PROGRESSIONUM GEOMETRICARUM

PARS SECUNDA

Terminum cuiuscunque progressionis in infinitum continuata designare.

PROPOSITIO LXXV.

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                                    L          M
   A              B        C      D   E              K
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Prop.75. Fig. 1.

Si fuerit magnitudo AB, ad magnitudinem BK, ut magnitudo BC ad magnitudinem CK.

Dico proportionem AB ad BC, sine termino continuati actu posse intra magnitudinem AK, ita ut numquam ad K perveniatur.

Demonstratio.

Fiat enim ut AB ad BC, sic BC ad L. quia igitur AB est ad BK, ut BC ad CK, erit alternando ut AB ad BC, id est ut BC ad L, sic BK ad CK : & rursum alternando, ut BC ad BK, sic L ad CK : quare cum BC ex datis, minor sit, quam BK, erit etiam L, minor quam CK : poterit ergo ipsi L, ex CK sumi aequalis CD : erant autem AB, BC, L, tres continuae proportionales; ergo & AB, BC, CD tres sunt continae. Fiat iam his tribus magnitudinibus continue proportionalibus AB, BC, CD, quarta proportionalis continua M : quoniam igitur paulo ante ostendi esse BK ad BC ut CK est ad L, sive CD, erit dividendo & invertendo, BC ad CK, ut CD ad DK : eodem plane discursu ostendam, M esse minoram ipsa DK, quo ante L ostendi esse minorem ipsa CK : poterit ergo ipsi M, ex DK, abscindi DE, aequalis. Sunt igitur quatuor magnitudines AB, BC, CD, DE continuae proportionales. Atque ita demonstrabimus proportionem AB ad BC, intra lineam AK sine termino posse actu continuari , ita ut nunquam ad K perveniatur. Quod erat demonstratum.

The terms of any progression can be shown to be continued indefinitely.

## *L2.§2.*   *PROPOSITION 75.*

If the length AB is to the length BK, as the length BC is to the length CK, then I say that it is possible for the proportion of AB to BC to be continued by acting within the length AK without reaching K.

Demonstration.

For indeed the ratio BC to L can be made equal to the ratio AB to BC. Therefore since the ratio AB to BK is as BC is to CK, on rearranging, so AB is to BC or BC to L, thus BK is to CK : and again on rearranging, as BC to BK, thus L to CK. Whereby from what is given, BC is less than BK, and hence also L is less than CK : therefore L can be taken to equal a length CD in the interval CK : but as AB, BC, and L are continued proportionals; so therefore AB, BC, CD are three continued proportionals. Now a fourth continued proportional M can be constructed from these three lengths AB, BC, and CD: for we have just shown that BK to BC is as CK to L, or CD; hence on dividing and inverting, BC is to CK, as CD is to DK : from the same clear argument it can be shown that M is less than DK, from which before L was shown to be less than CK : hence M can be set equal to DE, less than DK. AB, BC, CD, and DE are hence four lengths in continued proportional. And thus we will show [in the next proposition] that the proportion AB to BC can be made to act within the line AK, thus without reaching K. Q.e.d.

[Set BC/L = AB/BC; again, as AB/BK = BC/CK, AB/BC = BK/CK = BC/L, or BC/BK = L/CK.
Now, BC < BK, hence L < CK; CD is made equal to L, and then AB, BC, CD are in continued proportion,
etc.]

PROPOSITIO LXXVI.

$$\underline{\text{A} \qquad\qquad \text{B} \qquad \text{C} \qquad \text{D} \quad \text{E} \ \text{F G} \qquad \text{K}}$$
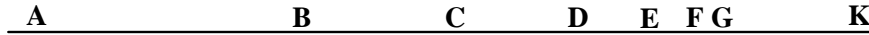
Prop.76. Fig. 1.

Si fuerit magnitudo AB, ad magnitudinem BK, ut magnitudo BC ad magnitudinem CK; & proportio AB ad BC, continuetur in magnitude AK, per plures terminos CD, DE, EF, &c.

Dico etiam CD, fore ad DK, & DE ad EK, &c. sic deinceps, ut AB est ad BK & BC ad CK, &c.

Demonstratio.

Quando quidem AB est ad BK, ut BC ad CK; erit alternando AB ad BC, hoc est ex datis BC ad CD, ut BK ad CK; & rursum alternando ac invertando: KB ad BC , uti KC ad CD; & dividendo ac invertendo, ut BC ad CK; sic CD ad DK: non aliter ostendimus ut CD ad DK; sic esse DE ad EK; atque ita deinceps in infinitum. Quod erat demonstratum.

L2.§2.   *PROPOSITION 76.*

If the length AB is to the length BK, as the length BC is to the length CK; and the proportion AB to BC is to be continued within the length AK through many terms CD, DE, ER, etc, then I say that CD also to DK, and DE to EK, etc, and thus henceforth, shall be as AB to BK, and BC to CK, etc.

Demonstration.

Since AB is to BK as BC is to CK; AB to BC is on interchanging terms, or from what is given, BC to CD, is as BK to CK; and again on alternating and inverting: KB to BC, as KC to CD; and by dividing and inverting, as BC to CK; thus CD to DK: in the same way we can show that as CD is to DK, thus DE is to EK; and thus henceforth indefinitely. Q.e.d.
[Since AB/BK = BC/CK; AB/BC = BC/CD = BK/CK ; KB/BC = KC/CD giving BC/CK = CD/DK, etc.]

[96]

PROPOSITIO LXXVII.

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad \text{L} \qquad\quad}$$

$$\underline{\text{A} \qquad\quad \text{B} \quad \text{C} \qquad \text{D} \qquad \text{E} \qquad\qquad \text{K}}$$

Prop.77. Fig. 1.

Data sit proportio quaevis minoris inaequalitatis AB ad AC.
Dico si haec continuetur, exhibendam magnitudinem quaevis data maiorem.

Demonstratio.

Detur enim magnitudo quaevis L : manifestum est si BC, excessus secundae magnitudinis AC, supra primam AB, aliquoties sumatur, maiorem fore magnitudine L : debeat ergo sumi BC quater, ut excedat L : continuetur ratio AB ad AC per quinque terminos AB, AC, AD, AE, AK; atque ita habebimus quatuor differentiae BC, CD, DE, EK. quoniam autem est ut a DA ad CA, sic DC ad CB, & cum DA maior sit CA,

erit quoque DC maior quam BC : similiter, erit ED maior quam CD, & KE quam ED. ergo KB ex quatuor differentis composita maior erit quam BC quater sumpta. Quare cum BC quater sumpta maior ponatur quam L : erit KB multo maior quam L, ideaque AK adhuc multo quam L maior erit : constat igitur quod fuerat demonstratum.

## *L2.§2.*   *PROPOSITION 77.*

    The proportion is given of some inequality AB to AC less than one.
    I say that if this ratio is continued a number of times then a magnitude greater than some given magnitude can be shown.

Demonstration.

For some magnitude L is given, it is clear that if BC, which is the increase in length of the second magnitude AC over the first AB, is taken a number of times, then it shall be greater than the magnitude L. Hence [in our example] BC has to be taken four times to exceed L : the ratio AB to AC is taken for the five terms AB, AC, AD, AE, AK; and thus we have the four differences BC, CD, DE, EK; but since [a] DA is to CA, thus as DC is to CB, and as DA is greater than CA, DC also is greater than BC, and similarly ED is greater than CD, and KE than ED. Hence KB arising from the four differences is greater than BC taken four times. Whereby if BC taken four times is placed greater than L, then KB is many times greater than L, and besides AK is many times greater than L: hence what had to be shown has been agreed upon.

PROPOSITIO LXXVIII.

A   B   C   D   E   K

L   M   N   O   P   R
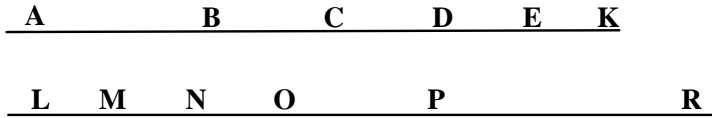
Prop.78. Fig. 1.

A magnitudine AK auferatur quaevis pars AB, & a residuo BK auferatur BC, ea lege ut sicut est AB ad BK, ita sit BC ad CK.
Dico si haec ablatio semper fiat, relinqui ex AK quantitatem data minorem *est universalis prima decimi*.

Demonstratio.

Detur enim quantitas LM aut alia quantumvis parva : dein ut KB ad KA, sic LM fiat ad LN: atque haec proportio per tot terminos continuetur, donec LR maior sit quam AK, hoc autem aliquando futurum est per praecedentem. Deinde quoties in M, N, O, P, divisa est LR, toties in ratione AB ad BK subdividatur AK in B, C, D, E. Quoniam ergo ex constructione: LM, LN, LO, &c. sunt continuae, erit ut LM ad LN, id est ut BK ad KA, sic LP ad LB. quare invertendo, ut AK ad BK, sic RL ad PL, & permutando ut AK ad RL, sic BK ad PL. Atqui ex constructione AK minor est quam RL, & ergo BK quam PL minor erit, deinde quoniam ex constructione est AB ad BK, ut BC ad CK; & CD ad DK. patet componendo omnes AK, BK, CK, EK esse continuas : itaque BK est ad CK, ut AK ad BK, id est ex constructione ut NL ad ML : sed PL est ad OL, ut NL ad ML; quia omnes RL, PL, OL, &c. sunt ex constructione continuae; ergo BK est ad CK , ut PL ad OL; & permutando ut BK est ad PL, sic CK est ad OL. atqui iam ostendimus BK minorem esse quam PL, ergo & CK quam OL minor erit. similiter demonstrabimus DR minorem esse quam NL, ac tandem EK esse minorem quam ML, quae erat data quantitas minor; ergo relinquitas quantitas: quod erat demonstratum.

L2.§2.   ***PROPOSITION 78.***

From a magnitude AK some part AB is taken, and from the remainder BK is taken BC, according to the rule AB is to BK as BC is to CK.

I say that if this is always done by subtraction, then the amount to be left from the given AK *is always less than the first quantity to be taken away*.

Demonstration.

For some quantity LM can be given which is however a little smaller than the other quantities given : then as KB is to KA, thus LM is made to LN: and this proportion is continued for all the terms, while LR is greater than AK, as this is soon to be the case from the preceding proposition. Hence as often as M, N, O, P divide the length LR, so AK is sub-divided by B, C, D, E in the ratio AB to BK. Therefore, since from the construction: LM, LN, LO, etc. are in continued proportion, as LM is to LN, or as BK to AK, thus LP is to LR. Whereby on inverting, as AK is to BK, thus RL is to PL, and on interchanging, as AK to RL, thus BK to PL. But from the construction AK is less than RL, and hence BK is less than PL, hence by construction AB is to BK as BC is to CK; and CD to DK. It is apparent from placing all these together that AK, BK, CK, DK, EK are in continued proportion : and thus BK is to CK, as AK is to BK, which by construction is as NL to ML : but PL is to OL as NL is to ML; since all of RL, PL, OL, etc. are in continued proportion from construction; hence BK is to CK, as PL is to OL; and on interchanging as BK is to PL, thus CK is to OL. But now we have shown that BK is less than PL, and hence CK is less than OL. similarly we can show that DK is less than NL, and finally that EK is less than ML, which were the given smaller quantities; hence the quantities are relinquished : Q.e.d.

[It is observed that the proportions on the upper line in Fig. 78 are diminishing towards K, while in the lower line, the corresponding proportions are increasing away from L.

Scholium.

Nota : dum in propositioe dicitur, si haec ablatio semper fiat, dico relinqui ex AK quantitatum data minorem : *sensum propositionis non esse, relinqui ex AK quantitatem data minorem, post ablationem terminorum in infinitum continuatum; sint post totam seriem absolutam, relinqui adhuc quantitatem data minorem; sed auferendo terminos ex AK, in ratione ante dicta, aliquando tot auferendo, ut residua pars totius AK, minor sit quantitate data:* quod in gratiam quarundam dictum sit.

Scholium.

Note : while in the said proposition, **if this is always done by subtraction, I say that the amount to be left is always less** : *the sense of the proposition is not that the quantity to be left from AK is to be less after an infinite number of terms have been taken away after continuing indefinitely; after the whole series has been summed, there is still to be taken a given smaller quantity; but terms are to be taken from AK in the ratio mentioned above, that finally are to be taken from the whole amount, as the remaining part of the whole of AK is less than any given amount*: **for any amount that may be said**.

PROPOSITIO LXXIX.

A		B	C	D	E F		K
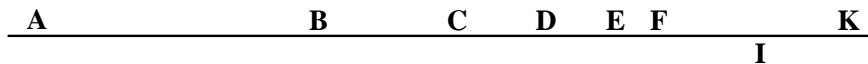
Prop.79. Fig. 1.

Data sit magnitudo quaecumque AK : si fuerit

AB ad BK, ut BC ad CK.

AB ad AK, ut BC ad BK.

vel   AB, BK, CK continuae proportionales.

AB ad BC, ut BK ad CK.

AB ad BC, ut AK ad BK.

Dico magnitudinem AK aequalem esse toti progressioni magnitudini, num continue proportionalium, rationis AB ad BC in infinitum continuatae terminum esse K.

Demonstratio.

Cum AB sit ad BK, ut BC ad CK, poterit ratio [a] AB ad BC, intra magnitudinem AK semper continuati, ita numquam perveniatur ad K, id est AK maior erit quaecunque serie finita terminorum; ergo AK non est minor serie tota rationis AB ad BC. Deinde quia AB est ad BK, ut BC ad CK, si ratio AB ad BC semper est continuetur, erit ut AB ad b BK, sive ut BC ad CK, sic CD ad DK, & DE ad EK, atque ita deinceps in infinitum : Itaque si continuetur semper ratio AB ad BC, relinquetur c tandem ex AK magnitudino quavis data minor. Quare AK nequit esse maior , serie rationis AB ad BC : nam si maior esset deberet aliquo excessu esse maior, ponatur is IK; igitur AI seriei rationis AB ad CD aequalis erit: ergo ratio AB ad BC quantumvis continuata non transiliet unquam I. ergo relinquetur ex AK magnitudo semper maior quam IK. ergo non minor quavis data, contra iam demonstrata. non erit igitur AK maior serie rationis AB ad BC: Quare cum neque minorem esse antea sit ostensum, aequalis sit necesse est. Quod erat demonstrandum.

Relique quorum hypothesum demonstrationes ad primam reducuntur. nam si fuerit AB ad AK, ut BC ad BK, erit dividendo AB ad BK, ut BC ad CK, ergo per primam demonstrationem, rationis AB ad BC semper continuatae terminus est in K.

Deinque si fuerit AB ad BC, ut BK ad CK, vel AK ad BK, erit permutando vel AB ad BK, ut BC ad CK, vel AB ad AK, ut BC ad CK. Unde iterum per primam demonstrationem conficitur proposition.

a 73 huius; b 74 huius; c 78 huius.

L2.§2.   *PROPOSITION 79.*

Some magnitude AK is given: if either of these ratios is true:

> AB to BK, as BC to CK.
> AB to AK, as BC to BK.
> AB, BK, CK are continued proportionals.
> AB to BC, as BK to CK.
> AB to BC, as AK to BK.

or

I say that the magnitude AK is equal to the magnitude of the whole progression: for if the proportions in the ratio AB to BC are continued indefinitely then K is the final amount.

Demonstration.

Since AB is to BK, as BC to CK, the ratio [a] AB to BC, within the length AK can always be continued, thus never reaching K, that is AK is larger than any finite series of terms ; hence AK is not less than the total series of the ratio AB ad BC. Hence as AB is to BK, as BC to CK, if the ratio AB to BC is always to be continued, then as AB to [b] BK, or as BC to CK, thus CD to DK, amd DE to EK, and thus henceforth to infinity. And thus if the ratio AB to BC is always continued, the length from AK that is left is at last smaller than any given length whatever [c]. Whereby AK is unable to be larger for the series of ratios AB to BC : for if AK is greater, then there is some length IK by which it is greater; therefore AI is equal to the series of ratios of AB ad CD: therefore the ratio AB to BC can be continued as far as you like without jumping beyond I at any time. Hence the remainder of the terms from the magnitude AK is always larger than IK, and therefore not less than any given, this is in contradiction to what has been shown. Therefore AK is not greater than the series of ratios AB to BC: whereas before it has been shown that it is not smaller either, then it shall be equal by necessity. Q.e.d.

The rest of these hypothesis can be demonstrated by being reduced to the first. For if AB to AK is as BC to BK, then on division AB is to BK as BC to CK, hence by the first demonstration, with the ratio AB to BC always to be continued, the end is in K.

And then if AB is to BC, as BK to CK, or AK to BK, by interchanging it will be either AB to BK, as BC to CK, or AB to AK, as BC to CK. Hence again the proposition is agreed upon by the first demonstration. *a 73 huius; b 74 huius; c 78 huius.*

PROPOSITIO LXXX.

| A | M | | B | C | D | E | F | | K |

Prop.80. Fig. 1.

Data serie continue proportionalium AB, BC, CD, &c. quiuscumque proportionis, & quocumque in generis quantitatis; invenire magnitudinem quae omnibus terminis totius seriei datae in infinitum continuatur, sit aequalis:

[98]

Constructio prima.

Sit AM, differentia primorum duorum terminorum : fiatque ut AM ad BC secundum terminum, sit BC ad tertium quempiam CK. Dico CK, magnitudinem cum primo AB, & secundo termino BC, aequalem esse seriei universae AB, BC, CD, &c.

Constructio secunda.

Fiat ut AM duorum primorum terminorum differentia, ad AB primum terminum, ita secundus terminus BC, ad tertiam aliquam magnitudinem BK. Dico magnitudinem BK, cum primo termino, exhibere quantitam aequalem tot seriei.

Constructio tertia.

Differentiae duorum primorum terminorum AM, & primo termino AB, tertia proportionalis fiat AK. Dico AK totam seriam exhibere.

Demonstratio.

Prima constructionis; AM est ad BC, ut BC ad CK; igitur componendo AM cum BC, hoc est AB, erit ad BC, ut BK ad CK : & permutando, AB ad BK, ut BC ad CK. Quare *a* AK magnitudo, hoc est CK, cum AB & BC primis terminis, toti seriei aequalis est.

Secunda constructionis; AM est ad AB, ut BC ad BK ex constructione : igitur dividendo, ut AM est ad MB, id est ut AM ad BC, sic BC ad CK, itaque per demonstrationem primae constructionis, AK (hoc est BK una cum primo termino AB) aequatur toti seriei.

Tertia constructionis; Cum erit ex constructione AM ad AB, ut AB ad AK, erit invertendo, dividendo, rursusque invertendo AM ad MB, ut AB ad BK : & componendo AB ad MB, hoc est BC, ut AK ad BK. Quare AK b toti seriei aequalis est. Factum igitur est quod petebatur.

Prima igitur constructione exhibere tota series praeter primum & secundum terminum. 2. constructione habetur seriei tota praeter primum terminum. 3. constructione simul tota producitur series. Huius autem problematis, aequalis esse aliquis eisdem universalitatem, amplius deductam habet in propositione 123, huius & corollario quarto ibidem.

a 79 huius; b 78 huius.

L2.§2.   *PROPOSITION 80.*

Given a series of continued proportionals AB, BC, CD, etc. of some proportion, and for some quantities in general; to find the magnitude that is equal to all the terms of the given series continued to infinity.

First construction.

Let AM be the difference of the first two terms : and as AM is to BC the second term, BC is made to some third term CK. I say that the magnitude CK, together with the first AB and second term BC is equal to the whole series AB, BC, CD, &c.

Second construction.

As AM the difference of the first two terms, is made to AB the first term, thus the second term BC is to some other third BK. I say that the length BK, together with the first term, furnishes an amount equal to the whole series.

Third construction..

The difference of the first two terms AM, and with the first term AB, then the third of the proportionals AK is made. I say that AK presents the whole of the series.

Demonstration.

First construction: AM is to BC, as BC is to CK; therefore by adding AM with BC, or AB is to BC as BK ad CK : and by interchanging, AB is to BK, as BC is to CK. Whereby *ᵃ* the magnitude AK, or CK with AB and BC of the first terms, is equal to the whole series.

Second construction: AM is to AB, as BC is to BK by construction : therefore on dividing, as AM is to MB, or as AM is to BC, thus BC is to CK, and thus by the demonstration of the first construction, AK (or BK together with the first term AB) is equal to the whole series.

Third construction: as by construction AM is to AB, as AB is to AK, on inverting and dividing, and inverting again, AM is to MB, as AB is to BK : and on adding, AB is to MB, or BC, as AK is to BK. Whereby AK *ᵇ* is equal to the whole series. Therefore what was sought has been achieved.

1. Therefore the whole series except the first and second terms are shown by the first construction.

2. The whole of the series except the first term is shown by the construction.

3. The whole series is produced at the same time from this construction. However this problem of producing the whole series has otherwise been more fully deduced in Proposition 123 of this book, and in the fourth corollary of the same. a 79 huius; b 78 huius.

PROPOSITIO LXXXI.
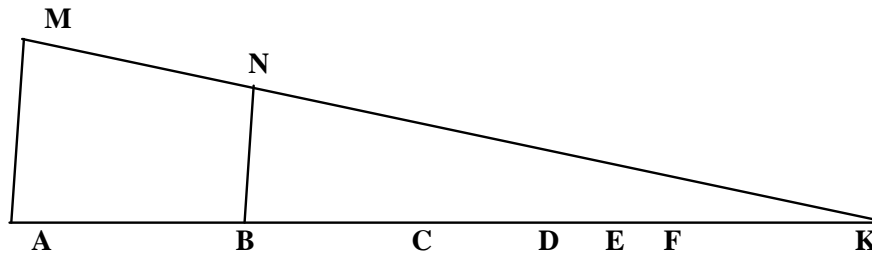
Prop.81. Fig. 1.

Quia vero ultimo ratio quantumcumque quantitatum ad lineas reduci potest, hinc etiam sequenti rationis lineis toti seriei ....poterunt aequalia reductia [???? text in a dreadful condition, with print from the other side of page showing through.], reperiemus aequale.

Constructio &demonstratio.

Ex punctis A & B, erige ad quantum angulum, parallelas AM , BN, & quarum AB, BC ipses sint proportionales; & per punctis M & N ductur recta MN, concurrens cum ABC in K. Dico AK, toti seriei AB, BC, CD, &c. aequalem esse.

Quod autem MN, occurrere debeat ABC productae, patet ergo, quod BC ex hypothesi, minor sit quam AB;

and proinde etiam BN: cum AM, BN proportionales sint ipsis AB, BC; & igitur concursus in K; erit ut AB ad BC, sic MA ad NB; sed ut MA ad NB, sic AK ad BK, ergo AB est ad BC, ut AK ad BK. Unde [a] AK toti seriei aequalis est. Factum igitur est quod perebatur.

a Ibid.

L2.§2. *PROPOSITION 81.*

Since indeed the last ratio of some quantity can be reduced to a line, hence also a sequence of ratios on a line can be reduced to be equal to the sum of the whole series.

Construction and demonstration.

From the points A and B, you can raise parallel lines AM and BN to an angle of some size, and to which AB and BC are proportional; and through the points M and N the line MN is drawn crossing with ABC in K. I say that AK is equal to the sum of the series AB, BC, CD, etc.
For since MN ought to cross ABC produced, it is apparent that BC from hypothesis is less than AB; and hence BN also is less than AM, since AM, BN are in proportion with AB and BC themselves. Hence the line MN meets the line ABC in K: thus, AB is to BC as MA is to NB; but as MA is to NB, thus AK to BK, hence AB is to BC, as AK is to BK. Hence [a] AK is equal to the sum of the whole series. Hence what was required has been accomplished.
[Thus, one of the most important propositions in the book is reduced to a state of almost total illegibility by poor printing workmanship in the copy from which the microfiche was made.]

a Ibid.

L2.§2. *PROPOSITIO LXXXII.*

A	N		B	C	D	E	F	G		M	K	M

Prop.82. Fig. 1.

Sit magnitudo AK : series tota rationis AB ad BC continuatae in infinitum.

