

[21]
PARS SECUNDA.

De Triangulis, earumque proprietatibus.

PROPOSITIO XIX.

Esto ABC triangulum Isosceles, habens AB, AC, latera aequalia ; ducanturque ; ex A & B normales AE, BD, ad opposita latera lineae occurrentes sibi mutuo in F. Dico esse AF ad BC , ut AD ad DB.

Demonstratio.

Quoniam anguli AEB, BDC per constructionem recti sunt, & angulus EBF, communis triangulis EBF, DBC. erunt EBF, DBC triangula inter se similia. Eodem modo ostenditur triangulum AFD, simile triangulo DBC : unde ut AD ad DB, sic AF ad BC. Quod erat demonstrandum.

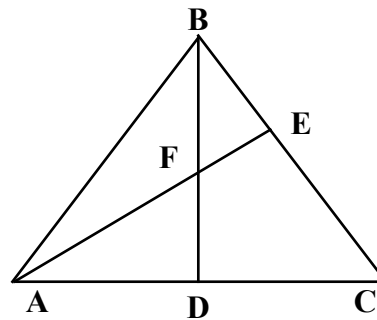
PART TWO.

Concerning triangles and their properties.

BOOK I.§2.

PROPOSITON 19.

Let ABC be an isosceles triangle, having the sides AB and AC equal. The normal lines AE and BD are drawn from A and B to the opposite sides cutting each other in F. I say that as AF to BC , so AD to DB.



Prop.19. Fig.1

Demonstration.

Because the angles AEB and BDC are right by construction, and the angle EBF is common to the triangles EBF and DBC, the triangles EBF and DBC are similar to each other. In the same way it can be shown that the triangle AFD is similar to the triangle DBC : hence as AD to DB, thus AF to BC. Q.e.d.

PROPOSITIO XX.

Ex quovis puncto baseos trianguli ABC Isosceles eductae DF, DE exhibeant angulos aequales EDC, FDA, iunganturque AE, CF Dico AED, CFD triangula esse inter se aequalia.

Demonstratio.

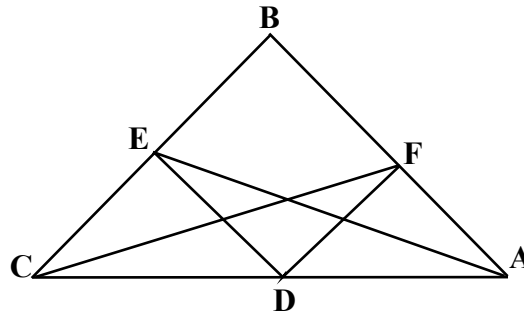
Cum enim tam angulus FDA, angulo EDC per constructionem , quam angulus FAD, angulo EXD aequalis sit, erunt ADD, CES triangula inter se similia. Quare ut AD ad DC, sic FD ad ED. Rursum cum angulus FDA, aequetur angulo EDC, addito communi angulo EDF, erit angulus EDA, aequalis angulo CDF. Unde

cum & latera, aequales angulos continentia, reciproce sint proportionalia ,^a erunt AED, CFD triangula inter se aequalia. Quod erat demonstrandum. ^{a 15 Sexti.}

BOOK I.§2.

PROPOSITON 20.

From some point on the base of the isosceles triangle ABC the lines DF and DE are drawn, showing the angles EDC and FDA equal. The lines AE and CF drawn. I say that the triangles AED and CFD are equal to each other.



Prop.20. Fig.1

Demonstration.

Since indeed the angle FDA is equal to the angle EDC by construction, so the angle FAD is equal to the angle EXD. The triangles AFD and CED are similar to each other. Whereby as AD to DC, thus FD to ED. Again since the angle FDA is equal to the angle EDC, with the common angle EDF added, the angle EDA, is equal to the angle CDF. Thus with the sides bordering equal angles, the corresponding sides shall be in proportion^a, and the triangles AED and CFD shall be similar to each other. Q.e.d. ^{a 15 Sexti.}

PROPOSITIO XXI.

Sint duo triangula Isoperimetra ABC, ADC, super eadem basi AC constituta, divisisque, AB, CD lateribus proportionaliter in E & G, ducantur EF, GH parallelae basi AC:

Dico Isoperimetria quoque fore EBF, DGH triangula.

Demonstratio.

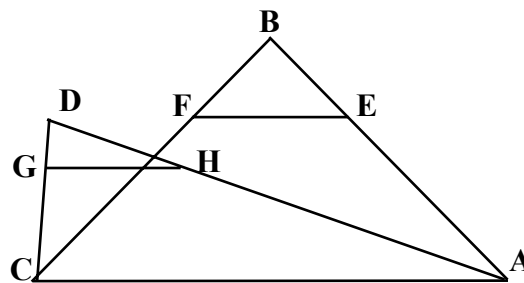
Quoniam in triangulis ABC, ADC proportionaliter sectae sunt lineae super basi erectae erunt ut ABC simul sumptae, ad EBF simul sumptas, ita ADC simul sumptae ad DGH simul sumptas, & permutado ut ABC latera ad latera ADC, sic EFB ad HDG simul sumpta ; sed ex suppositione aequales sunt lineae ABC, rectis ADC : Ergo etiam EBF, rectis HDG. Sed & recta FE aequalis est lineae GH. Igitur Isoperimetra quoque sunt triangula EBF & HDG. . Quod erat demonstrandum.

BOOK I.§2.

PROPOSITON 21.

There are two isoperimetric triangles ABC and ADC situated on the same base AC. The sides AB and CD are divided in proportion by the points E and G. The lines EF and GH are drawn parallel to the base AC.

I assert that the triangles EBF and DGH are also isoperimetric.



Prop.21. Fig.1

Demonstration.

Because the lines erected on the common base AC in triangles ABC and ADC are cut in proportion, the sides of triangle ABC will be proportional to the corresponding sides of the triangle EBF, and thus the sides of triangle ADC will also be proportional to the corresponding sides of triangle GDH. By changing the ratios around, the sides of triangle ABC are to the sides of triangle ADC, thus as the sides of triangle EBF are to the corresponding sides of triangle HDG. But the sum of the lengths of the sides making up triangles ABC and ADC are supposed equal : Therefore the lengths of the sides of EBF and HDG taken together are equal too. But the line FE is equal to the line GH [i. e. The proportionality between the sides of triangles BAC and BFE is the same as between triangles DCA and DGH]. Therefore the triangles EBF and HDG are isoperimetric triangles too. Q.e.d.

Triangulorum Isoperimetrorum super eadem basi constitutorum , maximam habet altitudinum Isoscelium.

PROPOSITIO XXII.

Demonstratio.

Sint ABC, ADC triangula isoperimetra, & ABC quidem isoscelium ; dico ABC maioris esse alitudinis quam sit ADC. Habeat enim si fieri possit triangulum ADC, eandem cum ABC triangulo altitudinem: productur CB in E, ut BE linea sit aequalis lineae CB; iunganturque BD, ED. Quoniam AC, BD lineae ex suppositione sunt paralleli, erit angulus ABD, aequalis angulo CAB, id est angulo ACB, id est angulo EBD ; sunt autem latera duo BE, BD, duobus AB, BD lateribus aequalia ; igitur triangulum ABD aequale triangulo EBD, & AD latus, lateri ED aequale. Sed ED, DC latera simul sumpta, maiora sunt latere EC, hoc est lateribus AB, BC simul sumptis; igitur & AD, DC latera simul sumpta, maiora sunt lateribus AB, BC. Quod est contra hypothesin. unde triangulum ADC, eandem non habet altitudinem cum triangulo ABC. Habeat iam ADC triangulum , maiorem altitudinem quam ABC; ducatur ex B linea BE, parallela basi AC, occurrens AD lateri in E; iunganturque CE. Quoniam AEC triangulum, eandem habet altitudinem cui triangulo ABC, & ABC sit Isoscelium; erunt AE, CE latera simul sumpta , maior lateribus AB, BC simul sumptis per primam partem huius propositionis; sed AD, DC latera maiora fuit lateribus AE, EC; cum E cadat infra D, igitur & latera AD, DC, multo maiora sunt lateribus AB, BC. Quod est contra hypothesim. igitur triangulum ADC, maiorem non habet altitudinem triangulo ABC; sed neque aequalem habet; igitur triangulorum isoperimetrorum super eadem base constitutorum, &c. Quod erat demonstrandum.

Propositio hac aliter demonstratur libro nostro de Ellipsi.

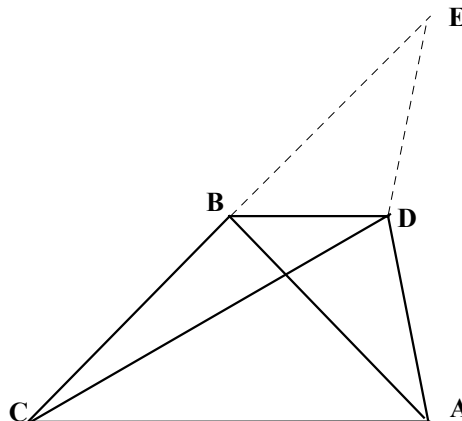
BOOK I.§2.

