[51]

QVADRATVRÆ
CIRCULI
LIBER SECUNDUS.
DE
PROGRESSIONIBUS GEOMETRICIS.
AD LECTOREM.

Praesens liber, quem de progressionibus Geometricis inscribimus, omnino necessarius est ad sternendam viam, quam inimus circulo ad quadratum reducendo: non ita tamen hoc velim intelligas, ut omnes omnino propositiones, quae in toto eius decursu reperiuntur, ad eum finem requiri credas; sed quod sine usu huius libri, quoad partes maxime principales, difficulter ad scopum pervenire quis possit: exigebat autem libri argumentum, ad doctrinae formam vel leviter saltem concinnari, & cognatis materiis exornari, ne faetum imperfectum ederemus. Idem de sequentibus libris iudicium derre dignabere.

TO THE READER.

The present book, in which we write about geometric progressions, is all the more necessary as we establish in it the path we must follow to undertake the reduction of the circle to a square. I would not have you believe that all of the propositions presented in this discourse are needed to reach that end; however, without the use of this book, especially as far as the main parts are concerned, it would be more difficult for anyone to be able to reach that goal. Thus the main subject matter of the book is prepared according to the basic principles of geometry lest we should publish an abortive attempt, while other parts are dealt with more lightly and furnished with related material. The same applies to the rest of the books in this work that are deemed worthy to be brought forward for your judgement.

ARGVMENTVM.

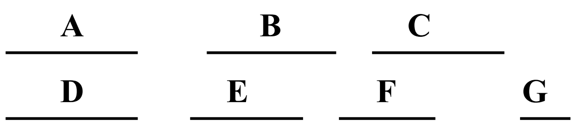
Plurimi de hoc argumento Speculationes non tantum Theorematicas, verum etiam Problematicas conscripserunt : sed omnes, prout ex libris huc usque conscriptis cognoscere potui, Arithmeticas materias concernantes, in medium attulerent : quas authoribus suis omnino intactas relinquo; mei enim instituti est progressiones tractare Geometricas, non Arithmeticas : & per illas cognoscere quantitatum omnis speciei magnitudines, sine illae in lineis, sine superficiebus, vel etiam solidis exhiberi debeant. Occasionem huic considerationi subministrarunt nonnulla, cum in Archimede, tum in Euclide loca, quae iubent in constructione Geometrica, auferri (verbi gratia) ab aliqua quantitate dimidium , & huius iterum dimidii dimidium, & pro clausula adfertur, & hoc semper fiat. *Titillavit me haec particula, & coegit morosiore cogitatione circa haec versari : tandem post longas vexas intellectus, ea quae mihi inciderent, tibi benigne Lector communico, ut quae huic materiae desunt supplere digneris. Illam etenim* *solum direxi ad cognoscendas quantitates, quas instituto meo necessarias esse arbitrarus sum: communi enim Geometria, quam a veteribus*

[52]

accipimus, non existimavi posse quempiam viam sibi sternere, ad problemata solvenda, quae a Mathematicis per tot saecula desiderantur: unde novas artes & methodos novas indicavi excogitandas, quae supplerent Geometriae antiquae defectus; dignaberis itaque benigne Lector, boni haec consulere, & cum adverteris , multa hic esse quae incudi reddi debeant, utpote male tornata, memineris velim : ita scientias in orbem esse ingressas; primo mutilas & inconcinnas, quas multorum tandem manus ad perfectum nitorem reduxerunt; facili etenim negotio inventis quidpiam addi potest, plura quippe vident oculi, quam oculus; qui domini sui iussu certis terminis se continere cogitur.

In quatuor partes partimur hunc tractatum, quo distinctius procedamus, & captui tyronum magis inserviamus.

Prima inserviet contemplationi progressionum inchoatarum , sine necdum terminatarum, quod terminus progressionum nondum in considerationem adducatur. quid autem sit Progressio Geometrica, quis eius terminus, & his similia, patebit ex sequentibus. Sed tribus verbis conabimur praesentis partis nomenclaturam explicare.



Datur quaevis ratio A ad B ;& petatur tertia proportionalis ad has duas quantitates, exhibeaturque tertia C : continuatio trium horum terminorum dicetur esse progressionis interminatae : eo quod possit ulterius procidi in eadem serie : nam si dentur tres quantitates D, E, F, & addatur quarta G , quae continuet eadem rationem D ad E, progressio terminorum D,E,F,G, ulterius est producta, quam sit progressio terminorum A, B, C: cum haec consistat in duabus rationibus; illa vero in tribus. Porro hac methodo procedendo, continuo auctis rationibus; augetur progressio ; manet interim semper interminata. Haec igitur pars primae non perget ulterius, & sistet in sola consideratione Progressionis huius, quam vocamus interminata, ad distinctionem alterius, quae tota exhaurietur, ac proinde terminabitur seipsa. variis igitur proprietatibus inchoatae progressionis in medium allatis, subsequitur.

Secunda pars, quae tota versabitur circa progressionem terminatam, absolutam, sine exhaustam; atque hoc universim in quocunque genere quantitatis : indagando scilicet cuiuscunque rationis, si in infinitum censeatur continuata, magnitudinem , seu quantitatem. neque velim quis in animum inducat, nos materiam ingredi, quae placitis philosophorum contradicat :imo ostendemus luce clarius, hac nostra methodo dissolui etiam gravissimas difficultates, quibus in Gymnasiis & Philosophorum Licaeis solent in materia quantitatis invicem esse molesti. Quod ut exemplo uno manifestius explicetur, subeat memoria argumenti, quod Achilles Zenonis nominatur, quo omnem motum eliminare ex orbe se posse contendebat: omnibusque studebat persuadere, falli oculos, dum quid loco moveri arbitrarentur; argumento ad id formato duorum, quae moverentur; Achillis scilicet velocissime currentis, & testudinis tardissime reptantis

Ponatur , inquiebat ille, Achilles cursor pernicissimus, ex A puncto, testudinem pergentem per semitam BC tardissimo motu, velle assequi suo curfsu : Quo tempore Achilles tendit ex A in B, mota est testudo ad aliquod spatium, perveniens in D, igitur necdum Achille assecutus est testudinem : iterum quo tempore

[53]

ex B Achilles currit, ut assequatur testudinem existentem in D, mota est testudo perveniens ad E punctum ; igitur nondum assecutus est testudinem, atque hoc in infinitum continget ; igitur nunquam assequetur Achilles testudinem. Argumentum hoc Zenonis facillime expedietur per ea, quae secunda huius libri parte adferemus ; nam ex doctrina illius partis, non solum manifestum fiet, Achillem perventurum ad testudinem , sed ipsum punctum assignabitur in quo apprehensio testudinis futura est.

Tertia Pars huius libri versabitur circa progressones terminatas, planorum praesertim similium, quae commodiora sunt ut ad doctrinae seriem reducantur : ut si datis duobus quadratis in ratione maioris inaequalitatis, quis requirat superficiem exhiberi, quae aequalis existat magnitudini, quam tota illa series quadratorum produceret, orta ex progressione rationis primi quadrati ad secundum, & huius secundi ad *tertium , & sic in infinitum : atque ut idipsum familiarius exponam, ponatur quis equum generosissimum velle a quopiam mercari, qui vulgi opinione mille aureis aestimatur: cumque mille aurei eidem non sint ad manum, hac sponsione contrahit, se post mensem centum aureos ei donaturum ; secundum vero mensem quinquaginta. post tertium vero vigintiquinque, atque ita deinceps : post singulos menses, perpetuo censu dimidium dimidii pretii pendat eius, quid postremo mense creditori persolverit: tandem pertaesus molestiarum, conetur pacisci cum eodem, offerendo ducentos aureos pro residuo, ut ab illo censu sese expediat, quis horum duorum tali contractu decipitur; emptorne an venditor? huius & similium quaestionum solutiones, doctrina tertiae partis huius libri, liquidissime experiet : & assignabit, quae post singulos menses summa sit residue debiti; imo & si quis postularet nosse qualis summa residua esset futura, post centesimum, imo millesimum mensem, Geometrica certitudine ex praesentis Partis scientiae cognoscet.*

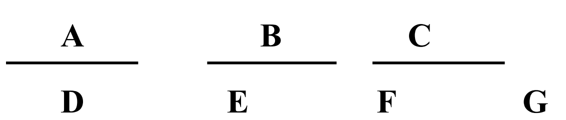
Quarta Pars totam doctrinam prioribus partibus explicatam corporibus solidisque applicabit : spero huius tractatus argumentum non fore illis ingratum, qui communi Geometria exculti sunt; admiranda enim Theoremata huius libri notitiae eruuntur, & Problemata (quae communi Geometria difficili methodo solvuntur) praxi commodissima experiuntur : sine adminiculo enim huius libri nulla esset huius operis lucubrationum certitudo, quae maxima ex parte huic fulcimento innixa est.

CONTENTS.

M*ost of the contents section for this book is concerned with speculations rather than the theorems or even problems that have been written down in it. All the material concerning arithmetical progressions, as taken from the books of others up to this point, can in the main be brought forwards : and which I leave with their authors entirely undisturbed. I have undertaken to investigate geometric progressions rather than arithmetical ones. Through these progressions I have found how to measure the sizes of all kinds of quantities* [*i.e. areas, volumes, and the like*] *that should be made known to everyone, whether the quantities are associated with lines, surfaces , or volumes. The opportunity to consider several problems of this kind has been provided to us both by Archimedes and Euclid, in which place instructions are given for geometric constructions in which, for example, half is to be taken from some quantity, and half of the half again, and so on, until some conclusion is produced, and this shall always be the case. This has always been a point of great interest for me, and finally I have brought to fruition many difficult thoughts that I have turned over in my mind concerning these things for a long period of time. Finally, gentle reader, I share with you these thoughts that befell me, in order that what is lacking in this material you may deem worthy to supply. My one and only concern has been directed towards understanding the quantities that I have considered necessary: indeed from the common geometry that we have accepted from the ancients, I have not been able to establish any way to solve the problems which mathematicians through the ages have desired to solve. From what I have said, new skills and methods need to be thought out which can be applied to the shortcomings of the geometry of the ancients. Thus, kind reader, you will consider it worthy to deliberate on these good things, and with your attention turned to them, there is much here badly turned out that needs to be returned and reshaped, as I would have you remember. Thus the knowledge of the world progresses: at first new knowledge is awkward and not properly formed, and which finally is brought to a state of perfection by the work of the hands of many; and indeed it is a relatively easy task to add something to such discoveries, for many eyes of course see possible additions that have escaped the eye of the master who thought through his own judgement to have contained the work within certain boundries.*

We have divided this tract into four parts, by which means we may procede in more definate directions, and to bring to task this terrible master we wish to serve.

The first part is concerned with considering the start of progressions, without yet thinking about the end, as the end of a progression has not yet been brought forward for examination. Exactly what the geometic progression shall be, and what its end shall be, will be made clear in the following sections. But with three terms only we try to explain the nomenclature of the present part. Some ratio A to B is given, and a third



proportion to these two quantites is sought, and then the third proportion C can be written down. A continuation beyond these three terms we will call the start of an unbounded progression, in which extra terms can fall into position beyond the third term in the same series. For if the three quantities D, E, and F are given, and a fourth is added continuing in the same ratio as D to E, then a progression of the terms D, E, F, and G has been produced. This goes further than the progression of the three terms A, B, C, first considered which could terminate after the two ratios, and this other which could indeed finish after three ratios. We can again continue with the preceding method, whereby the progression is augmented successively by terms in the given ratio, and meanwhile the progression remains unterminated. Thus the first progresson we examined does not procede further beyond three terms and stopped, and we are to distinguish that kind of progression from this other sort called non-terminating: the difference of the first being that it is totally finished and has thus terminated itself. Hence in what follows , a variety of starting conditions for progressions are examined along general lines.

The second part of the book is completely given over to these geometric progressions that end generally in some kind of final quantity, but with terms added indefinitely, through an investigation performed by means of some ratio of course, in the case where the final magnitude or quantity is considered with an indefinite number of terms added on. Now I do not wish to give you the idea that we are advancing material which is contrary to the pleas of the philosophers: and indeed we will show here quite clearly that by using our method even the gravest difficulties are resolved. Indeed in the grammar schools, and in turn in the schools of the philosophers as well, students are in the habit of experiencing much difficulty with such quantitative matters.

We will set forth our method planely by means of an example that goes straight to the history of the matter with an argument by Zeno, concerning Achilles and the tortoise. Zeno contended that all motion could be banished from the earth, and strove to convince everone by deceiving the eye, while the motion of a body from some place was being observed. The argument had the form of two bodies that could be set in motion: The athlete Achilles could of course run the fastest, while a tortoise crawled the slowest.

Achilles, said Zeno, the most nimble of runners, is set to run from point A, while the tortoise is set to go along the path from B to C by the slowest motion. Achilles desires to overtake the tortoise by running : but in the time taken for Achilles to go from A to B, the tortoise has moved some distance to arrive at D. Therefore Achilles has not yet caught up with the tortoise : again, by the time Achilles has ran from B to the place D where the tortoise was, so that he might catch up with it, the tortoise has moved further to arrive at some point E; and hence Achilles has not yet caught up with the tortoise, and this process can be continued indefinitely, and Achilles can never catch up with the tortoise [*as he is involved in an infinite sequence of catch-up steps of ever diminishing length.*] *Zeno's argument is readily dispensed with through the work we present in the second part of this book. For from what is taught here, not only do we show that Achilles will overtake the tortoise, but also the point will be found where this occurs.*

The third part of this book is based around geometric progressions that terminate, especially those associated with like planes, which are more suitable for reduction to a series by the method : for if two squares are given in the ratio of the larger inequality, from which it is required to show a surface, to which size it may prove to be equal, as the whole series of squares may produce, that has come from the progression of the ratio of the first square to the second, of the second to the third, and so on indefinitely.

In order that I may explain this in more familiar terms that may be put as follows : consider someone wishing to buy from somewhere a horse of the highest pedigree, which common opinion values at one thousand gold pieces. Now, since a thousand gold pieces are not at hand, this solemn contract is entered, that after a month the buyer will give the vendor one hundred gold pieces, after the second month he will give him fifty gold pieces, after the third month twentyfive, and so on henceforth. After several months, constantly assessing half of the half of the amount he should pay of this, which he will pay to the vendor at the end of the month. Finally wearied of the troubles, he tries to make a deal with the seller, by offering two hundred gold pieces for the remainder, in order that he might be freed from the bargain. Which of the two is cheated by the deal, the buyer or the seller? The third part of this book will demonstrate most clearly the solution of this problem and similar questions, and will assign what sum will be the remainder of the debt after several months. Indeed, someone might want to know also what the remaining sum should be after a *hundred or even a thousand months. The knowledge will be known with certainty from the geometry of the present part.*

The fourth part will applied all the methods set fourth in the previous parts to finding the volumes of solid bodies. I hope the subject of this tract will not be received in an unpleasant manner by those who have cultivated common geometry. Indeed the notions contained in the wonderful theorems of this book are to be brought to light, and (which are hard to solve by the methods of common geometry) can be proven with suitable practise. Indeed, without the help of this book there would be nothing of certainty in this laborious work, which has leaned on its support to a large extent.

[54]

DEFINITIO PRIMA.

Geometricam seriem voco quantitatem finitam, divisam secundum continuationem cuiuscunque rationis datae.

Explicatio.

Quamvis sensus, quem verba indicant obscurus non videatur, nihilominus mentem meam circa definitionem praesentem censui apponendam: ponatur itaque linea quaevis AB, divise in C, secundum quamcumque proportionem; & fiat quem admodum est tota AB ad CB, ita CB ad DB, & denuo ut CB ad DB, ita

DB ad EB, & hoc semper fiat. Oritur in hac ratione procedendi duplex consideratio; prima, rationis AB ad CB, & CB ad DB, & ulterius DB ad EB, atque ita deinceps : altera rationis AC ad CD, & CD ad DE, iterum DE ad EF, & ita consequenter ; & licet hae considerationes videantur diversae, in unum tamen scopum collimant: nam cum ratio AB ad CB, & CB ad DB, eadem sit cum ratione AC ad CD, & CD ad DE, atque ita ulterius, si fiat rectae AC aequalis HI & haec HI,

dividatur in punctis K, L, M, secundum rationem AB ad CB, & CB ad DB, &c. tunc apparebit ratio, qua duae istae considerationes in unam coalescunt : si quis igitur petat, quid velim intelligi nomine seriei? respondeo me nomine serie, totam illam continuationem linearum intelligere, quarum prima est AC, secunda CD, ita ut ratio AC ad CD, eadem sit cum illa quam obtinet CD ad DE, atque ita deinceps terminatam eodem termino, quo terminatur ratio AB ad CB, &c. Et quia in definitione, particula adiuncta est *finitam* ; hinc hoc loco explicare cogor seriem per rationes AB ad CB, & CB ad DB, & ita consequentur, cum necdum demonstratum sit quo puncto lineae A C D terminus existat seriei rationis AC ad CD. nam plusquam notum est, apud philosophos nunquam perveniri posse ad terminum continuationis, per partes proportionales AC ad CD, & CD ad DE, cum residua semper remaneat quantitas dividenda, secundum easdem rationes ; quare terminus, hoc tenore progrediendi acquiri nequit : sed si quis procedat iuxta considerationem continuationis AB ad CB, & CB ad DB, & sic in infinitum, semper includit terminum, ad quem per continuationem rationis AC ad CD, perveniri nequit ; seriem itaque voco, quantitatem finitam AB, divisam secundum continuationem rationis AC ad CD, & CD ad DE. quae eadem semper existit cum ea quae sit continuando rationes AB ad CB. Quotiescunque igitur occurret mentionem fieri continuationis alicuius seriei AC ad CD, & CD ad DE; iubeat animo continuationem hanc finitam esse, & totam illam terminorum infinitorum collectionem, alibi terminatam esse : quae collectio series alicuius rationis Geometricae nuncupatur.

FIRST DEFINITION.

A geometrc series can be formed from a fixed length by the process of continued division according to some given ratio.

Explanation.

Although I am of the opinion that the wording of the definition is not vague or obscure [note: as the translator, I have taken some liberties with the phrasing of this definition, to make it more accessible to the modern reader, as the original is a little obscure], I have it in mind nevertheless to add something to the present definition. Thus some line AB is put in place, which is divided at the point C, according to some given proportion; and the division certainly gives the ratio of the whole length AB to CB, thus again the remainder is divided as CB to DB, and as CB to DB, thus

DB to EB, and in this manner the ratio of subdivision always continues. Now, two series arise from the proceding ratios which are to be considered. The first series comes from the ratios AB to CB, CB to DB, DB to EB, and thus henceforth for further terms, as we have just seen. The other series comes from the ratio AC to CD, CD to DE, DE to EF, and so on for those that follow. Although these series are considered to be different, they do however aim [*i.e.* converge in modern terms] towards a single point: for with the ratio AB to CB, and CB to DB, the same situation shall be true for the ratio AC to CD, and CD to DE, and so on for further terms beyond.

For the line AC can be made equal in lenth to HI, and this line HI is divided by the points K, L, and M following the ratio AB to CB, CB to DB, etc. Then it will become apparent that the ratio formed from these two series merges into one : if someone so desired, by what name should the series be known ? I myself reply with a name for the series, that it must be understood to be the whole continuation of the lines, the first of which is AC, the second CD, thus in the ratio AC to CD, that will be the same as obtained from the ratio CD to DE, and so on for the rest of the terms with the same limit as that with which the ratio AB to CB, etc finishes [Here we use the word *limit* in its original sense, as a barrier or boundary].

Now, a small part is added to the definition, that is, *the end* of the series. Hence I have to give an explation about this place to which a series runs together, defined by the ratios AB to CB, CB to DB, and so on for the rest of the terms, since it has yet to be shown which point on the line ACD the end of the series formed by the ratio AC to CD shall prove to be. For, furthermore, it has been noted by the philosophers, that it is not possible for one to come to the end of the continuation of terms formed by the proportional parts AC to CD, CD to DE, etc, since there is always an amount left over from the division, according to the same ratios ; whereby the terminal point can never be attained by maintaining this course: yet if one could proceed to the consideration of the nearby continuations of AB to CB, CB to DB, and so on indefinitely, then the boundary would always be reached, but which one can never reach by a finite continuation of the ratios AC to CD.

Thus I give the name series to a finite quantity AB, divided according to a continuation of the ratios AC to CD, CD ad DE, etc, which shall always be the same as that formed by the continuation of the ratios AB to CB. However many times this occurs, there has to be a mention made of that other series of continued proportionals AC to CD, CD to DE, etc. One can consider that there is a limit to this continuation of ratios, and that for the total collection of an infinte number of terms, the limit is elsewhere [*i.e.* it is not one of the ratios] : any such collection of geometrical ratios is called a series.

DEFINITIO SECUNDA.

Progressio Geometrica est quotcunque terminorum secundum eandem rationem continuatio.

Explicatio.

Est itaque omnis progressio pars seriei , cum ut explicatum est, omnis series sit continuatio alicuius rationis eo usque producta, donec amplius protrahi nequeat, modo prius explicato; Progressio vero prout differt a serie, proprie interminata est ; ac proinde eo usque pars : ubi tamen inveneris in contextu sequentium, agi de tota progressione rationis alicuius continuata, de serie agi memineris, neque enim haec

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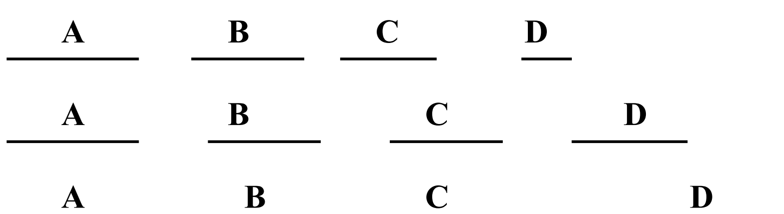
magis scrupolose observari volo, quam a Geometris omnibus scribentur huius continuatae proportionis, vel rationis; quarum licet a



altero in duobus terminis consistare, altera ......ribus, nihilominus pro arbitraque scribentis, passim ......ntur. Sint igitur haec praemissa, si non exactarum definitio cum loco, ... rhetoricis explicationis supplemento. Datis itaque A & B :vel ratio A ad B est maioris inaequalitatis, vel prorsus aequalitatis, vel minoris inaequalitatis: quod si fiant continuae proportiones A B C D huiusmodi continuatio progressio vocetur, quotcunque tandem termini existant.

Progressio vero Geometrica iam explicata duplex est; alia continua, alia discreta.Continua est cum omnes termini rationem connectentes, habent rationem antecedentis, & consequentis; ut in secumatque rationum explicatarum, si fuerit quemadmodum A ad B, ita B ad C; & quemadmodum B ad C, ita C ad D, atque ita in quovis tandem numero terminorum; huiusmodi progressionem continuam voco.

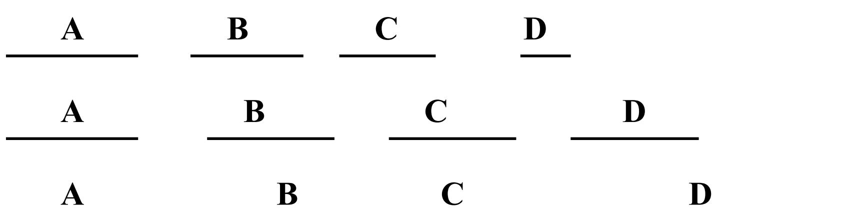
Discreta progressio est similium rationum secundum aliquam continuationem positio, ut consequentes non fiant antecedentes. exempli causa; si fiat quem quemadmodum A est ad B, ita C ad D, & B fuerit minor quam A, vel C, talem progressionem in quodlibet tandem terminis constituta sit, voco discretam; etiam in his terminis 1, 2. 5, 10. 3, 6. 8, 16. Ubi discreta ratio valde interrupta est, quia est continuatio similium rationum.

SECOND DEFINITION.

A geometric progression is a number of terms which are formed with the same continuing ratio.

Explanation.

Thus it is the case that the progression is part of the whole series: as has been explained, the whole series is a continuation of some ratio as far as that can be produced, and which cannot be extented further along the lines of the previous explanation. The progression is truly different from the series, which properly is unending, and the progression is thus part of this series : and you will however arrive at the progression in the following context that is taken from some whole series of continued ratios as you may recall, and indeed that I do not wish to observe more closely, since everything concerning the continued proportions or ratios can be written down from geometry, whereby from two given terms it is permissable to imply the others [the text is very difficult to read here], nevertheless in order to judge the writings [?], everywhere ....Therefore, with these premises, if the exact definition cannot be put in place, we can supplement with other rhetorical [?] definitions. Therefore with A and B given : either the ratio A to B is a greater inequality [i.e. A > B, corresponding to a decreasing progression], or they are equal [i.e. A = B], or it is a lesser inequality [i.e. A < B, corresponding to an increasing progression]: in which case they produce the proportions A B C D, in what is termed a continuous progression of this kind, until a certain number of terms shall arise.



Indeed the geometic progression has been set forth in a two-fold manner : the one continued and the other discrete. The continued one has all the terms connecting the ratio, the preceding terms have the same ratio as the subsequent, set forth according to the form of the ratio. If it were as A to B, thus B to C; and as B to C, thus C to D, ans thus in some final number of terms ; I call a progression of this kind a contining progression.

A discrete progession is composed of like ratios with the

position following some continuation, as a subsequent term is not made from the antecedent. For example, if the terms are made as A is to B, thus C to D, and B were less than A or C, such a progression that can then be set up with any terms I call discrete; now also with these terms 1, 2. 5, 10. 3, 6. 8, 16, where the individual ratio has been greatly interrupted, as it is a continuance of the same ratio.

DEFINITIO TERTIA.

Terminus progressionis est seriei finis, ad quem nulla progressio pertinget, licet in infinitum continuetur; sed quovis intervallo dato propius ad eum accedere poterit.

Explicatio.

Ponatur rect AB, divisa in C D E F G. ut continuent eandem rationem AB, CB, DB, EB, FB, G. Cum sit eadem ratio AC ad CD, & CD as DE, atque ita deinceps, cum ea, quae reperitur inter lineas AB, CB, DB, &c. similis quoque erit eiusdem rationis progressio AC ad CD, cum progressione AB, ad CB. terminus autem rationis AC ad CD, dicitur punctum B, sive illum intrinsecum velis, sive extrinsecum, per me licet; nam de re nobis est hic quaestio, non de verbo : ad quod punctum nulla progressio pertingere valet, cum omnis progressio interminata pars seriei existat : nihilominus tamen, poterit ad illum progressio per continuationem magis ac magis acccedere, ita ut vicinior ultimus terminus progressionis interminatae existat ipsi termino seriei, quam sit distantia quaecunque proposita.

An autem talis detur terminus progressionis, & quo pacto investigati debeat, libro secundo huius tractatos disceptabitur, illud interim insinuatum hoc loco desidero, huiusmodi terminum solum reperiri in iis progressionibus, quem proportionem maioris inaequalitaiis continuant, nam earum progressionum quae vel terminis semper aequalibus constant, vel certe terminis minoris inaequalitatis, nullum assignari posse terminum, manifestum nimis est, quam ut explicatione indigeat; quandoquidem dictae progressiones, si continuentur, magnitudinem quavis data maiorem exhibere natae sint. Itaque secundo libro, tertim et quarto, in quibus progressiones iam terminatae

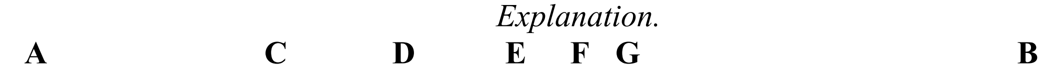
[56]

considerantur, nullae proportiones praeter eas, quae maioris erunt inaequalitatitis : in primo vero, quoniam illis a termina adhuc abstrahitur, etiam progressiones aequalitatis, & minoris aequalitatis contemplabimur. Terminus igitur progressionis talis est, quaemadmodum explicuimus, cum scilicet aggregatum, sive summa terminorum progressionis, quantumvis continuatae numquam excedit quandam magnitudinem; excedit vero omne minus llla magnitudine , atque ita posset etiam dici productum sive quantitas totius, datae progressionis, & magnitudo illa aequalis dicetur toti progressioni dati ; hoc est omnibus terminis proportionalibus simul sumptis. Idem igitur quoad rem erit, sive terminum quaerere progressionis , huc magnitudinem toti progressini parem , sive ipsammet integram seriem progressionis exhibere. Si quid praeteria pertinebit ad terminoram explanationem, in decursis operis exponetur.

THIRD DEFINITION.

The limit [terminus] of the progression is the end of the series, a point which no progression will reach even if it is allowed to continue indefinitely; but for any given interval around the end point, terms in the progression will be able to come nearer.

Explanation.



The line AB is put in place, divided in the points C D E F G, in order that they continue in the same ratio AB, CB, DB, EB, FB, GB. The ratio AC to CD, and CD to DE, etc, and thus henceforth, shall be the same as which is found between the lines AB, CB, DB, etc, and the ratio of the progression AC to CD will be similar too will be the the progression with the progression AB to CB. But the limit [terminus] of the ratio AC to CD is said to be the point B, either inside as you may wish, or outside, permitted by me; for concerning this thing this is the question for us, but not concerned with words : what is the point that no progression is able to reach, with the whole infinite part of the progression set out ? Yet nevertheless, more and more terms can be added to that progression by continuation, and thus as the final term of the progression gets closer to the limit of the infinite series itself, that shall be the proposed distance whatever.

However, just so many terms of a progression may be given, and the arrangement by which such a progression ought to be investigated will be discussed in the second book of this tract. Meanwhile, a number of these terms have insinuated themselves into the place I desire. A limit of this kind is only to be found in these progressions that continue with the larger inequality. For those progressions where the terms are either always equal or constant, or indeed for terms with the smaller inequality, nothing can be assigned to the limit, as is quite clear by way of any explanation needed, since the said progressions, if they are to continue, show some given magnitude become larger. Thus with the second, third, and fourth books, in which terminating progressions are now considered, there are no proportions except these from greater inequalities. In the first book, indeed, since for these so far only parts of the progressions separate from the limit will be contemplated, as with also with progression of equal or smaller ratios. The limit of a progression is therefore such, as we have explained, with a known collection or sum of terms of the progression, although continued to any will never exceed a certain amount. Truly it will exceed all magnitudes less than that amount, ans thus it may also be said to be the product or size of the whole given progression. This magnitude is said to be the total of the given progression ; that is , the sum of all the proportional terms taken together. Therefore it amounts to the same thing: either the limit of the required progression, here equal to the magnitude of the whole progression , or the sum of the whole series of the progression itself is shown. If there is anything else besides pertaining to the explanation of the limit, then it will be set out in the works as we proceed.

[57]

PROGRESSIONUM GEOMETRICARUM

PARS PRIMA.

Progressiones considerat indeterminatas.

PROPOSITIO PRIMA.

Continue proportionalium differentiae sunt in continua analogia eiusdem rationis suorum integrorum.

Prop. 1. Fig. 1.

Sunt magnitudines continuae proportionales AG, BG, CG, DG, EG, FG. ostendendum est AB, BC, CD, DE, EF differentias in continua esse analogia suorum integrorum, & contrae si AB, BC, CD, &c, fuerint continuae, & FG addatur ut AB, BG, AB, BC sint proportionales;

Dico & AG, BG, CG, &c. esse in continua analogia.

Demonstratio.

Quoniam AG est ad BG, ut BG ad CG : erit dividendo ut AB ad BG, sic BC ad CG :& permutando AB ad BC, ut BG ad CG, sive ut AG ad BG. Similiter BG est ad CG, ut CG ad DG : & dividendo BC ad CG ut CD ad DG, & pernutando BC est ad CD, ut CG ad DG, id est ut BG ad CG. id est ut AG ad BG. atqui erat ut AG ad BG, sic AB ad BC, ergo BC est ad CD ut AB ad BC. Continuae sunt igitur proportionales AB, BC, CD, & quidem in analogia suorum integrorum AG, BG, simili ratione ostendam reliquas cum his tribus esse continuas: quod erat prinum.

Sint iam AB, BC, CD, &c. continuae, quibus addita sit FG, sic ut AG sit ad BG ut AB ad BC; Dico omnes AG, BG, CG, DG, &c. esse in continua analogia differentiarum. Quia ut AB ad BC, sic AG ad BG, ergo permutando & dividendo, & rursum permutando ut AB ad BC, hoc est ex datis ut AG ad BG, sic BG ad CG. quoniam autem iam BC est ad CD, ut AG ad BG, hoc est ex datis ut AB ad BC, hoc est rursum ex datis ut BC ad CD, eodem plane discursu ostendemus BG, CG, DG esse continuas, &c. quidem in ratione BC ad CD, hoc est AB ad BC; Quare cum etiam AG, BG, CD sint in eadem ratione continuae, omnes quatuor AG, BG, CG, DG erunt in ratione AB ad BC continuae. similiter ostendemus & reliquas cum iisdem esse continuas. Patet igitur veritas propositionis.

GEOMETRIC PROGRESSIONS.

PART ONE.

Indeteminate progressions are considered.

L2.§1. *PROPOSITION 1.*

The differences of continued proportions are in the same continued ratio as their whole lengths.

Prop. 1. Fig. 1.

The magnitudes of the lines AG, BG, CG, DG, EG, FG are continued proportionals. It is to be shown that the differences AB, BC, CD, DE, EF are in the continued ratio of their whole lengths, and likewise if AB, BC, CD, etc, are continued proportions, and FG is added in order that AG, BG, AB, BC are proportionals, then I say that AG, BG, CG, &c. are in the continued ratio.

Demonstration.

Since AG is to BG, as BG to CG : by division it becomes as AB to BG, thus BC to CG: and on interchanging the terms, as AB to BC, so BG to CG, or as AG to BG. In the same way, as BG is to CG, so CG is to DG : and by division, BC to CG as CD to DG, and on interchanging the terms, BC is to CD, as CG to DG, that is as BG ad CG, which is as AG to BG. Yet as AG is to BG, thus AB to BC, hence BC is to CD as AB to BC. Therefore AB, BC, and CD are continued proportionals, and indeed in the ratio of their whole lengths AG to BG, and I can show that the rest are in continued proportion with the same ratio as these three: which is the first part.

Now AB, BC, CD, etc. are continued proportions, to which FG is added, thus as AG is to BG so AB to BC. I say that all the lengths AG, BG, CG, DG, etc. are in the continued ratio of the differences. Because as AB to BC, thus AG to BG, therefore by interchanging terms and dividing, and again by interchanging terms as AB to BC, that is from the given ratios as AG to BG, thus BG to CG. But now since BC is to CD, thus AG is to BG, that is from the given AB to BC as BC to CD. By the same clear discourse we can show that BG, CG, and DG are in continous proportion, etc. Indeed in the same ratio BC to CD as AB to BC.

Whereby as also AG, BG, and CG are in the same continued ratio, all four AG, BG, CG, and DG are in the continued ratio AB to BC. Similarly we can show the rest to be in the same continuous ratio with these. Therefore the truth of the proposition is apparent.

