[1]

QVADRATVRAE
CIRCULI
LIBER PRIMUS.
DE
LINEARVM POTENTIIS.
ARGVMENTVM.

Liber hic fere Lemmaticus est, quemadmodum & alter, qui de Circulorum variis proprietatibus tractat. Porrò quo magis materia Lectori admanum sint, omnem in tres partes dividere placuit.

Prima quidem maximè circa linearum proportionem versatur.

Secunda varias trianguli affectiones exhibet.

Tertia illas linearum contemplatur proprietates, quae earum potentias concernunt.

This book is generally concerned with a lemma [*i. e. the harmonic ratio, whereby a line segment is divided internally and externally in the same ratio*], *as with the following one which draws out various properties of circles. Again, so that more material should be at hand to the reader, it was determined to set everything out in three parts.*

The first part is mainly concerned with the proportions of lines.

The second shows the various uses of triangles.

The third considers these properties of lines which are concerned with powers.

PARS PRIMA.

De variae linearum inter se proportione.

PROPOSITO PRIMA.

Sint AB, BC,CD,DE, lineae in continua proportione; sit autum media FG, inter AB, BC; & inter AB, CD; media GH: Denique inter AB. DE. media sit HI. Dico AB. FG. GH. HL. lineas esse in continua analogia

Demonstratio.

Quadratum AB est ad rectangulum AB.C. [a] ut AB. ad BC. & ABC. rectangulum ad AB.CD. rectangulum est ut BC. ad CD. id est ut AB, ad BC. Eodem modo rectangulum super AB.CD ad AB.DE rectangulum, est ut CD ad DE ; id est AB ad BC. Quadratum autem AB, ad FG. quadratum, est ut AB ad BC. Igitur quadraturo AB. est ad quadratum FG, ut ABC rectangulum, ad rectangulum AB.CD. id est ad quadratum GH. Unde eandem rationem continuant quadrata AB.FG.GH.HI. ac proinde ipsae lineae; quod fuit demonstrandum.

[a] 1.Sexi.

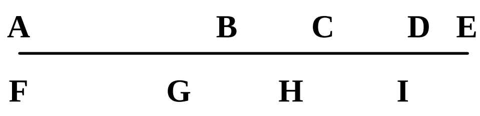
PART ONE.

Concerning the various proportions of line segments amongst themselves.

L1.§1. *PROPOSITON 1.*

Let AB, BC,CD, and DE be lines in continued proportion; moreover let FG be the mean proportion between AB and BC, and GH the mean proportion between AB and CD, and HI the mean between AB and DE.
I say that AB, FG, GH, and HL are analogous lines in continued proportion.

Demonstration.



Prop.1. Fig.1

The square on AB is to the rectangle AB.BC as AB to BC, and the rectangle AB.BC is to the rectangle AB.CD as BC to CD: that is, as AB to BC. In like manner, the rectangle AB.CD is to the rectangle AB.DE as CD to DE, that is as AB to BC. Moreover, the square on AB is to the square on FG as AB to BC, [as FG.FG is equal to AB.BC]. Therefore the square on AB is to the square on FG, as the rectangle AB.BC is to the rectangle AB.CD, that is to the square on GH. Hence the squares on AB, FG, GH, and HI continue in the same ratio, and so on for the rest of the points on the line; which was to be shown.

L1.§1. Prop. 1 Note:

We use the convention that for points such as A, B, C, D, ... in sequence, the segment AB = a, BC = b,..., etc, using the first letter only. For the given segments in proportion, we are given that : $a/b = b/c = c/d = d/e$, etc., from which it follows that $a^2/ab = a/b = ab/ac = b/c = ac/ad = c/d = ad/ae = d/e$; also, since $f^2 = ab, g^2 = ac, h^2 = ad,$ etc.; then $a^2/f^2 = a/b; f^2/g^2 = b/c; g^2/h^2 = c/d,$ etc. Thus, $a/f = f/g = g/h$, etc, and so these lines are also in continued proportion.

[2]

PROPOSITO II.

Circulum ABC contingant rectae DA, DC. & ductâ AC, ponatur altera DG occurrens in H. & circulo in G & I. Dico rectam DG. sectam esse in I & H. ut sit DG ad GH. quemadmodum est DI ad IH.

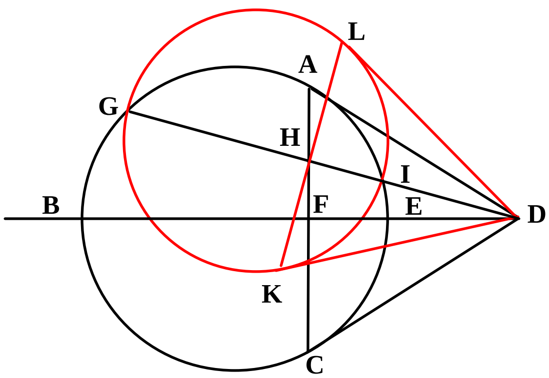
Demonstratio.

Super IG. diametro, describatur circulus GLK & per H punctum erigatur normaliter HL, vel HK, ad diametrum GI: ponaturque DL, erit itaque GHI rectangulo[a], hoc est AHC, aequale rectangulum LHK, sive LH quadratum ; pertinent igitur ad eundem [b] circulum puncta LAKC. iuncta igitur DL est aequalis tangenti DA, adeoque tangentinam LD quadratum est aequale quadrato LH; id est rectangulo GHI. id est [c] AHC unacum quadrato HD: id est quadratis HF, FD. unde LD quadrato rectangulum [d] IDG aequale; ac proinde cum IG. diameter sit circuli GLI. erit per Pappum [e] ut DG ad GH. ita DI ad IH. quod fuit demonstrandum.
a.35.Tertii; b.ibid.; c. 34.Tertii; d. 56. Sexti; e. Lib.7.pr.155.

L1.§1. *PROPOSITON 2.*

The lines DA and DC touch the circle ABC, and the line AC drawn, a line DG is drawn meeting the former line in H and the circle in G and I. I say that the line DG is divided by the points I and H so that [the ratio] DG to GH shall be as DI to IH.

Demonstration.



Prop.2. Fig.1

The circle GLK is described upon the diameter IG , and through the point H the line HL or HK is erected normal to the diameter GI. The line DL is put in place, and thus the rectangle GH.HI [a], that is AH.HC, is equal to the rectangle LH.HK or the square on LH ; hence the points L,A K, and C pertain to lie on the circumference of another circle[b]. Therefore the connected line DL is equal to the tangent DA [*i.e.* D is the centre of the third circle, as it lies on the intersection of the two perpendiculat bisectors of the chords AC and KL of this third circle.], and indeed the square on the tangent LD [from the symmetry of the diagram LD is also a tangent] is equal to LH squared, that is [the same as] the rectangle GH.HI or [c] AH.HC, together with the square of HD, that is also the sum of the squares on HF and FD. From which the square on LD is equal to the rectangle [d] ID.DG; and hence since IG is the diameter of the circle, the required ratio shall be according to Pappus [e] : as DG to GH, thus DI to IH. Q.e.d.

a.35.Tertii; b.ibid.; c. 34.Tertii; d. 56. Sexti; e. Lib.7.pr.155.

L1.§1. Prop. 2 Note:

The geometric proof makes use of well-known theorems. We present an algebraic proof as follows: Let HD = *b*, and GH = R + *a*, where GI = 2R, the diameter of the upper circle. Triangle GLI is right-angled in a semi-circle with centre O, not shown on the diagram, hence $LH^2 = GH.HI = (R + a)(R - a) = ab$, the final relation is true since OLD is also a triangle in a semi-circle : hence

$$R^2 = a(a+b), or \quad \frac{a+b}{R} = \frac{R}{a}; \frac{a+b}{R} \pm 1 = \frac{R}{a} \pm 1$$

$$\frac{a+b \pm R}{R} = \frac{R \pm a}{a}; hence \quad \frac{a+b+R}{R+a} = \frac{R}{a} = \frac{a+b-R}{R-a};$$

$$or \quad \frac{DG}{GH} = \frac{DI}{IH}.$$

Scholium.

Quoniam multoties in hoc opere recurret divisio lineae, prout praesenti propositione exposita est; operae pretium existimavi, aliquid illi peculiare nomen adiungere, quo illius cognito magnis innotescat; placuit autem nomenclaturam illi appropriare, sectionem lineae, secundùm mediam et extremam rationem proportionalem; quoniam similitudinem non exiguam habet, cum ea divisione, quae in elementis vocatur ratio mediae & extremae: differt tamen ab ea, quod illa solum sit aequalitatis, hac vero de qua agimus, omnem omnino proportionem admittat; solam aequalitatis excludens.

Note.

Since often in this work there will be recourse to the division of a line, as shown with the present proposition, I have considered it a valuable aid to add some special name by which the section may become better known. Moreover, it has been decided to appropriate the nomenclature for the sections of a line following that for the ratio of mean and extreme proportions, since the ratio we use has considerable similarity to that division, which in elementary work is called the ratio of the mean and extreme. However

our ratio is different from that ratio, which is for equality alone - indeed for the things we do here, all proportions are admitted; only equality being excluded.
[*In what follows, numerous situations where this ratio arises, and even its invaraince, are investigated : all of Gregory of St. Vincent's proofs of course are in the frame of Euclidean geometry. However, in some of the propositions we recognize properties from the realms of projective geometry , in which case we now talk about the invariance of the cross-ratio (AC/CB)/(AD/DB)*].

PROPOSITO III.

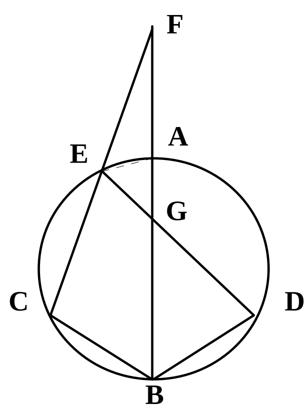
Ex termino diametri circuli ABC: sumantur arcus hinc inde aequales, BC.BD. & ducta qualibet CF, quae occurrat perimetro in E, ponatur DE intersecans AB in G.
Dico rectam FB. sectam esse in A & G. media & extrema ratione proportionali.

Demonstratio.

Demonstratum invenies libro nostro de Circulis, propos.101. rectam BF ad FA. eandem habere rationem, quam BG.GA. hoc est totam FB. ad alterum extremorum FA, eandem rationem obtinere, quam BG alterum extremum, habet ad medium AG ; ergo secta est recta FB. in A & G. media & extrema ratione proportionali. Secundam explicationem praecedenti Scholio factam.

L1.§1. PROPOSITON 3.

From the ends of a diameter of the circle ABC: the equal arcs BC and BD are taken, and any line CF is drawn, which crosses the perimeter in E, and DE is placed cutting AB in G. I say that the line FB is cut by A and G and that the mean and extreme ratios of the lengths are in proportion.



Prop.3. Fig.1

Demonstration.

You will find the demonstration in our book *Concerning Circles*, *Prop.101*. The lines BF to FA are in the same ratio as BG to GA. That is the whole length FB to the external line FA, maintains the same ratio as the internal line BG has to the middle line AG ; therefore the line FB cut by A & G has its points proportional in the mean and extreme ratio. Following the explanation made in the preceding Note.

L1.§1. Prop. 3 Note: EA/BD = AG/GB = EA/CB = FA/FB; as triangles EAG & DBG are similar, and likewise triangles FEA & FBC. Along the same lines as Prop. 2.

[3]

PROPOSITO IV.

Secta sit AB. in C & D. media & extrema ratione proportionali, & composita ex altera extremarum, & media, Verbi gratia CB, bifariam secetur in E.
Dico CE quadratum, rectangulo DEA, aequale exsurgere.

Demonstratio.

Producatur CA. in F, ut CA, AF aequales sint inter se : quoniam est ut BA ad AC ;ita BD ad DC ; erit componendo. BA cum AC. hoc est BF. ad FA. id est AC ut BC ad DC. sed totius BF, dimidia est AE, ipsius quoque BC, dimidia est EC, per constructionem.

Igitur AE, ad AC; est ut EC ad CD, & convertendo ut AE. ad CE ita est, EC ad DE. ergo AE.CE.DE. sunt in continue ratione. Unde & quadrato a CE. rectangulum AED aequale. Quod fuit demonstrandum.

a 36.*Sexti*.

L1.§1. *PROPOSITON 4.*

AB is divided by C & D with the mean and extreme ratios in proportion, and a line is formed by adding the extreme and the other mean length. For the sake of an example CB is bisected in E. In this case, the square CE comes to equal the rectangle DE.EA.

Demonstration.

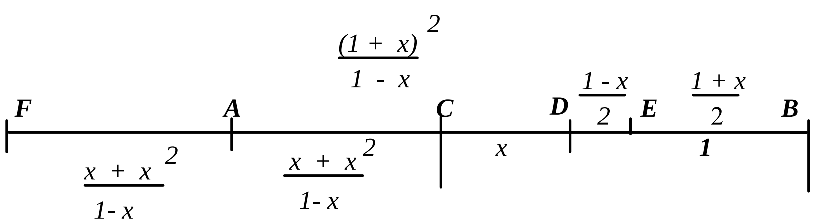
CA is produced to F in order that CA and AF are equal to each other. Since: BA to AC is as BD to DC. BA is to be added with AC. That is : BF to FA (or AC) as BC to DC; but AE is half of the whole length BF; and by construction, EC is half of BC too.

Prop.4. Fig.1

Therefore, we have AE to AC as EC to CD, & in turn, as AE to CE, thus EC to DE. Hence, AE, CE & DE are in a continued ratio. Thus the square on CE [a] is equal to the rectangle AE.ED. Q.e.d. a 36.*Sexti.*

L1.§1. Prop. 4 Note:

By definition, BA/AC = BD/DC, then (BA + AC)/AC = (BD + DC)/DC, or BF/AC = BC/DC. Then AE/AC = CE/DC, and AE/CE = AC/DC ; but also, (AE - AC)/AC = (CE - DC)/DC, or CE/AC =DE/DC, giving AC/DC = CE/DE = AE/CE, as required;
i. e. the lengths of the three consecutive segments AE, CE, DE are in proportion. In terms of algebra, this amounts to $(1 + x)^2 /2(1 - x)$, $(1 + x)/2$, and $(1 - x)/2$:
thus, starting from DE, the common ratio is $(1 + x)/(1 - x)$



Prop.4. Fig.2

PROPOSITO V.

Ponatur tres in continua ratione ED, EC, EA. & EB fiat aequalis EC , & ipsi CA. aequalis AF. dico AB. in C & D. divisam esse media & extrema ratione proportionali.

Demonstratio.

Quonium est ratio continuata, trium ED.EC.EA. erit quoque dividendo, DE ad DC ut EC ad CA. & componendo, ut CE ad DC sic AE ad AC. quare & BC ad DC ita BF. ad AC. hic est ad AF. & dividendo BD ad DC. ut BA ad AF. hic est , ad AC. Quod fuit demonstrandum.

L1.§1. *PROPOSITON 5.*

Three line segments ED, EC, EA are placed in a continued ratio. EB is made equal to EC, similarly CA is equal to AF. I say that AB is divided by C & D with the extreme and mean ratios to be in proportion.

Prop.5. Fig.1

Demonstration.

Since the ratios of the three segments ED, EC, EA are in continued proportion. By subtraction, the ratio DE to DC as EC to CA is given too. And by addition, as CE to DC thus AE to AC. Whereby as BC to DC thus BF to AC, that is as AF, & by division BD ad DC, as BA to AF, that is, to AC. Q.e.d.

L1.§1. Prop. 5 Note:

This theorem is the converse of the previous one. We are given : EC/ED = EA/EC ; hence (EC - ED)/ED = (EA - EC)/EC or CD/ ED = AC/EC, or DE/DC = EC/CA = EB/CA as required on inversion. Again, in a similar manner, CE/DC = (DC + DE)/DC = (EC + CA)/CA = AE/CA = AE/FA. Hence, CB/DC = BF/CA, as CB = 2.CE, and BF = 2.AE ; and by subtraction, (CB - DC)/DC = (BF - CA)/CA or BD/DC = AB/CA.

Corollarium.

Duo hinc consequuntur ; primum, rectangulum BDC. rectangulo ADE. esse aequale, alterum BAC. rectangulo aequale existere, EAD rectangulum. Primi demonstratio inde patet : quoniam BC. in E. secta est aequalia, & non aequalia in D. rectangulum sub inaequalibus segmentis totius BDC. unà cum [a] quadrato quosd sit ab intermedia DE. aequale est ei, quod à dimidia CE. describitur quadrato. Rursus cum AE sectasit in D. utcunque erit rectangulum AED. [b] aequale rectangulo ADE, & quadrato DE. sed iam ostensum est, AED rectangulum quadrato CE aequale. Igitur BDC rectangulum, unà cum DE quadrato, aequale est ADE rectangulo, unà cum DE quadrato. Sublato itaque communi DE quadrato, residuum erit BDC rectangulum ; rectangulo ADE aequale.

Alterum quoque sic manifestum erit. Quoniam BC in E, secta est bifariam , ipsique additur pars CA, erit rectangulum BAC, [c] unà cum quadrato CE, adquale quadrato AE; sed cum secetur AE in D, utcunque, erit quadratum AE, aequale duobus [d] rectangulis AED, & EAD. Igitur rectangulum BAC, unà cum quadrato CE, aequle est duobus rectangulis AED, EAD. sed horum alterum AED, ostendum est quadrato CE. esse [e] aequale. his itaque detractis, residium manebit BAC rectangulum, rectangulo EAD aequale.

a. 5.Secundi ; b. 5.Secundi ; c. 6.Secundi ; d. 4.Secundi ;e. 4.Huius.

Prop.5. Fig.1

Corollary.

There are hence two consequences ; first, the rectangle BDC is equal to the rectangle ADE, the other rectangle BAC proves to be equal to the rectangle EAD. The first demonstration is apparent in this way: since BC has been cut equally by E, and not equally by D, the rectangle BDC under the unequal division of the whole segment together with the square of the intermediate length DE is equal to the square described on the half segment CE. [i.e. $CE^2 = (CD + DE)^2 = CD^2 + 2.CD.DE + DE^2 = DE^2 + CD.(CD + 2.DE) = DE^2 + BD.DC$]. Again the section AE is cut in D, and the rectangle AED [b] is equal to the rectangle ADE and the square DE in anycase. [i. e. $AE.ED = AD.DE + DE^2$]. But now it has been shown above that the rect. AED is equal to the square CE [$AE.ED = CE^2$ by the common ratio property]. Therefore the rect. BDC,

together with the square DE, is equal to the rect. ADE, together with the square DE. Therefore by subtracting the common square DE, the rect. BDC will be left, equal to the rect. ADE .
[*i. e.* $DE^2 + BD.DC = CE^2 = AE.ED = AD.DE + DE^2$; hence $BD.DC = AD.DE$ as required.]
The other consequence is made plain too [*i.e.* rect. BAC = rect. EAD]. Since BC is bisected in E, and the part CA is added to it, the rect. BAC[c], together with the square CE will be equal to the square AE [i. e. $AE^2 - CE^2 = (AE - CE)(AE + CE) = BA.AC$ = rect. BAC]; but as AE is cut in D, the square AE will be equal to the sum of the two rectangles [d] AED and EAD in any case [i.e. $AE^2 = AE.(AD + DE)$] . Therefore the rect. BAC, together with the square CE, is equal to the sum of the two rectangles AED and EAD. But one of these AED, has been shown to be equal to the square CE[e] . And thus when the square is taken away, the remainder shows that the rect. BAC equal to the rect. EAD.
[*i. e.* rect. BAC $+ CE^2 = AE^2$ = rect. EAD + rect. AED = rect. EAD + CE^2, etc].

L1.§1. Prop. 5 Corollary Note:

This extra note is more by way of a neumonic to aid the memory than anything else.

In the first case, instead of considering the ratio BD/DC, we consider the rectangle BD.DC or BDC in the shortened version used in the text. This rectangle is found to equal the rectangle formed by the separation of the mean points A and D, and the distance of D from the mid-point of CB, *i. e.* BD.DC = AD.DE.

In the second case, correspondingly, the rectangle from the mean ratio is BA.AC, and this is equal to the rectangle formed from the separation of the mean points AD and the length from the other mean A to the mid-point E, i. e. EA.AD = BA.AC.

[4]

PROPOSITO VI.

Lineâ AB. sectâ in C & D. secundùm mediam & extremam rationem proportionalem, fiat circulus super AD. diametro, cuius centrum E, erigatur deinde BFG. normalis ad AB. ex puncto B.

Dico rectas omnes per C. ductas quae circulo in K. & L. occurrunt & FB. normali in aliquo puncto G. divisas esse secundum mediam & extremam rationem proportionalem.

Demonstratio.

Ex E. centro Circuli ALD, ducatur recta EH, normalis ad rectam KG; erit itaque KL. bifariam [a] divisa in H; & quia rectus est uterque angulorum ad H, & B, Circulus diametro GF descriptus, percurret puncta H, [b] & B. Rectangulum itaque BCE, [c] aequale est GCH rectangulo; est quoque rectangulum BCE, rectangulo ACD aequale. per Corollarium praecedentis propositionis; huic autem aequale est d rectangulum KCL. Igitur rectangulo GCH, rectangulù KCL, aequale erit; adde ergo utriq; rectangulo quod ex CH. fit quadratù, erit denuo rectangulum KCL, unà cum quadrato CH, aequale rectangulo GCH, unà cum quadrato CH; sed rectangulo KCL, una cum quadrato CH ostendum est aequari rectangulum GHC, sive GCH unà cum quadrato CH id est quadratum [e] HL. Itaque rectangulum GHC, aequale est HL quadrato : utque CH ad HL . ita eadem HL ad HG. quo fit, ut cum sit HL aequalis HK. Ipsa GH. [f] in C. & L. secta sit secundum rationem extremam & mediam proportionalem, per praecedentem propositionem. Quod fuit demonstrandum.

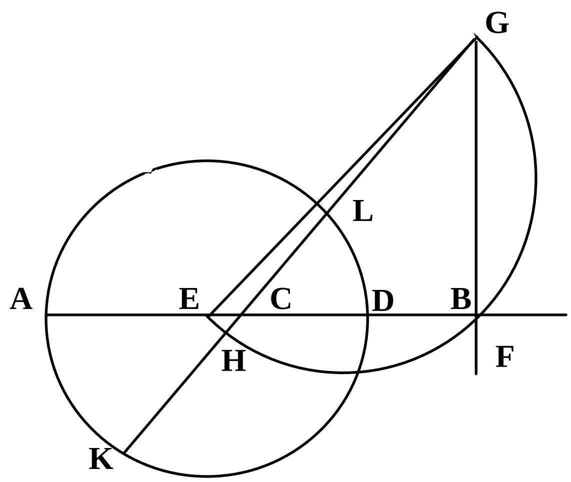
a. 3. Tertii ; b. 31.Tertii ; c. 34.Tertii ; d. ibid. ; e. 5.Secundi; f. 5.Huius.

L1.§1. *PROPOSITON 6.*

The line AB is cut in C and D in the ratio of mean and extreme proportions, and a circle with diameter AD is constructed with centre E. The normal BFG to AB is then erected from the point B.

I say that all the lines drawn through C which cross the circle in K and L and the normal FB in some point G, are also divided in this ratio.

[In this case, the ratio is AC/CD = AB/BD.]



Prop.6. Fig.1

Demonstration.

From the centre E of the circle ALD, the line EH is drawn normal to the line KG; and KL will thus be bisected [a] in H; and because both the angles at H and B are right, a circle with diameter GE [text has GF] can be described, that will pass through the points H and B [b]. And thus the rect. BCE [c] is equal to the rect. GCH; also, rect. BCE is equal to the rect. ACD, from the corollary to the preceding proposition; but the rect. KCL is equal to ACD [d] ; therefore rect. GCH is equal to rect. KCL; therefore the square CH is added to each rectangle ; it will be the new rect. KCL together with the square CH that is equal to the rect. GCH together with the square CH; but the rect. KCL[r] with the square CH is shown to be equal to the rect. GHC, or the rect. GCH together with the square CH, that is [both are equal] to the square [e] HL. And thus the rect. GHC is equal to the square HL: and thus as CH to HL, so HL to HG; from which it shall be, as HL is equal to HK. The section GH itself [f] shall be divided in C and L according to MERP, by the preceding proposition. Q.e.d.

L1.§1. Prop. 6 Note:

Starting from GCH = KCL, we then have GCH + CH^2 = (GC + CH).CH = GH.HC = GHC; again, KCL = (HL - CH)(HL + CH) = HL^2 - CH^2, or KCL + CH^2 = HL^2 ; thus GH.HC = HL^2 , and HC, HL, & HG are in continued proportion as required.

PROPOSITO VII.

Esto ABC trianguli basis AC divisa bifariam in D ; actâque per B lineâ BE, parallelâ basi AC, agatur per D linea quaecunque EF, occurrès trianguli ABC lateribus, in G.F.

Dico EF lineam in D & G punctis extrema & media ratione proportionali esse divisam: id est esse ut EF ad FD, sic EG ad GD.

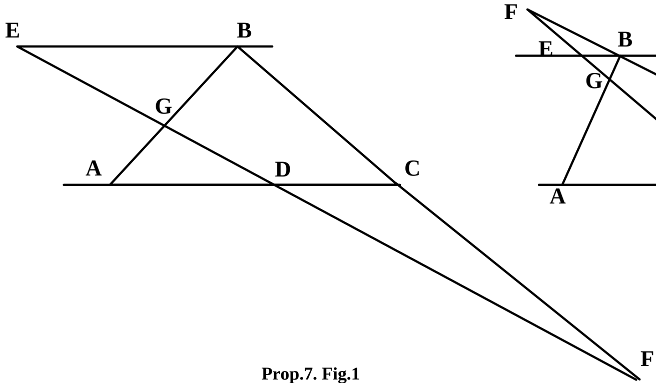
Demonstratio.

Quoniam EB, & AC lineae aequidistant, erit ut EB ad DC , sic EF ad DF; sed est ut EB ad AD, sic EG ad GD, igitur ut EF ad FD, sic EG ad GD. Quod erat demonstrandum. In 2. figura est FD linea in E & G divisa media & extreme proportionali.

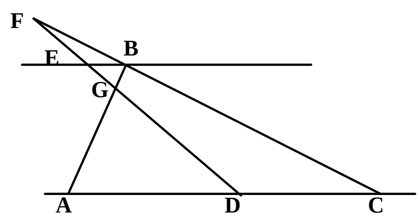
L1.§1. PROPOSITON 7.

Let the base AC of triangle ABC be bisected in D, and a line BE is drawn through B parallel to the base AC. Some line EF is drawn through D, crossing the sides of the triangle ABC in G and F.

I say that the line EF is divided in D & G in points in the extreme & mean ratio of proportion: that is they are as EF to FD, so EG to GD.



Prop.7. Fig.1



Demonstration.

As the lines EB and AC are parallel, EB is to DC as EF is to DF; but as EB is to AD, thus EG is to GD, therefore as EF is to FD, so EG is to GD. Q. e. d .

In figure 2, the line FD is divided proportionally in the mean and extreme ratio by E & G.

[5]

PROPOSITO VIII.

Sit angulus ABC divisus bifariam rectâ in BD, ad quam erecta ex B normali BE, agatur linea quaecunque EG per punctum D.

Dico EG lineam in F & D, extrema & media ratione proportionali esse divisam.

Demonstratio.

Ponatur per D, linea AC,parallela rectae BE: occurrens anguli ABC lateribus, in A & C. Quoniam igitur AC linea aequidistat, rectae EB. quae normalis ponitur ad lineam BD, erunt anguli ADB. CDB recti; sunt autem per constructionem, & anguli ABD, CBD inter se aequales; & BD linea communis, igitur triangulum ABD aequale est triangulo CBD, & AD basis, aequalis basi DC : igitur per praecedentem ut EG ad GD, sic EF ad FD, Quod erat demonstrandam.

In 2. figura linea GD divisa est, secundum mediam & extremam proportionalem.

L1.§1. PROPOSITON 8.

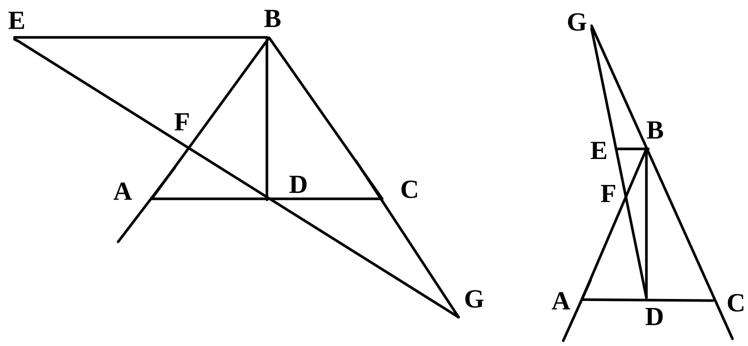
Let the line BD bisect the angle ABC, to which the normal BE has been erected from B, while some line EG is sent through the point D.

I say that the line EG is divided by F & D in the extreme & mean ratio of proportionality.

Demonstration.

Draw the line AC through D, parallel to the line BE: crossing the sides of the angle ABC in A & C. Since the line AC is parallel to the line EB, which has been placed normal to the line BD, the angles ADB and CDB are right; but the angles ABD and CBD are equal by construction, and BD is a common line,

therefore the triangle ABD is congruent [equal in text] to the triangle CBD, & the base AD is equal to the base DC. Therefore by the preceding as EG to GD, thus EF to FD. Q. e. d.
In figure 2, the line EG is divided by F & D in the extreme and mean ratio of proportionality.



Prop.8. Fig.1

PROPOSITO IX.