PROPOSITON 22.

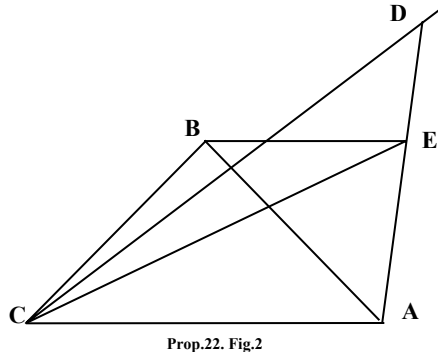
Of all isoperimetric triangles constructed on the same base, the largest shall have the altitude of the isosceles triangle.

Demonstration.

Triangles ABC and ADC shall be isoperimetric, and ABC indeed is an isosceles triangle. I assert that triangle ABC shall have a greater altitude than triangle ADC. Let triangle ADC have the same altitude as triangle: CB is produced to E, in order that the line BE shall be equal to the line CB; and the lines BD and ED are joined. Since the lines AC and BD are parallel from supposition, the angle ABD shall be equal to the angle CAB, that is to the angles ACB and EBD. But the two sides BE and BD are equal to the two sides AB and BD, hence the



Prop.22. Fig.1



Prop.22. Fig.2

triangle ABD is equal [*i. e.* congruent] to the triangle EBD, and the side AD is equal to the side ED. But the sum of the sides ED and DC taken together is larger than the side EC, that is with the sides AB and BC taken together. Therefore the sum of the sides AD and DC taken together is larger than the sum of the sides AB and BC. Which is contrary to the hypothesis: hence the triangle ADC cannot have the same altitude as triangle ABC.

The triangle ADC may have an altitude greater than triangle ABC. The line BE is drawn from B, parallel to the base AC, crossing the line AD in E, and the points C and E are joined. Since triangle AEC has the

same altitude as triangle ABC, and ABC is isosceles; the sides AE and CE taken together as a sum shall be greater than the sides AB and BC taken together according to the first part of this proposition. But the sum of the sides AD and DC shall be greater than the sum of the sides AE and EC; as E lies within AD, and hence the sum of the sides AD and DC shall be much greater than the sum of the sides AB and BC. Which is contrary to the hypothesis. Therefore the triangle ADC cannot have a greater altitude than triangle ABC, and neither can it be equal. Therefore of the isoperimetric triangles constructed on the same base, &c. Q.e.d.

This proposition is demonstrated otherwise in our book on the ellipse.

PROPOSITIO XXIII.

Esto ABC triangulum rectangulum, & ex B ad AC basim, demissa normalis BD. Dico ABC triangulum, ad tria laterum quadrata, eam habere rationem, quam BD linea ad quadruplum lineae AC.

[23]

Demonstratio.

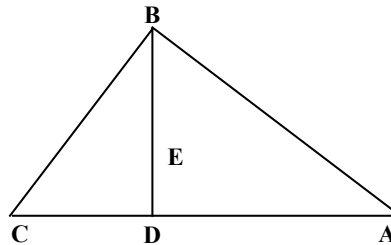
Quoniam Angulus ABC ponitur rectus, & BD normalis, erunt ABD, ABC triangula similia, & AB ad BD, ut AC ad CB, unde ABC^a rectangulum aequale rectangulo AC, BD, sed AC, BD rectangulum, est ad quadratum AC, ut BD lineae ad lineam^b AC; igitur α ABC rectangulum, est ad quadratum AC, ut BD lineae ad lineam AC; est autem quadratum AC, aequale quadratis^c AB, BC, igitur rectangulum ABC, est ad tria laterum AB, BC, CA quadrata ut idem rectangulum, ad quadratum AC bis sumptum; id est ut BC lineae ad AC lineam bis sumptam. sed ABC triangulum, dimidium est rectanguli ABC. Igitur triangulum ABC, ad AC quadratum bis sumptum, id est ad tria laterum quadrata, illam habet rationem, quam BD lineae, ad quadruplum lineae AC. Quod erat demonstrandum. a 17.Sextii; b 1.Sextii; c 47. Primi.

BOOK I.§2.

PROPOSITON 23.

Let ABC be a right-angled triangle, and from B the normal BD is sent to the base AC.

I say that the area of triangle ABC, to the squares of the three sides, has the same ratio as the line BD to four times the line AC.



Prop.23. Fig.1

Demonstration.

Because the angle ABC is made right, and BD the normal, the triangles ABD and ABC are similar, and as AB to BD, so AC to CB. Hence the area of the rectangle ABC^a [i.e. AB.BC] is equal to the area of the rectangle AC, BD; but the rectangle AC, BD is to the square AC, as the line BD is to the line^b AC. Therefore the rectangle ABC is to the square AC, as the line BD to the line AC; but the square AC is equal to the sum of the squares^c AB and BC. Therefore the rectangle ABC is to the square of the three sides AB, BC and CA as the same rectangle to the square of AC taken twice. That is as the line BC to the line AC taken twice. But the area of triangle ABC is half of the rectangle ABC. Therefore the area of triangle ABC to the square of AC taken twice, or the sum of the squares of the three sides, has the same ratio as the line BD to four times the line AC. [i. e. $(\frac{1}{2}AC.BD)/2AC^2 = BD/4AC.$] Q.e.d. ^{a 17.Sexti ; b 1.Sexii ; c 47. Primi.}

PROPOSITIO XXIV.

Iisdem positis, dividatur BD bifariam in E.
 Dico ABC triangulum, ad quadratum BC, eam habere rationem, quam habet ED linea, ad lineam DC.

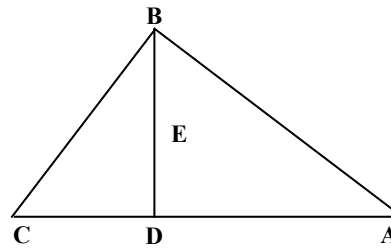
Demonstratio.

Quoniam linea BD in E divisa est bifariam, erit ABC triangulam, aequale rectangulo AC DE; nam in praecedenti propositione ostensum est, rectangulum ABC, aequale rectangulo super AC & BD: rursum cum ABC angulus sit rectus, & BD normalis, erit ACD rectangulo, aequale quadratum BC; quare ABC triangulum, est ad quadratum BC, ut AC DE rectangulum, ad rectangulum ACD, id est ut ED linea, ad lineam DC. Quod erat demonstrandum.

BOOK I.§2.

PROPOSITON 24.

With the same figure in place, BD is bisected in E.
 I say that the ratio of the area of triangle ABC to the square BC is the same as the line ED to the line DC.



Prop.24. Fig.1

Demonstration.

Since the line BD is bisected by E, triangle ABC is equal in area to the rectangle AC.DE, for it has been shown in the previous proposition that rectangle ABC is equal to the rectangle on AC and BD. Again when the angle ABC is right and BD the normal, the rectangle AC.CD will be equal to the square BC, whereby the area of triangle ABC is to the square BC as the rectangle AC.DE is to the rectangle AC.CD, that is as the line DE is to the line DC. Q.e.d.

PROPOSITIO XXV.

Lateri AB, anguli ABC, aequidistet DE. oportet per F punctum intra angulim DEC, rectam ponere AG, ut AD ad FG, datam habeat rationem H ad L.
 ED linea, ad lineam DC.

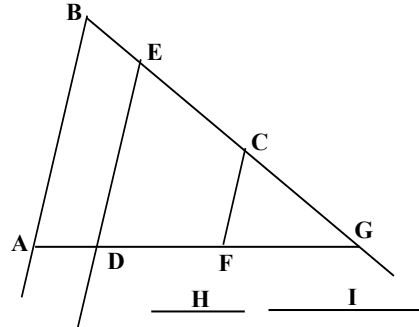
Constructio & Demonstratio.

Ducatur FC, quae aequidistet ipsi AB; & fiat ut G ad I sic BE ad CG: dein ponatur GFA. Dico factum quod requiritur, cum enim sit tam EB ad AD quam CG ad FG, ut EG ad DG, (ob AB, ED, CF parallelas) erit EB ad AD, ut CG ad FG; & permutando AD ad FG ut BE ad CG; id est per constructionem ut H ad I. igitur &c. . Quod erat demonstrandum.

BOOK I.§2.

PROPOSITON 25.

The line DE shall be parallel to the side AB of the angle ABC. It is required to draw the line AG through the point F within the angle DEC, in order that AD to FG shall have the given ratio H to I.

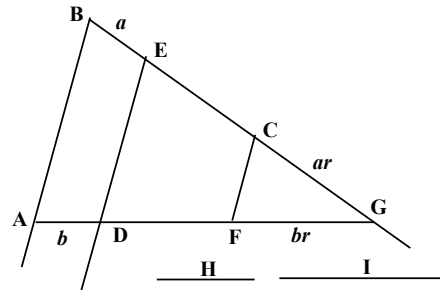


Prop.25. Fig.1

Construction and Demonstration.