L2.§1. Prop. 1 Note:

$AG/BG = BG/CG$: hence $AG/BG - 1 = BG/CG - 1$ or $AB/BG = BC/CG$: and $AB/BC = BG/CG = AG/BG$. In the same way, as $BG/CG = CG/DG$: so $BC/CG = CD/DG$, and $BC/CD = CG/DG = BG/CG = AG/BG$. As $AG/BG = AB/BC$, hence $BC/CD = AB/BC$. Therefore AB, BC, and CD are continued proportionals, and indeed in same ratio as AG to BG.

Now $AB/BC = BC/CD = \ldots DE/EF = EF/FG$ are continued proportions, to which FG is added. Hence, if we assume $AG/BG = AB/BC$, then all the lengths AG, BG, CG, DG, etc. are in the continued ratio of the differences. Since $AB/BC = AG/BG$, then $AG/AB - 1 = BG/BC - 1$ giving $BG/AB = CG/BC$ and $AB/BC = BG/CG = AG/BG$. Again from $BC/CD = BG/CG$ we can show that BG, CG, and DG are in continous proportion, and hence all four AG, BG, CG, and DG are in the continued ratio AB to BC.
Note also that we could have started an addition process from the right-hand end and arrived finally with the initial assumption that $AG/BG = AB/BC$.

[58]

PROPOSITIO II.

Si quatuor fuerint quantitates in continuata analogia, erunt aggregata ex prima & secunda, ex secunda & tertia, ex tertia & quarta, in continua proportione.

Prop. 2. Fig. 1.

Demonstratio.

Sunto quatuor in continua analogia AB, BC, CD, DE. Dico etiam AC, BD, CE, esse continue proportionales. cum enim ut AB ad BC, ita BC ad CD, erit [a] utraque antecedens AC, ad utramque consequentem BD, ut AB ad BC, una antecedens , ad unam consequentium : simili modo quia BC ad CD, est ut CD ad DE, erit utraque antecedens BD, ad CE utramque consequenem, ut BC ad CD, id est ut AB ad BC, hoc est ut AC ad BD. Sunt igitur AC, BD, CE in continua analogia. quod fuit demonstrandum.
a 11 Quinti.

L2.§1. PROPOSITION 2.

If there are four quantities in continued proportion, then the sums of the first and the second, of the second and the third, and of the third and the fourth terms are also continuously in proportion.

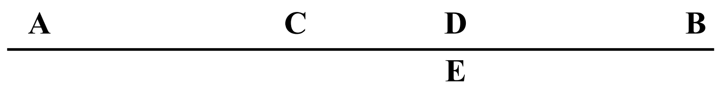
Demonstration.

Let the four terms in continued proportion be AB, BC, CD, and DE. I say that the terms AC, BD, and CE are also in continued proportion. For indeed as AB is to BC, thus BC is to CD, [a] both the preceeding terms AC, to both the following terms BD, shall be as AB to BC, the one preceeding term, to the one following term : in the same manner as BC to CD is as CD to DE, both the preceeding terms BD, shall be to both the following terms as CE as BC to CD, that is as AB to BC, that is as AC ad BD. Hence AC, BD and CE are continued ratios. q.f.d.
a 11 Quinti.
[From $AB/BC = BC/CD$ it follows that $AB/BC + 1 = BC/CD + 1$, that is $AC/BC = BD/CD$, etc.]

PROPOSITIO III.

Sit AB divisa in C & D ut ratio AB ad AC duplicata sit eius, quam habet BD ad DC. Dico AC, AD, AB tres esse in continua ratione.

Demonstratio.



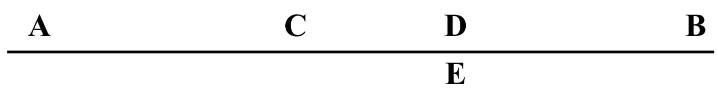
Prop. 3. Fig. 1.

Ponatur AE, media inter AB, & AC; igitur tres erunt continae quantitates AC, AE, AB quare per primam huius erit ratio AB ad AE, eadem cum ratione BE ad EC. sed ratio AB ad AC, duplicata est rationis AB ad AE, igitur & ratio AB ad AC duplicata est rationis BE ad EC. quare divisa est CB in E, ut divisa est eadem CB in D : ac proinde punctum D, unum idemque; est cum puncto E. Unde cum sint tres continuae proportionales AC, AE, AB, ex constructione, erunt quoque in continuata ratione AC, AD, AB. Q.E.D.

L1.§1. *PROPOSITION 3.*

Let AB be divided in C and D in order that the ratio AB to AC is equal to the square of that which BD has to DC. I say that AC, AD and AB are three terms in a continued ratio.

Demonstration.



Prop. 3. Fig. 1.

Put the mean AE between AB and AC; then the three quantities AC, AE, and AB are in continued proportion, whereby from the first PROPOSITION of this book, the ratio AB to AE is the same as the ratio BE to EC. But the ratio AB ad AC is equal to the square of the ratio AB ad AE, and therefore the ratio AB to AC is equal to the square of the ration BE ad EC. Whereby CB has been divided in E, as the same CB has been divided in D ; and hence the point D, is one and the same as the point E. Thus with three continued proportionals AC, AE, AB, from the construction, are also in the continued ratio AC, AD, AB. Q.E.D.

[AC/AE = AE/AB from the definition of the mean; Now, from the first proposition, {or from AC/AE + 1 = AE/AB + 1, that is EC/AE = EB/AB}, AB/ AE = BE/EC; but AB/AC = (AB/AE).(AE/AC) = $(BE/EC)^2$; and AB/AC = $(BD/DC)^2$ is given; hence CB is divided by E in the same way that it is divided by D. Hence D and E co-incide. This result is thus far easier to obtain by regarding AB, AD, and AC as a geometric progression, than by using means of ratios, etc, which is the point of this proposition.]

PROPOSITIO IV.

Quod si AB, AC,AD, sint quantitates proportionales:
Dico AD ad AB, rationem eius habere duplicatam, quam habere DC ad CB.

Demonstration.

Prop. 4. Fig. 1.

Ratio enim AD ad AB, duplicata est rationis AD ad AC, sit per primam huius ut AD ad AC, ita quoque est DC ad CB, igitur ratio AD ad AB duplicata est rationis DC ad CB. Q.E.D.

L2.§1. PROPOSITION 4.

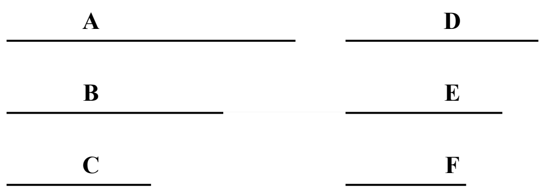
For if AB, AC, AD are proportional quantities, then I say that the ratio AD to AB is equal to the square of the ratio DC to CB.

Demonstration.

Indeed the ratio AD to AB is equal to the square of the ratio AD to AC, which is by the first PROPOSITION of this book, as AD to AC, thus too is DC to CB. Hence the ratio AD to AB is the square of the ratio DC to CB. Q.E.D.

PROPOSITIO V.

Sint tres continuae proportionales A, B, C, sit autem ratio A ad B, triplicata eius quam habet D ad E : ratio quoque B ad C, triplicara rationis E ad F. Dico D, E, F quantitates in continua analogia.



Prop. 5. Fig. 1.

Demonstratio.

Cum ratio A ad B, sit eadem cum ratione B ad C; igitur etiam ratio B ad C triplicata est rationis D ad E. est autem & ratio B ad C ex suppositione triplicata eius quam habet E ad F; igitur ratio D ad E, est eadem cum ratione E ad F. Sunt igitur in continuata proportione D, E, F. Quod demonstrandum fuit.

L2.§1. PROPOSITION 5.

Let there be three continued proportionals A, B, and C. Moreover, the ratio A to B is the cube of the ratio D to E : and also, the ratio B to C is the cube of the ratio E to F. I say that the quantities D, E, and F are in a continued ratio.

Demonstration.

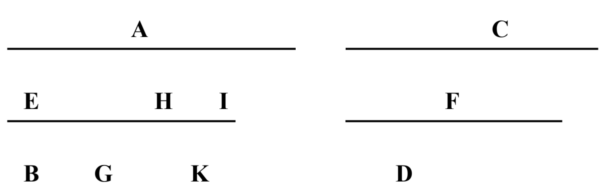
As the ratio A to B, shall be the same as the B to C; therefore the same ratio B to C is the cube of the ratio D to E. Moreover the ratio B to C from supposition is the cube of the ratio E to F; therefore the ratio D to E is the same as the ratio E to F. Therefore D, E, F are in a continued proportion . Quod demonstrandum fuit.

[59]

PROPOSITIO VI.

Sit A ad BK in minore ratione, quam C ad D.

Dico A ad EI mediam proportionalem inter A & BK, esse in minori ratione, quam sit C ad F mediam inter C & D.



Prop. 6. Fig. 1.

Demonstratio.

Fiat enim A ad BG in ratione C ad D, erit [a] BG minor quam BK. Et fiat inter A & BG media EH, erit quoque EH minor quam EI. Rursam quia A ad BG, duplicatam habet rationem A ad EH, & C ad D eamdem ex constructione habet rationem, quam A ad BG; habebit etiam C ad D rationem duplicatam eius, quam habet A ad EH : sed & C ad D duplicatam habet eius, quam C ad F; igitur ratio C ad F, eadem est cum ratione A ad EH: sed EI ostensa est maior quam EH, igitur A ad EI [b] minorem habet rationem, quam A ad EH, id est quam C ad F. Quod erat ostendendum.

a 10 Quinti; b 8 Quinti.

Corollarium.

Simili modo demonstrabitur si plures mediae inter primam & secundam, & inter tertiam & quartam ponantur, fueritque in prioribus maior proportio vel minor primae ad secundam , quam in posterioribus tertiae ad quartam; fore etiam in prioribus primae ad primam mediarum, vel secundam, aliamque quamcumque maiorem vel minorem proportionem, quam teriae ad similem ordine mediam inter posteriores.

L2.§1. *PROPOSITION 6.*

Let the ratio A to BK be less than the ratio C to D.

I say that A to EI, the mean proportion between A and BK, is less than the ratio C to F, the mean proportional between C and D.

Demonstration.

Indeed let A to BG be made in the ratio C to D, and [a] BG will be less than BK. And let the mean between A and BG be EH, which will also be less than EI. Again, since A to BG is the square of the ratio A to EH, and C to D by construction has the same ratio as A to BG; also C to D will have the same square ratio of that which A has to EH : but C to D has the square of that, which C has to F. Therefore the ratio C to F is the same as the ratio A to EH. But EI has been shown to be larger than EH, therefore A to EI [b] has the lesser ratio than A to EH, that is C to F. Which had to be shown.

a 10 Quinti; b 8 Quinti.

[$A/EH = EH/BG$; $A/BG = A/EH \,.\, EH/BG = (A/EH)^2 = C/D = (C/F)^2$; hence $A/EH = C/F$. But $EI > EH$, hence $A/EI < A/EH = C/F$.]

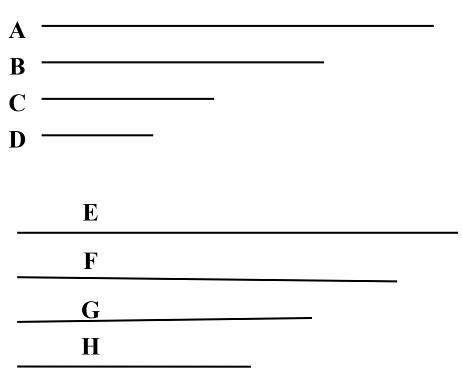
Corollory.

In the same way it will be shown that if more means are places between the first and the second terms, and between the third and the fourth terms, then the first to second terms will be in the initial greater or less proportion as in the final third to fourth terms; indeed the ratio in the initial first term to the first or second mean, or some other larger or smaller proportion, than the ratio of the third term to a similar mean between the terms of the second ratio.

PROPOSITIO VII.

Sint duo ordines continues proportionalium A, B, C, D, & E, F, G, H: & maior sit ratio A ad B, quam E ad F.

Dico maiorem quoque esse rationem A ad C tertiam, vel D quartam, quam E ad G tertiam, vel H quarta.



Prop. 7. Fig. 1.

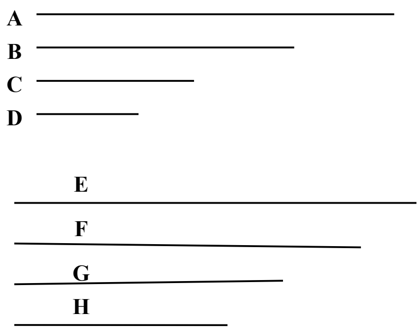
Demonstratio.

Cum enim sit eadem ratio A ad B, quae B ad C, & C ad D ; Similiter F ad G ratio, & G ad H eadem, quae est E ad F, sitque A ad B maior ratio, quam E ad F : erit etiam tam B ad C quam C ad D maior ratio, quam F ad G, aut G ad H : & proinde erit A ad c C maior ratio, quam E ad G : & similiter B ad D maior ratio, quam F ad H; & A ad D maior quam E ad H : quae erant demonstranda.

L2.§1. *PROPOSITION 7.*

Let there be two continued orders of proportionals A, B, C, D, and E, F, G, H: and the ratio of A to B shall be greater than the ratio of E to F.

I say that the ratio of A to the third proportion C, or to fourth proportion D is also larger than the ratio of E to the third proportion G or the fourth H.



Prop. 7. Fig. 1.

Demonstration.

Since indeed the ratio of A to B is the same as the ratio of B to C and also of C to D; similarly the ratio F to G , and the ratio G to H, is the same as the ratio E to F. The ratio of A to B is greater than the ratio of E to F: also the ratio of B to C and C to D is greater than the corresponding ratios F to G and G to H : and hence A to C [c] is greater than the ratio E to G : and similaly the ratio B to D is greater than the ratio F to H; and A to D is greater than E to H : which were to be shown.

[60]

PROPOSITIO VIII.

Sint tres magnitudines AC, AB, AF; & FE aequalis sit CB.

Dico si AC minima trium, & CB minor differentia, & BE, utriusque differentiae, differentia sint continue proportionales, ipsas quoque magnitudines AC, AB, AF esse in continua ratione.

Prop. 8. Fig. 1.

Demonstratio.

Rectangulum FAC [a] aequatur rectangulo FCA una cum quadrato CA, rectangulum autem FCA aequatur [b] rectangulo ACB, & rectangulo AC BE (id est [c] quadrato CB, cum AC, CB, BE ponantur continuae) ac praeterea rectangulo AC EF; id est ACB, quia aequales sunt CB, EF : igitur rectangulum FAC aequatur quadratis AC, CB, & rectangulo ACB bis; hoc est quadrato [d] AB; sunt [e] igitur tres proportiones magnitudines AC, AB, AF. Quod erat demonstrandum. *a 3. secundi; b 3 Secundi; c 17 Sexti; d 4 secundi; e 17 Sexti.*

Aliter.

Ut BC ad CA, sic FB ad BC, id est EF : igitur componendo ut BA ad CA, sic BF ad EF, hoc est BF ad BC ergo permutando & componendo FA ad BA, ut BA ad CA. Quod erat demonstrandum.

Corollarium.

Prop. 8. Fig. 2.

Si autem AC, CB, EF fuerunt proportionales, & BE aequalis CB minori differentiae, erunt rursum AC, AB, AF continuae proportionales. Demontratio eadem est, quae propositionis iam positae.

L2.§1. *PROPOSITION 8.*

AC, AB, and AF are three magnitudes ; and FE is equal to CB.

I say that if AC the smallest of the three, and both CB the least difference, and the other difference BE, are successive proportionals of differences, then the magnitudes themselves AC, AB, AF are also in a continued ratio.

Demonstration.

The rectangle FA.AC [a] is equal to the rectangle FC.CA together with the square CA; but the rectangle FC.CA is equal to [b] the rectangle AC.CB and the rectangle AC.BE (that is [c] to the square CB, since AC, CB, and BE are placed in continued proportion) and to the rectangle AC.EF in addition, that is AC.CB, since CB and EF are equal. Therefore FA.AC is equal to the squares AC and CB together with twice the rectangle AC.CB; that is to the square [d] AB. The three magnitudes AC, AB and AF are therefore [e] in proportion. Q.E.D. *a 3. secundi; b 3 Secundi; c 17 Sexti; d 4 secundi; e 17 Sexti.*

[FA.AC = FC.CA + AC^2 = AC.CB + AC.BE + AC.CB + A C^2 = CB^2 + CA^2 + 2.AC.CB = $(AC + CB)^2$ = AC^2 . Hence AC, AB, and AF are in continued proportion.]

Otherwise.

As BC is to CA, thus EB is to BC, that is EF : therefore by addition, as BA to CA, thus BF to EF, that is BF to BC. Hence by interchanging and additiom, FA is to BA, as BA is to CA. Q.E.D.

[BC/CA = EB/BC, giving BA/CA = EC/BC = BF/BC of BF/BA = CB/AC giving AF/AB = AB/AC.]

Corollary.

But if AC, CB, and EF are proportionals, and BE is equal to the smallest of the differences CB, again AC, AB, and AF will be continued proportionals. The demonstration is the same as that for the proposition just considered.

[CB^2 = AC.EF; BE = CB; consider AC.AF = AC.(AC + CB + BE + EF) = AC^2 + 2.AC.BC + BC^2 = AB^2.]

PROPOSITIO IX.

Sint in continua analogia AB, AC, AD, & minori differentiae aequalis sit ED.
Dico minimam AB, & minorem differentiam BC, deinde & CE, utriusque differentiae differentiam, esse proportionales.

Prop.9. Fig. 1.

Demonstratio.

Rectangulum DAB [f] aequatur rectangulo DBA, & quadrato AB : rectangulum autem DBA & aequatur rectangulis ABC, & AB ED, id est rursus rectangulo ABC (sunt enim ED, BC lineae aequales) & rectanguloAB CE: ergo rectangulum DAB, aequatur quadrato AB, & rectangulo ABC bis, & praeterea rectangulo ABCE; quadratum vero AC, aequatur quadrato AB, rectangulo ABC bis, & quadrato BC. Atque cum AB, AC, AD ponantur continuae proportionales, erit rectangulum [h] DAB aequale quadrato AC ; ergo quadratum AB, rectangulum ABC bis, & rectangulum ABCE simul sumpta, aequantur quadrato AB, rectangulo ABC bis, & quadrato BC simul sumptis. Itaque demptis communibus quadrato AB, & rectangulo ABC bis, aequalia remanent quadratum BC, & rectangulum ABCE; quare AB [g], BC, CE, sunt tres continuae proportionales. Q.e.d. *f 3. Secundi; g 1. Secundi; h 17. sexti; i ibidem*.

[61]

Aliter.

Quoniam DA, CA, BA ponuntur proportionales, ergo per primam huius DC ad CB est ut CA ad BA : quare (cum aequales sint DE, BC ex hypothesi) ut CD ad ED, sic CA ad CA. ergo dividendo CE ad ED, hoc est, CB, ut CB ad BA. Quod erat demonstrandum.

Corollarium.

Prop.9. Fig. 2.

Quod si positis continue proportionalibus AB, AC, AD, sumatur CE aequalis minori differentiae BC. erut AB, BC, ED in continua analogia : quod demonstrabitur prorsus eadem modo quo propositio iam posita.

L2.§1. *PROPOSITION 9.*

AB, AC, AD are [increasing] continued ratios, and ED is set equal to the smallest of the differences BC. I say that AB, the smallest difference BC, and thus CE, the difference of the other difference, are in proportion.

Demonstration.

The rectangle DA.AB[f] is equal to the sum of the rectangle DB.BA and the square AB : but the rectangle DB.BA is equal to the sum of the rectangles AB.BC, AB.ED, that is again equal to the rectangle AB.BC (for the lines ED and BC are equal) and the rectangle AB.CE: therefore the rectangle DA,AB is equal to the sum of the square AB and twice the rectangle AB.CC bis, and to the rectangle AB.CE besides; the square AC is truly equal to the sum of the square AB with twice the rectangle AB.BC and the square BC. And with AB, AC, AD placed in continued proportion, the rectangle [h] DA.AB is equal to the square AC ; therefore the sum of the square AB, twice the rectangle AB.BC and the rectangle AB.CE , is equal to the sum of the square AB, twice the rectangle AB.BC , and the square BC. Therefore with common square AB and twice the rectangle AB.BC taken away, there remains the square AB equal to the rectangle AB.CE; whereby AB [g], BC, CE, are three continued proportionals. Q.e.d.

f 3. Secundi; g 1. Secundi; h 17. sexti; i ibidem.

[DA.AB = DB.BA + AB^2 ; DB.BA = AB.BC + AB.ED (= AB.BC as ED = BC) + AB.CE. Therefore DA.AB = AB^2 + 2.AB.BC + AB.CE ; $AC^2 = AB^2$ + 2.AB.BC + BC^2 ; also, AC^2 = DA.AB from continued proportions. Therefore, AB^2 + 2.AB.BC + AB.CE = $AC^2 = AB^2$ + 2.AB.BC + BC^2 ; hence AB.CE = BC^2, and thus AB, BC, and CE are in continued proportion as required.
In modern terms, if AB = a, AC = ar, and AD = ar^2, then ED = BC = $a(r - 1)$,
CE = $ar^2 + a(1 - r) - ar = a(r - 1)^2$; hence a, $a(r - 1)$, $a(r - 1)^2$ are terms in the continued ratio $(r - 1)$]

Otherwise.

Since DA, CA, BA are placed as proportionals, it follows by the first proposition of this book, that DC to CB is as CA to BA : whereby (as DE is equal to BC by hypothesis) as CD to ED, thus CA to BA. Therefore on dividing, CE is to ED, that is CB, as CB to BA. Q.e.d.
[DA/CA = CA/BA giving CD/CA = CB/BA = DE/BA or CD/DE = CA/BA, giving CE/DE = BC/BA or CE/BC = BC/BA as required.]

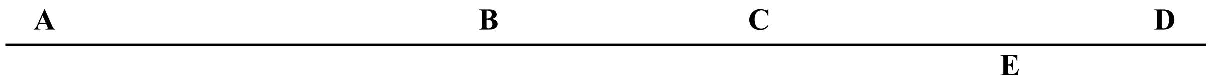
Corollory.

Prop.9. Fig. 2.

For if you take AB, AC, AD for the continued proportions, CE is taken equal to the smallest difference BC. AB, BC, ED are then in a continued ratio : which in short will be explained in the same way as the proposition now in place.

PROPOSITIO X.

Sint tres magnitudines AB, AC, AD, in continua analogia; ponantur autem maxima trium AD, & maior differentia CD, dein & tertia quaepiam ED, continuae proportionales.
Dico BC, CE, aequales esse lineas.



Prop.10. Fig. 1.

Demonstratio.

Cum tres ponantur continuae AB, AC, AD, erit DC ad [a] CB, ut DA ad CA: quare [b] rectangulum DCA rectangulo DA BC, aequale est. Similiter cum ponantur continuae AD, CD, ED, est AC ad CE [c] ut AD ad CD : ergo rectangulum idem ACD aequatur etiam rectangulo ADCE rectangula. rectangula ergo ADBC, AD, CE inter se aequantur ; unde BC, [d] CE aequales sunt. quod erat demonstrandum.

a 1. Huius; b 16 Sexti; c 1. Huius; d 1. Sexti.

Aliter.

Quandoquidem ponantur continuae AB, AC, AD, ergo per primam huius ut DA ad CA, sic DC ad CB. Iterum quoniam ponuntur continuae AD, CD, ED : per primam huius ut AD ad CD, sic AC ad CE ; & permutando ut AD ad AC, sic CD ad CE. Sed ut AD ad AC, sic ostendi esse CD ad BC; ergo CD est ad CE, ut CD est ad BC. aequales igitur sunt BC, CE. quod erat demonstrandum.

Corollarium.

Prop.10. Fig. 2.

Simili plane modo, si positis continuis AB, AC, AD, etiam AD trium maxima. CD maior differentia, & tertia quaepiam CE fuerint continuae proportionales.Dico BC, ED aequales esse. Demonstratio eadem estque propositionis decimae.

L2.§1. *PROPOSITION 10.*

The three magnitudes AB, AC, and AD are in a continued ratio; moreover the largest of the three AD, the larger difference CD, and some other third length ED are put in place in order that these three lengths are continued proportionals.

I say that in this case the lines BC and CE are equal in length.

Demonstration.

Since the three lengths AB, AC, and AD are put in a continued ratio, DC is to [a] CB as DA to CA: quare [b] rectangulum DCA rectangulo DA BC, aequale est. Similiter cum ponantur continuae AD, CD, ED, est AC ad CE [c] ut AD ad CD : ergo rectangulum idem ACD aequatur etiam rectangulo ADCE rectangula. rectangula ergo ADBC, AD, CE inter se aequantur ; unde BC, [d] CE aequales sunt. quod erat demonstrandum.

a 1. Huius; b 16 Sexti; c 1. Huius; d 1. Sexti.

[AC/AB = AD/AC, hence BC/AB = CD/AC = BC/AC = CD/AD; whereby DC.CA = DA.BC. Similarly, CD/AD = ED/CD, hence AC/AD = CE/CD and AC.CD = AD.CE. Hence AD.CE = AD.BC, and so CE = BC.
In modern terms again, the ratios are AD = ar^2, CD = $ar(r - 1)$, and ED = $a(r - 1)^2$.]

Otherwise.

Since the continued ratios AB, AC, AD are put in place, therefore by the first proposition of this book, DA is to CA, thus as DC is to CB. Again, since the lengths AD, CD, ED are placed in continued proportion : by the first prop. as AD is to CD, thus AC is to CE ; and by interchanging, as AD to AC, so CD to CE. But as AD is to AC, thus I have shown that CD is to BC; hence CD is to CE, as CD is to BC. Therefore BC and CE are equal. Q.E.D.

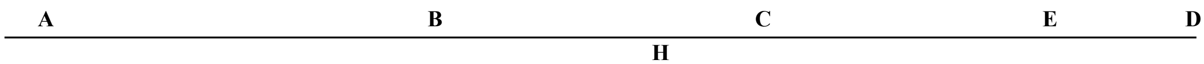
Corollary.

Prop.10. Fig. 2.

Clearly in the same way, if AB, AC and AD are placed in continuous proportion. Again, AD the greatest of the three, CD the greater difference, and the third some length CE are continued proportionals. I say that BC and ED are equal. The demonstration is the same as for proposition ten.

PROPOSITIO XI.

Sint AB, AC, AD in continua analogia, & minori differentiae BC aequalis sit CE. Dico ED, CD, AD in continua quoque analogia esse.



Prop.11. Fig. 1.

[62]

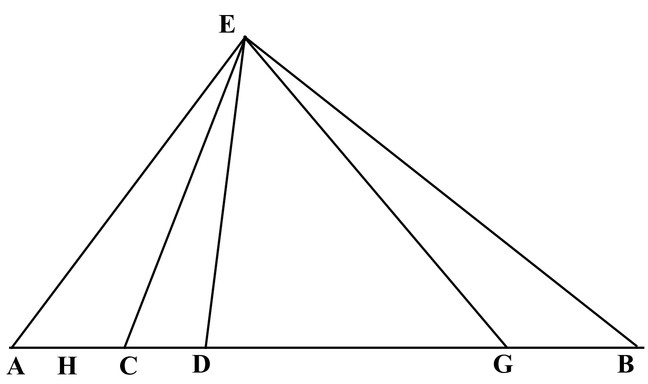
demonstratio.

Rectangulum ADE aequatur [a] rectangulo AED cum quadrato ED : rectangulum autem AED, aequatur rectangulo [b] DEC, & DECB, hoc est rursus rectangulo DEC (cum CE, CB sint aequales) & rectangulo DE BA. Sed per corollarium nonae huius, AB, BC, ED sunt continuae. ergo rectangulum ED AB aequale est quadrato BC, hoc est quadrato EC. rectangulum igitur AED aequatur rectangulo DEC bis, & quadrato EC, quare rectangulum ADE aequatur rectangulo DEC bis, & quadratis EC, ED, hoc est [c] quadeato CD. Unde AD, CD, ED sunt tres continuae proportionales. quod erat demonstrandum.

a 3. Secundi; b 1. Secundi; c 4 Secundi.

Aliter.

Quoniam DA, CA, BA ponuntur continuae, erit per primam huius DA ad CA, sicut DC ad CB, id est CE : quia aequales sunt ex datis BC, CE : Itaque dividendo ut DC ad CA, sic DE ad EC : & envertendo ut AC ad CD, ita CE ad ED, & componendo ut AD ad CD, sic CD ad ED. Q.e.d.



Prop.11. Fig. 2.

Corollarium.

Ex hac propositione educitur hoc Theorema. Sint AB, CB, DB proportionales, erigatur autem ad angulum quemcunque recta BE, aequalis ipsi CB, iunganturque puncta AE, CE, DE.Dico AEC, CED angulos esse aequales: & si AEC, CED anguli fuerint aequales, & CB linea aequetur rectae BE. dico AB, CB, DB esse proportionales : si vero AEC, CED anguli aequentur, & AB, CB, DB sint proportionales; dico CB, EB lineas esse aequales.

Demonstrationem.

Quoniam ponitur ut AB ad CB, hoc est BE, sic BE ad DB, & angulum ABE sit communis, erit DEB [d] triangulum, AEB triangulo simile, adeoque angulus DEB angulo EAB aequalis. Rursum cum CB, EB latera aequentur, erunt anguli CEB, ECB aequales; est autem angulus ECB, aequalis [e] duobus angulis aequalis EAC, AEC. Igitur angulus CEB aequatur duobus angulis AEC, EAC : sed DEB aequalis ostensus est angulo EAC; demptis igitur aequalibus EAC, DEB, manent AEC, CED reliqui aequales.

Sint iam anguli AEC, CED, & CB, BE lineae aequales, dico AB, CB, DB esse continuas. Quoniam angulus ECB hoc est CED aequetur angulis EAC, [f] ABC, & CED ponatur aequalis ipsi AEC, erit DEB reliquus reliquo EAC aequalis: est autem angulus ABE communis: ergo tertius tertio aequatur, adeoque & AEB, DEB triangula similia: unde ut AB ad EB hoc est CB, ita EB sive CB ad DB. Quod erat secundum.

Iam vero dint AEC, CED anguli aequales, & AB, CB, DB tres continuae proportionales : dico CB, EB lineas aequari : si enim non sint aequales, sit CB maior, fiatque CG ipse BE aequalis . erunt igitur per secundam partem huius propos. AG, CG, DG continuae proportionales. unde si fiat HC aequalis ipso CD, erunt tam & AG, AC, AH; quam AB, AC, AH continuae proportionales; igitur tam AGAH rectangulum, quam ABAH rectangulum, aequale quadrato AC mediae communi : quare AGAH, ABAH rectangula aequantur inter se. Unde ut AH ad AH h sic AB ad AG; sed AH sunt aequales, ergo AG,AB rectae aequantur, & G punctum idem est quod punctum B: & CB, CG. id est BE lineae aequales, eodem modo ostenditur CB non esse minorem ipsa BE.

L2.§1. *PROPOSITION 11.*

Let AB, AC, and AD be in a continued ratio, and the smallest difference BC is equal to CE. I say that ED, CD, and AD are in a continued ratio.

Demonstration.

The rectangle AD.DE is equal to the sum of the rectangle [a]AE.ED and the square ED : but rectangle AE.ED is equal to the sum of the rectangle [b] DE.EC and DE.CB, which is equal to the rectangle DE.EC (as CE and CB are equal) and also with rectangle DE.BA. But by the corollary of Prop. nine of this book, AB, BC, ED are continued proportions. Therefore rectangle DE.BA is equal to the square BC, that is also equal to the square EC. Therefore rectangle AE.ED is equal to the sum of twice the rectangle DE.EC and the square EC, whereby the rectangle AD.DE is equal to the sum of twice the rectangle DE.EC together with the squares EC and ED, that is to the square [c] CD. Hence AD, CD, ED are three continued proportionals. Q.e.d.

a 3. Secundi; b 1. Secundi; c 4 Secundi.

[AD.DE = AE.ED + ED^2 ; also, AE.ED = DE.EC + DE.CB (= DE.EC) + DE.BA. But AB, BC, and ED are in continued proportion, hence, BC^2 = DE.BA= EC^2; hence AD.DE = 2.DE.EC + EC^2 + ED^2 = CD^2 as required.]

Otherwise.

As DA, CA, and BA are in continued proportion, by the first proposition of this book, it follows that : DA is to CA, thus DC is to CB, *i.e.* CE, as BC and CE are given as equal to each other. Thus by division, as DC to CA, thus DE to EC : and on inverting, as AC to CD, thus CE to ED, and on addition, as AD to CD, thus CD to ED. Q.e.d.

[CA/DA = BA/CA giving DC/DA = CB/CA or DC/CB = DA/CA = DC/EC; and again, DC/CA = DE/EC, or AC/CD = CE/DE and AD/CD = CD/DE as required.]

Corollary.

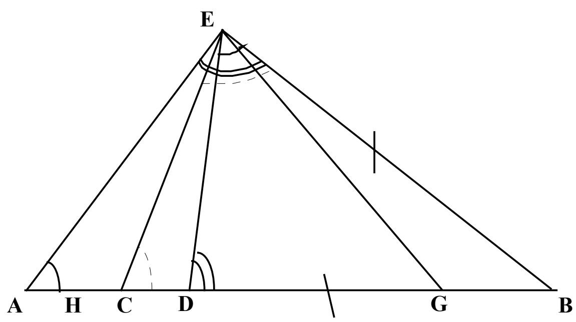
The following theorem can be established from this proposition. Let AB, CB, and DB be proportionals, and moreover a line BE equal to CB is erected at some angle. The points AE, CE, and DE are joined. I say that the angles AEC and CED are equal: and if the angles AEC and CED are equal, then the line CB is equal to the line BE.

I say that the lines AB, CB, DB are indeed proportionals if the angles AEC and CED are equal, and if AB, CB, DB are proportionals, then CB, EB are lines of equal length.

Demonstration.

Since the ratio AB to CB, or BE, is thus as BE to DB, and the angle ABE is common, then triangle DEB [d] is similar to triangle AEB, and hence the angle DEB is equal to the angle EAB. Again since the sides CB and EB are equal, then the angles CEB and ECB are equal. But the angle ECB is equal to the sum of the two angles EAC and AEC. Therefore angle CEB is equal to the sum of the angles AEC and EAC. But angle DEB has been shown to be equal [e] to the angle EAC ; therefore with the equal angles EAC and DEB taken away, there remains the equal angles AEC and CED. [Thus, $\angle$ CED = $\angle$ EDB - $\angle$ ECB = $\angle$ AEB - $\angle$ CEB = $\angle$ AEC.]