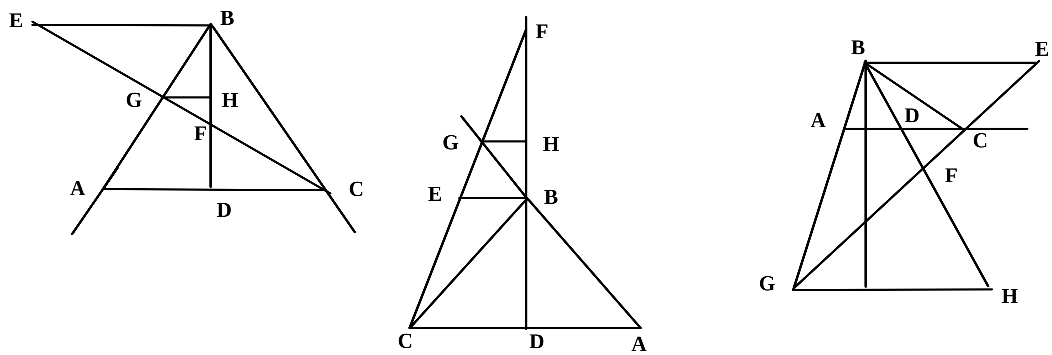
Esto ABC trianguli basis AC in D bifariam divisa , iunctisque BD, agatur per B linea BE, parallela basi AC; dein ex C, recta ducatur CE, occurrens utcunque lineae BE in E, BD rectae in F, & lateri AB in G. Dico in prima figura : lineam EC; in secunda lineam FC; in tertia rectam EG, extrema & media ratione proportionali esse divisam.

Demonstratio.

Ducatur GH linea, parallela basi AC: Quoniam AC, EB, GH, lineae aequidistant, erit AB ad BG ut CE ad GE; sed est ut AB ad GB, sic AD ad GH, hoc est DC ad GH, hoc est CF ad FG, igitur ut CE ad EG, sic CF ad FG. Q.e.d.

L1.§1. *PROPOSITON 9.*

Let the base AC of triangle ABC be bisected in D, and with BD joined, the line BE is sent through B, parallel to the base AC; then from C, the line CE is drawn, crossing the lines BE in E, BD in F, & the side AB in G, in some manner. I say that in the first figure : the line EC, in the second figure the line FC, and in the third figure the line EG, is cut so that the mean and extreme ratios are in proportion.



Prop.9. Fig.1

Demonstration.

The line GH is drawn parallel to the base AC. Since the lines AC, EB, and GH are parallel, AB is to BG as CE is to GE. But as AB is to GB, thus AD to GH, that is DC to GH, that is CF to FG. Therefore as CE to EG, thus CF to FG. Q.e.d.

[6]

PROPOSITO X.

Demittantur ex A puncto, lineae tres AB, AC, AD, quae angulos constituant BAC, CAD rectis minores. Ponatur autem quaevis ED occurrens ductis ex A. lineis, & divisa in B, & C. extrema & media ratione proportionali esse divisam.

Dico omnes lineas ex E ductas, occurrentes rectis AB, AC, AD, ab iisdem media & extrema ratione proportionali dividi.

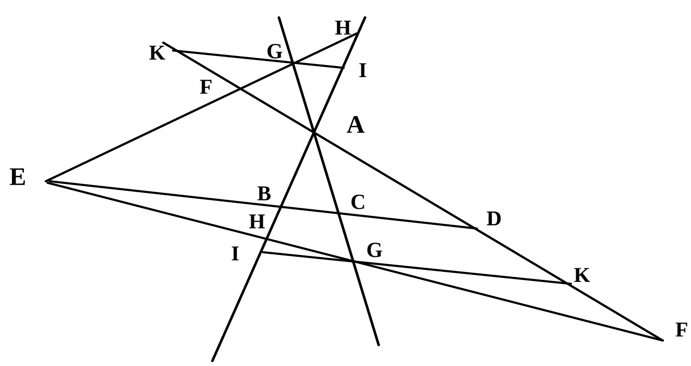
Demonstratio.

Ducatur ex E, quaevis linea EF, occurrens lineae AD in F, rectae AC in G, & AB lineae in H. dein per G, agatur linea IK, parallela rectae BD. Quoniam BD, IK lineae aequidistant; erit ut DC ad CB, sic KG ad GI; sed per hypothesim est, DC ad CB, ut DE ad EB, igitur erit ut DE ad EB, sic KG ad GI, & permatando ut DE ad KG, sic EB ad IG. est autem ut DE ad KG, sic EF ad FG, & ut EB ad IG, igitur ut EF ad FG, sic EH ad HG, & permutando ut EF ad EH, sic FG ad GH. Q. e. d.

L1.§1. PROPOSITON 10.

Three lines AB, AC, AD are drawn from the point A, of which the constituent angles BAC and CAD are less than right angles. Moreover some line ED is drawn, crossing the lines drawn from A, and divided at B and C in the extreme & mean ratio of proportionality.

I say that all the lines drawn from E, crossing the lines AB, AC, AD, are divided in the mean & extreme proportional ratio.



Prop.10. Fig.1

Demonstration.

Some line EF is drawn from E, crossing the lines AD in F, AC in G, & AB in H, then through G, the line IK is sent, parallel to the line BD. Since BD and IK are parallel [equidistant in text] lines; DC is to CB, thus as KG to GI. But by hypothesis, DC is to CB, as DE to EB, therefore as DE is to EB, so KG ad GI, and by interchanging, as DE to KG, so EB to IG; but as DE to KG, so EF to FG, & as EB to IG, thus as EF to FG, so EH ad HG, & by interchanging, as EF to EH, so FG to GH. Q.e.d.

L1.§1. Prop. 10 Note:

DC/ CB = KG/ GI; but by hypothesis, DC/CB = DE/EB, therefore DE/EB = KG/ GI, and by interchanging, DE/KG = EB/IG; but DE/KG = <u>EF/FG</u>, & EB/IG = <u>EH/HG</u>, & by interchanging the underlined terms, as EF/EH, so FG/GH, as required.

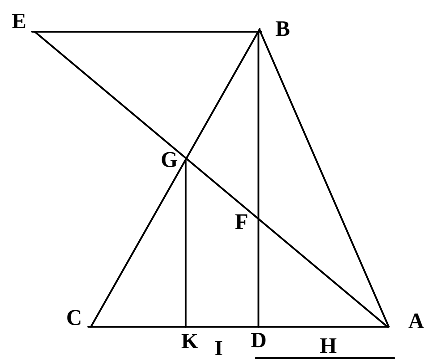
PROPOSITO XI.

Esto ABC trianguli basis AC, cui per B, verticum agatur aequidistans BE; oportet ex A rectam ducere AE, ut AE ad EG. datam habeat rationem maioris inaequalitatis H ad I.

Constructio & demonstratio.

Divisa AC bifariam D, fiat, ut H ad I sic AD ad DK, iunctisque punctis BD, erigatur ex K linea KG, parallela ipsi BD, occurrens BC lateri in G, tum ex A per G, agatur lina AE, secans BD lineam in F, & rectam EB in E; dico factum esse quod petitur. Quoniam FD GK lineae sunt parallelae erit ut AD ad DK, sic AF ad FG,sed AD est ad DK ut H ad I per constructionem igitur, ut H ad I sic AF ad FG est autem ut a AF ad FG sic AE ad EG, igitur ut H ad I sic AE ad EG. Duximus igitur ex A lineam, &c. Q. e. d.
a. IX huius.

L1.§1. *PROPOSITON 11.*



Prop.11. Fig.1

Let the base of triangle ABC be AC, to which the parallel line BE is drawn through the vertex B; it is required to draw the line AE from A, in order that AE to EG shall give the ratio of the larger inequality H to I.

Construction & demonstration.

AC is divided in D so AD to DK as H to I ; with the points B and D joined, the line KG is erected from K, parallel to BD itself, crossing the line BC in G; then the line AE is sent from A through G, cutting the line BD in F, & the line EB in E. I say that the construction is that sought. Because the lines FD & GK are parallel: AD is to DK, so AF to FG, but AD to DK as H to I by construction; therefore, as H to I so AF to FG, but as AF to FG by Prop.9 above so AE to EG, therefore as H to I so AE to EG. Therefore we have drawn the line from A, &c. Q. e. d.

PROPOSITO XII.

Sit AC linea utcumque divisa in D, oportet illi rectam quandam CE adiicere, ut AE tota, in D & C, divisa sit media & extrema ratione proportionali.

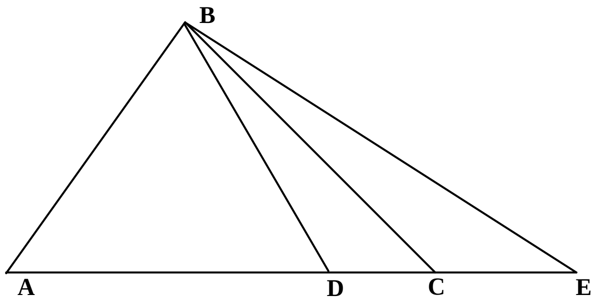
Constructio & demonstratio.

Super AC ut basi constituatur triangulum rectangulum ABC, habens ad B rectum angulum, & ex B demittatur linea BE constituens angulum EBC aequalem angulo DBC ; occurrens AC lineae in E ; dico factum esse quod petitur. Quoniam anguli DBC. CBE per constructionem sunt inter se aequales, & AB linea normalis ad rectam BC, [a] erit AD ad DC ut AE ad EC. datae igitur lineae AC adiecimus, &c. Quod erat postulatum.
a. Per 8. huius.

L1.§1. *PROPOSITON 12.*

Let AC be a line divided somehow at D, it is required that a certain line CE be added on so that the whole line AE is divided in proportion in the mean and extreme ratio at D & C.

[7]



Prop.12. Fig.1

Construction & demonstration.

On AC as base a right-angled triangle ABC is constructed, having the right angle at B, & from B the line BE is drawn making the angle EBC equal to the angle DBC, and meeting the line AC in E. I say that the desired result has been achieved. Because the angles DBC & CBE are equal to each other from the construction , & the line AB is normal to the line BC, by Prop. 8 above , AD is to DC as AE ad EC. Therefore we have added to the given line AC, &c. Which was postulated. [One may consider that the line EG containing the mean and extreme ratio in Prop. 8 has been moved to coincide with the base of the triangle]

PROPOSITO XIII.

Esto ABC trianguli basis AC, in D bifariam divisa, iunctisque BD, ponatur per B linea BE parallela basi AC : tum in EB linea, punctum sumatur quodvis E, ex quo linea demittatur EC, occurrens AB lineae in G, & rectae BD in F; dein ex C recta erigatur CH, secans orthonaliter lineam AB in H. iunganturque puncta HF EH. Dico angulos EHG, FHG esse inter se aequales.

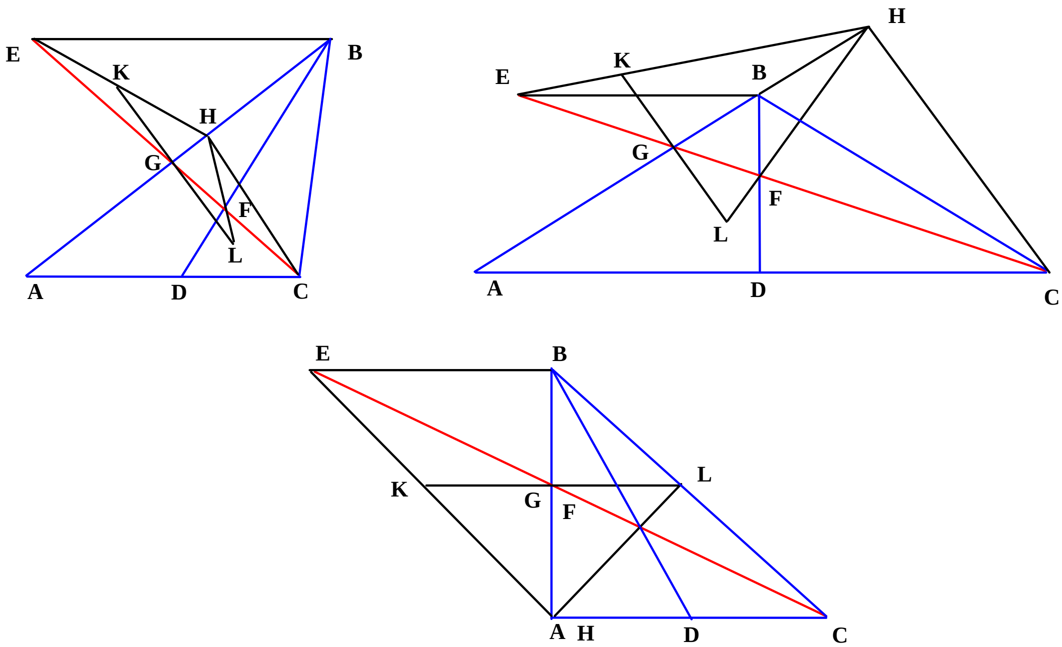
Demonstratio.

Ducatur per G linea KL, parallela rectae HC ;occurrens HE lineae in K, & FH rectae in L. Quoniam igitur HC KG, lineae sunt parallelae, erit ut HC ad KG, sic CE ad EG : sed ut CE ad EG, sic CF [a] est ad FG, igitur ut HC ad KG, sic CF ad FG : est autem ut CF ad FG, sic HC ad GL, igitur ut HC ad KG, sic HC est ad GL; quare KG. GL. lineae sunt inter se aequales. Rursum cum HC lineae, aequidistet linea KG, sit autem & HC normalis. ex hypothesi ad rectam AB, erit & KG linea perpendicularis ad lineam AB, adeoque anguli HGK, HGL recti ; igitur cum HG GL lineae duabus lineis HG GK sunt aequales, & anguli illis contenti recti, erunt HGK. HGL triangula inter se aequalis & similis, & anguli EHG, FHG aequalibus lineis subtensi, aequales. Q. e. d.

a.9. huius.

L1.§1. *PROPOSITON 13.*

Let the base AC of triangle ABC be bisected in D, and with the line BD joined, the line BE is drawn through B parallel to the base AC : then any point E is taken in the line EB, from which the line EC is drawn, crossing the line AB line in G, and the line BD in F. From C the line CH is erected, meeting the line AB at right angles in H. Then the points HF and EH are joined. I say that the angles EHG and FHG are equal to each other.



Prop.13. Fig.1

Demonstration.

The line KL is drawn through G, parallel to the line HC ; crossing the line HE in K, and the line FH in L. Because the lines HC and KG are parallel, HC will be to KG, as thus CE to EG. But as CE to EG, thus CF is to FG, by Prop. 9 of this section; and therefore as HC to KG, thus CF to FG. But as CF is to FG, thus HC to GL, therefore as HC to KG, thus HC is to GL, and hence the lines KG and GL are equal in length. Again, as the line HC is parallel to the line KG, and moreover by hypothesis the line HC is normal to the line AB, and since the line KG is perpendicular to the line AB, therefore the angles HGK and HGL are right; therefore the two lines HG GL are equal to the lines HG and GK, and HGK and HGL are equal and similar triangles, and the angles EHG and FHG are subtended by equal lines. Q.e.d.

[8]

PROPOSITO XIV.

Esto ABC trianguli basis AC, bifariam divisa in D, iunctisque BD, agatur per B linea BE parallela rectae AC, in qua assumpto puncto quovis E, ducatur recta EC: & ex C erigatur linea CF, orthona rectae AB, occurrens EB lineae in G ; dein ponatur EF intersecans BD in I, & BC in H. Ducaturque CI, occurrens lineae EB in K.

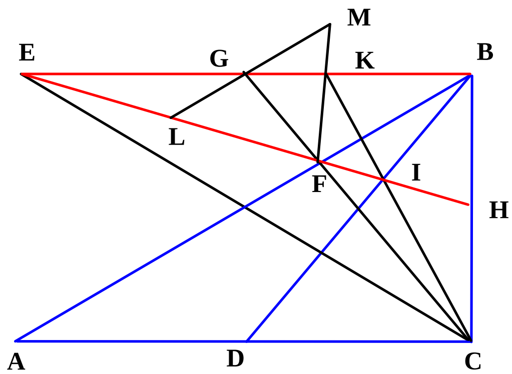
Dico lineam EB in G & K, divisam extrema & media ratione proportionali.

Demonstratio.

Agatur per G linea LM, parallela rectae AB: occurrens FK in M. Quoniam linea GC, per constructionem normalis est ad rectam AB, erunt anguli GFA, GFB recti ; est autem angulus KFB aequalis [a] angulo BFI, id est angulo AFL: igitur reliquus angulus GFL aequalis est reliquo GFK ; rursum cum LM linea aequidistet lineae AB normali ad rectam GC,erunt anguli FGL, FGM recti; igitur cum tam anguli FGM, FGL, quam GFM, GFL sint inter se aequales, & FG linea communis, erunt FGM, FGL triangula, adeoque & latera LG, GM inter se aequalia. Quare ut FB ad LG, sic FB ad GM : sed est ut FB ad LG, sic BE ad EG, & ut FB ad GM, sic BK ad KG, igitur ut BE ad EG, sic BK ad KG. Quod erat demonstrandum.

a. Per 13. huius.

L1.§1. *PROPOSITON 14.*



Prop.14. Fig.1

Let the base AC of triangle ABC be bisected in D, and with BD joined, the line BE is sent through B parallel to the line AC, in which some point E is taken, and the line EC is drawn. The line CF is erected from C, at right angles to the line AB, crossing the line EB in G, then EF is drawn intersecting BD in I, & BC in H. And CI is drawn, crossing with the line EB in K. I say the line EB is divided by G & K in the mean and extreme ratio of proportionality.

Demonstration.

The line LM is drawn through G parallel to the line AB, crossing FK in M. Since the line GC by construction is normal to the line AB, the angles GFA and GFB are right. Moreover, the angle KFB is equal to the angle BFI by Prop. 13, that is to the angle AFL: therefore the remaining angle GFL is equal to the remaining angle GFK ; again as the line LM is equidistant from the line AB to the normal to the line GC, the angles FGL and FGM are right; therefore as with the angles FGM and FGL, so the angles GFM and GFL shall be equal to each other, & the common line FG, the triangles FGM and FGL, and therefore the sides LG, GM are equal among themselves. Thus as FB to LG, so FB to GM : but as FB is to LG, thus BE to EG, & as FB to GM, thus BK to KG, therefore as BE to EG, thus BK to KG. Q. e. d .

PROPOSITO XV.

Constituant AB, BC rectum angulum, & AB maius sit latere BC. oportet AB latus producere in D, ut iunctâ DC. fiant proportionales DC, AD, & CB, AB.

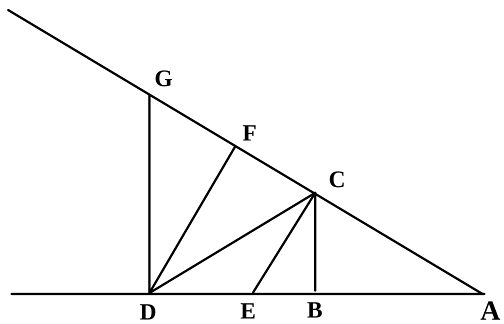
Constructio & Demonstratio.

Ductâ CA, demittatur ex C linea CE, normalis ipsi AC, & alia quedá CD, que angulum DCB constituat duplum anguli CAB. dico factum esse quod petitur. Erigantur ex D lineae DF, DG;& DF quidem aequidistet ipsi EC; DG verò, rectae CB. Quoniam angulus ACE per constructionem rectus est, & CB linea normalis ipsi AE, erunt ABC. EBC triangula similia. & angulun ECB aequalia angulo CAB ; quare & angulus DCB, rectâ CE bifatium est divisus. Rursum cum DG linea. per constructionem aequidistet rectae BC; & FD linea, ipsi EC; erit angulus GDC aequalis angulo DCB, & angulus FDC, aequalis angulo ECD;adeoque. GDC angulus, rectâ FD bifariam divisus; sunt autem & anguli DFG, DFC recti (cum FD linea aequidistet ipsi EC, normali ad AC) & FD linea communis, igitur triangulum DFC, aequale triangulo DFG, & DC latus, lateri GD aequale. Quare CD ad DA ut GD ad DA ; sed est GD ad DA, ut CB ad BA, igitur ut CB ad BA sic CD ad DA. Quoad igitur AB lineam produximus, &c. Quos erat faciendum.

L1.§1. *PROPOSITON 15.*

The lines AB and BC make a right angle, & with the side AB greater than BC. It is required that the side AB be produced to D, in order that DC joined shall make the lengths DC, AD ; CB, AB in proportion.

Construction and Demonstration.



Prop.15. Fig.1

With the line CA drawn, the line CE is drawn from C, normal to AC. A certain other line CD is drawn which makes the angle DCB double the angle CAB. I say that this is the sought construction.

The lines DF and DG are erected from D, and DF indeed shall be parallel [equidistant from] to EC itself, and DG truly parallel to the line CB. Since the angle ACE is right by construction, and CB is normal to AE, then ABC and EBC are similar triangles. The angle ECB is equal to the angle CAB, whereby the angle DCB is bisected by the line CE. Again, since the line DG by construction is parallel to the line BC, and the line FD parallel to EC itself, then the angle GDC is equal to the angle DCB, and also the angle FDC is equal to the angle ECD. Hence the angle GDC is bisected by FD, and moreover the angles DFG and DFC are right (as the line FD is parallel to the line EC, normal to AC) and FD is the common line, hence triangle DFC is congruent [equal] to the triangle DFG, and the side DC is equal to the side GD. Whereby CD is to DA as GD to DA ; but GD is to DA as CB to BA, therefore as CB to BA, thus CD to DA. As long as we have produced the line AB etc. Q.e.d .

[9]

PROPOSITO XVI.

Sint ABC, ADC triangula,inaequalis altitudinis, super eadem base AC constitura ; oportet ducere lineam EF, parallelam base AC, ut EL ad MF datam habeat rationem TV ad XV.

Constructio & Demonstratio.

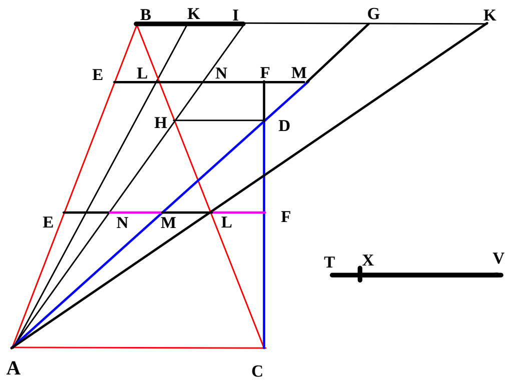
Constituantur per B & D lineae BG, DH parallelae basi AC; & DH quidem occurrat triangulo ABC in H ; dein AD producta, donec BG lineae occurrat in G, agatur per H, linea AHI, occurrens BG lineae in I : dividaturque BI in K, ut BI ad IK datam habeat rationem TV ad VX ; & ea K demittatur linea KA, occurrens BC lineae in L, tum per L ducatur linea EF parallela basi AC, occurrens ABC triangulo in E & L. rectae CD productae in F, & AD lineae in M.

Dico EL ad MF datam habere rationem in TV ad VX. EF recta occurrat Lineae AI in N. Quoniam HAD, HCD triangula super eadem basi HD & inter eadem parallelas constituta sunt, erunt LF, MN lineae inter se aequales : demptâ igitur communis NF, vel addica ML, erunt LN, FM . lineae aequales; igitur ut EL ad LN sic EL est ad MF, sed EL est ad LN, id est BK ad KI ut TV ad VX; igitur ut TV ad VX ;sic EL est ad MF; duximus igitur lineam EF, &c. Quod erat praestandum.

Idem patet evenire si dicta duo triangula, aequales habeant bases in directù positas.

L1.§1. *PROPOSITON 16.*

Let ABC and ADC be triangles with unequal altitudes, placed on the same base AC. It is reqiured to draw the line EF parallel to the base AC, in order that EL to MF is in the given ratio TV to XV.



Prop.16. Fig.1

Construction & Demonstration.

The lines BG and DH through the vertices B and D are drawn parallel to the base AC, and indeed DH meets the triangle ABC in H. AD is then produced, thus meeting the line BG in G, and the line AHI is drawn through H to meet the line BG in I. BI is divided by K, in order that BI to IK has the given ratio TV to VX. From this point K the line KA is drawn crossing the line BC in L, then the line EF is drawn through L parallel to the base AC, cutting the triangle ABC in E and L, while the line CD is produced to F and the line AD to M.

I assert that EL to MF has the given ratio TV to VX. The line EF crosses the line AI in N. Because HAD and HCD are triangles on the same base HD, and have been placed between the same parallel lines, the lines LF and MN are equal : therefore either by taking/adding each from the common line NF or by adding each to ML, the lines LN and FM are also equal. Hence, as EL to LN thus EL to MF, but as EL is to LN, so BK to KI and TV to VX. Hence, as TV to VX, so EL is to MF; therefore we draw the line EF, &c. Which was to be presented.

The same seems to come about if the two said triangles have equal bases placed in order.

PROPOSITO XVII.

Sit AB linea divisa utcunque in C, oportet illi rectam addere BD, ut tota AD sit ad AB, ut AC est ad BD.

Constructio & Demonstratio.

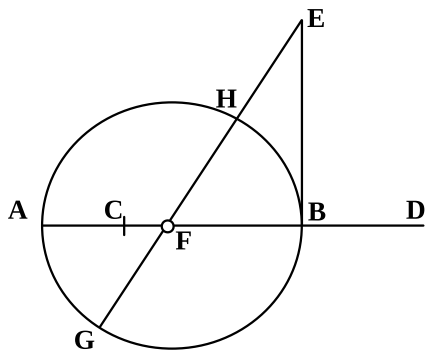
Descripto super AB ut diametro, circulo AHB, erigatur ex B tangens EB, quae sit media inter AB & AC: dein ex E per F centrum circuli AHB recta ducatur BG occurrens circulo in H, addaturque lineae AB quadam BD, aequalis ipsi EH. Dico factum esse quod petitur. Quoniam HE, BD lineae sunt aequales, erunt HEG, BDA rectangula inter se aequalia; sed HEG rectangulum est aequale quadrato EB, id est ex constructione rectangulo CAB; igitur CAB, BDA rectangula sunt inter se aequalia ; sed HEG rectangulum est aequale quadrato EB, id est ex constructione tectangulo CAB; igitur CAB, BDA rectangula sunt inter se aequalia; quare AD est ad AB ut AC ad BD, datae igitur lineae AB quandam adiecimus ut, &c. Quod erat postulatum.

L1.§1. *PROPOSITON 17.*

Let the line AB be divided at some point C. It is required to add the line BD to AB, in order that the total length AD shall be to AB, as AC is to BD.

Construction & Demonstration.

With the circle AHB described on AB as diameter, the tangent EB is erected from B, which shall be the mean between AB and AC [i.e. EB^2 is set equal to AB.AC or CAB]: then from E through F, the centre of the circle AHB, the line BG is drawn crossing the circle in H, and a certain line BD is added to the line AB, equal to EH. I assert that the required task has been performed.



Prop.17. Fig.1

Because the lines HE and BD are equal, the rectangles HEG and BDA are equal to each other; but the rectangle HEG is equal to the square EB, which is equal to the rectangle CAB by construction. Therefore CAB and BDA are equal rectangles; but the rectangle HEG is equal to the square EB, that is from the construction for the rectangle CAB; therefore CAB and BDA are equal rectangles from which [There appears to be a repetition here of the same argument.], AD is to AB as AC to BD. Therefore to the given line AB we have added a certain line, &c. Q. e. d. [HE = BD $\therefore$ rect. HEG = rect. BDA ; but rect. HEG = EB^2 = CAB (by construction)
$\therefore$ rect. CAB = rect. BDA; from which, AD/ AB = AC/BD].

PROPOSITO XVIII.

Datis duabus rectis AB, CD, rectas addere, vel detrahere, in data ratione E ad F; ut compositae vel reliquae datam habeant rationem GH ad HI.

[10]
Constructio & Demonstratio.

1. 1. Datae rectae AC CB sint inter se aequales. & tam ratio E ad F, quam GH ad HI, sit aequalitatis; addantur autem, vel detrahentur in ratione E ad F, lineae BK, DL, patet veritas propositionis.

2. Si fuerit AB maior quam DC; & ratio GH ad HI, maior quoque ratione AB ad CD, ratio autem E & F aequalitatis; Dico aequales addi non posse ut compositae sint in ratione data GH ad HI. Cum enim AB ponatur maior quam CD, fiat LB aequalis ipsi CD; addaturque rectae AB, quaevis BK; & si fieri possit, sit AK ad LK ut GH ad HI. erit igitur ratio AK ad LK maior ratione AB ad LB, id est ad CD; & quia ratio GH ad HI maior ponitur ratione AB ad LB, erit dividendo [a] quoque ratio GI ad IH maior ratione AL ad LB; sed est ut GH ad HI sic AK ad LK ex hypothesi, adeoque ut GI [b] ad IH, sic AL ad LK, igitur & ratio AL ad LK maior est ratione AL ad LB. Quod fieri non potest ; [c] cum LB minor sit ipsa LK. Quare lineae aequales addi non poterunt, ut, &c, Demi vero poterunt aequales, ut reliquae sint in ratione GH ad HI:facta enim LB aequali ipsi CD, fiat ut GI ad IH, sic AL ad LM, cadet M inter L& B; (cum AL ad LB ostensa sit minorem habere rationem, quam GI ad IH) dein rectae MB, fiat aequalis ND: Quoniam igitur per constructionem est ut GI ad IH, sic AL ad LM, erit componendo, AM ad LM, id est ad CN per constructionem ut GH ad HI. Quod erat demonstrandum.
a. 29. Quinti ; b. 17. Quinti ;c. 8.Quinti.