FC is drawn parallel to AB, and made such that the ratio BE to CG is equal to H to I, then the line GFA is put in place.

I assert that the required outcome has been achieved, since indeed as BE to AD so CG to FG, and as BE to AD so EG to DG, (from the parallel lines AB, ED, CF), thus CG to FG is as EG to DG. By changing the ratio this becomes AD to FG as BE to CG, that is by construction as H to I. Therefore, etc. Q.e.d.



Prop.25. Fig.2

BOOK I.§1. Prop. 25 Note: From the similar triangles it follows that $BG/EG = AG/DG$, and hence $BG/EG - 1 = AG/DG - 1$ giving $BE/EG = AD/DG$ and $BE/AD = EG/DG = CG/FG$. Now, $r = H/I$ and $CG/BE = r = FG/AD$, as required.

PROPOSITIO XXVI.

In dato triangulo ABC, parallelam uni laterum constituere, rectam DE: ut quadratum ED, aequale sit AEC, rectangulo.

[24]

Constructio & Demonstratio.

Ut AC quadratum, ad CB quadratum, ita fiat linea AE ad^a EC :& erigatur ED, quae aequidistet ipsi BC. Dico ED solvere problema; quoniam enim DE aequidistet rectae BC, erit quoque quadratum AE, ad ED, ut AC quadratum, ad CB: hoc est ut linea AE, ad EC: igitur sunt tres in continua ratione AE, ED, EC. cum ratio quadrati AE, ad DE quadratum, hoc est ratio AE, ad ED, duplicata sit rationis AE lineae, ad ED lineam; est igitur quadratum DE, aequale rectangulo AEC. Quod erat exhibendum.

BOOK I.§2.

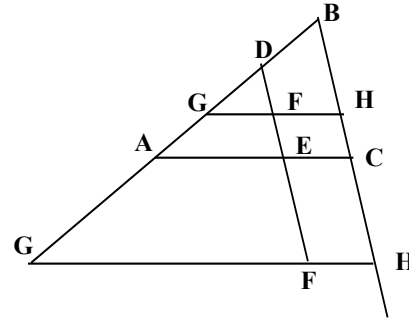
PROPOSITON 26.

In the given triangle ABC, a line DE parallel to one of the sides is to be constructed, in order that the square ED shall be equal to the rectangle AEC [i. e. $ED^2 = AE \cdot EC$].

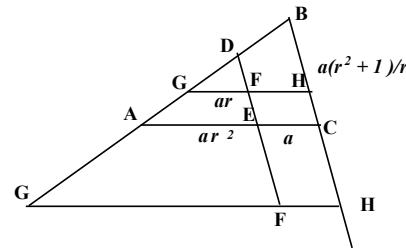
Construction and Demonstration.

As the ratio of the square AC to the square CB, so shall be made the ratio of the line AE to the line EC. The line ED is erected which shall be parallel to BC.

I assert that this construction has solved the problem, since indeed DE is parallel to the line BC, and the square of the ratio AE to ED will be as the square of the ratio AC to CB. That is, as the line AE is to EC, so therefore the three lines AE, ED, and EC are in continued proportion; since the ratio of the square of AE to the square DE is the same as the ratio of the lines AE to ED squared. Therefore the square DE is equal to the rectangle AEC [i. e. $ED^2 = AE \cdot EC$]. Q.e.d.



Prop.26. Fig.1



Prop.26. Fig.2

BOOK I.§1. Prop. 26 Note: The ratio is set: $AC^2/CB^2 = AE/EC = r^2$; which also equals $(AE/ED)^2$ and $(AC/CD)^2$, and $ED^2 = AE \cdot EC = a^2 r^2$.

PROPOSITIO XXVII.

Isdem positis constituatur GH, aequidistet base AC. Dico rectangulo GDH aequari recangulum FDE.

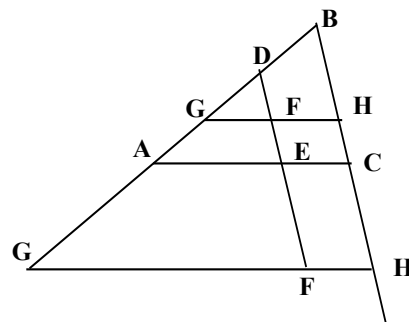
Demonstratio.

Rectangulum GFH, ad FDE rectangulum, rationem habet compositam, ex ratione GF, ad FD, hoc est AE ad ED; & ex ratione FH, ad ED, hoc est EC ad ED; si igitur fiat ut EC, ad ED, ita ED ad aliam, erit illa per praecedentem ipsa AE: igitur ut AE ad AE, ita rectangulum GFH ad FDE: patet igitur aequalia esse rectangula illa inter se. Quod erat demonstrandum.

BOOK I.§2.

PROPOSITON 27.

With the diagram, a line GH is drawn parallel to the base AC. I assert that the rectangle GFH is equal to the rectangle FDE. [i. e. $GF \cdot FH = FD \cdot DE$].



Prop.27. Fig.1

Demonstration.

The rectangle GFH to the rectangle FDE has a ratio composed from the ratio GF to FD, that is as AE to DE, and from the ratio FH to ED, that is EC to ED. Therefore if EC to ED is thus made equal to ED to some other length, by the preceding proposition that length will be AE : therefore as AE to AE, thus the rectangle GFH to FDE: it is apparent therefore that these rectangles are equal to each other. Q.e.d.

BOOK I.§1. Prop. 27 Note: The ratio :
 $GFH/FDE = GF.FH/FD.DE = (GF/FD).(FH/DE) = (AE/DE) . (EC/DE)$; but $EC/DE = DE/AE$, and hence the initial ratio is unity, making the rectangles equal in area.

PROPOSITIO XXVIII.

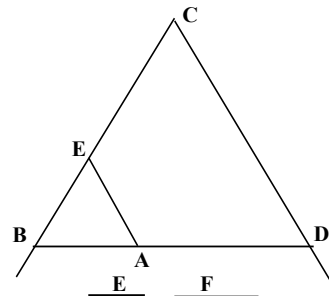
Dato A, puncto. intra angulum BCD, per illud lineam BAD ducere, quae divisa sit in A iuxta datam rationem E ad F.

Constructio & Demonstratio.

Ponatur AE, aequidistant ipsi DC : & fiat ut E ad F, ita CE ad EB; deinde ex B, per A ducatur recta BAD, quae pertingat in D. Dico factum quod quaritur: manifestam est ex elementis.

BOOK I.§2. PROPOSITON 28.

Given the point A within the angle BCD, to draw the line BAD through that point, which shall divide the line in the given ratio E to F as set down.



Prop.28. Fig.1

Constructio & Demonstratio.

The line AE is placed parallel to DC : and CE to EB shall be in the ratio E to F; then from B the line BAD is drawn through A , which is extended to D. I assert that the required outcome has been shown from these first principles.

PROPOSITIO XXIX.

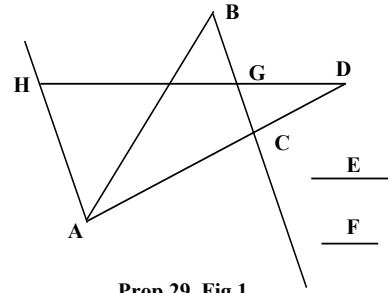
Adato puncto D extra angulum ABC, rectam ponere quae dividatur secundum datam rationem à lineis angulum constituentibus.

Constructio & Demonstratio.

Sit data ratio E ad F, & ex D puncto ducatur quaevis GH: Ita ut sit DG ad GH ut E ad F : quo facto ponatur HA parallela ipsi BG. & ducatur DA, occurrens BG productae in C. Dico DC ad CA, eandem rationem contineret, quae reperitur inter datas E, F. Cum enim aequidissent AH, CG erunt in eadem ratione DC, CA, cum rectis DG GH : hoc est E & F. praestitimus igitur quod requisitum fuit.

BOOK I. §2. PROPOSITON 29.

From the given point D outside the angle ABC, to draw the line which is divided in the following given ratio from the sides making up the angle.



Prop.29. Fig.1

Construction & Demonstration.

Let the given ratio be E to F, and from D some line GH is drawn thus so that DG to GH shall be as E to F: on doing this, HA is drawn parallel to BG, and DA is drawn cutting BG produced in C. I assert that DC to CA maintains the same ratio that occurs between the given E and F. For the lines AH and CG are parallel, and the lines DC and CA are in the same ratio as the lines DG and GH : that is E and F. Therefore we have performed that which was required.

[25]

PROPOSITIO XXX.

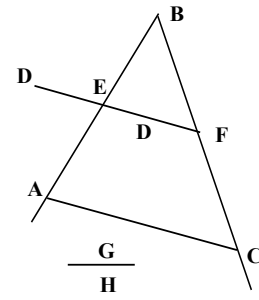
Dato angulo ABC, & D puncto extra lineas datas : rectam DF ponere quae rectas EB BF auferat, in data ratione G ad H.