Dico esse
- AB ad BK, ut BC ad CK.
- AB ad AK, ut BC ad BK.
- AK, BK, CK &c. continue proportionales.
- AB ad BC, ut BK ad CK.
- AB ad BC, ut AK ad BK.

Demonstratio.

Ex AB abscindatur NB, aequalis BC : si ergo non est AB ad BK, ut BC ad CK, neque permutando AB erit ad BC, id est BN, ut BK ad CK; ergo neque dividendo AN, erit ad NB, ut BC ad CK. Fiat igitur ut AN ad NB, sic BC ad aliam magnitudinem CM, maiorem vel minorem quam CK. Itaque componendo, AB erit ad NB, hoc est BC, ut BM ad CM: & permutando AB ad BM, ut BC ad CM. Quare totius seriei [b] rationis AB ad BC, terminus erit M; sive AM aequalis erit seriei rationes AB ad BC, quod est absurdum, cum AK maior vel minor quam AM, sit ex hypothesi aequalis seriei datae. non erit igitur alia ratios AB ad BK a ratione BC ad BK, ergo eadem, quod erat demonstrandum.

Reliquas autem assertionis partes ex prima deducemus. Cum enim iam demonstratum sit ex hypothesi theorematis, sequi AB esse ad BK ut BC ad CK, componendo erit AK ad BK, ut BD ad CK. quod erat secundum.

Et quoniam AK est ad BK, ut BK ad CK, igitur per constructionem rationis, AK est ad AB, ut BK ad BC; & invertendo AB ad AK, ut BC ad BK. quod erat tertium. rursum quoniam AB est ad BK, ut BC ad CK, erit permutando AB ad BC, ut BK ad CK, quod est quartium. denique quoniam ostensum est AB esse ad AK, ut BC ad BK, etiam permutando AB est ad BC, ut AK ad BK: quae omnia erant demonstranda.

Corollarium.

Ex quintae assertionis partem hoc theorema deducitur : data sit series magnitudinum continue proportionalium AB, BC, CD, &c. sine termino continuata. Dico esse ut una antecedentum, nempe AB, ad unam consequentium BC. sic omnes, hoc est infinitas antecedentes, sive AK, ad omnes sine infinitas consequentes, sive BK. *b 79 huius*.

PROPOSITION 82.

The magnitude AK is given: the whole series of ratios of AB to BC is continued to infinitity.

I say that

> AB is to BK, as BC is to CK.
> AB is to AK, as BC is to BK.
> AK, BK, CK &c. are continued proportionals.
> AB is to BC, as BK is to CK.
> AB is to BC, as AK is to BK.

Demonstration.

From AB, NB is cut off equal to BC : if therefore AB to BK is not as BC to CK, neither by interchanging will AB be to BC or BN, as BK is to CK; and therefore nor on dividing will AN be to NB, as BC to CK. Hence AN to NB is made thus as BC to another length CM, greater or less than CK. And thus on adding, AB is to NB or BC, as BM to CM: and on interchanging AB is to BM, as BC to CM. Whereby of the whole series of ratios [b] AB to BC, the end is M; or AM is equal to the series of ratios AB to BC, which is absurd, for AK which is greater or less than AM, by hypothesis is equal to the given series. Therefore the ratio AB to BK is the same as the ratio BC to BK, q.e.d.

We can be deduce the remaining parts of the proposition from the first part. For indeed thus can be show from hypothesis of the theorem that is follows that AB is to BK as BC is to CK, on adding AK is to BK, as BD is to CK. Which is the second part.

And since AK is to BK, as BK is to CK, therefore by construction the ratio AK is to AB, as BK is to BC; and on inverting, AB is to AK, as BC is to BK. Which is the third part. Again, since AB is to BK, as BC is to CK, on interchanging AB is to BC, as BK is to CK, which is the fourth part. Hence as it was shown that AB is to AK, as BC is to BK, also on interchanging AB is to BC, as AK is to BK: whereby all the parts have been demonstrated. *b 79 huius*.

Corollarium.

From the fifth part of the proposition, this theorem can be deduced : a series of magnitudes is given in continued proportion AB, BC, CD, etc. to continue without end. I say that as one preceding term to another following term, such as AB to BC, thus the same ratio holds for all the terms. That is for all the preceding terms, or AK, to all the following terms or BK. *b 79 huius*.

L2.§2.   PROPOSITIO LXXXIII.

A	**M**	**B**	**C**	**D**	**E**	**F**	**K**

Prop.83. Fig. 1.

Sit AK magnitudo producta ex ratione AB ad BC, in infinitum continuata; primi autem & secundi termini differentia sit AM;

Dico primo, differentiam AB, BC secundum terminum, CK totam seriem, (praeter duos primos terminos) in continua esse analogia.

[100]

Dico secundo, AM differentiam, ad primum terminum AB, esse ut BC secundus terminus, ad BK totam seriem, praiter primam terminum.

Dico tertio, differentiam AM, primum terminum AB, & totam seriem AK, in continua esse analogia.

Demonstratio.

A   M   B   C   D   E   F   K

Prop.83. Fig. 1.

Quoniam AK magnitudino producta est ex ratione AB ad BC in infinitum continuata, erit per praecedentem AB ad BK, ut BC ad CK: igitur permutando ut BK ad CK, sic AB ad BC, hoc est MB, & dividendo AM ad MB, hoc est BC, ut BC ad CK. quod erat primum.

Rursum cum sit ut AM ad BC, hoc est MB, sic BC ad CK, erit componendo invertendo, ut AB ad MB, sic BK ad CK; and convertendo invertendo ut AM ad AB, ita BC ad BK. quod erat secundum.

Iterum cum sit ut AM ad AB, sic BC ad BK, erit permutando, ut AM ad BC, hoc est MB, sic AB ad BK, & invertendo componendo, & iterum invertendo ut AM ad AB, sic AB ad AK. quod erat tertio loco demonstrandum.

PROPOSITION 83.

Let the length AK be produced from the ratio AB to BC continued indefinitely; also, the difference of the first and second terms is AM;

I say in the first case, that the difference AB, the second term BC, and the length CK of the total series (except the two first terms) are in continued proportion.

In the second place, I say that the difference AM, to the first term AB, is as BC the second term, to the whole series BK, except the first term.

In the third place, I say that the difference AM, the first term AB, and the whole series AK, are in continued proportion.

Demonstration.

A   M   B   C   D   E   F   K

Prop.83. Fig. 1.

Since the length AK has been produced from the ratio AB to BC continued indefinitely, then from the preceding proposition, AB is to BK, as BC is to CK: hence on rearranging, as BK is to CK, thus AB is to BC, and on dividing, AM to MB or BC is as BC is to CK. Which shows the first part.

[AB/BC = BK/CK; AM/BC = BC/CK].

Again as AM is to BC, or MB, thus BC is to CK, then on adding and inverting, as AB is to MB, thus BK is to CK; and converting and inverting, as AM to AB, thus BC to BK. Which shows the second part.

[AM/MB = AM/BC = BC/CK; AB/MB = BK/CK ; MB/AB = CK/BK; AM/AB = BC/BK.]

Again since AM is to AB, thus BC is to BK, then on interchanging, as AM is to BC, or MB, thus AB is to BK, and on inverting and adding, and again inverting, as AM is to AB, thus AB is to AK. Which was to be shown in the third place.

L2.§2.   *PROPOSITIO LXXXIV.*

A   B   C   D   E   K

M   N   O   P   R
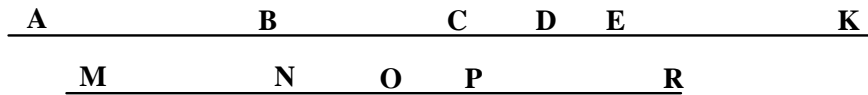
Prop.84. Fig. 1.

Datae sint binae series AK, MR (quas similes licebit appellare) magnitudinum continue proportionalium, in eadem proportione.

   Dico seriem totam AK, esse ad seriem MR ut AB primus terminus seriei AK, ad MN, primum terminum seriei MR.

Demonstratio.

   Per octavagesimam secundam huius, AK est ad BK, ut AB ad BC, hoc est , (cum ex hypothesi AB ad BC, MN ad NO, similes sint rationes) ut MN ad NO; sed per eandem etiam est MR ad NR, ut MN ad NO, igitur AK est ad BK ut MR ad NR. unde per conversionem rationis AK est ad AB, ut MR ad MN: & permutando series AK, est ad seriem MR, ut primus terminus AB primae seriei, ad primum terminum MN secundae seriei; quod erat &c.

PROPOSITION 84.

   Two series of magnitudes in the same continued proportion are given (which can be called similar).

   I say that the whole series AK is to the whole series MR, as the first term AB of the series AK, is to the first term of the series MR.

Demonstration.

   By the eighty-second proposition of this book, AK is to BK, as AB to BC, or , (since by hypothesis AB to BC and MN to NO, are similar ratios) as MN to NO; but from the same theorem also MR is to NR, as MN is to NO, therefore AK is to BK as MR is to NR. hence on converting the ratios, AK is to AB, as MR is to MN: and by interchanging, the series AK is to the series MR, as the first term AB of the first series, to the first term MN of the second series; quod erat &c.

   [AK/BK = AB/BC = MN/NO; MR/NR = MN/NO; hence AK/BK = MR/NR, giving BK/AK = NR/MR and hence AB/AK = MN/MR or AK/AB = MR/MN. Hence AK/MR = AB/MN as required.]

L2.§2.   *PROPOSITIO LXXXV.*

Quamquam ex iis, quae hactenus universaliter demonstravimus propositione octogesimam, & octagesimam primam, nota sit proportio totius seriei, ad primam

magnitudinem; placuit tamen exercitii causa, notissimis quibusdam proportionibus applicare, maxime cum se illis saepe mentio futura sit.

Primum igitur data sit, quocumque in genere quantitatis, proportio dupla, AB ad BC.
Dico totam seriem proportionis huius, sine termino continuatae, constituere magnitudinem, quae dupla sit primae magnitudinis.

[101]
Demonstratio.

A _____ I _____ B _____ C ____ D ____ E _____ K

Prop.85. Fig. 1.

Fiat enim ipsi BC, aequalis BI, & fiat ut AI ad AB, sic AB ad AK; erit K terminus ^{a} rationis AB ad BC, semper continuatae. & quoniam BA, dupla est BC, estque BI aequalia BC, erit quoque BA dupla IA; Quare cum sit ex constructione ut BA ad IA, sic AK ad AB, erit etiam AK, (id est tota series rationis AB ad BC) dupla BA, primae magnitudinis. Quod erat &c. *a 80 huius*.

Nevertheless from these propositions, which we have demonstrated so far entirely from propositions eighty and eighty-one, the proportion of the whole series to the first term is to be noted. The reason for this being the case is still to be determined; however from certain of the more noteworthy proportions to be considered, this one ratio is referred to most often.

PROPOSITION 85.

The first term is given, and the proportion of AB to BC is two to one for all the terms in the series.
I say that the sum of this series of terms in proportion, continued without end, is double in size to the first term.

Demonstration.

Indeed BC is made equal to BI, and as AI is to AB, thus AB is to AK; K is the end of the ratios AB to BC ^{a}, always continued. Since BA is double BC, and BI is equal to BC, BA is also double IA. Whereby as from construction, as BA is to IA, thus AK is to AB, and AK (which is the sum of the series AB to BC) is double BA, the first term. Q. e. d. *a 80 huius*.

[This proposition is merely an illustration of Prop. 81. It is of interest to compare the geometrical result of Gregorius with the algebraic infinite sum formula for a G. P.: $S = a/(1 - r)$, where a = AB, and the common ratio r = BC/AB; for AB/BC = AK/BK, hence BC/AB = BK/AK, and AB/AK = (AB - BC)/AB; or $a/S = a(1 - r)/a$,
giving $S = a/(1 - r)$; In the present case, AB - BC = AI, and AI/AB = $^1/_2$, hence AK = 2.AB as required.]
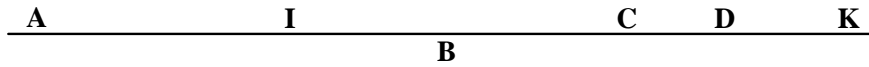
L2.§2.   *PROPOSITIO LXXXVI.*

Detur deinde proportio tripla, AB ad BC.
Dico totam seriem , fore sesquialteram primae magnitudinis.

Demonstratio.

```
A                  I                    C      D      K
                   B
```

Prop.86. Fig. 1.

Fiat enim secundae magnitudini BC, aequalis BI; erit ergo BI, tertia pars AB; & AI excessu, seu differentia primae & secundae; unde si fiat ut AI ad AB, sic AB ad AK; erit AK [b] aequalis toti seriei; & quia IB, tertia pars est BA, erit BA sesquialtera ipsius AI : quare cum sit ut BA AI, sic AK ad AB, erit quoque AK, sesquialtera primae magnitudinis AB. Quod erat demonstrandum. *b 80 huius*.

PROPOSITION 86.

In the next case, and the proportion of AB to BC is three to one.

I say that the sum of the series is one and a half of the first term.

Demonstration.

Indeed the second term BC is made equal to BI, and hence BI is equal to a third part of AB, and AI the difference of the first and second terms. Thus if AI to AB is made in the ratio AB to AK, then AK [b] is the sum of the whole series. Since IB is the third part of BA, BA is $^3/_2$ of AI : whereby as BA is to AI, thus AK is to AB, also AK is $^3/_2$ of AB. Q. e. d.

L2.§2.   *PROPOSITIO LXXXVII.*

Denique proportio data sit quadrupla, AB ad BC.

Dico totam seriem esse sesquitertiam primae magnitudinis : sive eam habere rationem ad primam magnitudinem, quam quatuor ad tria.

Demonstratio.

```
A                    I      B      C   D  E      K
```

Prop.87. Fig. 1.

Fiat enim BC aequalis BI; erit ergo BI quarta pars AB, & consequentur BA sesquitertia ipsius IA. Fiat igitur ut AI ad AB, sic AB ad AK; erit AK [c] tota seriei. Quare cum KA sit ad BA, ut BA ad IA, erit KA sesquitertia primae magnitudinis BA. Quod erat demonstrandum. *c 80 huius*.

PROPOSITION 87.

In the next case, the proportion of AB to BC is four to one.

I say that the sum of the series is one and a third of the first term : or the sum of the series to the first term has the ratio four to three.

Demonstration.

Indeed BC is made equal to BI, and hence BI is equal to a quarter part of AB, and AB is one and a third of IA. Thus if AI to AB is made in the ratio AB to AK, then AK [c] is the sum of the whole series. Whereby since KA is to BA, thus BA is to IA; AK is the four thirds part of BA. Q. e. d. *c 80 huius*.

[Again, AB/BC = AK/BK or (AB - BC)/AB = AB/AK; but BC = IB = $^1/_4$AB, hence AI = $^3/_4$AB and AK/AB = $^4/_3$ as required.]

Sed de his modo satis: poterit enim quilibet ex propositione ex octuagesima huius bene intellecta, data proportionis rationalis, seu numeri ad numerum, totam seriem, & consequnter rationem seriei, ad primum terminum exhibere.

<div align="center">

Scholium.

</div>

Quod si constructionem secundam octuagesimae propositionis adhibere placuerit, habebitur unica operatione proportio primae magnitudinis ad reliquam seriam: si vero prima constructione utamur, exhibebitur proportio primae & secundae magnitudinis simul sumptarum, ad seriem reliquam.

Praesens materia memorem me facit eius, quod in argumento huius libri praefatus sum, cum mentio incideret Zenonici discursus, quo se credebat omnem motus rationem e medio tollere posse: nucleus autem argumenti tanta apud auctorem authoritatis exstitit, ut eundem Achillis invictissimi Ducis nomenclatura dignaretur : persuasum habens, cum nucleum usque ad eo duro tortice futurum, qui par foret omnibus Philosophicorum Elenchorum malleis sustinendis.

Repetam discursum Zenonis, iisdem verbis, quibus in praefatione huius libri sum usus. consistebat ille in duobus, quae moverentur : primo Achillis, velocissime currentis, altero testudinis tardissime reptantis,

<div align="center">

[102]

</div>

A   B   D   E   C

<div align="center">

Prop.87. Fig. 2.

</div>

Ponatur inquiebat ille, Achilles cursor pernicissimus, ex A puncto testudinem reptantem per semitam BC, lentissimo motu, velle assequi; quo tempore Achilles tendit ex A ad B, mota est testudo ad aliquod spatium, per veniens in D : igitur necdum Achilles assecuties est testudinem; iterum quo tempore Achilles ex B currit, ut assequatur testudinem existentem in D, mota est testudo ad F punctum; igitur Achilles existens in D nondum assecutus est testudinem; atque hoc in infinitum eveniet ; quoniam continuum divisibile est in infinitum, unde numquam Achilles assequetur testudinem. Incumbit igitur nobis hunc nucleum effringere, ex doctrina huius libri; quod assecutos nos esse cognosces, cum ipsissimum punctum assignaverimus, quo Achilles testudinem apprehendre.

Ut nodum hunc Gordium, ex principiis huius libri dissoluamus, supponemus, non minus Achillem, quam testudinem in suo cursu uniformiter procedure; ita ut celeritas, prima parte motu assumpta, perseveret in eodem statu, usque ad ultimum temporis momentum quo suae spatia decurrunt; supponemus insuper, (quoniam omnis motu species est quantitatis) duos hosce motus, cum uniformes ponantur, insuis partibus, sortiri inter se aliquam proportionem, quod necesse est eveniat inter omnes quantitates, quae in eadem specie versantur; ut sunt duo motus recti & uniformes.

Ponatur igitur proportio duorum horum mobilium, secundum celeritatem, consistere in ratione dupla; ita ut Achilles duplo celerius, spatium decurrat, quam testudo : igitur quo tempore testudo ad quartem partem stadii promota fuerit, medium stadium confecerit Achilles. Eductae itaque lineae AC, DC, in ratione dupla ex C puncto, dividentur in B, F, H, &c, & E, G, I secumdum rationem duplam ;ut AC dupla, sit BC, & DC dupla EC. Item BC dupla FC, &c. EC dupla GC, &c.

A   B   F   H   C
         D   E   G   I

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Prop.87. Fig. 3.

</div>

Consistat itaque Achilles in A. sitque AC semitam repraesentans stadio longitudine aequalem; testudo vero constituta in dimidio stadii, in puncto B, vel D, posita DC aequali ipsi BC. Quoniam Achilles ex A moveri incipit, quo tempore ex D inchoat cursum testudo; igitur pervenerit Achilles ex A in B, quo tempore ex D, testudo pertinget in E : & quo Achilles ex B pertinget in F, eadem pervenerit testudo ex E in G. & sic consequenter: quia vero terminum progressionis rationis AB ad BF terminatur in C, prout propositione octuagesima quinta demonstratum est; similiter cum progressio, sucundum rationem BF, ad FH, vel DE ad EG, finem sortiatur in puncto C, secundum eandem proportionem: igitur concursus duorum horum

mobilium, Achillis scilicet & testudinis, continget in puncto C: Quod si loco proportionis duplae, assumatur proportio tripla, tunc assignabitur concursus per proportionem octaugesimam sextam hiuis. Si vero quadrupla, inserviet propositio octuagesima septima, & sic de reliquis.

Captiosus Zenonis discursus molestias creat non consideranti discrimen, quod in eo exsurgit, inter duplicem progressionem, qua argumentationis filum dubium facit; alia enim est progressio per partes aequales; alia per partes proportionales; hic utriusque cursus supponitur fieri per partes uniformes, sive per passus aequales, cum passus primus a secundo, vel tertio non discrepet, licet duos passus Achilles, verbi gratia, eadem contingant tempore quo unus passus testudinis; secundum vero hos passus sit utriusque

A   B   C   D   E

Prop.87. Fig. 4.

cursus : Zeno autem in decursu argumenti sui, distinguit motus cursorum per partes proportionales, secundum quas mobilia nullo modo moventur; ac proinde in idem eius discursus recidit, ac si dicat quis, eo tempore quo dividam lineam AE, in partes quatuor aequales, alius eam subdividet secundum aliquam seriem per partes proportionales, profecto citius assignabantur termini quatuor partium aequalium, qua infiniti termini partium proportionalium: Achilles enim & testudo decurrentes AE spatium, per partes aequales, suorum passuum aequalium terminum tandem acquirunt; Zeno vero dum hac contingunt, a cursoribus dividi vult spatium AE, in partes proportionales, secundum quas mobilia non succedunt.

[103]

Ad argumentum porro respondendum est, dum dicitur : Priusquam Achilles ex A perveniat ad punctum B, mota est testudo ex B in F:

A   B   F   H   C

Prop.87. Fig. 5.

Sensum huius propositionis coincidere cum hoc quo dicatur, prius debet Achilles assignare punctum B, quam notet punctum F. quid repugnat cursui secundum rationem motus; nam omnis assignatis in hac materia continet rationem subsistentiae, ut Mathematici sentiant, saltem secundem intellectam, ac proinde alicuis quietis, quae motui repugnat. Veram hac in gratium Philosophorum dicta sufficiant.

But enough has been said about this method : for anyone with a good understanding of Proposition 80 of this work can find the sum of the whole series, and consequently the ratio of the sum to the first term, either geometrically for a given ratio of proportion, or numerically in terms of one number to another.

Scholium.

For if one is resolved to put the second construction of the 80th proposition to use, one has in a single operation the proportion of the first term to the rest of the series : while indeed if we use the first construction, the proportion of the first two terms taken together to the rest of the series is found.

The present material brings to mind the argument that I have set out in the preface of the present book, concerning the discourse of Zeno, where he set out that all motion could be understood in terms of the average motion: but the heart of the argument really arose from the authority of the author, in order that a runner by the name of Achilles, the same as that most invincible leader, is deemed worthy to be persuaded to do battle with the hard- shelled one, for Zeno was an equal of the Elenctic philosophers [Those given to refuting or cross-examining matters], and sustained by their might .

[*Note: I have paraphrased somewhat here, as the original appears to be a pun on the word 'nucleus', which is the nut or kernel, while it is also the centre of the argument, and which cannot be reproduced sensibly.*]

Repeating Zeno's argument, in the same words that I used in the preface of this book. The argument consists of two opponents who are in motion : first there is Achilles, who is running swiftly, and with the other a tortoise who is crawling most slowly. It may be said that Achilles, that most nimble of runners, is placed at the point A initially, and wishes to overtake the tortoise who crawls along the path from BC most slowly. By the time Achilles has gone from A to B, the tortoise has moved some distance to come to the point D, and Achilles has not yet overtaken the tortoise. Again, by the time Achilles has run from B to D, in

order to overtake the tortoise present at D, the tortoise has moved to some point F; hence Achilles, now present at D has not yet overtaken the tortoise; and this will be the outcome indefinitely. Since the division process can be continued indefinitely, thus Achilles can never overtake the tortoise. It is incumbent therefore for us to break open this nut from the teaching of this book; since we know that Achilles overtakes the tortoise, we can assign the very point itself at which Achilles overtakes the tortoise.

In order that we can resolve this Gordian knot from the principles of this book, we can assume that not less than Achilles, the tortoise is to continue in its path at a uniform rate thus, so that the speed for the first part of the motion is assumed to continue in the same state until the final moment of time when their distances [from A] are equal. We assume above, since any kind of motion can be specified by a quantity, with these two motions made uniform in their own intervals, that some proportion can be chosen between the intervals, which necessarily comes about between all the quantities, as the two motions are along a straight line and uniform.

Therefore the proportion between these two moving intervals, following the speed, is taken in the ratio two; thus Achilles runs with twice the speed as the tortoise: hence in the time taken for the tortoise to move forwards by a quarter length of the course, Achilles has made it as far as the middle. Hence the lines AC and DC drawn from the point C, are divided in the ratio two to one by the points B, F, H, etc. and E, G, I, etc., in order that AC is double BC, DC is double EC; likewise, BC is double FC, EC is double GC, etc.