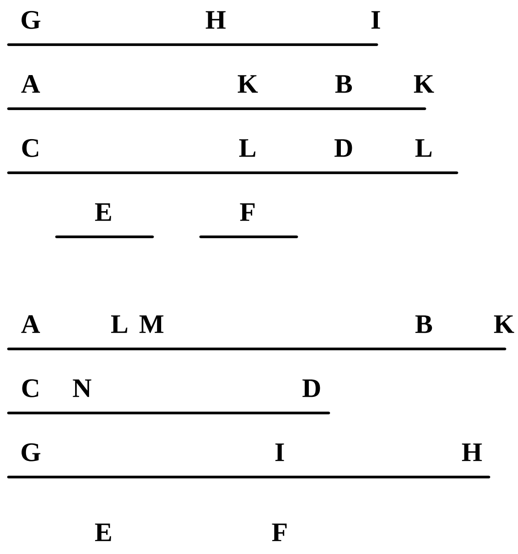
3. Si AB linea rursum fuerit maior recta CD, & ratio GH ad HI minor ratione AB ad CD, ratio autem E ad F aequalitatis, aequales addi poterunt ut compositae rationem habeant GH ad HI : fiat enim LB linea, aequalis rectae CD; quoniam per hypothesim, ratio AB ad LB, id est CD per constuct. maior est ratione GH ad HI, erit [a] dividendo, ratio AL ad LB, maior ratione GI ad IH; facto igitur ut GI, ad IH, sic AL ad LN, patet LN maiorem esse linea LB; nam ratio GH ad HI minor ponitur ratione AB ad LB, id est CD, adeoque & ratio GI [b] ad IH, id est AL ad LN minor est ratione AL ad LB; tum CD lineae addatur quaedam DO aequalis ipsi BN: Quoniam igitur est AL ad LN, ut GI ad HI, erit componendo. AN ad LN, id est CO, ut GH ad HI: Quod erat primum. a. 29. Quinti ; b. ibid.
Iisdem positis, aequales demi non poterunt, ut reliquae datam habeant rationem GH ad HI : auferantur enim aequales KB, MD; & si fieri possit, sit ut GH ad HI sic AK ad LK, id est ad CM. cum igitur per hypothesim, ratio GH ad HU minor sit ratione AB ad LB, id est ad CD,sit autem ut GH ad HU, sic AK ad LK, id est ad CM, erit ratio AK ad LK, minor quoque ratione Ab ad LB, & dividendo, a ratio AL ad LK, minor ratione AL ad LB, Quod fiere non b potest cum LB linea, maior sit linea LK. quare hoc casu demi aequales lineae non poterunt ut reliquae, &c.
a. 29. Quinti ; b. 8.Quinti.

L1.§1. *PROPOSITON 18.*

Lines are to be added or taken from the two given lines AB and CD, in the given ratio E to F; in order that the sum or difference shall have the given ratio GH to HI.

[10]

L1.§1. Prop. 18 Preliminary Note: In general, the lengths E and F are to be added/subtracted algebraically to AB and CD in order that the resulting ratio has the given value GH/HI; hence the ratio of the original lines becomes $(a \pm e)/(c \pm f) = g/h$; where AB = a , CD = c, etc. The inverse or inverted ratio F to E is also occasionally invoked, referred to as the reciprocal ratio by Gregorius. Thus a means is established for constructing a harmonic ratio from a given ratio. Gregorius now embarks on a long series of 17 geometric demonstrations of different cases.



Prop.18. Fig.i & ii

Construction & Demonstration.

1. The given lines AB and CD are equal to each other; and the ratio E to F is equal to GH to HI; moreover the lines BK, DL are added or taken in the ratio of E to F; [i. e. $a = c$; $e/f = g/h$.] the truth of the proposition is shown.

L1.§1. Prop. 18 Note 1: In the first trivial case, similar lengths are made equal : AB = CD; E = F ; GH = HI; AB is divided internally and externally by K ; AB - E = AK or AB + E = AK ; similarly, CD - F = CL, or CD + F = CL, then AK/CL = GH/HI = (AB ± E)/(CD ± F). If $e = f$ then $1 = g/h = e/f = (a \pm e)/(a \pm f)$, for inner and outer division.

2. If AB is greater than DC and also the ratio GH to HI is greater than the ratio AB to CD, but the ratio E to F is still one of equality, then I assert that it is not possible to add the equal quantities E and F in order that the sums shall be in the given ratio of GH to HI.

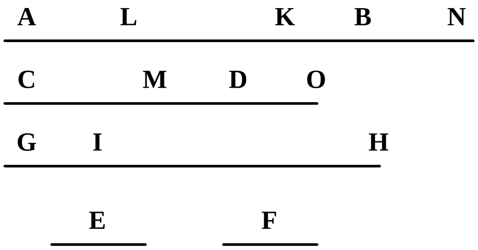
[i. e. given $a > c$; $a/c < g/h$; $e = f$, then $(a \pm e)/(c \pm f) \neq g/h$.]
For when AB is greater than CD, a length LB equal to CD itself is conveniently placed on the same line [The largest amount that may be taken away from AB], and some amount BK can be added to the line AB. If it is possible, AK will be to LK [*i.e.* CD + F] as GH to HI. Therefore the ratio AK to LK will be greater than the ratio AB to LB (or CD). Because the ratio GH to HI is set larger than the ratio AB to LB, the ratio GI to IH is also be greater than the ratio[a] AL to LB
[*i. e.* internal division is effected by GH/HI - 1 > AB/LB - 1, giving GI/HI > AL/LB];
but as GH is to HI thus AK to LK by hypothesis, and thus as GI [b] to IH, thus AL to LK on internal division, and therefore the ratio AL to LK is greater than the ratio AL to LB. Which cannot happen[c] when LB shall be less than LK itself: hence equal lines cannot be added, in order that, &c....

a. 29. book 5 ; b. 17. Book 5 ;c. 8. Book 5.

[Algebraically, the problem amounts to : $(a + e)/(c + e) = g/h$, where $g/h > a/c > 1$; we see that $g/h > a/c > (a + e)/(c + e) > 1$, and so the ratio is moving away from the required ratio for $e > 0$.]

The equal amounts [E or F] can truly taken away, so that the remainders will be in the ratio GH to HI: indeed from the fact that LB is equal to CD itself, the ratio becomes as GI to IH, thus AL to LM, for some M falling between L and B; (as AL to LB has been shown to have a smaller ratio than GI to IH) then the line MB [= E] shall be made equal to ND [= F]: Since indeed by construction as GI to IH, so AL is to LM, which can be added to give, AM to LM, that is equal to CN per construction, as GH to HI.
Q. e. d.

[In the subtractive mode, AB is initially divided internally by L as above : AB - CD = AL ; a new point M is placed on LB so that MB = E, & similarly, CD - MB = CN; then GI/IH = (AB - E)/(CD - F) = (AB - MB)/(CD - ND) = AM/CN .
Algebraically: $g/h \geq (a - e)/(c - e) > a/c > 1$, and $e = (gc - ah)/(g - h) > 0,$ as $gc/h > a,$ for equality.]



Prop.18. Fig.iii

3. If the line AB is again longer than the line CD & the ratio GH to HI is smaller than the ratio AB to CD, whereas the ratio of E to F is still one of equality, equal amounts can be added in order that the terms have the composite ratio GH to HI. Indeed the line LB can be made equal to the line CD, since by hypothesis the ratio AB to LB, that is equal to CD by construction, is greater than the ratio GH to HI, and by division [a] the ratio AL to LB, will be greater than the ratio GI to IH
[$AB/LB > GH/HI \therefore (AB - LB)/LB > (GH - HI)/HI$, or $AL/LB > GI/HI$; the products will therefore be as GI to IH, so AL to LN, LN is shown to be greater than the line LB; for the ratio GH to HI is smaller than the ratio AB to LB, that is CD, & hence the ratio GI [b] to IH, that is AL to LN is less than the ratio AL to LB; then to the line CD a certain line DO is added equal to BN itself: therefore since AL to LN, as GI to HI, and by addition, AN will be to LN, that is CO, as GH to HI: which was the first part. a. 29. Book 5 ; b. ibid.
[Algebraically: $a/c > (a + e)/(c + e) \geq g/h > 1$; equality holds when $e = (ah - gc)/(g - h) > 0$.]

With the same positions, equal amounts cannot be taken away, in order that the remainders have the given ratio GH to HI : for if the equal amounts KB and MD are taken away; and it was possible, then the ratio would be as GH to HI so AK to LK, (*i.e.* CM). When therefore by hypothesis, the ratio GH to HI is less than the ratio AB to LB (or CD), whereas it should be as GH to HI. Thus AK to LK (*i.e.* CM), the ratio will be AK to LK, which is less too than the ratio AB to LB, and by division[a] the ratio AL to LK is less than the ratio AL to LB. Which is not [b] possible when the line LB is greater than the line LK. Wherefore in this case the equal lines cannot be taken away, in order that the remainders, &c.
a. 29. Book 5 ; b. 8.Book 5.

[Algebraically: $a/c > (a - e)/(c - e) \geq g/h > 1$ has no solution for $e > 0$.]

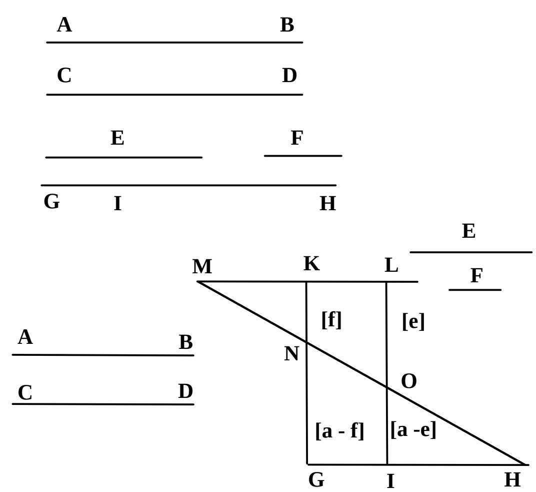
[11]

[11]

4. Si fuerint AB, CD lineae aequales, & ratio E ad F, inaequalitatis , & eadem cum ratione GH ad HI. Dico addi non posse lineas in ratio E ad F. ut compositae datam habeant rationem GH ad HI. Rationi enim GH ad HI non potest addi ratio aequalitatis, quin producatur ratio minor illa, quam habet GH ad HI. Ergo nec rationi aequalitatis potest addi ratio GH ad HI, quin producatur ratio minor illa, quam habet GH ad HI, cum eaedem sint quae resultant.

Iisdem positis, demi rectae poterunt, secundem rationem E ad F , ut residuem obtineant rationem GH ad HI reciprocem: id est , ut utriusque rationis antecedentes non sint in eadem linea : erigantur enim ex G & I parallelae GK, IL, aequales ipsis AB, CD; & ex L per K agatur linea LKM aequalis rectae GH, ducaterque linea MH occurrens GK, IL rectis in N & O. Quoniam igitur NK OL lineae sibi mutuo aequidistant, erit OL ad NK ablata ad ablatam ut LM ad MK, id est ut GH ad HI, id est ut E ad F per hypothesim; & ut GH ad IH, sic NG est ad OI, residua ad residuam : igitur lineas abstulimus secundum rationem E ad F ut reliquae datam obtineant rationem reciprocem GH ad HI.

4. If AB and CD are lines of equal length, & the ratio of E to F is one of inequality, in the same ratio as GH to HI.
I say that it is not possible to add the lines in the ratio E to F, in order that the sum has the given ratio GH to HI. For indeed a ratio equal to the ratio GH to HI cannot be reached by addition, as the ratio produced is less than GH ad HI. Therefore a ratio equal to GH to HI cannot be produced by addition, for such a ratio on being formed is less than GH is HI.
[Algebraically: It is required that: $(a + e)/(a + f) = g/h = e/f$, where $e \neq f$, which has no solution.]



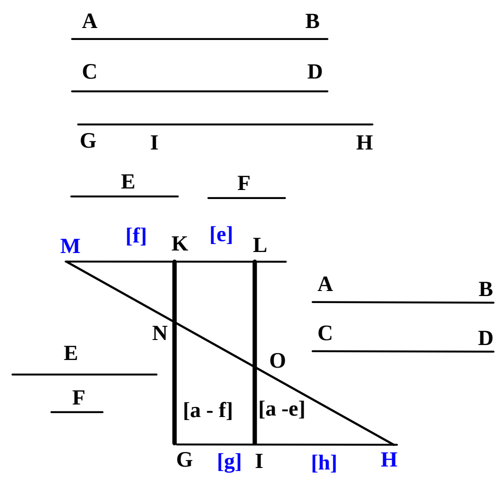
Prop.18. Fig.iv

With the same lengths in place, lines can be taken away, following the ratio E ad F, in order that the ratio of the complements of GH to HI are obtained from the remainders. In order to do this, as each of the preceding ratios cannot be in the same line, the parallel lines GK and IL are erected from G and I equal to AB and CD. The line LKM is drawn from L through K equal to the line GH, and the line MH is drawn crossing the lines GK and IL in N and O. Since the lines NK and OL are parallel, OL is taken to NK as LM to MK, *i.e.* as GH ad HI, *i.e.* as E ad F by hypothesis. Thus, as GH to IH, so NG is to OI, difference to difference, therefore we have taken the lines according to the ratio E to F in order that the differences maintain the inverted ratio GH to HI given.
[*i.e.* on a line, the ratio $(a - e)/(a - f) = g/h = e/f$, has the solution $e = f$, which is not allowed. but geometrically in a plane as in diagram (iv), GH/HI = $(a - f)/(a - e)$, where $a - f$ is the complement of f, etc.]

5. AB, CD lineae sint rursum inter se aequales, & ratio E ad F minor ratione GH ad HI.
Dico addi non posse lineas secundum rationem E ad F, ut compositae rationem obtineant GH ad HI. patet: cum ratio GH ad HI ponatur minor ratione E ad F; ratio vero E ad F aucta ratione aequalitatis, fiat minor ratione E ad F.

Iisdem positis : auferri poterunt lineae, in ratione E ad F, ut reliquae rationem habeant GH ad HI, reciprocem : erigantur enim ex G & I, paralleles GK, IL, aequales ipsis AB, CD ; dein ex L per K, agatur linea LKM : factaque LM ad KM ut E ad F; ducatur recta MH occurrens GK, IL lineis in N & O: erit igitur LO ad NK, ablata ad ablatam, ut LM ad MK, id est ut E ad F; & NG ad OI, reliqua ad reliquam, ut GH ad IH.

5. Again, let AB and CD be lines equal to each other, and let the ratio E to F be less than the ratio GH to HI.
I say that it is not possible to add the lines following the ratio E to F, so that the sums have the ratio GH to HI. It is apparant that when the ratio GH to HI is compared with the ratio E to F, the given equal ratio is truly increased by the ratio of the inequality, but it is smaller than the ratio E to F. [Algebraically: It is required that: $(a + e)/(c + f) = g/h > e/f$, where $e > f$, whereas it will always be less than e/f.]



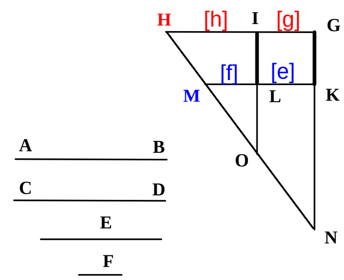
Prop.18. Fig.v

With the same lengths in place, lines can be taken away in the ratio E to F, in order that the remainders shall have the ratio derived from the inverted ratio AB - E to CD - F as GH to HI . For the parallel lines GH and IL are erected from G and I equal to AB and CD, then from L through K the line LKM is drawn, and the ratio LM to KM is made as E to F. The line MH is drawn crossing with the lines GK and IL in N and L: therefore LO will be to NK, difference to the difference, as LM to MK, *i. e.* as E to F, and NG to OI, remainder to the remainder, as GH to IH.

6. Sint iterum AB, CD lineae aequales, & ratio E ad F , maior ratione GH ad HI;
Dico addi posse lineas, in ratione E ad F, ut compositae habeant rationem GH ad HI. Demittantur enim ex G & I, parallelae GK, IL, datis AB, CD lineis aequales ; & ex K per L agatur parallela GH recta KM, ut KM sit ad LM , sic ut E ad F; tum ex H per M recta agatur HN; occurret illa lineis IL, GK in O & N: Cum enim ratio E ad F, id est KM ad LM, maior ponatur ratione GI ad IH; erit dividendo [a] , ratio KL ad LM. maior ratione GI ad IH; adeoque LM recta minor [b] ipsa HI, Quare & HM producta secabit lineas IL, GK in O & N: unde NK est ad LO, addita ad additam, ut KM ad LM, id est E ad F, & GN ad IO, tota ad totam, ut GH ad HI. Quod erat primum.

[12]

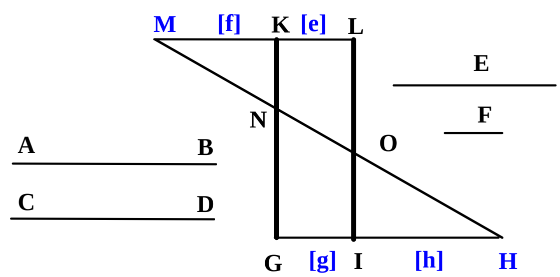
Iisdem positis : poterunt auferri lineae, in ratione E ad F, ut residuae datam habeant rationem GH ad HI, reciprocem. Erigantur enim ex G & I, parallelae GK, IL, aequales rectis AB, CD ; & ex L per K, ducatur linea LKM : ut LM sit ad KM quemadmodam est E ad F; ducaturque recta MH occurrens GK, IL lineis in N & O: patet LO ad NK, reliquam ad reliquam esse, ut LM ad MK, id est ut E ad F; & NG ad OI, reliquam ad reliquam, ut GH ad IH. Quod erat demonstratum. a 29. Quinti ; b 10. Quinti.



Prop.18. Fig.vi a.

6. Again AB and CD shall be equal line, and the ratio E to F larger than the ratio GH to HI;
I say that it is possible to add the lines in the ratio E to F, so that the sums shall have the ratio GH to HI. For the parallel lines GK and IL are sent from the points G and I, given equal with the lines AB and CD. From K through L the line KM is drawn parallel to GH, in order that KM shall be to LM as E to F. The line HN is drawn from H through M that meets the lines IL and GK in O and N. For when the ratio E to F, *i. e.* KM to LM, is made larger than the ratio GI to IH, the ratio KL to LM will be divided[a] in a ratio larger than the ratio GI to IH, and thus the line LM will be less than [b] HI, And because the line HM produced will cut the lines IL and GK in O and N, then NK is to OL, sum to sum, as KM to LM, that is E to F, and GN to IO, total to total, as GH to HI. Which was the first case.
[Algebraically, we have: $e/f > (a + e)/(a + f) = g/h$, where $e > f$, which can always be satisfied.]

[12]



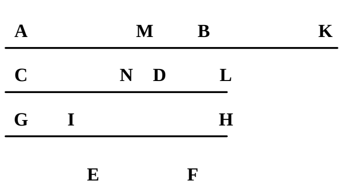
Prop.18. Fig.vi b.

With the same arrangements: the lines can be taken away in the ratio E to F, in order that the remainders shall have the given inverted ratio of GH to HI. For the parallel lines GK and IL can be erected from G and I, equal to the lines AB and CD ; and from L through K, the line LKM is drawn: in order that LM shall be to KM as E is to F; and the line MH is drawn crossing the lines GK and IL in N and O: it is apparent that LO to NK, which is the remainder to the remainder, shall be as LM to MK, that is as E to F; and NG to OI, remainder to remainder, as GH to IH. Q. e. d.
[i. e. $(a - f)/(a - e) = g/h$].

[12]

7. Sit AB maior recta CD, habeatque E ad F eandem ratioem quam AB ad CD, quae & eadem sit cum ratione GH ad HI. Addi poterunt lineae in ratione E ad F, ut compositae, rationem habeant GH ad HI. Addantur enim BK DL linea, secundum rationem E ad F: Quoniam igitur per hypothesin, AB est ad CD ut E ad F , id est per constructionem ut BK ad DL, erit [a] AK ad CL, ut AB ad CD, id est GH ad HI.

Iisdem positis : demi poterunt lineae, in ratione E ad F, ut reliquae rationem habeant GH ad HI: demantur enim BM, DN. Quoniam igitur est ut AB ad CD, sic E ad F, id est per constructionem BM ad ND, erit AM [b] ad CN, reliqua ad reliquam, ut AB ad CD, id est ut GH ad HI. Quod erat demonstrandum. *[a] 19. Quinti; b ibid.*



Prop.18. Fig.vii.

7. AB shall be longer than the line CD, and E to F shall have the same ratio as AB to CD, and which shall be the same with the ratio GH to HI. Lines can be added in the ratio E to F, in order that the sum can have the ratio GH to HI. For the lines BK and DL may be added, following the ratio E to F: Since indeed by hypothesis, AB shall be to CD as E to F, that is by construction, as BK to DL, [a] AK will be to CL, as AB to CD, that is GH to HI.
[For the sum, (the difference is similar): $e/f = a/c > 1$; if $g/h < a/c$ then e and f can be found such that $a/c > (a + e)/(c + f) = g/h$.]

With the same arrangements: the lines can be taken away in the ratio E to F, in order that the remainders shall have the given ratio GH to HI : for BM and DN may be taken away. Since therefore as AB is to CD, thus E to F, that is by construction BM to ND, will be as AM [b] to CN, remainder to remainder, as AB to CD, that is as GH to HI. Q. e. d. *[a] 19. Five; b ibid.*

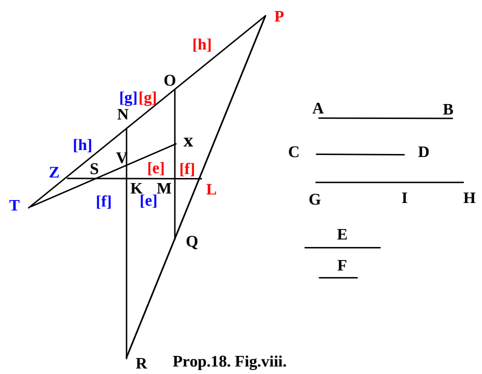
8. Sit ratio AB ad CD inaequalitatis ; & maior quidem, ratione GH ad HI ; sit autem ratio E ad F maior ratione AB ad CD.

Dico addi posse lineas in ratione E ad F, ut compositae, habeant rationem GH ad HI. Facto enim KL ad ML ut E ad F, erigantur ex K & M parallelae, KN, MO, aequales ipsis AB, CD ; iunctisque N, O, fiat ut GH ad HI, sic NP ad OP; & ex P per L linea agatur PR quae (cum ratio NP ad OP, id est GH ad HI, minor sit ratione KL ad ML, id est E ad F, adeoque PL non aequidistet ipsi OM) conveniet cum OM, NK lineis, in Q & R. Unde patet MQ ad KR additam ad additam esse, ut KL ad LM, id est per constructionem ut E ad F, & OQ ad NR, totam ad totam, esse ut NP ad OP, id est GH ad HI. Iisdem positis :

Dico demi posse lineas in ratione E ad F, ut reliquae obtineant rationem GH ad HI. Producatur enim ON, donec cum MK linea, conveniat in Z : factoque OT ad NT, ut GH ad HI, & sumpta KS aequali ipsi ML, erit NT maior ipsa NZ, & KS minor recta KZ, ut ostendam; quare si ex T per S agatur linea occurrens NK OM lineis, in V & X, factum erit quod petitur. Quoniam enim ratio OZ ad NZ , id est OM ad NK, id est per constructionem AB ad CD,maior ponitur ratione GH ad HI, id est OT ad NT, erit dividendo, ratio ON ad NZ, [a] maior ratione ON ad NT: unde NZ minor est linea NT. eodem modo cum ratio E ad F, id est MS ad KS, ex hypothesi sit maior, ratione AB ad BC, id est OM ad NK, id est MZ ad KZ, erit quoque dividendo, ratio MK ad KS, maior ratione MK ad KZ, quare & KS linea [b] minor linea KZ : & recta TS,

[13]

secabit lineas OM, NI unde patet XM adde ad VK, ablatam ad ablatam, ut MS ad KS, id est ut E ad F; & OX esse ad NV, residuum ad residuum, ut OT ad NT, id est ut GH ad HI. [a. 17. Quinti; b. 10 Quinti.]



Prop.18. Fig.viii.

8. The ratio of AB to CD is one of inequality, and greater indeed than the ratio GH to HI. Moreover the ratio E to F shall be greater than the ratio AB to CD.

I say that it is possible to add lines to AB and CD in the ratio E to F, in order that the sum has the ratio GH to HI. For the ratio KL to ML is as E to F, and the parallel lines KN and MO are set up from K and M equal to CD and AB. From the line joining the points N and O, the ratio NP to OP is made equal to the ratio GH to HI; and from P through L the line PR is drawn that meets the lines OM and NK in Q and R (where the ratio NP to OP, or GH to HI, is less than the ratio KL to ML, or E to F, and therefore PL is not parallel to OM [in the triangle NPR with base PF]). Thus it is apparent that the ratio MQ to KR is to be added sum to sum in the ratio, in order that from the construction KL is to LM as E to F, and OQ to NR, total to total, is as NP to OP, that is GH to HI.

[i.e. given $e/f < g/h < a/b < 1$, then $(a + k.e)/(b + k.f) = g/h$ for some k; a similar result follows for the subtractive case.]

With the same ratios in place: I say that lines in the ratio E to F can be taken away, in order that the remainders may be obtained in the ratio GH to HI. For ON is produced to meet the line MK in Z : and the ratio is OT to NT is made the same as GH to HI. With KS supposed equal to ML, NT is greater than NZ, and KS is less than the line KZ as shown: whereby if from T through S, a line is sent crossing the lines NK and OM, in V and X, the required ratio can be found.

For since the ratio OZ to NZ , or OM to NK (equal to AB to CD by construction), is put larger than the ratio GH to HI or OT to NT [*i. e.* $a/b > g/h$], giving the ratio ON to NZ (by division of the line[a]) greater than the ratio OT to NT: from which NZ is less than NT. In the same way, for the ratio E to F or MS to KS, by hypothesis the ratio AB to CD or OM to NK is the greater *i. e.* MZ to KZ. Also by division , the ratio MK to KS, will be greater than the ratio MK to KZ, and whereby the line KS [b] will be less than the line KZ. The line TS cuts the lines OM and NK, from which it appears with XM and VK to be taken in the ratio MS to KS or E to F, that OX is to NV, remainder to remainder, as OT to NT, that is as GH to HI. [a. 17. Five; b. 10. Five.]

[13]

9. Si fuerint AB, CD lineas inaequales ; & ratio E ad F eodem cum ratione GH ad HI ; quae minor sit ratione AB ad CD.

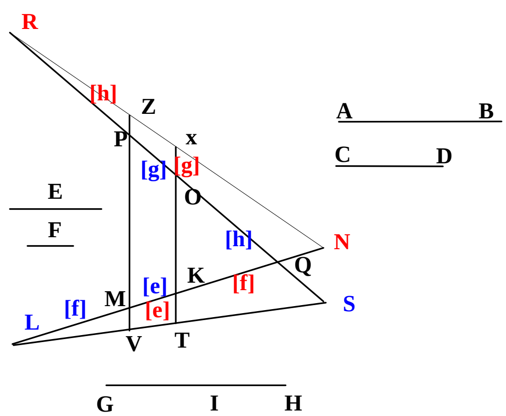
Dico addi posse lineas in ratione E ad F, ut compositae rationem habeant GH ad HI reciprocem. Fiat enim ut E ad F sic KL ad ML, & MN ad KN; erigantur ex M &K parallelis, KO, MP, aequalibus ipsis AB, CD ; iungantur OP, fiatque ut GH ad HI, sic OR ad PR, & PS ad OS, & PO producta occurrat MN lineae in Q, erit PS maior quam PQ. & MQ minor quam MN, ut ostendam; tum recta ducatur SL; quae secet MP OK lineas in V & T; Quoniam igitur MQ est ad KQ, ut PM ad OK, id est per constructionem ut AB ad CD, & MN ad KN, ut E ad FL sit autem ex hypothesi ratio AB ad BC, maior ratione E ad F, erit ratio MQ ad KQ, maior ratione MN ad KN, & dividendo, [c] ratio MK ad KQ , maior ratione MK ad KN, quare MQ linea minor est lineae MN ; eadem modo ostenditur PQ recta minor recta PS. Unde KT est ad MV addita ad addita ut KL ad ML; id est E ad F, & PV ad OT tota ad totam, ut PS ad OS, id est GH ad HI. Similiter additio fiet reciproca, si recta ducatur NR occurrens OK, PM lineis in X & Z : erit enim XO ad ZP, addita ad additam, ut OR ad PR, id est per constructionem ut GH ad HI, id est ex hypothesi ut E ad F; & ZM ad XK; tota ad totam, reciprocem, ut MN ad KN, id est per constructionem ut E ad F, id est ex hypothesi ut GH ad HI. c. 29. Quintii.

Iisdem positis: Dico addi non posse lineas in ratione E ad F, ut composita sit ad compositam, in ratione GH ad HI ordinate ; id est ut utriusque rationis antecedentes sint in eadem linea : addantur enim secundum rationem E ad F, lineae BK, DL, & si fieri posset, sit AK ad CL, ut GH ad HI: erit igitur ut AK ad CL sic BK ad DL, (quia ex hypothesi ratio E ad F, eadem est cum ratione GH ad HI) adeaque & AB ad CD, ut AK ad CL, id est ut GH ad HI; quod est contra Supposotum : nam ratio AB ad CD maior ponitur, ratione GH ad HI: quare ordinata additio non continget.