Now with the angles AEC and CED equal, and the lines CB and BE equal, I can say that the lines AB, CB, and DB are in continued proportion. Since the angle ECB, or CEB, is equal to the sum of the angles EAC and AEC [f], and the angle CED is set equal to the angle AEC, then the remaining angle DEB is equal to the angle EAC : but the angle ABE is common: therefore the third angle to angle is equal , and thus the triangles AEB and DEB are similar: thus as AB to EB or CB, thus EB or CB to DB. Which establishes the second part.



Prop.11. Fig. 3.

Now indeed the angles AEC and CED are set equal, and the lines AB, CB, and DB are three continued proportionals : in this case I say that the lines CB and EB are equal. For indeed if they are not equal, then let CB be the greater, and BE is set equal to CG. Therefore by the second part of this proposition, it follows that AG, CG, and DG are continued proportionals. Thus if HC is set equal to CD, as AG, AC, and AH, so AB, AC, AH are continued proportionals. Therefore, both rectangles AG.AH and AB.AH are equal to the square AC of the common mean. Whereby the rectangles AG.AH and AB.AH are equal to each other. Thus as AH to AH [h] so AB to AG; but the AH's are equal, therefore the lines AG and AB are equal, and the point G is the same as the point B: and the lines CB, CG or BE are equal. In the same way it can be shown that CB cannot be less than BE.

[63]

PROPOSITIO XII.

Dentur tres lineae in continua ratione AB, BC, CD, ita ut BC sit maior AC, & fiat rectae AC aequalis BE.

Dico AB, BE, ED esse proportionales.

Prop.12. Fig. 1.

Demonstratio.

Cum ponantur in continua analogia AB, BC, CD; igitur erit ut BC [a] ad CD ita AC, hoc est ex constructione BE, ad BD. Ergo dividendo ut BD ad DC, ita ED ad DB; & invertendo uti CD ad BD, ita se habet BD ad DE. Quare [b] BD quadratum aequatur EDC rectangulo; quadratum autem [c] BE aequale est quadratis BD, DE, & rectangulo EDB bis sumpto : atque rectangulum ABED aequatur iisdem : nam rectangulum ABED, aequatur EDC [d] rectangulo (hoc est , ut ostendi, quadrato BD) & rectangulo EDAC (hoc est rectangulo BED), id est rursum[e] quadrato ED cum rectangulo EDB, & rectangulo insuper EDB, rectangulum itaque ABED, dum iisdem aequale sit, aequabitur BE quadrato: patet igitur AB, [f] BE, BD esse continua analogia. quod demonstrandum fuit.

a. 1. huius; b 17. Sexti; c 4. Secundii; d 1 Secundi; e 3. Secundi; f 17. Sexti.

L2.§1. *PROPOSITION 12.*

The three lines AB, BC, and CD are given in a continued ratio, with BC greater than AC, and AC equal to BE.

I say that the lines AB, BE, and ED are in proportion.

Demonstration.

As AB, BC, and CD are put in a continued ratio, then as BC[a] is to CD thus as AC, or BE by construction, is to BD. Hence on dividing, as BD to DC, thus ED to DB; and on inverting, as CD to BD, thus BD to DE. Whereby [b] the square BD is equal to the rectangle ED.DC ; but the square [c]BE is equal to the sum of the squares BD and DE, together with twice the rectangle ED.DB : and the rectangle AB.ED is equal to the same : for the rectangle AB.ED is equal to the sum of the rectangle ED.DC [d] (that is, as shown, to the

square BD) together with the rectangle ED.AC (that is to the rectangle BE.ED, that is again equal to the square [e] ED with the rectangle ED.DB), and with the above rectangle ED.DB. Thus the rectangle AB.BD, is equal to the same, is equal to the square BE: therefore it is apparent that AB, [f] BE, BD are in continued proportion. Which was to be shown.
a. 1. huius; b 17. Sexti; c 4. Secundii; d 1 Secundi; e 3. Secundi; f 17. Sexti.

[AB/BC = BC/CD is given; hence AC/BC = BD/CD or BC/CD = AC/BD = BE/BD; again, BC/CD - 1 = BE/BD - 1 or BD/CD = ED/BD and CD/BD = BD/ED on inverting. Hence, BD^2 = ED.DC; also, $BE^2 = BD^2 + DE^2$ + 2.BD.DE; again, AB.ED = ED.DC (or BD^2) + ED.AC (or BE.ED = ED^2 + ED.DB) + ED.DB : Thus BE^2 = AB.ED as required.]

PROPOSITIO XIII.

Ponantur AB, BC, CD, tres lineae in continua proportione, & fiat BC aequalis DE.
Dico AE, EC, CD, esse in continuae analogia.

Demonstratio.

Quandoquidem ponuntur esse continuae proportionales lineae AB, BC, CD, etiam erunt continuae proportionales AB, DE, CD, & BC mediae aequalis DE erit quadratum DE rectangulo ABCD aequale, quadratum autem [g] CE aequatur CD, DE quadratis, & rectangulo CDE bis sumpto : quibus etiam aequatur rectangulum AECD nam rectangulum AECD aequatur ECD rectangulo (hoc est quadrato [h] CD cum rectangulo CDE) & rectangulo DCB. hoc est (quia ex constructioneBC, DE aequantur) iterum rectangulo CDE & rectangulo insuper ABCD, (id est, ut modo ostendimus, quadrato DE) quare rectangulum AECD cum iisdem aequale sit, quadrato CE aequare necesse est : adeoque CD, CE, EA esse in continua analogia. Q. e. d. *g 4. Secundi; h 3. Secundi.*

Prop.13. Fig. 1.

L2.§1. *PROPOSITION 13.*

The three lines AB, BC, and CD are placed in continued proportion, and DE is made equal to BC.
I say that the three lines AE, EC, and CD are also in continued proportion.

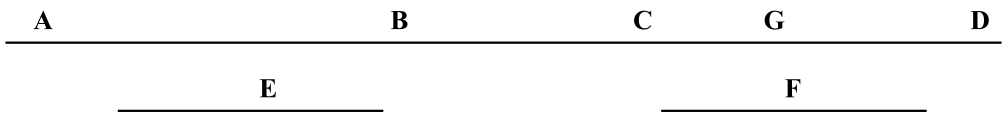
Demonstration.

Since the lines AB, BC, and CD are placed in continued proportion, the lines AB, DE, and CD are also in continued proportion, as the means BC and DE are set equal. The square DE is equal to the rectangle AB.CD, but the square [g] CE is equal to the sum of the squares CD and DE , together with twice the rectangle CD.DE: which is also equal to the rectangle AE.CD. For the rectangle AE.CD is equal to the sum of the rectangle EC.CD (that is, to the square [h] CD with the rectangle CD.DE) together with the rectangle DC.CB, that is (as BC is equal to DE by construction) again equal to the sum of the rectangle CD.DE, and the above rectangle AB.CD, (or, as we have shown to the square DE); whereby the rectangle AE.CD is equal to the same, is necessarily equal to the square CE: thus CD, CE, and EA are in continued proportion. Q. e. d.
g 4. Secundi; h 3. Secundi.
[DE^2 = AB.CD; also $CE^2 = CD^2 + DE^2$ + 2.CD.DE , which is also equal to AE.CD as AE.CD = EC.CD (*i.e.* CD^2 + CD.DE) + DC.CB (or DC.DE) + AB.CD (or DE^2): for, on adding the terms in brackets, we have the same sum for CE^2. Hence, CE^2 = AE.CD, and so AE, CE, and CD are also in continued proportion.]

PROPOSITIO XIV.

Sint AB, AC, AD, in continua analogia ; & fiat ut AB ad BC, ita BC ad E lineam; & ut AD ad DC, sic DC fiat ad F.
Dico E & F esse inter se aequales.

Demonstratio.



Prop.14. Fig. 1.

Quoniam supponuntur continuae AB, AC, AD, si fiat DG aequalis BC, erunt continuae [i] AB, BC, CG, ergo cum ex datis etiam fiat in continua ratione,AB, BC, & E ,

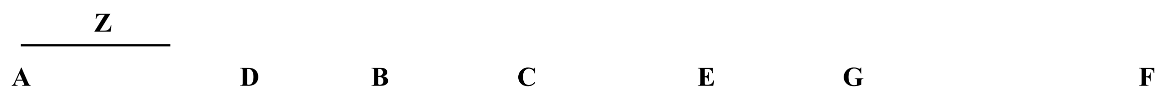
[64]

erit linea E aequalis CG : deinde quai continuae sunt AB, AC, AD, si fiat BC aequalis GD, erunt quoque in continua analogia a CG, DC, DA : ponuntur autem continua: AD, DC & F ; erunt igitur continuae F, DC, AD; est igitur F aequalis CG : unde F & E quoque sunt aequales, utpote ipsi CG aequales. Quod erat propositum demonstrare.

i 9. Huiua; a. 11. huius.

Lemma.

Dati sint duo ordines trium quantitatum CA, BA, DA & CF, EF, GF; & sit ut CA ad DA, sic CF ad GF, sitque item ut BA ad DA, sic CF ad EF.
Dico etiam esse CA ad BA, ut EF est ad GF.



Prop.14. Fig. 2.

Demonstratio.

Si enim non est CA ad BA, ut EF ad GF, erit ergo aliqua maior ver minor, quam CA, nempe Z ad BA, ut EF est ad GF; quoniam ergo ut CF prima ad EF secundam, ita in altero ordine BA secunda est ad DA tertiam, & ut EF secunda ad GF tertiam, ita in ordine altero Z prima, est ad BA secundam , ergo ex aequo in proportione perturbata, ut CF ad GF, ita Z [b] ad DA: atque ex hypothesi CF ad GF, ergo ut CA ad DA, sic Z (maior vel minor quam CA) est ad DA, quod est [c] absurdum: non est igitur CA ad BA, in alia pooportione, quam EF ad GF. *b 23. Quinti; c 8 Quinti.*

L2.§1. ***PROPOSITION 14.***

Let AB, AC and AD be in continued proportion ; and the ratio AB to BC is made as BC to the line E ; and as AD to DC, so DC to F.
I say that E and F are equal to each other.

Demonstration.

Since the lines AB, AC, and AD are considered to be in continued proportion , if DG is made equal to BC, then AB, BC, CG are continued proportions [i], therefore from that given, AB, BC, and E are also in a continued ratio. The line E is thus equal to CG : hence as the lines AB, AC, AD are in continued proportion, if BC is made equal to GD, CG, DC, DA are also in continued proportion [a] : but AD, DC and F are placed in continued proportion; therefore F, DC, AD are in continued proportion ; therefore F is equal to CG : hence F and E are equal too, as CG itself is equal to E. Which was the PROPOSITION to be shown.

i 9. Huiua; a. 11. huius.

Lemma.

Two orders of three quantities CA, BA, DA and CF, EF, GF are given as follows: as CA is to DA, thus CF is to GF, and also as BA is to DA, so CF is to EF.
I say that CA is to BA, as EF is to GF.

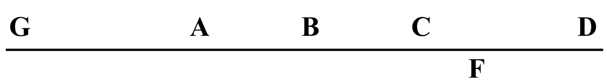
Prop.14. Fig. 2.

Demonstration.

For indeed, if the ratio CA to BA is not the same as EF to GF, then it will be either a larger or smaller ratio, namely Z to BA, rather than CA to BA, that is the same as EF to GF. Therefore, since the first term CF is to the second term EF, thus in the other order the second term BA is to the third term DA, and as the second term EF is to the third term GF, so in the other order the first term Z is to the second term BA, therefore from the equality in the disturbed proportion, as CF to GF, thus Z [b] to DA: and from the hypothesis, CF is to GF, thus as CA is to DA, thus Z (greater or less than CA) is to DA, which is [c] absurd: therefore the ratio CA to BA is not in any other proportion than EF ad GF. *b 23. Quinti; c 8 Quinti.*
[If CA/DA = CF/GF and BA/DA = CF/EF then CA/BA = CA/DA . DA/BA = CF/GF. EF/CF = EF/GF as required.]

PROPOSITIO XV.

Sint proportionales AB, AC, AD; & linea BD composita ex utraque differentia BC, CD, dividatur bifarium in F, ac mediae proportionali AC, aequalis sit producta AG.
Dico FC, FB, FG in continuata esse proportione.



Prop.15. Fig. 1.

Demonstratio.

Cum enim GC bisecta sit in A, & BD in F, erit CG ad AG, ut BD ad FD. praeterea cum AB, AC, AD ponantur continuae, erit per primam huius BC ad CD, ut AB ad AC, id est AG; & componendo ut BG ad AG, ita BD ad CD: sed ante ostenderamus esse ut CG ad AG, ita BD ad FD. Quare per lemma praecedens etiam erit ut CG ad BG, ita CD ad BF, hoc est FD; & dividendo CB ad BG, ut CF ad FD, hoc est FB; Itaque invertendo, permutando, componendo erit ut GF ad BF, ita BF ad CF; ergo GF, BF, CF sunt proportiones. Quod erat demonstrandum.

Corollarium.

Prop.15. Fig. 2.

Ex hac propositione hoc Theorema deducimus. Sint tres proportionales AB, BC,CD, divisaque sit AB bifariam in E. Dico AE, EC, & compositam ex AE & lineis BC, CD bis sumptis, esse proportionales, & si AE, EC, & composita ex AE & BC, CD lineis bis sumptis, sint continuae; dico AB,BC, CD esse in continua analogia.

[65]

Demonstratio.

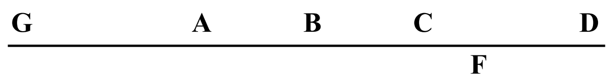
Producatur CD in F, ut DF linea aequetur BD compositae : BA vero producatur in G, ut GA, BC aequentur. Quoniam GA, BC lineae sunt aequales, erunt [a] DC, DB, DG continuae proportionales. Rursum cum GA aequetur rectae BC, & AE ipsi EB sit aequalis, erit GC in E bifariam quoque divisa: est autem DF aequalis ipsi BD per constructionem ; ergo EB, EC, & [b] EF, hoc est AE, EC, & composita ex AE & BC, CD bis sumptis, sunt continae proportionales. quod erat primum.

2. Quoniam EB, EC, EF sunt continuae, & EC lineae aequatur EG, & BF linea ex construc. sit in D bifariam divisa, erunt DC, DB, [c] DG continuae : quare & AB, BC, CD tres erunt [d] continuae proportionales. quod erat demonstrandum. *a. 8 huius; b 15 Huius; c Ibid; d 9 huius.*

L2.§1. *PROPOSITION 15.*

AB, AC, and AD are proportionals; and the line BD, which is the sum of both the differences BC and CD, is bisected in F. The line AG produced is equal to the mean proportional AC.

I say that FC, FB, and FG are in continued proportion.



Prop.15. Fig. 1.

Demonstration.

Since GC is bisected in A, and BD in F, CG is to AG, as BD is to FD. Furthermore, as AB, AC, and AD are placed in continued proportion, by the first proposition of this book, BC to CD, is as AB and AC, or AG; and by adding: BG to AG, is thus as BD to CD; but we have shown before that CG is to AG, thus BD is to FD. Whereby by the preceding lemma, also as CG to BG, thus CD to BF, or FD. On division, CB is to BG, as CF to FD, or FB. Thus on inverting, exchanging, and addition, GF is to BF, thus as BF is to CF; hence GF, BF, and CF are proportions. Q.e.d .

[CG/AG = BD/FD; from AB/AC = AC/AD it follows that BC/CD = AB/AC = AB/AG. On adding, this gives BG/AG = BD/CD; also, CG/AG = BD/FD; hence by the previous lemma, or from BG/AG . AG/CG = BD/CD . FD/BD, we have BG/CG = FD/CD or CG/BG = CD/FD on inverting. Hence, CB/BG = CF/FD = CF/FB, from which BG/CB = FB/CF, and BG/FB = CB/CF, giving GF/FB = FB/CF as required.

Using algebra, if AB = a, AC = ar, and AD = ar^2, then BD = $a(r^2 - 1)$ and BF = $a(r^2 - 1)/2$; FC = BF - BC = $a(r - 1)^2/2$; and FG = FA + AG = $a(1 + r)^2/2$.
Hence, $FB^2 = a^2(r^2 - 1)^2/4 = a(r + 1)^2/2 \,.\, a(r - 1)^2/2$ = FG.FC.]

Corollarium.

Prop.15. Fig. 2.

We can deduce these theorems from this proposition:

1. Let AB, BC, and CD be three proportionals, and AB divided in two equal parts by E. I say that AE, EC, and the sum of AE with twice the sum of the lines BC and CD are in continued proportion.

2. If AE, EC, and the sum of AE and twice the sum of the lines BC and CD are in continued proportion, then I say that AB, BC and CD are in a contined ratio.

Demonstratio.

CD is produced to F, in order that the line DF is equal to the lines making BD : BA truly is produced to, in order that GA and BC are equal. Since the lines GA and BC are equal, [a] DC, DB, and DG are continued proportionals. Again as the line GA is equal to the line BC, and AE is equal to EB, GC is likewise divided in two by E : furthermore, DF is equal to BD from the construction ; hence EB,EC, and [b] EF , that is AE, EC, and the sum of AE and twice BC and CD, are continued proportionals. Which establishes the first theorem. *a. 8 huius; b 15 Huius; c Ibid; d 9 huius.*

[BC^2 = AB.CD, or BC/AB = CD/BC or AB/BC = BC/CD, then GB/BC = BD/CD, or GB/BD = BC/CD, giving GD/BD = BD/CD or BD^2 = GD.CD; i. e. DC, DB, and DG are continued proportionals. Also, as GA = BC and AE = EB, then GE = EC. Again, AE/BC = BC/2.CD giving EC/BC = (BC + 2.CD)/2.CD, hence AE/BC = EC/(BC + 2.CD) or BC/AE = (BC + 2.CD)/EC giving EC/AE = (AE + 2.BC + 2.CD)/EC, as required.]

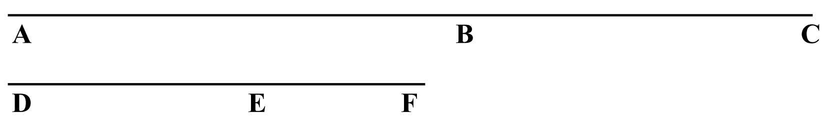
2. Since EB, EC and EF are continued proportionals, and the line EC is equal to the line EG, and the line BF from the construction is cut in two equal parts by D, then the lines DC, DB, and [c] DG are in continued proportion : whereby AB, BC and CD are [d] three continued proportionals. Q.e.d.

[The converse of 1. AE + 2.BC + 2.CD = AE + BF = EF and AE = EB, hence EC/EB = EF/EC from 1; BC/EB = CF/EC;
EC/EB = CF/BC; hence BC/EB = AB/BC (take) and AC/EB = BF/BC (add); hence AC/AB = BD/BC and BC/AB = CD/BC (take) as required.]

PROPOSITIO XVI.

Si sit prima ad secundam, ita tertia ad quartam, erit ut quarta [prima in text] cum tertia ad tertiam, ita omnes quatuor ad primam cum tertia; vel ut prima cum secunda ad secundam, ita omnes quatuor ad secundam cum quarta.

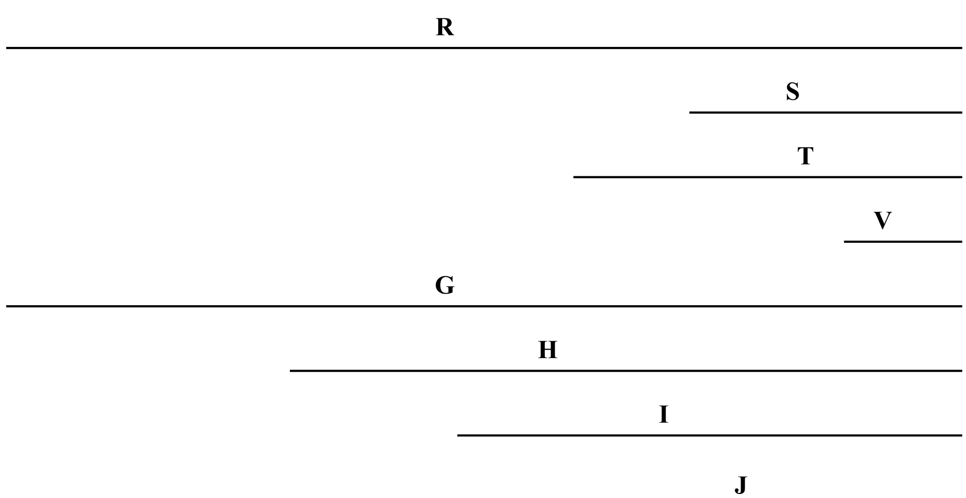
Demonstratio.



Prop.16. Fig. 1.

Esto AB ad BC, ut DE ad EF, dico esse AC ad AB, ut aggregatum ex AC & DF ad aggregatum ex AB, DE. Cum enim sit AB ad BC , ut DE ad EF, erit invertendo & componendo AC ad AB, ut DF ad DE: qua propter erit etiam veraque antecedens AC, DF, [e] ad utramque consequentem AB, DE, ut una antecedens AC ad AB unam consequentem. quod erat propositum. *e 11 Quinti.*

Corollarium.



Prop.16. Fig. 2.

Si igitur ex quatuor proportionalibus fiat una aequalis omnibus quatuor, verbi gratia R: & fiat altera S aequalis primae, ac tertiae, & tandem recta T exhibeatur aequalis primae & secundae : denique si ponatur V aequalis primae, erit R ad S, ut T ad V : similiter si omnibus quatuor fiat aequalis G, & secundae cum quartam aequalis H, & prima cum secunda aequalis I, & secundae aequalis K, erit ut G ad H, sic I ad K, res haec a Pappo

[66]

lib. tertio propositione decimaseptima in tribus continue proportionalibus proposita fuit; quae cum univeralis sit, censuimus etiam hoc ipsum verbo indicandum ad ampliorem usum.

L2.§1. *PROPOSITION 16.*

If the first term is to the second term, thus as the third is to the fourth, then the fourth and the third is to the third, thus as all four is to the first and the third; or as the first and the second to the second, thus as all four to the second and the fourth.

Demonstration.

Let AB be to BC as DE to EF, I say that AC to AB is as the sum of AC and DF to the sum of AB and DE. For indeed as AB to BC, thus DE to EF, and on inverting and adding, AC is to AB thus as DF is to DE: which is indeed the sum of the first terms AC and DF, [e] to the other subsequent terms AB and DE, as the one preceeding term AC to the one subsequent term AB. quod erat propositum. *e 11 Quinti.*

[The first part : AB/BC = DE/EF, then BC/AB = EF/DE and AC/AB = DF/DE giving AC/DF = AB/DE and (AC + DF)/DF = (AB + DE)/DE; hence (AC + DF)/(AB + DE) = DF/DE = AC/AB.
For the second part : we are to show that (AC + DF)/(BC + EF) = AC/BC. From AB/BC = DE/EF we have AC/BC = DF/EF and BC/AC = EF/DF or DF/AC = EF/BC and (AC + DF)/AC = (BC + EF)/BC, giving the required result.]

Corollary.

If therefore from four terms in proportion, there is one term equal to the sum of all the terms, for example R: and another S is equal to the first and the third, and the final term the line T is shown equal to the sum of the first and the second, then if V is put equal to the first term: R is to S, as T is to V.

[For (AC + DF)/(AB + DE) = AC/AB becomes R/S = T/V.]

In the same way, if all four terms are set equal to G, the second with the fourth equal to H, the first and second equal to I, and the second equal to K, then G is to H as I is to K.

[For (AC + DF)/(BC + EF) = AC/BC becomes G/H = I/K.]

This result comes from Pappus, Book III, Prop. 17, where the original theorem is proposed for three continued proportionals. We consider that it is of more general use as the example indicates.

PROPOSITIO XVII.

Sit A aequalis B, & C aequalis D; omnibus autem A, B, C, D, ponatur aequalis E, duabus vero B, D, ponatur aequalis F: similiter duabus C, D , aequalis G; & rectae D, aequalis H.

Dico rationem E ad F, & G ad H esse duplam. Quod si rectis E, F, G, H, aequalis ponatur I, & duabus F & H, aequalis statuatur K, & lineis G, H, aequalis ponatur L, denique residuae H, fiat M aequalis;

Dico rationem L ad K, & L ad M esse triplam.

Demonstratio.

A cum B, dupla est ipsius B, & C cum D, ipsius D dupla est: ergo etiam A, B, C, D simul sumptae , duarum B, D, simul sumptoram duplae sunt. Sed lineis A, B, C, D aequalis est E, & lineis B, D, aequalis F, ergo E dupla est F : similiter C, D, simul sumptae, hoc est G, ipsius D, id est H, sunt duplae. Quod erat primum. Secundam partem ita expedimus. E continet rectam A bis, & C bis; ipsa vero F continet A semel, & C semel : ipsa tandem G continet C bis; & H aequalis est C; igitur omnes E, F, G, H, , hoc est recta L continet tertio, & rectam C sexies : recta vero K, continet ex hypothesi F, & H; sed F continet B & D, hoc est ipsam A semel , & semel rectam C, igitur si addatur H, hoc est C, recta K erit semel A, & C bis sumptata. sed A sumpta semel, cum C bis, est tertia pars A lineae ter sumptae, & lineae C sexies sumptae: ratio ergo I ad K tripla est. Eodem modo ostendemus quod L, sit tripla lineae M, nam L aequalis est G, H; ipsa vero G continet bis C lineam, cui si adiungatur H, ipsi C aequalis, erit G cum H, hoc est L, tripla rectae C, hoc est H, hoc est M; patet ergo veritas propositione exposita.

Scholium.

Pappui lib. 3 propositine 18, tribus continuis quantitatibus applicuit hanc materiam, quae scilicet eandem continuant rationem; quia vero adventi non solum continuis rationibus convenire hanc proportionalitas proprietatem, verbum etiam discretissimus, modo rationes similes assumantur: hinc opera pretium duxi, hoc insinuare. Notatum autem dignum existimo, si quis ulterius

[67]

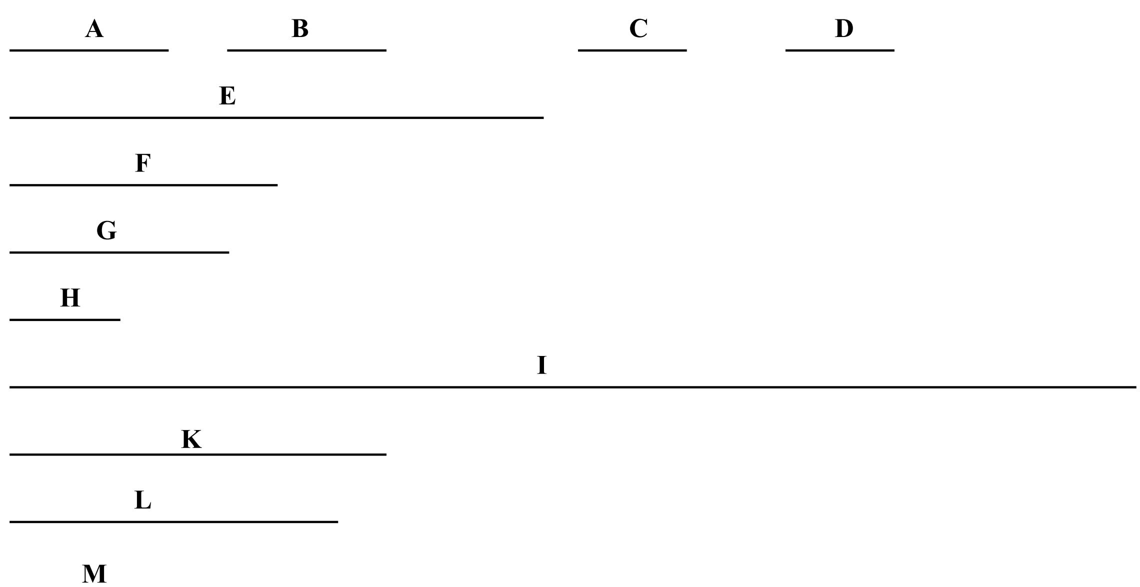
proportiones ita convertat, quemadmodum hac propositione secimus, in infinitum repariet plures ac plures subdivisiones rationum, sola praxis in propositione posita observatione.

L2.§1. *PROPOSITION 17.*

A is equal to B, & C is equal to D; moreover E is set equal to the sum of A, B, C, and D, while F is the sum of the two B and D : similarly G is the sum of the two C and D, and the line D equals H.

I say that the ratio E to F and the ratio G to H is two.

While if I is put equal to the sum of the lines E, F, G, H, and K is set equal to the two lines F and H, and L is placed equal to the lines G and H, and then H set equal to the remaining line M. In this case, I say that the ratio L to K and the ratio L to M is three.



Prop.17. Fig. 1.

Demonstration.

The sum of A and B is twice B, and the sum of C and D is twice D: therefore also, the sum of A, B, C, and D is twice the sum of B and D. But E is equal to the sum of the lines A, B, C, and D, and F is equal to the sum of the lines B and D, therefore E is twice F : similarly G is the sum of C and D, which is twice D or H. Which shows the first part to be true.

We second part we thus expedite as follows. The line E is composed of the line A twice and the line C twice; while F is composed of A and C once: finally G is composed of the line C twice; and H is equal to C. Therefore the sum of the lines E, F, G and H, that is the line L, contains the line A three times and the line C six times. The line K by hypothesis is composed of the lines F and H; but F is composed of B and D, that is A itself once and the line C once; therefore if H or C is added, then the line K is composed of A once and twice C. But A taken once with C taken twice is the third part of the sum of A taken three times and of C taken six times. Therefore the ratio L to K is three. In the same way we can show that L is three time the line M, for L is equal to the sum of G and H; and G is composed of twice the line C, to which if H or C is added, then L or the sum of G and H, is three times the line C or H or M. Hence it is apparent that the truth of the proposition has been shown.

[Using small letters and algebra, we have $a + b = 2b$; and $c + d = 2d$; hence, $a + b + c + d = 2b + 2d$. Now, $e = a + b + c + d$, and $f = b + d$; hence $e = 2f$, and $g = c + d = 2d$ or $2h$. (first part)
Again, $e = 2a + 2c$, and $f = a + c$; $g = 2c$ and $h = c$. Hence, $e + f + g + h = l = 3a + 6c$; $k = f + h$, but $f = b + d$; hence $k = a + 2c = 1/3.(3a + 6c)$, hence $l/k = 3$. Similarly $l = 3m$, for $l = g + h$. Again, $g = 2c$, and $l = g + h = 3c = 3h = 3m$. (second part)].

Scholium.

This material is an application of Proposition 18, Book 3 of Pappus for three continued quantities, which of course give the same continuing ratio. Because indeed not only is this a proportionality property to be found for continuous ratios considered together, but also for the most separate numbers; moreover similar kinds of ratios are assumed :and thus in this way I have exdeavoured to iintroduce the value of the work. Moreover I can judge their worth by observation, for if further proportions are thus inverted, in this way we can follow more and more subdivisions of the ratio, and these can be produced indefinitely by this proposition , put into the proposition for practise in observing the outcome.

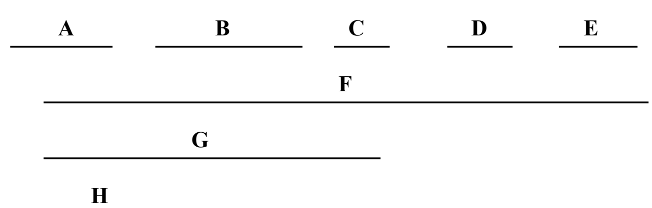
PROPOSITIO XVIII.

Sint quaecumque & quotcumque magnitudines A, B, C, D, E ; ponaturque F omnibus (praeter ultimam) bis sumptis, ac ultimae E semel sumptae aequalis; G vero aequalis omnibus simul sumptis, denique H aequalis ultimae E.

Dico F, G, H arithmeticam analogiam continuare.

Demonstratio.

Linea F, id est A, B, C, D, bis sumptae, una cum E semel, excedit recta G, id est A, B, C, D, E semel sumptas, excedit A, B, C, D, semel sumptis. Sed eodem excessu excedit linea G, ipsam H, quae aequalis ponitur rectae E, igitur F, G, H lineae arithmeticam continuant proportionem. Quod erat demonstrandum.



Prop.18. Fig. 1.

Scholium.

Quare miram videri non debuit Frederico Commandino, Pappum, cum lib. 3 ex Geometrica proportione, analogias, ac meditatas eruit, Arithmeticas neglexisse; quando quidem non ex Geometrica tantum sed quacumque quantitatum serie producatur; quod Pappum in caeterum sagacissimum latere potuisse vix credi potest; vel certe opere misset Commandinum dum Pappi defectum, ut vocat, eodem libro propositione

19. supplere mititur, continuitatem Geometricam assignarem ex qua, cum qua plane ratione Arithmeticam analogiam deduxisset, non ideo caeteris non proproportionalibus quantitatibus commune esse, posset demonstrari : si quidem a problematici nitaris splendore alienum videtur, certum & determinatum quid in construectionem adhibere, cum obviam quodlibet fuerit sufficiens. Sed haec ingratiam antiquaratis dicta sint, quam venerari omnes deberent: illius enim saeculi virorum labores, & ingeniorum partis haec usque non vidi a recentioribus adequata.

L2.§1. *PROPOSITION 18.*

Let A, B, C, D, E be some number of unspecified magnitudes ; and F is put equal to twice the sum of all of them except the last one term E, which is taken only once. G is taken as the sum of all the terms taken together, and then H is equal to the final term E. I say that F, G, and H continue in an arithmetic ratio.

Demonstration.

The line F, that is the sum of twice A, B, C, and D together with E taken once, exceeds the line G, that is the sum of A, B, C, D, and E, that in turn exceeds the sum of A, B, C, and D. But the line G exceeds the line H or E by the same amout. Hence the amounts F, G, H are line in continued arithmetical proportion. Q.e.d. [For 2G = F + H, and the common difference is A + B + C + D.]

Scholium.

*Does it not appear astonishing for Frederico Commandino [*1509 - 1575*: an Italian savant who translated several of the works of the ancient Greek masters, including the Collections of Pappus, published in 1565] to disregard arithmetic ratios, when Pappus in book 3 on geometrical proportion has considered the topic so carefully ?* [*Print hard to read in this section.*] *Indeed some quantity of interest can be produced not so much from geometry, but rather from a series; which fact Pappus is able to obscure from Commandino in a most able manner that can scarcely be believed. In this regard at least, the work published by Commandino is so poor regarding Pappus. For he invokes in his published work, according to PROPOSITION 19 of the same book, a continuitation of the geometrical assignation from which, planely the arithmetical analogy for the ratio may be deduced; however instead he asserts that for the rest there is no common proportionality that can be demonstrated. For indeed it seems that if you should struggle with the wonderful problems of others, then you should be able to overcome any difficulties by adhering to sure and tried methods of proof. This ingratitude to the ancients must be mentioned, as they should all be held in veneration:indeed the labours of the men of that age, and especially these with a share of ingenuity, have not seen an equal up to the present time.*

PROPOSITIO XIX.