So it is agreed that Achilles is at the point A, and AC is equal to the length of the course, representing the path. The tortoise is set in place in the middle of the course at B or D, where DC is equal to BC. Achilles begins to move from A at the same time as the tortoise starts the course from D. Therefore in the time taken for Achilles to go from A to B, the tortoise reaches E from D; and in the time Achilles reaches F from B, the tortoise arrives at G from E, and thus consequently. Since the end of the progression of the ratio AB to BF ends in C, as has been shown in Prop. 85; similarly with the progression following the ratio BF to FH, or DE to EG, the end is chosen at the point C for the intervals following the same proportion. Therefore the two motions of Achilles and the tortoise run together to meet at the point C. But if in the place of the duplicate proportion, the triplicate proportion is assumed instead, then the point of concurrence is assigned by Prop. 85 of this book, and if the proportion is the quadruple, then Prop. 87 takes care of the point, and thus for the remainder.

Zeno's deceptive discourse gives rise to troubles by not discriminating between two series that arise in it, which leads to a doubtful thread in the argument. For one series is a progression by equal parts [or intervals], while the other is by proportional parts. Here each course is supposed to be made in uniform intervals, or in equal steps from the first to the second, or the third without being out of step. For example, Achilles is allowed to have two steps that occur in the same time the tortoise has one step [of equal size], and the other course is indeed following these steps. However Zeno in the discourse of his argument, has distinguished the motion of the courses by proportional parts, following which the moving objects are not moved in any way [in that the time intervals tend towards zero]; and according to his discourse the same interval is diminished, and if one should say by what [amount], it is initially by that time in which the line AE has been divided into four equal parts, while the other is subdivided following some series of proportional parts; indeed the four boundaries of equal parts can be assigned more quickly than an infinite number of proportional parts. For Achilles and the tortoise are hurrying across the distance AE in steps of equal interval, and at last reach the end point, each in terms of their own equal steps: but Zeno, while they [actually] touch here, wants their courses in the distance AE to be divided into proportional parts, following which the objects in motion do not succeed in meeting.

When it is said: Before Achilles arrives at the point B from A, the tortoise has moved from B to F [Fig.5]; the response to the argument should be: The understanding of this proposition is to agree with that which is stated. Before Achilles can be assigned to the point B, the point F is noted; which disagrees with a course that follows [only] the ratio of the motion. For all intervals assigned in this matter contain a sustaining ratio, [i.e. the difference of the distances gone], as mathematicians think they have an understanding of the second method, and hence are quiet about other things which are in disagreement with the motion. Truly, with the grace of the philosophers, enough has been said here.

[It is not the business of the translator to interject his own ideas into the discourse of Gregorius. However, we should note that the paradoxes of Zeno are still with us, and lie at the heart of the limiting process. They were produced far in advance of a time when any proper understanding could be forthcoming; for Zeno was upset by the idea that an infinite number of subdivisions had to be gone through before the moving objects could meet. Gregorius some 2000 years later dismissed these objections, and

said essentially: Look, you are going about this problem in the wrong way: here is how you should think about it, in terms of a geometric progression; I can even sum the series of infinite intervals for you and find the point where the bold Achilles overtakes his adversary. Now we realise that both men had a valid point to make in an understanding of the limiting process.]

L2.§2.   *PROPOSITIO LXXXVIII.*

Data AB, cuius tertia pars sit CB; fiat ut tota AB, ad CB tertiam sui partem, ita CB ad CD, & CD ad DE, & DE ad EF, atque ita semper.

Dico K terminum huius progressionis, bifariam dividere propositam magnitudinem AB.

Demonstratio.

A   **KF E D   C**   **B**

Prop.88. Fig. 1.

Erit enim ex hypothesi AB cum BK, aequalis toti seriei proportionis triplae. atqui tota series rationis triplae, [a] sesquialtera est magnitudinis primae, ergo AB cum BK, sesquialtera est primae magnitudinis AB; ergo BK, est eiusdem dimidia. Quod erat demonstrandum. *c 86 huius*.

PROPOSITION 88.

Given the line AB, the third of which is CB; the progression is constructed such that the whole length AB to CB is three parts to one, and thus CB to CD, CD to DE, DE to EF, and so on in the same ratio.

I say that the end point K of this progression divides the magnitude AB in two equal parts.

Demonstration.

For indeed AB and BK by hypothesis is equal to three times the sum of the whole series. But the whole series of ratios tripled [a] is one and a half times the first, hence AB plus BK is one and a half of the first magnitude AB; hence BK is the half of AB. Q. e. d. *a 86 huius*.

[From Prop. 86, the sum of the whole series is $1^1/_2$ times the first term; BK or S = $^3/_2$BC; for: AB + BK = 3.BC + BK = 3.BC + $^3/_2$. BC = $^9/_2$. BC = 3.S. Hence AB + S = 3.S, or S (or KB) = AB/2.]

L2.§2.   *PROPOSITIO LXXXIX.*

Detur quantitatis AB, cuius quarta pars sit BC; fiat ut tota AB, ad quartam sui partem BC, ita BC ad CD, & CD ad DE, & DE ad EF, atque ita semper continuando, punctum K hiuis progressionis terminus emergati:

Dico BK esse tertiam partem propositae quantitatis AB.

Demonstratio.

| A | | KFED   C | | B |

Prop.89. Fig. 1.

Nam ex hypothesi AB cum BK, est tota series proportionis quadruplae. Atqui tota series rationis quadruplae, [b] est ad primam magnitudinem duem, ut quatuor ad trias, ergo AB cum BK ; est ad AB , ut quatuor ad tria. & dividendo KB ad AB, ut unum ad tria, hoc est KB tertia pars est ipsius AB. Quod erat demonstrandum. *b 87 huius*.

PROPOSITION 89.

Given the amount AB, the quarter of which is BC; the total is constructed such that the whole length AB to BC is four parts to one, and thus BC to CD, CD to DE, DE to EF, and always to be continued thus, until the end of the progression appears.

I say that BK is the third part of the proposed quantity AB.

Demonstration.

For by hypothesis AB and BK is four times the sum of the whole series of proportions. But the whole series of ratios quadrupled [b] to the first term is in the ratio two to one, hence AB plus BK is to AB, as four is to three, and by division, KB to AB, as one is to three. Hence KB is the third part of AB. Q. e. d.

a 87 huius.

[From Prop. 87, the sum of the whole series is $1^{1}/_{3}$ times the first term, or BK or $S = {}^{4}/_{3}BC$; for: $AB + BK = 4.BC + BK = 4.BC + {}^{4}/_{3} . BC = {}^{16}/_{3}. BC = 4.S.$ Hence $AB + S = 4.S$, or S (or KB) $= AB/3$.]

L2.§2.   *PROPOSITIO XC.*

Datam magnitudinem AK, in duobus punctis B, C, dividere, ut AB ad BC, habeat rationem datam, G ad H. Ex proportionis AB ad BC continuatae progressio terminetur in K.

[104]
Constructio &Demonstratio.

	G	
	H	

| A | | B | C D E F | | K |

Prop.90. Fig. 1.

Divide AK in B, ita ut sit AK ad BK, ut G ad H; & ut AK ad BK, sic AB fac ad BC. Dico factum esse quod petebatur. cum enim sit ut AK ad BK, sic AB ad BC, erit & reliquum BK [a] ad reliquum CK, ut AK ad BK: unde rationis AB ad BC [b] terminus est K : est autem AB ad BC ratio eadem cum ratione AK ad BK, it est G ad H. constat igitur propositum. *a 19 Quinti; b 79 huius*.

Corollarium.

Ex hac propositione manifestum est, omnem magnitudenem, omnes rationum seriei continere.

PROPOSITION 90.

The given magnitude AK is divided by the two points B and C, in order that AB to BC has the given ratio G to H. The progression of the continued proportion AB to BC is terminated in K.

Construction &Demonstration.

Divide AK in B, so that the ratio AK to BK is as G to H; and as AK is to BK, thus make AB to BC. I say that what is sought has been done. For indeed as AK is to BK, thus AB is to BC, and the remainder BK [a] to CK, as AK to BK: hence the termination of the ratio AB ad BC [b] is the point K : but the ratio AB to BC is the same as the ratio AK to BK, that is G to H. Therefore the proposition is agrees upon. *a 19 Quinti; b 79 huius*.

Corollary.

It is observed that every magnitude of all ratios of a series are contained in this proposition .

L2.§2.　　　　　　*PROPOSITIO XCI.*

Datis quotcumque rationibus A ad B, C ad D, E ad F, datam magnitudinem LO, oporteat dividere in tot partes, quot datae sunt rationes, nempe in LM, MN, NO, ita ut partes illae eandem habeant rationem, quam primi datarum rationum termini, A, C, E, & praeter ea singulae partes LM, MN, NO singulis rationibus, sine termine continuatis, sint aequales.

Constructio &Demonstratio.
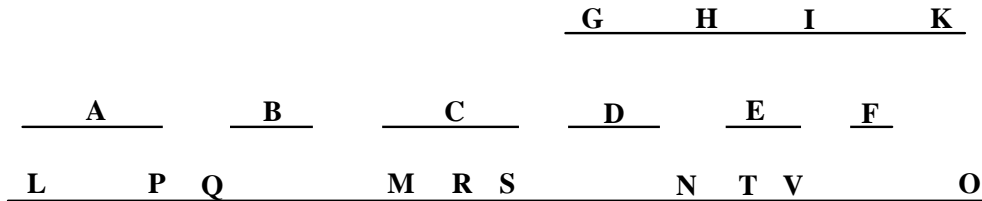
Prop.91. Fig. 1.

Fiat GHIK, omnibus A, C, E, aequalis, & ut divisa est GK, sic divide LO in M & N; denique per 90 huius LM ita divide in P & Q, ut sit LP ad PQ sic ut A ad B: & rationis LP ad PQ, terminus sit M, similiter MN, NO divide in punctis R, S, T, V secundum rationes C ad D, E ad F, ita ut rationum MR ad RS, NT ad TV, termini sint N & O. Dico factum quod petebatur. Demonstratio ex ipsa constructione est manifesta.

PROPOSITION 91.

For some given ratios A to B, C to D, E to F, it is required to divide a given magnitude LO into as many parts as the ratios give, surely in LM, MN, NO, thus as the parts in each section have the same ratio as the first of the given terms of the ratio A, C, E, and beyond that the individual parts are equal to the individual ratios within LM, MN, and NO which are to be continued without end.

Construction &Demonstration.

Let GHIK be made equal to all the first parts of the ratios A, C and E. Thus as GK is divided, so divide LO in M and N; hence by Prop. 90 of this book, LM is thus to be divides in P and Q, in order that LP to PQ thus is as A is to B: & the termination of the ratios LP to PQ is M; similarly MN and NO are to be divided in the points R, S, T, V following the ratios C to D, and E to F, thus in order that the terminations of the ratios MR to RS, and NT to TV, are the points N and O, I say that what is sought has been done. The demonstration is clear from the construction itself.

L2.§2.   *PROPOSITIO XCII.*

Datis quotcumque rationibus AB ad BC, DE ad EF, GH ad HI, &c. magnitudinem invenire, quae omnes harum rationum, series progressionum adaequet.

 A   B   C   D   E   F   G   H I

Prop.92. Fig. 1.

Constructio &Demonstratio.

Per octogesimam huius invenientur magnitudines, quae singularum rationum series adequent, atque omnibus illis magnitudinibus, ita fiet aequalis, hac ut patet, adeaequibit omnes series datarum rationum.

PROPOSITION 92.

For some given ratios AB to BC, DE to EF, GH to HI, etc. , to find the magnitude of the sum of the series of the progressions formed from these ratios.

Construction &Demonstration.

By Prop. 80 of this work, the magnitudes can be found to which the individual series of ratios are equal, and the sum of all of these magnitudes, as is apparent, is made equal to the sum of the series of the given ratios. [Again, the print is almost entirely illegible.]

[105]

L2.§2.   *PROPOSITIO XCIII.*

Datas quotcumque series diversarum rationum ita constituere, ut sint in continua analogia datae proportionis.

 A   B   C

 D   E   F

 G   H   I

    Y   Z

 K Q T   L   M   R V   N   O S Z   P
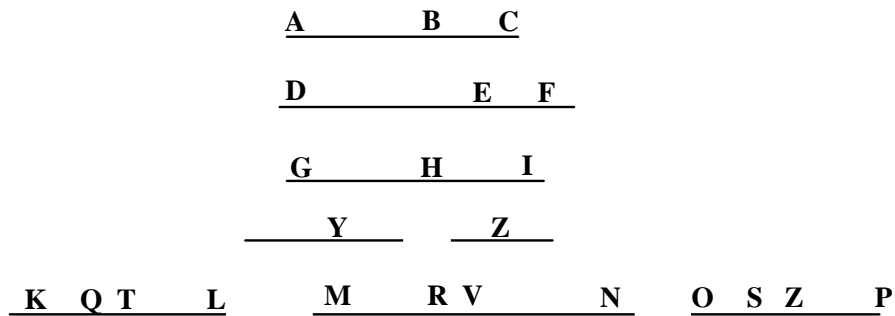
Prop.93. Fig. 1.

Constructio &Demonstratio.

Rationes diversae sint AB ad BC, DE ad EF, GH ad HI. & alia dato ratio Y ad Z. Fiant tres magnitudines
KL, MN, OP continue proportionales, in ratione Y ad Z, & KL ita *ᵃ* dividatur in QT, &c. ut seriem
constituunt rationis AB ad BC; & MN ita dividatur in RV, &c. ut adaequet seriem rationis DE ad EF : at
demum dividatur similiter OP in S & Z, ut rationis GH ad HI constituunt seriem, factumque erit quod
petebatur. *a 90 huius*.

PROPOSITION 93.

You are given some series of different ratios thus to be put in place, so that they are
continued ratios of a given proportion.

Construction &Demonstration.

The different ratios are AB to BC, DE to EF, GH to HI; and with the other given ratio Y to Z. Three
magnitudes KL, MN, and OP are made in continued proportion, in the ratio Y to Z; KL is thus *ᵃ* divided in
Q, T, etc, in order that AB to BC constitute a series of the ratio ; again MN is thus divided in R,V, etc, in
order that DE to EF is equal to a series of the ratio : and finally OP is similarly divided in S and Z, in order
that GH to HI constitute a series of ratios. What was required has been accomplished. *a 90 huius*.
[In modern terms, let the common ratios for the three lines be *r, s*, and *t*; while the fourth ratio is *R*. The line
KL can be chosen at will, but MN = *R*.KL, and OP = R^2.KL; KL is the sum of the series with ratio *r*, and
likewise MN and OP are the sums for the ratios *s* and *t*. Thus, the first terms KQ, MR, and OS are chosen to
accommodate the ratios *r*, *s*, and *t*, and the sums KL, MN, and OP]

L2.§2.   *PROPOSITIO XCIV.*

Datam magnitudinem AE, semel sectam in B, ita rursum secare in C & D, ut tam
progressio rationis AB ad BC, quam progressio rationis AD ad DB, terminetur in E.

| A | D | B | C | E |

Prop.94. Fig. 1.

Constructio &Demonstratio.

Reperiatur primo ipsius AE, BE, tertia proportionalis CE. Dico progressionem AB, BC terminari in E. patet
ex septuagesima nona huius. Deinde inter AE, BE inveniatur media DE. patet rursum progressionem AD,
DB terminari in E, per eandem propositionem : fecimus ergo quod petebatur.

PROPOSITION 94.

The given magnitude AE is cut once in B, and thus cut again in C and D, so that the
progression of the ratio AB to BC, as for the progression of the ratio AD to DB, is
terminated in E.

Construction &Demonstration.

First the third proportional CE of AE and BE is itself found. I say that the progression AB, BC is to
terminate in E, as is apparent from Prop. 79 of this work. Then the mean DE is found between AE and BE;
and again it is apparent that the progression AD, DB is to end in E, by the same proposition : we have done
what was asked. [In this case, there are two progressions: the first is produced from the ratio
BE/AE = CE/BE from AB/BC = AE/BE; while the second arises from the ratio DE/AE = BE/DE, etc.]

L2.§2.   *PROPOSITIO XCV.*

Data magnitudine DG, & proportione maioris inaequalitatis A ad B, itemque alia
magnitudine C: examinare quomodo series progressionis datae rationis A ad B, cuius
primus terminus sit C, se habeat ad datam DG magnitudinem.
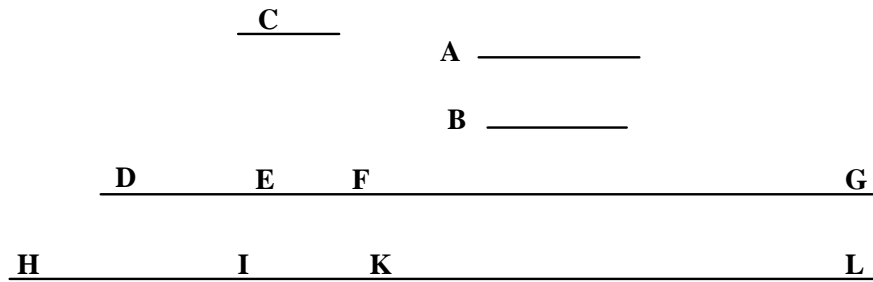
Prop.95. Fig. 1.

[106]
Constructio &Demonstratio.

Fiat ut A ad B, sic DG ad EG. Primo igitur data magnitudo C, quae primus terminus progressionis esse debet, aequalis sit DE, erit series rationis A ad B habens primum terminum C aequalis datae magnitudini DG nam fiat DE (id est C ad EF, ut A ad B). quoniam igitur DG est ad EG ut A ad B, erit quoque DE ad EF ut DG ad EG. ergo progressio *a* rationis DE ad EF, (id est progressio rationis A ad B, habens primum terminum C) constituet magnitudinem DG.

Secundo si magnitudino C, quae debet esse primas terminus, maior sit quam DE, erit progressio AB, habens primum terminum C, maior quam DG. Fiat enim ipsi C aequalis HI, utque A est ad B, sic HI sit ad IK, & progressionis HI ad IK *b* reperiatur terminus L. Igitur *c* ut HI ad DE sic HL tota series progressionis HI, IK ? ad DG totam seriem progressionis DE, EF. atqui HI maior est quam DE, ergo HL maior est quam DG.

Si denique C minor sit quam DE, erit quoque progressio rationis A ad B, habens primum terminum C, minor quam DC; quod eodem modo ostendimus, quo primum. Fecimus igitur quod poscebatur. *a 79 huius; b 80 huius; c 84 huius.*

PROPOSITION 95.

For the given magnitude DG, and with the proportion of the inequality A greater than B, and likewise with another magnitude C given: to examine in what manner the series of a given progression in the ratio A to B, of which the first term is C, can itself have the given magnitude DG.

Construction &Demonstration.

Thus DG to EG is made in the ratio A to B. In the first place therefore the magnitude C, which must be the first term of the progression, is equal to DE . There is thus a series of the ratio A to B having the first term C, to which DE is equal, which is equal to the given magnitude DG (i. e. C to EF, is as A to B). Therefore since DG is to EG as A to B, also DE is to EF is as DG to EG. Therefore the progression of *a* the ratio DE to EF, (i. e. the progression of the ratio A ad B, having the first term C) constitutes the magnitude DG. [DG is given, as is C and the ratio A/B; hence A/B = DG/EG = DE/EF = C/EF].

In the second place the magnitude C, which must be equal to the first term of the series, is greater than DE; in this case there is a progression in the ratio A to B, having the first term C, which is greater than DG. For C is itself made equal to HI, and thus A is to B, as HI is to IK, and the last point or terminus L is found of the progression HI to IK *b* . Therefore *c* as HI to DE thus HL to the whole series of the progression DG: HL to DG, the sum of the series of the progression DE to EF. But HI is greater than DE, hence HL is greater than DG.

Hence if C is less than DE, there is also a progression of the ratio A ad B, having the first term C, which is less than DG; which we can show by the same method as the first. We have done what was requested. *a 79 huius; b 80 huius; c 84 huius.*

L2.§2.   *PROPOSITIO XCVI.*

<u>A   F G   B   C   H I   D   K   L K   E</u>

Prop.96. Fig. 1.

Data sit progressio rationis AF ad FG, terminata in B, & alia magnitudino CE: Oportet in magnitudine CE, ita utrimque ad C & E continuere progressionem rationis AF ad FG (progressiones, nempe CH, HI, &c. & EK, KL, &c.) ut eundem habeant terminum D, qui ita dividat CE, ut AB, CD, DE, sint in continua analogia.

Constructio &Demonstratio.

Rectum CE ita divide per trigesimam sextam huius ut AB, CD, DE sint continue proportionales. Deinde per nonagesimam huius ita divide CD, in H & I, ut ratio CH ad HI, eadem cum ratione AF ad FG, ac simul progressio CH, HI, terminatur in D. Idem facito in ED. Factumque erit quod petebatur.

PROPOSITION 96.

The progression of ratios AF to FG is given, to terminate in B, and also another magnitude CE: it is required in the length CE from C and E, to continue a progression of the ratio AF to FG thus from both ends of the line (the progressions are namely CH, HI, etc. & EK, KL, etc.) in order that they have the same termination D, which thus divides CE so that AB, CD, DE are in the continued ratio.

Construction &Demonstration.

The line CE is thus to be divided by Prop. 36 of this work in order that AB, CD, and DE are continued proportionals. Then by Prop. 90 of this work, CD is to be divided in H and I, in order that the ratio CH to HI is the same as AF to FG, and at the same time the progression CH, HI is terminated in D. In the same way make a series in ED. What was sought has been done.

L2.§2.   *PROPOSITIO XCVII.*

<u>   A   </u>   <u>   C   </u>   <u> E </u> <u> G </u>   <u> K </u>   <u> M </u>

<u>   N   </u>   <u> L </u> <u> I </u>   <u> H </u> <u> F </u> <u> D </u> <u> B </u>

Prop.97. Fig. 1.

Si datum seriem AM, BN termini A, B, C, D, E, F, G, &c. & termini in continua sint analogia.

Dico illas inter se eam proportionem habere, quam primi termini.

Demonstratio.

Quoniam A, B, C sunt tres continuae proportionales; ergo A est ad C, in duplicata ratione A ad B; iterum cum C, D, E fior tres continuae, C erit ad E, in duplicata ratione C ad D; hoc est, ut patet ex datis A ad B: ergo cum rationes A ad C;

[107]

& C ad E eiusdem duplicatae sint, erunt A, C, E tres continue proportionales in ratione duplicata A ad B. Quare series AM est series rationis duplicatae rationis A ad B : deinde quia B, C, D sunt continuae proportionales, erit ratio B ad D, duplicata rationis B ad C. similiter ostendemus, rationem D ad F, duplicatam esse rationis B ad C. continuae proportionales sunt igitur B, D, F. unde series BN, est series duplicatae rationis B ad C, id est ex datis, A ad B: similes igitur series sunt AM & BN. quare sunt [a] inter se, ut primi termini A & B. quod erat demonstrandum. *a 84 huius.*

PROPOSITION 97.

If the given series AM, BN of terms A, B, C, D, E, F, G, &c. are terms in a continued ratio.

I say that [corresponding terms of] these series have the same proportion between each other as the first terms [of the two series].
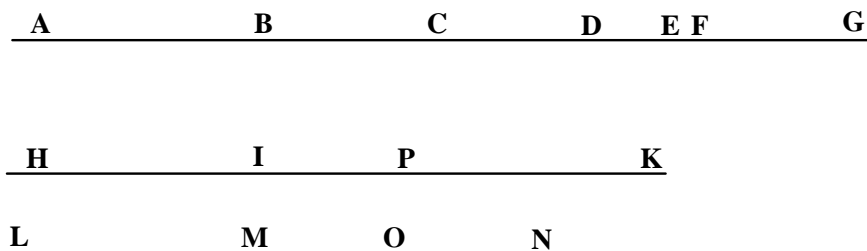
Construction &Demonstration.

