, &c. in continua analogia, ac differentiae illarum nempe AB, BC, CD, DE, &c. bisectae sint in α, β, γ, &c. Dico etiam αK, βB, γC, &c. esse continuas.

Demonstratio.

Quia AK, BK, CK, &c. sunt continuae per primam huius, AB est ad BC, hoc est Aα est ad Bβ, ut AK ad BK. cum ergo ablatum Aα sit ad ablatum Bβ ut totam AK ad totam BK, erit & reliquam αK ad reliquam βK, ut totam AK ad totum BK. similiter ostendemus βK esse ad γK ut BK est ad CK; hoc est ex hypothesi ut AK ad BK, hoc est ex demonstrationis ut αK ad βK. Sunt igitur αK, βB, γC continuae. Quod erat demonstrandum.

L2.§3. PROPOSITION 132.

Let MK be a series of similar rectilinear plane figures similar placed as before; and the terminus is K. The line MO is drawn through any two of the place figures AM and CO as you wish.

I say that this line produced ends in the terminus K and also passes through the vertices N, O, P, Q, etc. : or the whole plane figure is inscribed within the angle AKM.

Demonstration.

The discussion of the demonstration clearly is the same as that for the preceding proposition. For since there we demonstrated the adjacent proposition from the lemma of that proposition, thus here we can conclude the truth of the proposition by an application of the next lemma.

Lemma.

AK, BK, CK, DK, &c. are in continued proportion, and the differences of these are surely AB, BC, CD, DE, etc., which are bisected in α, β, γ, etc. I say that αK, βB, γC, etc. are also in continued proportion.

Demonstration.

Since AK, BK, CK, etc. are in continued proportion from the first part of this prop., then AB is to BC, or Aα is to Bβ, as AK is to BK. Therefore when Aα is taken and Bβ is taken in the ratio of the whole length AK to the whole length BK, thus the remainder αK is to the remainder βK, as the whole length AK to the whole length BK. In the same way we can show that βK is to γK as BK is to CK; or from hypothesis as AK is to BK, or from the demonstration as αK is to βK. Therefore αK, βB, γC are in continued proportion. Q. e. d.

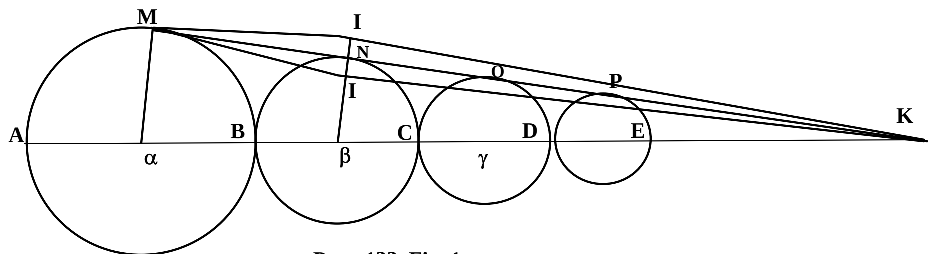
[For $A\alpha/B\beta = AK/BK$ gives $A\alpha/AK = B\beta/BK$ from which $\alpha K/AK = \beta K/BK$ or $\alpha K/\beta K = AK/BK$, etc.]

PROPOSITIO CXXXIII.

Esto circulorum progressio diametris in directum positis, terminum habens longitudinis punctum K: & ex K ducta linea KM tangat quemvis datae seriei circulum, verbi gratia circulum AMB.

Dico lineam KM totam circulorum seriem contingere.

Demonstratio.



Prop.133. Fig. 1.

Ex centro α ad contactum ducatur αM linea, cui ex centro β parallela sit βX secans circulum BNC in N. Recta igitur MK occurrit ipsi βX vel in puncto N, vel supra aut infra N in I. non autem supra aut infra posse occurrere enim, sic demonstro, occurrat emim, si fieri potest, supra vel infra N in I. quoniam igitur dantur circuli in continua analogia, etiam per 124 huius AB, BC, CD, &c. sunt continuae : ergo AB est ad BC, ut BC ad CD, hoc est ut Bβ ad Cγ : sed & AB est ad BC, ut αβ ad βC; ergo ut AB est ad BC, sic αβ est ad βγ. Praeterea quoniam K terminus est progressionis basium AB, BC, &c. per corollarium 125, huius erunt per 82, huius continuae proportionales AK, BK, CK, DK, quare per lemma etiam αK, βK, γK erunt continuae. ergo per primam huius ut αK ad βK, sic est αβ ad βγ. hoc est sicut ante ostendimus AB ad BC, hoc est αM ad βN. Atqui eiam est ut αK ad βK, sic αM ad βI : sunt enim ex

[136]

constructione lineae αM, βI parallelae; ergo αM est ad βI, ut αM ad BN maiorem aut minorem quam BI, quod est absurdum. Non igitur recta MK occurret rectae βI supra vel infra N: ergo in N. Quoniam autem ex centro ad contactum angulis KMα [a] rectus est; quare cum ex constructione βN parallela sit αM, angulus quoque KNβ rectus erit: ergo [b] linea KM tangit circulum BNC. simili ratiocinatione demonstrabimus KM reliquos etiam omnes circulos contingere. Quod erat demonstratum. *a 17 quinti; b 16 tertii.*

Corollarium.

Quod in hoc Theor. de circulis demonstravimus, etiam de similibus Ellipsibus, Parabolis, Hyperbolis demonstrari potest.

L2.§3. PROPOSITION 133.

A progression of diameters of circles is set out along a line, having the terminus at a distance K. The line KM is drawn to touch one or other of the given series of circles, such as the circle AMB, for example.

I say that the line KM touches all the series of circles.

Demonstration.

From the centre α to the point of contact the line αM is drawn, to which the line βX is drawn parallel from the centre β, cutting the circle BNC in N. Therefore the line MK crosses βX either at the point N, or above or below N in I. I demonstrate as follows that it is not possible for the point to be above or below N. Indeed if we assume that it is cut above or below N in I, then as the circles are given in continued proportion, it follows by Prop. 124 that AB, BC, CD, etc. are in continued proportion : hence AB is to BC as BC is to CD, or as Bβ is to Cγ : but AB is to BC as αβ is to βC; hence as AB is to BC, thus αβ is to βγ. In addition, since K is the terminus of the progression of the bases AB, BC, etc., by the corollary of Prop.

125 and Prop. 82, AK, BK, CK, DK are continued proportionals. Whereby by the lemma αK, βK, γK are also continued proportionals. Hence as αK is to βK, thus αβ is to βγ. or as we showed before, as AB is to BC, or αM to βN. But also as αK is to βK, so αM is to βI : for the lines αM, βI are parallel from the construction; hence αM is to βI as αM is to BN, larger or smaller than BI, which is absurd. Therefore the line MK does not cut βI above or below N: and thus it cuts in N. But as the angle KMα [a] from the centre of the circle to the point of contact is right; since from the construction βN is parallel to αM, the angle KNβ is also right: hence [b] the line KM is a tangent to the circle BNC. By similar reasoning we can show that KM is also a tangent to all the other circles. Q. e. d. *a 17 quinti; b 16 tertii.*

Corollary.

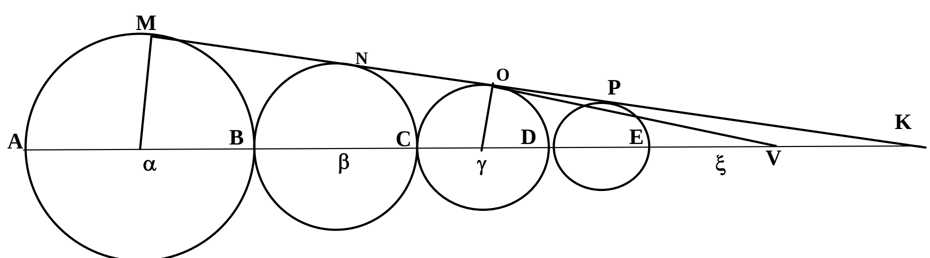
What we have shown for circles in this theorem can also be demonstrated for similar ellipses, parabolas, and hyperbolas.

PROPOSITIO CXXXIV.

Iisdem positis duos quoscumque circulos seriei datae circulos, verbi gratia AMB, COD tangat recta MO.

Dico hance productam cadere in terminum longitudinis K, & reliquos circulos omnes contingere.

Demonstratio.



Prop.134. Fig. 1.

Centra circulorum sint α, β, γ, ξ, etc., tum ex α ac γ ad contactus ducantur αM, γO. si igitur negetur assertio, tangens MO producta occurret lineae AK cis vel ultra K in V. & quoniam per corollarium propositionis 125. huius terminus progressionis basium est in K, erunt [c] AK, BK, CK, DK, EK, &c. ac proinde per lemma propositionis praecedentis etiam αK, βK, γK, in continua analogia. Quare patet ex aequo etiam αK, βK, γK esse continuas, ergo αK est [d] ad γK, ut αγ est ad γξ. Atqui αγ est ad γξ, ut AC ad CE (quod simili modo ostendemus quo in praecedenti ostendimus AB esse ad BC ut αB ad Bγ) & AC est ad CE in [e] duplicata rationis AB ad BC, id est ut AB ad CD; & AB est ad CD ut αM ad γO, ergo αK est ad γK ut αM ad γO. Deinde cum αM, γO ex centris ad contactus ductae sunt, erunt perpendiculares ad MV, ideoque parallelae inter se. Quare MV erit ad OV, ut αM ad γO, hoc est per iam demonstrata ut αK ad γK, quod est absurdum. Non igitur MO occurret ipsi AK cis vel ultra sed in K. Quod erat primam. Quo demonstrato, patet per praecedentem secunda pars, quae erant demonstranda.

Quod si loco circuli COD assumator circulus BNC, ita ut linea duos vicinos circulos contingat, demonstratio longe erit facilior, quam proinde omisimus. *c 82 huius; d 1 huius; e 106 huius.*

[137]

Corollarium.

Hoc quoque Theorema non tantum circulis, sed aliis similibus sectionibus applicari potest.

L2.§3. *PROPOSITION 134.*

With the same points in position, any two circles of the given series, for example, AMB and COD are tangents to the line MO.

I say that this line produced passes through the terminal point at a distance K, and that the rest of the circles in the series have the line MO as a tangent.

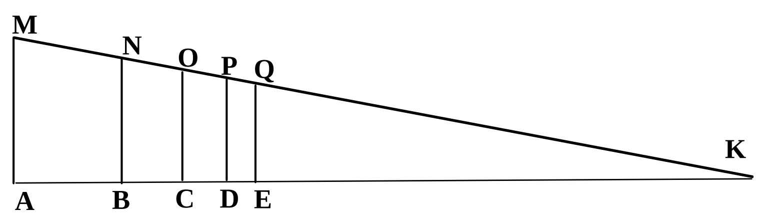
Demonstration.

The centres of the circles are α, β, γ, ξ, etc., then from α and γ to the point of contact the lines αM and γO are drawn. Therefore, if the proposition is not true, then the tangent MO produced meets one side or the other of the line AK in V. Since, by the corollary of proposition 125, the terminus of this progression of bases is K, the lines [c]AK, BK, CK, DK, EK, &c. are in a continued ratio, as also are the lines αK, βK, γK, by the lemma of the previous proposition. Whereby it is apparent from the equality that αK, βK, γK are in continued proportion, hence αK is [d] to γK, as αγ is to γξ. But αγ is to γξ, as AC is to CE (as we can show by a similar way as we have shown for the preceding that AB is to BC as αB to Bγ) and AC is to CE in [e] the square of the ratio AB to BC, or as AB ad CD; and AB is to CD as αM is to γO, hence αK is to γK as αM is to γO. Hence as αM and γO have been drawn from the centres of the circles to the points of contact, they are perpendicular to MV, and therefore parallel to each other. Whereby MV is to OV, as αM to γO, or as just demonstrated, as αK to γK, which is absurd. Therefore the line MO does not meet the line AK in one side or the other, but in K itself, which proves the first part. The second part is apparent from the first, and these had to be shown.
For if the place of the circle COD is taken by the circle BNC, since the line is a tangent to two neighbouring circles, the demonstration is much easier, and hence has been omitted. *c 82 huius; d 1 huius; e 106 huius*

Corollary.

This theorem can also be applied to other figures, as well as circles, with similar sections.

PROPOSITIO CXXXV.



Prop.135. Fig. 1.

Linearum AB, BC, CD, &c. progressio terminatur in K: erectaque ad quemvis angulum recta AM, ducatur MK: Deinde ex singulis punctis in progressione AK repertis, ad AM, parallelae erigantur BN, CO, DP, atque ita semper.

Dico ex triangulo AMK relinquendum tandem triangulum aliquod quavis data superficie minus.

Demonstratio.

Quoniam similia sunt triangulia AMK, BNK, COK, &c. erit duplicata eorum proportio, proportionis laterum AK, BK, CK, &c. quia autem progressionis AB, BC terminus est K, erunt [a] AK, BK, CK, &c. continuae proportionales. unde similiter triangula AMK, BNK, &c. etiam sunt in ratione continua & dividendo trapezium AM NB ad triangulum BNK, ut trapezium BN OC ad triangulum COK, & trapezium CP ad triangulum DPK: atque ita semper. Igitur [b] relinquetur tandem triangulum dato minus. Quod erat demonstrandum. *a 82 huius; b 78 huius.*

L2.§3. PROPOSITION 135.

The progression of the lines AB, BC, CD, etc. terminates in K: and the line AM is erected at right angles to AK at some angle to the line AK, and the line MK is drawn. Hence, from the individual points of the progression AK, the lines BN, CO, DP are drawn parallel to AM, indefinitely.

I say that by taking away parts of triangle AMK, a triangle is given at last with a surface less than some given area.

Demonstration.

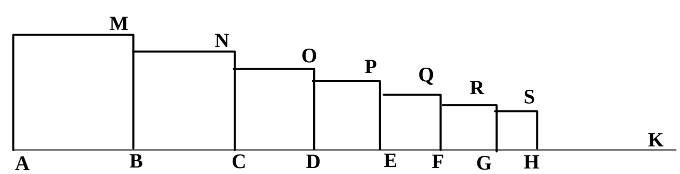
Since the triangles AMK, BNK, COK, etc. are similar, the areas are in the square ratio to the sides of the triangles AK, BK, CK, etc. But since the terminus of the progression AB, BC is K, [a] AK, BK, CK, etc. are continued proportionals. Hence, the triangles AMK, BNK, etc. also are similarly in a continued ratio, and on dividing, the trapezium AM NB is to the triangle BNK as the trapezium BN OC is to the triangle COK, & the trapezium CP is to the triangle DPK: and thus always indefinitely. Therefore [b] at last there remains a triangle less than some given surface. Q. e. d. *a 82 huius; b 78 huius.*
[ΔAMB/ΔBNK = ΔBNK/ΔCOK, etc., from which trap. AMNB/ΔBNK = trap. BNOC/ΔCOK, etc.]

PROPOSITIO CXXXVI.

Esto planorum series lateribus homologis in directum constitutis, terminus autem longitudinis sit K.

Dico ex ablatione continuata planorum AM, BN, CO, &c. relinqui residium seriei, quovis dato plano minus.

Demonstratio.



Prop.136. Fig. 1.