Datis duobus progressionibus terminorum in continua ratione Arithmetica A, B, C, D, E, F; ex utriusque seriei terminis A, D, B, E, C, F, in unum constatis constituntur tertia series G, H, I.

Dico etiam G, H, I esse in continua Arithmetica analogia.

Demonstratio.

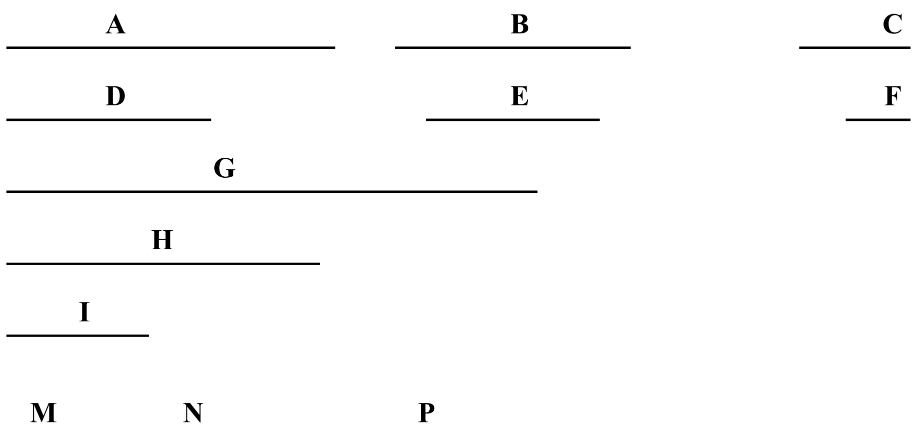
Terminorum seriei A, B, C, mutuus excessus sit M; N. vero excessus alterium M, autem & N sibi additi, faciant P : quoniam igitur A superat B excessu M, & D superat E excessu N, A, & D simul sumpti, hoc est G, superabunt B & E simul sumptos, hoc est H, excessibus M & N simul sumptis, hoc est excessu P : similiter ostendemus H superare I excessit P. sunt igitur G, H, I in Arithmerica analogia. Quod erat demonstrandum.

L2.§1. PROPOSITION 19.

A, B, C; & D, E, F are two given continued arithmetic progressions. From both series of terms A, D, B, E, C, F, it is agreed to set up the third series G, H, I.

I say that the terms G, H, and I are also in a continued arithmetic ratio.

Demonstratio.



Prop.19. Fig. 1.

The mutual excess of the terms of the series A, B, & C is M; while N is the excess of the other series, and moreover the sum of N and M is P. Therefore as A is greater than B by M, and D is greater than E by N, then the sum of A and D, or G, is greater than the sum of B and E, or H, by the sum of the excesses M and N, that is the excess P : similarly we can show that H is greater than I by the excess P. Hence G, H, and I are in arithmetic progression. Q. e. d.

PROPOSITIO XX.

Esto AB ad BC, ut BD ad DE; & fiat ipsi BC aequalis EF, ubicumque tandem cadat punctum E.

Dico esse ut AB ad AC, sic AD ad AF, & BD ad BE.

Demonstratio.

Quoniam est ut AB ad BC, ita BD ad DE, erit componendo, convertendo ut AB ad AC, ita BD ad BE: ulterius, cum sint EF, BC aequales, erit tota AF aequalis quatuor proportionalibus AB, BC, BD, DE : & AD aequalis primae & tertiae, uti & AC primae ac secundae : quare ut a AC ad AB, ita AF ad AD: & invertendo ut AB ad AC, ita AD ad AF, & ut ante ostendimus, ita BD ad DE. Quod erat demonstrandum. *a 16 Huius.*

L2.§1. PROPOSITION 20.

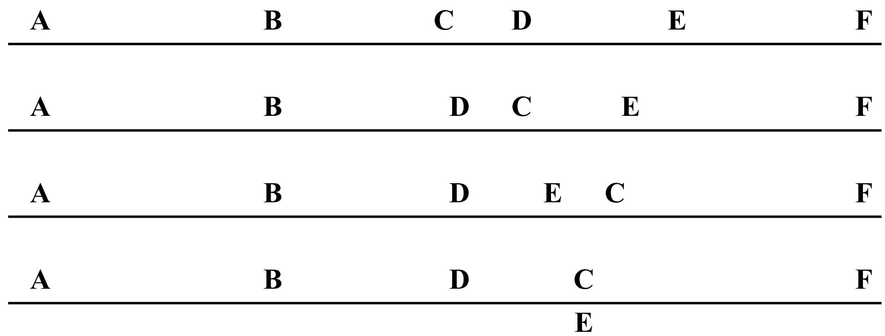
The ratio AB to BC shall be as BD to DE; and EF is made equal to BC, and finally the position of the point E is variable .

I say that as AB is to AC, thus AD to AF, and BD to BE.

Demonstration.

Since AB is to BC, thus as BD is to DE, by adding and rearranging as AB to AC, thus as BD to BE: further, as EF is equal to BC, the whole length AF is equal to the sum of the four ratios AB, BC, BD, and DE : AD is equal to the sum of the first and the third, as AC is equal to the first and the second. Whereby as [a] AC is to AB, thus AF is to AD: & invertenso ut AB ad AC, ita AD ad AF, & ut ante ostendimus, ita BD ad DE. Quod erat demonstrandum. *a 16 Huius.*

[BC/AB = DE/BD, giving AC/AB = BE/BD on adding, and AB/AC = BD/BE on inverting. Again, the



Prop.20. Fig. 1.

sum of the four ratios is AB + BC + BD + DE = AB + BD + EF + DE = AF; while AD = AB + BD and AC = AB + BC. From EF/AB = DE/BD we have EF/DE = AB/BD, giving DF/DE = AD/BD and DF/AD = DE/BD gives AF/AD = BE/BD = AC/AB and on inverting, AB/AC = AD/AF = BD/BE as required.]

PROPOSITIO XXI.

Datae sint quatuor proportionales, minina AB, secunda AC, tertia AD, quarta AE.

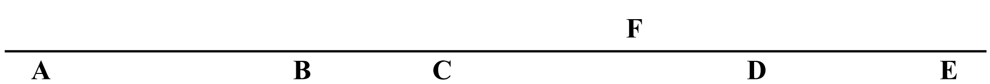
Dico primo DB differentiam primae & tertiae, minorem esse EC, differentia secundae & quartae.

[69]

Secundo si BD minori auferatur aequalis EF, ex CE maiori , ut AB ad BD, vel ut AC ad CE, sic BC differentiam primae & secundae, fore ad CF differentiam differentiarum BD & CE.

Demonstratio.

Cum ex hypothesesi AB sit ad AC, ut AD ad AE: igitur permutando ut AB ad AD, sic AC ad AE; & dividendo ut AB ad BD, sic AC ad CE; igitur permutando ut AB ad AC, sic BD ad CE; atque AB minor est quam AC, igitur & BD minor est quam CE. Quod erat primum. deinde ex discursu iam facto, ut AB ad AC,



Prop.21. Fig. 1.

sic est BD ad CE; quare cum EF ex hypothesi sit aequlis ipsi BD. erit etiam AB ad AC, ut EF ad CE; ac dividendo ut AB ad BC, sic EF ad FC, & permutando ut AB ad EF, id est ut AB ad BD, sic BC ad CF: atque etiam ut AB ad BD, sic est AC ad CE, (ut ante ostendi) ergo ut AB ad BD, vel AC ad BD, vel AC ad CE, sic est BC ad CF. Quod erat demonstrandum.

L2.§1. PROPOSITION 21.

There are four proportionals given, AB is the least, AC the second, AD the third, and AE the fourth.

I say in the first place, that the difference DB between the first and the third, is less than the difference EC between the second and the fourth.

In the second place, if the smaller difference BD is taken from the greater difference CE, equal to EF, as AB to BD, or as AC to CE, thus the difference BC between the first and second, will be as CF, the difference of the differences BD and CE.

Demonstration.

For by hypothesis, AB is to AC, as AD to AE: therefore on interchanging, as AB to AD, thus AC to AE; and on division, as AB to BD, thus AC to CE; therefore on interchanging, as AB to AC, thus BD to CE; and AB is less than AC, and hence BD is less than CE. Which establishes the first part of the proposition.

Further, from what has been established, as AB is to AC, thus BD is to CE; whereby as EF by hypothesis is equal to BD, then also AB is to AC, as EF is to CE; and on division, as AB to BC, thus EF to FC, and on interchanging, as AB to EF, or AB to BD, thus BC to CF: and also as AB to BD, thus AC is to CE, (as shown before) hence as AB to BD, or AC to CE, thus BC is to CF. Q. e. d.

[$AB/AC = AD/AE$, giving $AB/AD = AC/AE$ and on inverting, $AD/AB = AE/AC$ and on taking, $BD/AB = CE/AC$ or $AB/BD = AC/CE$ and finally, $AB/AC = BD/CE < 1$.
Again, from $AB/AC = BD/CE = FE/CE$, and on inverting, $AC/AB = CE/FE$, and on taking, $BC/AB = CF/FE$ giving $AB/BC = FE/CF$ and on interchanging, as $AB/EF = BC/CF = AB/BD$; also, as $AB/BD = AC/CE$ from above, thus, $AB/BD = AC/CE = BC/CF$.]

PROPOSITIO XXII.

Datae sint quatuor proportionales, minina AB, secunda AC, tertia AD, quarta AE.

Dico primo, differentiam primae & secundae BC, minorem esse differentia DE, tertiae & quartae:

Secundo si ex ED maiori tollatur EF, aequalis minori BC, fore ut AD ad DE, sic BD differentiam primae & tertia, ad DF differentiam differentiarum EC, DE.

Demonstratio.

Cum sit AB ad AC, ut AD ad AE: igitur invertendo, dividendo rursumque invertendo, ut AB ad BC, sic AD ad DE: Itaque permutendo ut AB ad AD, sic BC ad DE. sed AB minor est AD, ergo & BC minor est quam DE. Quod erat primum.

Prop.22. Fig. 1.

Deinde cum modo ostensum sit, esse AB ad AD , ut BC est ad DE; & cum FE ponatur aequalis BC, etiam AB est ad AD, ut EF ad DF. Itaque invertendo, ac convertendo ut AD ad BD, sic DE ad FE, ac deinum permutando ut AD ad DE, sic BD ad DF. Atqui etiam ut AD ad DE, sic est AB ad BC. (ut ante ostendi) ergo ut AD ad DE, sic est AB ad BC , sic est BD ad DF. Quod erat demonstrandum.

L2.§1. *PROPOSITION 22.*

There are four proportionals given, AB is the least, AC the second, AD the third, and AE the fourth.

I say in the first place, that the difference BC between the first and the second is less than the difference DE between the third and the fourth.

In the second place, if an amount EF equal to the lesser difference BC is taken from the greater difference ED, then the ratio AD to DE thus will be as the difference BD between the first and third proportionals, to DF, the difference of the differences EC and DE.

Demonstration.

For as AB is to AC, so AD is to AE: therefore on inverting, dividing, and again inverting, as AB to BC, thus AD to DE; and on interchanging, as AB to AD, thus BC to DE; but AB is less than AD, and hence BC is less than DE, as AB to AC. Which establishes the first part of the proposition.

Further, in this way that it can be shown that AB is to AD as BC is to DE, and with FE put equal to BC, also AB is to AD as EF is to DF. For on inverting and converting, as AD is to BD, thus DE is to FD, and then on interchanging, as AD to DE, thus BD to DF. But also, as AD is to DE thus AB is to BC, (as shown before) hence as AD to DE, thus AB to BC, thus BD is to DF. Q. e. d.

[AB/AC = AD/AE, giving AC/AB = AE/AD , BC/AB = DE/AD , and AB/AD = BC/DE < 1 as required. Again, from AB/AD = BC/DE = FE/DE, and on inverting, AD/BD = DE/FD, and on interchanging, AD/DE = BD/DF. Also, AD/DE = AB/BC, giving AD/DE = AB/BC = BD/DF as required.]

PROPOSITIO XXIII.

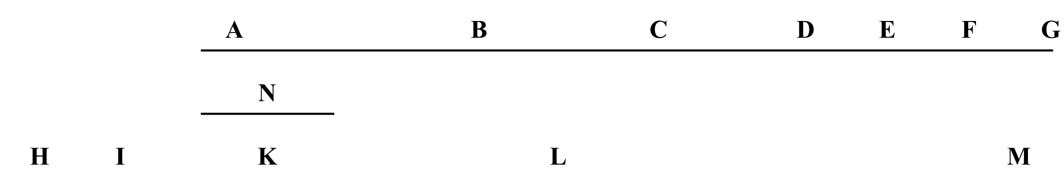
Sint continui proportionalium processus, AB, BC, CD, DE, EF: assumta vero quavis alia N, fiat ut AB ad N, sic N ad HI: utque BC ad N, sic N fiat ad IK, & ut CD ad N, sic N ad KL, & sic deinceps.

Dico HI, IK, KL, &c. esse in continua analogia.

[70]

Demonstratio.

Ex hypothesi lineae AB, N, HI, sunt continuae proportionales; item BC, N, IK; ergo tam rectangulum ABHI quam rectangulum BCIK [a] aequatur quadrato N. ac proinde aequalia sunt inter se rectangula; ergo ut AB ad [b] BC, sic IK ad HI, similiter rectangula BCIK, & CDKL, aequantur inter se, quia eidem quadrato N



Prop.23. Fig. 1.

aequalia sunt; igitur ut BC ad CD, sic KL ad KI : atqui ex datis BC est ad CD, ut AB ad BC. hoc est, ut ostendi, sicut IK ad HI, ergo KL est ad IK, ut IK ad HI: continuae sunt proportionales igitur HI, IK, KL. Quod erat demonstrandum. *a. 17 Sexti; b. 16 Sexti.*

L2.§1. *PROPOSITION 23.*

AB, BC, CD, DE, and EF is a procession of continued proportionals : assume that there is some other length N, constructed so that as AB is to N, thus N is to HI; and as BC is to N, thus N is made in the ratio IK, and as CD to N, thus N to KL, and thus henceforth.

I say that HI, IK, KL, etc are in a continued ratio.

Demonstration.

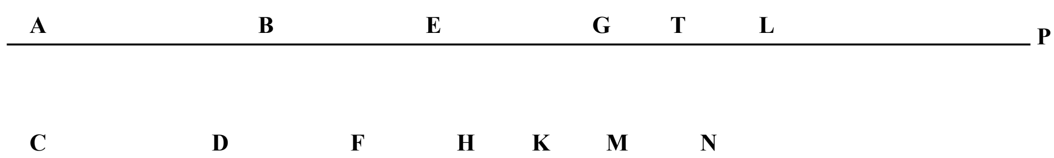
By hypothesis, the lines AB, N, and HI are in continued proportion; likewise for BC, N, and IK. Hence the rectangles AB.HI and BC.IK are equal to the square N [a]. Hence the rectangles are equal to each other; hence as AB is to BC, thus IK is to HI; similarly the rectangles BC.IK and CD.KL are equal to each other as both are equal to the same square N. Thus BC is to CD, thus KL is to KI; but as BC to CD is given as equal to AB to BC, as shown, thus IK to HI, as KL is to IK, as IK to HI. Hence HI, IK, KL are continued proportions. Q. e. d.

PROPOSITIO XXIV.

Si continue proportionalium rectangulorum bases sint in continua analogia: erunt & altitudines in continuata proportione.

Demonstratio.

Sint AB, BE, EG, GI, IL, insuper & rectangula AB.CD, BE.DF, EG.FH &c.in continua analogia . Dico etiam CD,DF, FH, &c. esse continue proportionales. Rectanguli enim AB.CD proportio ad rectangulum BE.DF, [c] componitur ex rationibus AB ad BE, & CD ad DF : item rectanguli BE.DF proportio ad EG.FH



Prop.24. Fig. 1.

rectangulum , componitur ex rationibus BE ad EG, & DF ad FH: quare cum ex hypothesi rectangulorum AB.CD, BE.DF, EG.FH, aequales sive eaedem proportionales sint, erunt quoque rationes AB ad BE, & CD ad DF, simul sumptae aequales, sive eaedem cum rationibus BE ad EG, & DF ad FH simul sumptis . Sunt autem ex hypothesi etiam aequales rationes AB ad BE, & BE ad EG; Quare si ab aequalibus rationibus, nempe a composita ex rationibus AB ad BE, & CD ad DF; itemque a composita ex rationibus BE ad EG, & DF ad FH, auferas aequales rationes, AB ad BE, & BE ad EG : patet reliquas proportiones CD ad DF, & DF ad FH fore aequales : ut constat ex definitione compositionis proportionum. Continue sunt igitur proportionales CD, DF, FH. Simili discursu etiam reliquae HK, KM, &c. cum praecedentibus eandem continuabunt ratione. Constat ergo propositium erat demonstrare. c. *15 Sexti.*

Corollarium.

Eodem genere discursus, si rectangula ABCD, BEDN, EGFN, &c. itemquebases AB, BE, EG, &c. sint in continua analogia, demonstrabimuseorum altitudines CN, DN, FN, &c. continuam quoque seruare analogiam.

L2.§1. *PROPOSITION 24.*

If the bases of rectangles in continued proportion are in continued proportion, then the altutudes are also in continued proportion.

Demonstration.

The bases AB, BE, EG, GI, IL as well as the rectangles ABCD, BEDF, EGFH , etc, are in continued proportion. I say that the altitudes CD, DF, FH, etc, are also in continued proportion. For the proportion of the rectangle ABCD to the rectangle BEDF, [c] is composed from the ratios AB to BE, and CD to DF : likewise the proportion of the rectangle BEDF to the rectangle EGFH is composed from the ratios BE to EG and DF to FH. Wereby by hypothesis, if the rectangles ABCD, BEDF, EGFH, are all in the same proportion, then the ratios AB to BE and CD to DF are equal for the ratios taken together, as likewise they are for the ratios BE to EG and DF to FH taken together. But by hypothesis, the ratios AB to BE, and BE to EG are also equal; whereby if from the equal ratios, truly composed from the the ratios AB to BE, and CD to DF, and likewise from the rectangles composed from ratios BE to EG, and DF to FH, you take away the equal ratios, AB to BE, and BE to EG : it is apparent that the proportions left, CD to DF, and DF to FH will be equal : in agreement with the definition of proportion. Therefore CD, DF, and FH are continued proportions. By a similar discussion the remaining terms HK, KM, etc also are in the same continued ratio with the preceeding. This is in agreement with the proposition that was to be shown. c*. 15 Sexti.*

Corollary.

From the same discussion, if the rectangles AB.CN, BE.DN, EG.FN, etc and likewise the bases AB, BE, EG, etc. are in continued proportion, then we can show that the altitudes CN, DN, FN, etc of the same are also in continued proportion.

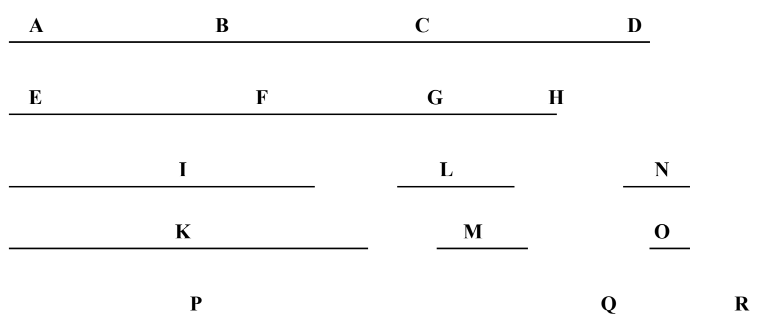
PROPOSITIO XXV.

Ponatur denuo AB, BC, CD proportionales, uti etiam EF, FG, GH, & ratio AB ad EF, continuetur quomodocunque in I, K, P, & similiter ratio BC ad FG producatur in L, M, Q: & ratio CD ad GH, pergat in N, O, R, &c.

Dico etiam I, L, N, & K, M, O, item P, Q, R, continuare suam rationem.

[71]

Demonstratio.



Prop.25. Fig. 1.

Quadratum enim EF est aequale rectangulo sub AB & I; & quadratum FG, aequale est rectangulo sub BC & L : uti etiam quadratum GH, rectangulo sub CD & N contento : igitur ut quadrata EF, FG, GH inter se sunt , ita etiam rectangula sub lateribus AB & I, sub BC & L, item sub CD, & N, contenta: sed quadrata sunt in continuata analogia, (cum latera super quibus fiunt ex hypothesi sint in continua analogia) igitur etiam rectangula, sub lateribus AB & I, BC & L, CD & N, sunt continue proportionalia : cum autem ipsorum bases ponantur in continuata ratione AB, BC, CD; etiam I, L, N, altutudines erunt in continuata analogia per praecedentem. Eodem modo quia ex hypothesi EF, FG, GH, & ex demonstratis modo I, L, N, sunt continuae; itemque ex hypothesi EF, I, K : & FG, L, M; item GH, N, O, continuam servant analogiam : demonstrabimus K, M, O esse continuas. similiter quoque procedetur in lineis P, Q, R, atque ita in infinitum. constat ergo veritas proportionis. *a. 27 Sexti.*

L2.§1. *PROPOSITION 25.*

The proportionals AB, BC, and CD are set in place anew, as also are the proportionals EF, FG, and GH, and the ratio of AB to EF is set to continue in some manner in successive proportionals I, K, and P. Similarly the ratio BC to FG proceeds as to the proportionals L, M, Q; and the ratio CD to GH goes on as N, O, R, etc.

I say that I, L, N, and K, M, O, and likewise P, Q, and R are also continued proportionals in their own ratio.

Demonstration.

The square EF is indeed equal to the rectangle under AB and I; and the square FG is equal to the rectangle under BC and L; as also the square GH is equal to the rectangle contained under CD and N. Thus as the squares EF, FG, and GH are self-contained , so also are the rectangles contained under the sides AB and I, BC and L, and likewise CD and N. But the squares are in a continued ratio (since the sides of the squares upon which they are formed are in a continued ratio by hypothesis). Hence the rectangles under the sides AB and I, BC and L, and CD and N, are also in continued proportion: but since the bases themselves are placed in the continued ratio AB, BC, and CD; thus the altitudes I, L, and N also are in a continued ratio from the preceeding proposition.

By the same method, whereby if by hypothesis EF, FG, GH are in continued proportion, then I, L, N are shown to be in continued proportion; it then can be shown in the same manner that EF, I, K are in a continued ratio: as are FG, L, M; and likewise GH, N, O. We can then show that K, M, O are in a continued ratio, and similarly the method can be applied to the lines P, Q, R, and thus indefinitely, hence in agreement with the truth of the proposition. *a 17 Sexti.*

[$EF^2 = AB.I$; $FG^2 = BC.L$; $GH^2 = CD.N$; as EF, FG, GH are in continued proportion, as are AB, BC, and CD, it follows that I, L, N are also in continued proportion. We can then set $I^2 = EF.K$; $L^2 = FG.L$; $N^2 = GH.O$ and so establish that K, M, O are in a continued ratio, etc.]

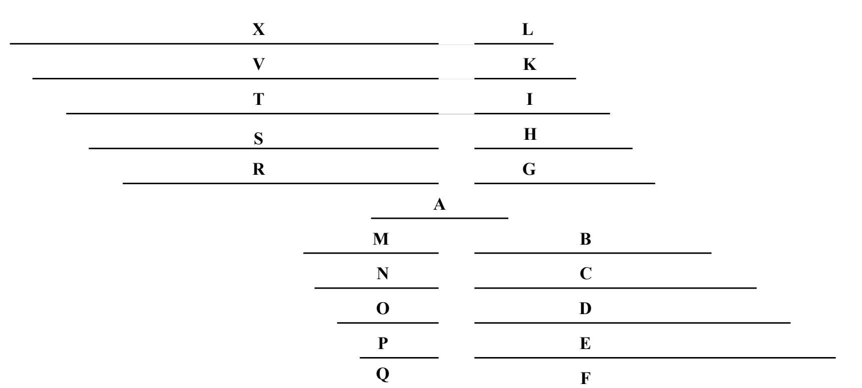
PROPOSITIO XXVI.

Sint series continuae proportionalium, habentes primum terminum A, communem; A, B, C, D, E, F, & A, M, N, O, P, Q; continuata autem serie utriusque , fiat ut B ad A, ita A ad G, &c. & ut M ad A, ita A ad R, sic ut omnes F, E, D, C, B, A, G, H, I, K, L: item omnes Q, P, O, N, M, A, R, S, T, V, X sint in continuata analogia.

Dico esse B ad M, ut R ad G, & C ad N, ut S ad H, & D ad O, ut T ad I, &c.

Demonstratio.

Cum B, A, G sint tres continuae, item M, A, R, erit tam [b] rectangulum BG, quam rectangulum MR, quadrato A, ideoque & inter se aequalia. Quare ut B ad M, sic [c] R ad G: similiter quoniam C, B, A, G, H, sunt continuae, ita ut mediam A,



Prop.26. Fig. 1.

aequalis utrimque proportionalium numerus cingat: patet ex elementis A esse mediam proportionalem, inter C & H : rectangulum igitur HC aequatur quadrato A; eodem modo rectangulum SN quadrato A aequale erit: ergo inter se aequantur rectangula HC, SN. Quare [d] ut C ad N, sic S ad H, simili discursu erit ut D ad O, sic T ad I, & c sic de caeteris. Quod erat demonstrandum. *b. 17 Sexti; c. 16 Sexti.*

[72]

L2.§1. *PROPOSITION 26.*

There are continued series of proportionals having the first term A in common : A, B, C, D, E, F; and A, M, N, O, P, Q. Moreover, a further series in continued proportion is constructed according to B to A, thus A to G, etc.; and as M to A, thus A to R, in order that all of F, E, D, C, B, A, G, H, I, K, L, and likewise Q, P, O, N, M, A, R, S, T, V, X are in continued proportion.
I say that as B is to M, so R to G, and as C to N, to S to H, as as D to O, so T to I, etc.

Demonstration.

Since B, A, and G are three terms in continued proportion, as well as M, A, and R, it follows [b] that the rectangles B.G and M.R are equal to each other and to the square A. Whereby as B to M, thus [c] R to G : similarly, since C, B, A, G, H are in continued proportion, in order that the mean A is surrounded by an equal term from both sides of the proportion, it is apparent from this principle that A is the mean proportional between C and H. Therefore the rectangle H.C is equal to the square A, and in the same way the rectangle S.N is equal to the square A. Hence the rectangles H.C and S.N are equal to each other. Whereby [d] as C is to N, thus S to H, and by a similar discourse, as D is to O, thus T is to I, and so on for the rest. Q.e.d. *b. 17 Sexti; c. 16 Sexti.*

[72]

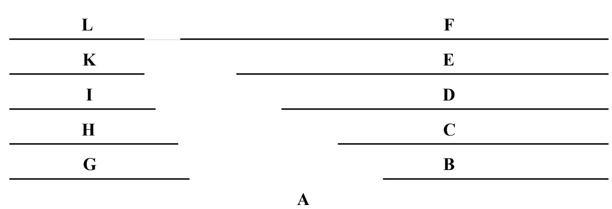
PROPOSITIO XXVII.

Sint duo ordines continuae proportionalium eandem habentes primum, A, B, C, D, E, F, & A, G, H, I, K, L.

Dico proportionem H ad C, esse duplicatum proportionis G ad B, & rationem I ad D, rationis G ad B esse triplicatam : & rationem K ad E, quadruplicatam : L vero ad F quintuplicatam eiusdem rationis G ad B: & sic in infinitum.

Demonstrat hanc Euclid.lib.14.pr.28 de quatuor continue proportionalibus. Nos eandem de quotcunque, & alia plane methodo demonstribimus.

Demonstratio.



Prop.27. Fig. 1.

Quoniam tam A, G, H quam A, B, C, sunt continuae proportionales, rectangulum HA, quadrato G , & rectangulum CA, quadrato [a] B aequales erit. unde rectangulum HA, ad rectangulum CA, est ut quadratum G, ad quadratum B: itaque cum quadrata G, B, sint [b] in duplicata ratione basium G ad B, erit & rectangulorum proportio duplicata, rationis G ad B : sed ratio rectanguli HA, ad rectangulum CA, est ratio [c] H ad C; ergo proportio H ad C; ergo proportio H ad C est duplicata rationis G ad B. Deinde quoniam tam G, H, I , quam B, C, D , sunt continuae, erit rectangulum IG, quadrato H, & rectangulum DB, quadrato C aequale, itaque ratio rectangulorum IG, DB, hoc est ratio [d] composita ex proportionibus laterum I ad D, & G ad B, aequalis est rationi quadrati H ad C: atqui ratio quadratorum H, C , est quadruplicata rationis G, B; (est enim ratio quadratorum H, C duplicata rationis H ad C, quae ostensa modo est duplicata rationis G ad B:) ergo proportio composita ex rationibus I ad D, & G ad B, est quadruplicata rationis G ad B. ex quo patet rationem I ad D solam, esse triplicatam rationis G ad B. Simili discursu demonstrabimus rationem K ad E,

esse quadruplicatam; & rationem L ad F, quintuplicatam rationis G ad B. constat ergo veritas propositionis.
a. 17 Sexti; b. 20 Sexti; c. 1 Sexti; d. 23 Sexti.

L2.§1. PROPOSITION 27.

There are two orders of continued proportionals having the same first term : A, B, C, D, E, F; and A, G, H, I, K, L.

I say that the proportion of H to C is double the proportion of G to B, and the ratio I to D to be the triple of the ratio G to B: and the ratio K to E, the quadruple; L to F the quintuple of the ratio G to B, and thus indefinitely.

This is shown by Euclid in book 14, prop. 28 for four continued proportionals. We show the same for any number of proportionals by another clear method.

Demonstration.

Since A, G, and H as well as A, B, C are in continued proportion, the rectangle H.A is equal to the square G, and likewise the rectangle A.C is equal to the square [a] B. Thus, the rectangle H.A is to the rectangle C.A, as the square G is to the square B. Thus as the squares G and B are in the ratio of the squares of the bases G and B, the ratio of the rectangles will be the square of the ratio G to B. But the ratio of the rectangles H.A and C.A is the ratio [c] H to C: hence the ratio H to C is the duplicate [i.e. square] of the ratio G to B. Hence, since G, H, and I, as well as B, C, D are in continued proportion, the rectangle I.G is equal to the square H, and the rectangle D.B is equal to the square C. Thus the ratio of the rectangles I.G to D.B , that is composed from the proportions of the sides I to D and G to BM, is equal to the ratio of the squares H and C. But the ratio of the squares H and C is the quadruple of the ratio G to B; (the ratio of the squares of H to C is indeed the duplicate of the ratio H to C, which has been shown to be the duplicate of ratio G to B); hence the ratio composed from the ratios I to D and G to B is the quadruple of the ratio G to B. From which it is apparent that the ratio I to D alone is the triplicate of the ratio G to B. By a similar argument we can show that the ratio K to E is the quadruple, and the ratio L to F the quintuple of the ratio G to B, in agreement with the truth of the proposition. *a. 17 Sexti; b. 20 Sexti; c. 1 Sexti; d. 23 Sexti.*

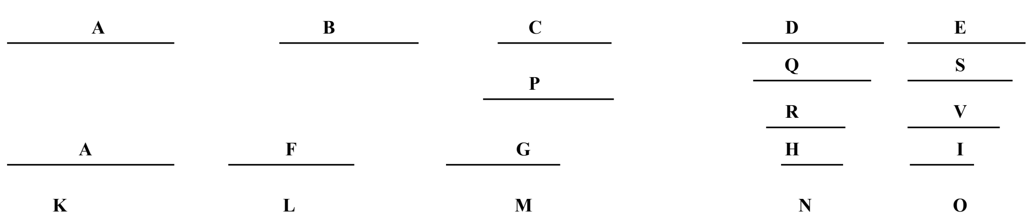
[$G^2 = H.A$; $B^2 = C.A$; hence $G^2/B^2 = H/C$. Again, $H^2 = I.G$, and $C^2 = D.B$, $I.G/D.B = I/D.G/B = H^2/C^2 = G^4/B^4$, from which it follows that $I/D = G^3/B^3$. Again, $K.H = I^2$ and $E.C = D^2$; hence, $K/E.H/C = I^2/D^2 = G^6/B^6$; hence, $K/E = G^4/B^4$ and $L/F = G^5/B^5$, etc. Note: the reader should bear in mind that the terms duplicate, triplicate, etc, refer to squares, cubes, etc, of ratios.]

PROPOSITIO XXVIII.

Sint duo ordines continuarum A, B, C, D, E; & A, F, G, H, I eandem nacti primam A : ponatur autem tertius ordo continue proportionalium, K, L, M, N, O, secundo ordini similis; ita tamen ut A & K, sint inaequales. Deinde inter tertias C & G, ponatur media P; & inter quartas D & H. ponantur duae mediae Q & R; tandem inter quintas ponantur tres mediae S, T, V, & ita deinceps.

Dico esse ut L ad B, ita M ad P, & N ad R, ut O ad V, &c.

Demonstratio.



Prop.28. Fig. 1.

[73]

Quoniam utriusque sereri, primus terminum idem est A, ergo per praecedentem G est ad C, in duplicata ratione F ad B; sed G etiam est ad C, in duplicata ratione G ad P, igitur G est ad P, ut F ad B: sed ex supposito F ad G est ut L ad M; igitur alternando M est ad G, ut L ad F. est igitur ex aequo M ad P, ut L ad B. Simili modo per praecedentem H ad D, triplicatam, habit ratonem, F ad B; sed etiam H ad D, habet triplicatam eius , quam habet H ad R; ergo ut F ad B, sic H ad R: deine est F ad H, ut L ad N, unde alternando, & ex aequo N ad R ut L ad B. Eadem prorsus ratione demonstrabitur esse O ad V, ut est L ad B. Quare patet veritas propositionis.

Hinc etiam patet B, P, R, V esse continuas cum L, M, N, O, sint continuae, exquidem in eadem analogia in qua sint lineae A, F, G, H, I, ut patet.

. *a. 17 Sexti; b. 20 Sexti; c. 1 Sexti; d. 23 Sexti.*

L2.§1. *PROPOSITION 28.*

There are two series of continued proportionals obtained from the same first term A; A, B, C, D, E; and A, F, G, H, I. A third series of continued proportinals K, L, M, N, O, is put in place, similar to the second series but with A not equal to K. Subsequently, a mean P is placed between the third terms C and G ; and two means Q and R is placed between the fourth terms D and H ; finally, three means S, T, and V are placed between the fifth terms, and henceforth.

I say that as L is to B, thus M is to P, and N is to R, as O to V, etc.

Demonstration.

Since the first term of the second series is also A, it follows by the preceeding proposition that G is to C in the duplicate [or square] ratio F to B. However, G is to C in the square ratio G to P, and therefore G is to P as F is to B. Moreover, from supposition, F is to G as L is to M; therefore alternatively, M is to G as L is to F. Therefore on equating, M is to P, as L is to B. In the same way, by the preceeding theorem, H is to D, in the triplicate ratio F to B; but also H to D is in the triplicate ratio of that which H has to R; therefore as F to B, thus H to R: then as F to H, so L to N, from which otherwise, and on equating, N is to R as L is to B. In short for the same ratio it can be shown that O is to V as L is to B. Whereby the truth of the proposition is apparent.