Constructio & Demonstratio.

Fiat ut G ad H, ita BA ad BC. iunctaeque AC ducatur DF aequidiatans: patet ex elementis factum esse quod petitur.

BOOK I. §2. PROPOSITON 30.

From the given angle ABC, and the point D beyond the given lines, to place the line DF from which the lines EB and BF can be taken away in the given ratio G to H [from the sides of the angle].



Prop.30. Fig.1

Construction & Demonstration.

The ratio BA to BC shall be as G is to H. The points A and C are joined and the line DF is drawn parallel to AC. It is apparent from first principles that what is sought has been done.

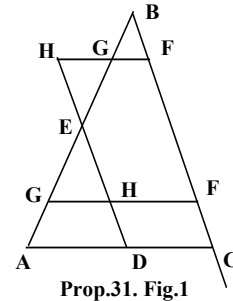
PROPOSITIO XXXI.

Esto ABC trianguli basis AC; quâ divisâ in D, ut DE recta aequidistans lateri BC, media quoque sit, inter AD, DC: ducatur quaevis GF, parallela bse AC, occurrens ED lineae in H.

Dico GHF rectangulum, aequari rectangulo DEH.

Demonstratio.

Est enim ut AD ad DE, sic GH ad HE: quia AD, GH lineae aequidistant ; sed ut AD ad DE, sic DE est ad DC, ex hypothesi; ergo etiam ut GH ad HE, sic DE ad DC : est autem ut HF ipsi DC^a aequalis, igitur ut GH ad HE , sic DE ad HF: & GHF rectangulum,^b aequale rectangulo DEH. Quod fuit demonstrandum. ^a 34. Primi ; ^b 17 Sexti.



Prop.31. Fig.1

BOOK I.§2.

PROPOSITON 31.

Let the base of the given triangle ABC be AC, divided in D, in order that the line DE is parallel to the side BC, and DE is the mean proportion between AD and DC. Some line GF is drawn parallel to the base AC, cutting ED in H.

I assert that the rectangle GHF is equal to the rectangle DEH.

Demonstration.

Indeed as AD to DE, thus GH is to HE: because the lines AD and GH are parallel ; but as AD is to DE, thus DE is to DC, by hypotheses. Hence also as GH to HE, so DE to DC : but as HF is equal to DC^a itself, therefore as GH to HE, thus DE to HF: and the rectangle GHF,^b is equal to the rectangle DEH. Which was to be shown. ^a 34. Primi ; ^b 17 Sexti .

PROPOSITIO XXXII.

Esto ABC trianguli basis AC, bifariam divisa in D; actaque per D linea EF, occurrente trianguli ABC lateribus, in E & F, ducatur quaedam GH parallela rectae EF, ut HI illius dimidia, media quoque sit inter ED & DF.

Dico GBH triangulum aequale esse triangulo ABC.

Demonstratio.

Acta per DKL, erigantur ex I...M, DN: & IM quidemin DN vero AB lateri parallela. Ut FD ad GI, sic GI est ad DE per hypothesim, sed ut FD ad GI, sic DK est ad IK, & ut IH ad DE, sic IL ad DL, igitur ut DK ad IK, sic IL ad DL, & dividendo ut DI ad IK, sic DI ad DL; quare IK, DL lineae sunt inter c se aequales. ac proinde cum (ut ex constructione patet,) triangula KMI, DNL, singula triangulo KBL similia sint, erunt & inter se similia, adioque (cum IK, DL rectae aequentur) erunt & inter se aequalia ; latusque KM,

[26]

lateri DN, & MI latus, NI. lateri aequale: Rursum est ut KI ad IL , sic KM ad MB, & ut LD ad DK , sic LN ad NB. sed ut KI ad IL, sic LD ad DK, (ex antedictis) igitur ut KM sive DN ad MB, sic LN sive IM ad NB : sed quia CA dupla ponitur ipsius CD, erit & AB ipsius DN, & CB ipsius CN, id est rectae NB dupla : similiter quia GH linea, ipsius GI dupla est, erit quoque BH dupla lineae MI : uti & GB ipsius MB ; sunt autem DN, MB, IM, NB ostendae proportionales, ergo & AB, BG, BH, BC illarum duplae quoque sent proportionales: & ABC, GBH , triangula inter se aequalia, cum circa communem angulum ABC, latera habeant reciproca proportionalia. Quod fuit demonstrandum.

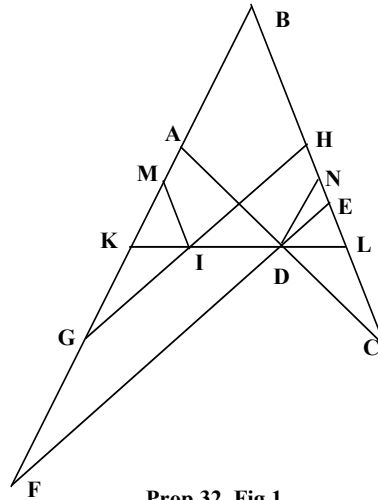
c 9. Quinti, a. 16. Sexti .

BOOK I.§2. PROPOSITON 32.

Let the base AC of the given triangle ABC be bisected in D, and a line DF be sent through D cutting the sides of the triangle in E and F. A certain line GH is drawn parallel to the line EF, in order that HI is half of GH and also HI is the mean proportion between ED and DF. I assert that the triangle GBH is equal to the triangle ABC.

Demonstration.

Sent through D, the line KL is drawn through I, and erected from I, IM, DN: and IM parallel to BC and DN truly parallel to the side AB. As FD to GI, thus GI is to DE by hypothesis [as GI is the mean proportional, etc], but as FD is to GI, thus DK is to IK, and as IH to DE, thus IL to DL, therefore as DK to IK, thus IL to DL, and on being divided, as DI to IK, thus DI to DL; whereby IK and DL are lines equal to each other^c. Hence as (which is apparent from the construction) the triangles KMI and DNL are similar to the triangle KBL, they are similar to each other, and hence (as the lines IK, DL are equal) they are equal to each other. KM is equal to the side DN, and MI to the side NL. Again as KI to IL, thus KM to MB, and as LD to DK, thus LN to NB. but as KI to IL, thus LD to DK, (from what has been said before) therefore as KM or DN to MB, thus LN or IM to NB. But CA is double CD, AB is double DN, & CB is double CN and NB. Similarly GH is double GI, BH is double MI : and GB of MB too. But DN and MB, IM and NB have been shown to be in proportion, and therefore AB, BG, GH and BC the doubles of these are in proportion too. Triangles ABC and GBH are equal among themselves [i. e. similar], as the sides are in proportion around the common angle ABC. Which had to be demonstrated.



Prop.32. Fig.1

^c 9. Quinti; a. 16. Sexti

BOOK I.§1. Prop. 32 Note: Due to the dense nature of the presentation of the argument, we have set it out in more modern terms as follows:

$FD/GI = GI/DE$ by hypothesis [as GI is the mean proportional, etc], but
 $FD/GI = DK/IK$, and $GI/DE = IH/DE = IL/DL$ [from sim.Δ's and $GI = IH$ given],

$\therefore DK/IK = IL/DL$, and $DK/IK - 1 = IL/DL - 1$,
 giving $DI/IK = DI/DL$; whereby $IK = DL$.

Hence, as ΔKMI and ΔDNL are similar to ΔKBL by construction, they are similar to each other, and hence (as $IK = DL$) they are congruent: thus $KM = DN$, and $MI = NL$.

Again, $KI/IL = KM / MB$, and $LD/DK = LN/NB$;
 but $KI/IL = LD/DK$;

$\therefore (KM \text{ or } DN)/MB = (LN \text{ or } IM)/NB$.

But $CA = 2 CD$ (given) $\therefore AB = 2 DN$, and $CB = 2 CN = 2 NB$.

Similarly $GH = 2 GI$ (given) $\therefore BH = 2 MI$, and $GB = 2 MB$.

But $DN/MB = IM/NB$,

$\therefore AB/BG = BH/BC$.

$\therefore \Delta ABC$ and ΔGBH are also congruent, with sides in proportion around the common angle ABC.

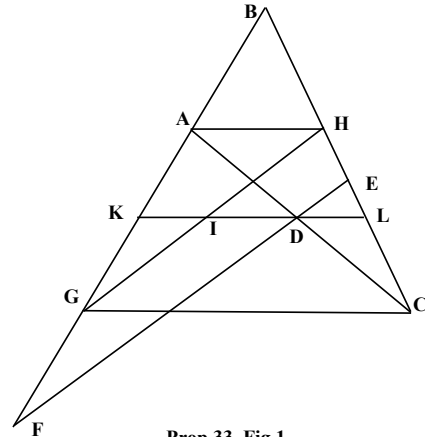
PROPOSITIO XXXIII.