Per octuagesimam secundam huius, AM est ad reliquam seriem planoram NK, ut planum BN est ad reliquam seriem OK : & per eandem planum BN est ad reliquam seriem OK, ut planum CO, est ad seriem reliquam PK: Atque ita in infinitum, ergo ex perpetua planorum AM, BN, &c. ablatione residuum seriei propositae [c] erit tandem qouvis dato plano minus. Quod erat demonstrandum. *c 78 huius.*

[138]

L2.§3. PROPOSITION 136.

A series of plane figures with sides in correspondence is set up along a straight line, while the terminus of the length of the line is K.

I say that by the successive removal of the plane figures AM, BN, CO, etc, a part of the series is left smaller than some given plane figure.

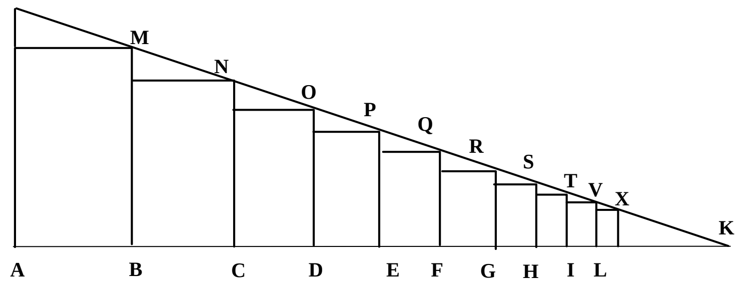
Demonstration.

By Prop. 82, the ratio of AM to the rest of the plane series NK, as the same manner BN is to the rest of the series OK, as the plane figure CO is to the rest of the series PK. Thus continued indefinitely, hence from the endless nature of the supply of plane figures, the remaining part of the proposed series is finally by subtraction less than some given plane figure. Q. e. d. *c 78 huius*

PROPOSITIO CXXXVI.

Detur series planorum similitium homologis basibus in directum collocatis, terminum habens longitudinis punctis K: sumpto autem quovis plano CO, numerentur alia plana in infinitum EQ, GS, IV, &c. totidem semper intermissis quot inter planum CO & primum seriei planum AM intercedunt. Petuntur omnia plana AM, CO, EQ, GS, &c. ex proposita serie auferri.

Constructio & Demonstratio.



Prop.137. Fig. 1.

Quoniam ex hypothesi plana AM, BN, CO, DP, EQ, FR, GS, sunt in continua ratione, etiam ex aequo plana AM, CO, EQ, IV, in continua sunt analogia. Igitur per propositionem 126 huius, seriei planorum continue proportionalium AM, CO, EQ, &c. inveniatur planum aequale. hoc si aufers ex plano, quod per eandem propositionem 126, factum fuerit aequale seriei datae MK, habebitur propositum.

L2.§3. *PROPOSITION 137.*

A series of similar plane figures with bases in a given continued ratio are set out together along a line, the terminal point being at a distance K: by taking some plane CO, the other planes EQ, GS, IV, &c. are counted indefinitely by always putting the same total between the planes as there are placed between the plane CO and the first plane AM. It is required to show that all the [odd] planes AM, CO, EQ, GS, &c. are taken from the proposed series.

Demonstration.

Since by hypothesis the planes AM, BN, CO, DP, EQ, FR, GS, etc. are in a continued ratio, then from the equality of the ratio, the planes AM, CO, EQ, IV, etc., are also in a continued ratio. It follows from proposition 126 that a plane figure can be found equal to the sum of the series of plane figures in continued proportion AM, CO, EQ, etc. If this plane figure is taken from the sum of all the plane figures, which by prop. 126 is equal to the sum of the given series MK, then the proposition is shown to be true.
[Prop. 126 asserts that (first figure AM)/(whole series of figures MK) = (first base AB)/(sum of odd bases);

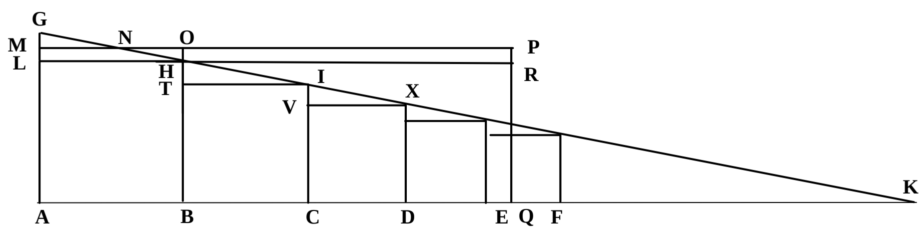
PROPOSITIO CXXXVIII.

Detur progressio quadratorum habens bases in directum, & terminum longitudinis punctis K: & ex K per H ducatur recta KH, quae per propositionem 131 huius contingat omnia serie quadrata & concurrat cum AL in G.

Dico triangulum AGK, toti trapeziorum GB, HC, ID, &c. sive utrique quadratorum AH, BI, &c. ac triangulorum LGH, THI, &c. progressioni, aequale esse.

[139]

Demonstratio.



Prop.138. Fig. 1.

Per corollarium propositionis 125. huius series basium AB, BC, CD terminatur in K; unde AK, [a] BK, CK, &c. sunt continuae proportionales, & triangula [b] AGK, BHK, CIK, &c. in continua sunt analogia. Quare toti seriei proportionis quam habet trapezium GB ad trapezium HC, sint statu continuatae, triangulum AGK aequale est. Atqui trapezia ID, XE, &c. continuant rationem trapezii GB, ad trapezium HC; (cum enim triangula AGK; BHK; CIK; &c. sint continue proportionalia, eorum quoque differentiae nempe dicta trapezia erunt continua) ergo triangulum AGK, trapezium seriei universae est aequale: Quod erat demonstrandum. *a 82 huius; b 124 huius; c 89 huius.*

L2.§3. PROPOSITION 138.

A progression of squares having bases along a line is given, and a terminus of length K: a line KH is drawn from K through H, which by Prop. 131 touches the whole series of squares and meets AL in G.

I say that the triangle AGK of the sum of all the trapeziums GB, HC, ID, etc. or of the sum of the progression of each of the squares AH, BI, etc. and the triangles LGH, THI, etc. are equal.

Demonstration.

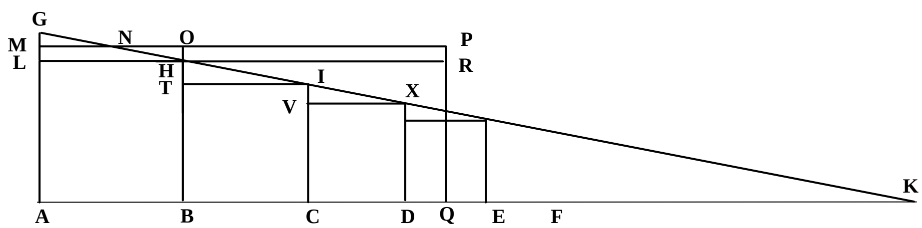
According to the corollary of Prop. 125, the series of the bases AB, BC, CD terminates in K; hence AK, [a] BK, CK, &c. are ratios in continued proportionals, and the triangles [b] AGK, BHK, CIK, etc. are in a continued ratio. Whereby the sum of the series of proportions that continues without end in the ratio of the trapezium GB to the trapezium HC, is equal to the triangle AGK. But the trapeziums ID, XE, etc. are continued in the ratio of the trapezium GB to the trapezium HC; (as the triangles AGK; BHK; CIK; etc. are continued proportionals, the differences of these too, surely the said trapeziums, are in continued proportion). Hence the triangle AGK is equal to the sum of the series of the trapeziums: Q. e. d. *a 82 huius; b 124 huius; c 89 huius.*

PROPOSITIO CXXXIX.

Esto quadratorum series HK; bases habens in directum, & terminum longitudinis punctis K: & ex K per H ducatur recta KH, tangat per propositionem 131 huius omnia seriei quadrata, ac concurrat cum AL in G; inventa deinde per propos. 80.huius linea AQ, quae aequalis sit seriei basium imparium AB, CD, EF, divisaque; bifarium LG in M, fiat rectangula QAMP.

Dico hoc triangulo AGK, aequale esse.

Demonstratio.



Prop.139. Fig. 1.

Producatur LH in R, & BH in O; linea MNO parallela est ex datis ad AQ, quae aequidistat LH; ergo MNO parallela est ad LH. & quia GM aequalis est ML, etiam GN aequalis erit NH; sunt autem & OH, ML, id est MG aequales, & anguli OHN, MGN, aequales : ergo aequantur triangula MGN, NOH: additoque communi LMNH, triangulum LGH, rectangulo LO aequale est : quare quadratum AH est ad triangulum LGH, ut idem quadratum, ad rectangulum LO, hoc est ut linea AL ad lineam LM. Praeterea triangula LGH, THI, VIX, similia sunt, & homologa latera LH, TI, VX; adeoque in duplicata ratione laterum LH, TI, VX, &c. Igitur cum quadrata AH, BI, &c. sint in eorundem laterum duplicata ratione; patet triangula dicta, quadratis proportionalia esse. unde permutando ut quadratum AH ad triangulum LGH, sic quadratum BI ad triangulum THI; & quadratum CX, ad triangulum VIX: atque ita in infinitum. Ergo ut quadratum AH ad triangulum LGH, hoc est ex ante demonstratis ut AL ad LM,

[140]

[a] sic omnia quadrata, ad omnia triangula. Atqui ut AL ad LM sic etiam est rectangulum AR ad rectangulum LR. Igitur ut omnia quadrata, ad omnia triangula, sic rectangulum AR ad rectangulum LP. Atqui rectangulum AR, aequale est toti [b] quadratorum seriei, (est enim AQ aequalis seriei basium imparium AB, CD, &c. & AL, AB) ergo cum series quadratorum ostensa sit aequalis esse rectangulo AR; etiam tota triangulorum series rectangulo LP [c] aequalis erit. quare totum rectangulum AP, utrique seriei quadratorum ac triangulorum, hoc est seriei trapeziorum GB, HC, ID, &c. aequabitur : sed trapeziorum series, [d] constituit triangulum AGK. ergo rectangulum AP triangulo AGK aequale est. Quod erat demonstrandum. *d 1 sexti; e 19 sexti; a 12 quinti; b 79 huius; c 14 quinti; d 136 huius .*

Corollorium.

From the discourse of the demonstration it can be gathered, that the rectangle MLRP is equal to the series of triangles GLH, HTI, &c. : truly the rectangle GLR, is twice the sum of the progression of the triangles.

L2.§3. *PROPOSITION 139.*

Let HK be a series of squares, having bases along a line, and the terminus of the points at a distance K. The line KH is drawn from K through H , touching all the terms in the series of squares, according to Prop. 131, and is concurrent with the line AL in G. Hence, by Prop. 80, a line AQ is found, which is equal to the sum of the series of the odd bases AB, CD, EF, etc, and bisecting LG in M, from which the rectangle QAMP is made.

I say that this rectangle QAMP is equal to the triangle AGK.

Demonstration.

LH is extended to R, and BH to O; from the given, the line MNO is parallel to the given line AQ, to which LH is equidistant ; hence MNO is parallel to LH; and since GM is equal to ML, also GN is equal to NH. Moreover OH and ML (or MG) are equal, and the angles OHN and MGN are equal : hence the triangles MGN and NOH are equal: and on adding to the common LMNH, the triangle LGH is equal to the rectangle LO. Whereby the square AH is to the triangle LGH as the same square is to the rectangle LO, or as the line AL is to the linea LM. In addition the triangles LGH, THI, and VIX are similar, and the homologous sides are LH, TI, and VX; and [the areas are] in the square ratio of the sides LH, TI, VX, &c. Therefore since the squares AH, BI, &c. are in the square ratio of the same sides, it is apparent that the said triangles are in proportion with the squares. From which on interchanging, as the square AH is to the triangle LGH, thus the square BI is to triangle THI; and the square CX is to triangle VIX: and thus indefinitely. Hence as the square AH is to triangle LGH, (or from the previous demonstration, as AL is to LM,)[a], so the sum of all the squares is to the sum of all the triangles. But as AL is to LM thus also rectangle AR is to rectangle LP. Therefore, as the sum of all the squares is to the sum of all the triangles, thus the rectangle AR is to the rectangle LP. But the rectangle AR is equal to the sum of all the squares in the series [b] of squares, (as AQ is equal to the sum of the series of the odd bases AB, CD, &c. & AL is equal to AB) hence as the series of squares is shown to be equal to the rectangle AR; so also the whole series of triangles is equal to the rectangle LP [c]. Whereby the whole rectangle AP is equal to the sum of the series of squares and triangles, or of the series of trapeziums GB, HC, ID, etc. : but the series of trapeziums [d] constitutes the triangulum AGK. Hence the rectangle AP is equal in area to the triangle AGK. Q. e. d. *d 1 sexti; e 19 sexti; a 12 quinti; b 79 huius; c 14 quinti; d 136 huius .*

Corollary.

From the discourse of the demonstration it can be gathered that the area of the rectangle MLRP is equal to the sum of the progression of triangles GLH, HTI, etc.: and the rectangle GLR is indeed equal to double the area of the same progression of triangles.

[The lines MNO, LHR, and AQ are parallel; GM = ML, GN = NH, and OH = ML = MG; the angles OHN and GNM are equal; hence the triangles MGN and ONH are equal, and ΔMGN + trap.LMNH = ΔLGH = ΔONH + trap.LMNH = rect.LO; sq.AH/ΔLGH = sq.AH/rect.LO = AL/LM (*).

In the same manner, since Δ's LGH, THI, and VIX are similar, with homologous sides LH, TI, and VX, then the areas are in the ratio LH^2 to TI^2, and TI^2 to VX^2, etc, (as the altitude of each is in the same proportion to the base); also, the squares AH, BI, etc are also in the same square ratio of the bases, and hence the triangles are in proportion to the squares. Hence: sq.AH/ΔLGH = sq.BI/ΔTHI = sq.CX/ΔVIX, etc, and sq.AH/ΔLGH = (sum of squares)/(sum of triangles), from proportionality, which can be extended indefinitely.

Also from (*), AL/LM = AL.AQ/LM.AQ = rect.AR/rect.LP = sq.AH/ΔLGH = (sum of squares)/(sum of triangles). But as rect. AR = sum of squares, (since AQ = AB + CD + EF + ..., and AL = AB); so also rect.LP = sum of triangles, from which the whole rectangle AP = sum of squares + sum of triangles = sum of trapeziums GB, HC, ID, etc, = ΔAGK.]

PROPOSITIO CXL.

Esto, ut prius, quadratorum series HK; inscripta triangulo AGK; sitque AQ aequalis seriei basium imparium AB, CD, &c.

Dico rectangulum sub AQ, & lineis GA, AB, tamquam una contentum, rectangulo GAK aequale esse.

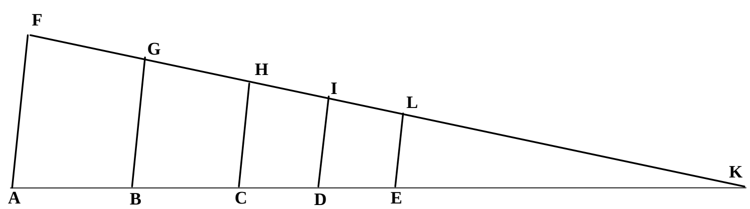
Demonstratio.