Hence, it can also be seen that B, P, R, V are in continued proportion with L, M, N, O; and these indeed are apparent to be in the same ratio as the lines A, F, G, H, I.

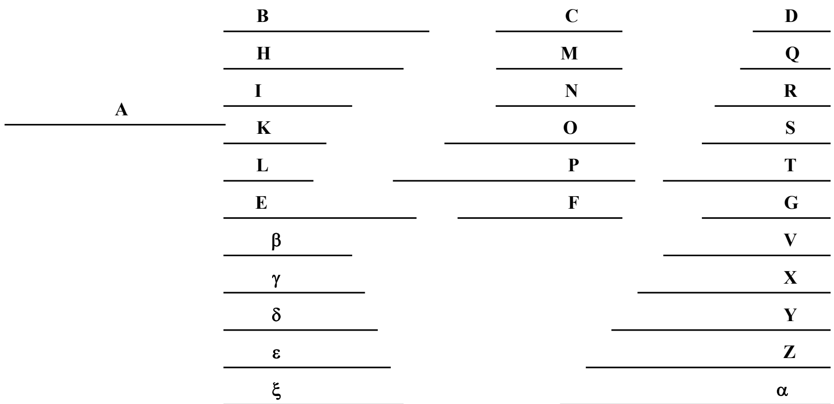
[$G/C = F^2/B^2$. Again, $G^2/P^2 = G/C$; hence, $G/P = F/B$. But $F/G = L/M$, or $M/G = L/F$ and from $G/P.M/G = F/B.L/F$, it follows that $M/P = L/B$.
Again, $H/D = F^3/B^3$, but in this case from the three means we have $H/R = R/Q = Q/D$ or $H/D = H^3/R^3$, from which $H/R = F/B$ or $F/H = B/R$; also, $F/H = L/N$, hence $N/R = L/B$. Again, $O/V = L/B$, etc. (Thus, the power associated with a mean between like terms in a series of proportions can take the place of the power associated with the first ratio, for any term in a series of proportions).]

PROPOSITIO XXIX.

Ponatur duae series continuarum A, B, C, D ; & A, E, F, G, communem habentes primam A : & inter secundas B, E, sint quotvis mediae H, I, K, L, totidem quo inter tertias interponantur; M, N, O, P, Similiter inter quartas D & G ponantur mediae Q, R, S, & T.

Dico A, H, M, Q; item A, I, N, R; & A, K, O, S, & A, L, P, T esse in continua analogia.

Demonstratio.



Prop.29. Fig. 1.

Ponantur enim inter A & M, media proportionalis V ; & inter A & N media X; similiter Y, Z, α mediae ponantur inter A, O, & A, P, & A, F. Quoniam ergo A, B, C, & A, V, M sunt continuae proportionales, erit [a] ratio C ad M, duplicata eius, quam habet B as V. uterius cum A, V, M & A, X, N etiam sint continuae proportionales, erit [b] M ad N, duplicata eius quam habet V ad X : eodem pacto ostenditur, rationes N ad O, & O ad P, & P ad F, esse duplicatas rationum X ad Y, & Y ad Z, & Z ad α : Quare cum C, M , N, O, P, F ponantur esse continuae, etiam B, V, X , Y, Z , α patet esse continuas. Est igitur ratio B ad α, quintuplicata rationis B ad V, & quia tam α, quam E, mediae sunt inter A & F, inter se aequales erunt ideoque & ratio B ad α, & ratio B ad E, eadem est; ergo ratio B ad E, quintuplicata est rationis B ad V, quia autem B, H, I, K, L, E ponantur continuae, etiam ratio B ad E, est quintuplicata rationis B ad H; est igitur B ad V, ut B ad H, unde aequales sunt H & V. Similiter ostenditur I & X, K & Y, L & Z aequales esse. Quare cum ex constructione A, V, M; A X N; A Y O; A, Z, P, sint continuae; etiam A, H, M; A, I, N; A, K, O; A, L, P continuae erunt. Non alia ratione ostendemus, etiam ipsas A, H, M, Q; item A, I, N, R, &c. esse in continuata analogia;

[74]

(Si nempe ipsis A, H, M inveniamus quartam β, & ipsis A, I, N quartam , γ, & ipsis A, K, O quartam δ, & ipsis A, L, P, ε, denique ipsis A, E, F quartam ξ) demonstrabimus esse ut prius ipsas β, γ, δ, ε, ξ, ipsis Q, R, S, T, G aequales; ac proinde omnes quatuor A, H, M, Q; A, I, N, R; &c. esse continuas. Quod erat demonstrandum.

a. 27 huius; b. ibid.

L2.§1. *PROPOSITION 29.*

There are two series of continued proportionals A, B, C, D ; and A, E, F, G ; put in place having the same first term A. Between the second terms B and E certain means H, I, K, L are inserted; and the same number of means are inserted between the third terms C and F: M, N, O, P . Similarly, the means Q, R, S, T are inserted between the fourth terms D and G.

I say that the series A, H, M, Q; as well as A, I, N, R; A, K, O, S ; and A, L, P, T are in continued ratios.

Demonstration.

For the mean V of the proportionals A and M can be placed between A and M ; and the mean X between A and N ; similarly the means Y, Z, and α are placed between A, O; A, P; and A, F. Hence, as A, B, C and A, V, M are continued proportionals, the ratio [a] C to M is the duplicate [i. e. square] of the ratio B to V. Further, as A, V, M and A, X, N also are continued proportionals, [b] M to N, is the duplicate of V to X : in the same manner it can be shown that the ratios N to O, O to P, and P to F, are the duplicate ratios of X to Y, Y to Z, & Z to α. Whereby as C, M , N, O, P, F are placed in a continued ratio [the 5th order geometric mean between C and F], so also B, V, X , Y, Z , α have been shown to be continued proportionals [the 2^{nd} order geometric mean between A and the series just mentioned]. Therefore the ratio B to α is the quintuplicate [i.e. B/V raised to the 5^{th} power] of the ratio B to V; and as for α, so also for E, which is the mean of A and F, and the ratio B to α is the same as the ratio B to E: hence E and α are equal to each other; therefore the ratio B to E is the quintuplicate of the rato B to V. Since in addition, B, H, I, K, L, E are placed in continued proportion, the ratio B to E also is the quintuplicate of the ratio B to H; therefore B to V is as B to H, and hence H and V are equal. In the same manner it can be shown that I and X, K and Y, and L and Z are equal. Whereby as A, V, M; A, X, N; A, Y, O; A, Z, P are in continued proportin by construction , also A, H, M; A, I, N; A, K, O; A, L, P are in continued proportion. We do not show others further, but A, H, M, Q; and likewise A, I, N, R, &c. are continued ratios. (For A, H, M themselves we can find the fourth mean β, and likewise for A, I, N, γ; A, K, O, δ; A, L, P, ε; A, E, F, ξ) . We can show that as before β, γ, δ, ε, ξ, are equal to Q, R, S, T, G; and hence all four A, H, M, Q; A, I, N, R; etc. are in continued proportion. Q.e.d.

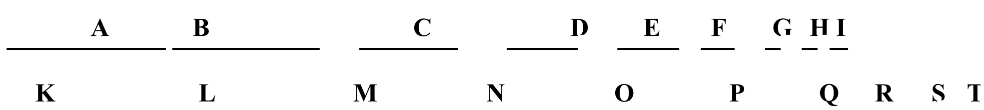
[We have the continued proportionals A, B, C; A, V, M; A, X, N; A, Y, O; A, Z, P; A, α, F.
Now, $C/M = B^2/V^2$; and similarly, $M/N = V^2/X^2$; as $N/O = X^2/Y^2$, $O/P = Y^2/Z^2$, $P/F = Z^2/\alpha^2$. Hence as C, M , N, O, P, F are in continued proportion, (as they are the successive 5^{th} order mean proportionals between C and F), so also B, V, X , Y, Z , α are in a continued ratio, and $B/\alpha = B^5/V^5$; $B/\alpha = B/E$ giving $\alpha = E$. Again, in the same way, $B/E = B^5/V^5$; as B, H, I, I, L, E are in a continued ratio, $B/E = B^5/H^5$: B/V = B/H and V = H. In the same way, I = X; K = Y; L = Z. Hence, as A, V, M; A, X, N; A, Y, O; A, Z, P are in continued proportin by construction , also A, H, M; A, I, N; A, K, O; A, L, P are in continued proportion, etc.]

PROPOSITIO XXX.

Dentur binae series continuae proportionalium in diversis rationibus: A, B, C, D, &c; K, L, M, N, &c. ita tamen ut A, K, L, B sint etiam continuae proportionales.

Dico omnes A, K, L, B, N, O : C, Q, R, D, & sic deinceps, (omisso in serie K L tertio quoque termino M, P, S) esse in continua analogia.

Demonstratio.



Prop.30. Fig. 1.

Quia A, K, L, B ponantur continue, ergo ut K ad L, sic L ad B; sed etiam est ex hypothesi ut K ad L, sic L ad M; ergo L ad B, & M eadem habet rationem; aequales [a] igitur sunt B & M. ideoque [b] rationes B ad C, M ad C aequales sunt. Quoniam autem A, K, L, B sunt quatuor continuae proportionales, erit ratio A ad B, id est ex hypothesi ratio B ad C, id est ex demonstratione ratio M ad C, triplicatata rationis A ad K: sed ratio A ad K, ex hypothesi est ratio K ad L, id est ratio M ad N : ergo ratio M ad C, triplicata est rationis M ad N : est autem ratio M ad P, triplicata rationis M ad N, (sunt enim M, N, O, P continuae) igitur ut M ad C, sic M ad P: unde & aequales sunt C & P. quare cum loco M, in serie statuatur illi aequalis B, & loco P, illi aequalis C. erunt A, K, L, B, N, O, C continuae proportionales. similiter ostendemus seriem hanc per terminos Q, R, D, &c. continuati in infinitum. Quod erat demonstrandum.

a. 9 Quinti; b. 7 Eius.

Corollarium.

Hinc sequitur : rationem A ad B, triplicatam esse rationis K ad L : & B ad C, triplicatam ipsius L ad M : item C ad D, triplicatam ipsius M ad N : & sic de ceteries, nam ratio A ad B continuatur, estque illa triplicata rationis K ad L, quam per reliquos terminos continuatur.

L2.§1. *PROPOSITION 30.*

Two series of continued proportionals are given with different ratios: A, B, C, D, etc; K, L, M, N, etc. that also however have the series A, K, L, B, etc in continued proportion.

I say that all the terms A, K, L, B, N, O : C, Q, R, D, and thus henceforth, (with the terms K L and also the terms M, P, S disregarded in the third series) are in a continued ratio.

Demonstration.

Since A, K, L, B are placed in a continued ratio, thus K is to L, as L is to B; but also from hypothesis as K to L, thus L ad M. Hence L to B and L to M are both equal to the same ratio K to L; and therefore [a] B and M are equal, and likewise the ratios [b] B to C and M to C are equal. As A, K, L, B are four continued proportionals, the ratio A to B, (that is by hypothesis equal to the ratio B to C, that has been shown to be equal to the ratio M to C), is the triplicate of the ratio A to K: but the ratio A to K, from hypothesis is equal to the ratio K to L, that is the ratio M to N : hence the ratio M (or B) to C is the triplicate of the ratio M to N. But the ratio M to P is the triplicate of the ratio M to N, (for M, N, O, and P are in a continued ratio): therefore as M is to C, thus M is to P: and thus C and P are equal. Whereby B can be set up equal to and in place of M in the second series, and C is set in place of P. The terms A, K, L, B, N, O, C are hence continued proportionals. Thus in the same way we can show that the series can be continued by the terms Q, R, D, etc., indefinitely. Q.e.d.

a. 9 Quinti; b. 7 Eius.

Corollary.

Hence it follows that the ratio A to B is the triplicate of the ratio K to L : and B to C is the triplicate of L to M itself : likewise C to D is the triplicate of M to N : and so on for the rest, for the ratio A to B on being continued, is the triplicate of the ratio K to L for the remainder of the terms in the progression.

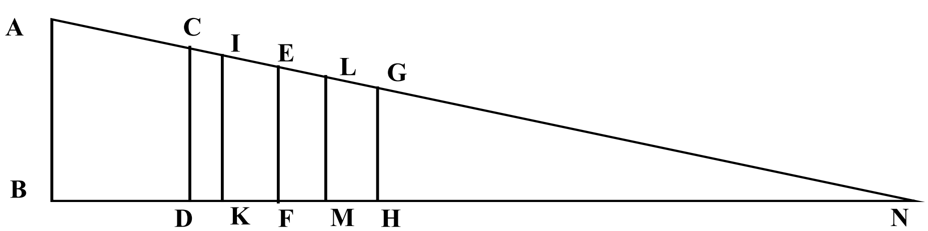
[For K and L are the cubic mean proportonals of A/B, etc. Thus, in the second series, B, C, D, ... can replace the terms which are multiles of three, i.e. M, P, S, ...]

PROPOSITIO XXXI.

In triangulo quovis ABN quatuor ponantur parallelae AB, CD, EF, GH in continua analogia : ac inter AB, GH, media sit IK, inter EF vero & GH, media sit LM.

Dico rationem AB ad IK triplicatam esse rationis LM ad GH.

Demonstratio.



Prop.31. Fig. 1.

Ratio AB ad GH, duplicatum est ex hypothesi, rationis AB ad IK : & ratio EF ad GH, duplicatam rationis LM ad GH; ergo cum iterum, ex hypothesi ratio AB ad GH triplicata sit rationis EF ad GH. erit ratio AB ad

[75]

IK,quae dimidiata est rationis AB ad GH; triplicata rationis LM ad GH; quae dimidiata est rationis EF ad GH. Quod erat demonstrandum.

L2.§1. *PROPOSITION 31.*

In some triangle ABN four parallel lines AB, CD, EF, GH are set in a continued ratio. The mean between AB and GH is IK, and the mean between EF and GH is LM.

I say that the ratio AB to IK is the triplicate of the ratio LM to GH.

Demonstration.

The ratio AB to GH by hypothesis is the duplicate [square] ratio AB to IK : and the ratio EF to GH is the duplicate of the ratio LM to GH. Hence in the same way from hypothesis, the ratio AB to GH is the triplicate [cube] of the ratio EF to GH, and the ratio AB to IK, which is square root of the ratio AB to GH, is the triplicate of the ratio LM to GH, which is square root of the ratio EF to GH. Q.e.d.

PROPOSITIO XXXII.

Quantitas ex quotvis continuae proportionalibus composita, ad aliam ex pari numero terminolum eiusdem seriei productae constatam, multiplicem rationem habet proportionalis primae ad secundam, ex quot terminis alterutra quantitatum componitur.

Demonstratio.

Prop.32. Fig. 1.

Ponatur series rationis alicuius, constituere quantitatem AB, & series ulterius producta conficiat quantitatem BC ; hoc tamen pacto ut utraque pari numero constet terminorum continue proportionalium eiusdem seriei productae; verbi gratia si numeret AB quantitas terminos octo continue proportionales, & BC totidem eiusdem seriei producae; dico AB ad BC, octuplicatam habere rationem uniformiter eius qua AD ad DE. Cum enim tota serieis rationis AD ad DE, pergat uniformiter & continue ex supposito usque ad C, & sint tot termini eiusdem seriei in AB, quot suntin BC, exempli causa in singulis octo; habebit ergo prima AD ad BF, octuplicatam rationem eius, quam habet AD a ad DE : similiter ratio DE ad FG, octuplicata est rationis DE ad EH, id est rationis AD ad DE : quare cum rationes AD ad BF, & DE ad FG, octuplicatae sint eiusdem rationis AD ad DE, erit ut AD ad BF, sic DE ad FG : eodem modo erit ut DE ad FG, sic EH ad GI, & sic de ceteris. Quare omnes [b] antecedentes, id est octo termini qui constituunt AB, se habent ad omnes consequentes, hoc est ad octo terminos qui constituunt BC, ut una antecedens AD, ad unam consequentem BF. Quare cum AD ad BF, rationem habeat octuplicatam rationis AD ad DE. habebit quoque AB ad BC octuplicatam eius, quam habet AD ad DE. Quod erat demonstrandum.

a Defin. 5 Sexti ; b 11 Quinti.

L2.§1. *PROPOSITION 32.*

A quantity [of terms in continued proportion] is composed from some continued proportionals, and is to have certain terms in common with that series, for which the quantity is a multiple of the ratio of the first to the second proportional, from which another of the terms of the new series is composed.

Demonstration.

A series is formed from some ratio, initially to give the AB, and the series is produced further to give the term BC ; this term corresponds to another term of the same series produced in continued proportion. For example, if eight terms in continued proportion are to be enumerated in the initial length AB, and in BC an equivalent number of terms are enumerated for the whole series produced ; I say that AB to BC is in the eight-fold ratio of AD to DE uniformly set our. For indeed the whole of the series of ratios AD to DE appear uniformly and continue from supposition as far as C, and there is the same number of terms of the series in AB, as there are in BC, which is eight in each for the example. Hence, the first ratio AD to BF is eight-fold the ratio of AD [a] to DE : and similarly the ratio DE to FG, is the eight-fold of the ratio DE to EH, that is also the ratio of AD to DE. Whereby as the ratios AD ad BF, and DE ad FG, are the eight-fold of the same ratios AD to DE, as AD to BF, thus DE is to FG : in the same manner, DE is to FG thus as EH is to GI, and so on for the rest. Whereby all the eight terms in the preceeding interval that constitute [b] AB, are held with all the following eight terms that constitute the interval BC, in order that one preceeding term AD to a following term BF has the same ratio. Whereby AD to BF has the eight-fold ratio of AD to DE. Also, AB to BC is the eight-fold of that which AD has to DE. Q.e.d.

a Defin. 5 Sexti ; b 11 Quinti.

[This result can easily be found using algebra: e.g. Let AD = a, DE = ar, EG = ar^2, etc.; then AB = $a(1 + r + r^2 .... + r^7)$; BC = $a(r^8 + ... + r^{15})$; giving BC/AB = r^8, etc. Gregorius's geometric proof seems a little intuitive.]

PROPOSITIO XXXIII.

Inter tres continue proportionales AD, BD, CD, mediae sint GD, ED : rursum inter AD, GD, media sit FD, & inter BD, ED, media sit HD: & hoc semper fiat:

Dico esse ut AB ad BC, sic AG ad BE, & AF ad BH, &c.

Prop.33. Fig. 1.

Demonstratio.

Cum inter tres continuas AD, BD, CD, mediae sint GD, ED; erunt AD, GD, BD, ED, CD, (ut patet ex elementis) omnes continuae proportiones: proindeque etiam [c] AG, GB, BE, EC erunt continuae proportionales: ergo AG ad GB, ut BE ad EC : & componendo ac per conversionem rationis AB ad AG, ut BC ad BF. Igitur alternando AB ad BC ut AG ad BE: Deinde quoniam AD, GD, BD, ED, sunt continuae, AD est ad GD, ut BD ad ED : Quare si inter AD, GD mediae sit FD, & inter BD, ED, media HD, patet ex elementis esse AD ad FD, ut BD ad FD, atqui cum AD, FD, GD, sint continuae, AF est ad FG, ut AD ad FD. per 1. huius. Et cum BD, HD, ED sint continuae etiam BH est ad HE, ut BD ad HD, hoc est (sicut iam ostendi) ut AD ad FD, ergo AF est ad FG, ut BH ad HE. Quare componendo ac per conversionem

[76]

rationis ut AG est ad AF, sic BE ad BH: & permutando ut AG ad BE, hoc est (sicut ostendi) ut AB ad BC, sic AF ad BH. Constat ergo propositionis veritas.

L2.§1. *PROPOSITION 33.*

Between three continued proportionals AD, BD, and CD, the means are GD and ED. Again, between AD and GD, the mean is FD, and between BD and ED the mean is HD; and this shall always be the case.

I say that as AB is to BC, thus AG is to BE, and AF is to BH, etc.

Demonstration.

Since between the three continued proportions AD, BD, CD, the means are GD, ED, then all the terms AD, GD, BD, ED, CD, (as is apparent from elementary considerations) are continued proportions: and hence also [c] AG, GB, BE, EC are continued proportionals: thus AG is to GB as BE is to EC : and on addition and conversion of the ratio, AB is to AG as BC is to BF. Therefore on re-arranging, AB is to BC as AG is to BE. Hence, since AD, GD, BD, ED, are in a continued ratio, AD is to GD, as BD is to ED : Whereby if the mean between AD and GD is FD, and between BD and ED, the mean is HD, then it is apparent from elementary considerations that AD is to FD, as BD is to FD, because as AD, FD, GD, are in a continued ratio, AF is to FG, as AD is to FD. by Prop. 1. of this book. And since BD, HD, ED are in continued proportion, also BH is to HE, as BD is to HD, that is (as has been shown) as AD is to FD, thus AF is to FG, as BH is to HE. Whereby by adding and by the converse of the ratio, as AG is to AF, thus BE is to BH: and on permuting, as AG is to BE, that is (thus as shown) as AB to BC, thus AF to BH. Thus the truth of the proposition is agreed upon.

PROPOSITIO XXXIV.

Sint in continua analogia AB, CB, DB, & inter AB, CB media sit EB; inter CB vero & DB sit media FB; deinde inter EB, CB, ac CB, FB mediae sint GB, HB : Denique inter GB, CB, & CB, HB sint mediae IB, KB, & sic deinceps.

Dico rationem AC ad CD, duplicatam esse rationis EC ad CF; quadruplicatam autem rationis GC ad CH, & octuplicatam rationis IC ad CK : atque ita in infinitum.

Prop.34. Fig. 1.

Demonstratio.

Cum AB, CB, DB sint continuae proportionales, erit AC ad CD [a] ratio eadem cum ratione AB ad CB: item quia continuae sunt AB, EB, CB, erit ratio AE ad EC eadem cum ratione AB ad EB : & quia inter tres continuas AB, CB, DB mediae sunt EB, FB, patet ex elementis AB, EB, CB, FB, DB omne esse continue proportionales: quare ratio AB ad CB, id est ut iam ostendi, ratio AC ad CD, duplicata est rationis AB ad EB; id est ut ostendi, rationis AE ad EC, quia autem continuae proportionales sunt AB, EB, CB, FB, etiam erunt AE, EC, [b] CF continuae; ergo ut AE ad EC, sic EC ad CF : ergo ratio AC ad CD, etiam duplicata est rationis FC ad CF; quod erat primum. Eodem discurrendi modo demonstrabimus, rationem EC ad CF, duplicatam esse rationis GC ad CH : adeoque rationem AC ad CD, quadruplicatam esse rationis GC ad CH; denique ostendemus etiam simili ratiocinatione, rationem GC ad CH, duplicatam esse rationis IC ad CK. Unde manifestum est rationem AC ad CD, eiusdem octuplicatam esse. Quae erant demonstranda.

a ibid; b 1. Huius.

L2.§1. *PROPOSITION 34.*

Let AB, CB, and DB be in a continued ratio, and let the mean between AB and CB be EB; and likewise between CB and DB the mean is FB; between EB, CB, and CB, FB the means are GB and HB : Further, between GB, CB, and CB, HB the means are IB, KB, and so on.

I say that the ratio AC to CD is the duplicate [square] of the ratio EC to CF; the four-fold [fourth power] of the ratio GC to CH, and the eight-fold [eighth power] of the ratio IC to CK : and so on indefinitely.

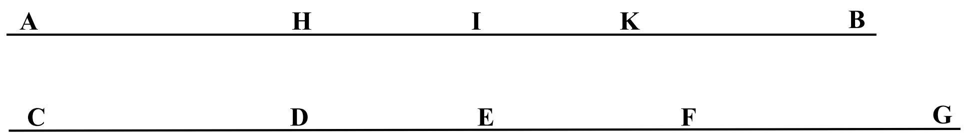
Demonstration.

Since AB, CB, and DB are continued proportionals, the ratio AC to CD [a] is the same as the ratio AB to CB: furthermore since AB, EB, and CB are continued proportionals, the ratio AE to EC is the same as the ratio AB to EB : and since between the three continued proportionals AB, CB, and DB the means are EB and FB, it is apparent from elementary considerations that AB, EB, CB, FB, DB are all continued proportionales: whereby the ratio AB to CB, that is as thus shown, the ratio AC to CD, is the duplicate of the ratio AB ad EB; or as shown, of the ratio AE to EC; but as AB, EB, CB, and FB are continued proportionals, AE, EC, and [b] CF are also continued proportionals; hence as AE is to EC, thus EC is to CF : hence the ratio AC to CD is indeed the duplicate of the ratio FC ad CF; which establishes the first part of the proposition. We can show by the same kind of discussion that the ratio EC to CF is the duplicate of the ratio GC to CH: and thus the ratio AC to CD is the four-fold of the ratio GC to CH; and then we can also

show by similar reasoning that the ratio GC to CH is the duplicate of the ratio IC to CK. From which it can be showm that the ratio AC to CD is the eight-fold ratio of the same. Q.e.d.
a ibid; b 1. Huius.

PROPOSITIO XXXV.

Datam lineam secare in quotvis continue proportionales, secundem datam rationem.



Prop.35. Fig. 1.

Constructio & Demonstratio.

Data sit linea AB dividenda sicut postulat progressio, in quatuor continuas, secundum rationem datam CD ad DE. Continuatur toties ratio CD ad DE, quot continuas numero desiderat in linea data AB, nempe CD, DE, EF, FG : tum divide lineam c AB, sicut divisa est linea CG in punctis H, I, K. Dico AB, esse divisam prout exigit propositio : patet demonstratio ex constructio.
c 10. Sexti.

L2.§1. *PROPOSITION 35.*

To cut a given line in some number of continued proportionals, following a given ratio.

Construction & Demonstration.

The progression demands that the given line AB is to be divided in four continued ratios following the given ratio CD to DE. The ratio CD to DE is to be continued in the given line AB as many times as the desired number, namely CD, DE, EF, FG : then divide the line [c] AB as the line CG has been divided, in the points H, I, and K. I say that AB has been divided as required by the proposition, as is apparent from the construction.
c 10. Sexti.

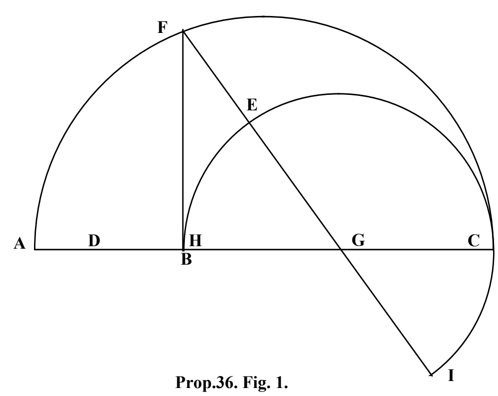
PROPOSITIO XXXVI.

Datarum linearum alteram ita secare, ut partes lineae sectae, cum insecta, sint in continua analogia.

[77]

Constructio & Demonstratio.

Datae sint duae lineae AB & BC, quarum alteram scilicet AB, oporteat dividere in D puncto, ut AD, DB, BC sint in continua analogia. pro constructione , super BC diametro describatur circulus, item super linea AC, composita ex duabus datis AB & BC, qui sint BEC & AFC; & ex puncto B erigatur perpendicularis BF, ad rectam AC; ducaturque ex F puncto, linea FI. per centrum minoris circuli G, fiat denique rectae GF, aequalis GD. Dico factum esse quod imperatum fuit : quoniam quadrato FB iam aequatur rectangulum ABC, quam [a] EFI, hoc est BDC, erunt BDC, ABC, rectangula aequalia inter se : sed ABC rectangulo aequantur

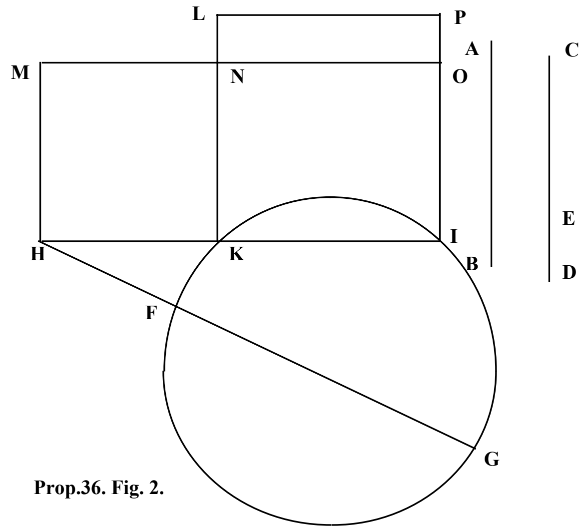


Prop.36. Fig. 1.

rectangula ADBC, DBC; & BDC rectangulo aequatur rectanglum DBC, una cum quadrato DB; dempto igitur communi rectangulo DBC, manet DB quadrato, aequale rectangulum ADBC. igitur AD,DB, BC sunt continuae.

Aliter.

Datae sint AB, CD, quarum una CD, ita secanda sit in E, ut AB, CE, ED sint in analogia continua.



Prop.36. Fig. 2.

Constructio & Demonstratio.

Assumpta FG linea maiori quam AB, fiat super FG tanquam diametro circulus FIG: deinde (quod ex elementis facile perficitur) rectangulum GHF aequale fiat rectangulo ABCD; & ex puncto H ita ducatur HKI, ut KI aequalis sit AB (quod ab aliis factum, & nos libro nostro de circulis alia atque alia methodo praestabimus) denique ex K erecta normali KL, aequali ipsi CD, praescio datur ex KL (ostendam enim esse maiorem) linea KN, aequalis ipsi HK. Dico factum quod petebatur.

Nam rectangulum [b] IHK aequatur rectangulo GHF, id est ex constructione rectangulo ABCD, id est rursum ex constructione IKL, ergo ut IH ad KI, ita est reciproce Kl ad KH : atqui IH maior est quam IK, ergo & LK, quam KH maior erit, quod assumptum fuerat in constructione: perficiantur iam rectangula IHK, IKL, auctis per N & L parallelis ad HI, & normalibus HM, IOP. Quoniam igitur rectangula IM, IL, aequalia sunt, ablato communi KO, reliqua OL, KM aequalia erunt. Quare cum KM (ut ex condtructione patet) sit quadratum, erunt NO, KN, LN tres continuae. Atqui NO est KI, hoc est AB: & KL est CD. Factum igitur est quod petebatur. *a 35 Tertii; b 36 Sexti.*

[78]

Scholium.

Hanc eandem propositionem nonnullis alii & proposuerant. & feliciter solverunt. Uti ? A. P.Clausius Magister meam (amicus per plures annos familiaris & domesticus auditor fui) Peletarius?, & Guido Ubaidius?, sed exercitis causa, etiam meo eam Matre ? expedire volui, ut alienis plumis me exornare vello videar : nosquam cuius propositionem statui meis lucubrationibus interserere, quam diverse discursis demonstratione meam non fecero; vel authoris nomen in publicum non protulere. Rogo benigne Lector huius res ,memor esse velis, dum in posterum simile occurret.

L2.§1. *PROPOSITION 36.*

To cut one of two given lines into two parts in order thus that the cut parts of the line together with the other section are in a continued ratio.

Construction & Demonstration.

Let AB and BC be two given lines, it is required to divide one of these, such as AB, by the point D in order that AD, DB, and BC are in a continued ratio. By way of construction, a circle BEC is described with diameter BC, and likewise a circle AFC is described with diameter AC, where AC is the sum of AB and BC. From the point B a perpendicular BF is erected to the line AC; and from the point F the line FI is drawn through the centre of the smaller circle G, and thus the lines GF and GD are made equal. I say that the required task has been performed : For the square FB thus is equal to the rectangle AB.BC, [a] since the rectangles EF.FI, BD.DC [as EF = DB and FI = DC] and AB.BC are equal to each other. But the sum of the rectangles AD.BC and DB.BC is equal to the rectangle AB.BC, and the rectangle BD.DC is equal to the sum of the rectangle DB.BC and the square DB; therefore by removing the common rectangle DB.BC, the rectangle AD.BC remains equal to the square DB. Therefore AD, DB, and BC are in continued proportion.

An alternative method.

Given the lines AB and CD, of which one CD is to be cut thus in E in order that AB, CE, and ED are in continued proportion.

Construction & Demonstration.

It is assumed that the line FG is longer than the line AB, and the circle FIG is constructed with FG as diameter : hence (and which can easily be established from first principles) the rectangle GH.HF is made equal to the rectangle AB.CD; and thus from the point H the line HKI is drawn, in order that KI is equal to AB (which has been done by the other method, and from our book on the circle and the other method we shall succeed one way or another) and hence from K a normal is erected KL, equal to CD, and from what is already known, the line KN from KL (which I will show to be the greater) is equal to HK. I say that what was demanded has been done.

For the rectangle [b] IH.HK is equal to the rectangle GHF, which is from the construction equal to the rectangle AB.CD, that is again from the construction equal to IK.KL, thus as IH is to KI, thus reciprocally as Kl to KH : but IH is greater than IK, and therefore LK is greater than KH, which was assumed in the construction: thus the rectangles IH.HK and IK.KL can be established, by raising parallel lines to HI through N and L, and with the normals HM and IOP. Hence, as the rectangles IM and IL are equal, with the common rectangle KO taken away, the remainders OL and KM are equal. Whereby as KM is a square(as is apparent from the condtruction) , the lines NO, KN, and LN are three continued proportionals. But NO is equal to KI, or AB: and KL is equal to CD. Hence that which was demanded has been done.

a 35 Tertii; b 36 Sexti.

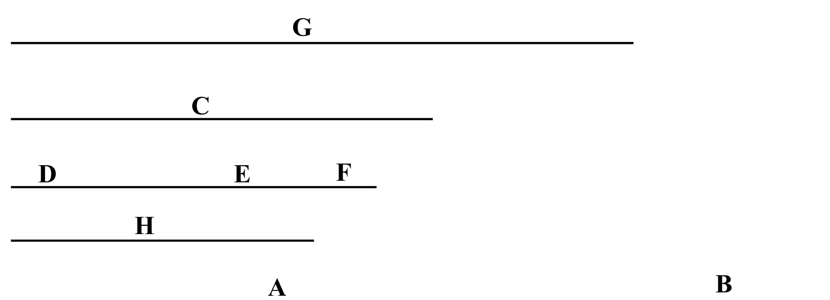
Scholium.

Thus the same proposition and some others have been proposed and happily solved by me. This problem was posed to me by my teacher R. P.Clavius (with whom I was a close family friend and listener for many years), as an exercise, so that I would have a more rounded education : indeed I have not stated the propositions composed from my lucubrations elsewhere, as I do not wish to make my demonstrations from different discourses; or my name would not be brought forward in public. I ask kind reader, that you may remember this, as later the same kind of thing will occur. [Translation paraphrased in parts due to poor quality of text on microfiche

PROPOSITIO XXXVII.