Sed neque hoc casu detractio ordinata, aut reciproca continget. Detrahantur enim ordinatim lineae BM, DN, in ratione E ad F: & si fieri possit, sit AM ad CN, ut GH ad HI. Quoniam igitur AB ad CD, maiorem habet rationem , quam BM ad DN, id est E ad F, erit AM reliqua, ad reliquam CN, in maiori ratione quam AB ad CD; adeoque multo maiore quam E ad F, id est GH ad HI: quare ordinata detractio nulla fiet.

De reciproca detractione sic constabit: fiat ut E ad F sic LD ad KB;& si fieri possit, sit AK ad CL, ut GH ad HI. Cum igitur ratio AB ad CD, ex hypothesi maior sit ratione AK ad CL, id est GH ad HI , fiat AK ad CM, ut AB ad CD; eritque CN linea, [b] minor ipsa CL : quare cum sit ut AB ad CD, sic AK ad CM, erit quoque [c] KB ad ND, residua ad residuam ut AB ad CD; adeoque KB maior, quam MD, & multo maior quam LD. Quod est contra hypothesim non igitur detractio, reciproca continget.

a. 33. Quintii; b. 10. Quintii; c. 19. Quintii.



Prop.18. Fig.ix a.

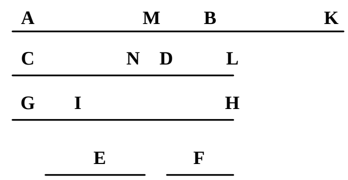
9. If AB and CD are lines of unequal lengths ; and the ratio E ad F is the same as the ratio GH to HI ; which is less than the ratio AB to CD, [*i.e.* $a/c > g/h = e/f$] then :
I say that the lines can be added in the inverted ratio E ad F, in order that the sums have the ratio of GH to HI. [*i. e.* $(a + k f)/(c + k e) = g/h$].

For by the following construction the ratios KL to ML and MN to KN can be made equal to E to F [See Fig. ix a]. From M and K the parallel lines MP and KO are erected equal to AB and CD, the points O P are joined, and OR to PR is made in the ratio GH to HI, and also as PS to OS [i. e. OR/PR = g/h = PS/OS.]. Now PO producted crosses the line MN in Q, and PS is greater than PQ. Also MQ is less than MN as shown; then the line SL is drawn; which cuts the lines MP and OK in V and T.

The result is demonstrated as follows: Since MQ is to KQ as PM is to OK, therefore as AB is to CD from the construction, & MN to KN, as E to F ; but the ratio AB to BC is larger than the ratio E to F by hypothesis, and the ratio MQ to KQ is larger than the ratio MN to KN [*i. e.* MQ/KQ = PM/OK = AB/CD = a/c > MN/KN = e/f ;] ; and by division of the line[c], the ratio MK to KQ is larger than the ratio MK to KN, whereby the line MQ is shorter than the line MN [*i. e.* MQ/KQ - 1 > MN/KN - 1, or MK/KQ > MK/KN]. In the same manner it can be shown that the line PQ is shorter than the line PS. From which KT is to MV, sum to sum, as KL to ML: that is, E to F; and PV to OT, total to total, as PS to OS, that is GH to HI. [*i. e.* KT/MV = KL/ML = e/f; and PV/OT = $(a + kf)/(c + ke)$ = PS/OS = g/h, for some $k > 0$.] c. 29. Five.

Similarly the inverted ratio can be formed from addition: if the line NR is drawn crossing the lines OK and PM in X and Z ; for now XO is to ZP, sum to sum , as OR to PR, which is by construction as GH to HI, which is by hypothesis as E to F; & ZM to XK; total to total, with the inverted ratio, as MN to KN, which is by construction as E to F, that is by hypothesis as GH to HI. [*i. e*. XO/ZP = OR/PR = $g/h = e/f$; & ZM/XK = $(a + k f)/(c + ke) = g/h$.]

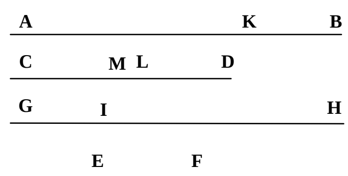
With the same ratios in place [*i.e.* $a/c > g/h = e/f$]: I say that it is not possible to add the lines in the ratio E ad F, in order that the ratio [in the form] sum to sum shall be accordingly GH to HI ; both of the preceding ratios should be in the same line: for the lines BK and DL are added following the ratio E to F, and if it were possible to make the ratio AK to CL as GH to HI: therefore AK to CL would be thus as BK to DL, (since by hypothesis the ratio E to F is the same as the ratio GH to HI) and so AB to CD, as AK to CL, as GH to HI; which is contrary to the supposition : for the ratio AB to CD is put larger than the ratio GH to HI: whereby the ordered addition will not happen. [*i.e.* $(a + ke)/(c+kf) \neq e/f$]



Prop.18. Fig.ix b.

But neither subtraction with the ordered or inverted ratio can give the desired ratio either. Indeed, the lines BM and DN being taken away in the order E ad F, if it were possible, would become AM to CN as GH ad HI. Since therefore AB to CD, has a greater ration than BM to DN, that is E to F, the remainder AM will be, to the remainder CN, in a greater ratio than AB ad CD; and thus in a much greater ratio than E to F, that is GH to HI: whereby no ordered subtraction will be possible.

Concerning the inverted subtraction it will be agreed thus : E to F will become thus as LD to KB; and if it were to become possible, AK to CL would be thus as GH to HI. Therefore when the ratio AB to CD, shall be greater than the ratio AK to CL from hypothesis, that is GH to HI, AK to CM shall become as AB to CD; and the line CN [b] shall be less than the line CL : whereby when AB thus shall be to CD as AK to CM, and [c] KB to ND also, remainder to remainder as AB to CD; and hence KB will be greater than MD, & much greater than LD. Which is contrary to the hypothesis and therefore subtraction with the inverted ratio cannot happen.



Prop.18. Fig.ix c.

a. 33. Five; b. 10. Five; c. 19. Five.

[14]

10. Sint AB, CD lineas inaequales ; & ratio E ad F eodem cum ratione GH ad HI ; quae maior sit ratione AB ad CD.

Dico additionem, neque ordinatam, neque reciprocam, fieri posse. Addantur enim ordinatim in ratione E ad F, lineae BL, DM, & si fieri possit, sit AL ad CM, ut GH ad HI. Quoniam est BL ad DM, ut AL ad CM, igitur erit quoque [a] ratio AB ad CD, eadem cum BL ad DM, hoc est E ad F, vel GH ad HI quodcontra suppositum; ponitur enim ratio AB ad CD minor ratione E ad F, sed neque reciproca additio continget : fiat enim ut E ad F, sic DL ad BK, addita ad additam, & si fieri possit, sit AK ad CL ut GH ad HI. Fiatque; ut GH ad HI, sic AB ad CM [b] erit CM minor quam CD, quia ratio AB ad CD minor est ratione GH ad HI, id est AB ad CM. igitur cum dit ut AK ad CL, id est GH ad HI, sic AB ad CM, erit [c] BK ad ML, residuum ad residuum ut AB ad CM ; adeaque; BK maior ipsa ML, & multo maior quam LD. Quod est contra hypothesin: quare nec additio reciproca continget.

Iisdem positis, neque ordinata detractio continget : detrahantur enim in ratione E ad F, lineae KB LD, & si fieri possit sit AK ad CL, ut GH ad HI. Quoniam igitur ratio AB ad CD, ex hypothesi minor est ratione E ad F, id est KB ad LB, ablatae ad ablatam, erit [d] ratio AK ad CL, reliquae ad reliquam, minor ratione AB ad CD, adeoque multo minor ratione KB ad LD, id est GH ad HI; quod est contra suppositum: quare detractio ordinata non continget.

Continget vero detractio reciproca hoc modo. Erigantur duae parallelae KM, LN, aequales ipsis AB CD ;iunctisque punctis MN, KL fiat ut E ad F, sic MO ad NO, & NP ad MP, item ut GH ad HI, sic KQ ad LQ, & LR ad KR: cadent R & P intra concursum linearum OP.QR., ut ostendam: quare rectae ducantur RO, PQ; & PQ quidem secet lineas KM, NL in S & T. RO vero lineam PQ in V, & rectas MK, LN in Z & X conveniant OP. RQ lineae in β. Quoniam ratio AB ad CD, id est per constructionem MK ad NL, id est Nβ ad Mβ, minor est ratione E ad F, id est per constructionem PN ad MP, erit dividendo, [e] quoque ratio NM ad

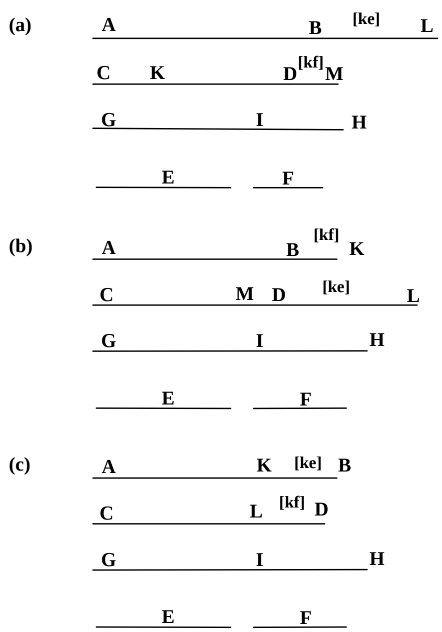
Mβ, minor ratione NM ad MP; adeoque MP linea [f] minor quam Mβ: eadem modo ostenditur KR linea minor quam Kβ : quare iunctae RO, PQ secabunt lineas MK, NL ut dictum est : unde NT est ad MS, ablata ad ablatam, ut NP ad MP, id est E ad F, & SK est ad TL, reliqua ad reliquam, ut SQ ad TQ, id eat GH ad HI : similiter erit MZ ad NX, ablata ad ablatam, ut MO ad NO;

[15]

id est E ad F, & XL ad ZK, reliqua ad reliquam, ut LR ad KR,id est GH ad HI.

a Ibid. ; b 10. Quintii. ; c 19. Quintii. ; d 33. Quintii. ; e 29. Quintii. ; f 10. Quintii.

10. Let AB and CD be lines of unequal lengths. Also, the ratio E to F shall be the same as ratio GH to HI, which shall be larger than the ratio AB to CD [*i.e.* $a/c < g/h = e/f$].



Prop.18. Fig.x.

I assert that neither ordinary nor inverted addition of the E to F ratio can be used to give an equal ratio between AB to CD and GH to HI. [*i.e.* $(a + ke)/(c+kf) \neq g/h$].

For the regular addition of the lines BL and DM in the ratio E to F, if it were possible, would give the ratio AL to CM equal to the required ratio GH to HI [See Fig. (x; a)]. But since BL to DM is in the same ratio as AL to CM, then AB to CD also is as BL to DM, or GH to HI : which is against the supposition that the ratio AB to CD is less than the ratio E to F.

[If $(a + ke)/(c+kf) = g/h = e/f$, then $a/c = e/f$ also, contrary to the original hypothesis.]

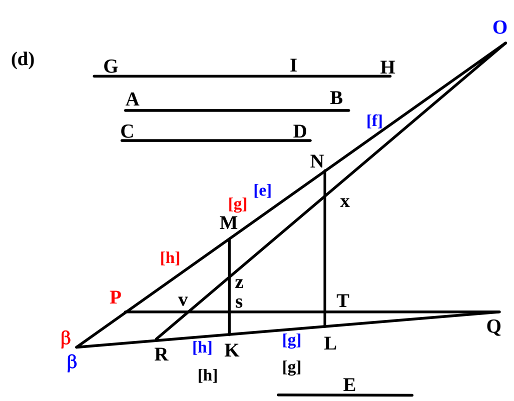
Neither will equality occur if the inverteds are added [*i.e.* again $a/c < g/h = e/f$ and we show that $(a + kf)/(c+ke) \neq e/f$; see Fig. (x; b) ;]:

For as E to F, thus DL to BK, with the ratio of AB to CD augmented by addition. If it were possible, then AK to CL shall be as GH to HI. However, in truth as GH to HI, so AB to some CM [b] where CM is less than CD, because the ratio AB to CD is less than the ratio GH to HI [for $a/c < g/h$ is given, hence g/h = AB/CM]. Therefore, according to what is being said, as AK to CL, *i. e.* GH to HI, thus AB to CM will be as [c] BK to ML, remaider to remainder as AB to CM. Hence BK is itself larger than ML, and much larger than LD. Which contradicts the hypothesis: whereby addition of reciprocals cannot be used in this case.

[*i. e.* we suppose $(a + kf)/(c+ke) = e/f$. In this case a length CM is assumed that insures the equality of the ratio $g/h = e/f$ = AB/CM ; but in this case the remaining ratio BK/ML must also have the value e/f or AB/CM and so BK > ML > DL, which is not so as DL > BK as $e > f$ is given].

With the same ratios, neither will ordinary subtraction give equality of the ratios:

[*i.e.* again $a/c < g/h = e/f$ and we show that $(a - ke)/(c - kf) \neq e/f$; see Fig. (x; c) ; The method of the additive case could be used.]: for if the lines KB and LD are taken in the ratio E to F from AB and CD, and it is assumed that the resulting ratio AK to CL is as GH to HI. Because the ratio AB to CD by hypothesis is less than the ratio E to F, that is KB to LB, then the ratio of the remainder to the remainder will give the ratio AK to CL, which is less than the ratio AB to CD [as a larger quantity is taken from the numerator than the denominator], and thus much less than the ratio KB to LD, that is GH to HI; which is contrary to the supposition: whereby ordinary subtraction cannot be used.



Prop.18. Fig.x .

Reciprocal subtraction gives the required ratio using the following construction:

Two parallel lines KM and LN are erected equal to CD and AB, and the points MN and KL are joined.

Then the ratio MO to NO is set equal to E to F; and as NP to MP is made equal to GH to HI, so Lβ to Kβ, & LR to KR also are set in the ratio GH to HI. The points R and P lie on the concurrant lines OP and QR., as shown. Whereby the lines RO and PQ are drawn, and indeed PQ cuts the lines KM and NL in the points S and T. RO cuts the line PQ in V, and the lines MK and LN in Z and X .

Reciprocal subtraction now gives the required ratio as follows: The lines OP and QR meet in β. Because the ratio AB to CD by construction is equal to NL to MK, then Nβ to Mβ is less than the ratio E to F. By construction, PN to PM is to be divided[e] in the ratio NM to Mβ too, less than the ratio NM to MP. Thus the line MP[f] is shorter than the line Mβ: in the same way it can be shown that the line KR is shorter than the line Kβ. Whereby the points RO and PQ are joined that cut the lines MK and NL as mentioned. From which NT is to MS, taken to taken, as NP to MP, *i. e.* as E to F; and SK is to TL, remainder to remainder, as SQ to TQ, *i. e.* GH to HI. Similarly MZ to NX, taken to taken, will be as MO to NO; *i. e.* E to F, and XL to ZK, remainder to remainder, as LR to KR, *i. e.* GH to HI.

[$AB/CD = NL/MK = a/c = N\beta/M\beta < e/f$; by construction, $PN/PM = NM/M\beta = g/h < NM/MP$;
From which $NT/MS = NP/MP = e/f$; and $SK/TL = SQ/TQ$, *i. e.* g/h.
Similarly $MZ/NX = MO/NO = e/f$; and $XL/ZK = LR/KR$, *i. e.* g/h.
Hence, $XL/ZK = (a - kf)/(c - ke) = LR/KR = g/h$.]

[15]

11. Sint iterum AB CD lineae inaequales, & ratio GH ad HI minor ratione AB ad CD; ratio autem E ad F minor ratione GH ad HI.

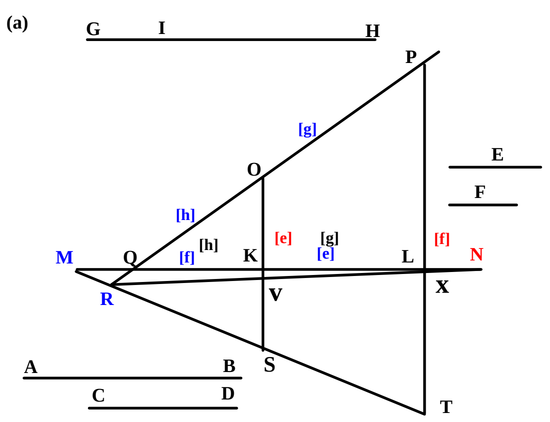
Dico tam ordinatim quam reciprocem addi posse lineas, in ratione E ad F, ut compositae rationem habeant GH ad HI, fiat enin ut E ad F sic LM ad KM, & KN ad LN : erectisque ex K & L parallelis KO, LP, quae rectis AB CD, sint aequales, iungantur puncta PO, fiatque ut GH ad HI, sic PR ad OR; occurrat autem PO recta ipsi LM in Q , erit OR maior recta OQ. & LQ minor ipsa LM : ut ostendam : quare ducta ex M per R linea MR, occurret OK, PL lineis in S & T: iunctaque RN, easdem secabit in V& X. Quoniam ergo ratio LM ad KM , id est per constructionem ratio E ad F minor est ratione LQ ad KQ, id est LP ad OK, id est AB ad CD, erit dividendo, [a] quoque ratio LK ad KM, minor ratione LK ad KQ , adeoque LM linea, b maior recta LK : eadem modo ostenditur OR recta maior recti OQ, Unde LT ad KS, ordinatum addita ad additam est ut LM ad KM, id est E ad F; & TP est ad SO, composita ad compositam, ut PR ad OR, ed est GH ad HI. igitur ordinatim lineas adiecimus, &c.

Reciprocè vero posse rectas adiici, in ratione E ad F, &c. sic ostendo. VK est ad LX addita ad additam, ut KN ad LN, id est per constructionem ut E ad F ; & XP est ad OV totam ad totam, ut PR ad OR; id est GH ad HI; igitur, &c.

Iisdem positis : Dico detractionem tam ordinatam, quam reciprocatam fieri non posse. Detrahantur enim ordinatim lineae KB LD, in ratione E ad F: & si fieri possit, sit AK ad LC, ut GH ad HI: Quoniam igitur ratio AB ad CD maior ponitur ratione E ad F, id est KB ad LD, erit quoque ratio AK ad CL residui ad resuduum, [c] maior ratione AB ad CD, quod est contra hypothesim ; cum ratio AK ad CL id est GH ad HI, minor ponatur ratione AB ad CD, quare ordinata detractio non continget.

De reciproca detractione sic constabit. Detrahantur reciprocem LD KB, in ratione E ad F: & si fieri possit, sit AK ad CL, ut GH ad HI; fiat deinde ut AB ad CD, sic AK ad CM,erit recta CM minor quam CL, quia ratio AB ad CD, id est AK ad CM, maior ponitur ratione GH ad HI, id est AK ad CL. igitur cum sit ut AB ad CD,sic AK ad CM,ablata ad ablatam, erit KB ad MD,d reliqua ad reliquam, ut AK ad CM : & KB linea, maior lineam MD: adeoque multo maior recta LD. Quod est contra hypothesim: quare detractio reciproca non continget. a 29: *Quinii* ; b 10: *Quinii* ; c 33: *Quinii* ; d 19: *Quinii* .

11. Let AB and CD again be lines of unequal length, but in this case the ratio GH to HI is less than the ratio AB to CD; also the ratio E to F is less than the ratio GH to HI. [*i. e.* $e/f < g/h < a/c$.]

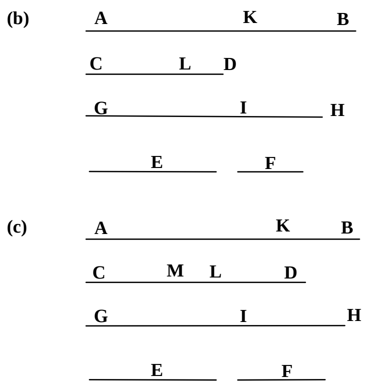


Prop.18. Fig.xi.

I say that for both regular and inverted addition it is possible to add the lines in the ratio E to F, in order that the composite ratio shall be GH to HI.

Construction: Indeed LM to KM is thus set as E to F, and also KN to LN. The parallel lines LP and KO are erected from K and L, and these lines are set equal to AB and CD. The points P and O are joined, and PR to OR is set in the ratio GH to HI. Now, PO crosses the line LM in Q , and OR is a longer line than OQ, and LQ smaller than LM : as shown in Fig. (xi;a). Whereby the line MR drawn from M through R, crossing the lines OK and PL in S and T. RN is joined, and it cuts the same lines in V and X.

Demonstration: Because the ratio LM to KM is equal from the construction to the ratio E to F which is less than the ratio LQ to KQ, that is LP to KO, that is AB to CD, then by division the ratio LK to KM is less than the ratio LK to KQ [i. e. given $g/h < a/b$; then $(LM/KM - 1) < (LQ/KQ - 1)$; hence $KL/KM < KL/KQ$] , and hence the line LM[b] is longer than LK [should this be KM > KQ ?] : in the same manner it can be shown that the line OR is longer than the line OQ. Hence LT to KS by ordinary addition to addition is as LM to KM, that is as E to F. Also, TP is to SO, sum to sum, as PR to OR, that is as GH to HI. Hence we may add the lines together in the ordinary way to give the required ratio, etc.



Prop.18. Fig.xi.

[i. e. $g/h = PT/SO$ (from ΔPRT) $= (PL + LT)/(OK + KS)$ (from Δ's MLT & PQL) $= (a + ke)/(b + kf)$].

The reciprocal lines can indeed be added in the ratio E to F, etc. as I show thus . VK is to LX, sum to sum, as KN to LN, that is by the construction as E to F ; and XP is to OV total to total, as PR to OR; that is as GH to HI; therefore, &c.

[*i. e.* $g/h = PR/OR$ (from ΔPRT) $= (PL + LX)/(OK + KV)$ (from Δ's PQR & NKV) $= (a + kf)/(b + ke)$].

With these ratios set as before : I declare that subtraction either of the regular or of the inverted ratio to give the required ratio is not possible. Indeed the lines KB and LD [see Fig. xi;b] may be taken away in the ordinary manner in the ratio E to F: and if it were possible, AK to CL would be as GH to HI. Because the ratio AB to CD is larger than the ratio E to F, *i. e.* KB to LD, the ratio AK to CL, remainder to remainder, will be larger than the ratio AB to CD[c] , which is contrary to the hypothesis, when the ratio AK to CL i. e. GH to HI, is set less than the ratio AB to CD, whereby ordinary subtraction cannot be considered.

[*i. e.* $a/c > e/f$ then $AK/CL = (a - ke)/(c - kf) > a/c$, as a larger number is taken from the denominator than from the numerator, all quantities remaining positive*:* but $g/h < a/c$, hence the equality cannot hold.]

The taking of reciprocals will be considered next thus, [see Fig. xi;d]. The inverted ratios LD and KB are taken in the ratio E to F: and if this were possible, the ratio AK to CL would thus be as GH to HI. Hence AB to CD would thus be as AK to CM, where CM is less than CL, as the ratio AB to CD, (*i. e.* AK to CM), is set larger than the ratio GH to HI, which is AK to CL. Therefore when AB to CD is thus as AK to CM, taken to taken, KB to MD[d] is, remainder to remainder, as AK to CM : and the line KB will be greater than the line MD: and thus much greater than the line LD. Which is contrary to the hypothesis: whereby subtraction of the reciprocals cannot be considered.

[*i. e.* $a/c > e/f$ and $AK/CL = (a - kf)/(c - ke) = g/h$, by assumption. Now, if $AK/CL = g/h < a/c$, then $a/c = AB/CD = AK/CM$, where $CL > CM$. In this case, $(AB - KB)/(CD - MD) = AK/CM$, and $KB > MD$ as a greater length must be taken from the numerator than the denominator to give equality: but this cannot be the case, as $F < E$. Thus, the original supposition is false.]

12. Sint AB CD lineae inaequales , & ratio E ad F minor ratione AB ad CD, maior vero ratione GH ad HI.

Dico neque ordinatim, nec reciproce lineas posse detrahi, in ratione E ad F, ut reliquae obtineant rationem GH ad HI: dematur enim reciproce in ratione E ad F,

[16]

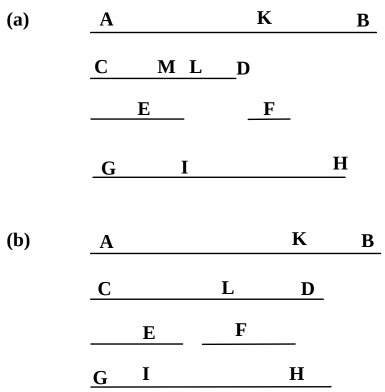
rectae LD KB, & si fieri possit, sit AK ad CL ut GH ad HI, fiatque ut AB ad CD,sic AK ad CM,erit CM minor ipsa CL, quia ratio AB ad CD, id est AK ad CM, maior est ratione GH ad HI, id est AK ad CL, cum igitur sit ut AB ad CD, sic AK ad CM, erit & KB ad MD, residuum ad residuum, ut AK ad CM : adeoque KB maior ipsa MD, & multo maior recta LD. Quodest contra hypothesim : quare detractio reciproca non continget. Sed neque ordinata detractio, casu hoc continget.

Detrahantur enim in ratione E ad F, lineae KB LD; & si fieri possit, sit AK ad CL, ut GH ad HI cum igitur ratio AB ad CD, maior ponatur ratione E ad F, id est KB ad LD,erit & ratio AK ad CL, a maior ratione AB ad CD ; adeoque multo maior ratione GH ad HI, id est AK ad CL. Quod est contra hypothesim. Igitur neque ordinata detractio fieri potest.

Iisdem positis, neque additio ordinata continget. Addantur enim in ratione E ad F, lineae BL, DM: & si fieri possit, sit AL ad CM ut GH ad HI. Quoniam igitur ratio AL ad CM, id est GH ad HI, minor ponitur ratione E ad F, id est BL ad DM, erit [b] ratio AB ad CD, minor ratione AL ad CM, id est GH ad HI. Quod est contra hypothesim ; quare nec ordinata additio continget.

Iisdem positis, additio reciproca fieri poterit: fiat enim ut E ad F, sic KL ad ML, & MN ad KN, erectisque ex M & K parallelis KO, MP, quea rectis AB, CD sint aequales, iungantur OP, & fiat OQ ad PQ , item PR ad OR, ut GH ad HI, occurratque PO linea, rectae MN in S; patet ex saepe dictis, SO lineam minorum esse recta OR, & SK minorem ipsa KN ; quia ratio AB ad CD,id est MP ad OK, id est PS ad OS, maior est ratione GH ad HI, id est PR ad OR; item MS ad KS,maior ratione MN ad KN, id est E ad F. Quare iuncta RL secabit rectas, OK PM, in T & V. Unde KT est ad MV,addita ad additam, ut KL ad ML, id est E ad F, & PV ad OT, composita ad compositam, ut PR ad OR, id est per constructionem GH ad HI.

Addi iterum poterunt lineae in ratione E ad F, ut OK minor utriusque rationis antecedentes habeat : ducatur enim in eadem figura, ex Q per L recta; occurret illa lineis PV OT in Z, & X;quia ratio OQ ad PQ, id est GH ad HI, minor ponitur ratione KL ad LM, id est E ad F; nam si LQ linea non concurreret cum PV, OT lineis, sed illis aequidistaret, esset ratio KL ad ML. eadem cum ratione OQ ad PQ. Quod est contra hypothesim ; occurret igitur LQ linea, rectis PV, OT in Z & X. Unde KX est ad MZ , addita ad additam, ut KL ad ML. id est E ad F, & OX ad PZ, ut OQ ad PQ. id est GH ad HI.



Prop.18. Fig. xii.

a 33: *Quintii* ; b *ibid.*

12. AB and CD are two unequal lines with AB > CD. The ratio E to F is less than the ratio AB to CD, but greater than the ratio GH ad HI.

[*i. e. g/h* < *e/f* < *a/c*.]

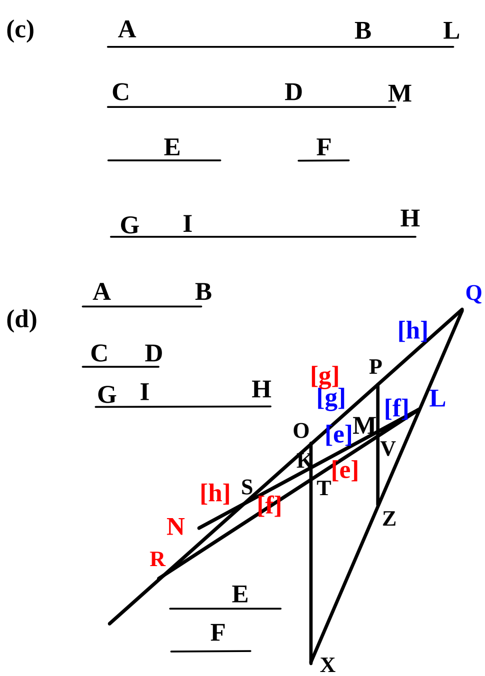
I declare that the lines cannot be taken in either the ordinary or the inverted ratio E to F, in order that the remainders obtained will be in the ratio GH to HI.

The lines LD and KB shall indeed be taken reciprocally in the ratio E to F,

[16]

and if it were possible, AK to CL shall be as GH to HI, and AB to CD thus becomes as AK to CM [On being 'scaled down', but still in the ratio *a/c*]. CM will be less than CL itself, as the ratio AB to CD (*i. e.* AK to CM) is greater than the ratio GH to HI [for *a/c* > *g/h* by the hypothesis], that is AK to CL.

Whereby AB to CD shall be thus as AK to CM, and KB to MD shall be, remainder to remainder, as AK to CM [on being similarly scaled down, but still in the ratio *a/c*]: and thus KB is larger than MD itself, and even larger than LD [*i. e.* F > E]. Which is contrary to the hypothesis, whereby subtraction of the reciprocals is not tenable.