Dividatur enim LG bifariam in M: igitur linea composita ex GA, AB, bis continet lineas LM, LA. quare composita ex GA, AB dupla est linea AM; ergo rectangulum sub AQ & composita ex GA, AB, duplum est [c] rectanguli sub AQ & AM. Atqui rectangulum sub AQ & AM, aequatur [f] triangulo AGK: ergo rectangulum sub AQ & GA, AB tanquam una linea, duplum est trianguli AGK. Quare cum & rectangulum GAK, eiusdem trianguli GAK sit duplum, aequabuntur inter se, rectangulum sub AQ & composita ex GA,AB, & rectangulum GAK. Quod erat demonstrandum. *e 1 sexti; f 139 huius .*

Lemma.

Latus AK trianguli AFK, sit divisum per lineas lateri AF parallelas , in continue proportionales AK, BK, C, etc.

Dico trapezia FB, GC, HD, &c. esse similia.



Prop.140. Fig. 2.

[141]

Demonstratio.

Ob linearum AF, BG, CH aequidistantiam, angulum FAB, angulo GBC; & GFA, angulo HGB, & BGF, angulo CHG, & ABG, angulo BCH, aequalis est. Deinde quia AK, BK, CK, &c. sunt continuae proportionales, erit AB ad BC, [a] & BC ad CD, ut AK ad BK. Cum igitur etiam FA sit ad GB, ut AK ad BK, erit FA ad GB, ut AB ad BC, & permutando FA ad AB, ut GB ad BC: similiter cum AB sit ad BC [b] ut AK ad BK, hoc est ex hypothesi, ut BK ad CK, hoc est ut BG ad HC; erit permutando AB ad BG, ut BC ad CH : non aliter etiam ostendamus BG esse ad GF, ut CH ad HG, & GF ad FA, ut HG ad GB. quare cum trapeziorum FB, GC, & anguli omnes sint aequales, & latera circa aequales angulos proportionalia. Trapezia [c] FB, GC & omnia reliqua, erunt similia. Quod erat demonstrandum. *a 1 huius; b ibid; c Def. sexti.*

L2.§3. *PROPOSITION 140.*

As before, let HK be a series of squares inscribed in the triangle AGK; and let AQ be equal to the series of odd bases AB, CD, &c.

I say that the rectangle formed from AQ and the lines GA and AB taken as one line, is equal to the rectangle GAK.

Demonstration.

Since LG is bisected in M, it follows that the line formed from the sum of GA and AB is twice the sum of the lines LM, LA. Whereby the sum of the lines GA and AB is twice the line AM; hence the rectangle under AQ and the sum of GA and AB is twice the [c] rectangle under AQ and AM. But the rectangulum under AQ and AM is equal to [f] the triangle AGK : hence the rectangle under AQ and GA, AB as one line, is twice the triangle AGK. Since the rectangle GAK is twice the same triangle GAK, then the rectangle under AQ and the sum of GA and AB, is equal to the rectangle GAK. Q. e. d. *e 1 sexti; f 139 huius .*

[GA + AB = 2.(LM + LA) = 2.AM, since LM = MG;
hence rect.AQ.(GA + AB) = 2.rect.AQ.AM; but
AQ.AM = ΔAGK, by the previous proposition;
hence rect.AQ.(GA + AB) = 2.ΔAGK = rect.AGK, as required.]

Lemma.

The side AK of the triangle AFK is divided by lines parallel to the side AF in the continued proportionals AK, BK, C, etc.
I say that the trapeziums FB, GC, HD, &c. are similar.

Demonstration.

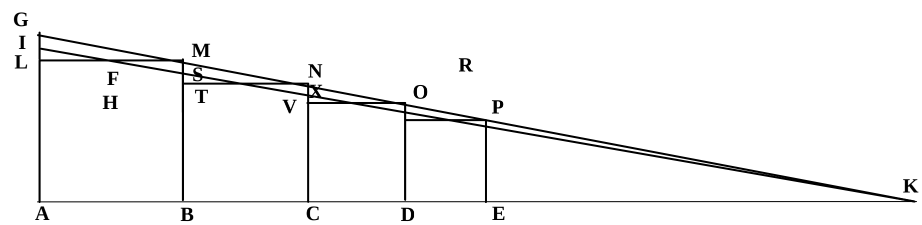
On account of the equidistant lines AF, BG, and CH, the angles FAB and GBC; GFA and HGB, BGF and CHG, ABG and BCH are equal. Hence since AK, BK, CK, etc. are continued proportionals, AB is to BC, [a] and BC to CD, as AK to BK. Therefore also as FA is to GB as AK is to BK, then FA is to GB as AB is to BC, and on interchanging, FA is to AB as GB is to BC. Similarly, since AB is to BC [b] as AK is to BK, that is by hypothesis, as BK is to CK, or as BG is to HC, then on interchanging, AB is to BG as BC is to CH : we can also show in the same way that BG is to GF as CH is to HG, and GF is to FA as HG is to GB. Whereby since the sides FB, GC of the trapeziums and the angles are all equal, and the sides around equal angles are in proportion. The trapeziums [c] FB, GC and all the others are similar. Q. e. d. *a 1 huius; b ibid; c Def. sexti.*

PROPOSITIO CXLI.

Data sit quadratorum series HK; habens bases in directum, & terminum longitudinis K; inscripta triangulo AGK; ac primi quadrati latere LM bisecto in F, per F ducatur recta FK, occurrens ipsi AG in I.

Dico triangulum AIK seriei quadratorum; triangulum vero IGK, seriei triangulorum LGM, TMN, &c. aequale esse.

Demonstratio.



Prop.141. Fig. 1.

Cum LM bisecta sit in F, & SM, IL, ex hypothesi sint parallelae, patet triangula IFL, SFM aequalia esse, ac proinde addito communi ALFSB, trapezium AISB quadrato AM aequale. Deinde cum per corollarium propositionis 125. huius, progressionis basium AB, BC terminus sit K, erunt [d] AK, BK, &c. continuae: quare cum etiam AI, BS, CX, &c. sint parallelae, erunt per lemma, trapezia IB, SC, XD, &c. similia inter se. unde & [e] trapezia IB, SC, XD, &c., sunt in duplicata ratione laterum homologorum AB, BC, CD, &c. atqui & quadrata AM, BN, CO, &c. sunt in dictorum laterum duplicata ratione, ergo trapezia sunt quadratis proportionalia, & ut primum [f] trapezium IB, ad primum quadratum, ita tota trapeziorum series, ad quadratorum seriem. Atqui primum trapezium, ostendimus primo quadrato aequale esse, ergo trapeziorum etiam & quadratorum series aequales sunt. Deinde series trapeziorum [g] GB, MC, ND, constituit triangulum AGK, ergo & series trapeziorum IB, SC, triangulo AIK aequalis erit; Quare triangulum AIK, quadratorum seriei aequabitur: Quod erat primum; ex quo etiam patet secundum.
Quod erat demonstrandum. *d 82 huius*; *e 20 sexti; f 12 sexti; g 138 huius* .

L2.§3. PROPOSITION 141.

A series of squares HK is given; having bases arranged on a line in a progression as usual, and the terminus of the series at a length K. The squares are inscribed in the triangle AGK; and with the side of the first square bisected in F, through F is drawn the line FK, crossing AG in I.

I say that the triangle AIK of the series of squares, and triangle IGK formed from the series of triangles LGM, TMN, &c. are equal.

Demonstration.

Since LM is bisected in F, and SM, IL, from hypothesis are parallel, it is apparent that the triangles IFL and SFM are equal, and hence by adding the common area ALFSB, the trapezium AISB is equal to the square AM. Following the corollary of proposition125, since the terminus of the progression of bases AB, BC, etc. is K, then [d] AK, BK, &c. are in continued proportion. Whereby since AI, BS, CX, &c. are also parallel, the trapeziums IB, SC, XD, &c are similar to each other according to the lemma. Hence the [e] trapezia IB, SC, XD, &c., are in the square ratio of the homologous sides AB, BC, CD, &c. But the squares AM, BN, CO, &c. are in the square ratio of the said sides, and hence the trapezia are proportional to the squares. As the first [f] trapezium IB is to the first square, thus the whole sum of the series of the trapezium series is to the sum of the series of squares. But the first trapezium has been shown to be equal to the first square, hence the sum of the series of trapeziums is also equal to the sum of the squares. Then the series of trapeziums [g] GB, MC, ND, constitute the triangle AGK, and hence the series trapeziums IB, SC, is equal to the triangle AIK. Whereby the triangle AIK is equal to the series of squares. Which demonstrates the first part, from which the second part follows. Q.e.d. *d 82 huius*; *e 20 sexti; f 12 sexti; g 138 huius* .

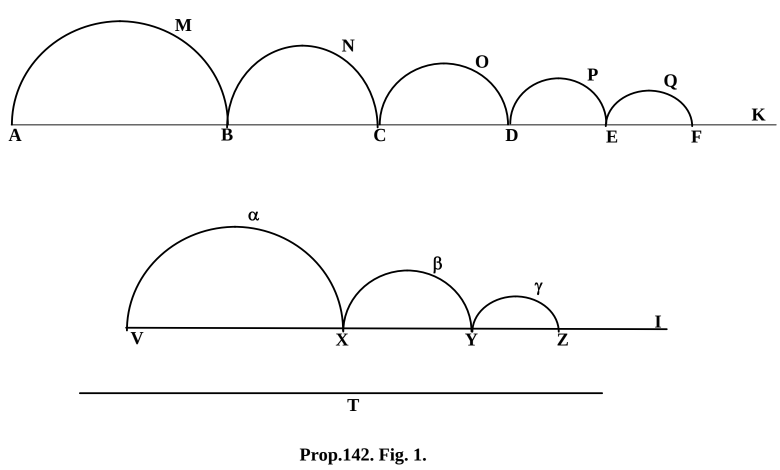
PROPOSITIO CXLII.

Data sit planorum similium series habens bases homologas in directum, & terminum longitudinis punctum K; sitque seriei rationis AB primae, ad tertiam CD, aequalis linea VI: linea vero T seriei rationis AB primae, ad EF quintam sit aequalis;

[142]

Dico seriem totam planorum AM, BN, CO, DP, &c. esse ad seriem planorum imparium AM, CO, EQ, &c. ut linea VI est ad lineam T.

Demonstratio.



Prop.142. Fig. 1.

Series basium imparium AB, CD, EF, etc. constituatur separatim, ut VX ipsi AB, & XY ipsi CD, & YZ ipsi EF sit aequalis : itemque plana super his facta planis AM, CO, EQ, aequalia sint, & similia. patet igitur series rationis AB ad EF, & rationis VX ad YZ, eadem esse. quare cum T aequalis sit seriei AB, EF, etiam seriei VX, YZ, aequalis erit. Deinde series [a] planorum AM, BN, CO, est ad planum AM, ut linea VI ad AB : & series planorum Vα, Xβ, &c. est ad planum VX, id est ad planum AM, ut T ad VX, id est AB: Atqui ut VI ad AB, sic rectangulum sub VI & AB, ad quadratum AB: & ut T ad AB, sic rectangulum sub T & AB, ad quadratum AB: ergo series planorum AM, BN, &c. est ad planum AM ut rectangulum sub VI & AB, ad quadratum AB: & series planorum Vα, Xβ, &c. est ad planum Vα, hoc est AM, ut rectangulum TAB ad quadratum AM. Igitur permutando series AM, BN, est ad rectangulum sub VI, AB; ut planum AM, ad quadratum AB: itemque permutando series Vα, Xβ est ad rectangulum TAB, ut idem planum AM ad idem quadratum AB. ergo series AM, BN est ad rectangulum sub VIAB ut series Vα Xβ ad rectangulum TAB: & permutando series AM, BN, est ad seriem Vα, Xβ, id est ex constructione ad seriem AM, CO, EQ, ut rectangulum sub VI AB ad rectangulum TAB, hoc est ut linea VI ad lineam T: Quod erat demonstrandum. *a 126 huius* .

L2.§3. *PROPOSITION 142.*

A series of similar plane figures is given, having homologous bases arranged on a line [in a progression as usual], and the terminus point of the series at a distance K. The line VI is equal to sum of the series of the first term AB to the third term CD, etc., and the line T is equal to the sum of the series of the first term AB to the fifth term EF, etc.

I say that the sum of the whole series of plane figures AM, BN, CO, DP, &c. is to the sum of the series of odd plane figures AM, CO, EQ, &c., as the line VI is to the line T.

Demonstration.

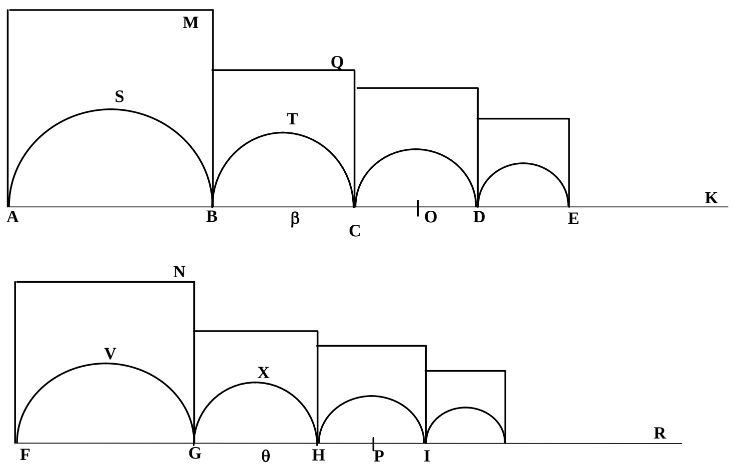
The series of odd bases can be set up separately, so that VX is equal to AB, XY is equal to CD, and YZ is equal to EF. In the same manner, the plane figures constructed on these lines are equal to AM, CO, EQ, and similar to each other. It is therefore apparent that the series formed from the ratios AB to EF and VX to YZ are the same. Whereby, since T is equal to the sum of the series AB to EF, it is also equal to the sum of the series VX toYZ. So the series [a] of plane figures AM, BN, CO, is to the plane figure AM, as the line VI is to AB : and the series of plane figures Vα, Xβ, &c. is to the plane figure Vα, (or to the plane figure AM), as T to VX, (or AB) [original text has Vα and VX interchanged]. But as VI is to AB, thus the rectangle VI.AB is to the square AB: and as T is to AB, thus the rectangle T.AB is to the square AB: hence the series of plane figures AM, BN, &c. is to the plane figure AM as the rectangle VI.AB is to the square AB: and the series of plane figures Vα, Xβ, &c. is to the plane figure Vα, or AM, as the rectangle T.AB is to the square AB. Therefore on interchanging, the series AM, BN, is to the rectangleVI.AB, as the plane figure AM is to the square AB. Likewise, on interchanging, the series Vα, Xβ is to the rectangle T.AB as the same plane figure AM to the same square AB. Hence, the series AM, BN is to the rectangle VI.AB as the series Vα, Xβ is to the rectangle T.AB: and on interchanging, the series AM, BN is to the series Vα, Xβ, (or from the construction to the series AM, CO, EQ), as the rectangle VI.AB is to the rectangle T.AB, or as the line VI is to the line T: Q.e.d. *a 126 huius* .

[From previously, S(all areas on AK)/AM = $1/(1 - r^2)$ = S(odd terms of AK)/AB = VI/AB; in the same manner, S(areas on line VI)/(plane figure Vα) = $1/(1 - r^4)$ = S(odd terms of VI)/VX = T/AB.
Following Gregorius: VI/AB = VI.AB/AB^2 ; and T/AB = T.AB/AB^2 .
Hence S(all areas on AK)/AM = VI/AB = rect.VI.AB/sq. AB
and S(areas on line VI)/(plane figure Vα) = T/AB = rect.T.AB/sq.AB.
Consequently, S(all areas on AK)/rect.VI.AB = AM/sq.AB ;
and likewise, S(areas on line VI)/rect.T.AB = (plane figure Vα or AM)/sq.AB.
Hence, S(all areas on AK)/rect.VI.AB = S(areas on line VI)/rect.T.AB, or
S(all areas on AK)/S(areas on line VI) = rect.VI.AB /rect.T.AB = VI/T as required.]