Datarum duarum rectarum alteram ita secate, ut rectangulum sub insecta, & parte lineae sectae, ad residum lineae quadratum, datam habeat rationem.

Constructio & Demonstratio.



Prop.37. Fig. 1.

Data sit ratio A ad B, deinde datae lineae sint C & DF, postulat propositio, fecari rectam DF in E, ut rectangulum sub C & EF, ad quadratum DE, eam rationem habeat, quam data B ad datam A. Fiat C ad G, ut B ad A, & per praecedentem secetur DF in puncto E, ut sint tres continuae proportionales lineae, G, DE, EF. Dico factum quod petebatur. fiat enim ut B ad A, sic H ad DE: erit ergo, ut G ad C, sic DE ad H: & permutando ut G ad DE, ita C ad H ; sed ut G ad DE, ita DE ad EF ex constructione; ergo ut DE ad EF, ita C ad H : rectangulum igitur super lineis C & EF, aequale est [a] rectangulum sub lineis DE & H constructum, porro rectangulum sub lineis DE & H consturctum, ad quadratum DE, eam habet proportionem quam lineae ipsae, scilicet quam habet [b] H ad DE: ergo & rectangulum C & EF, est ad quadratum DE, ut H ad DE: id est ut B ad A : igitur datarum rectarum alteram, &c. Quod fuit praestandum.

a 26?. Sexti. ; b 2 Sexti.

L2.§1. *PROPOSITION 37.*

To divide one of two given lines thus, in order that the rectangle formed from the one line and with a part of the line divided by the intersection, may have a given ratio to the square of the rest of the line divided.

Construction & Demonstration.

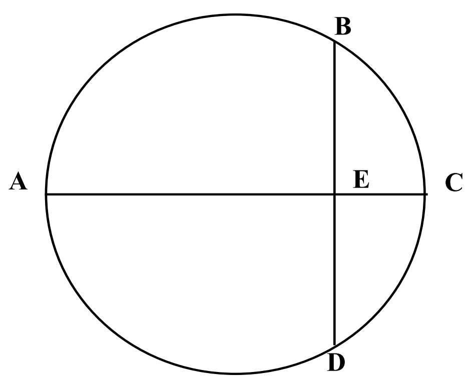
The ratio A to B is given, as well as the lines C and DF, and according to the proposition, the line DF is to be diveded in E, in order that the ratio of the rectangle under C and EF to the square DE has that given ratio A to B. The ratio C to G is made as B to A, and by the preceding proposition, DF is divided by the point E in order that the three lines C, DE, and EF are in continued proportion. I say that what is required has been done. For the ratio H to DE is made as B is to A,: hence as G is to C, thus DE is to H. On re-arranging, we have as G to DE, thus as C to H; but as G is to DE is thus as DE to EF by construction; hence as DE is to EF, thus C is to H. Hence the rectangle on the lines C and EF is equal to [a] the rectangle formed under the lines DE and H; again the ratio of the rectangle constructed under the lines DE and H to the square DE, has the proportion [b] as lines themselves that H has to DE. Thus the rectangle formed from C and EF, is to the square DE as H is to DE, or B to A. Therefore the proposition follows as required.

a 26?. Sexti. ; b 2 Sexti.

PROPOSITIO XXXVIII.

Data media trium in continua ratione existentium, & aggregato trium linearum primam & tertiam constituentium, exhibere primam & ultimam.

Constructio & Demonstratio.



Prop.38. Fig. 1.

Super aggregato tanquam diametro, circulus describatur ABC; in quo recta accommodetur BD, quae aequalis sit duplae datae mediae: quod fieri posse eo fiat ex datis & ex constructione : dein agatur per centrum linea AC, secans orthogonaliter in E, lineam BD. Dico factum esse quod petitur, est enim AC linea aequalis aggregato, & BE ([c] dimidia ipsius BD) datae mediae aequalis : quare cum AC orthogona sit ad BE, erit AEC [d] rectangulum aequale quadrato BE: igitur data media trium continue proportonalium, illarumque aggregato; exhibuimus primam & tertiam. Quod erat faciendum.

c 3. Tertii. ; d 33 Tertii.

[79]

Lemma.

Si fuerint tres BA, DA, CA in continua analogia, rectangulum super maxima AB, & excessu CD secundae supra tertiam, non erit rectus quadrato dimediae ipsius AB.

Demonstratio.

Prop.38. Fig. 2.

Quoniam enim BA, DA, CA sunt continuae proportionales, erit ut BA [a] ad DA, sic BD ad DC. Quare rectangula [b] BACD & BDA aequantur. Sed rectangulum BDA, non est maius quadrato dimidiae AB [c] ut patet. ergo neque rectangulum BACD maius erit quadrato dimidiae AB. Quod erat demonstrandum.
a 7 Huius; b 16 Sexti; c 5 Secundi.

L2.§1. *PROPOSITION 38.*

Given the mean arising from three lines in a continued ratio, and given the sum of first and third of the three lines put in place, to show the first and the last of the lines.

Construction & Demonstration.

The circle ABC is described on the diameter taken as the sum of the first and last terms, in which the line BD is established which is equal to twice the given mean: which can be done from the given sum by construction. The line AC is drawn passing through the centre, and cutting the line BD at right angles in E. I say that what was required has been accomplished. For indeed the line AC is equal to the sum of the terms, and BE (equal to half of BD [c]) is equal to the given mean, whereby as AC is perpendicular to BE, the rectangle AE.EC [d] is equal to the square BE: therefore given the mean of three lengths in continued proportion, and the sum of the first and last of these, we have shown the first and the third. *c 3. Tertii. ; d 33 Tertii..*

[If $a, b,$ and c are the lengths of three lines in continued proportion, with the mean satisfying $b^2 = ac$. The sum of the first and third proportionals is $a + c$, which is set equal to the diameter AC; while BE is set equal to the mean b. From geometry, $BE^2 = AE.EC$, or $b^2 = a.c$.]

Lemma.

If BA, DA, and CA are three lines in continued proportion, the rectangle with the greatest area on AB, and in excess of the second CD on the third, is not larger than half the square of AB itself.

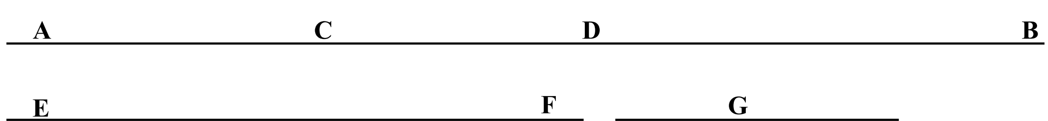
Demonstration.

For since BA, DA and CA are continued proportionals, BA [a] will be to DA, thus as BD to DC. Whereby the rectangles [b] BA.CD & BD.DA are equal. But the rectangle BD.DA is not greater than the square of the half of AB [c] as is apparent from the proposition. Therefore also the rectangle BA.CD cannot be greater than the square of half AB. Q.e.d. *a 7 Huius; b 16 Sexti; c 5 Secundi.*

PROPOSITIO XXXIX.

Data maxima trium continuarum, & excessu quo media superat minimam, exhibere mediam & minimam.

Constructio & Demonstratio.



Prop.39. Fig. 1.

Data sit AB linea maxima trium proportionalium, & G equalia excessui quo media superat minimam : oporteat igitur invenire mediam & minimam. fiat rectangulo ABG aequale quadratum EF: quoniam ergo per lemma praecedentes quadratum EF, id est rectangulum ABG, non maius est quadrato dimidiae AB, patet ita [d] fecari posse AB, ut rectangulum sub partibus, aequale sit quadrato EF. Itaque dividatur recta AB, in puncto D, ut rectangulum ADB aequale sit quadrato EF. Dico punctum D esse quod problema soluit : fiat enim rectae G, aequalis linea DC : erit itaque rectangulum ABCD aequale rectangulo ADB: cum utrumque aequale sit quadrato EF; ergo ut AB ad AD [e] , ita est BD ad DC: si iam a rectangulo ABCD, auferis rectangulum BDC, remanebit rectangulum ADC; si vero a rectangulo ADB, auferas item BDC, reliquum erit rectangulum ACDB: atqui tota ABCD, ADB sunt aequalia , itaque oblato communi, erunt & reliqua rectangula ADC, ACDB aequalia inter se; igitur ut BD ad DC, id est [f] (sicut iam ostendi) ut BA ad DA, sic DA ad CA. Sunt igitur tres in continua analogia AB, AD, AC; & CD aequalis G, est excessus, quo media AD, excedit minimum AC. Factum igitur est quod requirabatur.

d 5 Secundi; e 16. Sexti ; f 14 Sexti.

L2.§1. PROPOSITION 39.

From the given maximum of three lines in continued proportion, and the excess of the mean over the mininum, to find the mean and the minimum.

Construction & Demonstration.

The maximum AB is given of the three lines in proportion, and G is equal to the excess of the mean over the mininum: it is required to find the mean and the minimum. The rectangle AB.G is set equal to the square EF: therefore, according to the previous proposition, the square EF, or the rectangle AB.G, is not greater than the square of half of AB, it is thus apparent that AB can be divided in order that the rectangle under the sections is equal to the square EF. Thus the line AB is divided in some point D in order that the rectangle AD.DB is equal to the square EF. I say that the point D is the solution to the problem. For the line G is made equal to the line DC: and thus the rectangle AB.CD is equal to the rectangle AD.DB, as each is equal to the square EF; therefore as AB is to AD[e], so BD is to DC. If now from rectangle AB.CD you take away the rectangle BD.DC then the rectangle AD.DC remains; if indeed from rectangle AD.DB likewise BD.DC is taken away, then the remainder is the rectangle AC.DB: but the whole amounts AB.CD and AD.DB are equal, and hence on taking away the common amount, the remaining rectangles AD.DC and AC.DB are equal to each other. Therefore as BD is to DC, or [f] as BA is to DA (as now shown), thus DA to CA. The three lines AB, AD, and AC are hence in continued proportion, and CD equal to G, is the excess by which the mean AD exceeds the minimum AC. Hence what was required has been done.

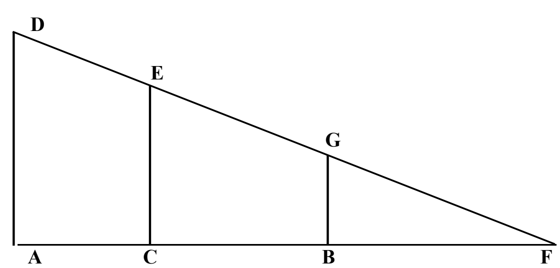
d 5 Secundi; e 16. Sexti ; f 14 Sexti.

[Essentially the argument is the converse of AC/AD = AD/AB giving CD/AD = DC/AB and AD.DB = EF^2 = AB.CD.]

PROPOSITIO XL.

Datis duobus excessibus, trium magnitudinum in continua analogia existentium, exhibere tres continuas.

Constructio & Demonstratio.



Prop.40. Fig. 1.

Dati sint excessui AC, CB, qui ponantur in directum, & erigantur AD, CE parallelae, quae inter se eandem rationem servent quam AC ad CB; & ducta per puncta D & E recta DE, conveniat cum AB producta in puncto quodam F. Dico factum quod postulatur :

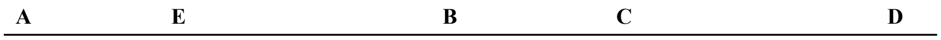
[80]

nam AD ad CE, eandem habeat rationem, quam linea AF ad CF. Sed quam rationem habet DA ad CE, eandem per constructione

habet AC ad CB; igitur quam rationem habet tota AF ad totam CF, eandem habet AC ablata ad CB ablatam. Ergo ut AF tota ad CF totam ac quoque [a] CF reliqua ad reliquam BF. Quare rectae AB adiuncta est linea BF, quae factiat AF, CF, BF in continua proportione; quod petebatur.
a 19 Quinii.

Aliter.

Cum huic constructio solis lineis conveniae subiungamus aliam, quae in omni genere quantitatis locum habeat:



Prop. 40. Fig.2.

Dati igitur sint excessus AB : BC : fiat AE differentia datorum excessuum, ipsisque AE, AB inveniatur tertia proportionalis continua AD : Dico AD absuluere problemata. Cum enim ex constructione AE sit AB, ut AB ad AD, erit dividenfo AE ad EB, ut AB ad BD; & componendo AB erit ad BE, hoc est BC ut AD ad BD; & permutando AB ad AD, ut BC ad BD; & dividendo AB ad BD ut BC ad CD; componendo igitur AD, BD, CD sunt continuae. Fecimus ergo quod petebatur. *a 19 Quinti.*

L2.§1. PROPOSITION 40.

With the two differences given of three magnitudes present in continued proportion, to show the three continued proportions.

Construction & Demonstration.

For the differences AC and CB are given, which are placed along a line, and the parallel lines AD and CE are erected which maintain the same ratio AC to CB between themselves; and the line DE is drawn through the points D and B and meets the line AB produced in some point F. I say that what was required has been done: for AD to CE has the same ratio as AF to CF. But DA to CE is in the same ratio as AC to CB by construction; therefore as the whole has the ratio AF to CF, AC taken to CB taken has the same ratio. Hence as the total AF to the total CF also the ratio of the remainder CF to the remainder BF. Whereby the line BF is added on to the line AB, which makes AF, CF, and BF in continued proportion, as sought.
a 19 Quinii.
[From CE/AD = CF/AF and CE/AD = CB/AC we have AF/CF = AC/CB or AF/AC = CF/CB, giving CF/AC = BF/CB or CB/AC = BF/CF = AF/CF as required.]

In another way.

As to this construction involving a single line, we can add another line for which there is a place for all kinds of quantities.

The ratio of the differences AB : BC is therefore given, and AE is the difference of the given differences. From AE and AB themselves the third of the continued proportionals AD can be found. I say that AD solves the problem. As indeed from construction, AE is to AB as AB is to AD; by subtraction, AE is to EB , as AB is to BD ; and by addition, AB is to BE, or BC , as AD is to BD; and on interchanging, AB is to AD as BC is to BD; and on subtraction, AB is to BD as BC is to CD; on addition, AD, BD, and CD are therefore in continued proportion. Thus, we have accomplished what was sought.

[For, if a, b, and c are three numbers in continued proportion with $a > b > c > 0$, then $a/b = b/c$, and $(a - b)/b = (b - c)/c$ or $d_1/b = d_2/c$ i.e. $d_1/d_2 = b/c$; then again, $(d_1 - d_2)/d_2 = (b - c)/c = d_2/c = d_1/a$. In this case, AB = d_1, BC = d_2, and AE = $d_1 - d_2$: hence AE/BC = BC/AD, giving AD; or, if you wish, $d_2/d_1 = b/a$; from which AE/AB = AB/AD as in the *aliter*, again giving AD. It then follows that EB/AB = BD/AD giving BD, while AB/AD = EB/BD = BC/BD, as EB = AB - AE = AB - AB + BC; etc.]

PROPOSITIO XLI.

Prop. 41. Fig.1.

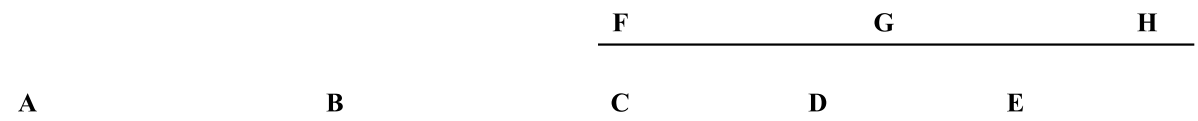
Datam magnitudinem AE semel sectam in C, proportione maioris inaequalitatis, ita in duobus aliis punctis B & D subdividere, ut quatuor partes AB, BC, CD, DE sint in continua proportiones; quae dimidiata sit rationis AC ad CE, ex prima divisione ortae.

Constructio & Demonstratio.

Per quadragesimam huius addatur EF, ut AF, CF, EF sint proportionales; tum inter AF, CF, media ponatur BF : inter CF quoque & EF, media DF. Dico factum quod petebatur. cum enim, inter tres continuas AF, CF, EF, mediae sint BF, DF, paret ex elementis omnes AF, BF, CF, DF, EF, esse continuae proportionales. Quare etiam AB, BC, CD, DE sunt b continuae, & quidem in ratione AF ad BF, quae ex constructione dimidiata est rationis AF ad CF, hoc est AC ad CE. Factum igitur est quod petabatur.

b 7 Huius; c Ibid.

Aliter.



Prop. 41. Fig.2.

Inter AC, CE inventam mediam FH ita divisae, ut FG sit ad GH, ut AC ad FH, seu FH ad CE : fiantque CB, CD ipsis FG, GH aequales. Dico factumquod petitur. Cum enim sit AC ad FH, ut FG ad GH, id est BC ad CD, erit quoque AC ad BD (quae ex constructione aequalis est FH) ut BC ad CD: & permutando AC ad BC ut BD ad CD: & dividendo AB ad BC, ut BC ad CD. Sunt igitur AB, BC, CD tres continuae proportionales in ratione BC ad CD. Similiter cum FH sit ad CE, ex constructione ut FG ad GH, erit quoque BD ad CE, ut BC ad CD, ac permutando convertendo BD ad CD, ut CB ad DB: Ideoque dividendo BC est ad CD, ut CD ad DB. Sunt igitur tres continuae BC, CD, DE in ratione BC ad CD : ac proinde omnes quatuor sunt continuae proportionales in ratione BC ad CD, id est FG ad GH, id est AC ad FH, quam ex

[81]

constructione dimidiata est rationis AC ad CE; fecimus ergo quod fuerat propositum.

Corollarium.

Ex hoc problemate licebit praxim desumere, non solum subdividendi duas in quatuor continuas, sed etiam in sex continuas; imo quotvis datas in duplo plures continuas, & quidem in ratione dimidiatis eius, in qua ipsae existunt.

L2.§1. *PROPOSITION 41.*

Given the magnitude AE cut once in C in a proportion greater than one, which is thus to be subdivided by two other points B and D, in order that the four parts AB, BC, CD, and DE are in continued proportion, in a ratio which is the root of the ratio AC to CE arising from the first subdivision.

Construction & Demonstration.

By the 40th proposition of this book, EF is added in order that AF, CF, and EF are proportionals; then the mean BF is placed between AF and CF, and the mean DF is placed between CF and EF. I say that what was required has been done. For indeed , between the three continued proportionals AF, CF, and EF, the means are BF and DF, and it is apparent from the fundamentals that all of AF, BF, CF, DF, and EF are continued proportions. Whereby also AB, BC, CD, and DE are continued proportionals [b] , and indeed these are in the ratio AF to BF, which by construction is the square root of the ratio AF to CF, or AC to CE. Thus what was required has been done.

b 7 Huius; c Ibid.

In Another Way.

The mean FH is found between AC and CE, thus as AC is to FH, so FH to CE : and thus by division, AC is to FH as FG is to GH and CB, CD are made equal to FG, GH themselves. I say that what was sought has been done. As indeed AC is to FH as FG to GH, or BC to CD; also AC to BD (which by construction is equal to FH) is as BC to CD : and by rearranging, AC is to BC as BD to CD. On subtracting, AB is to BC as BC is to CD. Therefore there are three proportionals AB, BC, CD in the ratio BC to CD. Similarly, as FH is to CE, by construction as FG is to GH, also BD is to CE as BC is to CD [For: BD/CE = BC/CD; BD/BC = CE/CD; CD/BC = DE/CD; BD/BC . BC/CD = CE/CD . CD/DE or BD/CD = CE/DE]; and on rearranging and multiplying, as BD is to CD, thus CE is to DE. Thus on subtracting, BC is to CD, as CD is to DE. There are hence three continued proportions in the ratio BC to CD : and thus all four are continued proportionals in the ratio BC to CD, or FG to GH, or AC to FH, which by construction is the square root of the ratio AC to CE; we have therefore done what was proposed.

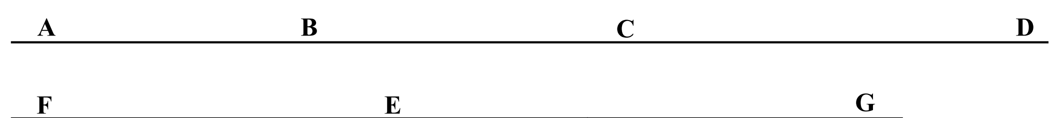
Corollory.

From this problem you can get some practise, not only in the subdivision by two into four continued proportions, but also into six; and indeed for whatever number into twice as many continued proportions, and they are in the ratio of the square root of that ratio in which these themselves are present.

PROPOSITIO XLII.

Sint AB, BC, CD in continua analogia; deinde secetur quaepiam linea FG in E, ut FE, ad EG, eadem habeat rationem, quam AB ad BC.

Dico rectangulum BCFG, aequale esse duobus ABEG & EFCD rectangulis.



Prop.42. Fig. 1.

Constructio & Demonstratio.

Cum enim sit ut AB ad BC, ita FE ad EG, rectangulum BCEF, [a] aequale est rectangulo ABEG. Similiter, quia ut BC ad CD, ita FE est ad EG, erit etiam rectangulum BCEG aequale rectangulo CDFE. Cum igitur BCGF, BCEG rectangula, id est [b] rectangulum BCFG, aequalia sunt duobus ABEG, CDFE rectangulis. Quod erat ostendendum. *a 16 Secundii; b 2 Secundii ?*

L2.§1. *PROPOSITION 42.*

AB, BC, and CD are in a continued ratio; then some line FG is cut in E, in order that FE to EG has the same ratio as AB to BC.

I say that the rectangle BC.FG is equal to the sum of the two rectangles ABEG and EFCD.

Constructio & Demonstratio.

Since indeed as AB is to BC as thus FE to EG, then the rectangle BC.EF, [a] is equal to the rectangle AB.EG. Similarly, since BC is to CD, thus as FE is to EG, also the rectangle BC.EG is equal to the rectangle CD.FE. Therefore as the sum of the rectangles BC.EF and BC.EG , that is the [b] rectangle BC.FG, is equal to the sum of the two rectangles AB.EG and CD.FE. Which it was required to show. *a 16 Secundii; b 2 Secundii ?*

PROPOSITIO XLIII.

Si fuerint quotvis continuae proportionales AB, CB, DB, EB, &c. Dico rectangula ABEF, CBDE, DBCD, EBAC esse inter se aequalia. Hoc est rectangula sub lineis seriei AB, CB, DB, &c. & sub residuis EF, DE, CD, AC esse inter se aequalia; modo retrograde coniungantur.

Prop.43. Fig. 1.

Demonstratio.

Quonium est ut AB ad [c] CB, sic DE ad EF, rectangulum ABEF aequatur [d] rectangulo CBDE. Similiter, quia ut CB ad DB, ita CD est ad DE, erit rectangulum CBDE aequale rectangulo DBCD. rursus quoniam DB est [e] ad EB ut AC ad CD, rectangulum DA/BCD? aequalia est ..... EB..AC: aequalia sunt igitur omnis inter se. Quod erat demonstrandum. *c 16 Huius; d 16 Sexti ; e 1 Huius.*

L2.§1. *PROPOSITION 43.*

AB, CB, DB, EB, etc. are some lines in continued proportion. I say that the rectangles AB.EF, CB.DE, DB.CD, and EB.AC are equal to each other. That is the rectangles under the lines of the series AB, CB, DB, &c. and under the differences EF, DE, CD, AC are equal to each other; joined together in a backward manner.

Constructio & Demonstratio.

Since indeed as AB is to [c] CB, thus DE ad EF, then the rectangle AB.EF is equal to the [d] rectangle CB.DE. Similarly, because CB is to DB, thus as CD is to DE, then the rectangle CB.DE is equal to the rectangle DB.CD. Again, since DB is to [e] EB as AC is to CD, then the rectangle DB.CD is equal to the restangle EB.AC: all the rectangles are therefore equal to each other. Q.e.d. *c 16 Huius; d 16 Sexti ; e 1 Huius.*

PROPOSITIO XLIV.

Iisdem positis:
Dico rectanguli EBAC, DBCD, CBDE esse inter se aequalia.

Demonstratio.

Prop.44. Fig. 1.

Cum enim ut CB ad DB, ita CD[f] sit ad DE ; igitur rectangulum CBDE aequale est[g] rectangulo DBCD. Sed rersum ut DB ad EB, sic AC ad[h] CD, quare etiam rectangulum DBCD, aequale erit rectangulo EBAC; ostensum autem fuit rectangulum CBDE, esse rectangulo DBCD aequale, quare & haec duo EBAC & CBDE, proindeque omnia tria inter se erunt aequalia. Quod erat demonstrandum. *f ibid; g 16 Sexti ; h 1 Huius.*

[82]

Pari ratione si ponatur ulterius produci series progressionis, ita ut AB, CB, DB, EB, FB, &c. ponatur continuae proportionales, demonstrari poterit quatuor rectangula FBAC, EBCD, DBDE, & CBEF inter sese aequalia esse.

L2.§1. PROPOSITION 44.

With the same points in place:

I say that the rectangles EB.AC, DB.CD, and CB.DE are equal to each other.

Demonstratio.

Since indeed as CB is to DB, so CD[f] is to DE ; and therefore the rectangle CB.DE is equal[g] to the rectangle DB.CD. But again, as DB is to EB, thus AC is to[h] CD, whereby also the rectangle DB.CD is equal to the rectangle EB.AC; but is has been shown that the rectangle CB.DE is equal to the rectangle DB.CD, and whereby these two EB.AC and CB.DE are equal, and hence all three are equal to each other. Q.e.d. *f ibid; g 16 Sexti ; h 1 Huius.*

If the series of the progression is produced further in the same ratio, thus in order that AB, CB, DB, EB, FB, &c. are continued proportionals, it can be show that the four rectangles FB.AC, EB.CD, DB.DE, & CB.EF are equal to each other.

PROPOSITIO XLV.

Prop.45. Fig. 1.

Sint AB, CB, DB, EB in continua analogia, &c.

Dico quinque rectangula inter se aequalia esse; quorum primum est illud, quod sub AB, BC tanquam una linea, & sub DE continetur : alterum, quod describitur a CB, & DB tanquam una, ac recta CD; tertium, quod a DB & EB tanquan una, & recta AC conficitur: quartum, quod ab EC, & CB; denique illud quod ab AD, DB describitur.

Demonstratio.

Rectangulum enim ABDE aequale est[a] rectangulo CBCD; & rectangulum CBDE aequale est[b] rectangulo DBCD : duo igitur rectangula ABDE, & CBDE, id est rectangulum sub ABCB tanquam una, & D... aequale sunt duobus CBCD & DBCD, id est contento sub CBDB tanquam una, & sub recta CD: eodem modo ostendam sub DBEB & AC contentum, aequari prioribus; de reliquis quoque idem simili discursu demonstrabitur. Constat ergo veritas propositionis. *a 43 Huius; b Ibid.*

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L2.§1. PROPOSITION 45.

The lines AB, CB, DB, EB are in continued proportion, etc.

I say that there are five rectangles equal to each other, the first of which is that which is contained under the sum of AB and BC regarded as one line, and DE; the second is described from the sum of CB and DB as one line, and the line CD; the third is made from the sum of DB and EB as one line, and the line AC; the fourth from EC and CB; and then that which is desctibed from AD and DB.

Demonstratio.

Indeed the rectangle AB.DE is equal to the [a] rectangle CB.CD; and the rectangle CB.DE is equal to the [b] rectangle DB.CD : therefore the two rectangles AB.DE, and CB.DE, that is the rectangle AB.CB as one line, and DE is equal to the two CB.CD and DB.CD, that is contained by CBDB as one line, and the line CD: in the same manner I can show that the rectangle contained by DB.EB and AC is equal to the previous rectangles; and concerning the remaining rectangles a similar argument can be shown. Hence the truth of the proposition can be agreed upon. *a 43 Huius; b Ibid.*

PROPOSITIO XLVI.

Prop.46. Fig. 1.

Sint continuae proportionales AB, CB, DB, EB, &c.

Dico rectangula ABCE, CBAD, & duo simul sumpta ACB, ACDB , denique trium aggregatum , scilicet quadrato CE, rectanguli ACE, & rectanguli CEB, esse inter se aequalia.

Demonstratio.

Quoniam est ut AC ad CD, ita CD ad DE, erit componendo & alternando, ut AD ad CE, sic CD ad DE: sed ut CD ad DE, ita est AB [c] ad CB : igitur ut AD ad CE, sic est AB ad CB, & rectangulum [d] ABCE aequalia erit rectangulo CBAD. Insuper quia est ut AB ad CB, ita AC ad CD, erit rectangulum ABCD, aequale rectangulo AC,CB. Quia vero ex aequo etiam est ut AC ad DE, ita AB ad DB, erit quoque rectangulum ACDB, aequale [e] sunt rectangulo ABCE: rectangulum igitur ABCE, aequale etiam est duobus ACB & AC DB. Deinde quia recta AB secta est in C & E, erit [f] ABCE aequale tribus CE in CA, & CE in CE ducto, (hoc est quadrato CE) & eidem CE in EB ducto: aequalia sunt igitur inter se. Quod fuerat demonstrandum. *c 1 Huius; d 16 Sexti; e 1 Secundi; f 1 Secundi.*

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L2.§1. *PROPOSITION 46.*

The lines AB, CB, DB, EB, &c. are continued proportionals

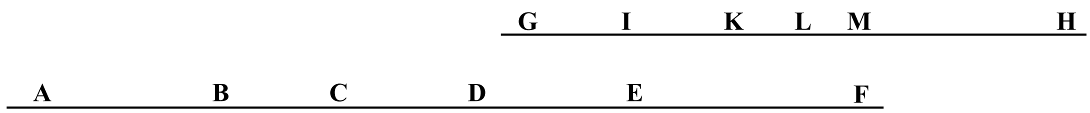
I say that the rectangles AB.CE, CB.AD, as well as the sum of the two rectangles AC.CB and AC.DB, and again the sum of the three rectangles AC.CE, AC.CE & CE.EB, are all equal to each other.

Demonstratio.

Since AC is to CD thus as CD is to DE then by adding and rearranging, as AD is to CE, thus as CD is to DE: but as CD is to DE, thus AB [c] is to CB : therefore as AD is to CE, thus AB is to CB, & the rectangle [d] AB.CE is equal to the rectangle CB.AD. In addition, since AB is to CB, thus as AC is to CD, then rectangle AB.CD is equal to rectangle AC.CB. Whereby also from equality, AC is to DE, thus as AB is to DB, and the rectangle AC.DB too is equal to the [e] to the rectangle AB.CE: therefore the rectangle AB.CE is also equal to the sum of the two AC.CB & AC.DB. Hence as the line AB is divided in C & E, [f] AB.CE is equal to the sum of the three CE by CA, CE by CE, (or the square CE) & CE by EB: these sums of rectangles are therefore equal to each other. Q.f.d. *c 1 Huius; d 16 Sexti; e 1 Secundi; f 1 Secundi.*

[From Prop. 1 of this book, as AC/CD = CD/DE; then AD/CD = CE/DE giving AD/CE = CD/DE ; again, AB/CB = CD/DE = AD/CE : hence <u>rect. AB.CE = rect. CB.AD</u>. Again, as AB/CB = AC/CD, then rect. AB.CD = rect. AC.CB. Again, as AC/DE = AB/DB, then rect. AC.DB = rect. AB.DE; but rect. AB.DE + rect. AB.CD = rect. AB.CE; hence also, rect. AB.CE = rect. AC.CB + rect. AC.DB (the under-lined pairs). Hence, as AB = AC + CE + EB, then rect. CE.AB = rect. CE.AC + rect. CE.CE + rect. CE.EB = rect. AC.CB + rect. AC.DB . The rectangles under-lined are those sought.]

PROPOSITIO XLVII.



Prop.47. Fig. 1.

Sint in continuae analogia minoris inaequalitatis AB, AC, AD, AE, AF, &c. & totidem aliae maioris inaequalitatis eiusdem seriei, GH, IH, KH, LH, MH, &c.

[83]

Dico rectangula ABGH, ACIH, item ADKH, AELH, AFMH, &c. esse omnia inter se aequalia.

Demonstratio.

Ex datis ut AB ad AC, sic IH ad GH, ergo rectangulum [a] ABGH aequatur rectangulo ACIH. rursum ex datis ut AC ad AD, sic KH ad IH; ergo rectangulum ACIH [b] rectangulo ADKH aequale erit. Simili discursu reliquorum aequalitatem ostendimus. est igitur quod fuerat demonstrandum. *a 16 Sexti; b Ibid.*

Corollarium.

Quoniam per primam huius positis continue proportionalibus AB, AC, AD, AE, AF, itemque GH, IH, KH, LH, MH, earum differentiae FE, ED, DC, &c. GH, IK, KL, &c. sunt etiam in ratione continua, manifeste patet eadem discurrendi methodo demonstrati rectangula FELM, DEKL, CDIK, BCGI, quoque inter se aequalia esse.

L2.§1. PROPOSITION 47.

Let AB, AC, AD, AE, AF, etc. be a series of lesser inequalities in continued proportion [i.e. from right to left], and GH, IH, KH, LH, MH, etc. the whole of another series of greater inequalities [i.e. from left to right] of the same series.

I say that the rectangles AB.GH, AC.IH, likewise AD.KH, AE.LH, AF.MH, etc. are all equal to each other.

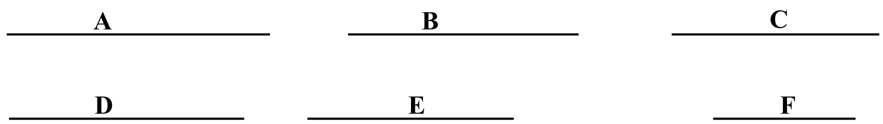
Demonstration.

From the given ratio, as AB is to AC, thus IH is to GH, and therefore the rectangle [a] AB.GH is equal to the rectangle AC.IH. Again, as AC is to AD, thus KH is to IH; hence the rectangle AC.IH [b] is equal to the rectangle AD.KH. We can demonstrate the rest of the equalities by a similar argument. This is what had to be shown. *a 16 Sexti; b Ibid.*

Corollary.

Since from the positions of the continued proportions from the first part of this proposition: AB, AC, AD, AE, AF; and in the same manner GH, IH, KH, LH, MH, then the differences FE, ED, DC, etc , and GI, IK, KL, etc of these, are also in a continued ratio. It is clearly obvious from the same kind of discussion that the rectangles FE.LM, DE.KL, CD.IK, BC.GI, can also be shown to be equal to each other.

PROPOSITIO XLVIII.



Prop.48. Fig. 1.

Sint duae series quotcumque continuarum eiusdem rationis, A, B, C, G, H; D, E, F, I, K: sit autem rectangulum AB, aequale DE rectangulo:

Dico etiam rectangula BC, EF, CG, FI, GH, IK; & sic deinceps aequalia esse.

Demonstratio.