Idem positisque suprà: si HBG triangulum aequale fuerit triangulo ABC:
 Dico quadratum GI, dimidia scilicet ipsius GH, aequale esse rectangulo FDE.

Demonstratio.

Iungantur puncta AH, GC. Quoniam ABC triangulum per
 hypothesin aequale est triangulo GBH, ablato communi
 triangulo ABH, aequalia remanent triangula AGH, ACH, unde
 & AH, GC lineae^b sibi mutuo aequidistant, & cum LK linea;
 bifariam secet rectas AC, HG, ex hypothesi erit & LK, ipsi AH
 parallela, & IK recta, aequalis rectae DL: quare ut DI ad IK sic
 ID ad DL, & componendo ut DK ad IK, sic IL ad DL: sed est
 ut DK ad IK, sic FD ad IG, & ut IL ad DL sic IH ad DE; igitur
 ut FD ad GI, sic HI sive GI ad DE: adeoque FDE rectangulum
 aequale quadrato GI, Quod erat demonstrandum.

b. 40. Primi



Prop.33. Fig.1

BOOK I. §2.

PROPOSITON 33.

With the same above diagram : if triangle HBG is equal to triangle ABC.
 I assert that the square GI, which is or course half of GH, is equal to the
 rectangle FDE.

Demonstration.

The points AH and GC are joined. Since the triangle ABC is equal to the triangle GBH by hypothesis, by
 taking away the common triangle ABH, there remain the equal triangles AGH and ACH. Hence the lines
 AH and GC are parallel^b, and as the line LK bisects the lines AC and HG, and by hypothesis LK and AH
 are parallel, and the line IK is equal to the line DL: whereby as DI to IK thus ID to DL, and by addition, as
 DK to IK, thus IL to DL. Also as DK to IK, thus FD to IG, and as IL to DL thus IH to DE. Therefore as FD
 to GI, thus HI or GI to DE: and hence the rectangle FDE is equal to the square GI. Q.e.d. ^{b. 40. Primi}

BOOK I. §1. Prop. 33 Note:

$\Delta ABC = \Delta GBH$ by hypothesis; hence

$\Delta ABC - \Delta ABH = \Delta GBH - \Delta ABH,$

giving $\Delta AGH = \Delta ACH.$

Hence AH is parallel to GC, [equal altitudes].

LK bisects AC and HG: $AD = DC$ and $GI = IH.$

LK and AH are parallel by hypothesis, and

$IK = DL:$ whereby

$DI/IK = ID/DL,$ and by addition,

$DK/IK = IL/DL.$

Also $DK/IK = FD/GI,$ and $IL/DL = IH/DE.$

$\therefore FD/GI = (HI \text{ or } GI)/DE:$ hence $FD.DE = GI.GI.$ Q.e.d.

This proposition is the converse of the proceeding one.

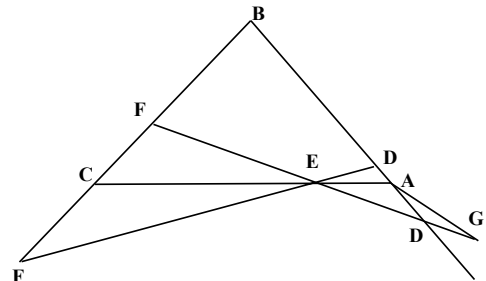
PROPOSITIO XXXIV.

Dato angulo ABC, & intra illum puncto E, oportet per E rectam ducere, occurrentem utrimque anguli lateribus, cuius segmenta minimum contineant rectangulorum quod segmentis cuiusvis lineae per E ductae contineri potest.

Demonstratio.

Constituatur recta AC, per E ducta Isoscelem ABC. dico factum esse quid petitur: agatur enim per E recta quaevis alia FED, & si CEA rectangulum non sit minimum, sit DEF rectangulum, vel aequale rectangulo AEC, vel minus: primo sit aequale. Quoniam igitur AEC, DEF rectangula sunt inter se aequalia, erit ut DE^c ad AE, sic EC ad EF, sunt autem anguli ad E oppositi inter se aequales; igitur triangula AED, FEC sunt inter se similia; adeoque angulus DAE id est BCA externus, aequalis angulo interno CFE. Quod^d fieri non potest; quare DEF rectangulum, aequale non est rectangulo AEC.

Sit igitur DEF rectangulum minus rectangulo AEC. producat FD linea in G, ut.....
 b. 40. Primi



Prop.34. Fig.1

[27]

BOOK I.§2.

PROPOSITIO 34.

For the given angle ABC, and a point E within the angle: it is required to draw a line through E, cutting both sides of the angle, in order that the segments contain the smallest possible of the allowed rectangles.

Demonstration.

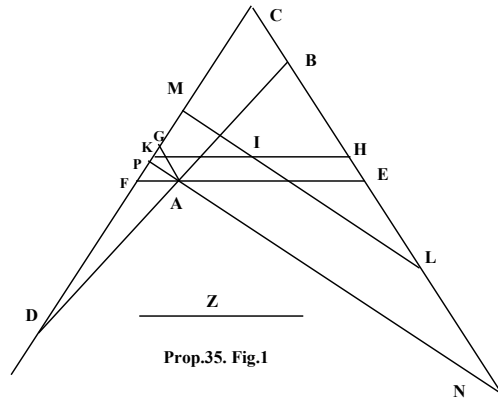
Construct the line AC, and through E draw the isosceles triangle ABC. I assert that the required task has been done: for indeed some another line FED is passed through E, and if the rectangle CEA is not the smallest possible, then let it be the rectangle DEF, which is either equal to the rectangle AEC or less. First let us consider that they are equal. Since the rectangles AEC and DEF are equal to each other, [i. e. $AE \cdot EC = DE \cdot EF$] DE to AE will be as EC to EF [FC in original text]^c. But the opposite angles at E are equal to each other; hence the triangles AED and FEC shall be similar to each other, and thus the angle DAE that is the external angle BCA, is equal to the internal angle CFE. Which cannot be the case^d; whereby the rectangle DEF is not equal to the rectangle AEC. Suppose now that the rectangle DEF can be less than the rectangle AEC [i. e. $DE \cdot EF < AE \cdot EC$], and let FD be produced in G, in order that.....[rest of text is missing! we continue to supply a proof in any case.] the equality is satisfied. In which case the triangles FEC and AEG are similar as above [i. e. $AE \cdot EC = DE \cdot EF$]. But in this case the angle AGE that corresponds to the angle CFE is actually less than this angle, which is the external angle of the triangle AGE at A. Hence again there is a contradiction and so the given line contains the minimum rectangle. Q.e.d.
 c. 14. Sexti, d. 16. Primi

PROPOSITIO XXXV.

Dato puncto A intra angulum BCD; per A rectam ducere pertingentem ad latera BC, CD, ut rectangulum quod sub segmentis continetur, aequale sit quadrato dato Z: quod oportet esse non minus minimo rectangulorum quae segmentis linearum per A ductarum continentur.

Demonstratio.

Quadratum Z vel aequale est minimo rectangulorum quae sub segmentis linearum per A describi possunt, vel maius. Si aequale, agatur per A linea EF exhibens ECF triangulum isosceles; patet per praecedentem EAF rectangulum aequare; quadrato Z. sit igitur quadratum Z maius minimo rectangulorum. Ducatur per A linea DB ut in A divisa sit bifariam (quod fiet si ducta ex A linea AG parallela lateri CB, fiat DG linea aequalis lineae GC: & ex D per A recta ducatur DB) dein ducatur linea HK, parallela rectae EF, ^a triangulum auferens HCK aequale triangulo BCD, eritque; HCK quoque isosceles, & HI quadratum ^b quod sit à dimidia lineae HK; aequale rectangulo EAF: quod per hypothesim minus est quadrato Z. dupla igitur rectae Z maior erit linea HK, poteritque auferre triangulum isosceles sub angulo C, maius triangulo HCK. Ducatur ergo LM, dupla rectae Z auferens ^c triangulum LCM aequale triangulum HCK; & per A recta agatur NP, parallela lineae LM. Dico factum esse quod petitur. cum enim recta BD in A divisa sit bifariam, & per A ducta quaedam NP, cuius parallela LM, triangulum aufert aequale triangulo HCK, id est per constructionem aequale triangulo BCD; erit NAP rectangulum aequale ^d quadrato dimidia ipsius LM, id est per constructionem quadrato Z duximus igitur per A lineam, &c. Q.e.d.



^a. 32. Huius. ^b 33. huius. ^c. 32. Huius. ^d. 33. Huius.

[27]

BOOK I.§2.

PROPOSITON 35.

Fora a given point A within the angle BCD, to draw a line through A extending to the sides BC and CD, in order that the rectangle which is contained by the segments shall be equal to the given square Z, which shall not be smaller than the least of the rectangles contained by the segments of the line drawn through A.

Demonstration.