PROPOSITIO CXLIII.

Datae sint quadratorum series binae, quae habeant bases in directum, & longitudinum terminos puncta K & R. sit autem AK divisa in O, in ratione AB ad BC; & FR divisa sit in P; secundum rationem FG ad GH.

Dico seriem quadratorum NK, ad seriem quadratorum NR, rationem habere compositam ex rationibus AB ad FG, & AO ad FP.



Prop.143. Fig. 1.

[143]

Demonstratio.

Series quadratorum MK, [a] aequatur rectangulo OAB, & series quadratorum NR, rectangulo PFG aequalis est. Atqui ratio rectangulorum OAB, PFG ex lateribus AB, FG, & AO, FP, rationibus componitur, ergo etiam serierum MK, NR, ex iisdem rationibus proportio componitur. Quod erat demonstratum. Sed hoc Theorema universale reddamus. *a 139 huius .*

L2.§3. *PROPOSITION 143.*

Two series of squares are given, which have bases on a line, and end-points of the series are of lengths K & R from the start. Moreover, AK is divided by O in the ratio AB to BC; and FR is divided by P, in the second ratio FG to GH.

I say that the sums of the series of squares NK to the squares NR has the ratio of the sum of the ratios AB to FG and AO to FP.

Demonstration.

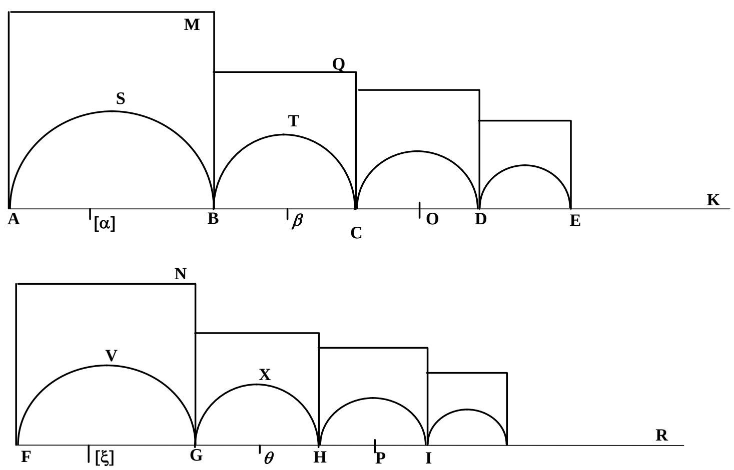
The series of squares MK [a] is equal to the rectangle OAB, and the series of squares NR is equal to the rectangle PFG . But the ratio of the rectangles OAB to PFG is composed of the ratios from the sides AB, FG, and AO, FP; hence also the ratio of the series MK, NR is composed from the same ratios. Q. e.d. But we can make this theorem general. *a 139 huius .*

PROPOSITIO CXLIV.

Eadem manente figura, datae sint series binae planorum similium, quarum longitudines sint AK, FR : sit autem AK divisa in O, in ratione AB ad BC, & FR in P, in ratione FG ad GH.

Dico seriem planorum MK, ad seriem planorum similium NR, habere rationem compositam ex rationibus AB ad FG, & AO ad FP.

Demonstratio.



Prop.144. Fig. 1.

Super iidem basibus fiant binae quadratorum series, & lineis CD, HI, aequales fiant αB, ξG: deinde ut Aα ad AB, & Fξ ad FG fiat quadratum AB, ad Aβ quadratum, & quadratum FG ad quadratum Fθ. itaque quadratum super Aβ aequalitur [b] seriei quadratorum MK: planum vero super Aβ simile planis AS, BT, FV, GX, &c. aequabitur seriei planorum SK : similiter ab altera parte quadratum super Fθ seriei quadratorum NR, & planum simile super eadem Fθ, planorum seriei VR aequalia erunt. Itaque series planorum SK est ad seriem planorum VR, ut planum super Aβ, ad planum Fθ: Item series quadratorum MK est ad seriem quadratorum NR, ut quadratum super Aβ, ad quadratum Fθ; (cum enim tam quadrata, quam plana ex constructione sint similia, utraque sunt in duplicata rationis Aβ ad Fθ:) ergo series planorum SK est ad seriem planorum VR, ut series quadratorum MK ad seriem quadratorum NR. Atqui per praecedentem, series quadratorum MK ad seriem NR, rationem habet compositam, ex rationibus AB ad FG, & AO ad FP, ergo & planorum series SK ad seriem planorum VR, rationem habet ex iisdem rationibus compositam: quod erat demonstrandum. *b 130 huius .*

L2.§3. *PROPOSITION 144.*

From the same figure as above, two series of similar plane figures are given, of which the lengths [or total sums] are AK, FR : while AK is divided by O in the ratio AB to BC, and FR in P in the ratio FG to GH.

I say that the series of plane figures MK to the series of similar plane figures NR has a ratio composed from the ratios AB to FG and AO to FP.

Demonstration.

Two series of squares are made upon the same bases, and with the lines CD, HI made equal to αB, ξG: Then, as Aα is to AB, and Fξ is to FG, the square AB is made to the square Aβ, and the square FG to the square Fθ. Hence the square upon Aβ is equal [b] to the series of squares MK: truly the square upon Aβ with the similar plane figures AS, BT, FV, GX, etc. is equal to the series of plane figures SK : similarly from the other part, the square upon Fθ of the series of squares NR, and the similar plane figure upon the same Fθ, are equal to the series of plane figures VR. Thus the series of plane figures SK is to the series of plane figuresVR, as the plane figure upon Aβ is to the plane figure Fθ: Likewise, the series of squares MK is to the series of squares NR, as the square upon Aβ to the square upon Fθ; (indeed as for for the squares, so the plane figures are similar from construction , and both are in the square ratio of Aβ to Fθ:); hence the series of plane figures SK is to the series of plane figures VR, as the series of squares MK is to the series of squares NR. But according to the preceding theorem, the series of squares MK to the series NR has the ratio composed from the ratios AB to FG and AO to FP. Hence the series of plane figures SK to the series of plane figuresVR has a ratio composed from the same ratios: q.e.d. *b 130 huius .*

PROPOSITIO CXLV.

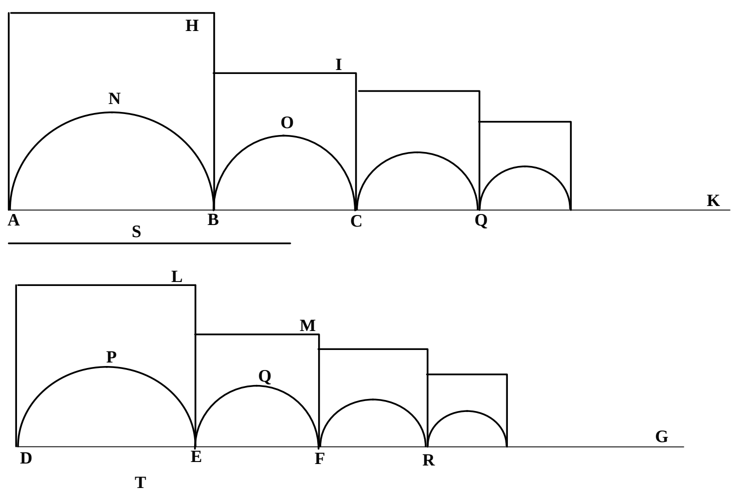
Datae sint quadratorum series HK, LG, in quibusvis rationibus, ab aequalibus quadratis incipientes: sitque linea S seriei rationis primae AB ad tertiam CQ, itemque linea T, seriei rationis DE primae ad tertiam FR aequalis.

Dico series quadratorum eam proportionem habere quam lineae, S, T.

[144]

Demonstratio.

Series HK [a] aequatur rectangulo SAB, item series LG aequatur rectangulo TDE, hoc est, (quoniam quadrata AH, DL, ideoque & lineae AB, DE, ex hypothesi aequantur) rectangulo TAB. eadem igitur est serierum, & SAB, TAB rectangulorum ratio. Atqui rectangulorum SAB, TAB, eadem est ratio [b] quae rectarum S, T, ergo & serierum eadem quae S, T, linearum est proportio. Quod erat demonstrandum. *a 128 huius; b 1 sexti.*



Prop.145. Fig. 1.

L2.§3. *PROPOSITION 145.*

The series of squares HK and LG are each given in any ratio, starting from equal squares : the line S is set equal to the ratio AB to CQ of the first series, and likewise the line T is equal to the ratio of the first DE to the third FR of the second series.

I say that the series of of squares have the same proportion as the lines S and T.

Demonstration.

The series HK [a] is equal to the rectangule SAB, and likewise the series LG is equal to the rectangule TDE, or (since the squares AH and DL, and thus the lines AB and DE are equal by hypothesis) to the rectangule TAB. Therefore the ratio of the series is the same as the ratio of the rectangles SAB and TAB. But the ratio of the rectangles SAB and TAB is the same as the ratio of the lines [b] S, T. Hence the proportion of the series is the same as of the lines S and T. Q.e.d. *a 128 huius; b 1 sexti.*

[For S(squares AH)/S(squares LG) = AH/sum(odd terms DE, FR, ...) × sum(odd terms AB, CQ, ...) /DL = AB.S/DL.T = S/T.]

PROPOSITIO CXLVI.

Eadem manente figura; dentur planorum similium binae series quarumvis proportionum NK, PG, quae ab aequalibus incipiant planis, sintque lineae S, T, aequales seriebus AB, CQ, &c. AB, FR, &c.

Dico eandem esse serierum & rectarum S, T, proportionem.

Demonstratio.

Ad huius Theorematis demonstrationem, eandem lector constructionem ac ratiocinationem si adhibeat, qua propositione praecedenti fuimus vsi, non aliter proportionis praesentis veritatem ex praecedenti deducet, quam propositionis 144, ex 143 huius deduximus.

L2.§3. PROPOSITION 146.

With the same figure kept; two series of any similar plane figures whatever are given in the proportion NK to PG, and for which with equal initial figures, the lines S and T are equal to the sums of the series AB, CQ, etc., and AB, FR, etc.

I say that the proportion of the series and the lines S and T is the same.

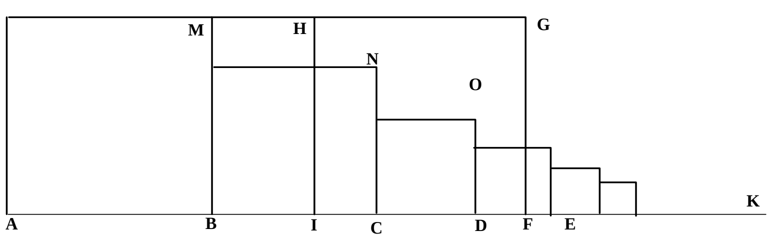
Demonstration.

The reader can use the same construction and reasoning for the demonstration of this theorem as for the previous proposition, and in the same manner the truth of the present proposition can be deduced from the previous theorem, as we deduced propositon 144 from proposition 143.

PROPOSITIO CXLVII.

Esto rectangulum altera parte longius AG, a quo AM quadratum ablatum sit. Petitur exhiberi series quadratorum, quae aequalis sit rectangulo AG, & incipiat a quadrato AM.

Constructio & demonstratio.



Prop.147. Fig. 1.

Ut AF ad BF, sic AB fiat ad BI : & ex I ducta IH, parallela ad BM rectangulo IBMH, aequale fac quadratum BN, basi BC in directum posita cum AB : cum progressionis quadratorum AM, BN, [c] continuatae inveniatur K terminus longitudinins. Dico seriem quadratorum MK, problemati satisfacere.

[145]

Nam cum ex constructione AB sit ad BI, ut AF ad BF; erit AF aequalis [a] seriei rationis AB ad BI. Deinde ex constructione rectangulum BH aequatur quadrato BN, ergo quadratum AM ad rectangulum BH, & quadratum BN, eadem habet rationem; unde cum quadratum AM, sit ad quadratum BN ut AB ad CD, erit quoque quadratum AM ad rectangulum BH, ut AB ad CD. atqui ut quadratum AM ad rectangulum BH, sic AB ad BI : ergo AB est ad CD, ut AB ad BI : aequantur igitur CD, BI, ergo AF etiam aequalis est seriei rationis AB ad CD : quare rectangulum FAB, id est rectangulum AG [b] aequale est seriei quadratorum AM, BN, CO, &c. hoc est seriei MK. factum igitur est quod postulabatur.

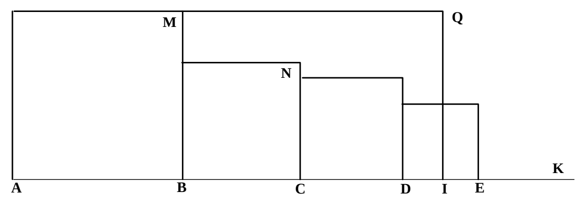
c 125 huius; a 79 huius; b 128 huius.

Demonstratio alia.

Tota series quadratorum MK est per 82 huius ad reliquam seriam seriem NK, ut quadratum AM ad quadratum BN, hoc est ex const. ad rectangulum BH. Atqui ex constructionis, rectangulum AG est ad rectangulum BG, ut rectangulum AM ad rectangulum BH, ergo series MK est ad seriem NK, ut AG ad BG. Igitur dividendo quadratum AM, est ad reliquam seriem NK, ut quadratum idem AM ad reliquum rectangulum BG: ergo series NK, & rectangulum BG aequantur. Quare communi addito quadrato AM, tota series MK, & rectangulum AG sunt aequalia.

Corollarium.

Esto linea AI secta in B. Petitur addi IK, ut AI sit ad IK, ut AK ad BK.



Prop.147. Fig. 2.

Constructio & demonstratio.

Super AI fiat in altitudine AB rectangulum AG, cui aequalis [c] inveniatur series quadratorum basibus in directum positis MNK; incipiens a quadrato AB, sive AM, & terminum habens longitudinis K: Dico factum quod petebatur. Cum enim rectangulum AG ex constructione sit aequale seriei MK, AI erit ad IK, ut AB ad BC, uti ex 129 huius facile demonstrari potest: & quia terminus longitudinis seriei quadratorum est K, progressionis etiam basium terminus [d] erit K; ergo AK est ad [e] BK, ut AB ad BC, hoc est per demonstrata ut AI ad IK. Factum igitur est quod petebatur.

c 147 huius; d 125 huius; e 82 huius.

L2.§3. *PROPOSITION 147.*

Let there be a rectangle with the longer side AG, from which the square AM is taken away. It is required to demonstrate a series of squares which is equal to the rectangle AG, and which starts from from the square AM.

Construction and Demonstration.

As AF is to BF, thus AB is made to BI: and from I the line IH is drawn parallel to BM; make the square BN equal to the rectangle IBMH, with base BC placed on the line with AB : since the terminus of the continued progression of squares AM, BN, [c] can be found at a distance K. I say that the series of squares MK solves the problem.