Proportio rectanguli AB, ad rectangulum DE, componitur [c] ex rationibus A ad D , & B ad E; sed rectanguli BC ad rectangulum EF, proportio quoque componitur ex rationibus B ad E, & C ad F, hoc est A ad D (nam cum ex datis & ex aequo sit A ad C, ut D ad F, erit permutando A ad D, ut C ad F) ergo ex iisdem rationibus componuntur proportiones rectangulorum AB, DE: BC, EF, adeoque eadem sunt. quare cum ratio rectangulorum AB, DE ponatur aequalitatis, rectangulorum quoque BC, EF aequalitatis; eodem modo reliqua reliquis ostendentur aequalia. Quod erat demonstrandum. *c 23 Sexti.*

L2.§1. *PROPOSITION 48.*

A, B, C, G, H; D, E, F, I, K are any two series in continued proportion with the same ratio: however, the rectangle A.B is equal to the rectangle D.E:

I say that the rectangles B.C, E.F, C.G, F.I, G.H, I.K and so on are also equal to each other.

Demonstratio.

The proportion of the rectangle A.B to the rectangle D.E, is composed [c] from the ratios A to D , & B to E; but the proportion of rectangle B.C to rectangle E.F, is also composed from the ratios B to E, & C to F, that is A to D (for from what is given and from the equality A is to C thus as D is to F, then on rearranging, A is to D as C is to F) hence the proportions of the rectangles A.B, D.E: B.C, E.F are composed from the same ratios. Whereby as the ratio of the rectangles A.B and D.E is put equal to one, the rectangles B.C and E.F are also equal to each other; and in the same manner the rest of what remains can be shown to be equal.
Q. e.d. *c 23 Sexti.*

. *a 16 Sexti; b Ibid.*

PROPOSITIO XLIX.

Prop.49. Fig. 1.

Si rectangula AB, BC, CG,GH; & DE, EF, FI, IK singula singulis sint aequalia:

Dico esse A ad C, ut D ad F, & B ad G, ut E ad I, & sic deinceps.

Demonstratio.

Cum rectangulum AB, aequale sit rectangulo DE, & rectangulum BC, rectangulo EF; ergo ut rectangulum AB ad DE, sic BC ad EF: & permutando

[84]

ut AB ad BC, sic DE ad EF aequi rectangulum AB ad BC, est ut A ad C, & DE ad EF est ut D ad F; ergo A ad C, ut D ad F. Eodem discursu erit B ad G, ut E ad I. Quod erat demonstrandum.

L2.§1. *PROPOSITION 49.*

If the individual rectangles in the two series AB, BC, CG,GH; & DE, EF, FI, IK are equal to each other term by term:

I say that A is to C as D is to F, & B is to G, as E is to I, & thus henceforth.

Demonstration.

Since rect. A.B is equal to rect. D.E, & rect. B.C is equal to rect. E.F; it follows that as A.B is to D.E, thus B.C is to E.F: & on interchanging, as A.B is to B.C, thus D.E is to E.F; and the ratio A.B to B.C is equal to the ratio A to C, & D.E to E.F is equal to D to F; hence A is to C, as D is to F. By a similar argument, B is to G, as E is to I. Q.e.d.

PROPOSITIO L.

Si rectangula AB, BC, CG,GH, &c. rectangulis DE, EF, FI, IK singula singulis sint aequalia:

Dico utramque laterum seriem A, B, C, G, H; & D, E, F, I, K, si ponantur esse continuae proportionales, esse quoque continuas eiusdem rationis.

Prop.50. Fig. 1.

Demonstratio.

Per praecedentem est A ad C, ut D ad F: quia autem tam A, B, C, quam D, E, F, sunt continuae proportionales ex datis, erit tam ratio A ad C, (id est D ad F), rationis A ad B duplicata; quam ratio D ad F (id est A ad C) duplicata sit rationis D ad E. Quare ut A ad B, sic D ad E; & B ad C, ut E ad F, &c. sunt igitur A, B, C, G, H, & D, E, F, I, K continuae eiusdem proportionis. Quod erat demonstrandum.

L2.§1. *PROPOSITION 50.*

If the rectangles A.B, B.C, C.G, G.H, &c. and the rectangles D.E, E.F, F.I, I.K are equal to each other term by term:

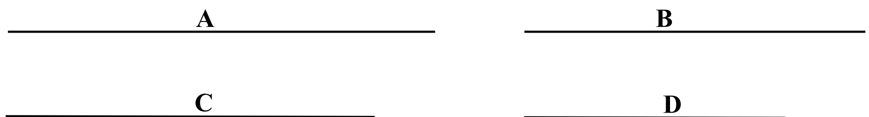
I say that each term of the series A, B, C, G, H; & D, E, F, I, K, if they are placed in continued proportion, are in the same ratio.

Demonstration.

By the preceding proposition, A is to C as D is to F: but since A, B, C, and D, E, F, are given as continued proportionals, so the ratio A to C, (or D to F) is the square of the ratio A to B ; just as the ratio D to F (or A to C) is the square of the ratio D to E. Whereby as A is to B, thus D is to E are in the same ratio; & so B is to C as E is to F, &c. Therefore A, B, C, G, H, & D, E, F, I, K are continued in the same proportion. Q.e.d.

PROPOSITIO LI.

Si prima A ad secundam B, eamdem habeat rationem, quam tertia C ad quartam D:
Dico tria rectangula ex hisce facta, esse in continuata proportione; nempe rectangula AB, BC, & CD.



Prop.51. Fig. 1.

Demonstratio.

Rectangulum AB, est ad rectangulum BC, ut A ad C. sed etiam rectangulum BC, ad [a] CD, (ob eandem rationem) est ut B ad D. cum igitur sit ratio A ad C eadem cum ratione B ad D; cum quoque rationem habet rectangulum AB, ad BC, quam habet BC ad CD rectangulum : sunt igitur in continuate rationis, prout erat demonstrandum. *a 1 Sexti.*

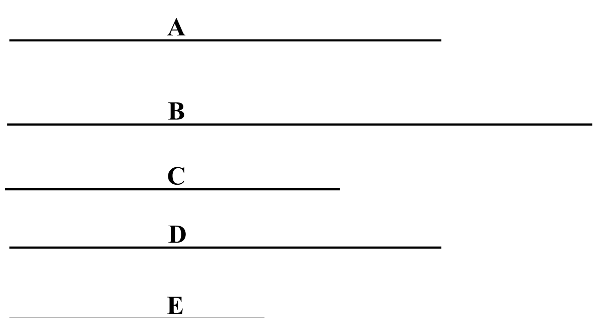
L2.§1. *PROPOSITION 51.*

If the first A has the same ratio to the second B, as the third C to the fourth B:
I say that the three rectangles made from these, namely AB, BC, & CD, are in a continued proportion.

Demonstration.

The rect. AB is to rect. BC, as A is to C; but also the rect. BC is to the rect. [a] CD, (on account of the same ratio) is as B to D. Therefore since the ratio A to C is the same as the ratio B to D; and since too the rect. AB has the same ratio to the rect. BC , as the rect.BC has to the rect.CD: hence the three rectangles are in continued proportion, as was to be shown. *a 1 Sexti.*

PROPOSITIO LII.



Prop.52. Fig. 1.

Si prima A ad secundam B eandem habeat rationem, quam tertia C ad quartam D, fiatque ut prima A ad tertiam C, ita tertia C ad quintam E Dico rectangula ex his lineis constituta, esse in continuata proportione; nempe rectangula AB, BC, CD, & DE.

Demonstratio.

Rectangulum AB ad rectangulum BC, habeat eandem rationem quam A ad C; sed ratio A ad C,

[85]

eadem est , cum ratione C ad E ex constructione, igitur ratio rectanguli AB, ad BC, eadem est cum ratione C ad E; sed quoniam ut A est ad B, ita C ad D, erit permutando A ad C, ut B ad D: ergo ratio rectanguli AB ad BC, est ratio B ad D: sed rectangulum BC ad CD, etiam est [a] ut B ad D ; ergo ratio AB rectanguli ad rectangulum BC, eadem est cum ratione rectanguli BC , ad CD rectangulum. est autem ratio B ad D , hoc est A ad C, eadem quae est C ad E; unde etiam ratio rectanguli CD ad DE rectangulum, eadem est cum ratione rectanguli BC, ad CD rectangulum; quocirca in continua analogia sunt rectanguli AB, BC, CD, DE, cum sunt in ratione A ad C. Constat igitur veritas propositionis. *a Ibid.*

L2.§1. *PROPOSITION 52.*

If the first A to the second B has the same ratio, as the third C to the fourth D, and the first A is made to the third C, thus as the third C to the fifth E :

I say that the rectangles constituted from these lines, surely the rectangles AB, BC, CD, & DE are in continued proportion.

Demonstration.

Rect. AB to rect. BC, has the same ratio as A to C; but ratio A to C is the same as the ratio C to E from the construction, therefore the ratio of rect. AB to rect. BC is the same as the ratio C to E; but since A is to B as C is to D, it becomes on interchanging, A to C as B to D: hence the ratio of rect.AB to rect.BC is in the ratio B to D: but rect.BC to rect.CD, is also as [a] B to D ; hence the ratio of rect. AB to rect.BC is the same as the ratio of rect.BC to rect.CD. But the ratio B to D, or A to C, is the same as that which C has to E; hence also the ratio of rect.CD to rect.DE, is the same as the ratio of the rect.BC to the rect.CD ; wherefore the rectangles AB, BC, CD, DE are in continued proportion, as they are in the ratio A to C. Therefore the truth of the proposition is agreed. *a Ibid.*

[AB/BC = A/C = C/E; also, A/B = C/D, giving A/C = AB/BC = B/D = BC/CD = CD/DE as required.]

PROPOSITIO LIII.

Sint A, B, C, tres in continua ratione. Sintque D, E, F, in eadem vel diversa continuata ratione:

Dico rectangula CD, CE, CF; item BD, BE, BF: item AD, AE, AF esse in continuata analogia.

Prop.53. Fig. 1.

Demonstratio.

Rectangulum CD, est ad rectangulum CE, ut D [b] linea est ad E : & CE rectangulum est ad rectangulum CF, ut E ad F : sed D, E, F ex hypothesi sunt continuae proportionales; ergo & rectangula CD, CE, CF sunt in continua analogia. Eodem modo probantur BD, BE, BF rectangula, item AD, AE, AF esse continue proportionalia. Quod erat demonstrandum. *b 1 Sexti.*

L2.§1. *PROPOSITION 53.*

A, B, and C are three quantities in a continued ratio; while D, E, F are three other quantities in the same or in another ratio.

I say that the rectangles CD, CE, CF; likewise BD, BE, BF: and likewise AD, AE, AF are in continued ratios.

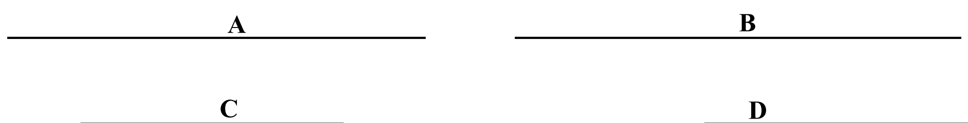
Demonstration.

Rect. CD is to rect. CE, as the line D [b] is to the line E : & rect. CE is to rect. CF, as E is to F : but D, E, F from hypothesis are continued proportionals; & hence the rectangles CD, CE, CF are in a continued ratio. In the same manner it can be agreed that the rectangles BD, BE, BF ; and likewise the rectangles AD, AE, AF are continued proportionals, which was to be shown. *b 1 Sexti.*

PROPOSITIO LIV.

Sit A prima ad B secundam, ut C tertia ad D quartam.

Dico quadratum sub prima, rectangulum sub secunda & tertia, & quadratum quartae, in continua esse analogia.



Prop.54. Fig. 1.

Demonstratio.

Ratio quadrati A ad rectangulum CB, [c] componitur ex ratione A ad B (hoc est C ad D) : & ex ratione A ad C. Sed ratio rectanguli BC ad D quadratum, composita est ex iisdem rationibus; nam ratio rectanguli BC ad D quadratum, componiter ex ratione B ad D, hoc est A ad C, & ex ratione C ad D, hoc est A ad B; ergo sunt continuae quantitates, quadratum A, rectangulum BC, & quadratum D. Quod erat demonstrandum.
c 23 Sexti.

L2.§1. *PROPOSITION 54.*

Let the first A be to the second B, as the third C to the fourth D.

I say that the square under the first, the rectangle under the second and the third, and the square under the fourth are in continued proportion.

Demonstration.

The ratio of the square A to the rectangle CB, [c] is composed from the ratio of A to B (that is C to D): & from the ratio A to C. But the ratio of the rectangle BC to the square D, is composed from the same ratios; for the ratio of the rectangle BC to the square D, is composed from the ratio B to D, that is A to C, & from the ratio C to D, that is A to B; hence the square A, the rect. BC, & the square D are continued quantities: which was to be shown.
c 23 Sexti.

PROPOSITIO LV.

Sint tres lineae AB, BC, CD in continua analogia.

Dico AB quadratum primae, rectangulum ABC, sub prima & secunda, quadratum BC sub secunda; rectangulum BCD sub secunda & tertia; CD quadratum tertia, esse in serie eiusdem rationis AB ad BC.

Prop.55. Fig. 1.

[86]

Demonstratio.

Ratio quadrati A ad rectangulum CB, [c] componitur ex ratione A ad B (hoc est C ad D) : & ex ratione A ad C. Sed ratio rectanguli BC ad D quadratum, composita est ex iisdem rationibus; nam ratio rectanguli BC ad D quadratum, componiter ex ratione B ad D, hoc est A ad C, & ex ratione C ad D, hoc est A ad B; ergo sunt continuae quantitates, quadratum A, rectangulum BC, & quadratum D. Quod erat demonstrandum.

c 23 Sexti.

L2.§1. PROPOSITION 55.

Let the first A be to the second B, as the third C to the fourth D.

I say that the square under the first, the rectangle under the second and the third, and the square under the fourth are in continued proportion.

Demonstration.

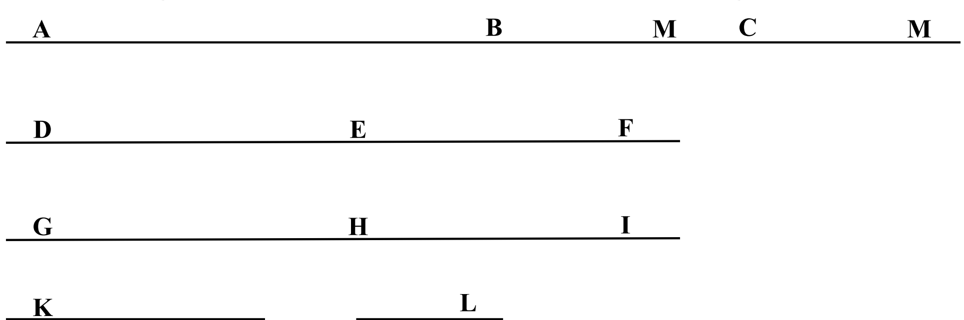
The ratio of the square A to the rectangle CB, [c] is composed from the ratio of A to B (that is C to D): & from the ratio A to C. But the ratio of the rectangle BC to the square D, is composed from the same ratios; for the ratio of the rectangle BC to the square D, is composed from the ratio B to D, that is A to C, & from the ratio C to D, that is A to B; hence the square A, the rect. BC, & the square D are continued quantities: which was to be shown.

c 23 Sexti.

PROPOSITIO LVI.

Lineae GI, DF, AC ita sint divisae in H, E, B, ut ratio DE, ad EF duplicata sit rationis GH ad HI; & ratio AB, ad BC triplicata rationis GH ad HI; sintque praeterea GH, AB, in continua analogia.

Dico & rectangula GHI, DEF, ABC, in continua esse analogia.



Prop.56. Fig. 1.

Demonstratio.

Fiat ut DE ad GH, sic GH ad K, & ut EF ad HI, sic HI ad L. igitur rectangulum sub DE & K quadrato GH, & rectangulum sub EF & L, quadrato HI [c], aequale est; ergo rectangulum DEK, est ad rectangulum EFL, ut quadratum GH ad HI. Ex hypothesi, autem DE est ad EF, in duplicata ratione GH ad HI, hoc est, DE est ad EF ut quadratum GH, ad quadratum HI: ergo rectangulum sub DE & K, est ad rectangulum sub EF & L, ut DE ad EF. Ergo [d] K ad L lineae sunt aequales: quia autem ex hypothesi GH, DE, AB, & ex constructione K, GI, DE, sunt continuae; patet omnes quatuor K, GH, DE, AB esse continuas. Iam alias quoque lineas L, HI, EF, BC, dico esse continuas; si enim non sint, fiat tribus lineis L, HI, EF, quae ex constructione sunt continuae, quarta continue proportionalis, quaevis BM, maior vel minor quam BC, habemus ergo duas continuarum series K, GH, DE, AB; L, HI, EF, BM quae incipiant ab equalibus terminis K & L. quare ratio AB ad [e] BM, erit triplicata rationis GH ad HI; Atqui ex hypothesi etiam ratio AB ad BC, erat triplicata rationis GH ad HI; est ergo ut AB ad BC, ita AB ad BM; quod est absurdum. ergo L, HI, EF, BC etiam sunt continuae. Atque ratio composita ex rationibus GH ad DE & HI ad EF, hoc est, ratio rectanguli GHI [f] ad rectangulum DEF, eadem est cum ratione composita ex rationibus DE ad AB, & EF ad BC, hoc est cum ratione rectanguli DEF, ad rectangulum ABC. rectangula igitur GHI, DEF, ABC sunt in continua analogia. Quod erat demonstrandum.

c 17 Sexti; d Ibid; e 27 huius; f 23 Sexti.

L2.§1. *PROPOSITION 56.*

The lines GI, DF, and AC are thus divided in H, E, and B, in order that the ratio DE, to EF is the square of the ratio GH ad HI; and the ratio AB to BC is the cube of the ratio GH to HI; and in addition GH, DE, and AB are in continued proportion.

I say that the rectangles GHI, DEF, and ABC are in continued proportion.

Demonstration.

The ratio DE to GH is thus made as GH to K, and EF to HI thus as HI to L. Therefore the rectangle under DE & K is equal to the square GH, & the rectangle under EF & L is equal to the square HI [c]; therefore rect. DE.K is to rect. EF.L, as the square GH is to the square HI. But from hypothesis, DE is to EF in the ratio of the square GH to HI; or DE to EF is as the square GH to the square HI. Therefore the rectangle under DE & K, is to the rectangle under EF & L, as DE to EF. Hence [d] the lines K to L are equal. Now, since from hypothesis GH, DE, and AB, and from construction K, GI, and DE are in continued proportion; it is apparent that all four lines K, GH, DE, AB are in continued proportion. Thus, I say that the other lines also: L, HI, EF, and BC are in continued proportion. For if they are not, then for the three lines L, HI, EF, which are in continued proportion, some line BM is made by construction as a fourth continued proportional, greater or less than BC. We have therefore series of continued proportionals : K, GH, DE, AB; and L, HI, EF, BM which begin from the equal terms K and L. Whereby the ratio AB to [e] BM, is the cube of the ratio GH to HI; but by the hypothesis the ratio AB to BC is also the cube of the ratio GH ad HI; hence as AB is to BC, so AB is to BM; which is absurd. Hence L, HI, EF, BC are indeed in continued proportion. The ratio composed from the ratios GH to DE & HI to EF, or, the ratio of rect.GHI [f] to rect. DEF, is the same as with the ratio composed from the ratios DE to AB, & EF to BC, that is with the ratio of rect.DEF to rect. ABC. Therefore the rectangles GHI, DEF, ABC are in a continued ratio. Q.e.d.

c 17 Sexti; d Ibid; e 27 huius; f 23 Sexti.

[DE/GH = GH/K and EF/HI = HI/L from construction; hence rect. DE.K = GH^2 and rect. EF.L = HI^2 ; hence rect.DE.K/rect.EF.L = GH^2/HI^2. But DE/EF = GH^2/HI^2; hence K = L. Now, GH/DE = DE/AB by hypothesis and K/GI = GI/DE by construction; hence K/GH = GH/DE = DE/AB; also, the other lines satisfy L/HI = HI/EF = EF/BC: for if the final length is some other magnitude BM, then BM/AB = K/DE/GH . HI/L.EF = HI/GH . EF/DE = $(HI/GH)^3$ or AB/BM = $(GH/HI)^3$; but by hypothesis, AB/BC = $(GH/HI)^3$ and so BM = BC. Again, GH/DE . HI/EF = GH.HI / DE.EF = rect. GHI/rect. DEF = DE/AB . EF/BC = rect. DEF/rect. ABC; hence rect. GHI/rect. DEF = rect. DEF/rect. ABC as required.]

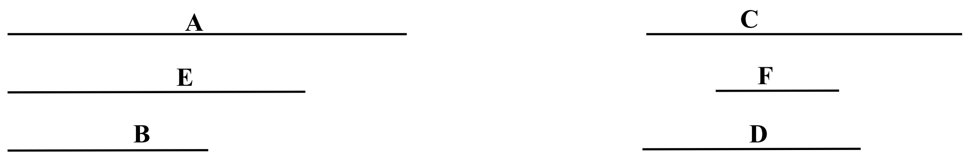
PROPOSITIO LVII.

Sint ratio A ad B, eadem cum ratione C ad D, & inter utramque tam A, B quam C, D interponantur quaevis lineae: E quidem inter A, B; F vero inter C, D.

[87]

Dico rectangulum AE, ad EB rectangulum, eandem habere rationem, quam habet rectangulum CF ad FD rectangulum.

Demonstratio.



Prop.57. Fig. 1.

Ratio rectanguli AE ad EB rectangulum, est ea [a] quam habet A ad B; sed ut A ad B, ita ponitur C ad D; ergo rectangulum AE ad EB, est ut C ad D : sed rectangulum CF, ad FD rectangulum, etiam est ut linea [b] C ad D. igitur rectangulum AE, ad EB rectangulum, eandem habet rationes, quam habet CF ad FD rectangulum, Quod demonstrare oportebat.

a 1 Sexti; b Ibid.

L2.§1. PROPOSITION 57.

Let the ratio A to B be the same as the ratio C to D, and between each, as for A and B, so for C and D: some line E is placed between A and B and some line F between C and D.

I say that the rectangle AE to the rectangle EB has the same ratio as the rectangle CF to the rectangle FD.

Demonstration.

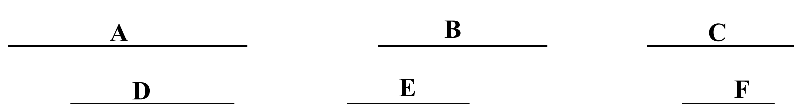
The ratio of rect.AE to rect.EB, is that which [a] A has to B; but as A to B, thus C to D is put; hence the rect. AE to the rect. EB is as C is to D : but rect.CF to rect.FD, is also as the line [b] C to D. Hence rect.AE to rect.EB, has the same ratios as rect.CF has to rect. FD. Which it was necessary to show.

a 1 Sexti; b Ibid.

PROPOSITIO LVIII.

Ponantur tres lineae A, B, C; & aliae tres D, E, F; ut tam primae, quam secundae, suam analogiam licet diversam continuent.

Dico etiam rectangula AD, BE, CE esse in continua analogia.



Prop.58. Fig. 1.

Demonstratio.

Rectangulum sub A & D, ad rectangulum sub B & E habet [c] rationem compositam ex rationibus A ad B, & D ad E : proportio quoque rectanguli sub B & E ad rectangulum sub C & F composita ex rationibus B & C, hoc est A ad B, & ex ratione E ad F, hoc est D ad E; unde ratio rectanguli sub B & E, ad rectangulum CF,

componitur ex iisdem, ex quibus ratio rectanguli sub A & D, ad rectangulum sub B & E est composita : Quare cum proportiones ex iisdem rationibus compositae eaedem sint, erunt rectangula AD, BE, CF in contina analogia. Quod erat demonstrandum.
23 Sexti.

L2.§1. PROPOSITION 58.

Three lines A, B, and C; and three other lines D, E, and F, are put in position so that each can continue in its own ratio.
I say that the rectangles AD, BE, and CE are in a continued ratio.

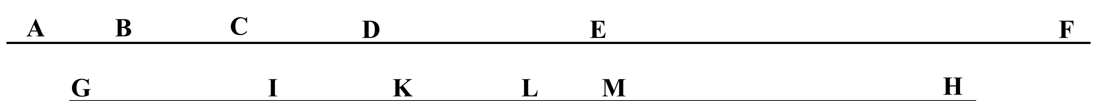
Demonstration.

The rectangle under A & D to the rectangle under B & E has the ratio [c] composed from the ratios A to B, & D to E : the proportion also of the rectangle under B & E to the rectangle under C & F is composed from the ratio B & C, or A to B, & from the ratio E to F, or D to E; hence the ratio of the rectangle under B & E, to the rectangle CF, is componsed from the same, from which the ratio has been composed of the rectangle under A & D to the rectangle under B & E. Whereby as the proportions are composed from the same ratios, the rectangules AD, BE, and CF are in contined proportion. Q.e.d.
c 23 Sexti.

PROPOSITIO LIX.

Sint duae diversarum rationum series continuè proportionalium AB, AC, AD, AE, AF; GH, IH, KH, LH, MH.
Dico rectangula ABGH, ACIH, ADKH, AELH, AFMH esse in continua analogia.

Demonstratio.



Prop.59. Fig. 1.

Rectangulum enim sub ABGH, ad rectangulum sub ACIH rationem habet compositam ex ratione AB ad AC, & ex ratione GH ad IH : & rectangulum ACIH ad rectangulum ADKH rationem compositam ex rationibus ex AC ad AD & IH ad KH. Quare cum ex datis sit ut AB ad AC, sic AC ad AD, & ut GH ad IH, sic IH ad KH: igitur ratio rectangularum ACIH & ADKH, ex iisdem rationibus componitur, ex quibus rectangulorum ABGH, ACIH, sunt igitur rectangula ABGH, ACIH, ADKH in contina analogia : idemque ita reliquis eodem discursu ostendetur.

[88]

Corollarium.

Porro cum per primium huius, etiam continuè proportionalium, differentiae sint in continua analogia suorum integrorum : manifestum est rectangula quoque ABGI, BCIK, CDKL, DELM, &c, esse continuè proportionalia.

L2.§1. PROPOSITION 59.

There are two series in continued proportion with different ratios AB, AC, AD, AE, AF; and GH, IH, KH, LH, MH.
I say that the rectangles ABGH, ACIH, ADKH, AEFH, ALMH are in a continued ratio.

Demonstration.

Indeed the rectangle under AB.GH, to the rectangle under AC.IH, has a ratio composed from the ratios AB to AC, & GH to IH : & the rectangle AC.IH to the rectangle AD.KH has a ratio composed from the ratios AC to AD & IH to KH. Whereby as from what is given, AB is to AC, thus as AC is to AD, & as GH is to IH, so IH is to KH: therefore the ratio of the rectangles ACIH & ADKH, is put together fro, the same ratios, from which the rectangles ABGH & ACIH are composed. Hence the rectangles ABGH, ACIH, and ADKH are in acontinued ratio: and thus the remainder can be shown by the same discussion.

Corollary.

Again from the first proposition of this book, for continued proportions, the differences also of the wholes are in a continued ratio : the rectangles AB.GI, BC.IK, CD.KL, DE.LM, &c, too, can be shown to be in continued proportion.

PROPOSITIO LX.

Sint in continua analogia AB, AC, AD.

Dico rectangulum ABC, ad ACD rectangulum, duplicatam habere rationem eius, quam habet AB ad AC lineam.

Demonstratio.

Prop.60. Fig. 1.

Quoniam DA, CA, BA ponuntur continuè proportionales, erit CD ad BC, ut CA ad BA, & invertendo BC ad CD, ut [a] AB ad AC : itaque cum rectangulorum ABC, [b] ACD ratio componatur ex laterum rationibus AB ad AC, & BC ad CD; quoque iam ostensae sunt aequales, constat eam esse duplicatam, rationis AB ad AC. Quod erat demonstratum. *a 1 Huius; b 23 Sexti.*

L2.§1. *PROPOSITION 60.*

AB, AC, AD are in a continued ratio.

I say that the rectangle ABC, to the rectangle ACD, is in the square ratio of line AB to the line AC.

Demonstration.

Since DA, CA, and BA are put in continued proportion, CD is to BC, as CA is to BA, & on inverting, BC is to CD, as AB is to [a] AC : and thus the ratio of the rectangles ABC [b] and ACD is composed from the ratio of the sides AB to AC, & BC to CD; thus shown to be equal also, and it is agreed that the ratio is the square of the ratio AB to AC. Q.e.d. *a 1 Huius; b 23 Sexti.*

[DA/CA = CA/BA by hypothesis; hence CD/BC = CA/BA and BC/CD = AB/AC.
Rect. AB.BC/rect AC.CD = AB/AC . BC/CD = $(AB/AC)^2$ as required.]

PROPOSITIO LXI.

Iisdem positis :

Dico ABC rectangulum ad ADC rectangulum, triplicatam habere rationem eius, quam habet AB ad AC lineam.

Demonstratio.

Ex datis ratio AB ad AD, duplicata est rationis AB ad AC : & ut patet ex praecedente & per primam huius ratio BC ad CD, aequalis est rationi AB ad AC; ergo ratio composita ex rationibus AB ad AD, & BC ad CD, triplicata est rationis AB ad AC. Quare cum rectangulorum ABC, ADC ratio ex proportionibus laterum AB ad AD, & BC ad CD c componatur, patet eam esse triplicatam rationis AB ad AC. Quod erat demonstratum. *c 23 Sexti.*

L2.§1. PROPOSITION 61.

With the same positions.

I say that the rectangle ABC, to the rectangle ADC, is the cube of the ratio of line AB to the line AC.

Demonstration.

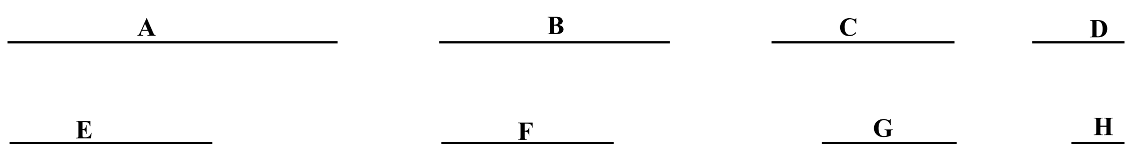
From what is given, the ratio AB to AD is the square of the ratio AB to AC. It is apparent from the preceeding proposition and from the first proposition of this book, that the ratio BC to CD is equal to the ratio AB to AC. Hence the ratio composed from the ratios AB to AD and BC to CD, is the cube of the ratio AB to AC. Whereby since the ratio of the rectangles ABC and ADC is composed from the proportions of the sides AB to AD and BC to CD [c]. It is apparent that it is the cube of the ratio AB to AC. Q.e.d. *a 23 Sexti.*
[$AB/AD = AB/AC \,.\, AC/AD = (AB/AC)^2$. Also, $BC/CD = AB/AC$;
hence $AB/AD \,.\, BC/CD = (AB/AC)^2 \,.\, AB/AC = (AB/AC)^3$ = rect. ABC /rect. ADC].

PROPOSITIO LXII.

Ponatur duae series quatuor continuarum A, B, C, D ; & E, F, G, H.

Dico rectangulum AH, ad ED rectangulum, triplicatam rationem habere eius, quam habet BG rectangulum, ad rectangulum FC.

Demonstratio.



Prop.62. Fig. 1.

Rectanguli sub A & H, ad rectangulum sub E & D, ratio est composita, [d] ex ratione A & D, H & E : ratio vero rectanguli BG, ad rectangulum FC, composita est ex ratione B ad C, & G ad F : ratio autem A ad D, triplicata est ratione B ad C, & ratio H ad E etiam triplicata est rationis G ad F; igitur ratio rectanguli AH ad DE, triplicata est rationis eius, quam habet BG rectangulum, ad FC. Quod erat demonstratum. *c 23 Sexti.*

L2.§1. *PROPOSITION 62.*

Two series in continued proportion are put in place: A, B, C, D ; & E, F, G, H.
I say that the ratio of rectangle AH to rectangle ED is the cube of the ratio that rectangle BG has to rectangle FC.

Demonstration.

The ratio of the rectangle under A & H, to the rectangle under E & D, is made from [d] the ratio A & D, and H & E : also, the ratio of rectangle BG, to rectangle FC, is made from the ratio B to C, & G to F. But the ratio A to D, is the cube of the ratio B to C, & the ratio H to E also is the cube of the ratio G to F; hence the ratio of rectangle AH to DE, is the cube of that which rectangle BG has to FC. Q.e.d. *c 23 Sexti.*

[Rect.A.H/rect E.D = A/D . H/E; also, rect. B.G/rect F.C = B/C . G/F; but A/D = $(B/C)^3$, and H/E = $(G/F)^3$; hence rect.A.H/rect E.D = $(\text{rect. B.G/rect C.F})^3$ as required.]

[89]

PROPOSITIO LXIII.

Datae sint tres continuae proportionales AC, CD, DE ; & DE bifarium sit in B.
Dico quadratum AB, aequale esse quadratis AD, CB.

Demonstratio.

Prop.63. Fig. 1.

Quia AC, CD, DE sunt in continua proportione, quadratum CD [a] aequatur rectangulo ACDE ; hoc est (quoniam DE bisecta ponitur in B) rectangulo ACDB bis sumpto. Quare si utrisque/verisque ? commune addatur rectangulum CDB bis , erit quadratum CD, cum rectangulo CDB bis, aequale rectangulo ACDB bis, cum rectangulo CDB bis; quae quatuor rectangula constituunt [b] rectangulum ADB bis. Rursum ergo communi addito quadrato DB, erit quadratum CD, cum rectangulo CDB bis, & quadrato DB, id est [c] quadratum CB, aequale rectangulo ADB bis, cum quadrato DB : itaque communi addito quadrato AD, erit rectangulum ADB bis, cum quadratis DB, AD, id est [d] quadratum AB, aequale quadratis CB, AD : Quod erat demonstratum. *a 27 Sexti; b 2 Sexti ?; c 4 Secundi; d Ibid;*

L2.§1. *PROPOSITION 63.*

Three continued proportionals AC, CD and DE are given ; & DE is bisected in B.
I say that the square AB is equal to the sum of the squares AD and CB.

Demonstration.

Since AC, CD and DE are in a continued proportion, then the square CD [a] is equal to the rectangle AC.DE ; or (as DE is bisected in B) to twice the rectangle AC.DB. Whereby if indeed twice the rectangle CDB is added in common, then the square CD with twice the rectangle CDB, is equal to the sum of twice the rectangle AC.DB and twice the rectangle CDB; which four [b] constitute twice the rectangle ADB. Again, therefore with the common square DB added, erit quadratum CD, cum rectangulo CDB bis, & quadrato DB, id est [c] quadratum CB, aequale rectangulo ADB bis, cum quadrato DB : itaque communi addito quadrato AD, erit rectangulum ADB bis, cum quadratis DB, AD, id est [d] quadratum AB, aequale quadratis CB, AD : Quod erat demonstratum. *a 27 Sexti; b 2 Sexti ?; c 4 Secundi; d Ibid;*

[Rect.A.H/rect E.D = A/D . H/E; also, rect. B.G/rect F.C = B/C . G/F; but A/D = $(B/C)^3$, and H/E = $(G/F)^3$; hence rect.A.H/rect E.D = $(\text{rect. B.G/rect C.F})^3$ as required.]