The square Z is either equal to the smallest of the rectangles which can be described by the segments of the line through A, or it is larger. If it is equal, [*i. e.* $Z^2 = EA.AF$] then draw a line EF through A giving the isosceles triangle ECF. It is apparent from the preceding propositions that the rectangle EAF is equal to the square Z. If the square Z is larger than the smallest of the rectangles, then a line DB is drawn through A in order that A divides that line in two (which can be done if AG is drawn from A parallel to the side CB, when the length of DG becomes equal to GC, and from D the line DB is drawn through A) then the line HK is drawn, parallel to EF ^a [which is taken to be horizontal]. The triangle HCK takes away an area equal to triangle BCD, then HCK will be isosceles too, and the square HI ^b which is constructed from half the line HK is equal to the rectangle EAF [Prop. 32]: which is less than the square Z by hypothesis. Therefore double the line Z will be greater than HK, and an isosceles triangle can be taken away under the angle C greater than the triangle HCK. Hence LM is drawn, double the line Z ^c making the triangle LCM

equal to the triangle HCK; and the line NP is sent through A, parallel to the line LM. I assert that what has been done is what was required .

For indeed the line BD is bisected in A, and a certain line NP is drawn through A, of which the parallel LM, a triangle equal to triangle HCK is taken, that is by construction equal to BCD. Rectangle NAP shall be equal^d to the square of half LM, that is by construction equal to the square Z we have considered.

Therefore the line through A , &c. Q.e.d. ^{a. 32. Huius, b 33. huius, c. 32. Huius, d. 33. Huius.}

PROPOSITIO XXXVI.

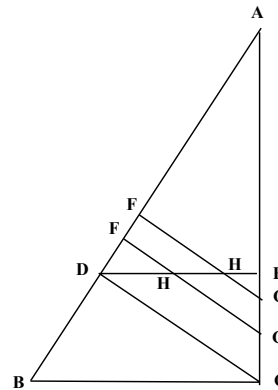
Esto triangulum ABC scalenum ductaque, ex C linea CD, ad oppositum latus, quae angulum ACD aequalem faciat angulo ABC, ducatur ex D linea DE, parallela ipsi CB ; quam in H secent rectae quotcunque FG, parallelae ipsi CD, occurrentes CAD trianguli lateribus in F & G. Dico DHE rectangla, aequari rectangulis FHG.

Demonstratio.

Angulus AGF, id est ACD, per constructionem est aequalis angulo ABC, id est ADE : sunt autem & anguli FHD, EHG ad verticam oppositi inter se aequales, igitur DFH, HEG triangula sunt similia. Unde FH ad HD, ut HE ad HG, adeoque DHE^a rectangula aequala rectangulis FHG. Q.e.d.

^{a. 17. Sexti}

[28]



Prop.36. Fig.1

BOOK I.§2.

PROPOSITON 36.

Let ABC be a scalene triangle, and a line CD be drawn from C to the opposite side, which makes the angle ACD equal to the angle ABC. The line DE is drawn from D parallel to CB which any number of lines parallel to CD cut in H, and meeting the sides of the triangle in F and G.

I assert that the rectangle DHE is equal to the rectangle FHG.

Demonstration.

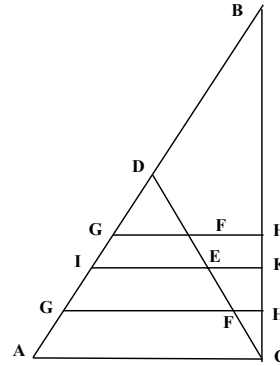
The angle AGF, i. e. ACD, is equal to the angle ABC, i. e. ADE. by construction. But the angles FHD and EHG are vertically opposite and so are equal to each other. Therefore the triangles DFH and HEG are similar to each other. Hence FH to HD as HE to HG, and thus the rectangle DHE is equal to the rectangle FHG. Q.e.d. ^{a. 17. Sexti}

PROPOSITIO XXXVII.

Esto ABC triangulum ductaturque, ex C recta quavis CD, secans AB latus oppositum in D; divisa autem CD bifariam in E, agatur per E linea IK, parallela basi AC, cui & alia quevis GH, ducatur aequidistans, occurrens CD lineae, in F. Dico IEK rectangulum, maius esse rectangulo GFH.

Demonstratio.

Ratio IEK rectanguli, ad rectangulum GFH, ^b composita est ex ratione IE ad GF, id est DE ad DF, & ex ratione EK ad FH, id est EC ad FC : sed ex iisdem quoque ^c composita est ratio rectanguli DEC, ad rectangulum DFC. Igitur ut DEC rectangulum, ad rectangulum DFC, sic IEK rectangulum, est ad rectangulum GFH. est autem DEC rectangulum maius rectangulo ^d DFC (cum DC in E divisa sit bifariam) igitur & IEK rectangulum maius est rectangulo GFH.
 Q.e.d. ^{b. 23. Sexti . c. ibid . b. 23. Sexti . Papp. L 7. Prop. 13}



Prop.37. Fig.1

BOOK I.§2.

PROPOSITON 37.

Let ABC be a triangle, and from C some line CD is drawn cutting the opposite side in D. Also the line CD is bisected in E, and a line IK is sent through E, parallel to the base AC. Some other lines are drawn parallel to AC, and crossing the line CD in F.

I assert that the rectangle IEK is greater than the rectangle GFH. Q.e.d.

Demonstration.

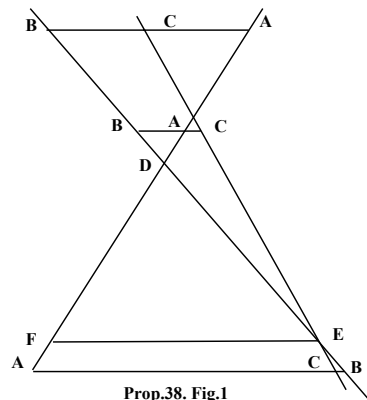
The ratio of the rectangle IEK to the rectangle GFH, ^b has been composed from the ratio IE to GF, *i. e.* DE to DF, and from the ratio EK to FH, *i. e.* EC to FC. But from the same ratios ^c the ratio of the rectangle DEC to the rectangle DFC has been composed. Therefore as rect. DEC to rect. DFC. thus rect. IEK to rect. GFH. But the rect. DEC is greater than the rect. DFC ^d (since DC has been bisected in E [*i. e.* it is the square]), and therefore IEK is greater than the rectangle GFH. Q.e.d. ^{b. 23. Sexti ; c. ibid. b. 23. Sexti . Papp. L 7. Prop. 13.}

PROPOSITIO XXXVIII.

Ocurrent AB lineae quotcunque inter se parallelae, rectis EBD, EC, EF : quas omnes secet linea FAD. Dico ABC rectangulum, ad rectangulum ABC, eam habere ratio, quam habet DBE rectangulum, ad rectangulum DBE.

Demonstratio.

Ratio rectanguli ABC, ad rectangulum ABC, ^d componitur ex ratione AB ad AB, id est DB ad DB, & ex ratione BC ad BC, id est BE ad BE : sed ex iisdem composita est ratio rectanguli DBE, ad rectangulum



Prop.38. Fig.1

DBE. Igitur rectangulum ABC, ad rectangulum ABC, tam habet rationem, quam DBE rectangulum ad rectangulum DBE.
 Q.e.d. ^{d. 23. Sexti}

BOOK I.§2.

PROPOSITON 38.

Any number of lines AB lie parallel to each other and cross the lines EBD, EC, and EF [?]: all these lines cross the line FAD.
 I assert that one rect. ABC to another rect. ABC has the same ratio that the corresponding rect. DBE has to the corresponding rect. DBE . Q.e.d.

Demonstration.

The ratio of the rect. ABC to another rect. ABC, ^d has been composed from the ratio of one AB to another AB, *i. e.* one DB to another DB, and from the ratio BC to BC, *i. e.* BE to BE. [i. e. all the line segments labeled AB have a common vertex D, while those labeled BC have the common vertex E.] But the ratio of the rect. DBE to the rect. DBE has been composed from the same ratios. Therefore as rect. ABC to rect. ABC, so the rect. DBE to rect. DBE. Q.e.d. ^{d. 23. Sexti}.

PROPOSITIO XXXVIX.

Esto ABC trianguli basis AC, bifariam divisa in D, ut recta ducta BD.

[29]

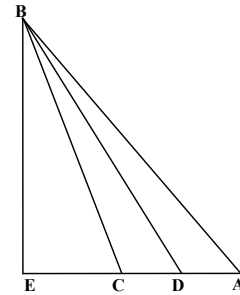
Dico quadrata AB, BC simul sumpta, aequalia esse quadratis AD, DB bis sumptis.

Demonstratio.