Prop.18. Fig.xii.

But neither is the result true for ordinary subtraction, with this case now touched on.

The lines KB and LD are to be taken in the ratio E to F, and if it were possible, the remainders AK to CL should be as GH to HI. When the ratio AB to CD is set larger than the ratio E to F, (*i. e.* KB to LD), then the ratio AK to CL[a] is larger than the ratio AB to CD ; and thus much larger than the ratio GH to HI, i.e. AK to CL. Which is contrary to the hypothesis. Therefore ordinary subtraction of the ratios can be effected either to produce the given ratio. [*i. e.* if *a/c* >*e/f* ; we wish to show that *(a - e)/(c - f)* > *a/c*. Algebraically, assume this is true, then *a*(1 - *e/a)/c(1 - f/c)* > *a/c*, giving 1 - *e/a* >1 - *f/c*, *e/a* < *f/c,*
and *e/f* < *a/c,* which is true. Note: The *k* in the *e/f* ratio will be taken as understood henceforth.]

With the same ratios in place [*i. e. g/h* < *e/f* < *a/c*] , ordinary addition cannot be used to give the required ratio [Fig.xii;c]. Indeed the lines BL and DM may be added in the given ratio E ad F, and if it were possible, AL to CM would be as GH to HI. Since the ratio AL to

CM (*i. e.* GH to HI) is therefore made less than the ratio E to F, (*i.e.* as BL is to DM), the ratio[b] AB to CD is less than the ratio AL to CM, (GH to HI). Which is contrary to the hypothesis ; whereby ordinary addition will not produce the required ratio.

With the same ratios in place , inverted addition does give the required ratio [Fig.xii;d]. Construction:

Indeed KL to ML is made in the same ratio as E to F, and also MN to KN. The parallel lines MP and KO are erected from K and M equal in length to the lines AB and CD. The points O and P are joined, and both OQ to PQ and PR to OR are made in the ratio GH to HI, and the lines PO and MN cross at S.

Demonstration: It is apparent from what has often been said, that OS is less than OR, and KS is less than KN. Since the ratio AB to CD (*i. e.* MP to KO, and also PS to OS), is greater than the ratio GH to HI, *i. e.* PR to OR, and likewise MS to KS is greater than the ratio MN to KN, *i. e.* E to F. Whereby the line RL will cut the lines OK and PM in T and V. Then KT is to MV, the ratios to be added, are as KL to ML, i. e. E to F, and PV to OT, the sum to the sum, is as PR toOR, that is by construction GH to HI.

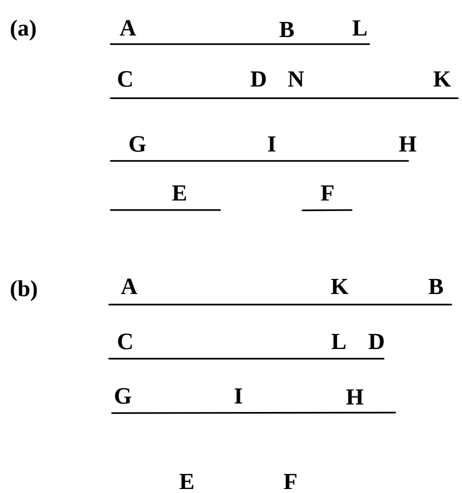
Again the lines can be added directly in the ratio E to F, as OK is the smaller of the two lines in the preceeding ratio. Indeed in the same figure, a line can be drawn from Q through L that crosses the lines PV and OT in Z and X. Since the ratio OQ to PQ is as GH to HI, which is made less than the ratio KL to ML, (E to F); for if the line LQ does not intersect with the lines PV and OT, but should be parallel to them, the ratio KL to ML will be the same as the ratio OQ ad PQ, which is contrary to the hypothesis. Hence the line LQ cuts the lines PV and OT in Z and X; then KX is to MZ, the ratio to be added, shall be as KL to ML, i. e. E to F, and the required ratio OX to PZ, as OQ to PQ, i. e. GH to HI.

[17]

13. Sint AB CD lineae inaequales; ratio vero GH ad HI , minor ratione E ad F ; quae maior sit ratione AB ad CD. fieri non poterit additio reciproca. addantur enim DK, BL in ratione E ad F, & si fieri possit sit AL ad CK ut GH ad HI : dein fiat ut AB ad CD, ita BL ad DN; erit DN linea minor recta DK, cum ipsa BL minor ponatur recta DK, adeoque tota CN minor, CK. igitur cumsit ut AB ad CD, sic BL ad DN, & componendo , ut AB ad CD, sic AL ad CN, erit ratio AL ad CN, minor ratione GH ad HI, id est per constructionem ratione AL ad CK. quid fieri nin potest; cum CN linea minor sit recta CK : quare additio reciproca non continget.

Neque etiam ordinata detractio fieri poterit : demantur enim lineae KB, LD in ratione E ad F, & si fiere possit , sit AK ad CL, ut GH ad HI; Quonim igitur ratio AB ad CD, minor ponitur ratione KB ad LD, sive E ad F, erit ratio AK ad CL, id est GH ad HI, [a] minor ratione AB ad CD, quod est contra Suppositum: quare ordinata detractio non continget.

Detrahi tamen poterunt lineae, in ratione E ad F, ut reliquae reciprocè habeant rationem GH ad HI. Fiat enim ut E ad F, sic KL ad ML, & MN ad KN; erectisque ex K & M parallelis KO, MP, quae rectis AB, CD, sint aequales iungantur OP; & fiat ut GH ad HI, sic OQ ad PQ, & PR ad OR : ostendetur ut in casu decimo huius propositionis puncta N & R, esse intra concursum linearum LN, QR, quare ductae RL, NQ secabunt lineas OK, PM : & N1 quidem illas secet in ST, recta vero RL eisdem occurrat in V & X. patet TM esse ad SK, ablatam ad ablatam , ut MN ad KN, id est per constructionem ut E ad F, & SO esse ad PT, residuam ad residuam, ut IQ ad PQ, id est GH ad HL.



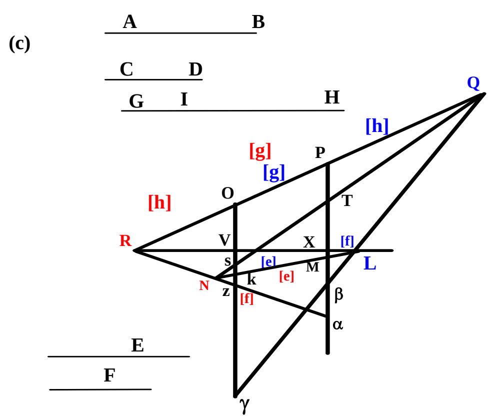
Prop.18. Fig.xiii.

Addi etiam poterunt ordinatim lineae in ratione E ad F, ut compositae rationem habeant GH ad HI, agatur enim ex R per N linea RN, occurret illa rectis IK, PM: quid saepius ostensum est, cum ratio XR ad VR, id est PR ad OR, id est per constructionemGH ad HI, minor ponatur ratione E ad F, id est MN ad KN. occurrat igitur in Z & α. erit Mα ad KZ, addita ad additam , ut MN ad KN , id est E ad F, & Tα ad OK, composita ad compositam, ut PR ad OR, id est GH ad HI.

Rursum addi possunt lineae ut minor OK, utriusque rationis, habeat antecedentes; ducatur enim ex Q per L recta, quae cum OK, PM lineis conveniet ut ostésum saepius; quia ratio OQ, ad PQ minor est rationeKL ad ML : conveniat igitur in β & γ erit Kγ ad Mβ, addita ad additm, ut KL ad ML, id est per constructionem ut E ad F, & Oγ ad Pβ, tota ad totam, ut OQ ad PQ, id est GH ad HI. [a]. 33. Quinii

13. AB and CD are lines of unequal length, while the ratio of GH to HI is truly less than the ratio E to F, which in turn is larger than the ratio AB to CD. [*i. e.* $a/c < g/h < e/f$ and $a > c$.]
It is not possible to add lines in the inverted ratio.
Indeed let DK and BL be added in the ratio E to F in order that AL to CK is in the ratio GH to HI, if it shall be possible [See Fig.xiii;a]. Now, as AB is to CD, thus BL is to DN, where DN is a line less than DK as BL is itself established as a line smaller than DK, and therefore the total CN is less than CK [*i. e.* BL/DN= a/c]. Therefore as AB to CD, thus BL is to DN; and by adding the ratios, as AB to CD, thus AL to CN. [For the numerators and denominators of like ratios can be added and equality maintained.] The ratio AL to CN is smaller than the ratio GH to HI [For $a/c < g/h$] , that is by construction set equal to the ratio AL to CK. which cannot be the case, as CN is a smaller line than CK : whereby the required ratio cannot be formed by the addition of the reciprocals.

Neither indeed will regular subtraction of the ratio be possible : Let the lines KB and LD are taken in the ratio E to F from AB and CD, and if possible, the ratio AK to CL is as GH to HI [Fig.xiii;b]. Because the ratio AB to CD is put less than the ratio KB to LD, or E to F, [For $a/c < e/f$], the ratio AK to CL, that is by assumption equal to the ratio GH to HI[a], shall be less in turn than the ratio AB to CD, which is contrary to the supposition: whereby ordinary subtraction cannot be used to produce the required ratio.



Prop.18. Fig.xiii.

However the lines can be taken in the ratio E to F, so that the remainders are reciprocally in the ratio GH to HI [See Fig.xiii;c].

Construction: Let KL to ML and MN to KN indeed be made in the ratio E to F. The parallel lines KO and MP are erected from K and M, which are equal to the lines AB and CD, the points O and P are joined; and the points Q and R are chosen so that the ratios OQ to PQ and PR to OR are made in the ratio GH to HI : it is made clear in the tenth case of these propositions that the points N and R, lie on the joins of the lines LN and QR, whereby the lines drawn RL and NQ cut the lines OK and PM : and NQ indeed cuts these lines in S and T, the line RL cuts the same in V and X. It is apparent that TM is to SK, taken to taken, as MN is to KN, that is by construction as E is to F, and SO is to PT, remainder to remainder, as OQ to PQ, that is GH to HL. [Thus, (PM - XM)/(OK - VK) = $(c - e)/(a - f)$ = PX/OV = RP/RO = g/h.]

The lines can also be added in the ordinary ratio E to F, in order that the sums are in the ratio GH to HI. [Recall that $a/c < g/h < e/f$ and $a > c$.] Indeed the line RN can be sent from R through N, that crosses the lines OK and PM: which more often has been shown, when the ratio XR to VR, that is PR to OR, that is by construction GH to HI, which is put less than the ratio E to F, that is MN to KN. The line RN crosses these lines in Z and α. The ratio Mα to KZ, sum to sum, shall be as MN to KN, that is E to F, and Pα [text has Tα] to OK, sum to sum, as PR to OR, that is GH to HI.

[Thus, (PM +Mα)/(OK + KZ) = $(a + e)/(c + f)$ = Pα/OZ = PR/OR = g/h.]

Again the lines can be added, as happens with the smaller of both ratios preceeding. Indeed a line is drawn from Q through L, which meets the lines OK and PM as has often been shown. Because the ratio OQ to PQ is less than the ratio KL to ML, the line QL meets the lines PM and OK in β and γ. The ratio Kγ to Mβ, sum to sum, shall be as KL to ML, that is by construction as E to F, and Oγ to Pβ, total to total, shall be as OQ to PQ, that is GH to HI.

[Thus, (OK +Kγ)/(PM + Mβ) = $(c + e)/(a + f)$ = Oγ/Pβ = OQ/PQ = g/h.]

[18]

14. Sint iterum rectae AB, CD inaequales, ratio vero E ad F maior ratione AB ad CD; quae eadem sit, cum ratione GH ad HI: neque addi, neque demi ordinatim poterunt lineae, in ratione E ad F, ut compositae vel reliquae, rationem habeant GH ad HI. addantur enim ordinatim lineae BK, DL, in ratione E ad F; & si fieri possit, sit AK ad CL, ut GH ad HI. Quoniam igitur AK est ad CL, ut AB ad CD, erit [a] reliqua BK ad DL

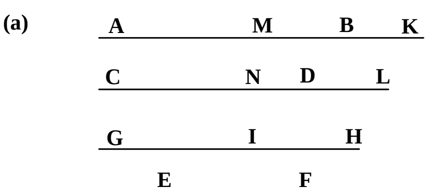
reliquam, id est E ad F, ut AK ad CL, id est ex hypothesi GH ad HI. Quod est contra Supposită. Quare ordinatim addi non poterunt lineae, &c. Neque demi possunt lineae in ratione E ad F ut residuae rationem obtineant GH ad HI, auferantur enim lineae MB, ND in ratione E ad F, & si fieri possit, sit ut GH ad HI, sic AM ad CN, cum igitur sit ut AB ad CD, sic GH ad HI, id est per constructionem AM ad CN, erit [b] MB reliqua ad reliquam ND, ut AB ad CD ; Quod est contra hypothesin. Quare neque ordinatim demi poterunt lineae, etc.

Iisdem positis; neque additio reciproca fieri poterit addantur enim lineae DK, BL, secundum rationem E ad F, & si fieri possit, sit AL ad CK ut GH ad HI. erit igitur AL ad CK, ut AB ad CD, & BL reliqua ad reliquam DK, ut AB ad CD : quare BL maior quam DK. Quod est contra hypothesin. Unde additio reciproca fieri non poterit.

Poterunt ramen detrahi reciprocem lineae in ratione E ad F, ut reliquem rationem obtineant GH ad HI, fiat enim KL ad ML, ut GH ad HI, erectisque ex K & M parallelis KO MN, quae aequales sint lineis AB, CD, ducatur recta ON; occurret illa lineae KL IN L; quia OL est ad NL, ut OK ad NM, id est ut AB ad CD, id est GH ad HI, id est per constructionem ut KL ad ML : dein OP facta aequali ipsi NL fiat ut E ad F, sic KQ ad MQ, ducaturque recta QP : occurret illa lineis OK, NM, quia ratio OL ad NL, id est GH ad HI, id est ratio KL ad ML, minor ponitur ratione E ad F, id est KQ ad MQ:adeoque MQ linea [c] minor recta ML :secet igitur PQ linea, rectas OK, NM in V &X : patet VM esse ad XK ablatam ad ablatam, ut KQ ad MQ, id est E ad F : & NV esse ad OX , residuum ad residuum, ut NP ad Opid est GH ad HI.

Addi etiam poterunt lineae in ratione E ad F, ut minor utriusque rationis habeat antecedentes. Facta enim KR aequali ipsi MQ, ducatur recta PR : occurret illa lineis OK, NM in S & T: quia ratio MR ad MR, id est E ad F, maior ponitur ratione GH ad HI, id est NP ad OP. Unde patet MT esse ad KS, additam ad additam ut MR ad KR, id est E ad F, & NT esse ad OS, compositam ad compositam, ut NP ad OP, id est GH ad HI. [a]

a 19. Quinti ;b 19. Quinti.; c 10. Quinti.



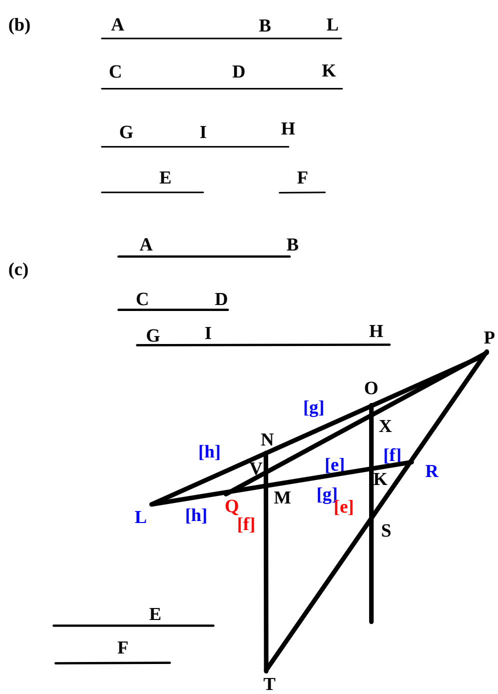
Prop.18. Fig.xiv.

14. AB and CD are again lines of unequal length, with the ratio of E to F truly greater than the ratio of AB to CD, which is the same as the ratio GH to HI: the lines cannot be added or taken in the regular manner in the ratio E to F, in order that the sum or difference has the ratio GH ad HI.
. [*i. e. g/h = a/c < e/f* and *a > c*.]

Indeed the lines BK, DL can be added in the ratio E ad F; and if it were possible, AK to CL will be as GH to HI [See Fig.xiv;a]. Since AK is therefore to CL as AB to CD, the remainder[a] BK to the remainder DL, that is E to F, shall be as AK to CL. that is by hypothesis as GH to HI. This is contrary to the supposition. Whereby regular addition of the lines cannot be performed, etc.

Neither can the lines be taken in the ratio E to F in order that the remaining ratio GH to HI is obtaind, Indeed the lines MB and ND can be taken in the ratio E ad F, and if it were to be the case, GH is to HI as AM is thus to CN. Since AB is to CD, so GH is to HI, that is by construction AM to CN, hence the remaindef[b] MB to the remainder ND, as AB to CD: which is contrary to the hypothesis. Whereby ordinary subtraction of the lines cannot be effected, etc.

With the same lines in position, neither by addition of the reciprocals can the ratio be made to work. [See Fig.xiv;b]. Indeed the lines DK and BL can be added, following the ratio E to F, and if it were to be possible, AL to CK will be as GH to HI. Therefore AL to CK will be as AB to CD, and the remainder BL will be to the remainder DK, as AB to CD : Whereby BL is larger than DK. Which is contrary to the hypothesis. Hence inverted addition is not possible.



Prop.18. Fig.xiv.

However the reciprocal of the lines may be taken in the the ratio E to F, in order that the resulting ratio gives the ratio GH to HI.

[Recall: $g/h = a/c < e/f$ and $a > c$.]

Construction: [See Fig.xiv;c.] Indeed, KL to ML, shall be made in the ratio GH to HI, and the parallel lines KO and MN erected from K & M, which shall be equal to the lines AB and CD, and the line ON is drawn. That line cuts the line KL in L; because OL is to NL as OK to NM, that is as AB to CD, it is as GH to HI, that is by construction as KL to ML. Then OP is made equal to NL itself; and as E to F, thus KQ to MQ. The line QP is drawn.

Demonstration. Because the ratio OL to NL, that is equal to the ratios GH to HI and KL to ML, is put less than the ratio E ad F, that is KQ to MQ. Indeed the line MQ[c] is less than the line ML: therefore the line PQ cuts the lines OK and NM in V and X. It is apparent that VM is to XK, in the taken to taken ratio, as KQ to MQ, that is as E to F: amd NV to OX, remainder to remainder, as NP to OP. That is as GH to HI.

Also the lines can be added in the ratio E to F, as happens with the smaller of both ratios preceeding. Indeed KR is made equal to MQ, and the line PR is drawn and that line cuts the lines OK and NM in S and T. Because the ratio MR to MR, that is E to F, is put larger than the ratio GH to HI, that is NP to OP. Hence it is apparent that MT is to KS, in the ratio of sum to sum, as MR is to KR that is E to F, and NT to be to OS, sum to sum, as NP to OP, that is GH to HI.

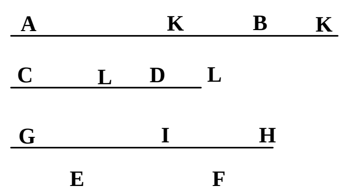
15. Sit iterum ratio AB ad CD, eadem cum ratione GH ad HI, quae maior sit ratione E ad F ; dico ordinatim addi, nec demi posse lineas, in ratione E ad F, ut compositae, vel reliquae, rationem habeant GH ad HI; addantur enim lineae BK, DL in ratione E ad F, & si fieri possit, sit AK ad CL, ut GH ad HI : erit igitur ut AK ad CL, sic AB ad CD ; unde & BK ad DL, id est per constructionem E ad F, ut AK ad CL, id est GH ad HI: Quod est contra hypothesim : igitur ordinata additio non continget.

Eodem modo detrahantur ordinatim BK, LD in ratione E ad F; & si fieri possit sit AK ad CL, ut GH ad HI; erit igitur AK ad CL, ut AB ad CD, quare & BK ad LD, id est E ad F, ut AB ad CD,quod est contra hypothesim, igitur, etc.

[19]

Neque etiam addi, vel demi reciprocem poterunt lineae in ratione E ad F, ut residuae vel compositae rationem habeant GH ad HL. addantur enim DL, BK in ratione E ad F, & si fieri possit sit AK ad CL, ut GH ad HI : erit igitur ut AK ad CL, sic AB ad CD. quare & BK ad DL, est ut AB ad CD, ideoque BK maior ipsa DL. Quod est contra hypothesim: ergo, &c. eodem modo ostenditur detractionem reciprocam fieri non posse.

(a)



Prop.18. Fig.xv.

15. Again the ratio AB to CD shall be the same as the ratio GH to HI, which shall be larger than the ratio E to F ; I say that the lines cannot be added or subtracted in the regular manner in the ratio E ad F, in order that the sum or difference shall be in the ratio GH to HI.

[i. e. $g/h = a/c > e/f$ and $a > c$.]

Indeed the lines BK and DL can be added in the ratio E to F, and we may assume that AK to CL is in the ratio GH to HI. [See Fig.xv;a.] Therefore as AK to CL, thus AB to CD. Hence BK to DL, that is by construction E to F, is thus as AK to CL, that is GH to HI. Which is contrary to the hypothesis : therefore the required ratio cannot be effected by regular addition.

In the same way the ratio BK to LD may be taken in the ratio E to F; and we may assume that AK to CL is in the ratio GH to HI. Therefore AK to CL is as AB to CD, whereby BK to LD, that is E to F, shall be as AB to CD, which is contrary to the hypothesis, therefore, etc,

The inverted or inverted ratio can neither be added nor taken in the ratio E to F, in order that the sum or difference gives the ratio GH to HL. Indeed the ratio DL to BK can be added in the ratio E to F, and if this were possible, then AK to CL would be as GH to HI. Hence, as AK to CL, thus AB to CD. Whereby BK to DL is then as AB to CD, and therefore BK shall be greater than DL, which is contrary to the hypothesis: therefore, etc. It is shown in the same way that subtraction of the reciprocals is not possible. [a 33. Quinti.]

16. Sit ratio E ad F eadem cum ratione AB ad CD, minor autem ratione GH ad HI : dico ordinatim addi nec demi posse lineas in ratione E ad F, sic ut compositae vel reliquae , rationem habeant GH ad HI, addanur enim BK, DL in ratione E ad F, & si fieri possit sit AK ad CL, ut GH ad HI : erit igitur ratio AK ad CL, id

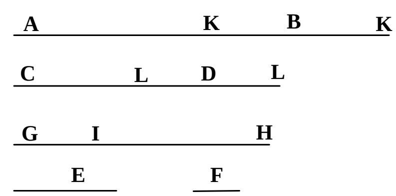
est GH ad HI, maior ratione BK ad CL. id est E ad F; quare & ratio AB ad CD [a] maior ratione AK ad CK, id est GH ad HI. Quod est contra hypothesim:igitur ordinata additio non continget.

Eodem plane modo ostenditur, ordinatam detractionem fieri non posse:

Detrahi tamen reciprocem poterunt lineae in ratione E ad F, ut reliquae rationem habeant GH ad HI. Fiat enim ut E ad F, sic XL ad ML & M1 ad XQ, erectisque ex X & M , parallelis XO, MN, quae lineis AB, CD, sint aequales; ducaur ON linea, quae occurret XL lineae in L, quia OX est ad NM , id est AB ad CD, ut XL ad ML, id est E ad F; dein fiat ut GH ad HI, sic OP ad NP. id est GH ad HI, maior ponitur ratione OL ad NL, id est XL ad ML, id est E ad F ut saepius ostensum :quare iuncta PQ secabit lineas OX, NM in R & S : eritque SM ad RX ablata ad ablatam, ut MQ ad XQ, id est E ad F, & reciproce OR ad NS, ut OP ad NP, id est GH ad HI.

Addi vero reciproce lineae non poterunt in ratione E ad F, &c; addantur enim lineae DL, BK in ratione E ad F, & si fieri possit, sit AK ad CL ut GH ad HI. Fiatque ut AB ad CD, ita BM ad DL. erit BM maior quam DL, adeoque multo maior ipsa BK cum igitur sit ut AB ad CD, sic BM ad DL, & componendo, AM ad CL ut BM ad DL, id est AB ad CD, erit ratio AM ad CL minor ratione AK ad CL, id est GH ad HI: Quod fieri non potest, cum AM recta maior sit recta AK. Quare reciproca additio non continget.

(a)



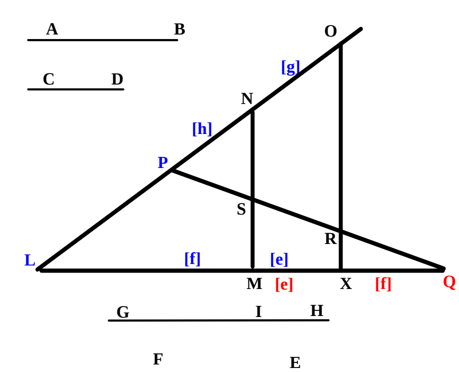
Prop.18. Fig.xvi.

16. AB to CD shall be in the same ratio as E to F, but smaller than the ratio GH to HI: I say that neither regular addition nor subtraction of the lines in the ratio E to F, as sums or differences, can give the ratio GH to HI. [i. e. $a/c = e/f < g/h$ and $a > c$.]

Indeed BK to DL can be added in the ratio E to F, and if it were possible AK to CL is thus as GH to HI. [See Fig.xvi;a.] Therefore the ratio AK to CL, that is GH to HI, is greater than the ratio BK to DL; that is E to F. Whereby the ratio AB to CD [a] is greater than the ratio AK to CK, that is GH to HI. Which is contrary to the hypothesis. Therefore regular addition cannot be used to obtain the result.

Clearly it can be shown in the same way that regular subtraction cannot be used to obtain the result.

(b)



Prop.18. Fig.xvi.

However the inverted ratio of the lines can be taken in the ratio E to F, in order that the differences are in the ratio GH to HI. [See Fig.xvi;b.] Indeed XL to ML and MQ to XQ are made in the ratio E to F thus, and the parallel lines XO and MN are erected from X and M, which are equal to the lines AB and CD. The line ON is drawn which crosses the line XL in L. Because OX is to NM as AB to CD, and as XL to ML, that is E to F. Then GH to HI shall be made thus as OP to NP. That is GH to HI is put greater than the ratio OL to NL, i. e. XL to ML, or E to F as is often made clear : whereby the points P and Q are joined and the line will cut the lines OX and NM in R and S. SM to RX in the taken in the ratio will be as MQ to XQ, that is E to F, and reciprocally OR to NS, as OP to NP, that is GH to HI. [*i. e.* $(a - f)/(b - e) = g/h$.]

The lines cannot be added in the inverted ratio E to F, etc. [See Fig.(xvi, c)] Indeed the lines DL and BK can be added in the ratio E to F, and if it were possible, AK to CL would be as GH to HI. The ratio BM to DL can be made as AB to CD, where BM is greater than DL, and thus much greater than BK: as therefore AB to CD, thus BM to DL, and by addition, as AM to CL thus BM to DL, i. e. AB to CD, and the ratio AM to CL will be smaller than the ratio AK to CL, that is GH to HI, as postulated: but this is not possible, as the line AM is greater than the line AK. Whereby addition of the inverted ratio cannot be considered.

[20]

17. Sit iterum ratio E ad F inaequalitatis & eadem cum ratione AB ad CD, quae maior sit ratione GH ad HI, dico ordinatum lineas in ratione E ad F demi vel addi non posse, ut reliquae vel compositae rationem habeant GH ad HI. demantur enim in ratione E ad F, lineae BK LD, sitque si fieri possit AK ad CL, ut GH ad HI: cum igitur sit ut AB ad CD, sic E ad F, id est KB ad LD, ablata ad ablatam ; erit & AK ad CL reliqua

ad reliquam, ut AB ad CD; quod est contra hypothesim: cum ratio AK ad CL, id est per constructionem GH ad HI, minor ponatur ratione AB ad CD. Quare ordinata detractio non continget.

Eodem modo probatur ordinatam additionem fieri non posse.

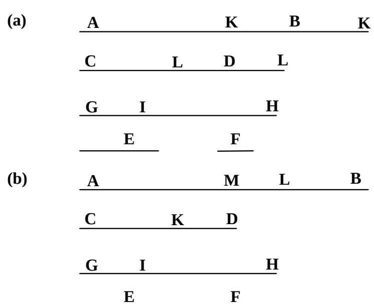
Sed neque detractio reciproca continget: demantur enim in ratione E ad F, lineae KD, LB; & si fieri possit, sit ut GH ad HI, sic AL ad CK : fiat dein ut AB ad CD, sic MB ad KD, erit MB maior, quam KD, adeoque multo maior recta LB. Igitur cum sit AB ad CD, sic MB ad KD, erit & AM ad CK, ut AB ad CD, sed ratio AM [a] ad CK, minor est ratione AL ad CK, id est GH ad HI: Quod est contra hypothesim. Quare nec reciprocem lineae detrahi poterunt in ratione E ad F, &c.

Poterit tamen fieri additio reciproca, &c. Fiat enim ut E ad F, sic KL ad ML, erectisque ex K & M parallelis KN, MO, quae AB, CD lineis sint aequales; ducatur recta ON, fiatque OP ad NP, item NR ad OR, ut GH ad HI : occurratque OP lines rectae KM in Q ; erit NQ linea minor recta NP. quia ratio OP ad NP, id est GH ad HI, minor est ratione OQ ad NQ, id est OM ad NK, sive AB ad CD: quare & iuncta PL secabit lineas NK, OM, in S & T. Unde KS est ad MT, addita ad additam, ut KL ad ML, id est E ad F, & OT ad NS, ut OP ad NP, id est GH ad HI.