Since A, B, C are three continued proportionals, then A to C is the square ratio of A to B; again with the terms C, D, E, I can make three continued proportions, and C to E is the square ratio of C to D. That is, the ratios A to C and C to E are the same squared ratios of A to B. Hence A, C, and E are three continued proportionals in the square ratio of A to B. Whereby the series AM is a series of the squares of the ratio A to B: then since B, C, D are continued proportionals, the ratio B to D is the square of the ratio B to C. Similarly we can show that the ratio D to F is the square of the ratio B to C. Therefore B, D, F are continued proportionals. Hence the series BN is a series of the squares of the ratio B to C, or of the given ratio A to B: the series AM and BN are therefore similar, [a] whereby the corresponding terms of the two series are in the ratio of the first terms of the two series A and B. *a 84 huius.*

[A/B = B/C= C/D=D/E=....= $1/r$, the common ratio in modern terms is r; then A/B .B/C = A/C = A^2/B^2 = $1/r^2$; similarly, C/E = C^2/D^2 = $1/r^2$; etc. Thus the original series $a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, ...$ is regrouped as AM or $a, ar^2, ar^4, ar^6, ...$ and BN or $ar, ar^3, ar^5, ar^7, ...$, in which the common ratio r is maintained between corresponding terms.]

L2.§2.   *PROPOSITIO XCVIII.*

A   **B**   **C**   **D**   **E F**   **G**

H   **I**   **P**   **K**

L   **M**   **O**   **N**

Prop.98. Fig. 1.

Continuetur ratio AB ad BC, habeatque terminum G. deinde fiat seriei rationis AB ad CD, aequali magnitudo HK: seriei autem rationis AB ad DF, aequalis LN.

Dico HK magnitedinem maiorem esse quam sit LN.

Demonstratio.

Quia HK est series AB, CD, &c. & LN est series AB, DE, &c.. sumantur ex HK, partes HI, IP aequales ipsis AB, CD: ex LN vero partes LM, MO aequales ipsis AB, DE. Quoniam igitur seriei HI, IP terminus est K, erit HK [b] ad IK, ut HI ad IP : & quia seriei LM, MO, &c. terminus est N, erit LN [c] ad MN, ut LM ad MO. sed HI ad IP minorem habet rationem, quam LM, id est HI ad MO. Ergo HK ad IKM minorem habet, quam LN ad LM, & per conversionem rationis KH ad HI, maiorem habet quam LN ad LM. ergo permutando HK ad LN, maiorem habet, quam HI ad LM. Quare cum ex const. HI, LM sint aequales, necesse est HK maiorem esse quam LN. Quod erat demonstrandum.

a 84 huius.

PROPOSITION 98.

The ratio AB to BC is continued [indefinitely] and has the termination [or endpoint] G. Then a series of ratios AB to CD is made to be equal to the magnitude [or length] HK; and also a series of the ratio AB to DE equal to the length LN.

I say that the magnitude HK is greater than LN.

Construction &Demonstration.

Since HK is the series AB, CD, &c. & LN is the series AB, DE, &c., the parts [or terms] HI and IP taken from HK are themselves equal to AB and CD: from LN likewise the terms LM and MO themselves equal to AB and DE. Therefore since the end point of the series HI, IP is K, HK [b] is to IK, as HI to IP : & since the end point of the series LM, MO, &c. is N, LN [c] is to MN, as LM is to MO. But HI to IP is a smaller ratio than LM (, or HI as both equal AB) to MO. Therefore HK to IK is less than LN to MN , & by the converse of the ratio KH to HI, is greater than LN to LM. [see note immediately below.] Therefore by interchanging, HK to LN is greater than HI to LM. Since from the construction, HI and LM are equal, it is necessary that HK is greater than LN. Q.e.d. *a 84 huius*.

[HI/IP = AB/CD = HK/IK ; and LM/MO = AB/DE = LN/MN ; but since CD > DE and LM = HI = AB, hence: the first set of equalities is less than the second set, and HK/IK - 1 < LN/MN - 1, or IH/IK <   LM/MN, inverting, we have IK/IH > MN/LM, and HK/IH > LN/LM, giving HK > LN as required.]

L2.§2.   *PROPOSITIO XCIX.*

A	B	C	D	E F G H I KL	M

Prop.99. Fig. 1.

Series rationis AB ad BC, terminetur in M; assumatur autem ex serie AM, terminus quicumque DE.

Dico rationis AB ad DE seriem, una cum serie rationis BC ad EF, ac serie rationis CD ad FG, aequari seriei rationis AB ad BC.

Demonstratio.

Quandoquidem omnes AB, BC, CD, DE, &c. in continua sint analogia, patet ex elementis AB, DE, GH, & sic in infinitum esse continue proportionales. Similiter BC, EF, HI, &c. itemque CD, FG, IK esse continue proportionales; ergo series trium rationum AB, DE, BC, EF, CD, FG, continuantur perpetuo; intra seriem AB, BC, CD, &c. ita ut in his tribus seriebus simul sumptis, nec plures, nec pauciores termini reperiantur, quam sint in serie AB, BC, CD, &c. manifestum igitur est has tres series, seriei AB, BC, &c. aequales esse. Quod erat demonstrandum.

PROPOSITION 99.

The series in the ratio AB to BC is to be terminated in M; but some term DE is taken from the series AM.

I say that the series of the ratio AB to DE, together with the series of the ratio BC to EF, and the series of the ratio CD to FG, is equal to the given series of the ratio AB to BC.
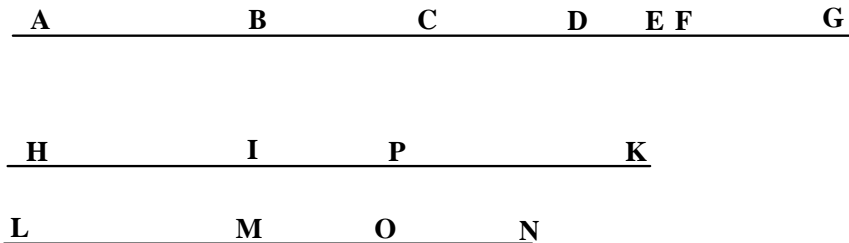
Demonstration.

Since all AB, BC, CD, DE, &c. are in continued proportion, it is apparent from elementary considerations that AB, DE, GH, & thus indefinitely are continued proportionals. Similarly, BC, EF, HI, &c. , and likewise CD, FG, IK are continued proportionals; therefore the three series of ratios AB, DE, BC, EF, CD, FG, are to continue indefinitely; within the series AB, BC, CD, &c. thus as with these three series taken together, neither more nor less terms can be found, than there are in the series AB, BC, CD, &c. It is therefore shown that the sum of the three series is equal to the series AB, BC, &c. Q.e.d.

L2.§2.   *PROPOSITIO C.*

Series rationis AB ad BC, terminetur in G; seriei autem rationis AB ad CD, aequalis sit HK; item seriei rationis BC ad EF, aequalis LN.

Dico seriem AG, duabus HK, LN, maiorem esse.

A	B	C	D	E F	G

H	I	P	K

L	M	O	N

Prop.100. Fig. 1.

Demonstratio.

Series BC, EF, &c. *ᵃ* minor est serie BC, DE, &c. ergo series BC, EF, &c. cum serie AB, CD, &c. minor erit, quam series BC,DE, &c. una cum serie AB, CD, &c. atqui per praecedentem series BC, DE, &c. cum series AB, CD, &c. constituit seriem AB, BC, CD, DE, &c. hoc est magnitudinem AG. ergo series BC, EF, &c. cum series AB, CD, &c. minorem constituit, quam AG: ergo HK, LN series aequales seriebus AB, CD, &c. BC, EF, &c. minores sunt, quam series AG. Quod erat demonstrandum. *a 98 huius.*

PROPOSITION 100.

The series of the ratio AB to BC is terminated in G; but HK is equal to a series of the ratio AB ad CD ; likewise LN is equal to a series of the ratio BC to EF.

I say that the series AG is greater than the two series HK and LN.

Demonstration.

The series BC, EF, &c. *ᵃ* is less than the series BC, DE, &c. Hence the series BC, EF, &c. summed with the series AB, CD, &c. is less than the series BC,DE, &c. together with the series AB, CD, &c. But by the

preceding proposition, the series BC, DE, &c. taken with the series AB, CD, &c. makes the series AB, BC, CD, DE, &c., or the magnitude AG. Hence the series BC, EF, &c. taken with the series AB, CD, &c. makes a smaller sum than AG: hence the series HK and LN summed together is equal to the series with the ratios AB, CD, and BC, EF, which is less than the sum of the series AG. Q.e.d. *a 98 huius.*

L2.§2.   *PROPOSITIO CI.*

Series continue proportionalium AB, BC, CD, &c. terminetur in L. Detur autem proportio α ad β, multiplicata rationis AB ad BC, iuxta datum aliquem numerum (quatuor exempli causa) & quot sunt unitates in dato numero, tot una minus fac magnitudines LM, OP, RS, aequales datae seriei terminis AB, BC, CD.

Dico series, rationis α ad β, quarum primi termini sint L, O, R, simul sumptas, constituere eandem magnitudenem, quam series rationis AB ad BC.

A	B	C	D E F G H I K L

L	M N	α β	(4)

O	P Q

R	S T
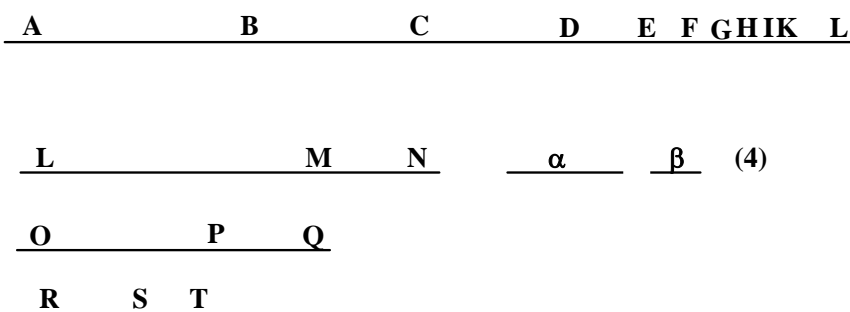
Demonstratio.

Fiat ut α ad β, sic LM ad MN, & OP ad PQ, & RS ad ST; erunt ergo omnes hae rationes quadruplicatae rationis AB ad BC : & quoniam LM, AB, aequales sunt, eandem habebunt rationem ad MN; ergo & ratio AB ad MN, quadruplicata est rationis AB ad BC; est autem & ratio AB ad DE, quadruplicata rationis AB ad BC. ergo ut AB ad DE, ita AB ad MN, aequantur igitur DE & MN: series igitur rationis LM ad MN, est series rationis AB ad DE. Similiter ostendam seriem rationis OP ad PQ, esse seriem rationis BC ad EF, & seriem rationis RS ad ST, seriem esse rationis CD ad FG, atqui tres *b* series simul sumptae, adaequant seriem rationis AB ad BC; ergo etiam & illae eandem adaequabunt. Quod erat demonstrandum. *b 99 huius.*

[109]
PROPOSITION 101.

The series of continued proportions AB, BC, CD, &c. is terminated in L. While a proportion α to β which is a multiple [or power] of the ratio AB to BC is given, such as some given number, e. g, four, and for whatever the given number chosen is, one less than this number gives e. g. the three magnitudes LM, OP, RS equal to the terms AB, BC, CD of the whole given series.

I say that the series in the ratio α to β, the first terms of which are L, O, R taken together, give rise to the same magnitude as the series with the ratio AB to BC.

Demonstration.

Thus LM to MN, & OP to PQ, & RS to ST are made in the ratio α to β ; therefore all these ratios are the quadruple [i. e. multiplied by themselves four times, or raised to the fourth power in modern terminology] of the ratio AB to BC : & since LM and AB are equal, they have the same ratio to MN; & hence the ratio to AB to MN is the fourth power of the ratio AB to BC; & whereas the ratio AB to DE is the fourth power of AB to BC, then it follows that AB is to DE thus as AB to MN, and hence DE & MN are equal. Hence

the series of the ratio LM to MN is the series of the ratio AB to DE. Similarly I can show that the series of the ratio OP to PQ is the series of the ratio BC to EF, & the series of the ratio RS to ST is the series of the ratio CD to FG, but the three [b] series added together are equal to the series of the ratio AB to BC; and hence also these are equal to the same. Q.e.d. *b 99 huius.*

L2.§2.   *PROPOSITIO CII.*

   Series rationis AB ad BC, continuatae, terminetur in K. Data autem sit LM aequalis AB, & quivis numerus (puta 3) deinde fiat ratio LM, ad MN multiplicata rationis AB ad BC, iuxta datum numerum ; sitque; rationis LM ad MN terminus O.
   Dico seriem AK, ad seriem LO, eandem habere proportionem, quam totidem termini seriei AB, BC, CD quot sunt unitates in dato numero, habent ad primum AB.

A	B	C	D	E F G H I Q K

L	M N	O	P	3
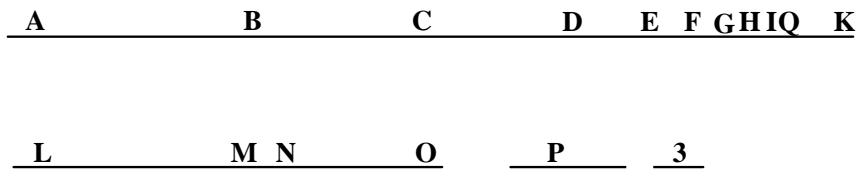
Prop.102. Fig. 1.

Demonstratio.

Cum, exempli causa, numerus datus ponatur ternarius; erit ratio LM ad MN; triplicata rationis AB ad BC; ostendendum nobis est, AK esse ad LO, ut tres primi termini DA ad primum AB. ex serie AK sume sex terminos AG: Igitur proportio AD [a] ad DG, triplicata est proportionis AB ad BC; aequalis igitur est rationi LM ad MN, hoc est [b] rationi LO ad MO. cum enim O sit terminus seriei LM, MN, erit LO ad MO, ut LM ad MN; Deinde quia K terminus est seriei AB, BC, erunt [c] tres AK, BK, CK continuae proportionales: adeoque omnes etiam sequentes erunt continuae: Quare & AK, DK, GK, inter quas par continue proportionalium numerus interijcitur, ex elementis patet esse continue proportionales. Unde AK, [d] est ad DK, ut AD ad DG, id est (quemadmodum iam ostendi) ut LO ad MO. Itaque per constructionem rationis AK est ad AD, ut LO ad LM, & permutando AK ad LO, ut AD ad LM, id est ex datis AB . Quod erat demonstrandum. *a 32 huius; b 82 huius; a 32 huius; c Ibid; d 1 huius.*

PROPOSITION 102.

   The series of the ratio AB to BC continued is terminated in K. While LM is given equal to AB, and some number, taken as 3, then makes the ratio LM to MN equal to this power of the ratio AB to BC, for this given number; and O is the terminus of the given ratio LM to LN.
   I say that the series AK has the same proportion to the series LO as the same number of terms AD (in the series AB, BC, CD, which is the number of units in the given number or power), has to the first term AB.

Demonstration.

For the sake of an example, the given number is put as three ; the ratio LM to MN is the triplicate of the ratio AB to BC [i. e. raised to the third power]. We must show that AK is to LO as the first three terms DA is to the first term AB. From the series AK six terms AG are taken: therefore the proportion AD [a] to DG is the triplicate [or third power] of the proportion AB to BC, which is therefore equal to the ratio LM to MN, or [b] to the ratio LO to MO; also, as O is the terminus of the series LM, MN: the ratio LO to MO is as LM to MN. Then, since K is the terminus of the series AB, BC, the three terms [c] AK, BK, and CK are in continued proportion: and thus also all the following terms are in continued proportion. Whereby AK, DK,

GK, between which the number of continued proportions is inserted, from elementary considerations are apparent to be continued proportionals. Thus AK ^{d} is to DK, as AD to DG, that is (as has now been shown) as LO to MO. Hence from the construction of the ratio, AK is to AD, as LO to LM, & on interchanging, AK is to LO, as AD to LM, or to the given AB. Q.e.d. *a 32 huius; b 82 huius; a 32 huius; c Ibid; d 1 huius.*

[We are given LM/LN = $(AB/BC)^3$; it is required to prove that AK/LO = DA/AB or DA/LM, or
(sum of whole series with ratio *r*) / (sum of series with ratio r^3) = (AB + BC +CD)/AB = $(a + ar + ar^2)/a =$
$1 + r + r^2$ in modern terms, from which the sum of the ratio r^3 follows.
Geometrically, AG/DG = $(AB/BC)^3$ = LM/MN = LO/MO; since K is the end-point of the whole series, all the terms are in continued proportion with a ratio *r*: AK/BK = BK/CK = CK/DK = ... = 1/*r*; also, AK/DK
= DK/GK = ... = $1/r^3$, from which AK/AD = DK/DG or AK/DK = AD/DG. By construction, AK/AD =
LO/LM, or AK/LO = AD/LM.
Algebraically, this equality corresponds to $(a/1 - r)$ /(sum of series with ratio r^3) = $a(1 + r + r^2)/ar^3$, or
the sum of series with ratio $r^3 = ar^3/(1 - r^3)$.
Note also that when these proportions are evaluated with the aid of triangles, then the triangle used for a
power of *r* is similar to the original triangle for the whole series, diminished by the factor $1/(1 + r)$, $1/(1 + r$
$+ r^2)$, etc., for the square, cube, etc., corresponding to the expansion $1 - r^n = (1 - r)(1 + r + r^2 + ...+ r^{n-1})$ in
general.]

L2.§2. *PROPOSITIO CIII.*
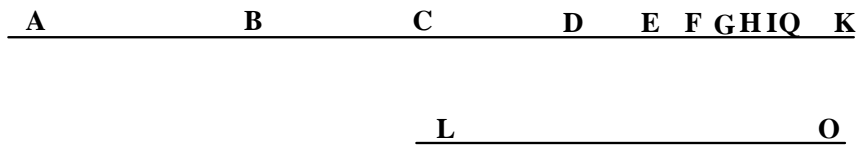
Prop.103. Fig. 1.

Ex serie continue proportionalium AK, sumatur quivis terminus ut HI.
Dico seriem rationis AB ad BC, habere proportionem ad seriem rationis AB ad HI,
quam habet HA (omnes nempe termini ipsum HI praecedentes) ad AB primum terminum.
Haec propositio, ut consideranti facile patebit, eadem est cum praecedenti, sed aliter
& commodius fortasse proposita. Quare eadem erit utriusque demonstratio.

Corollarium.

Ex hoc theoremate licebit praxim desumere, assignato quovis termino HI, in serie, AK, rationis AB ad BC;
reperiendi magnitudinem toti seriei rationis AB ad HI aequalem. Nam si fiat ut HA ad AB, sic KA ad aliam
LO, erit LO aequalis toti seriei rationis AB ad HI.
Fatior tamen opus non esse ad hanc praxim recurrere, cum universalem methodum, eamque facillimam
reperiendi magnitudinem, toti seriei cuiuscumque rationis aequalem, propositio 80 huius suppeditet.

[110]
PROPOSITION 103.

Some term such as HI is taken from the series of continuously proportional terms
I say that the [sum of the] series in the ratio AB to BC has the same proportion to the
[sum of the] series in the ratio AB to HI, as HA has to the first term AB (truly all the
terms preceding HI itself).
This proposition, as it should be considered easy to show, is the same as the preceding
one, although perhaps more easily established. Whereby the demonstration is the same as
the other.

[One cannot of course generalise geometrically as is readily done algebraically, which is what this proposition tries to do for the previous proposition : one is always stuck with a particular instance according to the diagram.]

Corollory.

From this theorem one might wish to choose an exercise: to designate some term HI in the series AK, of ratio AB to BC, to which the magnitude of the whole series in the ratio AB to HI can be found to be equal. For if the ratio is constructed whereby HA to AB is thus equal to the ratio of KA to a different magnitude LO, then LO is equal to the sum of the whole series of the ratio AB to HI.

Nevertheless there is no need to return to this exercise as I have composed a more general method by which the magnitude of any series of any ratio can be found , and which meets the needs of proposition 80 of this work.

L2.§2.　　　　　　*PROPOSITIO CIV.*

A　　　　B　　　C　　D　　E　　F　G　H　　　X

T　　　K　　　L　　M　　N　O　P　Q　　V
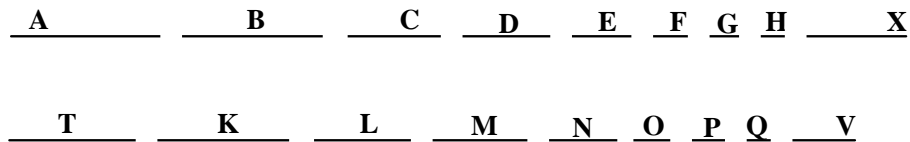
Prop.104. Fig. 1.

Data sint continue proportionalium series binae, AX , KV rationum diversarum , ita tamen ut A, K, L, B etiam sint continuae.

Dico seriem A, K, L, M, N, &c. terminatam in V, eam habere proportionem ad seriem A, B, C, D, &c. terminatam in X , quam A, K, L simul sumpti ad A primum terminum.

Demonstratio.

Addatur ipsi K terminus T in directum, aequalis ipsi A: ratio igitur (quod ex hypothesi colliges) T ad M, triplicata rationis T ad K. Quia autem A, K, L, B ponantur continuae proportionales, erit L ad B, ut K ad L: sed etiam L est ad M, ut K ad L; ergo L ad B, & M, eandem habet rationem: adeoque ^{a} B & M aequales sunt. Sunt vero etiam aequales AT, ergo ratio A ad B, eadem est cum ratione T ad M. quare ratio A ad B, triplicata est rationis T ad K. cum ergo utriusque seriei initium, idem sit terminus A, erit series T, K, L, &c. id est series A, K, L, M, &c., ad seriem A, B, C, D, &c. ut tres primi termini L, K, T, hoc est L, K, A, simul sumpti ad T, hoc est ad A, primum terminum : quod erat demonstrandum. *a 9 Quinti; b 102 Huius.*

PROPOSITION 104.

Two series of continuously proportional terms of different ratios are given, AX and KV, nevertheless in order that A, K, L, B are thus also in continued proportion.

I say that the series A, K, L, M, N, &c. terminating in V has the same proportion to the series A, B, C, D, &c. terminating in X, as the sum of A, K, L has to the first term A.

Demonstration.

A term T equal to A is insertd next to K on the second line : therefore the ratio (which by hypothesis are collected together [according to A or T, K, L, B in proporton]) T to M, is the three-fold or cube of the ratio T to K. However since A, K, L, B are placed in continued proportion, L is to B, as K is to L: but also L is to M as K is to L; hence L to B and L to M are in the same ratio : hence ^{a} B & M are equal. Also indeed, A and T are equal; hence the ratio A to B is the same as the ratio T to M. Whereby the ratio A to B is the cube of the ratio T to K. Hence likewise for the term A for the start of the other series . Hence the series T, K, L, &c. (or the series A, K, L, M, &c.) is in the same ratio to the series A, B, C, D, &c. as the sum of the three first terms L, K, T, or L, K, A, is to T (or A), the initial term. Q. e. d.   *a 9 Quinti; b 102 Huius.*

[T = A; A/K = K/L = L/B and K/L = L/M and hence M = B; hence T/K .K/L .L/M = T/M = $(T/K)^3$; again, T/M = A/B = $(T/K)^3$. It is required to show that TV is to AX as TL is to A: A/B = AX/BX, while T/M = TV/MV, hence AX/BX = TV/MV, and AX/A = TV/TL or TV/AX = TL/A as required.

Algebraically, let the top series be *a, ar, ar²*, ar^3, ... so that AX/BX = BX/CX = ... = *r*; and the second series is *b, bt,* bt^2, bt^3, ... so that KV/LV = LV/MV = ... = *t*. We are given, however, that *a/b = b/bt = bt/ar*, from A, K, L, B in proportion, hence *t = b/a* and *ar = bt²* or *r = t^3 = b^3/a^3*. Hence, maintaining the second series, the first one can be written as *a,* at^3, at^6, a^9, ...
The required ratio TV/AX reduces to 1 + t + t^2, and likewise for TL/A.]