PROPOSITIO LXIV.

Sint tres lineae in continuae analogia AB, BC, CD ; & dividatur CD bifarium in E. Dico quadratum AE, aequare quadratis AC, EB.

Demonstratio.

Prop.64. Fig. 1.

Cum sint continuae proportionales AB, BC, CD, erit ut BC [e] ad CD, sit AC ad DB. unde rectangula CBD, ACD aequantur. sed rectangulum CBD, est rectangulum CDB, cum quadrato DB, hoc est (quoniam ex datis CD bisecta est in E) rectangulum EDB bis, cum quadrato DB : rectangulum vero ACD, est rectangulum ACE bis; ergo rectangulum EDB bis, cum quadrato DB, aequatur rectangulo ACE bis, & communibus additis quadratis AC, CE, sive ED, erunt rectangulum EDB bis & quadrata BD, ED, AC simul sumpta, aequalia rectangulo ACE bis, & quadratis AC, CE : Atqui rectangulum EDB bis, cum quadratis DE, ED, AC, aequale est quadrato [f] EB, & quadrato AC; rectangulum vero ACE bis, cum quadratis AC, CE, aequantur [g] quadrato AE : ergo quadrata EB, AC aequantur quadrato AE. Quod erat demonstratum. *e 1 Huius; f 4 Secundi; g Ibid.*

L2.§1. *PROPOSITION 64.*

There are three lines in continued proportion AB, BC and CD ; & CD is bisected in E. I say that the square AE is equal to the sum of the squares AC and EB.

Demonstration.

Since AB, BC, and CD are continued proportionals , thus BC [e] to CD is as AC to DB. Hence the rectangles CBD and ACD are equal; but rectangle CBD is equal to the sum of rectangle CDB and the square DB, or (as CD is given bisected in E) to the sum of twice the rectangle EDB and the square DB : but rectangle ACD is twice the rectangle ACE; hence the sum of twice the rectangle EDB and the square DB is equal to twice the rectangle ACE, and with the common squares AC and CE or ED added, the sum of twice the rectangle EDB and the squares BD, ED and AC taken together, is equal to the sum of twice the rectangle ACE and the squares AC and CE. But twice the rectangle EDB with the squares DE, ED and AC is equal to the square [f] EB and the square AC; thus twice the rectangle ACE with the squares AC and CE is equal to the square [g] AE : hence the sum of the squares EB and AC is equal to the square AE. Q.e.d.
e 1 Huius; f 4 Secundi; g Ibid.

[AB/BC = BC/CD; AC/BC = BD/CD or BC/CD = AC/BD and rect. CB.BD or rect. CBD = rect. AC.CD or rect. ACD. But rect. CB.BD = (CD + DB).BD = rect. CDB + BD^2 = 2.rect EDB + BD^2 ; also, rect. ACD = 2.rect. ACE; hence 2.rect EDB + BD^2 = 2.rect. ACE; on adding AC^2 + ED^2 to both sides: 2.rect EDB + BD^2 + AC^2 + ED^2 = 2.rect. ACE + AC^2 + EC^2 ; but 2.rect EDB + BD^2 + ED^2 = BE^2, and] 2.rect. ACE + AC^2 + EC^2 = AE^2; hence BE^2 + AC^2 = AE^2, as required.]

PROPOSITIO LXV.

Continuae proportionales sint AB, AC, AE, &c.; & ex BC, sunti possit BD aequalis AC.

Dico rectangulum sub BA & sub CD, AE, tamquam unam lineam constructum, aequali quadrato AD.

Demonstratio.

Prop.65. Fig. 1.

Rectangulum BACD, [h] aequatur rectangulo BCD, (id est [i] quadrato CD, & rectangulo BCD) una cum rectangulo ACD, sed (quoniam aequales sunt positae AC.BD)

[90]

aequalia sunt rctangula BDC, ACD; ergo rectangulum ABCD, aequatur rectangulo ACD bis, cum quadrato, CD: & quia sunt continuae BA, CA, EA, rectangulum BAE, aequale est quadrato CA : Igitur si rectangulo ABCD addas rectangulum BAE: & rectangulo ACD bis, cum quadrato CD, addas quadratum CA, erunt rectangula BACD, BAE, (id est rectangulum [a] ex BA in CDAE tanquam unam lineam) aequalia quadratis CD, CA, & rectangulo ACD bis, id est [b] quadrato AD. Quod erat demonstratum. *a 1 Secundi; b 4 Sextii.*

L2.§1. *PROPOSITION 65.*

AB, AC, AE, etc. are continued proportionals, & from BC, BD is to be taken equal to AC.

I say that the rectangle under BA and under CD and AE, constructed as it were from one line, is equal to the square AD.

Demonstration.

The rectangle BACD [h] is equal to rectangle BCD, (that is [i] to the sum of the square CD and the rectangle BDC) together with rectangle ACD, but (since AC and BD are made equal) the rectangles BDC and ACD are equal; hence rectangle ABCD is equal to the sum of twice the rectangle ACD and the square CD: and since BA, CA and EA are in continued proportion, rectangle BAE is equal to the square CA. Hence if you add rectangle BAE to rectangle ABCD: and if you add the square CA to the sum of twice rectangle ACD and the square CD, then the sum of the rectangles BACD and BAE, (that is the rectangle [a] from BA by CDAE considered as one line) is equal to the sum of the squares CD and CA with twice the rectangle ACD , that is [b] to the square AD. Q.e.d. *a 1 Secundi; b 4 Sextii.*

[Rect. BA.CD = rect.BC.CD + rect. AC.CD, where rect.BC.CD = CD^2 + rect.CD.DB; But as AC = BD, rect.CD.DB = rect.AC.CD, then rect. BA.CD = 2.rect. AC.CD + CD^2 . Again, as BA/CA = CA/ EA, rect.BA.AE = CA^2 , then rect. BA.CD + rect.BA.AE = rect.BA.(AE + CD) = 2.rect. AC.CD + CD^2 + CA^2 = AD^2 as required].

PROPOSITIO LXVI.

Ponatur linea AB, divisa in tres proportionales AB, CB, DB; duabus autem CB, DB, in directum constituantur aequales, BF, BE.

Dico rectangulum CDF, rectangulo ADB aequale esse.

Demonstratio.

Prop.66. Fig. 1.

Quoniam AB, CB, DB, ponatur continuae, erit rectangulum ABD, (id est [c] rectangulum ADB cum quadrato DB) aequale quadrato CB; atqui etiam (cum ex datis CF bisecta sit in B) rectangulum CDF cum quadrato DB, aequatur [d] quadrato CB; ergo rectangulum ADB, cum quadrato DB, aequatur rectangulo CDF, cum quadrato DB. Quocirca ablato communi quadrato DB rectangulum, CDF, rectangulo ADB aequale erit. Quod erat demonstratum. *c 3 Secundi; d 5 Secundi.*

L2.§1. *PROPOSITION 66.*

The line AB is established and divided in three proportionals AB, CB, DB; but the lines BF and BE are placed in order equal to the two lines CB and DB.

I say that the rectangle CDF is equal to the rectangleADB.

Demonstration.

Since AB, CB, and DB are placed in continued proportion, the rectangle ABD, (that is [c]rectangle ADB plus the square DB) is equal to the square CB; but also (since from what is given, CF is bisected in B) the sum of rectangle CDF and the square DB, is equal to [d] the square CB; hence the sum of the rectangle ADB and the square DB, is equal to the sum of the rectangle CDF and the square DB. Whereupon, by taking the common square DB, the rectangle CDF is equal to the rectangle ADB. Q.e.d. *c 3 Secundi; d 5 Secundi.*

[Since AB/CB = CB/ DB, rect.AB.BD = CB^2 = rect. AD.DB + DB^2, given BC = BF and BD = BE; also rect.CD.DF + DB^2 = CB^2 , from the difference of the squares; hence rect. AD.DB = rect.CD.DF, as required].

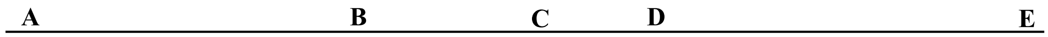
PROPOSITIO LXVII.

Sit AB ad AC, ut AD ad AE.

Dico primo rectangulum sub primo AB, & DE differentia quartae & tertiae, aequari rectangulo sub BC, differentiae primae & secundae, & sub AD tertia:

Secundo rectangulum sub prima AB, & BE differentia primae & quartae, aequari duobus rectangulis sub AC, BD, & sub AB, BC.

Demonstratio.



Prop.67. Fig. 1.

Cum sit ut AB ad AC, sic AD ad AE, itaque invertendo & dividendo ut CB ad BA, sic ED ad DA. quare rectangulum EDAB aequatur rectangulo DACB. quod erat primum. Deinde cum sit ut EA ad DA, sic CA ad BA, ergo rectangulum EAB aequatur rectangulo DAC, hoc est [e] rectangulo DCA cum quadrato CA. Quod sit autem a rectangulo EAB abstuleris quadratum AB, remanet rectangulum EBA : & si idem abstuleris a rectangulo DCA cum quadrato CA, remanent [f] rectangulum DCA, rectangulum CBA bis cum quadrato BC. Itaque cum tota fuerint aequalia, erunt ablato communi, aequalia adhuc reliqua; rectangulum nempe EBA, & rectangula DCA semel, CBA bis, & quadratum BC simul sumpta. atqui rectangulum ACB, aequale est rectangulo ABC cum quadrato BC : cui si addas rectangulum ACD, orietur [g] rectangulum ACBD, quod cum rectangulo ABC aequale erit rectangulo ABC bis cum quadrato BC, & rectangulo ACD : quae cum simul sumpta aequalia esse ostendeum rectangulo EBA, erit quoque rectangulum AC, BD cum rectangulo ABC aequale rectangulo EBA, quod erat demonstratum. *e 3 Secundi; f 4 Secundi ; g 1 Secundi.*

L2.§1. *PROPOSITION 67.*

The line AB is established and divided in three proportionals AB, CB, DB; but the lines BF and BE are placed in order equal to the two lines CB and DB.

I say that the rectangle CDF is equal to the rectangleADB.

Demonstration.

Since AB, CB, and DB are placed in continued proportion, the rectangle ABD, (that is [c]rectangle ADB plus the square DB) is equal to the square CB; but also (since from what is given, CF is bisected in B) the sum of rectangle CDF and the square DB, is equal to [d] the square CB; hence the sum of the rectangle ADB and the square DB, is equal to the sum of the rectangle CDF and the square DB. Whereupon, by taking the common square DB, the rectangle CDF is equal to the rectangle ADB. Q.e.d. *c 3 Secundi; d 5 Secundi.*

[Since AB/CB = CB/ DB, rect.AB.BD = CB^2 = rect. AD.DB + DB^2, given BC = BF and BD = BE; also rect.CD.DF + DB^2 = CB^2 , from the difference of the squares; hence rect. AD.DB = rect.CD.DF, as required].

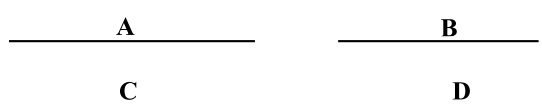
[91]

PROPOSITIO LXVIII.

Si quatuor lineae proportionales fuerint, maximae & minimae, quadrata simul sumpta, maiora sunt reliquarum quadratis simul sumptis.

Demonstratio.

Cum enim quatuor lineae ponantur proportionales, etiam [a] earum quadrata erunt proportionalia, ita tamen ut quadratum maximae lineae maximum sit, quadratum vero minimae, sit minimum, ut patet ex elementis; igitur [b] constat propositum. Theorema eadem fere posita demonstratione quibusuis planis & solidis similibus applicari poterit. quod erat demonstratum. *a 22 Sexti; b 25 Quinti.*



Prop.68. Fig. 1.

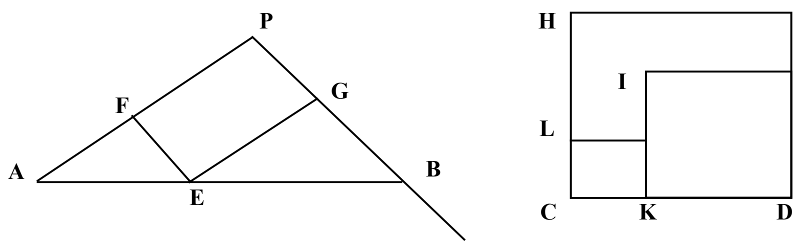
L2.§1. *PROPOSITION 68.*

If there are four lines in proportion, then the squares of the maximum and minimum are larger and smaller than the squares of the other proportions.

Demonstration.

Since indeed there are four lines placed in proportion, the squares of these are also in proportion, thus indeed as the square of the greatest line is the greatest, and the square of the least shall indeed be the least of the squares, as is apparent from elementary considerations. Therefore the truth of the propositon is agreed upon. It is possible for the same theorem to be applied generally for any similar demonstrations of planes and solids. *a 22 Sexti; b 25 Quinti..*

PROPOSITIO LXIX.



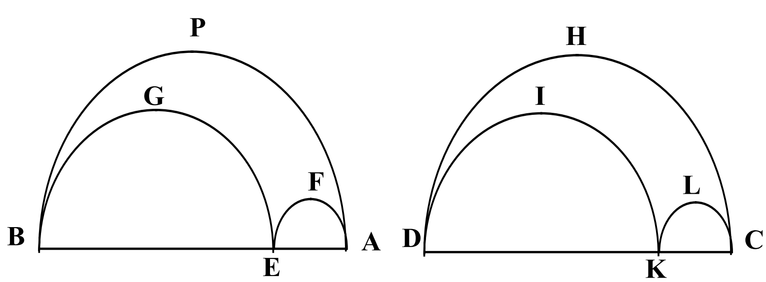
Prop.69. Fig. 1.

Duobus lineis AB, CD secundi eandem rationem divisis in E, & K, fiat super earum una AB quavis figura APB : & sub partibus duae similes ei quae fit a tota, nempe AFE, EGB. Deinde sub alia CD fiat quaecumque alia figura CHD, sive similis praecedentibus, sive dissimilis seu rectilinea seu curvilinea; sub partibus autem statuantur duae aliae CLK, KID, similes ei quae sit tota.

Dico ut APB ad CHD, sic duae AFE, EGB ad duas CLK, KID.

Demonstratio.

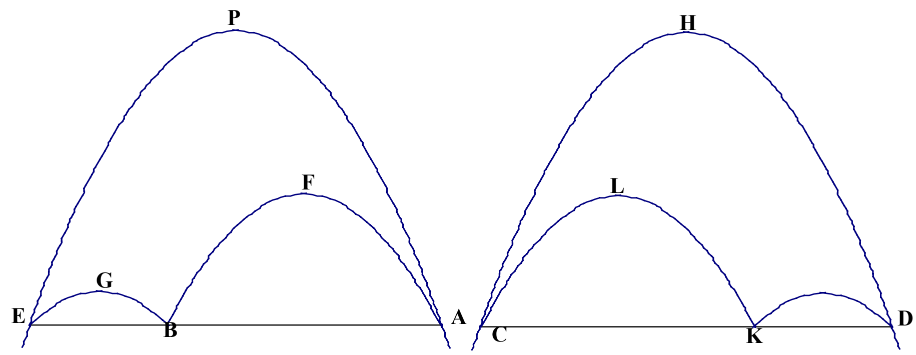
Comparimus primo rectilinea cum rectilineis : quia figurae similes sunt AFE, APB, erit AFE, ad APB, in duplicata ratione laterum [c] homologorum AE, AB : similiter quia CLK, CHD sunt figura similes, erunt in duplicata ratione CK ad CD, id est ex datis, in duplicata ratione AE ad AB : ergo AFE est ad APB, ut CLK ad CHD. igitur AFE d cum EGB, est as APB, ut CLK cum KID, ad CHD. Itaque permutando AFE est cum EGB, ad CLK cum KID, ut APB ad CHD.



Prop.69. Fig.2.

Comparimus deinde rectlinae cum curvilineis. Super CD constituatur segmentum circuli CHD: & super CK, KD, segmenta CLK, KID. similia segmento CHD: cum igitur similia circulorum segmenta, duplicatam habeant proportionalem subtensarum, si rectilinea lineae AB, cum segmentis lineae CD comparentor, eadem prorsus demonstratione concludetur propositum, qua usi fuimus in prima comparatione : Tertio si curvilinea est curvilineis eiusdem speciei conferantur, patet a fortiori propositum:

[92]



Prop.69. Fig.3.

Eodem modo si curvilinea, cum diversae speciei curvilineis, tres nempe parabolae similes, cum tribus hyperbolis similibus, conferantur, eadem his quoque demonstrandi ratio conveniet : cum tam parabolae similes, quam hyberbolae sint in duplicata ratione subtensarum. Constat igitur huius theorematis universalis veritas. *c 19 Sexti; d 24 Quinti.*

L2.§1. *PROPOSITION 69.*

For two lines AB and CD, following the same ratio of division in E and K, there is constructed on the one line AB some figure APB, and within the parts of the division two figures AFE and EGB are made similar to the whole figure. Then within the other CD some other figure is made, either similar to the first mentioned, or dissimilar, either rectilinear or curvilinear; however two other figures CLK and KID are established within the sections of the line, which are similar to the whole figure.

I say that as APB is to CHD [i.e. as areas], thus the sum of ADE and EGB is to CLK and KID.

Demonstration.

First we compare the rectilinear figure with the rectilinear figures : since AFE and APB are similar figures, the ratio of AFE to APB is in the square ratio of the homologous sides [c] AE and AB : similarly since CLK and CHD are similar figures, they are in the square ratio of CK to CD, that is from what is given, in square ratio AE to AB : hence AFE is to APB, thus as CLK is to CHD. Therefore the sum of AFE [d] and EGB, is to APB, as the sum of CLK and KID is to CHD. Thus by interchanging, as the sum of AFE and EGB is to sum of CLK and KID, thus APB is to CHD.

In the second case we can compare a rectlinear figure with a curvilinear one. Upon CD the segment of a circle CHD is set up: and upon the sections CK and KD, the segments CLK and KID are set up similar to the segment CHD. Therefore, since the ratio of the areas of similar segments of circles are as the squares of the subtended chords, if the rectilinear figures of the line AB are compared with the segments of the line CD, the same proposition can in short be conculuded, which we used in the first comparison.

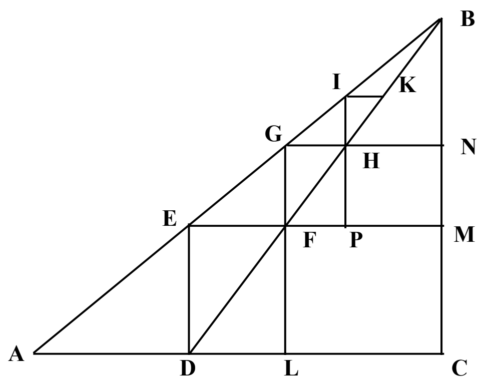
In the third case if a curvilinear figure is brought together with curvilinear figures of the same kind, the proposition is apparent from what has gone before : In the same manner, if a curvilinear figure is brought together with a different kind of curvilinear figure, indeed if three similar parabolas are brought together with three similar hyperbolas, for these too the same ratio can beshown to be agreed upon: for as with similar parabolas, so the areas under similar hyberbolas are in the ratio of the squares of the subtending chords. Thus the truth of this theorem can be agreed upon in all cases. *c 19 Sexti; d 24 Quinti.*

PROPOSITIO LXX.

Sit ABC triangulum divisum rectam lineam DB, ducanturque lineae DE, EF, FG, GH, HI, IK, basi AC, & lateri BC parallelae quot libverit.

Dico omnes AD, EF, GH, IK, item DE, FG, HI, &c. esse in eadem continuata analogia.

Demonstratio.



Prop.70. Fig. 1.

Producantur enim lineae GF in L, EF in M, GH in N, & IH in P. Igitur cum parallelae sint AC, EM, erit AD ad DC , ut EF ad FM: & componendo AC ad CD, ut EM ad FM, id est ut DC ad LC; sunt igitur in continua ratione AC, DC, LC; quare & AD [a] ad DL, id est EF ut AC est ad DC. Similiter ostendam esse continuas EM, FM, PM : unde rursus est ut EM [b] ad FM (id est ut AC ad DC, id est AD ad EF) sic EF ad FP, id est GH. continue proportinales sunt igitur AD, EF, GH : eodem modo ostendam IK, & alias quotcumque in eadem serie esse continuas. Deinde cum AD ipsiEF, & DE ipsi FG sit parallela, patet similia esse triangula ADE, EDG: ergo ut AD ad EF, ita DE ad FG : similiter ostendam esse ut EF ad GH, ita FG ad HI. quare erunt etiam DE, FG, IH, &c. continuae, & quidem in ea ratione in qua sunt AD, EF, GH, Quod erat demonstrandum. *a 1 Huius; b Ibid.*

L2.§1. *PROPOSITION 70.*

Triangle ABC is divided by the line DB, and the lines DE, EF, FG, GH, HI, and IK are drawn parallel to the base AC and to the side BC as often as it pleases.

I say that all the lines AD, EF, GH, IK, likewise DE, FG, HI, etc. are in the same continued ratio.

Demonstration.

For the lines are produced: GF in L, EF in M, GH in N, and IH in P. Therefore as the lines AC and EM are parallel, the ratio AD to DC is thus as EF to FM: and by adding AC is to CD as EM is to FM, or as DC is to LC; AC, DC, and LC are therefore in a continued ratio; and whereby AD [a] is to DL, or EF, as AC is to DC. Similarly it can be shown that EM, FM, PM are in continued proportion : thus again as EM [b] is to FM (or AC to DC, or AD to EF) thus EF is to FP, or GH. Hence AD, EF, anf GH are continued proportinals: in the same way it can be shown that IK and any othes in the same series are in proportion. Hence as AD to itself EF, & DE to FG itself are parallel, it is apparent that the triangles ADE and EDG are similar: hence as AD is to EF, thus DE is to FG : similarly it can be showm that as EF is to GH, thus FG to HI. Whereby DE, FG, IH, etc. are also in continued proportion, and indeed are in the same continued ratio as AD, EF, and GH. Q.e.d. *a 1 Huius; b Ibid.*

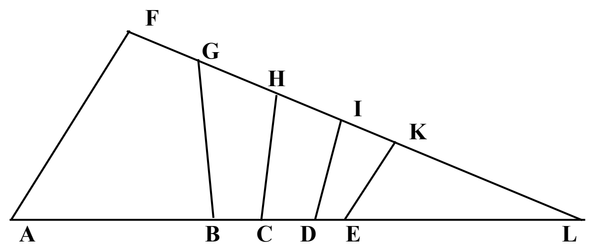
PROPOSITIO LXXI.

Duae lineae AL, FL angulum facientes, sectae sint in continuae proportionales, quarumcumque rationes AL, BL, CL, DL, EL, &c. item FL, GL, HL, &c. dein opposita sectionum puncta lineis AF, BG, CH, DI, &c. coniungantur.

Dico triangula AFL, BGL, CHL, & caetera in infinitum esse in continua analogia.

[93]

Demonstratio.



Prop.71. Fig. 1.

Cum AL, BL, CL, &c. ponantur continuae proportinales, est ut AL ad BL, sic BL ad CL, & CL ad DL, &c. similiter cum ponantur continuae FL, GL, &c. erit ut FL ad GL, ita GL ad HL, atque ita semper : igitur ratio composita ex rationibus AL ad BL, & FL ad GL, eadem erit cum ratione composita ex rationibus BL ad CL, & GL ad HL; & composita ex rationibus BL ad CL, & GL ad HL, eadem erit, cum composita ex rationibus CL ad DL, & HL ad IL: Atqui trianguli AFL, ad triangulum BGL proportio composita est ex rationibus AL ad BL, & FL ad GL; & ratio trianguli BGL ad triangulum CHL, composita est ex rationibus BL ad CL, & GL ad HL, ut ex Commandino demonstrat Clavius, ad propositionem 23 sexti : eadem igitur est ratio trianguli AFL, ad triangulum BGL, quae huius, ad triangulum CHL. Similiter ostendentur reliqua triangula esse in analogia continua. Quod erat demonstrandum.

Corollarium.

Hinc consequitur etiam Trapezia AG, BH, CI, &c. esse in continua analogia, sunt enim trapazia, triangulorum continuae proportionalium differentiae, unde ex prima huius patet corollarii veritas.

L2.§1. *PROPOSITION 71.*

The two lines AL and FL making the sides of an angle are cut in continued proportions, according to some ratio AL, BL, CL, DL, EL, etc., and likewise FL, GL, HL, etc., then the opposite points of the sections of the lines AF, BG, CH, DI, etc. are joined.

I say that the triangles AFL, BGL, CHL, and so on indefinitely are in continued proportion.

Demonstration.

Since the lines AL, BL, CL, etc. are placed in continued proportion, AL is to BL, thus as BL is to CL, and CL to DL, etc. Similarly, since FL, GL, etc, are put in proportion, FL is to GL thus as GL is to HL, and so on indefinitely: therefore the ratio composed from the ratios AL to BL, and FL to GL, is the same as that composed from the ratios BL to CL, and GL to HL; and that composed from the ratios BL to CL, and GL to HL, is the same as that composed from the ratios CL to DL, and HL to IL. But the proportion of triangle AFL to triangle BGL is composed from the ratios AL to BL, and FL to GL; and the ratio of triangle BGL to triangle CHL is composed from the ratios BL to CL, and GL to HL, as Clavius shows from Commandinus, by Proposition 23 of Book Six : hence, in the same manner is the ratio of triangle AFL to triangle BGL, and of that to triangle CHL. Similarly the rest of the triangles can be shown to be in continued proportion. Q.e.d.

[Note that in general the ratios on the two lines are different; however, $\Delta AFL/\Delta BGL = AL/BL \cdot FL/GL$, etc, insuring the truth of the proposition.

Corollary.

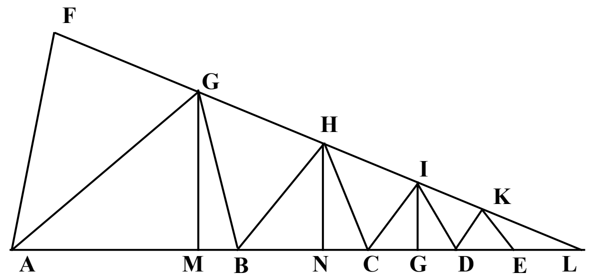
It hence follows that the trapezia AG, BH, CI, etc. also are in continued proportion, for the trapazia are the differences of the of the triangles in continued proportion, thus the truth of the corollary is apparent from the first part of this proposition .

PROPOSITIO LXXII.

Iisdem positis ducantur in singulis trapeziis diametri, AG, BH, CI, &c

Dico triangula inde nata AGB, BHC, CID, &c, itemque triangula FAG, GBH, &c. esse continuae proportionalia.

Demonstratio.



Prop.72. Fig. 1.

Ex punctis enim G, H, I, &c. demittantur normales GM, HN, IG, &c. ratio trianguli AGB, ad triangulum BHC, componitur ex [a] rationibus AB ad BC, & altitudinis GM, ad altitudinem HN : sed quia AL, BL, CL, &c. ponuntur continuae proportinales etiam AB, BC, CD, [b] &c. erunt continuae in ratione suorum integrorum AL, BL, CL, &c., atque? GM, HN, IG, &c. ad AL normales sint? interse parallela sunt, erit GM ad HN, ut GL ad HL, hoc est ex datis, ut HL ad IL, igitur ratio trianguli AGB, ad triangulum BHC, eo aequitur ex rationibus BC ad CD, & HL ad IL, simili plane dIscursu ostendemus rationem trianguli BHC ad triangulum CID : ex iisdem rationibus esse compositam : igitur triangula AGB, GHC, CID, sunt in continua analogia, similiter de aliis idem demonstrabimus. Patet veritas propositionis.

a Claudius ad 23 Sexti ; b 1 Huius.

L2.§1. *PROPOSITION 72.*

With the same lines and points in position, the diameters of the trapeziums AG, BH, CI, etc., are draw.

I say that the triangles thus produced AGB, BHC, CID, &c. and likewise the triangles FAB, GBH, &c. are in continued proportion.

Demonstration.

Normals GM, HN, IG, are sent from the points G, H, I, &c. The ratio of triangle AGB to triangle BHC is composed from the ratios [a] AB to BC and the altitude GM to the altitudine HN : but since AL, BL, CL, etc. are placed in continued proportion, AB, BC, CD, [b] etc. are also in proportionals in the ratio of the whole AL, BL, CL, etc. But GM, HN, IG, &c. are normal to AL and hence are parallel to each other, hence GM is to HN, as GL is to HL, or from what is given, as HL to IL. Therefore the ratio of triangle AGB to triangle BHC is equal to that from the ratios BC to CD and HL to IL. By a similar argument we can show that the ratio of triangle BHC to triangle CID is composed from the same ratios : therefore triangles AGB, GHC, and CID are in a continued ratio, similarly we can show the same for the other triangles, and the truth of the proposition is apparent.

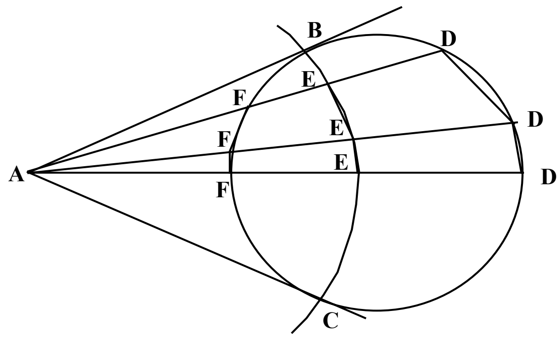
PROPOSITIO LXXIII.

Contingant circulum BCD duae lineae AB, AC, ex eodem puncto eductae ; & centro A intervallo B, C, describatur arcus BEC. deinde ex puncto A, ductis quotcumque lineis, secantibus AFED iungantur DD, EE, FF.

[94]

Dico triangula inde nata DAD, EAD, FAF, esse in totinua analogia.

Demonstratio.



Prop.73. Fig. 1.

Cum AB contingat circulum, erit [a] rectangulum DAF, aequale quadrato AB, hoc est quadrato AE. Unde DA, EA, FA, sunt continuae proportinales. Similiter reliquae omnes lineae DA, EA, FA, erunt in continua analogia. triangula igitur DAD, EAE, FAF [b] etiam in continua sunt analogia. Quod erat demonstrandum .
a 36 Tertii ; b 72 Huius.

L2.§1.

PROPOSITION 73.

Two lines AB and AC are drawn from the same point A and touch the circle BCD; with centre A, and within the interval BC the arc BEC is described, then from the point A some lines are drawn with secants AFED and DD, EE, FF are joined.

I say that the triangles thus produced DAD, EAD, FAF are in complete proportion.

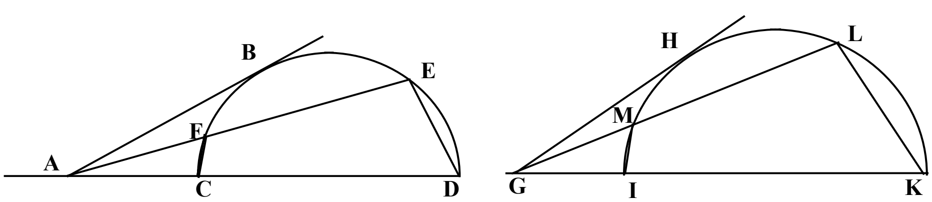
Demonstration.

Since AB touches the circle, the rectangle DAF [a] is equal to the square AB, or to the square AE. Hence DA, EA, and FA are in continued proportion. Similarly the rest of all the lines DA, EA, FA are in continued proportion. Therefore the triangles DAD, EAE, and FAF also are in continued proportion. Q.e.d.
a 36 Tertii ; b 72 Huius.

PROPOSITIO LXXIV.

Duos circulos inaequales CBD, IKL, contingant aequalea lineae AB, GH : & ex punctis A, G, educantur secantes ACD, AFE, GIK, GML continentes angulos aequales EAD, LGK : iunganturque FC, ED, MI, LK.

Dico reciprocam esse triangulorum EAD, KGL, MIG, FAC proportionem.



Prop.74. Fig. 1.

Demonstratio.

Quoniam tangentes AB, GH aequales sunt, patet [c] rectangula DAC, KGI aequalia esse: igitur & rationes AD ad KG, & IG [d] ad CA aequales sunt. Similiter cum rectangula EAF, LGM aequalium tangentium quadratis aequantur, inter se erunt aequalia: quare & rationes EA ad LG, MG ad FA eaedem sint, si igitur rationibus aequalibus AD ad KG, & IG ad CA, aequales addantur rationes, EA ad LG, & MG ad FA, erit ratio composita ex rationibus AD ad KG, & EA ad LG aequalis compositae ex rationibus IG ad CA, & MG ad FA; hoc est ratio trianguli EAD ad LGK triangulum aequalis rationi trianguli MGI, ad FAC triangulum; cum ob angulorum aequalitatum A G, rationem ex lateribus habeant compositam. Unde veritas patet propositionis. *c 36 Tertii ; b 24 Sexti.*

L2.§1. *PROPOSITION 74.*

Two lines of equal length AB and GH are tangents to the unequal circles CBD and IKL ; from the points A and G the secants ACD, AFE, GIK, GML are drawn that contain equal angles EAD and LGK : the lines FC, ED, MI, and LK are joined.

I say that the triangles EAD, KGL; MIG, FAC are in reciprocal proportion.

Demonstration.

Since the tangents AB and GH are equal, it is apparent that the rectangles [c]DAC and KGI are equal: and therefore the ratios AD to KG, and IG [d] to CA are equal. Similarly, since the rectangles EAF and LGM are equal to the squares of the tangents, they are equal to each other: and whereby the ratios EA to LG and MG to FA equal. Therefore, if the equal ratios AD to KG, and IG to CA, are added the equal ratios EA to LG, and MG to FA, the common ratio of AD to KG and EA to LG is equal to the common ratio of IG to CA, and MG to FA; that is the ratio of triangle EAD to triangle LGK is equal to the ratio of triangle MGI to triangle FAC; also as the angles subtended A and G are equal, the ratio is composed from the sides alone. Thus the truth of the proposition is apparent. *c 36 Tertii ; b 24 Sexti.*

[For $AB^2 = AC.AD = AF.AE = GH^2 = GI.GK = GM.GL$; and $AD/GK = IG/CA$ and $EA/LG = MG/FA$. Hence, $(AD/GK).(EA/LG) = EA.AD / LG.GK = \Delta EAD/\Delta LGK$;
while $(IG/CA).(MG/FA) = MG.IG / CA.FA = \Delta MIG/\Delta FAC$. Hence $\Delta EAD/\Delta LGK = \Delta MGI/\Delta FAC$