Additi etiam sic poterunt lineae in ratione E ad F, ut NK minor utriusque rationis habeat antecedentes, ducatur enim recta RL: occurret illa lineis OM, NK in V & X, quia ratio KL ad ML, id est E ad F maior ponitur ratione NR ad RO. Unde erit KX ad MV, addita ad additam, ut KL ad ML, id est E ad F, & NX, ad OV, composia ad compositam, ut NR ad RO, id est GH ad HI. [a 3. Quinti.]

Scholion.

Possent in hac propositione plures casus determinari, & aliqui etiam , quibus pooppositio absolui non potest ; sed ne taedium Lectori adferam, consultò abstinui; satis esse ducens, viam ad reliquas determinationes Geometriae studioso aperuisse.



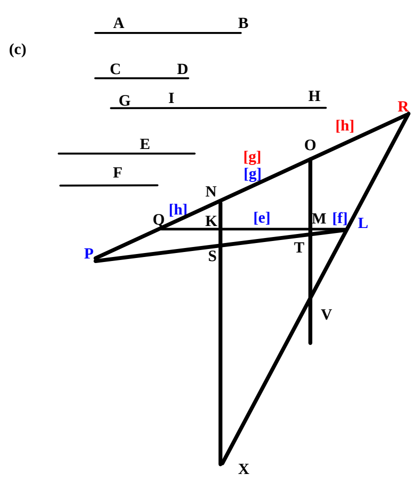
Prop.18. Fig.xvii.

17. Again the ratio E to F shall be unequal, and the same as the ratio AB to CD, which shall be larger than the ratio GH ad HI. I say that the lines in the ratio E to F cannot be added or taken in the regular manner, in order that the difference or sum shall be in the ratio GH ad HI. [i. e. $a/c = e/f > g/h$ and $a > c$.]

Indeed the lines BK and LD can be taken away in the ratio E to F, [See Fig.xvii;a], and if it were to be possible, AK to CL is as GH to HI. Therefore, since AB to CD is in the same ratio as E to F, this is KB to LD in the 'take to take' ratio. Thus AK to CL, remainder to remainder, shall be as AB to CD, which is contrary to the hypothesis: since the ratio AK to CL is by construction GH to HI, put smaller than the ratio AB to CD. Whereby regular subtraction cannot be effected.

In the same way ordinary addition cannot be approved.

Inverse subtraction does not hold either. Indeed the lines KD and LB can be taken in the ratio E to F [See Fig.xvii;b], and if it were possible, AL to CK would thus be as GH to HI. As AB to CD, MB to KD is then made in the same ratio, and MB is larger than KD, and thus much larger than LB. Therefore as AB to CD, so thus is MB to KD, and AM to CK, will be as AB to CD, but the ratio AM [a] to CK is smaller than the ratio AL to CK, that is GH to HI. Which is contrary to the hypothesis. Whereby lines cannot be taken in the inverse ratio E ad F, etc.



Prop.18. Fig.xvii.

However inverse addition can be performed, [See Fig.xvii;c]. Thus the ratio KL to ML can be made as E to F, and the parallel lines KN and MO erected from K and M which shall be equal to AB and CD. The line ON is drawn, and the ratio OP to NP, likewise NR to OR, shall be made as GH to HI. The line KM cuts the line OP in Q ; then NQ is shorter than NP, because the ratio OP to NP (that is GH to HI), is smaller than the ratio OQ to NQ (that is OM to NK), or AB to CD. Whereby the joined line PL cuts the lines NK and OM in S and

T. Hence KS is to MT, 'sum to sum', as KL to ML (that is E to F), and OT is to NS, as OP to NP (that is GH to HI).
[*i. e.* $(a + f)/(b + e) = g/h$.]
The lines can also be added in the regular ratio E to F, as NK is the smaller part of the foregoing ratio. The line RL is drawn that cuts the lines OM and NK in V and X, and the ratio KL to ML (that is E to F) is put greater than the ratio NR to RO. Hence KX to MV, added to added, shall be as KL ad ML (that is E to F), and NX to OV, sum to sum, as NR to RO, (that is GH to HI). [*i. e.* $(a + e)/(b + f) = g/h$.]

Note.

Many cases are determined in this proposition, and indeed others for which the proposition could not be solved ; but to spare the reader the tedium , I have purposely witheld proceeding further; to be satisfied to show the way, and to leave the rest of the determinations for the eager student of geometry.

[37]

PARS TERTIA.

De Rectangulorum inter se proportione.

PROPOSITIO LV.

Si AB linea, divisa fuerit utcunque in C & D.
Dico ACB, CDB rectangula, aequalia esse rectangulis BDA, DCA.

Demonstratio.

Quadratum AB, aequale est BA quadrato; sed AB [a] quadratum aequatur quadratis AC, CD, DB, una cum rectangulis ACB, CDB, bis sumptis, & BA quadratum aequatur quadratis BD, CD, CA, una cum rectangulis BDA, DCA bis sumptis. Igitur ablatuis communibus quadratis AC, CD, DB, remanent ACB, CDB rectangula, aequalia rectangulis BDA, DCA. a. 4 Secundi.

Corollarium.

Propositio haec quoque vera est, si AB linea, utcumque & quotcumque punctis dividatur. Eademque est methodus progrediendi, & demonstrandi, qua in propositione usi sumus.

[37]

PART THREE.

Concerning rectangles in proportion.

BOOK I.§3. *PROPOSITON 55.*

IF the line AB is divided in some way by the points C and D, then I assert that the sum of the rectangles ACB and CDB is equal to the sum of the rectangles BDA and DCA.

Prop.55. Fig.1

Demonstration.

The square AB is equal to the square BA; but the square AB [a] is equal to the sum of the squares AC, CD, and DB together with twice the rectangles ACB and CDB, and the square BA is equal to the sum of the

squares BD, CD, and CA together with twice the rectangles BDA and DCA. Therefore with the common squares AC, CD, and DB taken away, there remains the sum of the rectangles ACB and CDB equal to the sum of the rectangles BDA and DCA. a. 4 Secundi.
[Thus, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2.(ab + bc + ca) = a^2 + b^2 + c^2 + 2.bc + 2a.(b + c) = a^2 + b^2 + c^2 + 2.ab + 2c.(a + b)$, giving $bc + a.(b + c) = ab + c.(a + b)$: contiguous elements can be added.]

Corollary.

This proposition is also true if the line AB is divided in any manner by any number of points. We are to proceed and demonstrate the proposition by the same method as we have used in this proposition.

PROPOSITIO LVI.

Si fuerit AB ad BC, sic AD ad DF, & BC linea aequalis EF. Dico ABCE, rectangulam aequale esse rectangulo CBD.

Demonstratio.

Quoniam per constructionem est, ut AB ad BC, sic AD ad DF, erit permutando, invertendo, ut AD ad AB, sic DF ad FE, id est ad BC : & dividendo, ut AB ad BD, sic FE ad DE, id est BC ad DE; quare ABDE rectangulum, aequale est rectangulo CBD. Quod erat demonstrandum. a. 4 Secundi.

BOOK **I.§3.** *PROPOSITON 56.*

IF the ratio AB to BC is equal to AD to DF, and the lines BC and EF are equal, then I assert that the rectangle ABDE is equal to the rectangle CBD .

Prop.56. Fig.1

Demonstration.

Since by construction, AB to BC is in the same ratio as AD to DF, thus by inverting and interchanging, AD to AB is as DF to FE, that is BC: and on division, as AB to BD, thus FE to DE, that is BC to DE; whereby rect. ABDE is equal to rect. CBD. Q.e.d.
[Thus, AB/BC = AD/DF, and hence AD/AB = DF/BC = DF/FE.
Hence, AD/AB - 1 = DF/FE - 1, or BD/AB = DE/FE = DE/BC: giving AB.DE = CB.BD as required.]

PROPOSITIO LVII.

Si fuerit AB recta divisa in C & D ut AC DB, lineae sint inter se aequales; Dico CB quadrarum aequari quadrato AC una cum rectangulo ABCD.

Demonstratio.

Quadratum AB, aequale est quadratis AC, CB, una cum ACB rectangulo bis sumpto; sed AC quadratum una cum rectangulo ACB aequali est

[38]

rectangulo CAB, igitur quadratum AB, aequale est quadrato CB, una cum rectangulis ACB, CAB. Rursum AB quadratum, aequale est rectangulis ABCD, CAB, DBA : igitur quadratum CB, una cum rectangulis

ACB, CAB , aequale est rectangulis ABCD, CAB, DBA. quare dempto communi rectangulo CAB; aequalia remanent CB quadratum , una cum rectangulo ACB, rectangulis ABCD, DBA. id est rectangulis ABCD, BDA, una cum quadrato DB : ablatis igitur aequalibus rectangulis BDA, ACB, manet CB quadratum, aequale quadrato DB, id est AC, una cum rectangulo ABCD. Quod fuit demonstrandum.

BOOK I.§3. *PROPOSITON 57.*

IF the line AB is divided by the points C and D in order that the lines AC and DB are equal, then I assert that the square CB is equal to the sum of the square AC and the rectangle ABCE.

Prop.57. Fig.1

Demonstration.

The square AB is equal to the sum of the squares AC and CB together with twice the rect. AC.CB. But the square AC together with the rect. AC.CB is equal to rect. CA.AB, and hence the square AB is equal to the square CB together with the sum of the rectangles AC.CB and CA.AB. Again, the square AB is equal to the sum of the rectangles ABCD, CA.AB, and DB.BA. Therefore the square CB together with the rectangles AC.CB and CA.AB is equal to the rectangles ABCD, CA.AB, and DB.BA. Whereby with the common rect. CA.AB taken away, there remains the square CB with the rect. AC.CB equal to the sum of the rectangles ABCD and DB.BA, that is to the rectangles ABCD and BD.DA together with the square DB : with the equal rectangles BD.DA and AC.CB taken away, there remains the square CB equal to the square DB or AC together with the rect. ABCD. Q.e.d.

[Thus, $AB^2 = AC^2 + CB^2 + 2.AC.CB$. But $AC^2 + AC.CB = CA.AB$;
hence $AB^2 = CB^2 + AC.CB + CA.AB$.
Again, $AB^2 = AB.CD + CA.AB + AB.BD$;
hence $CB^2 + AC.CB + \cancel{CA.AB} = AB.CD + \cancel{CA.AB} + AB.BD = AB.CD + BD^2 + AD.DB$;
and as $AC.CB = AD.DB$, there remains $CB^2 = AB.CD + BD^2$. (This can be more easily found from the difference of the squares of CB and BD.)]

PROPOSITIO LVIII.

Si fuerit AD linea, divisa in B & C ut AB CD, lineae sint aequales, sumatur autem inter B & C, punctum quodvis E,
Dico AED rectangulum, aequale esse rectangulis CEA, EBA, una cum quadrato AB.

Demonstratio.

Rectangulum AED, aequale est rectangulis a ABEC, BEC, BECD, una cum quadrato AB; sed iisdem, aequalia sunt rectangula AECD ABE, una cum quadrato AB; (quia AEC, b aequatur rectangulis ABEC, BEC) igitur AED rectangulum, aequale est rectangulis CEA, EBA, una cum quadrato AB. Quod fuit demonstrandum. a 1. Secundi ;b 2 ibid.

BOOK I.§3. *PROPOSITON 58.*

IF the line AD is divided by the points B and C in order that the lines AB and CD are equal, also some other point E is taken between B and C, then I assert that the rectangle AED is equal to the sum of the rectangles CEA and EBA, together with the square AB.

Prop.58. Fig.1

Demonstration.

The rect. AE.ED is equal to the sum of the rectangles [a] AB.EC, BE.EC, BE.CD, together with the square AB; but this is also equal to the sum of the rectangles AE.EC, AB.BE, together with the square AB; (since rect. AE.EC b is equal to the sum of the rectangles AB.EC and BE.EC). Therefore the rect. AE.ED is equal to the sum of the rectangles CE.EA and EB.BA together with the square AB. Q.e.d. a 1. Secundi ;b 2 ibid.

[Thus, $AE.ED = (AB + BE).(EC + CD) = \underline{AB.EC} + BE.EC + \underline{BE.CD} + AB^2 = \underline{AE.EC} + AB.BE + AB^2$, as $AE.EC$ [b] $= \underline{AB.EC} + \underline{EB.BA}$, as the underlined rectangles are equal. Hence, $AE.ED = AE.EC + AB.BE + AB^2$ as required.]

PROPOSITIO LIX.

Si fuerit AC linea, utcunque divisa in D, B, & E ;
Dico rectangula ADC, AEC, DBE ; aequalia esse rectangulis ABC, ADB, BEC, ADEC.

Demonstratio.

Rectangulum ADC, aequale est rectangulis [c] ADB, ADBE, ADEC: & AEC rectangulum [d] aequale est rectangulis CEB, CEBD, CEDA; quare addito rectangulo DBE, erunt ADCL AEC, DBE rectangula, aequalia rectangulis ADB, ADBE, ADEC, CEB, CEBD, CEDA, DVBE. Sed & ABC rectangulum aequale est rectangulis DBE, ADBE, CEBD, CEDA; additis igitur rectangulis ADB, BEC, ADEC erunt ABC, ADB, BEC, ADEC rectangula, aequalis rectangulis ADB, ADBE, ADEC, CEB, CEBD, CEDA, DBE: sed & iisdem rectangulis, ostensa sunt aequalis, rectangula ADC, AEC, DBE; Igitur rectangula ADC, AEC, DBE , aequalia sunt rectangulis ABC, ADB, BEC, ADEC. Quod fuit demonstrandum. c ibid ;d ibid.

BOOK I.§3. *PROPOSITON 59.*

IF the line AC is divided by the points D, B and E in some manner, then I assert that the sum of the rectangles ADC, AEC, and DBE is equal to the sum of the rectangles ABC, ADB, BEC, and ADEC.

Prop.59. Fig.1

Demonstration.

The rect. AD.DC is equal to the sum of the rectangles [c] AD.DB, AD.BE, AD.EC; the rect. AE.EC [d] is equal to the sum of the rectangles CE.EB, CE.BD, and CE.DA. Whereby, with the rect. DB.BE added, the sum of the rectangles AD.DC, AE.EC, and DB.BE is equal to the sum of the rectangles AD.DB, AD.BE, AD.EC, CE.EB, CE.BD, CE.DA, and DB.BE . But rect. AB.BC is equal to the sum of the rectangles DB.BE, AD.BE, CE.BD, and CE.DA; therefore with the rectangles AD.DB, BE.EC, AD.EC added, the sum of the rectangles AB.BC, AD.DB, BE.EC, AD.EC is equal to the sum of the rectangles AD.DB,

AD.BE, AD.EC, CE.EB, CE.BD, CE.DA, and DB.BE. But the same rectangles have been shown to be equal to the sum of the rectangles AD.DC, AE.EC, and DB.BE. Therefore the sum of the rectangles AD.DC, AE.EC, and DB.BE is equal to the sum of the rectangles AB.BC, AD.DB, BE.EC, and AD.EC. Q.e.d. c ibid ;d ibid.
[Thus, AD.DC = AD.DB + AD.BE + AD.EC .
The rect. AE.EC = CE.EB + CE.BD + CE.DA
Whence, on adding: AD.DC + AE.EC + DB.BE = AD.DB + AD.BE + AD.EC + CE.EB + CE.BD + CE.DA + DB.BE . But rect. AB.BC = DB.BE + AD.BE + CE.BD + CE.DA;
hence: AD.DC + AE.EC + DB.BE = AD.DB + AD.BE + AD.EC + CE.EB + CE.BD + CE.DA + DB.BE = AD.DB + AB.BC + AD.EC + CE.EB.
Hence, AD.DC + AE.EC + DB.BE = AD.DB + AB.BC + AD.EC + CE.EB.
Using the algebraic notation introduced in Prop. 1, where AD = a, etc. this result amounts to:
$a(d + b + e) + e(a + d + b) + db = (a + d)(b + e) + ad + be + ae$.]

PROPOSITIO LX.

Si fuerit AB linea, divisa in quinque partes aequales, punctis C, D, E, F & ei quaevis in directum adijciatur GA ;
Dico GD quadratum, aequalia esse quadrato AG, una cum rectangulo GCB.

[39]

Demonstratio.

Quadratum GD, aequale est quadratis [a] AG, AD una cum rectangulum GAD bis sumpto : sed AD quadratum aequale est quadratis AC, CD una cum rectangulo ACD bis sumpto [b]; id est una cum quadratis DE , EF, & GAD rectangulum bis sumptum, aequalia est rectangulis GACE, GAEB, id est rectangulo GACB (ob CE, EB lineas aequales lineae CB) igitur quadratum GD, aequale est quadratis AG, AC, CD, DE, EF, una cum rectangulo GA, CB. Rursum rectangulum GCB aequale est rectangulis GACB, ACB. sed ACB rectangulum aequale est rectangulis ACD, ACDE, ACEF, ACFB, id est quadratis AC, CD, DE, EF, una cum rectangulo GACB; Igitur addito quadrato AG, erit GCB rectangulum, una cum quadrato AG, aequale quadrato GD. Quod erat demonstrandum. a 4. Secundi ; b Per eandem.

BOOK I.§3. *PROPOSITON 60.*

IF the line AB is divided by the points C, D, E and F into five equal parts adjoined in some manner along that line GA, then I assert that the square is equal to the sum of the square AG and the rectangle GCB.

Prop.60. Fig.1

[39]

Demonstration.

The square GD is equal to the sum of the squares [a] AG and AD together with twice the rect. GA.AD ; but the square AD is equal to the sum of the squares AC and CD together with twice the rect. AC.CD [b], that is with the squares DE and EF; and twice the rect. GAD is equal to the sum of the rectangles GACE and GAEB, that is to the rect. GACB (as the lines CE and EB are equal to the line CB). Therefore the square GD is equal to the sum of the squares AG, AC, CD, DE, and EF, together with the rect. GA.CB. Again, the rect. GCB is equal to the sum of the rectangles GA.CB and AC.CB . But the rect. ACB is equal to the sum of the rectangles AC.CD, AC.DE, AC.EF and AC.FB, that is to the sum of the squares AC, CD, DE, and EF and the rectangle GA.CB. Therefore the sum of the rect. GC.CB and the square AG is equal to the square GD. Q.e.d. a 4. Secundi ; b Per eandem.

[$GD^2 = AG^2 + AD^2 + 2.GA.AD$; $AD^2 = AC^2 + CD^2 + 2.AC.CD = AC^2 + CD^2 + DE^2 + EF^2$;
$2.GA.AD = GA.CE + GA.EB = GA.CB$; therefore
$GD^2 = AG^2 + AC^2 + CD^2 + DE^2 + EF^2 + GA.CB$. Again,
$GC.CB = GA.CB + AC.CB$;
but $AC.CB = AC.CD + AC.DE + AC.EF + AC.FB = AC^2 + CD^2 + DE^2 + EF^2$; hence
$GC.CB = GA.CB + AC^2 + CD^2 + DE^2 + EF^2$; and therefore
$GD^2 = AG^2 + GC.CB$

Using the algebraic notation introduced in Prop. 1, where GA = g, AC = a, etc. this result amounts to:
$(g + 2a)^2 = g^2 + 4a.(g + a)$.]

PROPOSITIO LXI.

Sicut AB in quoties partes utcunque in C, D, E, &c.
Dico quadrato AC, CD, DE, EB, simul cum rectangulis ACB, CDB, DEB bis sumptis, aequari quadrato totius AB.

Demonstratio.

Quadratum enim AB, aequatur quadratis AC, CB, & rectangulo [c] ACB bis sumpto : eodem modo quadratum CB aequale est quadratis CD, DB , & rectangulo CDB bis sumpto; denique & quadratum DB aequale est quadratis DE, EB, & rectangulo DEB bis sumpto. collectis igitur in unum quadratis AC, CD, DE, EB, & rectangulis ACB, CDB, bis sumptis; exsurget quantitas, quadrato AB aequalis
Quod erat demonstrandum. c 4. Secundi .

Corollarium.

Hinc colligere licet; quod de duabus rectis lineis, quomodocumque divisis etiam secundum dissimillimas rationes, iudicium ferre debeamus. Ex discursu enim posito in demonstratione huius propositionis constat i quadrata partium cuiuscumque lineae, cum rectangulis bis simul sumptis; quae sub partibus fiunt, secundum tenorem proportione contentum, aequalia esse quadrato totius; unde eam propositionem habere necessarium est quadrata partium ratios, simul cum rectangulis bis sumptis, ad quadrata omnia partium alterius, cum rectangulis suis bis sumptis, quam ipsam de quadrata totarum inter se obtinent. Quod admiratione non caret, cum una quantitatum, in paucissimas partes possit dividi, plaera vero in quam plurimas.
Hoc etiam quod subiungam, ignorantibus Geometriam maxime, videbitur parum credibile; si datum numerum verbi gratia 100 quis iubeatur, secum iacitus in plures pro libitu partes partiri, deinde singularum partium quadrata, in unam summam collecta, seponat; quae summae, ex multiplicationibus partium inter se, secundum sensum propositione contenum, coniuncta, certum quaedam numerum sibi computarit. Alter vero Geometrie gnarus, sponsione cum eo facta, certet se divinaturum etiam eum numerum, quem supputatione facta, in codicillis conscripserit. Ut res etiam ryronum captui magis accommodetur: eam fusius nonnihil deducam.

BOOK **I.§3.** *PROPOSITON 61.*

Let the line AB be cut a number of times by the points C, D, E, etc in some manner adjoined in some manner. I assert that the sum of the squares AC, CD, DE, EB taken together with twice the sum of the rectangles ACB, CDB, and DEB is equal to the square of the whole length AB.

Prop.61. Fig.1

Demonstration.

The square AB is indeed equal to the sum of the squares [c] AC and CB together with twice the rect. AC.CB; in the same manner the square CB is equal to the sum of the squares CD and DB together with twice the rect. CD.DB, and finally the square DB is equal to the sum of the squares DE and EB and twice the rect. DE.EB. Therefore with the squares AC, CD, DE, EB collected together and with the rectangles ACB, CDB taken twice, there quantities arise equal to the given square AB. Q.e.d. c 4. Secundi.

[$AB^2 = AC^2 + CB^2 + 2.AC.CB$; $CB^2 = CD^2 + DB^2 + 2.CD.DB$; $DB^2 = DE^2 + EB^2 + 2.DE.EB$; therefore $AB^2 = AC^2 + CD^2 + DE^2 + EB^2 + 2.AC.CB + 2.CD.DB + 2.DE.EB$.

Using the algebraic notation introduced in Prop. 1, where AC = *a*, etc. this result amounts to:
$(a + c + d + e)^2 = a^2 + c^2 + d^2 + e^2 + 2a(c + d + e) + 2c(d + e) + 2de$.]

Corollary.

Hence we are allowed to sum what is true for two lines, and we ought to make some comments concerning lines divided in any manner into unlike ratios. Indeed in the course of the demonstration of this proposition the position has been taken : find the sum of the squares of the arbitrary ratios of the parts of the line considered, together with twice the sum of the rectangles, and following the same course with the method applied to the sum of the squares and rectangles of any further subdivisions. The sum is equal to the square of the whole length. Thus, to have this proposition true, it is necessary to have the sum of the squares of the ratios of one subdivision, together with twice the sum of the rectangles, equal to the sum of the squares of some other subdivision with their rectangles taken twice, in order that each gives the square of the total length. For it is not without our admiration that a quantity can be divided into its smallest parts possible ....[Note: the microfilm I am using has some illegible words at the end of this section].
This section which I am going to add as well, with complete disregard for geometry, will make it seem a little more credible ; if a given number, say 100, is taken to be set out in several parts for argument's sake, then the squares of the individual parts are collected together in a single sum and set aside; which sums, by multiplication of the parts among themselves, following the argument of the proposition in contention, are added together and a certain number computed. The correct number is otherwise indeed known from geometry, and one might make a wager based on that fact that the same number is indeed written in your notebook from computation .

[40]

Ponatur numerus 100 expositus; divisus in quotuis partes; verbe gratia in AC, CD, DB. sitque AC partium 20 : CD, partium 30: residuum igitur partium 50. iubetur singulas partes multiplicare per seipsas; prodacent hae multiplicationes tres summam; quarum prima 400; secunda 900. tertia 2500. partes continebit. Ulterius iubetur AC, multiplicare per numerum CB. hoc est 20. per 80. & emergit summa 1600. quae bis sumpta, excrescit ad summam 3200. Denique hoc facto, etiam numerum CD ducere in DB, & exsurget summa 1500. quae bis sumpta efficit numerum 3000. tandem colligit hae producta in unam massam, cuius summa est 10000. quam Geometra discursu propositionis iam positae, sola multiplicatione numeri 100. per 100. factam illico manifestam habebit; scilicet 10000.

100

A **20** **C** **30** **D** **50**

400 **900** **2500**

1600 **1500**

3200 **3000**

400
900
2500
3000
3200

10000

The number 100 is set out divided into a number of parts; for argument's sake into the parts AC, CD, and DB. AC shall be 20 of the whole, CD 30, and the remainder DB 50. The individual parts are selected and squared, and these multiplications produce three sums, the first of which will contain 400, the second 900, and the third 2500 parts. Of the others to be selected is AC times by CB, that is 20 by 80, giving rise to the sum 1600, which taken twice is increased to the sum 3200. Briefly according to this, the number CD is taken with DB, and the sum 1500 arises, which taken twice gives rise to the number 3000. Finally collect these numbers into a single term, the sum of which is 10000. Since by the geometrical discourse of the proposition thus established, one need only multiply the number 100 by itself, which will of course show to be 10000 immediately.

PROPOSITIO LXII.

Si fuerit AB linea, divisa utcunque punctis CD. Dico rectangulum sub AB, & composita ex ACDB ; una cum rectangulo sub CD, & composita ex ACDB; equalia esse rectangulis BCA, DBC, ADB, CAD, simul sumptis.

Demonstratio.

Rectangulum super AB, & composita ex AC, DB, aequale [a] est rectangulis CAB, DBA; id est rectangulis ACB, ADB, una cum quadratis AC, DB ; id est aequale rectangulis CDB, ACDB, DCA, BDCA, una cum quadratis AC, DB. Rursum rectangulum super CD, & composita ex ACDB, aequale est rectangulis ACD, CDB: igitur rectangulum super AB, & composita ex ACDB, unu cum rectangulo super CD, & composita ex ACDB, aequale est rectangulis, CDB, ACDB, DCA, BDCA, ACD, CDB, una cum quadratis AC, DB. Iterum rectangulum BCA, aequale est rectangulis DCA, BDCA; item DBC rectangulum, aequale est rectangulo BDC, una cum quadrato DB; item ADB rectangulum ,aequale est rectangulis CDB, ACDB; denique rectangulumCAD, aequale est rectangulo ACD, una cum quadrato AC; igitur rectangula BCA, DBC, ADB, CAD, aequalia sunt rectangulo sub AB, & composita ex ACDB, una cum rectangulo sub CD, & composita ex ACDB. Quod erat demonstrandum. a 1. Secundi .

BOOK **I.§3.** *PROPOSITON 62.*

If the line AB were cut in some points CD. I assert that the rectangle under AB, and composed from ACDB is equal to the sum of the rectangles BCA, DBC, ADB and CAD.

Prop.62. Fig.1

Demonstration.

The rectangle on AB, and composed from AC and DB [a] is equal to the rectangles CAB and DBA, that is equal to the sum of the rectangles ACB and ADB together with the squares AC and DB. This is equal to the sum of the rectangles CDB, ACDB, DCA, BDCA, with the squares AC and DB. Again, the rectangle on CD, and composed from AC and DB, is equal to the sum of the rectangles ACD and CDB. Therefore the rectangle on AB, and composed from ACDB, together with the rectangle on CD, and composed from ACDB, is equal to the sum of the rectangles CDB, ACDB, DCA, BDCA, ACD, CDB, together with the squares AC and DB. Again the rectangle BCA is equal to the sum of the rectangles DCA and BDCA; likewise the rectangle DBC is equal to the rectangle BDC and the square DB; and likewise the rectangle ADB is equal to the rectangles CDB and ACDB. Finally the rectangle CAD is equal to the rectangle ACD, with the square AC; therefore the sum of the rectangles BCA, DBC, ADB, and CAD is equal to the rectangle under AB, and composed from ACDB, together with the rectangle under CD, composed from ACDB.
Q.e.d. c 4. Secundi.
[rect. on AB from AC and DB = $CA.AB + DB.BA = CA.CB + AD.DB + AC^2 + DB^2 = CD.DB + AC.DB + DC.CA + BD.CA + AC^2 + DB^2$;
Again, rect. on CD from AC and DB = $AC.CD + CD.DB$. Hence the sum of these rectangles is

= $\underline{CD.DB} + \underline{AC.DB} + \underline{DC.CA} + \underline{BD.CA} + \underline{AC.CD} + \underline{CD.DB} + \underline{AC^2} + \underline{DB^2}$;
Now rect. BC.CA = $\underline{DC.CA} + \underline{BD.CA}$; rect. DB.BC = $\underline{BD.DC} + \underline{DB^2}$; rect. AD.DB = $\underline{CD.DB} + \underline{AC.DB}$;
and rect. CA.AD = $\underline{AC.CD} + \underline{AC^2}$. Hence the sum of these rectangles:
BC.CA + DB.BC + AD.DB + CA.AD = rect. on AB from AC and DB + rect. on CD from AC and DB as required.
In algebraic terms:
$a.(c + d) + d.(c + d) + d.(a + c) + a.(a + c) = (a + d)(c + d) + (a + d)(a + c)$
$= (a + d)(a + 2.c + d) = (a + c + d).(a + d) + c.(a + d)$, where the first and the last terms are those in the theorem.]