For since from the construction AB is to BI, as AF is to BF; AF is equal to the sum of the series of ratios [a] AB to BI. Hence from the construction, the rectangle BH is equal to the square BN; hence the rectangle BH and the square BN have the same ratio to the square AM. Hence since the square AM is to the square BN as AB to CD, also the square AM is to the rectangle BH, as AB to CD. But as the square AM to the rectangle BH, thus AB to BI : hence AB is to CD, as AB to BI : therefore CD is equal to BI. Hence AF is equal to the series of ratios AB to CD also: whereby the rectangle FAB, or the rectangulum AG [b] is equal to the series of squares AM, BN, CO, &c. or to the series MK. Thus what was postulated has been done.

c 125 huius; a 79 huius; b 128 huius.

[AF/BF is the ratio of the lengths of sides of consecutive squares, of which F is the limit point. The ratio of consecutive squares is also constructed to be sq.AM/rect.BH = AB/BI = AB/CD ; from which BI = CD. Thus, BI is the same as the term CD in the series of odd lengths AB, CD, FE, etc. that sum to AF. Hence, starting from the square AM, consecutive rectangles of the form BH can be taken from the rect. AG, each of which is equal to a corresponding square such as BN, and in the limit, the sum of the squares is equal to the area of the rectangle AG.]

Another demonstration.

The sum of the series of square MK is to the sum of the rest of the series of squares NK by Prop. 82 as the square AM is to the square BN, (or by construction) as the rectangle BH. But from the construction, rect.AG is to rect.BG as rect.AM is to rect.BH, and hence the series MK is to the series NK as rect. AG is to rect.BG. Hence on subtraction, the square AM is to the remainder of the series NK, as likewise the square AM is to the remaining rect. BG: hence the series NK is equal to the rectangle BG. Whereby by adding the common square AM, the sum of the series MK and the rect. AG are equal.

Corollarium.

Let the line I be cut in B. The line IK is sought to be added in order that AI is to IK, as AK is to BK.

Construction & demonstration.

Upon AI the rectangle AG is conctructed with altitude AB, on which an equal series of squares with bases on the line MNK are placed, beginning with the square AB, or AM, and having the terminus point K at a distance K. I say that what was required has been effected. For since the rectangle AG by construction is equal to the series MK, AI is to IK, as AB to BC, as can be easily found from from Prop. 129: and because the terminus of the series of squares is at a length K, also the terminus of the progression of bases is [d] K; hence AK is to [e] BK, as AB is to BC, or as has been shown, as AI to IK. Thus what was sought has been accomplished. *c 147 huius; d 125 huius; e 82 huius.*

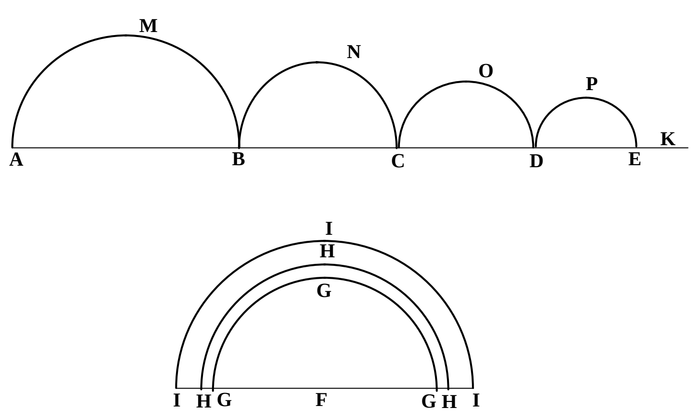
PROPOSITIO CXLVIII.

Esto figura plana quaecunque FI, a qua similis auferatur FG, petitur exhiberi series planorum similium, quae incipiat a plano ablato FG, & dato plano FI sit aequalis.

[146]

Constructio & demonstratio.

Ut planum FI ad segmentum GI, sic planum FG, fiat ad segmentum aliquod GH, deinde plano FG fac simile & aequale AM, & segmento GH aequale sit planum BN, simile autem planis AM, FG, FI.



Prop.148. Fig. 1.

Deindeque progressionis planorum AM, BN, continuarae [a] inveniatur terminus longitudinis K. Dico seriem planorum similium MK postulato satisfacere. est enim tota series planorum MK, ad reliquam seriem NK, ut [b] planum AM ad planum BN, hoc est ex constructione, ut planum FG ad segmentum GH; sed rursum ex constructione planum FI est ad segmentum GI, ut planum FG ad segmentum GH, ergo series MK, est ad reliquam NK, ut planum FI ad segmentum GI. Quare per conversionem rationis series MK, est ad planum AM, ut planum FI ad planum FG: sed ex constructione plana AM, FG aequalia sunt. Quod erat faciendum . *a 125 huius; b 82 huius.*

L2.§3. PROPOSITION 148.

Let FI be some plane figure, from which a similar figure FG is taken, it is required to exhibit a series of similar plane figures, which begin with the plane figure FG, and the sum of which is equal to the given plane figure FI.

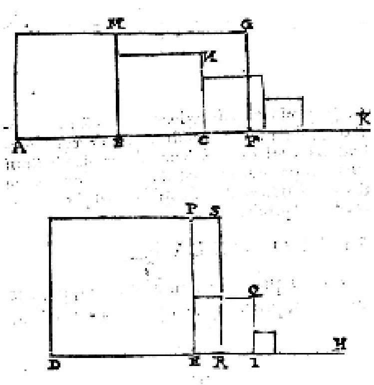
Construction & demonstration.

As the plane figure FI is to the annulus GI, thus the plane figure FG is made to some annulus GH [the text refers to GH as a segment, but this word now given another meaning]; then make the plane FG similar and equal to AM, in which case the plane figure BN in turn becomes equal to the annulus GH, and moreover similar to the figures AM, FG, and FI. Hence the terminus K of the continued progression of figures AM, BN [a] can be found. I say that the series of similar figures MK satisfy the postulate. For indeed the sum of the series of the figures MK is in the same ratio to the rest of the series NK, as [b] the figure AM is to the figure BN, or from the construction, as the figure FG is to the annulus GH; but again from the construction the figure FI is to the segment GI, as figure FG is to the segment GH, hence the series MK, is to the remainder of the series NK, as the figure FI is to the annulus GI. Whereby from the conversion of the ratio, the series MK is to the figure AM, as the figure FI is to the figure FG: but from the construction the figures AM and FG are equal. Which was to be established. *a 125 huius; b 82 huius.*

PROPOSITIO CXLIX.

Data sit progressio quadratorum MK, & quadratum aliud DP, quod minus esse debet serie MK. Petitur exhiberi alia series quadratorum, incipiens a quadrato DP, aequalis seriei MK.

Prop. 149, Fig. 1



Constructio & demonstratio.

Fiat rectangulum AG in altitudine AB, aequale seriei datae MK[c]. Deinde, (quoniam quadratum DP ponitur minus serie MK, id est ex constructione rectangulo AG,) auge quadratum DP, rectangulo ES, ut rectangulum totum DS, aequale sit rectangulo AG : tum rectangulo DS [d], inveniatur aequalis series quadratorum PQH, incipiens a quadrato DP. Dico seriem PQH solvere Problem. Nam ex constr. Series quadratorum PQH incipit a quadrato dato DP, & aequalis est rectangulo DS, hoc est ex constructione rectangulo AG, hoc est rursum ex constructione seriei datae MK. Factum igitur est quod petebatur.
c 129 huius; d 147 huius.

L2.§3. PROPOSITION 149.

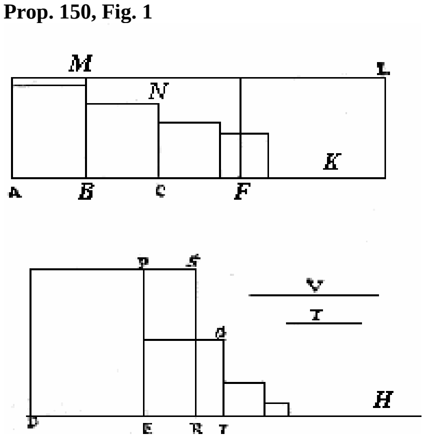
For a given progression of squares MK, & some other square DP, which should be less than the series MK. It is desired to show another series of squares, starting from the square DP, equal to the series MK.

Construction & demonstratin.

Let the rectangle AG be made equal in height to AB, equal [in area] to the given series MK[c]. Then, (since the square DP is put less than the series MK, that is from the construction to the rectangle AG,) increase the size of the square DP by the rectangle ES, in order that the total DS is equal to the rectangle AG : for then the rectangle DS [d] is found equal to the series of squares PQH, beginning from DP. I say that the series PQH solves the problem. For from the construction, the series of squares PQH that begins from the given square DP, and is equal to the rectangle DS, that is from the construction equal to the rectangle AG, again this is from construction equal to the given series MK. Therefore what was sought has been accomplished. *c 129 huius*; *d 147 huius*.

[p. 147]

Prop. 150, Fig. 1



PROPOSITIO CL.

Data sit iterum quadratorum series MK, & aliud quadratum DP, item ratio quaevis sive maioris sive minoris inaequalitatis V ad T, serie MK. Petitur exhiberi series quadratorum incipiens a quadrato DP, & habens ad seriem MK, rationem datam V ad T.

Constructio & demonstratio.

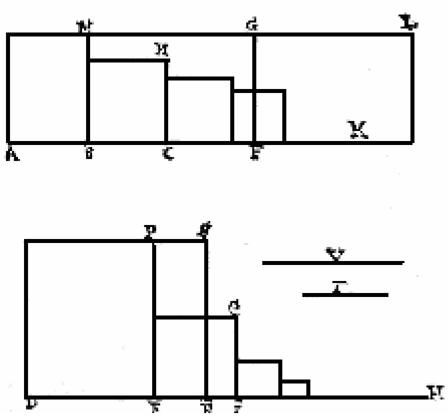
Fiat [a] rectangulum AG aequale seriei MK, & in eadem altitudine rectangulum AL, quod ad rectangulum AG, datam habeat rationem; si iam quadratum DP maius sit aut aequale rectangulo AL, impossibile est problem. Minus ergo sit oportet quadratum DP rectangulo AL. Itaque augeatur rectangulo ES, ita ut totum DS [b] rectangulo AL aequale sit. Tum rectangulo DS inveniatur aequalis series, quadratorum PQH incipiens a quadrato DP. Dico hanc solvere problema.

Ex constructione enim series PQH incipit a quadrato DP, & aequalis est rectangulo DS, id est ex constructione rectangulo AL; quare cum rectangulum AL ex constructione sit ad rectangulum AG, ut V ad T, etiam series PQH erit ad rectangulum AG, id est rursum ex constructione ad seriem datam MK, ut V ad T. Fecimus ergo quod petebatur. Nunc vero utramque propositionem praecedentem universalem faciamus. *a 129 huius*; *b 147 huius*.

L2.§3. PROPOSITION 150.

The series of squares MK is given again, and another square DP likewise with some other unequal ratio V to T, which is either greater or less than that for the original series. It is desired to find the series of squares beginning from the square DP, & and having the given ratio to the series V to T.

Prop. 150, Fig. 1



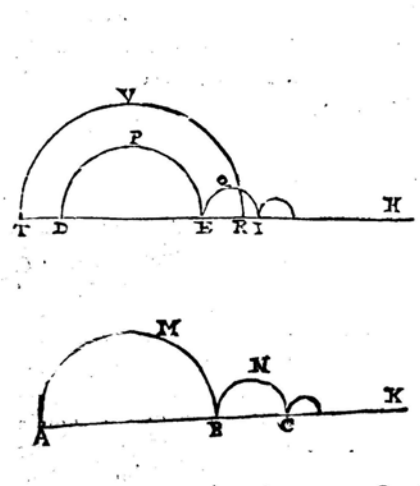
Construction & demonstration.

The rectangle AG is made equal to [the sum of] the series MK[a], & the rectangle AL is made with the same height, which has the given ratio to the rectangle AG ; now if the square DP is greater or equal to the rectangle AL, then the problem cannot be solved. Hence, it is necessary that the square DP is less than the rectangle AL. Hence it is augmented by the rectangle ES, thus in order that the rect. AL is equal to the rect. DS [b]. Then the rect. DS is found to be equal to the series of the squares PQH, beginning from the square PQ. I say that the problem is solved.

Indeed from the construction, the series PQH begins with the square DP, & is equal to the rectangle DS, that is by construction equal to the rectangle AL; whereby since the rectangle AL, from the construction, is to the rectangle AG, as V to T, and then also the series PQH is to the rectangle AG, (that is again by construction to the given series MK), as V to T. We therefore have constructed that which was desired. Now truly we can make general each of the preceding propositons [by considering other plane shapes].
a 129 huius; *b 147 huius*.

PROPOSITIO CLI.

Data sit planorum similium progressio ut prius disposita MK, & aliud planum DP simile planis seriei MK : Petitur exhiberi similium planorum series, incipiens a dato plano DP, aequalis vero seriei datae MK.



Prop. 151, Figure 1.

Constructio & demonstratio.

Planis [a] seriei MK aequale ac simile fiat planum TVR, si iam planum datum DP, aequale aut maius sit plano TVR, fieri problema non poterit : minus ergo sit necesse est. Itaque inveniatur series [b] planorum similium PQH, quae aequalis sit plano TVR, & incipiat a plano dato DP. Dico hanc solvere problema.

Nam ex constructione, series PQH incipit a dato plano DP, & aequalis est plano TVR, id est ex constructione seriei datae MK; Factum igitur est quod postulabatur. *a 130 huius*; *b 148 huius*.

L2.§3. *PROPOSITION 151.*

A progression of similar plane shapes MK is given in order as above, and another plane figure DP similar to the series of plane figures MK is given: It is desired to exhibit a series of plane figures, starting from the given plane figure DP, that is truly equal to the given series MK. [i.e. which has the same sum.]

Construction & demonstration.

The plane figure [a] TVR is made equal [to the sum of]and is similar to the series of plane figures MK, now if the given plane figure DP is equal to or greater than the plane figure TVR, then the problem cannot be solved : therefore it must be less. Thus the series of similar plane figures [b] PQH can be found, which is equal to the plane figure TVR, and which starts from the given plane figure DP. I say that this has solved the problem.

Now by the construction, the series PQH starts from the given plane figure DP, and [the sum of the series] is equal to the plane figure TVR, that follows from the construction of the given series MK ; therefore what was demanded has been done. *a 130 huius*; *b 148 huius*.

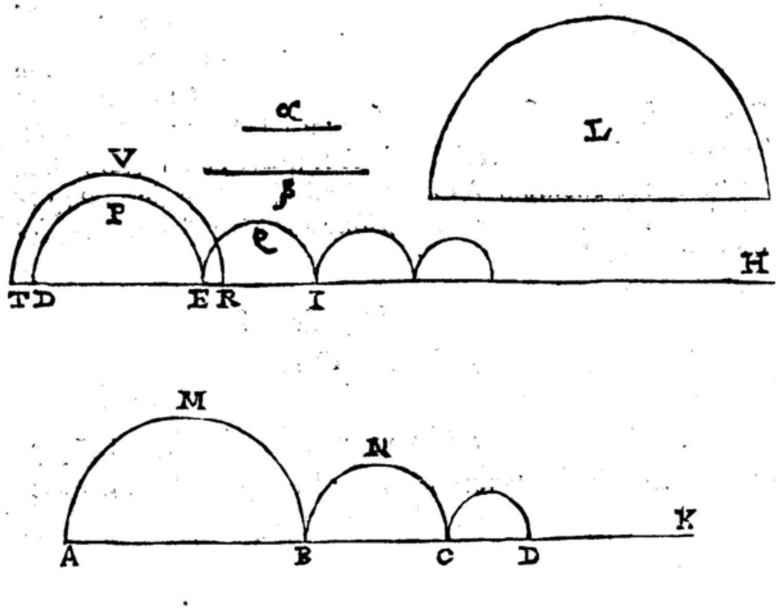
PROPOSITIO CLII.