L2.§2.   *PROPOSITIO CV.*

A   B   C D E   K F   G H I L   M

N   O   P Q R   S
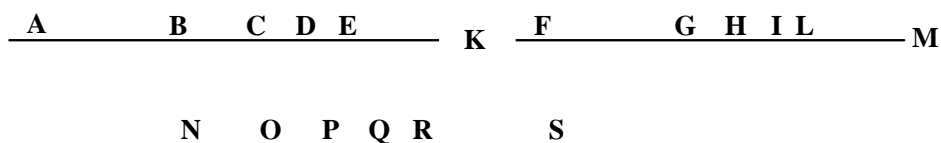
Prop.105. Fig. 1.

Data sint binae series continue proportionalium magnitudinum in diversis rationibus, terminatae in K & M ; & ab aequalibus terminis AB, FG incipiens. Fiat autem secundo termino BC, unius seriei aequalis NO; & GH secundo termino alterius seriei, aequalis OP; Deinde ratio NO ad OP (vel ratio OP ad NO, si OP maior sit quam NO) in infinitum continuetur.

Dico NO esse ad PQ, ut CD ad HI; & NO esse ad QR ut DE ad IL, atque ita in infinitum.

Demonstratio.

Cum series AK, FM incipiant ab aequalibus terminis, erit ratio ^{c} CD ad HI, rationis BC ad GH, hoc est, ex constructione, rationis NO ad OP, duplicata; Atqui etiam ratio NO ad PQ, duplicata est, ex datis, rationis NO ad OP, eaedem igitur sunt rationes CD ad HI, & NO ad PQ; similiter ratio DE ad IL, triplicata est rationis BC ad GH, hoc est rationis NO ad OP: Quare cum & ratio NO ad QR, eiusdem rationis NO ad OP, sit triplicata, eaedem erunt rationes DE ad IL, & NO ad QR. Atque ita in infinitum, similis demonstratione procedemus. Patet igitur Theorematis veritas. *c 27 Huius.*

[111]

PROPOSITION 105.

Two series of continuously proportional magnitudes in different ratios are given, terminating in K and M; and beginning from equal first terms AB and FG.   But the second term of the first series BC is made equal to NO, and the second term GH of the

other series is set equal to OP; from there on, the ratio NO to OP (or the ratio OP to NO, if OP is greater than NO) is continued indefinitely.

I say that NO is to PQ, as CD is to HI; & NO is to QR as DE to IL, and thus indefinitely.

Demonstration.

As the series AK and FM begin with equal terms, the ratio [c] CD to HI, is the square of the ratio BC to GH, or from the construction, the square of the ratio NO ad OP. But also the ratio NO to PQ is the square of the ratio NO to OP, from what is given, and therefore the ratios CD to HI, & NO to PQ are the same; similarly the ratio DE to IL is the cube of the ratio BC to GH, or of the ratio NO to OP. Whereby the ratio NO to QR is the cube of the same ratio NO ad OP, and & NO to QR is the same ratio as DE ad IL. And thus indefinitely, we can proceed with a similar demonstration. Therefore the truth of the theorem is made apparent. *c 27 Huius.*

[AB/BC = BC/CD = CD/DE = ...; FG/GH = GH/HI = HI/IJ = ...; AB = FG;
$AB/BC.BC/CD = AB/CD = (AB/BC)^2$ and $FG/GH.GH/HI = FG/HI = (FG/GH)^2$;
hence $CD/HI = (BC/GH)^2 = (NO/OP)^2$ by construction; also, since $NO/PQ = (NO/OP)^2$,
$CD/HI = NO/PQ$. Similarly, $AB/DE = (AB/BC)^3$ and $FG/IL = (FG/GH)^3$,
hence $DE/IL = (BC/GH)^3 = (NO/OP)^3 = NO/QR$, etc.]

L2.§2.   *PROPOSITIO CVI.*

Data sint duae rationes similes, AB ad CD, & BC ad DE, quae terminis sic alternatim positis, continuentur.

Dico utriusque rationis in infinitum continuatae eundem terminum futurum.

$$\underline{\text{A} \qquad\qquad \text{B} \qquad \text{C} \qquad\quad \text{D} \quad \text{E} \qquad \text{F} \; \text{G} \qquad\quad}\;\; \text{K}$$
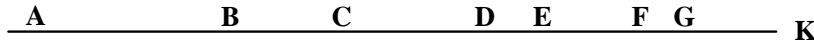
Prop.106. Fig. 1.

Demonstratio.

Quoniam ex hypothesi AB est ad CD, ut BC ad DE; erit permutando componendo, rursumque permutando AC ad CE, ut BC ad DE: similiter quia CD est ad EF, ut DE ad FG, erit permutando componendo, rursumque permutando, CE ad EG, ut DE ad FG; hoc est ex hypothesi ut BC ad DE, hoc est iam demonstratis, ut AC ad CE: sunt igitur AC, CE, EG continuae proportionales. Quod si rationes AB ad CD, & BC ad DE, in infinitum continuentur, ostendam pariter, rationem AC ad CE, in infinitum continuari, per terminos continue proportionales AC, CE, EG, &c. Inveniatur igitur [a] terminus seriei AC, CE, EG, &c. sitque K. Itaque non est punctum assignabile, inter puncta A & K, ultra quod non cadat aliquis terminus seriei AC, CE, EG. Quare cum rationum AB ad CD, & BC ad DE continuatarum, termini omnes ita contineantur in serie AC, CE, &c. ut singuli termini seriei AC, CE, &c. contineant unum terminum rationis AB ad CD, & unum terminum rationis BC ad DE; manifestum quoque est nullum punctum assignari posse inter A & K, ultra quos non cadat aliquis terminus, tam rationis AB ad CD, quam rationis BC ad DE; neutra igitur series terminabitur inter A & K: sed neque ulli dictarum rationum termini transilient K, cum perpetuo contineantur in serie AC, CE, EG, &c.(quae ex constructione non transilit unquam K) ergo binae series rationum AB ad CD, & BC ad DE, eundem habent terminum K. Quod erat demonstrandum. *a 8 Huius.*

PROPOSITION 106.

Two like ratios are given, AB to CD, and BC to DE, are put in line, the terms of which thus alternate in position.

I say that each ratio continued indefinitely comes to the same terminus.

Demonstration.

Since AB is to CD, as BC is to DE by hypothesis ; on interchanging and adding, and interchanging again, AC is to CE, as BC is to DE: similarly since CD is to EF, as DE to FG, on interchanging and adding and again interchanging, CE is to EG as DE is to FG; that is from hypothesis as BC is to DE, now by demonstration it is as AC to CE: therefore AC, CE, and EG are continued proportionals. For if the ratios AB to CD, & BC to DE, are continued indefinitely, I can show equally that the ratio AC to CE, to be continued indefinitely, by means of the terms in continued proportion AC, CE, EG, &c. Therefore a terminus of the seris[a] AC, CE, EG, &c. can be found and it is K. Hence there is no point assignable between the points A & K, beyond which some term of the series does not fall AC, CE, EG. Whereby since all the terms of the continued ratios AB to CD, & BC to DE are thus contained in the series AC, CE, &c. in order that individual terms of the series AC, CE, &c. contain one of the terms of the ratio AB to CD, & one of the terms of the ratio BC to DE. It is also the case that no point can be assigned between A & K, beyond which some term does not fall, either of the ratio AB to CD, of of the ratio BC to DE; neither series therefore terminates between A & K: and yet none of the terms of the said ratios can jump across K, as they are always contained in the series AC, CE, EG, &c.(which by construction cannot jump over K at any time) . Hence the two series of ratios AB to CD, & BC to DE, have the same terminus K. Q. e. d.   *a 8 Huius.*

[Since AB/CD = BC/DE = CD/EF = DE/FG = ... , then AB/BC = CD/DE, AC/BC = CE/DE on adding, <u>AC/CE</u> = BC/DE = AB/CD on interchanging; similarly, BC/DE = CD/EF = DE/FG = <u>CE/EG</u> , hence AC/CE = CE/EG, and AC, CE, EG are in continued proportion. The rest then follows.
In terms of algebra, we are dealing with a geometric progression split into the odd and even terms S_1 and S_2, and thus, in an obvious notation: AB = a; BC = ar; CD= ar^2; DE = ar^3; EF = ar^4; FG = ar^5; etc.
AB/CD = $1/r^2$ = BC/DE = etc.; while BC/DE = $1/r^2$ = CD/EF = etc.; thus both series have the same ratio.
The infinite sum of S_1 is $a/(1 - r^2)$ and the sum of S_2 is $ar/(1 - r^2)$; hence the sum of both series is
AK = $a/(1 - r^2) + ar/(1 - r^2) = a/(1 - r)$ as expected. On the other hand, the single equivalent progression considered AC, CE, EG, ... has the terms $a(1 + r)$, $ar^2(1 + r)$, $ar^4(1 + r)$,.....
for which the infinite sum is $a(1 + r)/(1 - r^2)$, or $a/(1 - r)$.]

L2.§2.   *PROPOSITIO CVII.*

<div align="center">

A   C   EGI F D   B
_____

</div>

Prop.107. Fig. 1.

Magnitudo AB bisecta sit in C; BC autem in D; & CD in E; & DE in F; & EF in G; & FG in I; atque hoc semper fiat:
 Dico alternae huius progressionis terminum fore in puncto, quo magnitudo AB, dividitur in partes, habentes rationem quam unum ad duo, sive terminum progressionis abscindere BG, tertium partem magnitudinis AB.
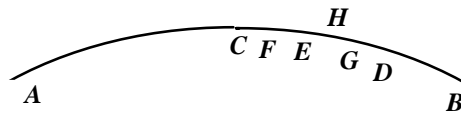
Demonstratio.

Quoniam DC dupla est CE, & BC dupla DC, erit BC, hoc est AC, quadrupla ipsius CE: similiter cum FE dupla sit GE, & DE dupla FE, erit & DE, hoc est CE, quadrupla EG; sunt igitur AC, CE, EG tres continuae in proportione quadrupla. Deinde cum DE ex hypothesi dupla sit DG, & CD dupla ED; erit iterum CD, hoc est BD, quadrupla DF: similiter quia GF dupla est FI, & EF dupla GF, erit EF, hoc est DF quadrupla FI: sunt igitur BD, DF, FI continuae in ratione quadrupla. Itaque si alterna illa bisectio sinc statu continuetur, constituerat utrimque progressio in infinitium proportionis quadruplae; & quoniam AC

quadrupla est CE, erit & BC eiusdem CE quadrupla : est vero & CE, quadrupla EG, atque ita in infinitium progressio igitur AC, CE, EG, &c. eadem est cum progressione

[112]

BC , CE, EG, &c. & eundem terminum habet : Atque progressionis quadruplae BC, CE, &c. terminus H, secat ^{a} CB in ratione unius ad duo, ergo etiam terminus progressionis AC, CE secat CB, in ratione unius ad duo. Quare CH dimidia est ipsius HB, & CB sesquialtera HB : ideoque AB tripla ipsius HB; ac denique AH dupla HB; ergo terminus progressionis AC, CE, &c. secat AB, in ratione unius ad duo. ulterius cum progressiones AC, CE, &c. D, DF, FI, &c., eiusdem sint rationis, nempe quadruplae, erit tota series progressionis AC, CE, EG, &c. ad totam seriem progressionis BD, DF, &c.^{b} ut AC ad BD : Quare cum AC dupla sit DB, erit quoque series progressionis AC, CE, &c. id est AH; dupla seriei BDM DF, &c. Atque AH iam ostendimus etiam duplam esse HB; ergo AH, ad seriem progressionis BD, DF, eandem habet rationem quam ad HB: unde progressionis BD, DF, series terminatur etiam in punct H. Quare per eadem procedens puncta, cum alterna illa bisectio constituat utramque progressionem, illius quoque terminus erit punctum H, quo dividitur AB in ratione unius ad duo: Quod erat demonstrandum.   *a 89 huius; b 89 huius.*

Prop.107. Fig. 2.

Corollarium.

Ex hocTheoremate reperietur arcus trisectio, si independenter a trisectione, alternae illius progressionis terminus inveniatur. Cum enim Theorema univerale sit, & in quavis magnitudine demonstratio allata valeat, si in arcu dato AB, similis alterna fiat bisectio, terminus quoque progressionis alternae H, abscindet tertiam arcus partem BH : proindeque reperto alia via ducto termino, arcus etiam dati trisectio reperietur.

PROPOSITION 107.

The magnitude AB is bisected in C; BC again in D; CD in E; and DE in F; and EF in G; and FG in I; and this is done indefinitely.

I say that the end [or terminus] of this alternating progression to be in a point, by which the magnitude AB is divided in parts having the ratio one to two, or the end of the progression cuts BG into a third part of the magnitude of AB.

Demonstration.

Since CD is twice CE, & BC double CD, BC or AC is four times CE itself: similarly as FE is twice GE, & DE twice FE, DE or CE is four times EG; therefore AC, CE, and EG are three lengths in four-fold continued proportion. Then since DE by hypothesis is double DG, & CD double ED; CD or BD is again four times DF: similarly since GF is twice FI, & EF twice GF, EF or DF is four times FI: therefore BD, DF, FI are continued proportions in a four fold ratio. Thus if this alternate bisection is continued without stopping, a progression is established indefinitely from both ends in the four fold proporton; and since AC is four times CE, and BC is the same four times CE : CE is truly four times EG, and thus AC, CE, EG, &c. are in the same infinite progression as BC, CE, EG, &c, and have the same terminus [or end-point]. But the terminus H of the four fold progression BC, CE, &c. cuts ^{a} CB in the ratio of one to two, hence the terminus of the progression AC, CE also cuts CB, in the ratio of one to two. Whereby CH is half of HB, & CB is one and a half of HB : thus AB is three times HB; and hence AH is twice HB; hence the terminus of

the progression AC, CE, &c., cuts AB in the ratio of one to two. Since the progressions on either side AC, CE, &c. BD, DF, FI, &c., are of the same ratio, truly the quadruple, the sum of the series of the progression AC, CE, EG, &c. to the sum of the series of the progression BD, DF, &c.[b] is as AC to BD. Whereby as AC is twice DB, the series of the progression AC, CE, &c. or AH is also twice the series BD, DF, &c. But we have shown that AH is also thus twice HB; hence AH has the same ratio to the series of the progression BD, DF as it has to HB: thus the series of the progression BD, DF is also terminated in the point H. Whereby the proceding point, from the other bisection gives rise to the other progression, the terminus of that too is the point H, by which we conclude that AB is divided in the ratio of one to two. Q.e.d.   *a 89 huius; b 89 huius.*

[This probem is analysed by Gregorius in terms of two series, whereas we would now consider a series of alternating terms with $r = -\frac{1}{2}$. Initially we follow the scheme of Gregorius, Fig. 3:

|					
1.	**A**				**B**
2.	**A**	**C**			**B**
3.	**A**	**C**	**D**		**B**
4.	**A**	**C**	**E**	**D**	**B**
5.	**A**	**C**	**E F D**		**B**
6.	**A**	**C**	**E GF D**		**B**
7.	**A**	**C**	**EGIF D**		**B**

Prop.107. Fig. 3.

AC, CE, and EG are segments leading to the right in lines 2, 4, 6 (with which we could associate the lines AC, AE, and AG); and these segments are in the progression AC, AC/4, AC/16. Again, BD, DF, and FI are segments leading to the left in lines 3, 5, 7 (with which we could associate the line BD, BF, and BI), and these lines are in the progression BD, BD/4, BD/16. It is now established that each progression has an end-point, and from the similar nature of the progressions, these are shown to be equal to H. The 'forwards' progression involves the even line numbers in Fig. 3, and following the geometrical nature of summation, we have AH/CH = AC/CE , or AC/CH = BE/CE, or CH = AC.CE/BE = AC.(1/8)/(3/8) = AB/6; Hence AH = 2AB/3. The 'backwards' progression similarly follows the terms BD, DF, FI, and agreement is reached that the two progressions give the same result as required. In modern terms, if A is taken as the origin on a number line, and AB = a, then $r = -\frac{1}{2}$, and the sum is $a/(1-r) = 2a/3$]

Corollary.

The trisection of an angle can be found from this theorem, if independently from the trisection considered above, another example of a termination of this progression can be found. For indeed the theorem is universal, and the demonstration can be brought to prevail for any magnitude. Hence, if in the arc of a given circle, the bisection is made by a similar alternate series [i. e. choosing one half of each section repeatedly in a forwards and reverse manner as above], then the terminus H of the alternate progressions also cuts off a third part of the arc : and hence by finding both ways that leads to the same terminus, the trisection of the given arc is found. [One has to perform an infinite number of divisions of line segments by two, using compasses and ruler.]

L2.§2.   *PROPOSITIO CVIII.*

A   1   B   3   E 5 H 7 K   I 8 G 6 F   4   D   2   C
                            P
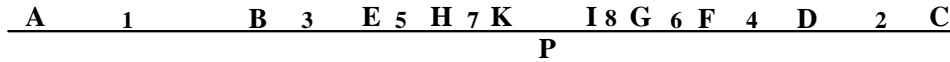
Prop.108. Fig. 1.

     Data sit magnitudo AC utcunque secta sit in B; Deinde fiat AC ad BC, sic BC ad BD; & BD ad ED ; & ED ad EF; & EF ad HF; atque sic altera divisio semper fiat;

     Dico utrimque constitutum in duas progressiones similes, magnitudinum AB, BE, EH, &c. & CD, DF, FG, &c. continue proportionalium in ratione duplicata proportionis AC ad BC.

Demonstratio.

Quoniam ex datis AC, BC, BD, ED, EF, &c. sunt continuae proportionales, erunt eurum differentiae [c] AB, CD, BE, DF, EH, FG, &c. etiam in continua analogia, & quidem eo ordine ut prima, tertia, quinta, septima, & sic deinceps (intermisso semper numero medio) constituant seriem A; secundo, vero quarta, sexta, octava, & sic deinceps (semper omisso numero medio) seriem C, conficiunt. igitur ut AB ad BE, prima ad tertiam, sic CD est ad DF, secunda ad quartam, & sic deinceps; adeoque rationes AB ad BE, & BE ad EH, &c. similes erunt rationibus

[113]

CD ad DF, & DF ad FG, &c. est autem AB ad BE, ut a AC ad BD, hoc est in ratione duplicata AC ad BC; & CD est ad DF, ut BC est ad ED secunda ad quartam, hoc est in ratione duplicata BC ad BD; hoc est AB ad BC; ergo duae illae progressiones, similes erunt, & in ratione duplicata AB ad BC. Quod erat demonstrandum.   *c 1 huius; a ibid.*

PROPOSITION 108.

     The magnitude AC is cut in some manner in B. Then as AC to BC, thus the ratio BC to BD; and BD to ED, and ED to EF, and EF to HF; and thus the division is always done for the following terms.
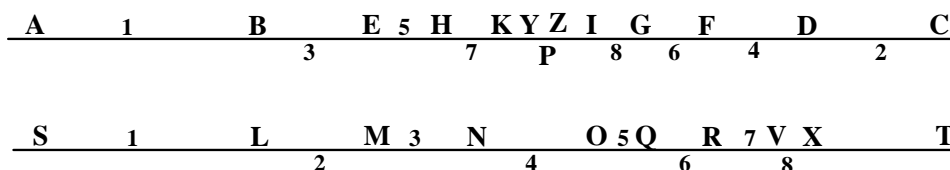
     I say that the series on either side establish two like progressions of magnitudes AB, BE, EH, etc. & CD, DF, FG, etc., of continued proportionals in the square of the ratio AC to BC.

Demonstration.

Since AC, BC, BD, ED, EF, etc. are continued proportionals from the given ratios, the differences of these [c] AB, CD, BE, DF, EH, FG, etc. are also in continued proportion, and indeed the terms in the order of the first , third, fifth, seventh, and thus henceforth (with the middle number always missing) constitute serie A; while the second, the fourth, sixth, eighth, etc., terms, and thus henceforth (with the middle number always missing) make the series C. Thus as AB is to BE, the first to the third term, thus CD is to DF, the second to the fourth term, and thus henceforth ; hence the ratios AB to BE, & BE to EH, etc. are similar to the ratios CD to DF, & DF to FG, etc. But AB to BE is as [a] AC to BD, that is in the square of the ratio AC to BC; & CD is to DF, as BC is to ED the second to the fourth, or in the square ratio of BC to BD; or AB to BC; hence these two two progressions are alike, & in the square ratio of AB to BC. Q.e.d.   *c 1 huius; a ibid.*

L2.§2.   *PROPOSITIO CIX.*

A   1   B   E 5 H   K Y Z I   G   F   D   C
                3   7   P   8 6   4   2

S   1   L   M 3 N   O 5 Q   R 7 V X   T
                2   4   6   8

Prop.109. Fig. 1.

Iisdem positis progressio utraque AB, BE, EH, &c. CD, DF, FG, &c.; ex alterna illa divisione nata, terminum habebit eandem in magnitudine AC.

Demonstratio.

Sumatur ST aequalis AC; & capiantur ex ea magnitudines LT, MT, NT, OT, &c. quae aequales sint continue proportionalibus BC, BD, ED, EF; &c.; erunt igitur SL, LM, MN, NO, &c. ipsis quoque AB, CD, BE, FD, &c. aequales; cum enim tota AC, ST, & ablata BC, LT, aequalia sint; necesse est etiam reliqua AB, SL esse aqualia: & rursum quia tota BC, LT, & ablata BD, MT aequalia sunt, patet quoque reliqua CD, LM aequalia esse. Similiter ostendam & BE ipsi MN, DF ipsi NO, atque ita in infinitum, reliqua reliquis aequalia esse. ergo utraque progressio A & C. progressioni SL, LM, aequales sunt simul sumptae. & quoniam ST, LT, MT, &c. aequantur continue proportionalibus AC, BC, BD, &c. etiam ipsae erunt continuae *b* ideoque SL, LM, MN, &c. sunt continuae proportionales. atque ita sine termino continuatur ratio, SL ad LM; ergo progressio c SL ad LM, terminatur in T, sive constituit magnitudinem ST : quarecum utraque progressio A & C simul sumptae, aequentur progressioni SL, LM; etiam constituent magnitudinem ST, hoc est AC ex constructione: eundem igitur terminum habeant in magnitudine AC necesse est. nam si diversos habeant, sint illi Y & Z. vel inter utrumque terminum Y, Z superit media quaedam magnitudo, quae ad neutram seriem pertineat, vel aliqua erit magnitudo, utrique seriei communis, eritque Z terminus seriei A, & Y terminus seriei minorem quam AC; & posito altero maiorem quam AC. Quod utrumque repugnat modo demonstratis; non igitur diversos habebunt terminos dictae progressiones, sed eundem. Quod erat demonstrandum. *b 1 huius; c 79 huius.*

PROPOSITION 109.

With the same points in place the progression AB, BE, EH, &c. and CD, DF, FG, &c. are given on each side; from that alternating form of division of the line AC another progression is produced ending in the same magnitude AC

Demonstration.

ST is taken equal to AC; & from that line the magnitudes LT, MT, NT, OT, etc are put in place which are equal to the continued proportionals BC, BD, ED, EF, etc; therefore SL, LM, MN, NO, etc and AB, CD, BE, FD, etc are respectively equal to each other; for indeed the whole lengths AC are equal ST, and if the equal lengths BC and LT are taken away, then it follows that the remainders AB and SL are also equal. Again, since the whole lengths BC and LT and the lengths taken BD and MT are equal, it is apparent that the lengths CD and LM are equal. Similarly I can show, for BE taken with MN, and DF with NO, and thus indefinitely, that the rest of the remainders are equal. Therefore both progressions A and C taken together are equal to the progression SL, LM. Since ST, LT, MT, etc. are equal to the continued proportions AC, BC, BD, etc. they are also in continued proportion. *b* and therefore SL, LM, MN, etc. are continuee proportionals. Thus the ratio SL ad LM can continue without end, and hence the progression *c* SL to LM can finish in T, or given the magnitude ST : whereby as both progressions A and C taken together are equal to the progression SL, LM; they give rise to the magnitude ST or AC from the construction: hence by necessity the two series have the same total length or termination given by the magnitude AC. For if they have different terminations, let these be the points Y and Z. Hence either between each of the terminations Y and Z there is present some magnitude in the middle to which neither series belongs, or there is another magnitude which both series have in common, and Z is the end of series A and both series are greater than AC, or Y the end of a series and both series are less than AC. Which are both shown to be in disagreement in this manner; therefore the said progressions do not have different end-points, but the same. Q.e.d. *b 1 huius; c 79 huius.*

L2.§2.   *PROPOSITIO CX.*

Iisdem positis; geminae progressionis AB, BE; & CD, DF, ex alterna sectione natae, communis terminus P, magnitudinem AC, dividet in ratione AC ad BC.