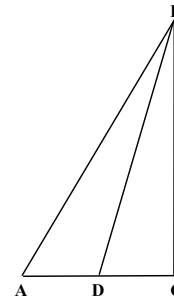
Demittatur ex B linea BE, normalis ad basim AC; & BE quidem 1. cadat extra triangulum ABC: Quoniam igitur BE normalis, cadit extra triangulum ABC formans angulum AEB rectum, paret angulos BAE, BDE, BCE acutos esse; quare AB quaeratum superat ^a quadrata AD, DB, rectangulo ADE bis sumpto. & BC ^b quadratum ab iisdem deficit rectangulo CDE, id est ADE bis sumpto: addendo igitur ad quadratum BC, excessum quo AB quadratum, superat quadrata AD, DB, sit ut AB, BC quadrata simul sumpta aequalia sint quadratis AD, DB bis sumptis.

Sit iam EB normalis, eadem cum latere BC, adeoque angulus ACB rectus; quadratum AB, superat ^c quadrata AD, DB, rectangulo ADC bis sumpto, id est quadratum AB, est aequale quadratis AD, DB, & DC quadrato bis sumpto: sed BC quadratum, deficit ^d a quadrato BD, quadrato DC; igitur si quadratum DC, id est dimidium excessum quo AB quadratum superat quadrata AD, DB, additur quadrato BC; patet AB, BC quadrata, aequari quadratis AD, DB bis sumptis.

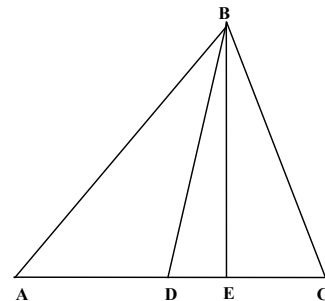
Cadat BE normalis, inter BC & BD lineas: Quoniam anguli AEB, CEB sunt recti, erunt AB, BC quadrata, aequalia quadrato BE bis sumpto; una cum quadratis AE, EC: sed AE, EC quadrata, ^e dupla sunt quadratorum AD, DE; igitur quadrata AB, BC. aequalis sunt, quadratis BE, AD, DE bis sumptis; est autem BD quadratum bis sumptum, aequale quadratis DE, BE bis sumptis; igitur quadrata AB, BC simul sumpta sunt aequalia quadratis AD, DB bis sumptis. Q.e.d. Est haec *Pappi* Lib.7. Prop. 122. a. 13. Secundi; b. 12 Secundi; c. ibid; d. 13 Secundi.



Prop.39. Fig.1



Prop.39. Fig.2



Prop.39. Fig.3

BOOK I.§2.

PROPOSITON 39.

Let the base of the triangle AC be divided in two equal parts in D by drawing the line BD.

I assert that the sum of the squares AB and BC is equal to double the sum of the squares AD and DB.

Demonstration.

The line BE is sent from B, normal to the base AC, and [in Fig. 39.1, the first of 3 cases considered] BE falls outside the triangle ABC. Since the normal BE falls outside the base of triangle ABC, forming the right angle AEB, it is apparent that the angles BAE, BDE, and BCE are acute. Whereby ^a the square AB is exceeded by the squares AD and DB by the rectangle ADE taken twice. Again the square BC is deficient from the same squares by the rectangle CDE, that is ADE, taken twice. Therefore the amount by which the square AB exceeds the squares AD and DB is to be added to the square BC, from which the sum of the squares AB and BC shall be equal to the sum of the squares AD and DB taken twice. Q.e.d. d. 23. *Sexti.*

[For $AB^2 = BE^2 + (ED + DA)^2 = BD^2 + AD^2 + 2.ED.DA$, while
 $BC^2 = BE^2 + (ED - CD)^2 = BD^2 + CD^2 - 2.ED.CD$, giving
 $BC^2 + 2.ED.CD = BC^2 + CD^2$; hence $BC^2 + AB^2 - (BD^2 + AD^2) = BC^2 + CD^2$; giving
 $AB^2 + BC^2 = 2(BD^2 + DA^2)$ as required].

Now the normal shall be CB [in Fig. 39.2], with the same side BC, and thus the angle ACB is right. The square AB is exceeds the squares ^c AD and DB by the rectangle ADC taken twice. That is the square AB is equal to the squares AD and DB and with the square DC taken twice [AD = DC]. But the square BC is less than the square ^dBD by the square DC. Therefore, if the square DC, that is half the excess by which the square AB exceeds the squares AD and BD, is added to the square BC, then it is apparent that the sum of the squares AB and BC is equal to double the sum of the squares AD and DB.

[For $AB^2 = BC^2 + (AD + DC)^2 = BD^2 + AD^2 + 2.AD.DC = BD^2 + 3.AD^2 = 2(BD^2 + AD^2) + AD^2 - BD^2$
 $\therefore AB^2 + BC^2 = 2(BD^2 + AD^2)$].

The normal falls between BC and BD [in Fig. 39.3]. Since the angles AEB and CEB are right, the squares AB and BC together are equal to the twice the square BE together with the squares AE and EC. But the sum of the squares AE and EC are twice the sum of the squares AD and DE ^e [as AD = DC], and hence the squares AB and BC are together equal twice the sum of the squares BE, AD, and DE. But twice the square of BD is equal to twice the sum of the squares of DE and BE. Therefore the sum of the squares AB and BC is twice the sum of the squares AD and DB. Q.e.d. Following *Pappus* Lib.7. Prop. 122; a. 13. Secundi; b. 12 Secundi; c. ibid; d. 13 Secundi.

[For $AB^2 + BC^2 = 2. BE^2 + AE^2 + EC^2$, also $AE = AD + DE$, $EC = AD - DE$;
 hence $AE^2 + EC^2 = 2. (AD^2 + DE^2)$. Hence $AB^2 + BC^2 = 2. BE^2 + 2. (AD^2 + DE^2)$
 $\therefore AB^2 + BC^2 = 2(BD^2 + AD^2)$, as required.]

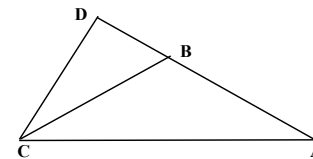
PROPOSITIO XL.

Esto ABC triangulum isosceles, & ex C, alterutro angulorum, ducta normalis CD, ad latus oppositum.

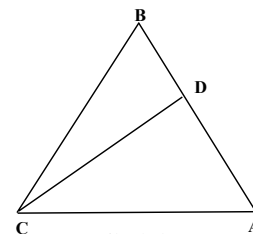
Dico quadrata tria laterum trianguli, ABC, aequari quadrato AD semel; quadrato DB bis, & DC quadrato ter sumpto.

Demonstratio.

Quoniam angulus CDB per constructionem rectus est, erit CB quadratum, aequale quadratis CD, DB : sed AB lineae ex



Prop.40. Fig.1



Prop.40. Fig.2

constructione est aequalis lineae CB. Igitur & quadratum AB; aequale est quadratis CD, BD. Rursum quadratum AC, aequale est quadratis CD, DA: Igitur tria laterum

[30]

trianguli ABC quadrata, sunt aequalia quadrato AD semel, quadrato DB bis, & CD quadrato ter sumpto. Q.e.d.

BOOK I.§2. PROPOSITON 40.

Let ABC be an isosceles triangle, and from one of the angles C draw a line normal to the opposite side.
 I assert that the sum of the squares of the three sides of the triangle ABC is equal the sum of the squares AD taken once, DB taken twice, and DC taken three times.

Demonstration.

Since the angle CDB is right by construction [see Fig. 40.1 and 2] , the square CB is equal to the sum of the squares CD and DB. But the line AB is equal to the line CB by construction. Therefore the square AB is equal to the sum of the squares CD and DB. Again the square AC is equal to the sum of the squares CD and DA. Therefore the sum of the squares of the three sides of the triangle ABC is equal to the square AD together with twice the square DB and three times the square CD. Q.e.d.
 [i.e. $AB^2 + BC^2 + CA^2 = 1.AD^2 + 2.DB^2 + 3.CD^2$.]

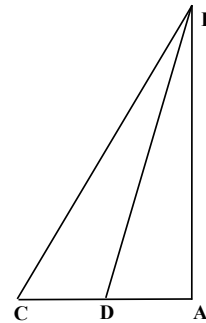
PROPOSITIO XLI.

Esto ABC triangulum rectangulum, & ex B, recta demittatur quaevis BD, ad oppositum latus AC.

Dico AD, BC quadrata simul sumpta, aequari quadratis AC, BD simul sumptis.

Demonstratio.

Quoniam angulus BAC rectus ponitur, erit BC quadratum, una cum quadrato AD, aequale tribus quadratis AB, AC, AD: sed iisdem aequale est quadratum BD una cum quadrato AC. Igitur AD, BC quadrata, aequalia sunt quadratis AC BD. Q.e.d.



Prop. 41. Fig.1

BOOK I.§2. PROPOSITON 41.

Let ABC be a right-angled triangle, and from B some line BD is sent to the opposite side AC.
 I assert that the sum of the squares AD and BC is equal the sum of the squares AC and BD.

Demonstration.