PROPOSITIO LXIII.

Rectum AB, divisam utcunque in CD, iterum in E dividere, ut AED rectangulum, aequale sit rectangulo BEC

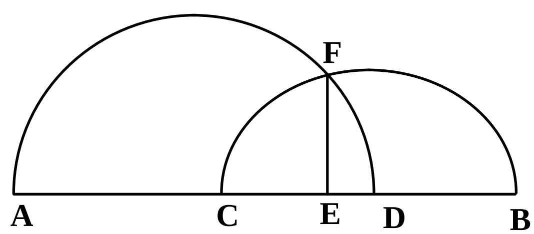
[41]

Constructio & Demonstratio.

Super AD, CB ut diametris, circuli describantur, AFD, DFC qui occurrant sibi mutuo in F: tum ex F recta demittatur FE, normalis ad lineam AB. Dico punctum E, satisfacere peritioni patet; cum tam AED, quam BEC recangulum aequale, [a] sit quadrato FE. igitur lineam AB, utcumque in C & D divisam, iterum secuimus in E; &c. . Quod erat faciendum a 33. Terti .

BOOK I.§3. *PROPOSITON 63.*

The line AB is divided in some way by the points C and D, and for a second time by E, in order that the rectangle AE.ED is equal to the rectangle BE.EC.



Prop.63. Fig.1

Construction & Demonstration.

Upon AB CD as diameters, the circles AFD and DFC are described which cut each other in F. Then from F the line FE is sent normal to the line AB. I assert that the point E satisfies the requirement: since the rect. AE.ED as rect. BE.EC are equal to the square FE [a]. Therefore the line AB, divided somehow by C and D, we have cut in E, etc. Q.e.d. a 35. Tertii.
[AE.ED = CE.EB = EF^2 .]

PROPOSITIO LXIV.

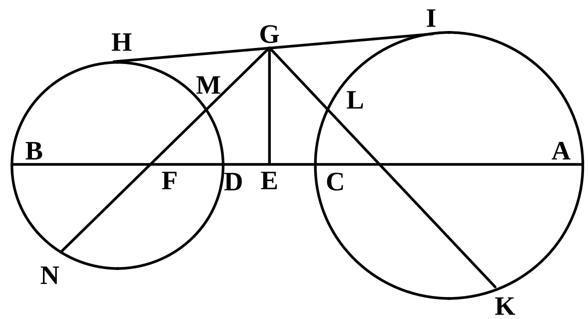
Lineam AB divisam utcunque in C, D, iterum dividere in E , ut CEA rectangulum, aequale sit rectangulo DEB.

Constructio & Demonstratio.

Describantur super AC, DB lineis , ut diametris, circuli AIC, BHD: quos in H & I contingit linea HI; quae bifariam in G divisa, demittatur ex G linea GE normalis ad rectam AB. Dico punctum E, esse quod queritur. secetur BE linea in F, ut EF quadratum sit aequale rectangulo DEB. & per F, ex G ducatur recta GN, occurrens circulo DHB in M, & N. & ex G ducatur altera GK, occurrens circulo AIC in L & K, facta constructione ut prius. Quoniam per constructionem, EF quadrartum, aequale ponitur rectangulo DEB; & HG recta est tangens, erit FG quadraum, aequale [b] rectangulo MGN; sed FG quadratum aequale est quadratis EG, EF. Igitur & MGN rectangulum aequale est quadratis, EG, EF: id est quadrato EG, una cum rectangulo DEB; eodem modo ostendetur LGK rectangulum , aequari quadrato EG, una cum rectangulo, AEC. Unde, cum aequalia sint rectangula LGK, MGN, dempto communi quadrato EG, erunt AEC, DEB rectangula, inter se aequalia. Divisimus igitur lineam AB in E, &c. Quod erat faciendum. [b] Pappi L7. Pr.159.

BOOK I.§3. *PROPOSITON 64.*

The line AB is divided in some way by the points C and D, and for a second time by E, in order that the rectangle EC.EA is equal to the rectangle ED.EB.



Prop.64. Fig.1

Construction & Demonstration.

The circles AIC and BHD are described on the lines AC and DB as diameters, and which the line HI touches the circles in H and I. HI is bisected in G, and from G the line GE is sent normal to the line AB. I assert that E is the point sought. The line BE is cut in F, in order that the square EF is equal to the rectangle DE.EB. From G through F the line GN is drawn, cutting the circle DHB in M and N. From G another line GK is drawn, crossing the circle AIC in L and K, according to the same construction as the first. Since by construction, the square EF is put equal to the DE.EB; and since the line HG is a tangent, the square FG is equal [b] to the rectangle MG.GN; but the square FG is equal to the sum of the squares EG, EF. Therefore the rectangle MG.GN is equal to the sum of the squares EG and EF: that is to the square EG, together with the rectangle DE.EB. In the same way is can be shown that the rectangle LG.GK is equal to the square EG, together with the rectangle AE.EC. Hence, since the rectangles LG.GK and MG.GN are equal with the common square EG taken away, the rectangles AE.EC and DE.EB are equal to each other. Therefore we can divide the line AB in E, &c. Quod erat faciendum. [b] Pappi L7. Pr.159.

PROPOSITIO LXV.

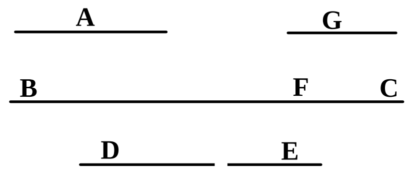
Data sint duae lineae A & BF. oportet BF lineae, quandam FC adjicere, ut quadratum A ad BCF rectangulum datam habeat rationem D ad E.

Constructio & Demonstratio.

Fiat ut D ad E, sic A quadratum, ad quadratum G; dein FB lineae quaedam adiungatur FC, ut BC, G, FC , ttres sint in continua analogia: quid fiet si data differentia extremarum BF, & media G. inveniantur extremae FC, CB per Andersonium & alios. Dico factum esse quod petitur.Quoniam BC, G, FC lineae sunt continuae proportionales, erit CF rectangulum, aequale quadrato G: igitur quadratum A est ad rectangulum BCF ut A quadratum, ad quadratum G; id est per constructionem ut D ad E. Igitur rectae BF quandam addidimus, &c. Quod erat faciendum.

BOOK I.§3. *PROPOSITON 65.*

Two lines are given A and BF, it is required to adjoin some line FC to the line BF, in order that the ratio of the square of A to the rectangle BC.CF is in the given ratio D to E.



Prop.65. Fig.1

Construction & Demonstration.

Thus the ratio of the square A to the square G shall be made in the ratio D to E; then the line FB is added to some line FC, in order that BC, G, and FC are three lines in analogous continued proportion, which can be done if the difference of the extremes of the ratio BF and the mean G are given. A method for finding the extremes of the ratio, FC and CB, has been found by Anderson et al. I assert that what was asked has been done. Since the lines BC, G, and FC are lines in continuous proportion, the rect. BC.CF is equal to the square G. Therefore the square A is to the rect. BC.CF

[42]

as the square A is to the square G; that is by construction as D to E. Therefore the line BF, &c. Q.e.d.
[$A^2 : G^2 = D : E$; $G^2 = BC.FC$; hence $A^2 : G^2 = A^2 : BC.FC = D : E$.
To explore the method would take Gregorius too far out of his way : Alexander Anderson, who came from Aberdeen (King's College library still has his original publications dating from 1615), during his sojourn in Paris, had become acquainted with Vieta's work, and went on to extend and edit Vieta's posthumous papers. In the brief work *Pro Zetetico Apollonii Redivivi*, Anderson investigated the solution of various proportionalities involving cubic equations in a primitive pre-algebraic form that originated with Vieta . Others including Snell were later to work on this problem. A modern reference for the ideas behind this work, which stemmed originally from Apollonius, is p.190 - 191, *A History of Greek Mathematics,* vol. ii, Sir Thomas Heath, Dover 1981.]

PROPOSITIO LXVI.

Datae lineae AB, inaequaliter in C divisae, quandam BD adiicere, ut BDA rectangulum , aequale sit quadrato CD.

Constructio & Demonstratio.

Fiat AE aequalis CB. & fiat quadrato CB aequale rectangulum ECBD. Dico factum esse quid petitur. Sunt enim per constructionem continuae DB, BC, CE, & quia AE aequalis est CB, erunt & BD,DC,DA [a] continuae proportionales ; unde CD quadrato aequale est rctangulum BDA. addidimus igitur rectam, &c. Quod erat faciendum. a 5. Huius.

BOOK I.§3. *PROPOSITON 66.*

For the given line AB, unequally divided in C, some line BD is to be added on to AB, in order that the rectangle BD.DA shall be equal to the square CD.

Construction & Demonstration.

The line AE is made equal to CB. And the square CB is made equal to the rectangle EC.BD. I assert that what was sought has been done. For indeed from the construction the

Prop.66. Fig.1

lines DB, BC, and CE, are in continued proportion. Since AE is equal to CB: BD, DC, and DA [a] are continued proportionals; hence the square CD is equal to the rectangle BDA. We can therefore the line, &c. Quod erat faciendum. a 5. Huius.
[We are given BC/CE = DB/BC ; then BC/DB = CE/BC and CD/DB = AC/BC , giving CD/AC = DB/BC, from which by addition once more, we have AD/AC = CD/BC, and AD/CD = AC/BC = CD/DB;
i. e. AD.DB = CD^2 as required. Geometrically, we jump from ratio to ratio, starting from the r.h.s. of the diagram with BC/BD....]

PROPOSITIO LXVII.

Si fuerit A ad B, ut C ad D, & E ad F, ut G ad H. Dico AH rectangulum, ad rectangulum BG, eam habere ratonem, quam habet CF rectangulum, ad rectangulum DE.

Demonstratio.

Ratio rectanguli AH, ad BG rectangulum, est composita ex ratione A ad b B, & ex H ad G. Sed etiam ratio CF rectanguli, ad rectangulum DE, composita est ex ratione C ad D, id est per constructionem A ad B; & ex F ad E, id est H ad G. Igitur rectangulum AH, ad rectangulum BG, eam habet rationem, quam CF rectangulum, ad rectangulum DE. Quod erat demonstrandum. b 23. Sexti.

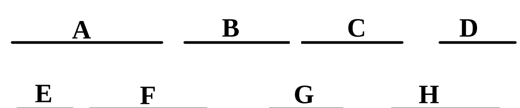
BOOK I.§3. *PROPOSITON 67.*

If the ratio A to B were as C to D, and E to F as G to H, then I say that the rectangle AH to the rectangle BG has the same ratio as the rectangle CF to the rectangle DE.

Demonstration.

The ratio of the rect. AH to the rect. BG, is composed of the ratio A to [b] B, and from H to G. But the same ratio of the rect. CF, to rect. DE, is composed from the ratio C to D, that is by construction A to B; and from F to E, that is H to G. Therefore the rect. AH to the rect. BG has the same ratio as the rect. CF to the rect. DE. Q. e. d. b 23. Sexti.

[AH/BG = A/B . H/G ; likewise, the corresponding ratio CF/DE =C/D . F/E, from which the result follows.]



Prop.67. Fig.1

PROPOSITIO LXVIII.

Si A, B, C, D, lineae fuerint in continua ratione, sint autem & alia quator E, F, G, H in continua analogia; Dico AH rectangulum, ad rectangulum DE, ratonem habere triplicatum eius, quam habet BG rectangulum, ad rectangulum CF.

Demonstratio.

Ratio rectanguli AH, ad rectangulum ED, [c] composita est ex ratione A ad D, id est , ex triplicata [d] ratione B ad C, quia A, B, C, D continuae sunt propotiionales & ex H ad E, id est ex triplicata ratione G ad F; sed

BG rectangulum, ad rectangulum CF, rationem habet compositam [e] ex B ad C, & ex G ad F; igitur rectangulum AH, ad rectangulum DE, rationem habet triplicatam , eius quam habet DE rectangulum, ad rectangulum CF. Quod erat demonstrandum. c 23. Ibid ; d 10 *Defin.* Quinti; e 23 Sexti.

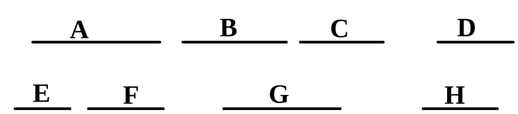
[43]

BOOK I.§3. *PROPOSITON 68.*

If A, B, C and D, were four lines in a continued ratio, and also E, F, G, and H were another four lines in continued proportion similarly, then I say that the rectangle AH to the rectangle DE has the triplicate ratio of that which the rectangle BG has to the rectangle CF.

Demonstration.

The ratio of the rect. AH to the rect. ED [c], is composed of the ratio A to D, that is from the triplicate ratio B to C, as A, B, C, D are continued proportionals and from H to E, that is from the triplicate ratio G to F; but rect. BG, to rect. CF, has a ratio composed [e] from B to C, and from G to F; therefore rect. AH, to rect. DE, has the triplicate ratio, of that which rect. CF has to rect. DE. Q.e.d. c 23. Ibid ; d 10 *Defin.* Quinti; e 23 Sexti.



Prop.68. Fig.1

[$A/B = B/C = C/D = r$ and $E/F = F/G = G/H = s$ are given. Then $AH/ED = A/D \,.\, H/E = r^3/ s^3$; the ratio $BG/CF = B/C \,.\, G/F = r/s$, hence $AH/DE = A/D \,.\, H/E = r^3/s^3 = (BG/CF)^3 = (CF/DE)^3$,as $CF/ DE= C/D.\ F/E = r/s$.]

PROPOSITIO LXIX (LXVII).

Si AB, AC, AD, lineae fuerint continuae, quibus aequales fiant AE, AF, AG, in directum;
Dico DCG rectangulum, ad rectangulum CFB, esse ut DA ad BA.

Demonstratio.

Centro A, intervallis AC, AD, semicirculi describantur DHG, CKF; erectaque ex C normali CH, ducatur recta AH, occurrens circulo CKF in K, iunganturque BK, ut AC ad AD, sic AK ad AH, sed ut AC ad AD sic AB est ad AC per constructionem; igitur ut AB ad AC, sic AK ad AH ; adeoque BK, HC lineae sunt parallelae, & BK recta, normalis ad lineam AC : quare DCG rectangulum, est ad rectangulum CBF, ut HC quadratum, ad quadratum KB, id est in duplicata ratione lineae HC ad KB, id est AC ad AB, id est ut AD, linea ad lineam AB. Quod erat demonstrandum.

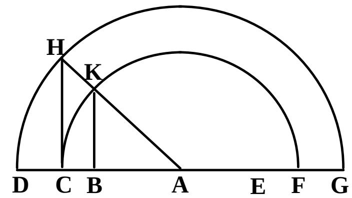
BOOK I.§3. *PROPOSITON 69 (67).*

I

f AB, AC, AD were lines in a continued ratio, from which the equal lines AE, AF, AG, are made in order. I say that the rectangle DCG is to the rectangle CFB as the square DA is to the square BA.

Demonstration.

With centre A, semi-circles DHG and CKF are described with radii AC, AD; and a normal CH is erected from C, and the line AH is drawn, crossing the circle CKF in K, and the points BK are joined. As AC is to AD, thus AK is to AH, but as AC to AD thus by construction AB is to AC; therefore as AB to AC, thus AK to AH ; and hence the lines BK and HC are parallel and the line BK, normal to the line AC : whereby the rect. DCG is to the rect. CBF, as the square HC is to the square KB, that is in duplicate ratio of the line HC to KB, that is AC to AB, that is as the line AD to the line AB. Q.e.d.



Prop.69. Fig.1

[AC/AD = AK/AH; and AC/AD = AB/AC, hence AB/AC = AK/AH and AC/AB = AH/AK. Hence, $DC.CG/CB.BF = CH^2/KB^2 = AC^2/AB^2 = AD^2/AB^2$ as required. Note: Prop. 69 is wrongly called Prop. 67 again, and so on to the end of the first book.]

PROPOSITIO LXX (LXVIII).

Si fuerint quotqunque lineae continuae proportionales AB, CD, DB, EB, FB, GB. Dico rectangula ABDE, CBDE, CBEF, DBEF, DBFG, EBFG in continua esse analogia; & quidem in ratione AB ad CB.

Demonstratio.

Rectangulum enim ABDE ad rectangulum CBDE, est ut AB [a] linea, ad lineam CB; & CBDE rectagulum ad rectangulum CBEF, est ut [b] DE ad EF, id est, ut AB ad CB; Rursum rectangulum CBEF, ad rectangulum DBEF; est ut CB ad DB, id est AB ad CB. quia AB, CB, DB, &c. ponuntur continuae, & DBEF rectangulum, ad rectangulum DBFG, est ut EF ad FG, id est iterum AB ad CB; & sic de ceteris. Igitur rectangula ABDE, CBDE, CBEF, &c. in continua sunt analogia, & quidem in ratione AB ad CB. Quod erat demonstrandum. a 1. *Sexti*; b 2. *Ibid*.

***BOOK* I.§3.** *PROPOSITON 70 (68).*

If there are a number of lines AB, CD, DB, EB, FB, G, B in continued proportion, then I assert that the rectangles AB.DE, CB.DE, CB.EF, DB.EF, DB.FG, EF.GB are similarly in continued proportion, and indeed in the ratio AB to CB.

Demonstration.

Indeed the ratio of rect.AB.DE is to rect. CB.DE as the line AB [a] is to the line CB; and the rect. CB.DE to the rect. CB.EF, is as [b] DE to EF, that is, as AB to CB; Again the rect. CB.EF to the rect. DB.EF is as CB to DB, that is AB to CB, since AB, CB, DB, &c. are placed in continued proportion, and the rect. DBEF to the rect. DB.FG, is as EF to FG, that is again as AB to CB; and so on for the others. Therefore the rectangles AB.DE, CB.DE, CB.EF, etc. are in continued proportion in analogy, and indeed in the ratio AB to CB. Q.e.d. . a 1. *Sexti*; b 2. *Ibid*.

Prop.70. Fig.1

PROPOSITIO LXXI (LXIX).

Si fuerint tres ordines continuae proportionalium A, B, C, D. E, F, G, H. I, J, K, L, M. & AF rectangulo aequale fiat quadratum N: & R quadratum aequale rectangulo EK; sit autem & IF rectangulo, aequale quadratum O, & EB rectangulo, quadratum S. dein & rectangulo CH, aequale quadratum P; & T quadratum , aequale rectangulo GM; denique rectangulo LH, aequale quadratum Q, & GD rectangulo, quadratum V.

Dico quadratum N esse ad quadratum P, ut est quadratum S, ad quadraum V: & R quadratum, ad quadratum T, ut O quadratum ad quadratum Q.

[44]

Demonstratio.

Quadratum enim N ad quadratum P, id est per constructionem, AF rectangulum, ad rectangulum CH rationem habet [a] compositam, ex A ad C, id est per hypothesim ex duplicata ratione A ad B; & ex F ad H, id est duplicata ratione E ad F; sed ratio quadrati S ad quadratum V, id est per hypothesim rectanguli EB, ad rectangulum GD, etiam componitur ex ratione B ad D, id est duplicata ratione A ad B, & ratione E ad G, id est duplicata ratione E ad F, igitur ut quadratum N ad quadratum P: sic quadratum S ad quadratum V. Eodem modo ostenditur quadratum R, ad quadratum T esse, ut quadratum O, ad quadratum Q.
Quod erat demonstrandum. a 23. *Sexti*.

BOOK **I.§3.** *PROPOSITON 71 (69).*

If there are three series of continued proportions A, B, C, D. E, F, G, H. I, J, K, L, M, and the square N is made equal to the rect. AF: and the square R made equal to the rect. EK; and also the square O is equal to the rect. IF, and the square S to the rect. EB, the square P to the rect. CH, the square T equal to the rect. GM, the square Q equal to the rect. LH, and the square V equal to the rect. GD.

I assert that square N is to square P, as square S is to square V: and square R to square T, as square O to square Q.

Demonstration.

A.	**B.**	**C.**	**D.**
E.	**F.**	**G.**	**H.**
I.	**K.**	**L.**	**M.**
N.	**P.**	**S.**	**V.**
R.	**T.**	**O.**	**Q.**

Prop.71. Fig.1

For square N to the square P, that is by construction as rect. AF to rect. CH has the composition ratio [a] from A to C, that is by hypothesis from the duplicate ratio A to B; and from F to H, that is the duplicate ratio E to F; but the ratio of square S to square V, that is by hypothesis of rect. EB to rect. GD, also put together from the ratio B to D, that is the duplicate ratio of A to B, and with the ratio E to G, that is the duplicate ratio E to F. Therefore as square N to square P: thus square S to square V. In the same way it can be shown that square R to square T to be as square O to square Q.

Q.e.d. a 23. *Sexti*.

[In modern terms:
A = a ; B = ar ; C = ar^2 ; D = ar^3. E = e ; F = es ; G = es^2 ; H = es^3; I = i ; K = it ; L = it^2 ; M = it^3.
N^2 = A.F = aes ; R^2 = E.K = eit ; O^2 = I.F = ies ; S^2 = E.B = ear ; P^2 = C.H = aer^2s^3 ; T^2 = G.M = eis^2t^3 ;
Q^2 = L.H = eis^3t^2 ; V^2 = G.D = aer^3s^2 .
$N^2/P^2 = aes/aer^2s^3 = 1/r^2s^2$ = A/C . F/H = $(A/B)^2.(E/F)^2$; $S^2/V^2 = ear/aer^3s^2 = 1/r^2s^2$ = E/G . B/D = $(A/B)^2.(E/F)^2$, from which the result follows.]

PROPOSITIO LXXII (LXX).

Sit AC linea divisa inaequaliter in B, oportet utrimque rectas aequales adijcere AD, CE, ut ABC rectangulum, ad rectangulum DBE, datam habeat rationem, R ad S.

Constructio & Demonstratio.

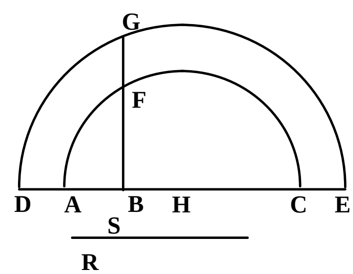
Descripto super AC, ut diametro, semicirculo AFC, erigatur ex B, normalis BF; & fiat ut R linea ad S. lineam, ita BF quadratum , ad quadratum BG, tum centro communi H, intervallo HG, describatur semicirculus DGE occurrens AC lineae productae in D & E. Dico AD, CE, lineas satisfacere petitioni. Rectangulum enin ABC, aequale est quadrato [b] FB, & GB quadrato, aequale est rectangulum DBE: igitur rectangulum ABC, est ad rectangulum DBE, ut FB quadratum , ad quadratum BG, id est linea R ad lineam S per constructionem. datae igitur lineae AC rectas aequales vtrimque adiecimus; &c. Quod erat demonstrandum.

BOOK **I.§3.** *PROPOSITON 72 (70).*

Let AC be a line divided unequally in B, it is required to add equal lines AD, CE, in order that the rectangle ABC to the rectangle DBE shall have the given ratio R to S.

Demonstration.

With a semi-circle AFC described on AC as diameter, a normal BF is erected from B; the square BF shall be made to the square BG in the ratio of the line R to the line S. Then with the common centre H, with radius HG, the semi-circle DGE is described cutting the line AC produced in D and E. I say that the lines AD and CE satisfy the requirements. Indeed the rect. ABC is equal to the square [b] FB, and the square GB is equal to the rect. DBE: therefore the rect. ABC is to the rect. DBE. as the square FB to the square BG, that is to the line R to the line S by construction. Therefore we may add to the line AC equally on both sides, etc. Q.e.d. b 35. *Tertii.*



Prop.72. Fig.1

PROPOSITIO LXXIII (LXXI).

Dato quadrato A, & linea DF, utcunque in E divisi, exhibere rectam, quae divisa secundum rationem DE ad EF, exhibeat quadratum sub tota, una cum rectangulo sub segmentis, ad quadratum A, in data ratione B ad C.

Constructio & Demonstratio.

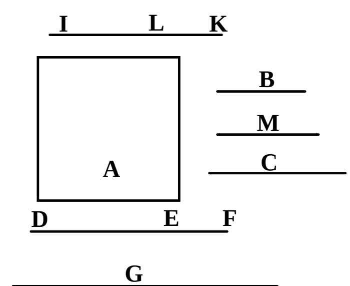
Inventa M, media inter B & C, fiat ut quadratum B ad quadratum M; ita DF quadratum una cum rectangulo DEF, ad quadratum G, dein ut G linea, ad DF lineam. sic latus quadrati, A fiat ad quandam IK; quae ita secetur in L, ut DF est devisi in E. Dico IK lineam esse quaesitam. Quoniam est ut G linea, ad lineam DE, sic latus quadrati A ad rectum IK; erit invertendo, permutando, DF ad IK, ut G ad latus quadrati A. Rersum cum IK, DF lineae proportionaliter sint

[45]

divisae, erit ut IK ad DF sic LK ad EF. Est autem ratio rectanguli ILK ad rectangulum DEF, a composita ex ratione IL ad DE, & LK ad EF, id est ex duplicata ratione IK ad DF, item quadratum IK ad quadratum DF, duplicatam habet rationem , eius quam habet IK linea, ad lineam DF; igitur, erit ILK rectangulum una cum quadrato IK ad rectangulum DEF una cum quadrato DF. in duplicata ratione IK ad DF. & quia est ut DF ad G sic IK ad latus quadrati A, erit ut rectangulum DEF una cum quadrato DF,ad quadratum G, sic ILK rectangulum una cum quadrato IK, ad quadratum A. sed per constructionem est DEF rectangulum una cum quadrata DF, ad quadratum G , ut quadratum B ad quadratum M , id est ut linea B ad lineam C, igitur rectangulum ILK una cum quadrato IK, est ad quadratum A, ut B ad C. Unde dato quadrato A, &c. Quod erat faciendum.

BOOK I.§3. *PROPOSITON 73 (71).*

Given the square A, and a line DF divided by E in some manner, to find a line which has been divided according to the ratio DE to EF, and which shows the ratio: sum of the square under the total length and the rectangle under the segments to the square A that is equal to the given ratio B to C.



Prop.73. Fig.1

Demonstration.

Find the mean between B and C, and thus the square B to the square M is made in the ratio of the sum of the square DF and the rect. DEF to the square G. Then as the line G is to the line DF, thus the side of the square A shall be made to some length IK; which thus is cut in L, as DF had been divided by E. I assert that the line IK is that sought. Since the line G is to the line DE, thus as the side of the square A to the line IK. By inverting and interchanging, the ratio of DF to IK, is as G to the side of the square A. Again, as the lines IK and DF will be divided proportionally as IK to DF thus LK to EF. But also the ratio of the rectangle ILK to the rectangle DEF, [a] put together from the ratio IL to DE, and LK to EF, that is from the square of the ratio IK to DF, likewise the square IK to the square DF, has the square ratio, of that which the line IK has to the line DF. Therefore the rect. ILK added to the square IK will be to the rect. DEF added to the square DF in the squared ratio IK to DF. And because DF is to G thus as IK is to the side of the square A: as the rect. DEF with the square DF, to the square G, thus the rect. ILK with the square IK, to the square A. But from the construction the ratio of the rect. DEF with the square DF, to the square G , shall be as the square B to the square M , that is as the line B to the line C, therefore the rect. ILK with the square IK, is to the square A, as B to C. Hence, from the given square A, &c. Q. e. f. Q.e.d. a 23. *Sexti.*

[$M^2 = B.C$; then $B^2/M^2 = (DF^2 + DE.EF)/G^2$ and $G/DF = a/IK$, where we have set $a^2 = A$,
$IL/LK = DE/EF$ and $\underline{IL/DE} = \underline{LK/EF}$. ILK is the required line.
For, as $DF/IK = G/a$, and as $IK/LK = DF/EF$; $IK/DF = \underline{LK/EF}$, then $IK/DF = LK/EF$. Also, rect. ILK /rect. $DEF = IL.LK/DE.EF = \underline{IL/DE} . \underline{LK/EF} = (\underline{LK/EF})^2 = (IK/DF)^2$ $(= a/G)$. Hence, $IL.LK/IK^2 = DE.EF/DF^2$ and $(IL.LK + IK^2)/(DE.EF + DF^2) = (IK/DF)^2 = a^2/G^2$.
Thus, $(IL.LK + IK^2)/a^2 = (DE.EF + DF^2)/G^2 = B^2/M^2 = B/C$ as required.]

PROPOSITIO LXXIV (LXXII).

Esto AB linea divisa utcunque in C, oportet eam augere recta BD, ut ACB rectangulum, ad rectangulum ADB, datam habeat rationem E ad F.

Constructio & Demonstratio.

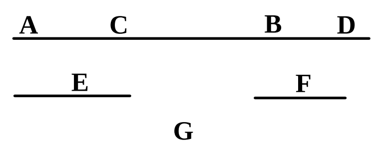
Fiat ut E ad F, sic ACB rectangulum ad G quadratum; deinde data media G, & excessu extremarum AB, inveniantur extremae AD, DB per Andersonium & alios. Dico factum esse quod petitur : est enim ut E ad F, sic ACB rectangulum, ad quadratum G, sed quadrato G, per constructionem aequale est rectangulum ADB, igitur ut E ad F, sic ACB rectangulum , ad rectangulum ADB; datae igitur lineae AB, quandum adiecimus, &c. Quod erat faciendum.

BOOK* I.§3. *PROPOSITON 74 (72).

Let the line AB be divided in C in some manner. It is required to add the line BD to this line, in order that the rectangle ACB to the rectangle ADB shall have the given ratio E to F.

Construction & Demonstration .