Demonstratio.

Prop.110. Fig. 1.

Per praecedentem ponatur P esse communis utriusque terminus, quoniam igitur similes sunt progressiones A & C, erit tota series progressionis A, hoc est AP, ad totam seriem progressionis C hoc est

[114]

CP, *ᵃ* ut AB ad CD. quia autem ex hypothesi AC, BC, BD sunt continuae proportionales, erit AB ad CD, *ᵇ* ut AC ad BC; est igitur AP ad CP, ut AC ad BC. terminus ergo P utriusque progressionis. dividit AC, in ratione AC ad BC: quod erat demonstrandum.   *d 108 huius; a 94 huius; b 1 huius.*

PROPOSITION 110.

With the same points in place, for both of the progressions AB, BE, EH, &c. and CD, DF, FG, &c. coming from alternate sections; common terminus P, divides the length AC in the ratio AB to BC.

Demonstration.

By the preceding theorem, P is put in place as the common terminus of both series, and therefore as the progressions A and C are like, then the whole series A, or AP is to the whole series B or CP *ᵃ*, as AB is to CD. But since from hypothesis, AC, BC, and BD are continued proportionals, AB is to CD, *ᵇ* as AC is to BC; therefore AP is to CP as AC is to BC: the end-point or the limit of both progressions divides AC in the ratio AC to BC. Q.e.d. *d 108 huius; a 94 huius; b 1 huius.*

[AB/AP = CD/CP ; given AC/BC = BC/BD, then AC/BC = AB/CD, and hence AC/BC = AB/CD = AP/CP as required. In terms of algebra, the left-hand series can be identified with a, ar^2, ar^4, ..., etc., while the right-hand series is given by ar, ar^3, ar^5, ..., etc. The sum of the first series $S_1 = a/(1 - r^2) = AP$, while the second series is $S_2 = ar/(1 - r^2) = CP$; hence $S_1 + S_2 = a/(1 + r) = AC$, while $AP/CP = a/ar = AB/CD = AC/BC = (a/1 - r)/(ar/1 - r)$.]

L2.§2.   *PROPOSITIO CXI.*

Iisdem positis; terminus alternae sectionis, sive progressionis AC, BC, BD, ED, EF, &c., dividet magnitudinem AC, in ratione AC ad BC.

Demonstratio.

A		B	E		P		F	D		C
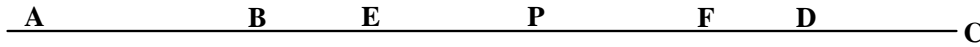
Prop.111. Fig. 1.

Utriusque progressionis AB, BE & CD, DF, termini simul sumpti sunt iisdem cum terminis progressionis AC, BC, BD, ED, &c. alternatim sumptis. ergo progressio alterna AC, BC, BD, &c., eundem habet terminum quem progressiones AB, BE & CD, DF. sed harum terminus per praecedentem, dividit AC, in ratione AC ad CB; ergo & alternae progressionis AC, BC, BD, &c., terminus in eadem ratione dividet magnitudinem AC. Quod erat demonstrandum.

PROPOSITION 111.

With the same points in place, the terminus or end-point of the alternate progressions AC, BC, BD, ED, EF, &c., divides the magnitude AC in the ratio AC to BC.

Demonstration.

The terms of each progression AB, BE & CD, DF are the same as of the progression AC, BC, BD, ED, &c., taken alternately. Hence the alternate progression AC, BC, BD, &c., has the same terminus as the progressions AB, BE and CD, DF. But the terminus of these by the preceding theorem, divides AC in the

ratio AC to CB; and hence the terminus of the alternate progressions AC, BC, BD, &c., divides AC in the same ratio. Q.e.d.

[AC/BC = BC/BD = BD/ED = ED/EF = ...etc., gives AB/BC = CD/BD = BE/ED = DF/EF = ...etc., or
AB/BE = BC/ED ..... for the right-hand series, is equal to CD/DF = BD/EF = ... for the left-hand series.
Hence both progressions have the same limit or terminus as required.
In terms of algebra:
AP is the series a, ar^2, ar^4, ..., and has the sum AP = S_L = $a/(1 - r^2)$; while
CP is the series ar, ar^3, ar^5, ..., and has the sum CP = S_R = $ar/(1 - r^2)$;
in which case AC = $S_L + S_R = a/(1 - r)$.
In terms of these lengths, BC = $ar/(1 - r^2)$; BD = $ar^2/(1 - r^2)$; ED = $ar^3/(1 - r^2)$; etc. Since this is an alternating series, the sum S = $(a/(1 - r)) / (1 + r) = a/(1 - r^2)$ = AP as required .
Hence, AP/PC = $1/r$ = AC/BC as required.]

L2.§2.   *PROPOSITIO CXII.*

Data sint continue proportionalium series, constituens magnitudinem AK: sit autem LN, aequalis AK, fiatque primae AB, aequalis LM; secundae vero BC, aequalis fiat NO; tertiae autem CD, sit aequalis MP, & quartae DE, aequalis OR : atque hoc alternatim semper fiat, ita ut omnes AB, CD, EF, GH, &c. sint ex parte L; omnes vero BC, DE, FG, HI, &c. sint ex parte N.

Dico terminum hiuis alternae progressionis, AB, NO, MP, OR, PQ, RS, &c.; dividere magnitudinem LN, in ratione AB ad BC.
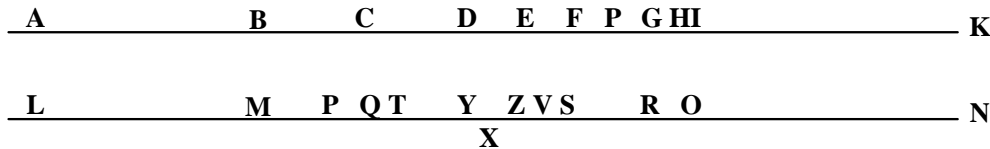
Prop.112. Fig. 1.

Demonstratio.

Quoniam AB, BC, CD, DE, sunt continuae proportionales, etiam AB, CD, EF &c. sunt continuae, & quidem in ratione duplicata AB ad BC; ut patet ex elementis. Similiter in eadem ratione duplicata AB ad BC, erunt continuae proportionales omnes BC, DE, FG, &c atqui omnes AB, CD, &c. sunt ex parte L, & omnes, BC, DE, &c ex parte N. Igitur in magnitudine LN per alternam illam progressionem, constituuntur duae series appositae similes, eiusdem nempe rationis duplicatase AB ad BC. Quare series tota LM, ^{c} MP, &c. est ad seriem totam NO, OR, &c. ut LM ad NO, hoc est AB ad BC. Deinde quia per series AB, d CD, EF, &c. & series BC, DE, &c. sumul sumptae, aequabuntur seriei AB, BC, CD, DE, &c series quoque L & N , simul sumptae aequabunter seriei AB, BC, CD, DE, &c. Quare cum haec ex hypothesi constituat magnitudinem AK, id est ex hypothesi LN, etiam series L & N magnitudinem LN constituent. ergo eundem

[115]

in magnitudine LN, habeant terminum necesse est : si enim diversos habeant ut Y, Z; vel inter utrumque terminum superit media queadam magnitudo, quae ad neutram seriem pertineat; vel aliquid erit utrique commune; ita ut terminus seriei L, sit Z, terminus vero seriei N, sit Y. neutrum autem fieri potest: nam primo dato constitueret utraque series magnitudinem minorem quam LN, alterio autem posito maiorem ; quod utrumque iam demonstratis repugnat. eundem igitur terminum X, habebunt series L & N: cum igitur ostendum prius sit, seriem L esse ad seriem N, ut AB ad BC, utriusque seriei terminus communis X dividet magnitudinem LN, in ratione AB ad BC. Atqui alterna illa magnitudinum LM, NO, MP, OR, &c. progressio, constituit series L & N; ergo ipsius quoque terminus erit X; dividens LN in ratione AB ad BC: quod erat demonstrandum. *c 84 huius; d 99 huius;*

PROPOSITION 112.

A series of continued proportionals is given, making the magnitude AK: moreover, let LN be equal to AK, and the first term AB is made equal to LM; the second truly BC is made equal to NO; the third CD is equal to MP, the fourth DE equal to OR: and this is always done alternately, thus in order that AB, CD, EF, GH, etc. are from the part L; and BC, DE, FG, HI, etc. are from the part N.

I say that the terminus of this alternating progression divides the magnitude LN in the ratio AB to BC.

Demonstration.

Since AB, BC, CD, and DE are continued proportionals, AB, CD, EF, etc., also are continued proportionals, and indeed are in ratio of the square AB to BC, as is apparent from elementary considerations. Similarly all the continued proportionale BC, DE, and FG etc. are continued proportions in the same square ratio AB to BC. But all AB, CD, etc. are from part L, and all of BC, DE, etc., are from part N. Hence in the length LN from the other progression, two similar series are set up opposed to each other which are indeed in the same square ratio of AB to BC. Whereby the whole series formed from LM, [c] MP, etc. is to the whole series from NO, OR, etc., as LM is to NO, or as AB is to BC. Hence, in accordance the series AB, [d] CD, EF, &c. and the series BC, DE, &c. summed together are equal to the series AB, BC, CD, DE, &c., and also the series L and N summed together are equal to the series AB, BC, CD, DE, &c. Whereby as this constitutes the magnitude AK from hypothesis, also by hypothesis the series L and N constitute the magnitude LN. Hence in the same magnitude LN, it is necessary to have the same terminus or limit : for indeed if they have different end-points, such as Y and Z; then either between both end-points there will be some middle length to which neither series belongs, or there will be some some length common to both series, so that the terminus of series L is Z, and the terminus of series N is Y. But neither is possible to be the case: for the first given gives rise to another series with magnitude less than LN, whereas the other put in place is greater; which both now disagree with the demonstration. Therefore the series L and N have the same terminal point X: hence as was to be shown before, the series L is to the series N, as AB to BC, and the common terminus X of the series divides the magnitude LN in the ratio AB to BC. But that other progression of magnitudes LM, NO, MP, OR, &c., constitute the series L and N; hence the terminus of that too is X; dividing LN in the ratio AB to BC: q.e.d.   *c 84 huius; d 99 huius;*

[This theorem is a re-run of the previous one, but with an extra line drawn, and a more detailed proof of a common limit point, which was assumed before.]
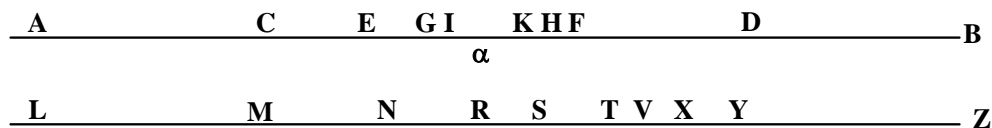
L2.§2.   *PROPOSITIO CXIII.*

Centesima undecim aliter demonstratio.

Data sit magnitudino AB utcunque divisa in C; fiat autem ut AB ad BC, sic BC ad CD, & CD ad DE, & DE ad EF, & EF ad FG: atque hoc semper fiat.

Dico alternae hiuis progressionis terminum α, dividere AB in ratione AB ad BC.

Prop.113. Fig. 1.

Demonstratio.

Sumatur enim LZ aequalis AB, & singulis in quas dividitur alternatim AB, continuis proportionalibus AB, BC, CD, DE, &c. aequalis fiant MZ, NZ, RZ, &c., erunt igitur etiam hae *ᵃ* ideoque & LM, MN, NR, &c. continuae proportionales; & progressionis huius LM, MN, &c. *ᵇ* terminus erit Z; & quoniam AB, LZ, & BC, M Z, aequantur, etiam AC, LM aequales erunt. rursus quia BC, MZ, & CD, NZ, aequales sunt, etiam BD, MN aequales erunt. Similiter ostendam CE, ipsi NR, & DF ipsi RS, & GE ipsi ST, & FH ipsi TV (& sic in infinitum) aequales esse : habemus igitur progressionem alternam magnitudinum AC, BD, CE, DF, &c. qualis in praecedente propositione proponebatur, cuius terminus X dividit AB in ratione LM ad MN. Igitur cum ex progressione alterna hic proposita AB, BC, CD, DE, EF, &c. illa altera oriatur, ita ut tam progressio AB, BC, CD, DE, &c. quam progressio AC, BD, CE, DE, &c. in punctis iisdem C, D, F, G, H, &c. dividant magnitudinem AB, huius quoque terminus erit α, dividens AB in ratione LM ad MN, *ᶜ* hoc est in ratione LZ ad MZ, hoc est ex constructionein ratione AB ad CB. Quod erat demonstrandum. *a 1 huius; b 79 huius; c 69 huius.*

Corollarium.

Si vero fiat ut AC ad CB; sic BD ad DC, & CE ad ED, & DF ad FE, atque ita semper; hiuis quoque alternae progressionis terminus, dividet AB in ratione AB ad BC. cum enim sit ut AC ad CB, sic BD ad DC, & CE ad ED, &c. componendo erit AB ad BC, ut BC ad CD, & CD ad DE, &c. atqui terminus progressionis AB, BC, CD, &c. dividit AB in ratione AB ad BC: ergo & progressionis AC, CB, BD, DC, &c. terminus, dividet AB in ratione AB ad BC cum enim veraque haec progressio in punctis semper iisdem secet AB, eundem utraque terminum habere debet.

[116]

PROPOSITION 113.
Another Demonstration of Proposition One hundred and Eleven.

The magnitude AB is given divided in some manner in C; moreover the division is made so that AB is to BC thus as BC is to CD, and CD to DE as DE to DF, and EF to FG: and this shall be done indefinitely.

I say that the terminus α of this alternating progression divides AB in the ratio AB to BC.

Demonstration.

For LZ is taken equal to AB, and for every term in which AB is alternately divided by the continued proportions AB, BC, CD, DE, etc., the equal terms MZ, NZ, RZ, etc. are set out, and therefore these terms are also in *ᵃ*continued proportion: LM, MN, NR, &c.; and the terminus of this progression LM, MN, &c. *ᵇ* is Z; and since AB and LZ, and BC and M Z are equal, and also [the differences] AC and LM are equal. Again since BC, MZ, and CD, NZ, are equal, BD and MN are also equal. Similarly I can show that CE is equal to NR, and likewise DF to RS, GE to ST, FH to TV (and so on indefinitely): we therefore have a alternating progression of magnitudes AC, BD, CE, DF, etc. such as are proposed in the preceding proposition, the terminus X of which divides AB in the ratio LM to MN [or AC to BD]. Therefore from the first progression proposed here: AB, BC, CD, DE, EF, etc., another is generated. Thus as the progression AB, BC, CD, DE, etc. , so also the progression AC, BD, CE, DE, etc., divide the magnitude AB in the same points C, D, F, G, H, etc., the terminus of which is α too. The point α divides AB in the ratio LM to MN, *ᶜ* or in the ratio LZ to MZ, or from the construction, in the ratio AB to CB. Q.e.d. *a 1 huius; b 79 huius; c 69 huius.*

Corollary.

If indeed the ratio AC to CB is thus made as BD to DC, and CE to ED, and DF to FE, and thus indefinitely ; the terminus of this alternating progression also divides AB in the ratio AB ad BC. For indeed as AC is to CB, thus BD is to DC, & CE to ED, &c. By addition, AB is to BC as BC to CD, & CD to DE, &c., but the terminus of the progression AB, BC, CD, &c. divides AB in the ratio AB to BC: and hence the

terminus of the progression AC, CB, BD, DC, &c. also divides AB in the ratio AB to BC; for indeed this progression always divides AB in the same points, and both must have the same terminus.

[This is easy to establish algebraically, but beware that the points are not assigned the same labels consistently between the diagrams for the different propositions!]

Lemma primum.

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A              C     E  G            D    F H                  B
                     I                    K
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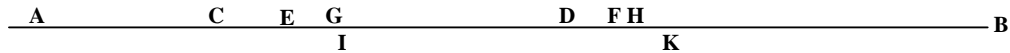
Prop.113. Fig. 2.

Data sit magnitudo AB sectam tres partes aequales, in I & K: & rursum aliter secta in C, inter A & I.
Dico bisectionem partis CB, cadere inter I & K in D; & bisectionem partis DA, cadere inter I & C in E; rursum bisectionem partis EB, contingere inter D & K, in F; ipsius autem FA bisectem, inter E & I in G : atque ita in infinitum.

Demonstratio.

Quoniam CB maior est, quam IB dupla AI; erit ipsius CB demidia, maior quam AI. ergo bisectio ipsius CB, cadit ultra I, versus B. Iterum CB plus est, quam dupla KB, adeoque ipsius CB dimidia, maior quam BK; quare bisectio CB, cadit ultra K, versus A : adeoque cadit inter I & K in D. Deinde cum CB plus sit quam duae tertiae, ipsius AB, erit CD eius dimidia, plus quam una tertia ipsius AB; sit autem AC ex datis minor, quam una teria; ergo CD maior est AC. & bisectio ipsius DA, cadet ultra C versus B. similiter cum AI etiam maior sit quam DI, cadet bisectio ipsius DA ultra I versus A, adeoque inter C & I, in E; non aliter ostendemus reliqua, quae in assertione proposuimus. Constat igitur veritas lemmatis.

Lemma secundum.

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A                          G   E          C    H F      D          B
                           I              K
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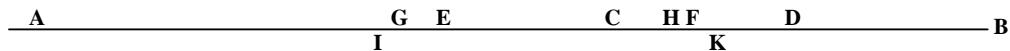
Prop.113. Fig. 3.

Data rursum sit AB sectam tres partes aequales, in I & K: & rursum aliter secta inter A & I, in C.
Dico bisectionem partis CB, cadere inter K & B in D; & bisectionem partis DA, cadere inter C & F in E; bisectionem autem partis EB, cadere inter K & D, in F; partis vero FA, bisectionem contingere inter E & I in G : atque ita in infinitum.
Demonstrato eadem prope quae lemmatis praecedentis.

First Lemma.

The magnitude AB is cut in three equal parts by the points I and K: and again cut by the point C in some other way.
I say that the bisection of the part CB lies at D between I and K; and the bisection of the part DA falls between I and C at E; again the bisection of the section EB lies at F, between D and K; moreover the bisection of FA lies at G between I and E, and thus indefinitely.

Demonstration.
Since CB is greater than IB, which is twice AI; then half of CB itself is greater than AI. Therefore the bisection of CB falls beyond I towards B at D. Again CB is more than twice KB, and hence half of CB is greater than BK; whereby the bisection of CB falls beyond K towards A: and thus D lies between I and K. Again, as CB is more than two thirds of AB, then half of this or CD is more than one third of AB; but it is

given that AC is less than one third of AB; hence CD is greater than AC, and the bisection of DA lies beyond C towards B at E. Similarly, also as AI is greater than DI, the bisection of DA lies beyond I towards A, and thus E lies between C and I ; we will not otherwise demonstrate the rest of the terms, which we have presumed in the statement of the lemms, the truth of which is now established.

 [Fig 2: AB is trisected by the points I and K into the equal sections AI, IK, and KB; AI is then cut in some manner at the point C. Now, CB > IB = 2.AI; hence CB/2 > AI, or the bisection point D viewed from the left end A lies beyond I towards B. Again, CB > 2.KB; hence CB/2 > BK, or the bisection point viewed from the right end B lies beyond K towards A. Hence D lies between I and K, or AI < AD < AK in modern terms. This bisection at D starts a series in the middle section IK.

For the next term, which starts a progression in the first section AI: Since CB > 2.AB/3 then CD = CB/2 > AB/3 ; and it is given that AC < AB/3; hence AC < CD gives 2.AC < AC + CD < 2.CD and hence the bisection of AD at E viewed from A lies beyond C towards B.
Again, as AI > DI, then 2.AI > AI + DI > 2.DI, or the bisection of AD at E is less than I viewed towards A. Hence AC < AE < AI, establishing another term E in the series in the first section AI following C. Subsequently, AD < AF < AK for next point F in the middle series, etc. ; and AE < AG < AI for the next point G in the first series, and so indefinitely for the two series.]

Second Lemma.

Again AB is given cut in three equal parts by the points I and K: and again cut by the point C in some way. I say that the bisection of the part CB lies at D between K and B; and the bisection of the part DA falls between C and F at E; again the bisection of the section EB lies at F, between D and K; and truly the bisection of FA lies at G between I and E : and thus indefinitely.
    This can be shown by almost the same method as the preceding lemma.
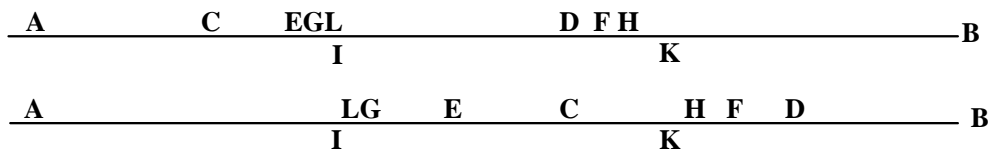

L2.§2.   *PROPOSITIO CXIV.*

    Ex magnitudine AB secta in tres partes aequales in I & K, sumatur AC minor vel maior tertia parte totius AB, & bifariam dividatur CB in D, & DA bifariam in E, & EB in F, & FA in G. Rursum GB bifariam in H, & HA in L: atque hoc semper fiat.
    Dico hiuis progressionis alternae terminos dividere magnitudinem AB in tres partes aequales.

Demonstratio.

Prop.114. Fig. 1.

Cum CB dupla sit DB, & EB dupla FB, erit CB ad DB, ut EB ad FB, ergo CE ad DF, [a] ut EB ad FB. Quare cum EB dupla sit FB, etiam CE, ipsius DF dupla erit. Deinde cum DA dupla sit EA, itemque FA dupla ipsius GA, erit

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DA ad FA, ut EA ad GA; ac proinde DF [a] ad GE, ut DA ad EA. Quare cum DA ipsius EA dupla sit, etiam DF dupla erit EG. unde CE quadrupla est ipsius EG. Similiter ostendemus EG duplam esse FH, ipsam autem FH duplam esse GL, proindeque EG ipsius GL quadruplam esse : atque ita continuando sine statu, per alternam illam bisectionem constitui progressionem magnitudinum CE, EG, GL, &c. proportionis quadruplae. eadem autem discursu quo prius usi fuimus, demonstrabimus DF esse quadruplam FH, & FH

quadruplam sequentis termini, ac proinde etiam hic progressione rationis quadruplae statui. Ulterius quoniam tam ratio CB ad DB, quam IB ad KB, dupla est, erit CB ad DB, ut IB ad KB, & [b] CI ad DK, ut IB ad KB. Itaque cum IB dupla sit KB, etiam CI ipsius DK dupla erit; similiter DK ipsius EI duplam esse demonstrabimus. Igitur CI quadrupla est EI: eadem methodo discurrendi, ostenditur [c] DK esse ad FK, ut DF est ad FH. Quare & progressionis DF, FH, &c. terminus erit K. dum igitur utraque progressio CE, EG, &c. DF, GH, &c. constituatur ab alterna illa bisectione, in propositione proposita, ipsius quoque termini erunt in I & K; ubi trifariam dividitur magnitudo AB. Quod erat demonstrandum. *a 19 Quinti; a ibid; b ibid; c 79 huius.*

Assumpsimus AC minorem aut maiorem tertia parte magnitudinis data AB ; quia si aequalis uni tertia foret, bisectiones alterna in eadem semper puncta I & D inciderent; uti manifestum est, assertionem propositionis consideranti.

PROPOSITION 114.