Data sit iterum planorum similium series MK, & aliud planum DP, simile datae seriei planis itemque ratio quaevis, sive maioris, sive minoris inaequalitatis α ad β. Petitur exhiberi series planorum similium, quae incipiat a plano DP, & ad seriem MK, datam habeat ratione.

Constructio & demonstratio.

Fiat planis seriei MK [c] aequale planum L, deinde ut α ad β, sic fiat planum simile TVR ad planum L, si iam planum datum DP, aequale vel maius est problema non poterit. Minus ergo planun DP sit opererteт, plano TVR : Itaque inveniatur [d] series planorum similium PQH, aequalis vero plano TVR. Dico hanc problema solvere, nam ex constructione series PQH, incipiat a plano dato DP, & aequalis est plano TVR, quare cum planum TVR ex constructione datam habeat rationem ad planum L, etiam series PQH ad planum L, hoc est rursum ex constructione ad seriem datam MK, habebit rationem datam : Factum igitur est quod petebatur.

c 130 huius; *d 148 huius*.



Prop. 151, Figure 1.

L2.§3. *PROPOSITION 152.*

The series of similar plane figures is given again, and another plane figure DP, similar to the given series of plane figures, and likewise for some either greater or less unequal ratio α to β. It is desired to show the series of similar plane figures, which starts from the plane figure DP, and which has the given ratio to the series MK.

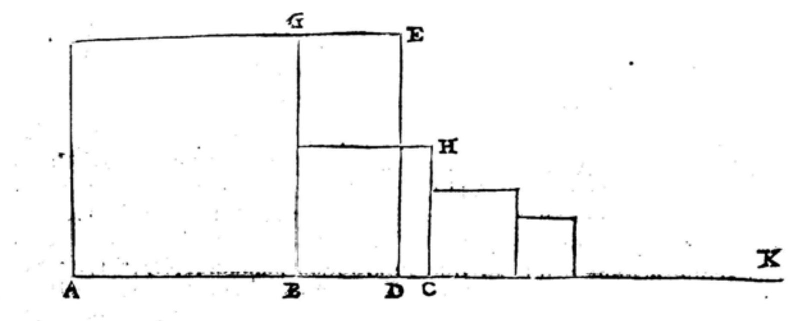
Construction & demonstration.

The plane figure L is constructed equal to the [sum of the] series of plane figures MK [c], thus the figure TVR is made in the ratio α to β to the plane figure L; if the plane figure DP is now given equal or greater than this, then the problem cannot be solved. Hence it is necessary that the plane figure DP is smaller than the figure TVR : Thus a series [d] of similar plane figures PQH can be found, truly equal [in area] to the plane figure TVR. I say that this problem has been solved; for from the construction , the series PQH, beginning from the given plane figure DP, is equal to the plane figure TVR, and whereby, since the plane figure TVR, from the given construction, is in the given ratio to the plane figure L, then also the series PQH has the same ratio to the plane figure L, that is again from the construction, to the given series MK. Therefore, what was sought has been accomplished. *c 130 huius*; *d 148 huius*.

PROPOSITIO CLIII.

Esto progressio quadratorum GHK, basibus in directum positis, & terminum longitudinis habens punctum K.

Dico quadratum super tota AK factum, ad seriem datam, proportionem habere compositam, ex ratione KA ad BA, & CA ad BA.



Prop. 153, Figure 1.

Demonstratio.

Fiat enim AD ad DK, ut AB ad BC. Igitur rectangulum DAB, sive AE, [a] aequatur seriei GK. Ergo quadratum AF, eandem ad seriem GK, & ad rectangulum AE habet rationem quoniam autem AD est ad DK, ut AB ad BC, erit invertendo ac componendo KA ad DA, ut CA ad BA. Ergo cum quadratum AF ad rectangulum AE, rationem habeat [b] compositam ex rationibus KA ad DE, & KA ad DA, habebit quoque quadratum AF, ad idem rectangulum, compositam ex rationibus KA ad DE, & CA ad BA : quare cum series GK ad rectangulum AE, aequalia sint, habebit quoque quadratum AF ad seriem GK rationem compositam ex rationibus KA ad DE, hoc est BA, & CA ad BA. Quod erat demonstrandum. Placet quoque theorema universaliter demonstrare.
a 129 huius; b 23 sextius;

L2.§3. PROPOSITION 153.

Let GHK be a progression of squares with bases placed along a line, and having the end point of length K .

I say that the square constructed upon AK, for the given series, has the proportion composed from the ratios KA to BA, & CA to BA.

Demonstration.

For let AD to DK be made as AB to BC. Therefore the rectangle DAB, or AE, [a] is equal to the sum of the series of squares GK. Hence the square AB has the same ratio to the series GK and to the rectangle AE; moreover since AD is to DK as AB is to BC, by inverting and putting the ratios together KA is to DA, as CA is to BA. Hence since the square AB to the rectangle AE, has the ratio [b] composed from the ratios KA to DE, & KA to DA, also the square AB to the same rectangle composed from the ratios KA to DE, & CA to BA : whereby since the series GK is equal to the rectangulum AE, also the square AB to the series GK has a ratio composed from the ratios KA to DE, that is BA, & CA to BA. Q.e.d. It is also pleasing to demonstrate the theorem generally. *a 129 huius; b 23 sextius;*
[The reader can refer back to the explanations around Prop. 129, or note that in algebraic terms, if r is the common ratio of the 1-D geometric progression of which the first term is a,

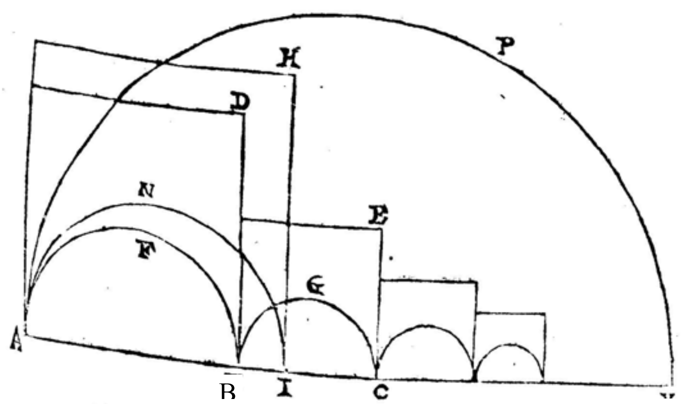
then the rectangle DAB $= \frac{a^2}{(1-r^2)}$, in which case the ratio AB^2 : sum of squares = a^2 : rect. DAB, from which the result follows. Thus, the sum for 2-D shapes in a geometric progression follows from that for 1-D line segments in a similar geometric progression.]

PROPOSITIO CLIV.

Data sit planorum similium quorumcunque progression FGK, habens bases homologas in directum, & terminum longitudinis K, datae progressionis.

Dico planum APK ad totam seriem FGK habere rationem compositam, ex rationibus KA ad BA, & CA ad BA. [p. 150]

Demonstratio.



Prop. 154, Figure 1.

Super iisdem basibus AB, BC, &c. construatur quadratorum series DEK : fiatque quadratum AD ad aliud AHI, ut AC ad AK, quod toti [a] seriei quadratorum DEK aequabitur. Super AI vero fac planum ANI siile plano AF. Itaque planum AF est ad planum ANI [b] ut quadratum AD ad quadratum AHI, hoc est ex constructione ut AC ad AK. Quare etiam planum ANI, seriei planorum similium FGK [c] aequale erit; ergo series FGK est ad planum ANI, ut series DEK ad quadratum AHI : atqui planum ANI est ad planum APK, [d] ut quadratum AHI ad quadratum totius AK : igitur ex aequalitate series FGK est ad planum APK, ut quadratum AK ad seriem DEK, sed quadratum AK [e] ad seriem DEK, proportionem habet compositam ex rationibus KA ad BA, & CA ad BA. Ergo & planum APK ad seriem FGK, proportionem habet ex iisdem rationibus compositam; quod erat demonstrandum.
a 130 *huius*; *b* 22 *sexti; c* 130 *huius; d* 22 *sexti; e* 153 *huius*.

L2.§3. PROPOSITION 154.

FGK is any progression of similar plane figures set out in order along a line, having homologous bases, and K is the terminal point of the given progression on the line.

I say that the ratio of the plane figure APK to the whole series FGK is composed from the ratios KA to BA & CA to BA.

Demonstration.

A series of squares DEK is constructed on the same bases AB, BC, &c. : and the square AD is made to another square in the ratio AD to AHI, which is as AC to AK, which is equal to the whole series of squares DEK [a]. Upon AI truly make the plane figure ANI similar to the plane figure AF. And thus the plane figure AF is to the plane figure ANI [b] as the square AD is to the square AHI, that is from the construction as AC

to AK. Whereby also the plane figure ANI, is equal to the series of plane figures FGK [c] ; hence the series FGK is to the plane figure ANI, as the series DEK is to the square AHI : but the plane figure ANI is to the plane figure APK, [d] as the square AHI is to the square of the whole length AK : therefore from the equality, the series FGK is to the plane figure APK, as the series DEK is to the square AK [this sentence has been inverted from the original], but the square AK [e] to the series DEK has the proportion composed from the ratios KA to BA, & CA to BA. Hence the plane figure APK to the series FGK, has a proportion composed from the same ratios; q.e.d.
a 130 *huius*; *b* 22 *sexti; c* 130 *huius; d* 22 *sexti; e* 153 *huius*.
[AD/AHI = AC/AK**; [**or $\frac{a^2}{rect.AHI} = \frac{a(1+r)}{a/(1-r)} = 1 - r^2$], hence rect. AHI [$= \frac{a^2}{(1-r^2)}$], which is the sum of the squares DEK to infinity.
Again, fig.ANI/fig.AF = sq.AD/rect.AHI = AC/AK; whence fig.ANI is the sum of the figures FGK to infinity.
Hence (sum of plane figures FGK)/ (plane figure ANI) = (sum of squares DEK)/(rect.AHI), as these are identically equal; Again, (fig.ANI/fig. APK) = (rect.AHI)/(AK^2), which amounts to (series FGK)/(fig. APK) = (series DEK)/(AK^2); but (series DEK)/(AK^2) [$= \frac{a^2}{(1-r^2)} / \frac{a^2}{(1-r)^2} = \frac{1-r}{1+r}$] = BA/KA× BA/CA; we have not inverted the ratios as Gregorius has done.**]**

PROPOSITIO CLV.

Esto quadratorum series habens bases in directum, & terminum logitudinis K, inscripta triangulo AGK, iuxta propositionem 131 huius, & completo rectangulo AI, latera quadratorum producantur in LS, &c. item in Q, O, R, &c.

Dico ex hac laterum productione perpetua, oriri progressionem rectangulorum MI, NL, &c. similium, & continue proportionalium, quae progressioni quadratorum quoque sit aequalis.

Demonstratio.

Quoniam ex hypothesi K terminus est longitudinis quadratorum seriei, etiam K termins [f] erit progressionis basium AB, BC, &c. igitur BK [g] est ad CK, ut [p. 151] AB ad BC, hoc est ut PM ad HN, hoc est ut GM ad MN, hoc est (quia QGM, HMN similia sunt triangula) ut QM ad MH, hoc est denique ut QM ad ON : a primo igitur ad ultimum, BK est ad CK, hoc est ML ad NS, ut QM ad ON : rectangula igitur MI, NL, proportionalia habent latera. Quare cum sint & aequiangula, erunt similia; quod erat primum.

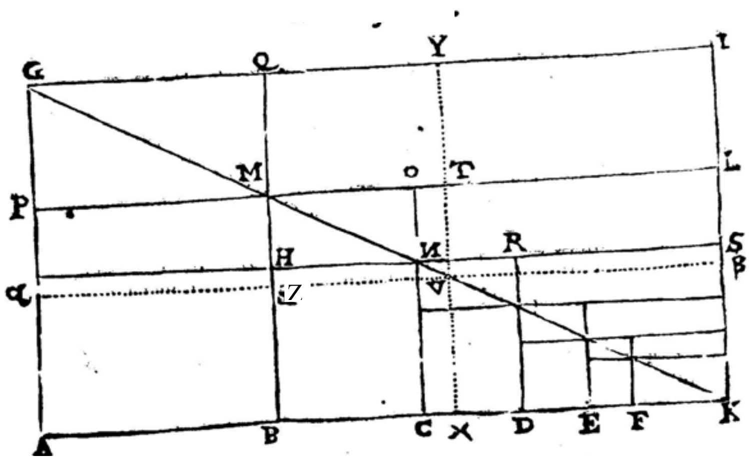
Deinde dicta rectangula complementa sunt eorum, que circa diametrum sunt. Ergo singula quadratis singulis datae seriei [a] aequantur. ergo & progressio tota toti progressioni aequalis erit; ex quo etiam secundum patet. Cum enim quadrata ex hypothesi sint in ratione continua, etiam complementa illis aequalia, in continua erunt analogia; quae erant demonstranda.

f 125 huius; g 82 huius; *a 43 primi.*

L2.§3. PROPOSITION 155.

Let a series of squares having their bases ordered along a line and a terminal point of length K, be inscribed in the triangle AGK, as in proposition 131 of this section, and on completing the rectangle AI, the sides of the squares are produced in LS, &c. likewise in Q, O, R, &c.

I say that from this continued extension of the sides, a progression of similar rectangules MI, NL, &c. arises in proportion, which is also equal to the progression of squares.



Prop. 155, Figure 1.

Since by hypothesis, the terminal point K is the length of the series of squares, K is also the terminal point of the progression of base lengths AB, BC, &c. ; therefore BK [g] is to CK, as AB to BC, that is as PM to HN, that is as GM to MN, that is (since QGM and HMN are similar triangles) as QM to MH, and that is hence as QM to ON : therefore from the first to the last, BK is to CK, that is ML to NS, as QM to ON : therefore the rectangles MI and NL have sides in proportion. Whereby since the have equal angles, then they are similar figures; this is true for the first two rectangles.

Then the said rectangles are the complements of these which lie around the diagonal of the main rectangle. Hence the individual rectangles are equal to the given squares of the series [a] . Hence the whole progression of rectangles is equal to the whole progression of squares ; from which the following is also apparent. Since indeed the squares are in a continued ratio by hypothesis, also the complements of these are equally in an analogous continued ratio; which it was required to show. *f 125 huius; g 82 huius*; *a 43 primi.*
[Thus, from sim. triangles, PM/ML = PG/KL; or PM^2 = …,etc.]

PROPOSITIO CLVI.

Iisdem positis fiat AX b aequalis seriei rationis AB ad CD, & ex puncto X, ducta normalis XY secet lineas PL, GK, in T & V.

Dico rectangulum sub BKVY, toti complementorum seriei MI, NL, &c. aequari.

Demonstratio.

Ducatur enim AK per punctum V parallela αβ. Igitur rectangula [c] VI, AV, itemque rectangula MY, αM, aequalia sunt inter se. Igitur figura MQIβV figurae MPAXV aequalis est. quare communi addito rectangulo ZT, erunt rectangular ZI, AT aequalia. Atqui rectangulum ZI est rectangulum sub Zβ, id est BK, & sub ZQ, id est VY; rectangulum vero AT seriei quadratorum, [d] est aequale, ergo rectangulum sub BKVY, seriei quadratorum, hoc est per praecedentem seriei complementorum est aequale. Quod erat demonstrandum. *b 80 huius*; *c 43 primi; d 128 huius.*

L2.§3. PROPOSITION 156.