Thus the ratio of the rect. ACB to the square G shall be made in the ratio E to F ; then from the given mean G, and from the excess of the extremes AB, the extremes of the ratio AD, DB can be found from the method of Anderson and others. I say that what was desired had been accomplished : for indeed as E is to F, thus the rect. ACB is to the square G. But the square G by construction is equal to the rect. ADB; therefore as E to F, thus the rect. ACB to the rect. ADB; therefore for the given line AB, we may add some line, &c. Q.e.f.



Prop.74. Fig.1

[$AC.CB/G^2 = E/F$; but $AD.DB = G^2$ also, from the construction of Anderson et al. Hence, $AC.CB = AD.DB$ as required.]

PROPOSITIO LXXV (LXXIII).

Datam rectam divisam in C utcunque, iterum in D secare , ut DAB rectangulum ad rectangulum DCB, datam habeat ratiomem E ad F.

Constructio & Demonstratio.

Fiat ut E ad F, sic AB linea ad lineam G; dein AB secetur in D, ut AD sit ad DC, sicut CB est ad G. Dico factum esse quod petitur.

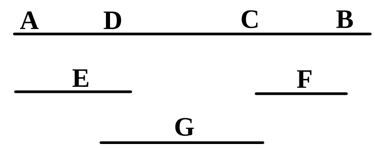
Ratio DAB rectanguli, ad rectangulum DCB, composita est ex ratione AB ad CB, & AD ad DC, id est per constructionem CB ad G. sed etiam ratio b AB ad G, id est E ad F, componitur ex ratione AB ad CB, & CB ad G, igitur rectangulum DAB ad rectangulum DCB, eam habet rationem quam AB linea ad G, id est E ad F; rectam igitur AB in D secuimus, &c. Quod erat faciendum. b *Defin. 5 Sexti.*

BOOK* I.§3. *PROPOSITON 75 (73).

The given line AB is cut by some point C, and again in D, in order that the rectangle DAB to the rectangle DCB shall have the given ratio E to F.

Construction & Demonstration .

Thus the line AB to the line G shall be made in the ratio E to F; then AB is cut in D, in order that AD shall be to DC, thus as CB is to G. I say that the task is done.
For the ratio of rect. DAB to rect. DCB, which has been composed from the ratio AB to CB, and AD to DC, that is by construction as CB to G. But also the ratio [b] AB to G, that is E to F, is made from the ratio AB to CB, and CB to G; therefore the rect. DAB to the rect. DCB, has that ratio which the line AB has to G, that is E to F; therefore we have cut the line AB in D, &c. Quod erat faciendum. b *Defin. 5 Sexti*.
&c. Q.e.f.
[DA.AB/DC.CB = AB/CB . AD/DC = AB/CB . CB/G = AB/G = E/F, as required.]



Prop.75. Fig.1

PROPOSITIO LXXVI (LXXIV).

Rectam AB divisam utcunque in C , iterum secare in D, ut ABD rectangulum ad quadratum CD, datam habeat rationem E ad F.

Constructio & Demonstratio.

Fiat ut E ad F, sic AB ad G; dein AB secetur in D, ut DB, DC & G sunt lineae continuae. Dico factum esse quod iubetur.

[46]

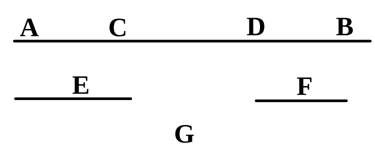
Ratio enim rectanguli ABD, ad quadratum CD, componitur ex ratione AB ad CD, & DB ad CD, id est ex ratione CD ad G. sed & ratio AB ad G (id est E ad F per constructionem) composita est ex ratine [a] AB ad CD, & ex CD ad G, igitur ABD rectangulum ad quadratum CD, eam habet rationem , quam E ad F. divisimus ergo AB lineam in D, &c. Quod erat faciendum. a *Defin. 5 Sexti*.

BOOK **I.§3.** *PROPOSITON 76 (74).*

The line AB is divided by some point in C, and again in D, in order that the rectangle ABD to the square CD has the given ratio E to F.

Construction & Demonstration .

Thus the line AB to the line G shall be made in the ratio E to F; then AB is cut in D, in order that DB, DC, and G are lines in continued proportion. I say that what was ordered has been done.
For the ratio of rect. ABD to the square CD, which is composed from the ratio AB to CD, and DB to CD, that is from the ratio CD to G. But also the ratio AB to G, (that is E to F by construction), is made from the ratio [a] AB to CD, and CB to G; therefore the rect. ABD to the sq. CD, has the ratio which the line AB has to G, that is E to F; therefore we have cut the line AB in D, &c. Quod erat faciendum. b *Defin. 5 Sexti*.
&c. Q.e.f.
[AB.BD/CD.CD = AB/CD. BD/CD = AB/CD . CD/G = AB/G = E/F, as required.]



Prop.76. Fig.1

PROPOSITIO LXXVII (LXXV).

L

ineam AB divisam utcunque in C & D , iterum dividere in E, ut AEC rectangulum ad rectangulum EBD, datam habeat rationem F ad G.

Constructio & Demonstratio.

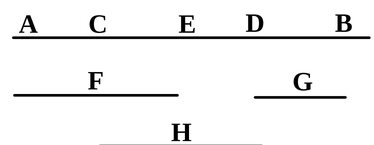
Fiat ut AC ad DB, sic F ad H; & AE ad EB, ut H ad G. Dico factum esse quod iubetur. Ratio enim lineae F ad G, [b] composita est ex, F ad H, & H ad G . sed ratio EAC rectanguli ad rectangulum EBD composita est ex ratione [c] AC ad DB, id est per constructionem F ad H, & ex AE ad EB, id est H ad G. igitur ut F ad G, sic EAC rectanguluum , ad rectangulum EBD. divisimus igitur lineam AB in E, &c. Quod erat faciendum. b *Defin. 5 Sexti*; c *23 Sexti*.

BOOK **I.§3.** *PROPOSITON 77 (75).*

The line AB is divided by some points in C and D, is again to be divided in E in order that the rectangle AEC to the rectangle EBD has the given ratio F to G.

Construction & Demonstration .

Thus the line AC to the line DB shall be made in the ratio F to H; and AE to EB as H to G. I say that what was ordered has been done.
For the ratio of rect. EAC to the rect. EBD has been made from the ratios F to H and H to G; but the ratio of the rect. EAC to the rect. EBD has been made from the ratio that is by construction F to H [c], and from AE to EB, that is H to G. Therefore as F to G, thus rect. EAC to rect. EBD. We have divided the line AB in E, &c. Quod erat faciendum. b *Defin. 5 Sexti* ; *c 23 Sexti*.
[EA.AC/EB.BD = AE/EB . AC/BD = H/G . F/H = F/G, as required.]



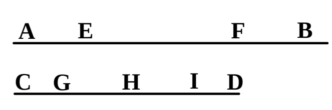
Prop.77. Fig.1

PROPOSITIO LXXVIII (LXXVI).

Sint AB, CD, divisae quomodocunque ; Dico quadrata partium AB, una cum rectangulis AEB, EFB bis sumptis, ad quadrata partium lineae CD, una cum rectangulis CGD, GHD, HID bis sumptis, eam habere rationem quam AB quadratum ad quadratum CD.

Constructio & Demonstratio.

Demonstratum est AB [d] quadratum aequare quadratis partium lineae AB una cum rectangulis AEB, EFB, bis sumptis. quemadmodum etiam de quadratis partium CD eiusque rectangulis CGD, GHD, HID bis sumptis: patet ergo ea inter se illlam obtinere rationem quae inter quadrata AB , CD reperitur. d *61 Huius*.



Prop.78. Fig.1

BOOK **I.§3.** *PROPOSITON 78 (76).*

The lines AB and CD are divided in some manner. I say that the ratio of the square of part of the line AB taken together with twice the sum of the rectangles AEB and EFB to the square of part of the line CD taken together with twice the sum of the rectangles CGD, GHD, and HID, is the same as the ratio of the square AB to the square CD.

Construction & Demonstration .

It has been shown that the square AB [d] is equal to the squares of the parts of the line AB together with twice the sum of the rectangles AEB and EFB, as also concerning the squares of the parts of CD and the sum of the rectangles of these CGD, GHD and HID taken twice. It is therefore apparent that the same ratio is obtained between these which is found between the squares AB and CD. d *61 Huius*.

PROPOSITIO LXXIX (LXXVII).

Datarum duarum alteram ita secare, ut sectae partes cum insecta, in continua sint analogia.

Propositionem hanc demonstratam invenies in libro nostro de progressionibus geomtrici props- 36 duplici methodo : luber alia tamen praxi hic eandem expedire.

Constructio & Demonstratio.

Sint AB, BC lineae quarum alteram BC, ita oporteat partire in D. ut sint in continua ratione, AB, BD, DC: divisam AB bifariam in E, fiat rectangulo super datis ABC contento, una cum quadrato dimidiae, EB aequale quadratum ED, cadet punctum D, inter C & B, cum EC quadratum sit aequale [a] quadrato BE una cum rectangulo ACB, quod maius est rectangulo ABC: adeoque & EC quadratum maius quadrato ED. Dico itaque peractum quod postulatur: constituto enim super AB semicirculo AFB, ducatur tangens FD, iungaturque FE ad centrum; erit itaque quadratum DE, quadrato DF, hoc est b rectangulo BDA: una cum quadrato EB, hoc est quadrato EF aequale. Adeoque rectangulum ADB, una cum quadrato EB, ipsi ABC cum eodem quadrat aequabitur, ablato igitur communi quadrato BE remanet ADB rectangulum, aequale ABC rectangulo, quare AD est ad AB, ut est BC ad DB, & ut AB ad BD, ita DB ad DC; sunt igitur CD, DB, BBS in continua ratione; igitur datarum duarum alteram ita secuimus, &c, Quod erat faciendum.

a *6 Secundi*.

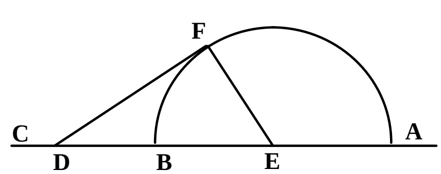
BOOK **I.§3.** *PROPOSITON 79 (77).*

To divide the second part of a given section thus, in order that the first part of the section with the cut parts of the second section are in continued proportion.

You will find the demonstration of this Proposition in our book on geometric progressions - proposition 36 - by a second method : however it is pleasing to give the same here in another way to be expedient .

Constructione & Demonstration.

The section shall be the lines AB and BC, and it is required to divide the line BC in D in order



Prop.79. Fig.1

that the lines AB, BD and DC are in a ratio of continued proportion. The line AB is bisected in E, with the given rect. contained upon ABC together with the square of the half EB made equal to the square ED ; as the point D falls between C and B, the square EC is equal to the [a] square BE, together with the rect. ACB, which is larger than the rect. ABC: and also the square EC is larger than the square ED. I say that what was demanded has been accomplished: Indeed with the semi-circle AFB set up on AB, the tangent FD is drawn, and FE is joined to the centre of the circle; and thus the square DE is equal to the square DF, that is to the rect. BDA [b], together with the square EB which is equal to the square EF. Thus the rect. ADB, together with the square EB, will be equal to the rect. ABC with the same square, therefore with the common square BE taken away there remains the rect. ADB equal to the rect. ABC, whereby AD is to AB, as BC is to DB, and as AB to BD, thus DB to DC; therefore the lines CD, DB, BA are in a continued ratio; therefore of the two given lines, we have thus cut the other, &c, Quod erat faciendum. *a 6 Secundi.*
[$EC^2 = EB^2 + AC.CB$, for $EC^2 = (EB + BC)^2 = EB^2 + (BC^2 + 2.EB.BC) = EB^2 + BC.(BC + AB) = BE^2 + CA.CB$;
$ED^2 = EB^2 + BD.DA$. , for $ED^2 = (EB + BD)^2 = EB^2 + (BD^2 + 2.EB.BD) = EB^2 + BD.(BD + AB) = BE^2 + DB.DA$;
Again, $DE^2 = DF^2 + FE^2 = DB.DA + EB^2$. Thus rect. ADB (or BD.DA) + $EB^2 = AB.BC + EB^2$ by construction, and hence $BD.DA = AB.BC$ giving $AD/AB = BC/DB$ and hence $AB/BD = BD/DC$, as $(AD - AB)/AB = (BC - DB)/DB$ gives $DB/AB = BC/AD = CD/DB$, from which $DB^2 = AB.CD$ as required.]

PROPOSITIO LXXX (LXXVIII).

Datarum duarum alteram ita partiri, ut rectangulum sub indivisa & altera parte divisae, ad quadratum residuae, datam habeat rationem.

Constructio & Demonstratio.

Propositionem quam prius particularem soluimus in ratione aequlitatis, conabimur quoque uniuersaliter solvere.
Sint igitur,AB, BC lineae, oporteatque dividere CD, in E, ut rectangulum AB DE, ad quadratum CE, rationem habeat datam H, ad I. Fiat ut H ad I, sic AB ad FG: dein CD dividatur in E, ut ED, EC, FG sint continuae per praecedente. Dico factum esse quid petitur. erit enim rectangulum super ABED, ad quadraum CE, ut idem ABED rectangulum, ad rectangulum FG ED, id est ut AB, ad FG, id est per constructionem ut H ad I. Divisimus igitur lineam CD, &c. Quod erat faciendum.
Huius Propositionis aliam invenies demonstrationem in libro nostro de progressionibus Propos. 37.

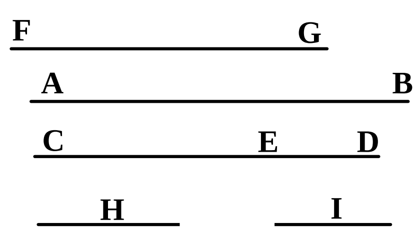
BOOK **I.§3.** *PROPOSITON 80 (78).*

One of two given lines to be divided in order that the ratio of the rectangle formed by the whole first line and one part of the other line to the square of the remaining part of the other line have a given ratio.

Constructione & Demonstration.

The particular proposition that we solved before with an equal ratio we will try to solve universally too.

The lines shall be AB and CD, and it is required to divide the line CD in E in order that the rectangle AB.DE to the square CE shall have the given ratio H to I. Thus AB to FG



Prop.80. Fig.1

shall be made in the ratio H to I: then CD is divided in E, in order that ED, EC, FG shall be in continued proportion by the preceding proposition. I say that the required task has been performed. Indeed the ratio of the rect. on AB.ED to the square CE, is the same as the rect. AB.ED to the rect. FG.ED, that is as AB to FG, that is by construction as H to I. Therefore we have divided the line CD, &c, Q. e. f.
We have found another demonstration of this Proposition in our book 'Concerning Progressions', Prop. 37.

[AB/FG = H/I , then CD is divided by E such that ED/EC = EC/FG;
The ratio $AB.ED/CE^2$ = AB/CE . ED/CE = AB/CE . EC/FG = AB/FG = H/I , as required.]

PROPOSITIO LXXXI (LXXIX).

Datum AB sectam in C & D, ita secare in E puncto , inter C & D constituto , ut rectangulum AED, ad CEB rectangulum, obtineat rationem quadrati F, ad G quadratum.

[48]

Constructio & Demonstratio.

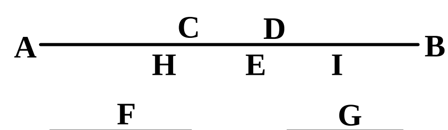
Dividatur AD in H ; & CB in I bifariam : & recta HI divisa sit in E, ut sit HE ad EI, sicut est F ad G. Dico rectangulum AED, ad CEB rectangulum habere rationem quadrati F, ad G , quadratum. .
Huius rei invenies demonstrationem reperies libro de parabola. Ex cuius proprietatibus est eruta.

BOOK **I.§3.** *PROPOSITON 81 (79).*

Given the line AB cut in C and D, thus is cut in the point E placed between C and D, in order that the ratio of the rectangle AED to the rectangle CEB equal to the ratio F to G holds.

Constructione & Demonstration.

The line AD is divided in H, and CB is bisected in I. The line HI is divided by E, in order that HE to EI shall be as F to G. I say that the ratio of the rect. AED to the rect. CEB, has the ratio of the square F to the square G. You will find the demonstration of this Proposition in our book about the parabola, from the properties of which it has been taken.



Prop.81. Fig.1

PROPOSITIO LXXXII (LXXX).

Datum iterum AB, sectam utcunque in C & D, denuo partiri in E puncto , ut rectangulum AEC, ad BED rectangulum, datam rationem contineat quadrati F, ad G.

Constructio & Demonstratio.

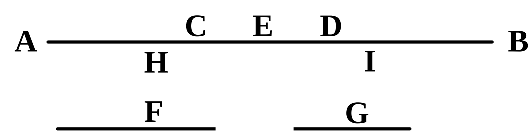
Dividantur ut prius rectae AD, CB, in H ; & I punctis bifariam : quo facto dividatur HI in E, secundum proportionem G, ad F. Dico rectangulum AEC, ad BED rectangulum, datam habere rationem quadrati F, ad G, quadratum. .
Huius quoque demonstrationem invenies eodem libro de parabola.

BOOK I.§3. *PROPOSITON 82 (80).*

Given again the line AB cut somehow in C and D, to be cut anew in the point E , in order that the ratio of the rectangle AEC to the rectangle BED equal to the ratio F to G holds.

Construction & Demonstration.

The lines AD and CB are divided as previously in H and I as the points of bisection: with which accomplished HI is divided by E, in the proportion G to F. I say that the ratio of the rect. AEC to the rect. BED, has the ratio of the square F to the square G.



Prop.82. Fig.1

You will find the demonstration of this Proposition too in our book about the parabola.

PROPOSITIO LXXXIII (LXXXI).

Datum denuo AB, divisam utcunque in C & D, iterum dividere in E, ut rectangulum ACE, ad BDE rectangulum, datam obtineat rationem quadrati F ad G.

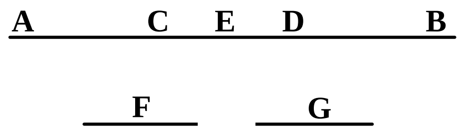
Constructio & Demonstratio.

Fiat ut AC ad DB; ita HI, ad KL; & ut F ad G, ita IM ad KN, tandem fiant in continuata analogia HI, ML, OI; & similiter KL, NK, PK: denique dividatur CD in E, secundum rationem IO ad KP; Dico factum quid requiritur. Rectangulum ACE, ad BDE, habet rationem compositum ex AC, ad DB, hoc est HI, ad KL, & ex ratione CE ad ED, hoc est OI, ad KP. Igitur rectangulum ACE, ad BDE eandem obtinet rationem, quam rectangulam, HIO ad LKP; sed HIO ad LKP, eandem habet rationem, quam quadratum IM ad KN, (ex constructione enim sint tres in continua analogia tam IO, IM, IH, quam KP, KN,KL) hoc est quam quadratum F, ad G, quadratum; Igitur rectangulum ACE, ad BDE, eandem rationem continet, quam F quadratum , ad G quadratum. Perfecimus igitur quod imperatum fuit. *a 23 Sexti*.

BOOK I.§3. *PROPOSITON 83 (81).*

G

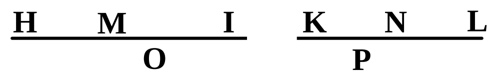
iven anew the line AB divided somehow in C and D, to be cut again in the point E , in order that the ratio of the rectangles ACE to BDE to the ratio of the squares F to G shall hold.



Prop. 83. Fig.1

Construction & Demonstration.

The ratio AC to DB shall be made as H I to KL; and thus IM to KN as F to G. However, HI, MI, and OI are made in continuous analogous proportion ; and similarly KL, KN, KP: and then CD is divided in E, following the ration IO to KP. I say that what was required has been done. Rect. ACE to rect. BDE [a] has a ratio composed from AC to DB, that is HI to KL, and from the ratio CE to ED, that is OI to KP. Therefore the rect. ACE to rect. BDE contains the same ratio as rect. HIO to rect. LKP; but HIO to LKP, has the same ratio as the square LM to the square IN, (from the construction there are indeed three lines in continuous analogous proportion as IO, IM, IH, so KP, KN,KL) that is as the square F to the square G. Therefore the rect. ACE to the rect. BDE, has the same ratio as the square F to the square G. We have completed what was required. *a 23 Sexti*.



Prop. 83. Fig.2

[$AC.CE/BD.DE = AC/DB . CE/DE = HI/KL . OI/KP = HI.IO/KL.KP = IM^2/KN^2$, (as $HI/MI = MI/ OI$ and $KL/KN = KN/KP$) $= F^2/G^2$.

PROPOSITIO LXXXIV (LXXXII).

Data basi, aggregato laterum, & altitudine trianguli; exhibere triangulum.

[49]

Constructio & Demonstratio.

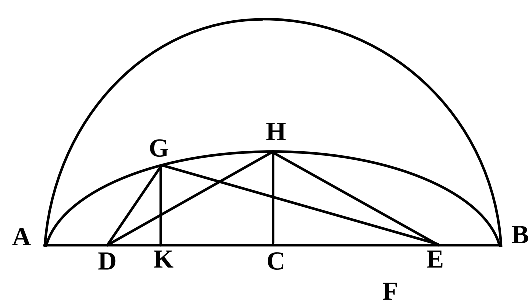
Dato laterum aggregato, aequalis ponatur AB quae bifariam divisam in C. fiat DE aequalis basi trianguli, bifariam divisae in G, & altitudini aequalis ponatur F; ex lateribus AG, CB,DE, fiat triangulum DHE: nam AC, CB simulsumptae maiores sunt DE. erit igitur DHE isosceles : deinde fiat ut HC quadratum, ad quadratum F, sic ACB rectangulum, ad rectangulum AKB. & erigatur KG aequalis ipsi F, parallela HC, iunganturque DG, GE. Dico DGE triangulum esse quaesitum.
demonstrationem huius invenies libro de ellipsi Propos. 121.

BOOK **I.§3.** *PROPOSITON 84 (82).*

For a given base, with the sum of the sides, and with the altitude of the triangle given, to exhibit the triangle.

Construction & Demonstration.

The line AB is put equal to the given sum of the sides which is bisected in C. DE is made equal to the base of the triangle, which is bisected in C, and F is set equal to the altitude. From the lines



Prop.84. Fig.1

AG, CB, and DE, the triangle DHE is made: for the sum of AC and CB is greater than DE. The triangle DHE will therefore be isosceles: then the ratio of the squares HC to F shall be made as the ratio of the rectangles ACB to AKB. KG is erected equal to F parallel to HC, and DG and GE are joined. I say that DGE is the triangle sought.
You will find the demonstration of this Proposition in the book of the ellipse, Prop. 121.

PROPOSITIO LXXXV (LXXXIII).

Datam AB sectam in C, dividere in D, ut quadratum AD, aequale sit CDB rectangulo.

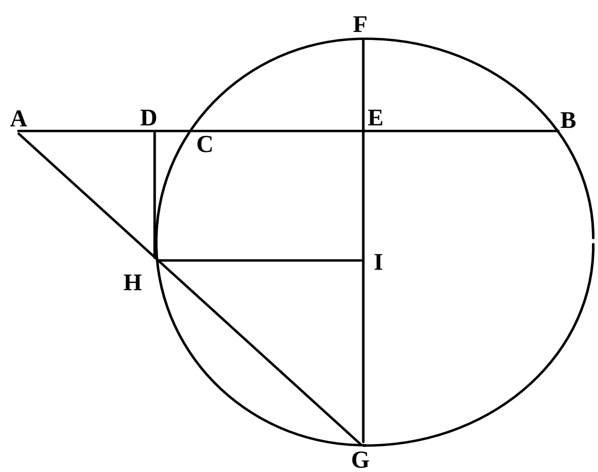
Constructio & Demonstratio.

Divisa CB in E bifariam, ponatur ex E normalis EG, aequalis ipsi AE : iunctisque AG, describatur per C, B, G puncta circulus CBG, occurrrens EG lineae in F : divisamque FG bifariam in I, agatur per I recta IH parallela ipsi CB, occurrens AG lineae in H puncto; ex quo normalis erigatur HD, occurrens CB lineae in D. Dico factum esse quod petitur : cum enim HI parallela sit ipsi AE, poniturque AE, EG lineae aequales, erunt & HI, IG quoque inter se aequales, & H communis intersectionis linearum HI, HG cum circulo. Unde & HD eundem contingitur in H : estque CDB [a] rectangulum aequale quadrato HD, id est AD, Divisimus igitur AB lineam , &c. Quod erat faciendum. *a 36 Tertii*.

BOOK I.§3. *PROPOSITON 85 (83).*

Given the line AB cut in C, that is to be divided in D so that the square AD shall be equal to the rectangle CDB.

Construction & Demonstration.



Prop.85. Fig.1

The line CB is bisected in E, a normal line EG is drawn from E equal to the line AE : the points A and G are joined, and the circle CBG is drawn through the points C, B, and G, crossing the line EG in F. FG is bisected in I, and the line IH is sent through I parallel to CB, crossing the line AG in the point H; from which the normal HD is erected, crossing the line CB in D. I say that what was sought has been done.
For indeed HI shall be parallel to AE, and the lines AE and EG are made equal to each other; the lines HI and IG will also be equal to each other, and the common point of intersection of the lines HI and HG with the circle shall be H. Hence HD is a tangent to the circle at H : and the rect. CDB [a] shall be equal to the square HD, that is AD. We have therefore divided the line AB, etc. Q.e.f. *a 36 Tertii*.

PROPOSITIO LXXXVI (LXXXIV).

Datam AB sectam in C, denuo partiri in D, ut quadratum AD, ad rectangulum BDC, datam obtineat rationem.

Constructio & Demonstratio.

Data sit ratio E ad F, & fiat rectangulum aliquod GIK, quod ad rectangulum LIM rationem eandem cuius ratone E ad F contineat : hoc facto ponatur quaedam IS, orthogonalis ad GM, & inventa IN media inter KI, IG; dividatur KG bifariam in O, & erigatur OT parallela IS, donec concurrat cum TS, quae aequidistet GM, in T : & iungatur TN, quae SI productae occurrat in P. Deinde inventa media IQ inter LI, & IM, ducatur recta PQ, occurrens TS protractae in R.

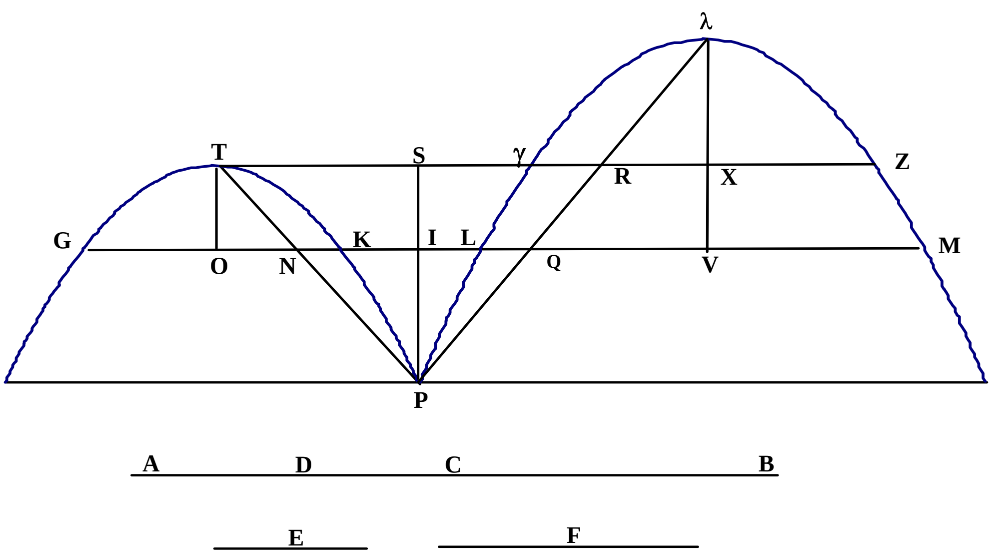
[50]

tandem dividatur ; M bifariam in V, & erigatur VX aequidistand TO, concurrens cum PQ in λ; & fiat ut Vλ quadratum, ad quadratum Xλ, ita quadratum VL, ad Xλ quadratum : Dico TZ lineam divisam esse in S, γ secundum rationem postulatam : unde si dividatur AB in D, & C, ut est divisa TZ in S, & γ : habebitur ratio quadrati AD, ad CDB rectangulum, in ratione E ad F. quod fuit postulatum. Vlterius non pergo in demonstratione huius rei, cum non sit huius loci; sed eam reperies libro de parabola Geometrice tractatam, & perfectam.

Libri primi finis.

BOOK I.§3. *PROPOSITON 86 (84).*

Given the line AB cut in C, to be divided in D anew, so that the square AD to the rectangle BDC shall be in equal to a given ratio.



Construction & Demonstration.

The ratio of two lines E to F shall be given, and some rect. GIK shall hold the same ratio E to F to another rect. LIM. Following this, some line IS is placed orthogonal to GM, and the mean IN found between KI and IG. KG is bisected in O, and the line OT is erected parallel to IS, then the points T and S are joined together to give a line TS equidistant from GM meeting OT in T. The line TN is drawn which crosses SI produced in P. Following this the mean IQ is found between LI and IM, and the line PQ is drawn cutting

TS produced in R. At last LM is bisected in V, and the line VX erected equidistant to TO, crossing the line PQ in λ. The square Vλ to the square Xλ is thus made in the ratio of the square VL to the square Xλ. I say that the line TZ has been divided by S and γ in the ratio required. Thus if AB is divided by D and C as TZ is divided by S and γ, then the ratio of the square AD to the rect. CDB will have the ratio E to F, which was required. I will not go on further with the demonstration of this, as this is not the place, and you can find this propostion worked out geometrically in the book on the parabola.

End of Book One.