From the magnitude AB cut into three equal parts by I and K, a section AC is taken either larger or smaller than the third part of the whole length AB, and CB is equally divided in D, and DA equally divided in E, and EB in F, and FA in G. Again GB is equally divided in H, and HA in L: and this bisection is made indefinitely.

I say that the alternate terms of this progression divide the magnitude AB into three equal parts.

Demonstration.

Since CB is twice DB, and EB is twice FB, then CB is to DB as EB is to FB, hence CE is to DF, [a] as EB is to FB. Whereby as EB is twice FB, also CE is twice DF. Hence as DA is twice EA, and likewise FA is the double of GA, DA is to EA as FA to GA; and hence DF [a] is to GE as DA to EA. Whereby as DA is the double of EA, also DF is the double of EG. Hence CE is four times EG. Similarly we can show that EG is twice FH, but FH is twice GL, and hence EG is four tims GL : and thus by continuation without stopping, the progression of the magnitudes CE, EG, GL, etc. of the quadruples of the proportions can be set up through bisecting the other section. Moreover by the same discourse which we used previously, we can show that DF is the quadruple of FH, and FH the quadruple of the following term, and thus also this progression of quadruple ratios is set up. Furthermore, since the ratios IB to KB and CB to DB are both 2, CB is to DB as IB to KB, and [b] CI to DK is as IB to KB. Thus as IB is twice KB, we can also show that CI is the double of DK; similarly DK is the double of EI. Therefore CI is four times EI: by the same kind of reasoning, it can be shown that [c] DK is to FK as DF is to FH. Whereby the terminus of the progression DF, FH, etc. is K, while each progression in the proposition proposed CE, EG, &c. DF, GH, &c. are thus established by the bisection of the sections of the one by the other, the end-points are also I and K, thus trisecting the magnitude AB. Q.e.d. *a 19 Quinti; a ibid; b ibid; c 79 huius.*

Assumpsimus AC minorem aut maiorem tertia parte magnitudinis data AB ; quia si aequalis uni tertia foret, bisectiones alterna in eadem semper puncta I & D inciderent; uti manifestum est, assertionem propositionis consideranti.

We have assumed that AC is either less or greater than the third part of the magnitude AB; for if it should be equal to a third, alternate bisections always occur at the points I and D; a useful vindication of the theorm considered.

[CB = 2.DB and EB = 2.FB; giving CB/DB = EB/FB from which CB/EB = DB/FB giving CE/EB = DF/FB and on re-arranging we have CE/DF = EB/FB, and so EB = 2.FB and CE = 2.DF;

Again: DA = 2. EA and FA = 2.GA and hence DA/EA = FA/GA or FA/DA = GA/EA

giving FD/DA = GE/EA or DF/GE = DA/EA; as DA = 2.EA and DF = 2.EG, then CE = 2.DF = 4.EG.

Similarly, EG = 2.FH and FG = 2.GL, hence EG = 4.GL; and thus the progression CE, EG, GL, ...is established where each term is $\frac{1}{4}$ of the previous term.

Again, DF = 4.FH; FH = 4....., etc; hence another similar progression can be set up.

Again, IB/KB = 2 and CB/DB = 2, then and CB/DB = IB/KB or CB/IB = DB/KB giving CI/DK = IB/KB = 2; thus IB = 2.KB and CI = 2.DK.

Similarly, DK = 2.EI and CI = 4. EI; since CI/EI = CE/EG = 4, and the limit of the progression is I.

Again, DK/FK = DF/FH , and thus the limit of the progression DF, FH, ... is K; hence the limit points I and K divide the line AB in the required ratio.

Analytically, we can set AB = 1 without loss of generality, and consider the point C to be at a distance x_0 from A, lying in the interval $0 < x_0 < \frac{1}{3}$. The point y_0, according to the construction, then lies at a distance $\frac{1}{2}(1 - x_0)$ from B: i. e. $y_0 = \frac{1}{2}(1 - x_0)$; subsequently, $x_1 = \frac{1}{2}(1 - y_0) = \frac{1}{4}(1 + x_0)$;
$y_1 = \frac{1}{2}(1 - x_1) = \frac{1}{2}(\frac{3}{4} - \frac{1}{2}x_0) = \frac{1}{8}(3 - x_0)$; $x_2 = \frac{1}{2}(1 - y_1) = \frac{1}{16}(5 + x_0)$;
$y_2 = \frac{1}{2}(1 - x_2) = \frac{1}{32}(11 - x_0)$; $x_3 = \frac{1}{2}(1 - y_2) = \frac{1}{64}(21 + x_0)$; $y_3 = \frac{1}{2}(1 - x_3) = \frac{1}{128}(43 - x_0)$, etc.

To generalise: the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_n = \frac{1}{2}(1 - y_{n-1})$ and $y_n = \frac{1}{2}(1 - x_n)$ for $n \geq 0$ and x_0 given in the interval $0 < x_0 < \frac{1}{3}$, obviously converge to the points I $= \frac{1}{3}$ and K $= \frac{2}{3}$.
For if we set lim $x_n = X$ and lim $y_n = Y$ for very large n, then $X = \frac{1}{2}(1 - Y)$ and $Y = \frac{1}{2}(1 - X)$, leading to $X = Y = \frac{1}{3}$.

In terms of the above ratios, CA $= x_0$; CB $= (1 - x_0)$; DB $= y_0 = \frac{1}{2}(1 - x_0)$; AD $= (1 - y_0) = \frac{1}{2}(1 + x_0)$;
AE $= x_1 = \frac{1}{2}(1 - y_0) = \frac{1}{4}(1 + x_0)$; EB $= 1 - x_1 = \frac{1}{4}(3 - x_0)$; FB $= y_1 = \frac{1}{8}(3 - x_0)$; etc.]

Lemma.

PARS PRIMA.

Data sit magnitudo AB secta in I & K secundum rationem V ad X: ita ut AK sit ad KB, ut BI ad IA. divisa sit deinde AB adhuc aliter inter A & I in C.

Dico si CB divitatur in ratione V ad X, sectionem fiere ultra K in D, item si DA dividatur in ratione V ad X sectionem cadere ultra I in E: rursum si EB divitatur in eadem ratione, sectionem contingere inter K & D in F; & si FA, sectionem fore inter I & E in G. atque in infinitum.

Demonstratio.
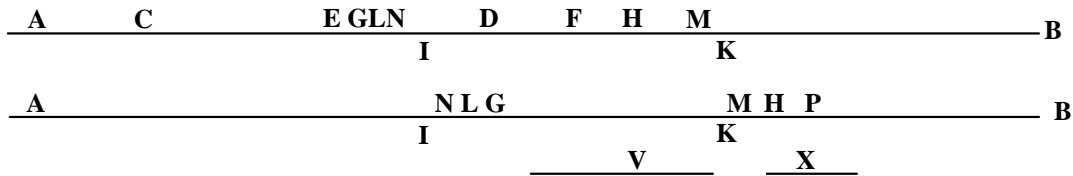
Prop.114. Fig. 2.

Cum AK sit ad KB, ut BI ad IA, erit componendo AB ad KB, ut AB ad IA; ergo KB, IA , additoque communi IK etiam AK, IB aequantur: unde cum AK sit ad KA , ut V ad X; utique IB ad KBm in eadem ratione erit: quare CB (ex datis maior quam IB) maiorem habet rationem ad KB, quam V ad X; ergo sectio ipsius CB, in ratione V ad X, cadit ultra K in D: similiter cum AK sit ad KB, id est IA, sic ut V est ad X, erit DA ad eandem IA, in minori ratione, quam V ad X: unde sectio ipsius DA in ratione V ad X, cadet ultra I in E: Quod autem sectio ipsius EB cadet ultra K, eodem quo prius modo ostendatur: Item quod ultra D, versus B sic ostendo; facta CB ad DB, in ratione V ad X erit EB ad DB, in minori ratione quam V ad X.

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ergo sectio ipsius EB, in ratione V ad X, cadit ultra D, versus B: ergo cum etiam ultra K versus A cadat, inter D & K, contingat necesse est, nempe in F; similiter sectionem ipsius FA inter I & E, futuram in G demonstrabimus, atque ita in infinitum; discursus enim idem omnibus divisionibus sequintibus quadrat.

PARS SECUNDA.

Iisdem positis; si C cadat inter I & K (na si inter B & K cadere, foret casus primae partis) simili plane discursu demonstrabimus eadem omnia contingere quae prius, hoc solum mutato, quod signa divisionum E, G, K, &c. DF, HM, &c. ad alterum latus ordine constituantur.

Lemma.
First Part.

The magnitude AB is given cut by the points I and K according to the ratio V to X :
thus as AK is to KB, and so also BI to IA are as V to X. AB is then divided again in some
manner at the point C lying between A and I.

I say that if [subsequently] CB is divided in the ratio V to X, then the section is made
beyond K at D; likewise if DA is divided in the ratio V ad X then the section falls beyond
I at E: again if EB is divided in the same ratio, then the section lies between K and D at F;
and if FA is divided, then the section is between I and E at G; and so on indefinitely.

Demonstration.

Since AK is to KB as BI is to IA, then the sum AB is to KB as the sum AB is to IA; therefore KB and
IA are equal and on adding the common length IK to each, AK and IB are also equal: thus as AK is to KB
as V is to X, then IB to KB is in the same ratio V to X. Whereby CB (which is given greater than IB) to
KB, has a larger ratio than V to X; hence the section of CB, in the ratio V to X, lies beyond K at D.
Similarly, AK is to KB (or IA) thus as Vis to X: hence DA is to the same IA in a smaller ratio than V to X,
and thus the section of DA in the ratio V to X falls beyond I at E. But concerning the section of EB that lies
beyond K at F, it can be shown in the same manner as established earlier, likewise as beyond D and
towards B, I can thus show that CB made to DB in the ratio V to X results in a ratio EB to DB smaller than
V to X that lies beyond D towards B . Hence the section of EB, in the ratio V to X, lies beyond D towards
B : it also lies beyond K towards A, and is hence between D and K, and so it lies at F. Similarly the section
of FA lies between I and E, we can show that it lies at G, and so on to infinity; indeed the same discourse
for all quadruple divisions follows.
[We are given initially that AK/KB = BI/IA = V/X from which AB/KB = AB/IA and hence AI = KB and
AK = IB; and hence BI/BK = AK/AI = V/X. We are given AC < AI initially, or equally, BA > BC > BI.
Now, BC > BI and hence BC/BK > V/X = BI/BK: hence the section of BC in the ratio V to X results in a
point D that lies beyond K away from B; i. e. BI > BD > BK or alternately AI < AD < AK.
In a like manner, AK/AI = V/X, and hence as AD < AK, then AD/AI < AK/AI = V/X, and the section of
AD in the ratio V/X results in a point E such that AC < AE < AI, or alternately, BC > BE > BI.
Continuing the subdivision of the alternate interval BE : BA > BC > BE > BI, and hence subdivision of BE
in the ratio V/X gives a point F such that BI > BD > BF > BK, etc.]

Second Part.

For the same positions; if C falls between I and K (for if it lies between B and K, it will be the case of the
first part.) by the same clear discussion we can show that all the points lie as previously, but with this
change only, that the marks of the divisions E, G, K, etc., and DF, HM, etc. are set up in order on the other
side.

L2.§2.   *PROPOSITIO CXV.*
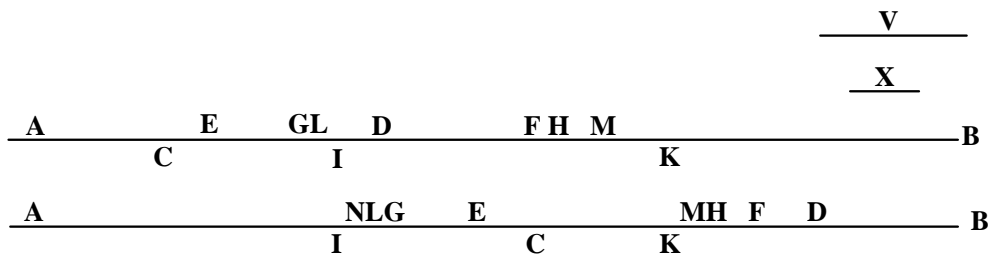
Prop.115. Fig. 1.

Data sit proportio V ad X, & magnitudo AB ita secta in I & K, ut AK sit ad KB, & BI ad IA, sicut V est ad X. Aliter deinde dividat ut AB in C; quocumque tandem loco cadat C, modo non incidat in I aut K:

Fiat autem CB ad DB, ut V ad X; & DA ad EA, ut V ad X: item EB ad FB, & FA ad GA, & GB ad KB, & HA ad LA, fuerint inter se ut V, est ad X: Atque hoc semper continuetur.

Dico alternae hiuis progressionis terminos, fore in I & K, ubi AB, dividitur in ratione V ad X.

Demonstratio.

Quoniam est ex constructione CB ad DB, ut EB ad FB (sunt enim utraque ad invicim in ratione V ad X) etiam CE reliquum [a], DF reliquum, erit ut CB ad DB, id est sicut V ad X: & quia DA est ad EA, ut FA ad GA (nempe in ratione V ad X) rursum erit DF ad EG, ut FA ad GA, hoc est ut V ad X : sunt igitur CE, DF, EG, tres continuae proportionales in ratione V ad X. ergo ratio CE ad EG duplicata est rationis V ad X. Similiter ostendimus EG, FH, GL esse continuae in ratione V ad X, ideoque rationem EG ad GL, duplicatam esse rationis V ad X. Cum ergo etiam ratio CE ad EG, sit rationis V ad X duplicata, erunt CE, EG, GL, in continua analogia; atque ita continuando sine statu alternam illam divisionem, demonstrabimus constitui progressionem magnitudinem CE, EG, GL, LN, &c. continue proportionalium in ratione duplicata V ad X, ab alterna vero parte, eodem plane discursu ostendemus DF, FH, HM, &c. esse continuas in ratione duplicata V ad X; ac proinde sic quoque constitui progresionem proportionis duplicatae V ad X. ulterius quia AK est ad KB, ut BI ad IA, componendo AB erit ad KB , ut AB ad AI; ideoque KB, AI aequantur: additaque IK communi, aequalis erunt IB, AK; ergo ut AK ad KB, id est ex constructione ut V ad X, sic IB ad KB: Quare cum & CB ad DB, sit ut V ad X, etiam CB erit ad DB, ut IB ad KB. allatis ergo IB, KB, CI erit ad DK, ut CB ad DB reliquum ad reliquum, hoc est ex constructione ut V ad X. similiter demonstrabimus DK esse ad EI, ut V ad X,

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erunt igitur CI, DK, EI continuae proportionales in ratione V ad X: ideoque ratio CI ad EI, duplicata erit rationis V ad X; quare cum & ratio CE ad EI, eiusdem ostensa sit esse duplicata, erit CI ad EI, ut CE ad EG, & permutando ut CE ad CI, sic EG ad EI, unde terminus [a] progressionis CE, EG, &c. est I. simili discursu ostentetur, etiam DK esse ad FK, ut DF est ad FH; quare huius quoque progressionis terminus erit K; Itaque cum utraque progressio CE, EG, &c. DF, FH, &c. ab alterna illa divisione constituatur; ipsius quoque termini erunt I & K; ubi magnitudo AB, dividitur in ratione V ad X. Quod erat demonstrandum.

a 19 Quinti; b ibid; a 79 huius.

Scholium.

Hic quoque voluimus punctum C non incidere in I aut K; eo quod si in alterutrum incideret, divisiones quoque alterna, in eadem semper puncta I & K, deberent incidere; ut patet consideranti statum Theorematis.

Caeterum qui hanc propositione cum priori contulerit, facile intelliget hanc universalem esse, illam vero particularem casum complecti. placuit enim id subinde tum hic, tum alibi factitare, tum quia in particularibus casibus eiusdem Theorematis veritas, clarius non raro atque illustrius emicat, tum quia a particularem casum cognitione, facilius ad percipienda, universalium Theorematum demonstrationem proceditur.

PROPOSITION 115.

The proportion V to X is given, and the magnitude AB is thus cut in I and K, in order that AK is to KB, and BI is to IA, as V is to X. Following this, AB is divided by C in some other way; yet whatever point C falls on, it is not I or K:

Moreover CB is made to DB as Vto X; and DA to EA, as V to X: likewise EB to FB, and FA to GA, and GB to HB, and HA to LA, are each as V to X : and this construction is continued indefinitely.

I say that the alternating terminating points of this progression are I and K, as AB is divided in the ratio V to X.

Demonstration.

Since from the construction CB is to DB, as EB is to FB (for each in turn is equal to the ratio V to X) also the difference CE [a], and the difference DF are as CB to DB, or as V to X: and since DA is to EA, as FA to GA (surely in the ratio V to X) again DF is to EG as FA is to GA, or as V to X : therefore CE, DF, and EG are three continued proportionals in the ratio V to X. Therefore the ratio CE to EG is the square of the ratio V ad X. Similarly we can show that EG, FH, GL are continued proportionals in the ratio V to X, and thus the ratio EG to GL is the square of the ratio V to X. Also, as the ratio CE to EG is therefore the square of the ratio V to X then CE, EG, GL are in continued proportion; and thus by continuing without ceasing this alternate division , we can show that a progression of magnitude CE, EG, GL, LN, etc. of continued proportions can be established in the ratio of the square of V to X.

Indeed from the other part, by the same clear discourse, we can show that DF, FH, HM, &c. are continued in the square ratio V to X; and hence thus also constitute a progresion in the proportion of V to X squared. Beyond which AK is to KB, as BI to IA, on adding AB is to KB as AB is to AI; and hence KB and AI are equal: and on adding to the common length IK, IB and AK are equal. Hence as AK is to KB, or from the construction as V to X, thus IB is to KB. Whereby as CB to DB is as V is to X, also CB is to DB as IB is to KB, therefore on bringing together IB and KB, CI is to DK as CB is to DB, remainder to remainder, from the construction, or as V to X. Similarly we can show that DK is to EI as V to X, and therefore CI, DK and EI are continued proportionals in the ratio V to X: and hence the ratio CI to EI is the square of the ratio V to X; and the ratio CE to EI can be shown to be the square of the same ratio; hence CI to EI as CE to EG, and on re-arranging, as CE to CI, thus EG to EI, hence the terminus [a] of the progression CE, EG, etc. is I. By a like discourse it can be shown that DK also is to FK as DF is to FH; whereby the terminus of the progression is K. Thus as each progression CE, EG, erc., and DF, FH, erc. is established by this alternate division with termini I and K, just as the length AB is divided in the ratio V ad X. Q.e.d. *a 19 Quinti; b ibid; a 79 huius.*

Scholium.

In the present circumstances we have wished the initial point C not to fall on I or K; but if the point C happens to fall on in either of these, then the divisions alternate too, but always they have to fall on the same points I and K, as is apparent from the statement of the Theorem.

Otherwise, whoever brings this proposition and the previous one together, can easily understand that it is of a more universal nature than the particular case included. Indeed it may please one, immediately upon doing this example, to do one with a different ratio, for the truth of the Theorem arises most clearly from a study of particular cases; for from a study of the particular case, the general case should be made easer to understand, and the demonstration of the Theorem can proceed.

[CB/DB = EB/FB = V/X; hence on subtracting, CB/EB = DB/FB, giving CE/EB = DF/FB and on re-arranging we have CE/DF = EB/FB = CB/DB = V/X.
Again, DA/EA = FA/GA = V/X is given; from which DA/FA = EA/GA and on subtracting,
DF/FA = EG/GA or DF/EG = FA/GA = V/X and (CE, DF, EG) are lengths in continued proportion.
Hence, CE/DF = DF/EG = V/X : and also (CE/DF).(DF/EG) = CE/EG = $(V/X)^2$.
Similarly, (EG, FH,GL) are in continued proportion in the ratio V/X, and EG/GL = $(V/X)^2$.
Also, from the underlined ratios: CE/EG = $(V/X)^2$ = EG/GL, (CE, EG, GL) are in continued proportion, and this sequence formed from the alternate divisions can be continued indefinitely in the ratio $(V/X)^2$.
We can thus show that a progression of magnitudes (CE, EG, GL, LN, ...) in the ratio $(V/X)^2$ can be established. Likewise, for the other series, (DF, FH, HM, ...) are continued in the ratio $(V/X)^2$.

We are also given AK/KB = BI/IA = V/X for the limit points I and K, from which on addition, AB/KB = AB/IA and hence KB = IA and also IB = AK on adding IK to each, as in the previous theorem. Hence AK/KB = IB/KB = V/X, and also CB/DB = V/X = IB/KB or KB/DB = IB/CB, and on subtraction, DK/DB = CI/CB or CI/DK = CB/DB = V/X . Again, DK/EI = V/X; for BI/IA = AK/ AI = V/X = DA/EA and so AK/DA = AI/EA giving DK/DA = EI/EA or DK/EI = DA/EA = V/X as required. Hence, (CI, DK, EI) are continued proportionals in the ratio V/X, and CI/EI = $(V/X)^2$. 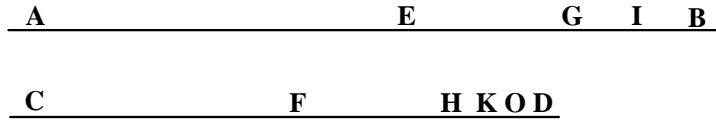From CI/EI = $(V/X)^2$ = CE/EG above, we have CE/CI = EG/EI = GL/GI = ... , and hence the termination of the series of ratios is I. In a similar manner, DF/DK = FH/FK = HM/HK = ..., and the termination of the other series of ratios is K.

Thus the series (CE, EG, GL, LN, ...) and (DF, FH, HM, ...) terminate in the points I and K, as does the length AB divided in the ratio V to X.

L2.§2.   *PROPOSITIO CXVI.*

A   E   G   I   B

C   F   H K O D

Prop.116. Fig. 1.

Sint duae quantitates AB, CD; sitque AB divisa in E & G, ita ut AE, sit non minor dimidio AB, & EG non minor dimidio EB; eodem modo divisa sit CD in F & H, sintque AE, EG; CF, FH proportionales: & hoc semper fieri possit.

Dico totam AB esse ad totam CD, ut est AE ad CF.

Demonstratio.

Si enim non est proportio AB ad CD aequalis proportioni AE ad CF, erit vel maior vel minor: sit primo minor. cum ergo ponatur AB ad CD, minorem habere rationem, quam AE ad CF, habebit AB ad aliquam [b] minorem quam CD; nempe ad CK, eandem proportionem, quam AE ad CF: & quoniam ex quantitatibus AB, CD, earumque residuis semper non minus dimidio aufertur, si continuetur haec ablatio terminos, verbi gratia per tres CF, FH, HO, relinquetur tandem OD [c] minor quam KD ; ideoque CO erit maior quam CK: si iam ex AB totidem partes ad mentem proportionis AE, EG, GI, tollantur, erit ex hypothesi AE ad EG, ut CF ad FH, & EG ad GI, ut FH ad HO: ideoque permutando ut AE ad CF, sic EG ad FH, & ut EG ad FH, sic GI ad HO. ergo [e] ut AE una antecedentium, ad CF unam consequentium, sic omnes antecedentes, id est linea AI ad omnes consequentes, id est lineam CO: sed ut AE ad CF, sic est ex constructione AB ad CK; ergo AI, est ad CO, ut AB ad CK; quod est absurdum; ut patet ex elementis. non est igitur proportio AB ad CD minor proportione AE ad CF.

A   E   G K   I   B   C   F   H O D

Prop.116. Fig. 2.

Sit iam, si fieri potest, proportio AB ad CD maior proportione AE ad CF : itaque aliqua E minor quam AK, habebit ad CD eandem rationem quam AE ad CF, & quoniam aufertur semper non minus dimidio, post aliquot partes, exempli gratia post tres AE, EG, GI, ablatas, relinquetur tandem IB minor quam AK; ideoque AI erit maior quam AK. Si iam totidem auferantur est quantitate CD, nempe partes CF, FH, HO, erit ex hypothesi, & permutando AE, ad CF, ut EG, ad FH,

[120]

item ut GI ad HO. ergo a ut AE una antecedentium, ad CF unam consequentium, ita omnes antecedentes, id est linea AI, ad omnes consequentes, nempe lineam CO. Atqui ex constructione ut AE ad CF, sic erit AK ad CD; ergo AI est ad CO ut AD ad CD, quod esse absurdum patet ex elementis. non est igitur ratio AB ad CD, maior ratione AE ad CF. patet ergo proportionis veritas. *b 8 Quinti; c 1 Decimi; d 8 Quinti ; e 8 Quinti; a 12 Quinti.*

Corollarium.