With the same figure in place, AX [b] is made equal to the sum of the series of ratios AB to CD, and from the point X, the normal line XY cuts the lines PL and GK in T and V.

I say that the rectangle under BKVY is equal to the whole of the complementary series of rectangles MI, NL, &c.

Demonstration.

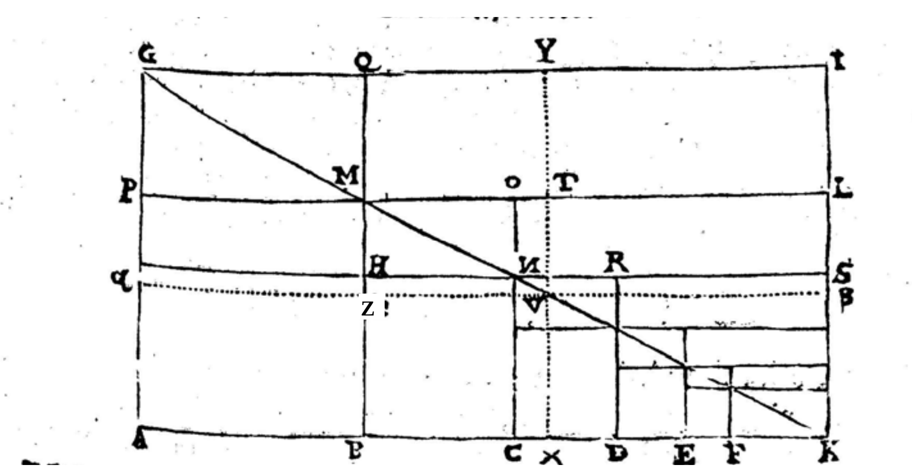
For $\alpha\beta$ is drawn through the point V parallel to AK. Therefore the rectangles [c] VI and AV, and likewise the rectangles MY and αM are equal to each other amongst themselves [from similar triangles]. Therefore the figure MQIβV is equal to the figure MPAXV. Whereby on adding the common rectangle ZT, the rectangles ZI and AT are equal. But the rectangle ZI is the rectangle under Zβ, that is BK, and under ZQ, that is VY; truly the rectangle AT is equal to the series of squares [d]. Hence the rectangle under BKVY is equal to the series of squares, that is by the preceding, to the series of complements. Q.e.d. *b 80 huius*; *c 43 primi; d 128 huius.*

[It is a tedious but elementary process to show that both these expressions XV.BK and the sum of the odd squares are equal to $\frac{abr}{(1-r^2)(1-r)}$, where AB = a; PG = b, and r is the common ratio for BC/AB, etc. Geometrically, it has been shown above in *d* that the rect. AT is the sum of the even squares, while here it is shown that the rect. AT = XV.BK.]

PROPOSITIO CLVII.

Iisdem positis, qua supra,

Dico ea quae circa diametrum sunt rectangula PQ, HO, & esse similia inter se & continue proportionalia; rectangulum vero sub AB & VY, toti eorum progressioni esse aequale.



Prop. 157, Fig. 1.

Ut PM est ad MO, sive HN, sic GP est ad ON, ob triangulorum GMP, MON, similitudinem : quare cum rectangula PQ, HO, & latera habeant proportionalia, & angulos aequales, erunt similia; ac proinde & reliqua omni eodem discursu similia erunt : quod fuit primum. Deinde cum similia sint dicta rectangula, erunt in duplicate homologorum laterum PM, HN, &c. ratione. Quare cum & quadrata sint in eorumdem laterum ratione duplicata, eadem erit rectangularum ac quadratorum proportio. Atqui haec ex hypothesi sunt in continua analogia, ergo & illa; quod erat alterum. Denique; rectangulum αM, aequatur [e] rectangulo MY. Addito [p. 152] igitur communi PQ, aequabitur rectangulum α Q, hoc est rectangulum sub AB & VY, rectangulo PY : Atque rectangulum PY, [a] duplum est progressionis triangulorum GPM, MHN, &c. ergo & rectangulum sub AB & VY, progressionis triangulorum est duplum : quare cum progressio rectangulorum PQ, HO, &c. etiam sit progressionis triangulorum dupla, rectangulum sub AB & VY, & progressio rectangulorum aequalia erunt : Quod postremum fuit eorum, quae erant demonstranda.

e 43 primi ; a 139 huius;

L2.§3. *PROPOSITION 157.*

With the same figure in place as above:
I say that regarding those rectangles PQ, HO, etc., which lie around the diagonal GK, which are similar and in continued proportion to each other; truly the whole sum of this progression is equal to the rectangle under AB by VY.

Demonstration.

As PM is to MO, or HN, thus GP is to ON [note that the 'N' has been reversed on the original diagram included here], on account of the similar triangles GMP and MON : whereby since the rectangles PQ and HO have sides in proportion, and the angles are equal, then they too are similar; and hence the remainder of the rectangles are similar by the same reasoning: which shows the first part of the proposition. Then when the rectangles PQ and HO are given as similar, then they are in the ratio of the squares of the sides PM and HN, &c. Whereby since the given squares are in the same ratio of the sides, then the rectangles and the squares are in the same proportion. But these by hypothesis are in continued proportion, and hence what is true for one is also true for the other. Finally : the rectangle αM is equal to the rectangle [e] MY. Therefore by adding the common rectangle [p. 152] PQ, then the rectangle αQ, or the rectangle formed by AB and VY, which is equal to the rectangle PY : And the rectangle PY, [a] is twice the progression of triangles GPM, MHN, &c. ; hence the rectangle formed by AB and VY is twice the progression of triangles : whereby since the progression of rectangles PQ, HO, &c. is also twice the progression of triangles, then the rectangle formed by AB and VY, and the progression of rectangles are equal : Which was the last of these, which were to be demonstrated.

e 43 primi ; a 139 huius;

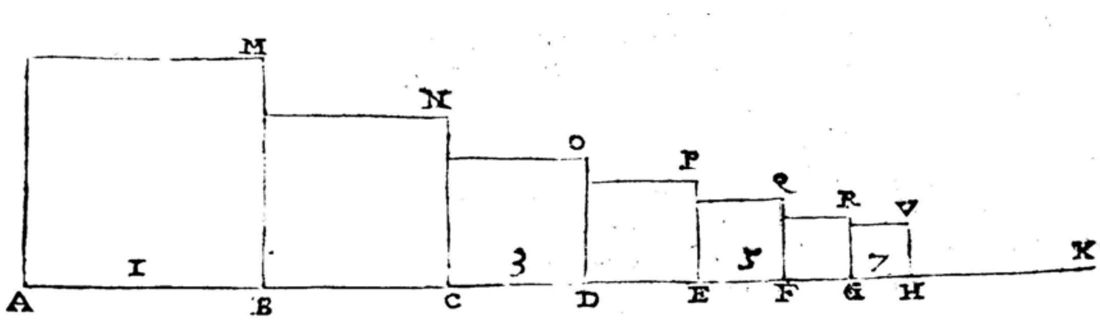
[Again, it is easy to show that both expressions are equal to $\frac{ab}{1-r^2}$. Geometrically, the rectangles PQ, HO, etc, are equal to the areas ab, abr^2, abr^4, etc, which are thus in proportion to the odd squares.]

PROPOSITIO CLVIII.

` Data sit quadratorum series, basibus in directum positis, & terminum habens longitudinis K : impares autem quadratorum bases AB, CE, EF, &c. notentur numeris imparibus 1.3.5.7. &c.

Dico, primum quadratum AM esse ad secundum BN, ut AB ad CD : & rursum primum quadratum AM esse ad tertium CO, ut AB ad EF; & ad quartum DP, ut AB ad GH. Atque ita in infinitum, bases notatae imparibus numeris, sunt primo quadrato cum subsequentibus comparato, proportionales.

Demonstratio.



Prop. 158, Fig. 1

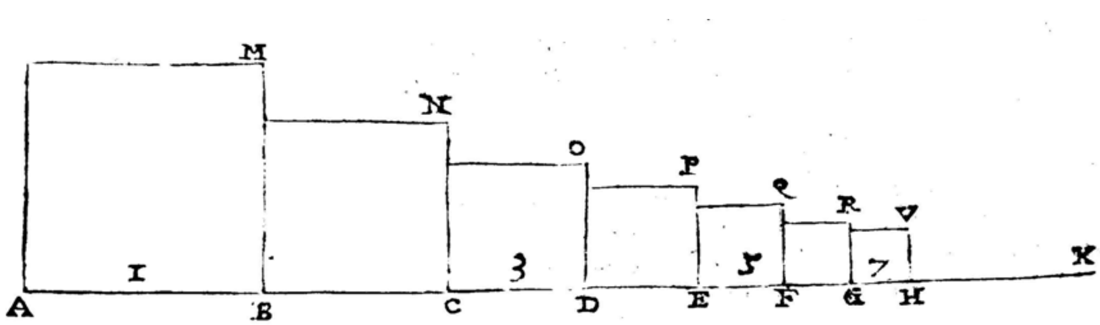
Comparimus exempli gratia quadratum AM, cum tertio CO; Quoniam quadrata omnia AM, BN, CO, &c. in continua sunt analogia, erunt[b] & bases continue proportionales ;ex aequalitate igitur etiam AB, CD, EF, erunt continuae; (cui inter ipsas aequalis continue proportionalium numerus intercedit) ergo quadratum AM est ad quadratum CO, ut AB ad EF; (cum ratione tam quadrati ad quadratum, quam lineae AB ad lineam EF, sint rationis AB ad CD duplicatae) eadem valebit demonstratio, si quadratum AM cum quovis alio comparetur, Constat ergo propositionis conclusio. *b 124 huius.*

L2.§3. *PROPOSITION 158.*

` A series of squares is given, with bases ordered along a line and with its terminal point at a distance K from A : moreover the odd squares bases AB, CD, EF, &c. are to be noted by the odd numbers 1.3.5.7. &c.

I say that the first square AM is to the second square BN, as AB to CD : and again the first square AM is to the third square CO, as AB to EF; and to the fourth DP, as AB to GH. And thus without end, the bases are to be noted by the odd numbers, are proportional to the first square by comparison with the following squares.

Demonstration.



Prop. 158, Fig. 1

For the sake of an example we compare the square AM with the third CO; Since all the squares AM, BN, CO, &c are in a continued ratio, and the bases are in an analogous continued proportion [b]; from the equality selected therefore also AB, CD, EF, etc, are in a continued ratio; (for the [odd] numbers interceding are themselves in continued proportion) hence the square AM is to the square CO, as AB to EF; (so the ratios of square to square, as of the line AB to the line EF, are of the square of the ratio AB to CD) the same demonstration will prevail, if the square AM is compared with any other square. Therefore the conclusion of the proposition is agreed upon. *b 124 huius.*

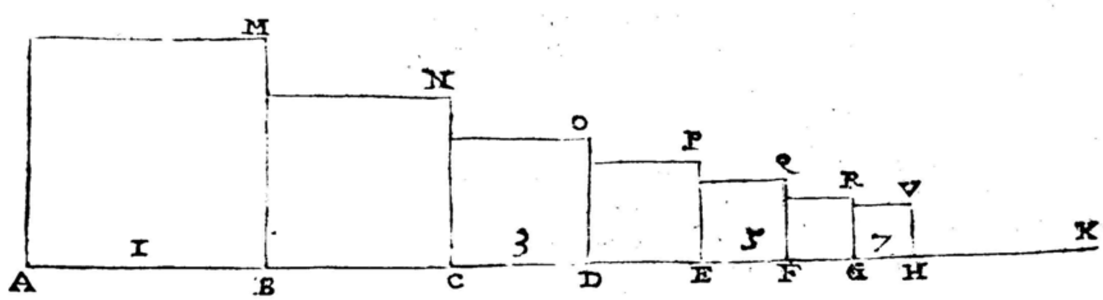
[If we let the geometric series on the line be $a, ar, ar^2, ar^3, ar^4, ar^5, ar^6$, ...,then the two dimensional series of squares are in the ratio $a^2, a^2r^2, a^2r^4, a^2r^6, a^2r^8, a^2r^{10}, a^2r^{12}$, ...; it is evident that sq.AM/sq.BN $= 1/r^2 =$ AB/CD; sq.AM/sq.CO $= 1/r^4 =$ AB/EF; sq.AM/sq.DP $= 1/r^6 =$ AB/GH; etc.]

PROPOSITIO CLIX.

Eadem posita figura; data sit planorum similium series MK, habens bases homologas in directum positis, & terminum longitudinis K.

Dico seriem MK ad nullam sui partem, verbi gratia ad seriem NK aut seriem OK, vel seriem PK, &c. eam habere rationem, quam inter se habent duae quacumque in hac basium serie, rectae lineae, inter quas par linearum numerus intercedit.

Demonstratio.



Prop. 158, Fig. 1

Series enim data MK, cum ea sui parte comparatur, ut inter utriusque primum terminum, vel par intercedat planorum numerus, vel impar; comparentur primo series MK & OK, inter quarum initia, impar terminorum numerus intercedit; quia igitur [p. 153] plana sunt in continua analogia, etiam bases AB, BC, &c. erunt a continuae proportionales. Quare ex aequo etiam AB, CD, EF, erunt continuae : ergo planum AM est ad planum CO, ut [b] AB ad EF. Atqui series MK [c] est ad seriem OK, ut planum AM ad planum CO, (sunt enim similium rationum series) ergo series MK est ad seriem OK, ut AB ad EF; inter quas impar numerus basium intercedit, nempe 3. Atqui in tota serie basium, non possunt reperiri duae aliae lineae, quae eandem rationem habeant, quam AB, EF, nisi illae inter quas idem ternarius linearum intercedit, uti ex elementis demonstratur; igitur nullae lineae ex serie basium, inter quas impar linearum numerus inter iicitur, eandem habent rationem, quam series MK ad sui partem OK.

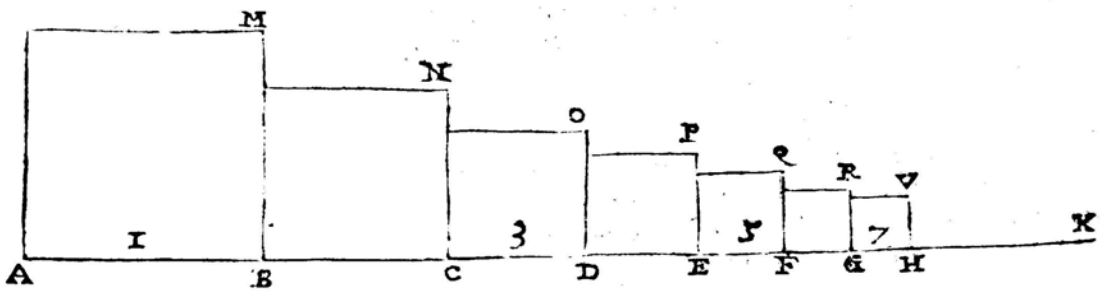
Comparentur modo duae series MK, PK, inter quarum initia par planorum sit numerus : Rursus igitur ostendemus uti prius seriem MK esse ad seriem PK, ut AB ad GH. quare cum inter AB & GH, impar linearum sit numerus, nempe 5; tota demonstratio primae partis huic etiam quadrat : unde patet proportionis veritas. *a 124 huius; b 20 sexti; c 84 huius*

L2.§3. *PROPOSITION 159.*

With the same figure in place; a series of plane figures MK is given, having bases ordered along a line and with its terminal point at a distance K from A

I say that the series MK, taken in comparison with another series formed from any part of itself, for instance either to the series NK, or the series OK or PK, &c., always has that ratio between any two corresponding terms, corresponging to the intercession of an odd number of lines.