A						E		G	I	B

C			F				H K O D			
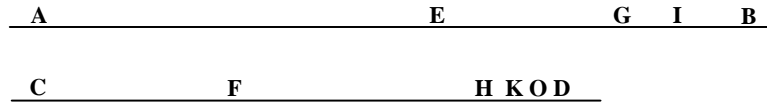
Prop.116. Fig. 3.

A duabus quantitatibus AB, CD, auferri possint AE, CF, aequalia, & non minora dimidio ipsarum AB, CD; & a residuis EB, FD rursum auferri possint EG, FH, aequalia & non minora dimidio residuorum : sic hoc semper fieri possit, aequales erunt quantitates AB, CD. Patet ex demonstratione propositionis.

Quamquam fateat hoc Theorema aliud non continere, quam particularem casum propositionis prioris: tamen quia in libris sequentibus non semel usui veniet, visus mihi sum operae pretium facturus, si facilitatis causa explicite hic apponerem.

Similiter hoc quoque Theorema eiusdem proportionis universalis casus erit : si fuerint duae quantitates, a quibus auferri semper possint non minora dimidio, sic ut ablata singula unius, dupla perpetuo sint singulorum ex altera ablatorum, erit una quantitas alterius dupla.

Quod si ablata unius, semper tripla fuerint ablatorum alterius; erit una quantitas, alterius quadrupla. Atque ita in infinitum per proportiones quadruplam, quintupla, &c. licebit procedere.

PROPOSITION 116.

There are two lengths AB and CD ; and AB is divided by E and G, so that AE is not less than half of AB, and EG is not less than half of EB; CD is divided by F and H in the same way, and AE, EG; CF, and FH are proportionals: and this is can be done indefinitely for both lines.

I say that the whole length AB to the whole length CD is as AE is to CF.

Demonstration.

If indeed the proportion AB to CD is not equal to the proportion AE to CF, it will be either greater or less than this ratio: first let us assume that it is smaller. Now, when AB to CD is made to have a smaller ratio than AE to CF, then AB will have the same ratio to some quantity less than [b] CD, surely CK, as AE to CF: and since from the quantities AB and CD, from these portions of the lines still remaining, a proportion always greater than a half is taken away, if these terms are continued in proportion by subtraction. For example, if the three terms CF, FH, and HO are constructed in the second series, and there remains finally a term KO less [c] than KD ; in this case CO is greater than CK. If now from the other series AB, the sum of the same terms according to the example of the proportion AE, EG, GI is taken away, then by hypothesis AE is to EG, as CF to FH, and EG is to GI, as FH to HO: and thus on interchanging, as AE is to CF thus EG is to FH, and as EG is to FH, thus GI is to HO. Hence, [e] the ratio of AE, the first term of the first series to CF, the first term of the second series, is thus to the sum of these terms of the first series or the linea AI, to the sum of the corresponding terms of the second series or the line CO: [i. e. AE/CF = AI/CO] but as AE is to CF, thus by construction AB is to CK [i. e. we have assumed that AE/CF = AB/CK]; hence AI is to CO as AB is to CK; which is absurd, as is apparent from basic principles. [i. e. AI/CO = AB/CK : the partial sum CO is greater than the whole assumed sum CK]. Therefore the proportion AB to CD is not less than the proportion AE to CF.

Now, if it is possible, the proportion AB to CD is greater than the proportion AE ad CF: and hence some [e] lesser length such as AK, has the same ratio to CD as AE to CF, and as the amounts taken away are always greater than half the amount left, after some number of terms, for example after the three terms AE, EG, and GI taken away, there remains finally a length IB less than AK; and thus AI is greater than AK. If now the same sum of terms is taken away from the quantity CD, surely the lengths CF, FH and HO, and by hypothesis and on interchanging AE is to CF as EG is to FH, and likewise as GI is to HO. Hence [a] as AE, the first term of the first series to CF the first term of the second series, thus the sum of the first series or the line AI, to the sum of the second series, surely the line CO. But from the construction as AE is to CF, thus AK is to CD; hence AI is to CO as AD to CD, which is absurd from elementary considerations.

hence the ratio AB to CD in not greater than the ratio AE to CF. The truth of the proportion is thus apparent. *b 8 Quinti; c 1 Decimi; d 8 Quinti ; e 8 Quinti; a 12 Quinti.*

Corollory.

From two quantities AB and CD, let it be possible to take away equal
amounts AE and CF, which are not less than half of AB and CD; and from the remainders EB and FD again take equal amounts EG and FH, not less than half of these remainders : if this can always be done, the quantities AB and CD are equal. This is apparent from the demonstration of the proposition.

Although I must confess that this Theorem does not contain anything other
than a particular case of the preceding proposition: nevertheless because in the following books it will be used not just once, it seems to me that I would improve the quality of the work, if for the sake of convenience, I were to expound it here explicitly.

Similarly too, this Theorem is an instance of the general proposition: if there
are two quantities from which it is always possible to take away not less than half, and the amount taken from the first quantity is twice the amount taken from the other, then one quantity is twice the other.

And likewise, if an amount is taken from the first, and three times as much can
always be taken from the other, or there is one quantity and four times from the other. Thus to infinity through the proportions four times, five times, etc, as you may wish to proceed.

L2.§2.   *PROPOSITIO CXVII.*
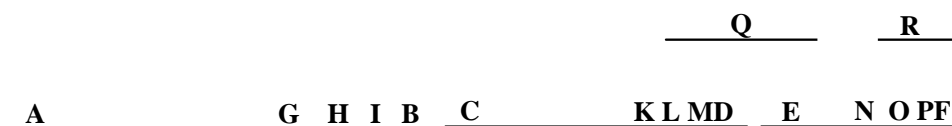
Prop.117. Fig. 1.

Data sint tres magnitudines, aut plures AB, CD, EF & a singulis auferri possit non minus dimidio, ita ut ablata AG, CK, EN sint in continua analogia Q ad R. Deinde a residuis auferri possit iterum non minus dimidio, ita ut ablata GH, KL, NO sint continue proportionalia in ratione eadem Q ad R: & hoc semper fieri possit.

Dico propositas magnitudenes AB, CD, EF esse in continua analogia.

Demonstratio.

Quoniam AG est ad CK, ex hypothesi ut Q ad R; & GH ad KL, & HI ad LM, ut Q ad R, erunt partes ablatae AG, CK, GH, KL; HI, LM, atque ita in finitum invicem proportionales : Quare cum hypothesi etiam singulae sint non *b* minores dimidiis suorum integrorum, erit AB ad CD, ut AG ad CK, hoc est ex datis ut Q ad R. Similiter ostendam CD esse ad EF, ut CK ad EN, hoc est ex datis ut Q ad R. erunt igitur AB, CD, EF, continuae proportionales magnitudines. Quod erat demonstrandum. *b 116 Huius.*

[121]

PROPOSITION 117.

There are three or more lengths given AB, CD, EF, and not less than half of each can be taken from the individual sections, in order thus that the parts taken AG, CK, EN are in the continued ratio Q to R. Then from the remaining lengths not less than half of each can be taken away, in order that the remaining lengths GH, KL, NO are in continued proportion in the given ratio Q to R : and this process can be continued indefinitely.

I say that the magnitudes AB, CD, and EF are in continued proportion.

Demonstration.

Since AG is to CK, from hypothesis as Q to R; and GH to KL, and HI to LM, as Q to R; then the parts taken AG, CK, GH, KL; HI, LM, and thus indefinitely are in turn proportionals. Whereby by hypothesis the individual terms also are not[b] less than half of their wholes, hence AB is to CD as AG to CK, that is given as Q to R. Similarly I can show that CD is to EF as CK to EN, or as Q to R from what is given. Therefore AB, CD, and EF are magnitudes in continued proportion. Q.e.d.   *b 116 Huius*

L2.§2.   *PROPOSITIO CXVIII.*

A	G	H I	B	C	K L MD	E	N O PF	Q	R

Prop.118. Fig. 1.

Propositae sint tres, aut plures magnitudines, AB, CD, EF & ratio Q ad R quaecumque, minoris inaequalitatis: Auferantur a singulis AG, CK, EN, ita ut ablata AG, CK, EN, sint ad sua tota, in ratione Q ad R & in eadem ratione Q ad R inter se continuae proportionalia;

Dico hoc semper fieri possit, propositas magnitudenes AB, CD, EF esse in continua analogia.

Demonstratio.

Quia ex hypothesi AG est ad AB, ut GH ad GB, erit etiam [a] reliquum GB ad reliquum HB, ut tota AB ad totam GB; sunt igitur AB, GB, HB & eodem discursu etiam IB reliquaeque in infinitum continuae proportionales : unde etiam [b] ablata AG, GH, HI, &c. sunt in continua analogia, & [c] terminus huius progressionis AG, GH, &c. est B. similiter ostendam ablata CK, KL, LM, &c. esse in continuae analogia, cuius terminus sit D. & quoniam ex hypothesi AG est ad AB, ut Q ad R; & CK ad CD, ut Q ad R, erit AG ad AB ut CK ad CD; & invertendo ac per conversionem rationis AB ad GB, ut CD ad KD; Atqui AG, GH, HI, &c, sint [d] continuae proportionales in ratione AB, ad GB; hoc est ut iam ostendi CD ad KD; & per eandem CK, KL, &c. sunt etiam continuae in ratione CD ad KD: Igitur AG, GH, &c. CK, KL, &c.similium rationum series sunt. Quare AB est ad CD [e], ut AG ad CK. simili prorsus discursu ostendam CD esse ad EF ut CK ad EN, sunt autem ex datis AG, CK, EN, tres continuae proportionales; ergo AB, CD, EF in continua sunt analogia. Quod erat demonstrandum.   *a 19 Quinti; b 1 Huius; c 79 huius; d 6 huius; e 84 huius.*

PROPOSITION 118.

Three or more magnitudes [or lengths] AB, CD, EF, &c. are proposed,and some ratio Q to R of the smaller inequality [less than one]: AG, CK, and EN are to be taken from the individual lengths in order that the terms taken AG, CK, EN are in the ratio Q to R to their own total lengths, and each series is in continued proportion in the same ratio Q to R;

   I say that this can always be accomplished, and the proposed magnitudes AB, CD, EF
are in continued proportion.

Demonstration.

Since from hypothesis AG is to AB, as GH to GB, also the remainder [a] GB is to the remainder HB, as the
total AB to the total GB; therefore AB, GB, HB, and by the same discourse IB and the rest also, are part of
an infinite series of continued proportionals : hence al so [b] the remainders AG, GH, HI, &c. are in
continued proportion, and [c] the terminus of this progression AG, GH, &c. is B. Similarly I can show that
the remainders CK, KL, LM, &c. are in a continued progression, the terminus of which is D. Since from
hypothesis AG is to AB as Q to R; and CK to CD as Q to R, then AG is to AB as CK to CD, on inverting
and conversion of the ratio AB to GB as CD to KD. But AG, GH, and HI, &c, are [d] continued proportions
in the ratio AB to GB; this is as now shown CD to KD; and in the same manner CK, KL, etc. are also
continued in the ratio CD to KD. Therefore AG, GH, etc. and CK, KL, etc. are series of similar ratios.
Whereby AB is to CD [e] as AG to CK. In short by a similar argument I can show that CD is to EF as CK to
EN;   but AG, CK, EN are given as three continued proportionals; hence AB, CD, and EF are in continued
proportion. Q.e.d.   *a 19 Quinti; b 1 Huius; c 79 huius; d 6 huius; e 84 huius.*

L2.§2.   *PROPOSITIO CXIX.*
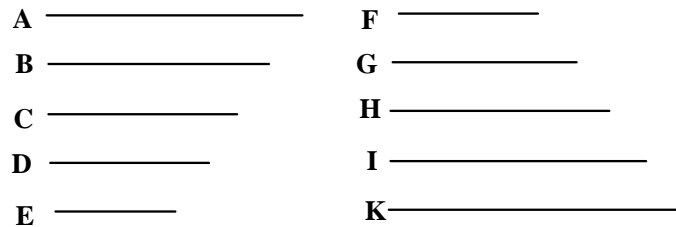
Prop.119. Fig. 1.

   Si data quaelibet proportio maioris inaequalitatis A ad B, continuetur perpetuo;
devenietur tandem ad magnitudinem data minorem.

   Dico hoc semper fieri possit, propositas magnitudenes AB, CD, EF esse in continua
analogia.

Demonstratio.

Ponatur enim magnitudo quaevis F; & fiat ad altam G ut B ad A; continueturque ratio F ad G, donec per
septuagesimam septimam huius habeatur K magnitudo, maior A magnitudine; & per totidem terminos
continuetur ratio A ad B. Dico E minorem esse quam F. est enim ut A ad B, sic G ad F; & ut B ad C, sic H
ad G, &c. ergo ex aequo in proportione perturbata, ut A ad E, sic K ad F; sed A minor est quam K, ergo &
E [f] quam F. Quod erat demonstrandum.   *f 14 Quinti.*

Scholium.

*Huc etiam pertineret propositio septuagesima septima, nisi illam, quod ad terminum progressionis
inveniendum esset necessaria, coacti essemus citeriori loco collocare.*

[122]

PROPOSITION 119.

   If some proportion is given of the greater inequality of A to B, and it is continued
repeatedly, then finally it will come to a given smaller magnitude.

Demonstration.

For some magnitude F is put in place to a greater G and made in the same ratio as B to A; and the ratio F to
G is continued, until according to Prop. 77 of thus book, it can have a magnitude K greater than the

magnitude A, and the ratio A to B is continued throughout the terms. I say that E is less than F: for as A is to B, thus G is to F; and as B is to C, thus H is to G, etc., and hence from the equality of the re-arranged proportions, as A is to E, thus K is to F; but A is less than K, and hence ^{f}E is less than F. Q. e. d. *f 14 Quinti.*

Scholium.
Up to this point too, the theorem is related to the 77th proposition, except that it was necessary there to continue to the terminus of the progression, and we are able to gather terms to a nearer place here.

L2.§2.   *PROPOSITIO CXX.*
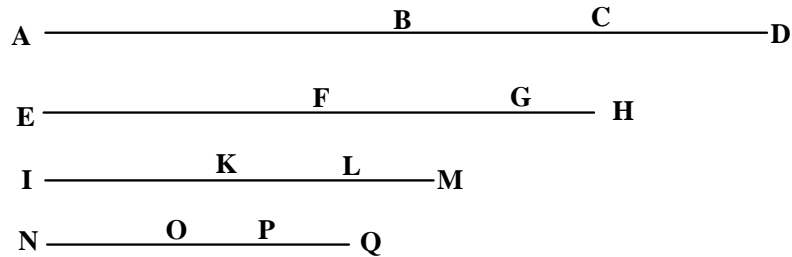
Prop.120. Fig. 1.

Sit magnitudo aliqua AD, secta in tres partes AB, BC, CD: ablata media BC vel alterutra extremarum, residuis AB, CD, fiat aequalis EH, quae dividatur in tres partes EF, FG, GH in eadem ratione qua secta est AD: se hoc continuetur;

Dico relinqui tandem magnitudinem data minorem.

Demonstratio.

Quoniam AB est ad BC, ut EF ad FG, & BC ad CD, ut FG ad GH; igitur permutando AB ad EF, ut BC ad FG, & CD ad GH: ergo ^{a} ut AB ad EF, sic AD ad EH: & permutando AB ad AD, ut EF ad EH, similiter demonstrabimus DC esse ad DA, ut HG ad HE: ergo AB ^{b} cum CD, ad AD, ut EF cum GH ad EH: & invertendo AD ad AB cum CD, id est ex hypothesi EH, ut EH ad EF cum GH, id est ex hypothesi IM. sunt igitur AD, EH, IM in continua ratione minoris inaequalitatis. non aliter demonstrabimus NQ, caeteraque residua in infinitum cum prioribus eandem proportionem minoris inaequalitatis continuare. Quare ^{c} relinquetur tandem magnitudo data minor. Quod erat demonstrandum.   *a 12 Quinti; b 24 Quinti; c 119 Huius.*

PROPOSITION 120.

Some magnitude AD is cut into three parts AB, BC, and CD; and the middle part BC or one of the end parts is taken away, and EH is made equal to the sum of the remaining parts AB and CD. This remaining part is divided into three parts EF, FG, and GH in the same ratio as which AD is cut: and this process is continued indefinitely among the parts.

I say that the magnitude can be diminished to any given [smaller] amount.

Demonstration.

Since AB is to BC as EF is to FG, and BC is to CD as FG is to GH; therefore on permuting, AB is to EF, as BC is to FG, as CD to GH: hence^a , as AB is to EF, thus AD is to EH: and on permuting, AB is to AD, as EF is to EH. Similarly, we can show that DC is to DA as HG is to HE: hence^b (the sum of AB and CD) to AD is as (the sum of EF and GH) to EH: and on inverting AD is to (the sum of AB and CD, or EH by hypothesis), as EH is to (the sum of EF and GH, or IM hypothesis). Therefore AD, EH, and IM are in a continued ratio of the smaller inequality [i. e. in the ratio < 1]. In the same way can we show that NQ, and the remaining remainders are to continue indefinitely with the same smaller proportion as before. Whereby^c at last a given small magnitude is left. Q. e. d.   *a 12 Quinti; b 24 Quinti; c 119 Huius.*

[AB/BC = EF/FG, and BC/CD = FG/GH; therefore on permuting, AB/EF = BC/FG = CD/GH = k: hence
AB = k.EF; BC = k.FG; CD = k.GH; and hence AD = k.EH giving AB/EF = AD/EH = k: and on permuting,
<u>AB/AD</u> = <u>EF/EH</u>.
Similarly, <u>CD/AD</u> = <u>GH/EH</u>: hence <u>(AB + CD)/AD</u> = <u>(EF + GH)/EH</u> on equating the underlined terms: and
on inverting AD/(AB + CD, or EH) = EH/(EF + GH, or IM). Hence, AD, EH, and IM are in the continued
proportion of the inequality less than one .
In modern terms, let AB = a; BC = ar; CD = ar^2 ; then AD = $a.(1 + r + r^2)$; EH = $a.(1 + r^2)$;
k = AD/EH = $(1 + r + r^2)/(1 + r^2)$; <u>GH</u> = CD/k = $ar^2.(1 + r^2))/(1 + r + r^2)$; <u>EF</u> = $a.(1 + r^2)/(1 + r + r^2)$;
IM = <u>EF + GH</u> = $a.(1 + r^2)^2/(1 + r + r^2)$ = EH^2/AD as required.
The lesser ratio is EH/AD = $a.(1 + r^2)/(1 + r + r^2)$.]

L2.§2.   *PROPOSITIO CXXI.*

A ——————————————————————— B
                          D   E   F   C
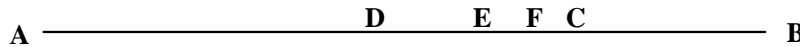
Prop.121. Fig. 1.

  Data sit magnitudo utcumque secta in C, ac inter AB, CB media proportionalis ponatur
DB; rursum inter DB, CB media sit EB, & hoc continuetur.
  Dico ex AC relinqui tandem magnitudinem data minorem.

Demonstratio.

Per primium huius ; AD est ad DC ut AB ad DB; atqui AB maior est quam DB, ergo etiam AD est maior
quam dimidia AC: Similiter quoniam DB, EB, CB sunt continuae, erit DE ad EC, ut DB ad EB: quare DE
maior est quam EC, ideoque & maior quam dimidis DC. Eodem modo probabitur EF esse plus dimidio ab
EC, atque ita in infinitum semper plus dimidio ab AC, eiusque; residuis auferetur. Quare [d] relinquetur
tandem magnitudo data minor. Quod erat demonstrandum.   *d 1 Decimi.*

[123]

PROPOSITION 121.

  The given magnitude AB is cut at C, and the mean of the proportions DB is placed
between AB and CB; again the mean proportional EB is placed between DB and CB :
and this process is continued among the parts.
  I say that the magnitude to be left at last from AC is less than some given amount.

Demonstration.

According to the first part of this above; AD is to DC as AB is to DB; but AB is greater than DB, hence
also AD is greater than the half of AC. Similarly, since DB, EB and CB are continued proportions, DE is
to EC as DB is to EB: whereby DE is greater than EC, and likewise greater than half DC. In the same way
it is agreed that EF is more than half from EC, and thus indefinitely always more than half from AC, and
the remainder is always taken from this amount. Whereby [d] at last the magnitude left is less than a given
magnitude. Q. e. d.   *d 1 Decimi.*
[As AB, DB, and CB are continued proportions, hence AB/DB = <u>DB/CB</u>; from which on subtraction,
AD/DB = DC/CB, giving <u>DB/CB</u> = AD/DC = AB/DB as required; as AB > DB then AD > DC or AD is
greater than half AC. Again, DB/EB = EB/CB, as DB, EB, and CB are continued proportions also; from
which as just demonstrated, DE/EB = EC/CB or EB/CB = DE/EC = DB/EB; since DB > EB then DE > EC
or DE is greater than half DC; and so similarly, EF is greater than half EC.]

L2.§2.   *PROPOSITIO CXXII.*

Inter duas magnitudines inaequales A, B, inveniatur media proportionalis C, & inter has tres A, C, B, inveniantur duae mediae D & E: rursum inter illas quinque, quatuor statuantur mediae, & hoc semper fiat.

Dico hac praxi tandem exhibendas lineas quae simul sumptae maiorem sint data quavis magnitudine.

A ————————————————————

D ————————————————

C ——————————————

E ————————————

B —————————
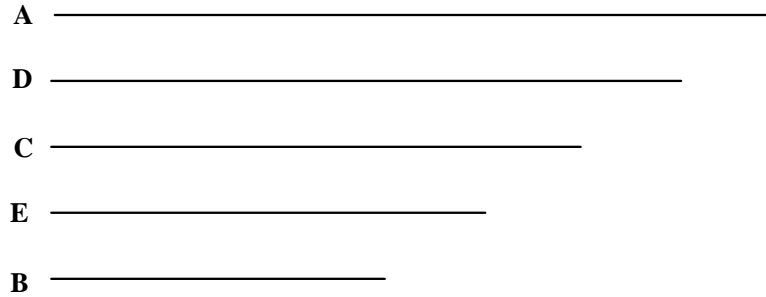
Prop.122. Fig. 1.

Demonstratio.

Quoniam C media est inter A & B , habebit A ad B maiorem rationem, quam ad C; ergo C maior est quam B; ergo tres magnitudines A, C, B maiores erunt quam tripla ipsius B. Similiter ostendam D & E maiores esse singulas, quam B: ac proinde A, D, C, E, B simul sumptas maiores esse, quam quintupla ipsius B; atque ita demonstrabimus si plures semper mediae reperiantur, summam magnitudinem, excessuram B magnitudinem determinatam, secundum quemvis numerum assignabilem. ex quo liquet magnitudines illas simul sumptas, futuras quavis data quantitate maiores.

PROPOSITION 122.

Between two unequal magnitudes A and B, the mean C of the proportions is found; and between these three A, C, B are found the two means D and E: again, betweem these five proportionals, four means are put in place, and this construction can always be made.

I say that this practise finally produces lines which have a sum greater than any given magnitude.

Demonstration.

Since C is the mean of A and B, the ratio A to B is greater than the ratio A to C; hence C is greater than B. and hence the sum of the three magnitudes A, C, and B is greater than three times B. Similarly, I can show that the further means D and E are each greater than B, and hence the sum of A, D, C, E, B is greater than five times B; and thus we can show that if always more means are found, then their sum surpasses the magnitude determined by B, according to whatever number is assigned for that quantity: from which it is proven that the sum of these magnitudes is soon greater than any given quantity.