Demonstration.



Prop. 159, Fig. 1

Indeed the given series MK is compared with that part of itself, in order that betwen the first term of the other [sub-series], either an even or odd number of plane figures intercede. In the first place, the series MK is compared with the series OK, between the start of which, an odd number of terms of the original series intercede; therefore since the plane figures are in analogous proportion, [p. 153] also the bases AB, BC, &c. are continued proportionals [a]. Whereby from the equality, also AB, CD, EF, are in continued proportion : hence the plane figure AM is to the plane figure CO, as [b] AB to EF. But the series MK [c] is to the series OK, as the plane figure AM is to the plane figure CO, (for they are both series of similar ratios) hence the series MK is to the series OK, as AB to EF; between which an odd number of of base lengths intercede, truly 3. But in the whole series of bases, no two other lines can be found, which have the same ratio that AB has to EF, except these between which likewise have three interceding lines, as is demonstrated from [Euclid's] Elements ; therefore no other lines from the series of bases, between which an odd number of lines can be placed between, have the same ratio as the series MK has to its own part OK.

In the same manner the two series MK and PK can be compared, between which there is an even number of plane figures : Therefore again we can show, that the prior series MK is to the series PK, as AB is to GH, whereby since between AB and GH, there is an odd number of lines, surely 5; the whole demonstration rests upon the squaring of the base element : thus the truth of the proposition is apparent. *a 124 huius; b 20 sexti; c 84 huius*

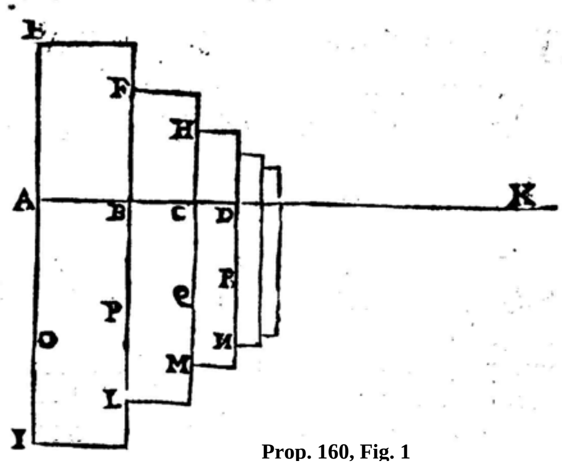
[If we let the geometric series on the line be $a, ar, ar^2, ar^3, ar^4, ar^5, ar^6$, ...,then as above, the two dimensional series of squares are in the ratio $a^2, a^2r^2, a^2r^4, a^2r^6, a^2r^8, a^2r^{10}, a^2r^{12}$, ... If we now consider the series OK that starts from the term OC, corresponding to the length ar^2 , and with the area a^2r^4 corresponds to the length EF, for which there are 3 interceding terms BC,DC, and DE. Again, PF has 5 interceding terms, etc. ; thus, the squared elements always lie on even values of the index *r*, leading to the odd number of spaces,]

PROPOSITIO CLX.

Data sit quadratorum series habens bases in directum & terminum longitudinis K. Deinde ex singulis punctis B, C, D, &c. erectae sint perpendiculares AI, BL, CM, &c. proportionales continuae in ratione dimidiata proportionis AB ad BC; & super illis perpendicularibus in altitudine linearum AB, BC, &c. fiant rectangula IB, LC, MD, &c.

Dico seriem rectangulorum IK, ad seriem rectangulorum LK, triplicatam habere proportionem rationis AI ad BL; cuius quadruplicatam habet series quadratorum EK, ad seriem quadratorum FK.

Demonstratio.



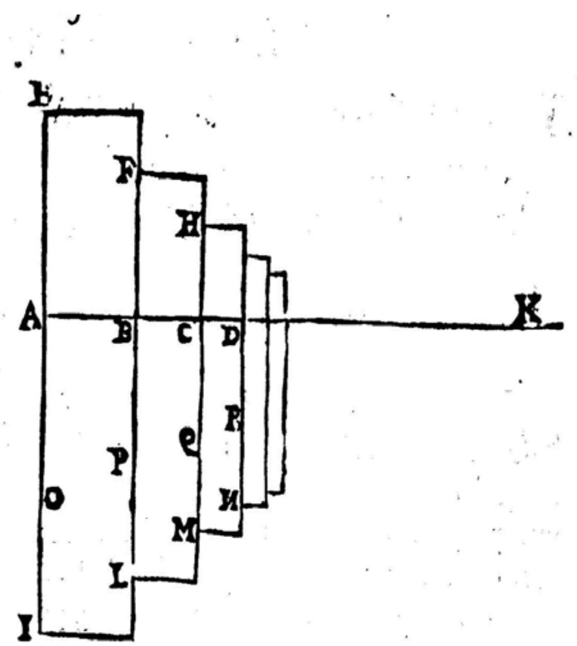
Prop. 160, Fig. 1

Primum enim rectangula IB, LC, &c. esse in continua analogia sic ostendo. Ratio rectanguli IB ad LC, coponitur ex rationibus AI ad BL, hoc est hypothesi BL ad CM, & AB ad BC, hoc est BC ad CD; Atqui etiam rectangulorum LC, MD, ratio componitur ex rationibus BL ad CM, & BC ad CD; ergo eadem est rectanguli IB ad LC, & LC ad MD ratio : ergo illa rectangula sunt in continua analogia, habeturque progressio continue proportionalium rectangularum I, L, M, N, K: quare series IK est [d] ad seriem LK, ut rectangulum IB, ad rectangulum LC : Atqui ratio rectanguli IB ad LC, componitur ex ratione AI ad BL, & ex ratione AB ad BC, quae ponitur esse duplicata rationis AI ad BL; ergo ratio rectanguli IB ad LC, hoc est sicut modo ostendimus, ratio seriei IK ad seriem LK, est triplicata rationis AI ad BL. Series autem quadratorum EK est ad seriem FK, ut quadratum BE ad quadratum CF, hoc est in duplicata rationis AB ad BC, [e] ut quadratum BE ad quadratum CF, hoc est (ut ex hypothesi colligitur) in duplicata rationis AI ad BL, cuius triplicata ratio seriei rectangulorum IK, ad seriem LK. Quod erat demonstrandum. *d 82 huius; e ibid.*

L2.§3. *PROPOSITION 160.*

A series of squares is given, having ordered bases set out along a line and the terminus of length K. Then from the individual points B, C, D, &c, perpendicular continued proportionals AI, BL, CM are erected in the ratio of the square of AB to BC; and upon these perpendiculars at the heights of the lines AB, BC, &c, the rectangles IB, LC, MD, &c are made.

I say that the series of rectangles IK, to the series of rectangles LK, to be in the cubic ratio to the ratio AI to BL; and the series of squares EK to the series of squares FK has the quadruple of this ratio.



Prop. 160, Fig. 1

Demonstration.

Indeed I show thus that the first rectangles IB, LC, &c. are in continued proporiton by analogy. The ratio of the rectangle IB to LC, is composed from the ratios AI to BL, that is by hypothesis as BL to CM, and from AB to BC, that is, from BC to CD; [Thus, rect.IB/rect.LC = AI/BL × AB/BC = AI/BL × BC/CD]

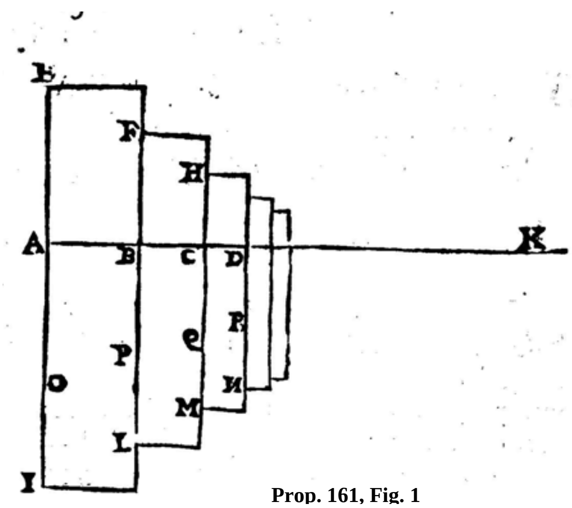
But also the ratio of the rectangles LC and MD, is composed from the ratios BL to CM, and BC to CD; hence the ratio is same for the rectangle IB to LC, and for the rectangles LC to MD : hence these rectangles are in analogous continued proportion, and a progression of rectangles in continued proportions is obtained : I, L, M, N, K; whereby the series IK is to [d] the series LK, as the rectangle IB is to the rectangle LC. But the ratio of the rectangle IB to LC, is composed from the ratio AI to BL, and from the ratio AB ad BC, which is put equal to the square of the ratio of AI ad BL; hence the ratio of the rectangle IB to LC, thus as we have shown in this way, or the ratio of the series IK to the series LK, is as the cube of the ratio of AI ad BL.
[Thus, series IK/series LK = rect.IB/rect.LC = AI/BL × BC/CD = $(AI/BL)^3$]
Moreover the series of squares EK is to the series FK, as the square BE is to the square CF, that is in the square ratio of AB to BC, [e] (as can be gathered from the hypothesis) or as the square BE to the square CF, or in the quadruple ratio of AI to BL, of which the cubic ratio is the ratio of the series of rectangles IK to the series of rectangles LK. Q.e.d. *d 82 huius; e ibid.*
[Thus, series EK/series FK = sq.BE/sq.CF = $(AB)^2/(BC)^2 = (AI/BL)^4$].

[p. 154]

PROPOSITIO CLXI.

Iisdem positis loco rectangulorum intelligatur super normalibus AI, BL, &c. construi series quadratorum.

Dico seriem quadratorum EK, ad seriem FK, duplicatam habere rationem eius, quam habet series quadratorum AI, BL, CM, &c. ad series quadratorum BL, CM, DN, &c.

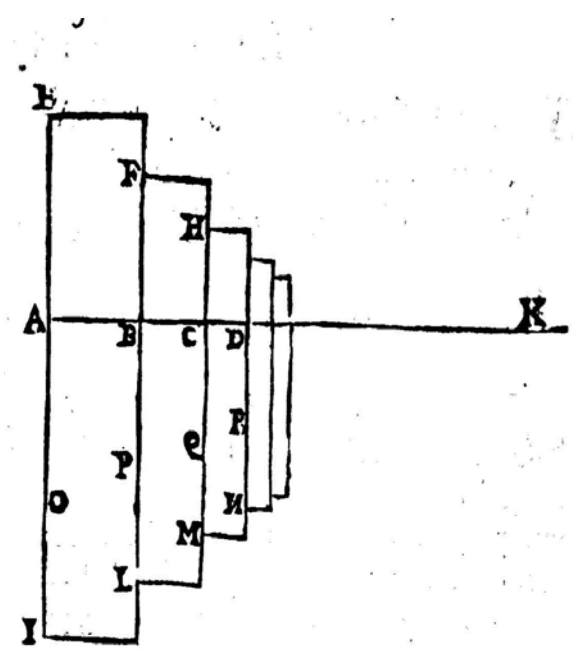


Prop. 161, Fig. 1

Demonstratio.

Series EK est ad seriem [a] FK ut quadratum BE ad quadratum CF, hoc est in duplicata rationis AB ad BC. Similiter ratio fieri quadratorum AI, BL, &c. ad seriem quadratorum BL, CM, &c. eadem est quae quadrati AI ad quadratum BL, hoc est ex hypothesi ratio seriei quadratorum AI, &c., ad seriem quadratorum BL, &c. Quare cum ostensum sit rationem seriei EK, ad seriem FK, esse duplicatam rationis AB ad BC, erit quoque duplicata rationis, quam habet series rectangulorum IK, ad seriem rectangulorum LK : Quod erat demonstrandum. *a ibid.*

L2.§3. *PROPOSITION 161.*



With the same figure above, and in place of the rectangles, a series of squares is understood to be constructed with normals AI, BL, etc.

I say that the series of squares EK to the series FK, has the square ratio of that which the series of squares AI, BL, CM, &c. has to the series of squares BL, CM, DN, etc.

Demonstration.

The series EK is to the series [a] FK as the square BE to the square CF, that is, in the ratio of the squares of AB to BC. Similarly the ratio of the squares AI, BL, etc., to the series of squares BL, CM, &c. is the same as that of the square of AI to the square of BL, that is, from the hypothesis, the ratio of the series of squares AI, etc. to the series of squares BL, etc. Whereby since it can be shown that the ratio of the series EK to the series FK, is the square of the ratio of AB to BC, it is also the square of the ratio, that the series of rectangles IK has to the series of rectangles LK : Q.e.d. *a ibid.*

PROPOSITIO CLXII.

Iisdem positis quae supra; inter BA, AI, CB, BL, DC, CM, &c. inveniantur mediae proportionales AO, BP, CQ, &c.

Dico quadrata AO, BP, CQ, &c. cubis AI, BL, CM, &c. ad series quadratorum BL, CM, DN, &c. esse proportionalia.

Demonstratio.

Quoniam BA, AO, AI, sunt continuae proportionales, erit quadratum AO, rectangulo BAI, [b] aequale : similiter reliqua quadrata BP, CQ, &c. reliquis rectangulis CBL, DCM, &c. erunt aequalia . Quare cum rectangula [c] dicta sunt continue proportionalia in ratione triplicata AI ad BL, quadrata quoque AO, BP, &c. erunt in dictorum laterum AI, BL, &c. triplicata ratione continue proportionalia. Atqui etiam cubi AI, BL, &c. sunt in laterum AI, BL, &c. [d] triplicata ratione; ergo quadrata AO, BP, &c. cubis AI, BL, &c. sunt proportionalia. Quod erat demonstrandum.

a ibid; b 17 sexti;c 160 huius ; d 33 undecimi.

L2.§3. PROPOSITION 162.

With the same in place as above; the mean proportionals AO, BP, CQ, &c. are found between BA, AI, CB, BL, DC, CM, &c.

I say that the squares AO, BP, CQ, &c. with the cubes AI, BL, CM, &c. to the series of squares BL, CM, DN, &c are in proportion.

Demonstration.

Since BA, AO, AI, are continued proportionals, the square AO is equal to the rectangle BAI [b] : similarly the remaining squares BP, CQ, &c. are equal to the remaining rectangles CBL, DCM, &c. Whereby since the said rectangles [c] are in continued proportion in the ratio of the cube of AI to BL, also the squares AO, BP, &c.are in continued proportion in the ratio of the cubes of the sides AI, BL, &c. But also the cubes AI, BL, &c. are in the cubic ratio of the sides AI, BL, &c. [d] ; hence the squares AO, BP, &c. are proportional to the cubes AI, BL, &c. Q.e.d.

a ibid; b 17 sexti;c 160 huius ; d 33 undecimi.

[Thus, from above, $\text{rect.IB}/\text{rect.LC} = (AI/BL)^3$, while $\text{rect.IB} = AO^2$ and $\text{rect.LC} = BP^2$;
hence $(AO/BP)^2 = (AI/BL)^3$].