Since the angle CDB is made right [see Fig. 41.1], the sum of the squares BC and AD is equal to the sum of the three squares AB, AC, and AD: but these are equal to the sum of the squares BD. Therefore the sum of the squares AD and BC is equal to the sum of the squares AC and BD. Q.e.d.
 [i.e. $AD^2 + BC^2 = AB^2 + BC^2 + BD^2 - AB^2 = AC^2 + BD^2 = AB^2 + AD^2 + AC^2$.]

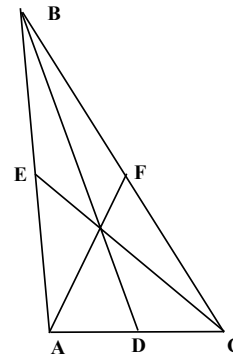
PROPOSITIO XLII.

Esto ABC triangulum, & ex singulis angulis ductae lineae BD, AF, CE secent latera opposita bifariam in D, E, F.

Dico AF, EC, BD, quadrata, ad quadrata tria laterum trianguli ABC, eam habere rationem, quam tria ad quatuor.

Demonstratio.

Quadrata AB, BC dimul sumpta, aequalia sunt quadratis BD, ^a AD bis sumptis; & AC, BC quadrata, aequalia sunt quadratis EC, AE bis sumptis; quadrata vero AB, AC aequantur quadratis AF, CF bids sumptis; igitur quadrata AB, BC, CA semel sumpta, aequalia sunt quadratis AF, CE, BD, AD, AE, CF semel sumpta. Sed AD, AE, CF quadrata, sunt quarta pars quadratorum AB, BC, AC; igitur quadrata reliqua AF, CE, BD, tres habent quartes, quadratorum AB, BC, AC. Q.e.d. a. 39 Huius.



Prop. 42. Fig.1

BOOK I.§2.

PROPOSITIO 42.

Let ABC be a triangle, and from the individual angles the lines BD, AF, and CE are drawn bisecting the opposite sides in D, E, and F. I assert that the sum of the squares of AF, EC, and BD to the sum of the squares of the three sides of the triangle has the ratio three to four.

Demonstration.

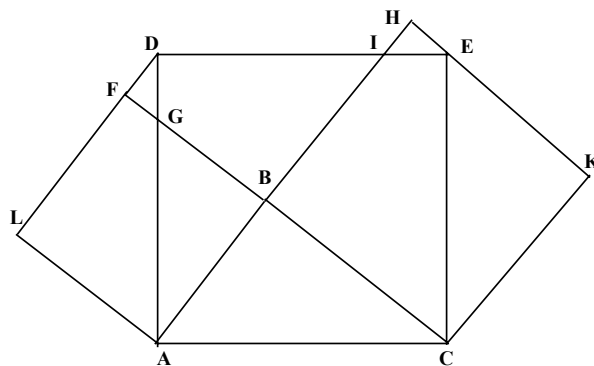
The sum of the squares AB and BC [see Fig. 42.1], is equal to twice the sum of the squares BD and AD ^a. Also, the sum of the squares of AC and BC is equal to twice the sum of the squares of EC and EA, and again the sum of the squares of AB and AC is equal to twice the sum of the squares of AF and CF. Therefore the sum of the squares AB, BC, and CA is equal to the sum of the squares AF, CE, BD, AD, AE, CF. But the sum of the squares AD, AE, and CF is the fourth part of the sum of the squares AB, BC, and CA. Therefore the sum of the remaining squares is three quarters of the sum of the squares AB, BC, and AC. Q.e.d. ^a. Prop.39 of this Book.
 [i.e. $AB^2 + BC^2 = 2.(BD^2 + AD^2)$; $AC^2 + BC^2 = 2.(EC^2 + EA^2)$; $AB^2 + AC^2 = 2.(AF^2 + CF^2)$. Hence on adding : $(AB^2 + BC^2 + CA^2) = BD^2 + EC^2 + AF^2 + \frac{1}{4} .(AB^2 + BC^2 + CA^2)$, giving the required result.]

PROPOSITIO XLIII.

In triangulis rectangulis quadratum, quid sit a latere rectum angulum subtendente, aequale est eis quae a lateribus rectum angulum continentibus, describuntur quadratis.

Demonstratio.

Sit ABC triangulum rectangulum, & laterum quadrata sint AE, AF, CH, & FB linea secet AD in G. Quoniam anguli KCB, ECA sunt inter se aequales dempto communi angulo ECB, erunt anguli ACB,



Prop. 43. Fig.1

ECK reliqui, inter se

[31]

aequales ; sunt autem EC, KC duo latera, aequalia duabus lateribus AC, CB; igitur reliquum latus trianguli CKE, reliquo lateri trianguli ABC est aequale; unde punctum E est in recta HK, & EKC triangulum, aequale triangulo ABC. Eodem modo ostenditur triangulum ABC aequale triangulo DLA, & punctum D esse in directum cum LF producta. Rursum cum DL latus, aequale sit lateri CK, hoc est KH, & FL aequale ipse AB, id est EK ; erit reliquum latus EH, aequale reliqui DF; est autem angulus FGD (ob BF, LA parallelas) aequalis angulo DAL, id est IAC id est angulo EIH; & angulus EHI, aequalis angulo DFG, igitur DFG, EHI triangula, & DG, IE latera sunt inter se aequalia. Iterum cum anguli ADI, GAC sint inter se aequalis, & AC, AG latera, aequalia lateribus AD, DI, erit IDA triangulum, aequale triangulo GAC ; unde dempto communi triangulo ABG, erit ABC triangulum, aequale Trapezio IBGDI. Igitur cum triangulum ABC, id est EKC aequale sit trapezio IBGDI, & DFG triangulum, aequale triangulo EHI, reliqua vero sint communia; erit AE quadratum, lateris rectum angulum subtendentis, aequale quadratis AF, CH, laterum rectum angulum continentium. Q.e.d.

Scholion.

Demonstrat hanc Clavius ad 47 primi Element. triplici alia methodi [-o in text]: secus quam Peletarius fieri posse existimavit : copiae tamen, non necessitatis ergo, etiam nos illam, alia via demonstrandam assumpsimus, maxime occasione sequentium duarum propositionum.

BOOK I. §2.

PROPOSITON 43.

In right-angled triangles, the square which is subtended by the side opposite the right angle is equal to the sum of the squares described by the sides containing the right angle.

Demonstration.

ABC is a right-angled triangle, and the squares of the sides are AE, AF, and CH; the line FB cuts AD in G. Since the angles KCB and ECA are equal to each other, with the common angle ECB taken away, the remaining angles ACB and ECK are equal to each other. Also, the two sides EC and KC are equal to the two sides AC and CB, and therefore the remaining side of the triangle CKE is equal to the remaining side of the triangle ABC. Hence the point E lies on the line HK, and triangle EKC is equal to triangle ABC. [*i.e.* AB = EK also.] By the same method it is shown that triangle ABC is equal to triangle DLA, and the point D lies on LF produced. Again, since DL is equal to CK, (*i.e.* KH), and FL is equal to AB, (*i.e.* EK), the remaining lines EH and DF are also equal. Also, the angle FGD (on account of the parallel lines BF and LA) is equal to the angle DAL, *i.e.* IAC or EIH. Again, the angle EHI is equal to the angle DFG. Therefore in triangles DFG and EHI, the sides DG and IE are equal to each other. Again, as the angles ADI and GAC are equal to each other, and the sides AC and AG are equal to the sides AD and DI, then the triangles IDA and GAC are equal. Hence with the common triangle ABG taken away, triangle ABC is equal to the trapezium IBGDI. Therefore since the triangle ABC (*i.e.* EKC) is equal to the quadrilateral [trapezium in text] IBGDI, and triangle DFG is equal to triangle EHI, the remainders indeed shall be related to each other: the square AE subtended by the right angle is equal to the sum of the squares AF and CH held by the sides of the right angle. Q.e.d.

Note.

Clavius demonstrates Prop. 47 of the First Book of the Elements by three other methods; and one can proceed otherwise according to the thoughts of Peletarius. There are hence a number of methods available, and it is not therefore by necessity that we have added yet another method to these, but especially for the purpose of demonstrating the two following theorems by another method.

[Clavius (Christopher Klau 1537 - 1612) edited the *Elements* of Euclid, and editions of his work were published from 1574 - 1612, he '...rewrote the proofs, compressing or adding to them, where he thought he could make them clearer. Altogether his book is a most useful work.' extract from p. 105. Vol. I. *Euclid...Elements*. Tr. by Sir Thomas Heath. Dover Publications. N. Y.

Peletarius (Jacques Peltier) provided demonstrations to *Euclid's Elements*, first published in 1557. *Ibid*, p. 104.]

[i.e. Sq. AF = Δ ABG + Δ ABC - Δ FGD;
 Sq. BK = quad. BCEI + Δ ABC + Δ HIE;
 Hence on adding :
 Sq. AF + Sq. BK = Δ ABG + Δ ABC - Δ FGD + quad. BCEIB + Δ ABC + Δ HIE = Sq. AE, the required
 result, as Δ ABC = quad. IBGDI.]

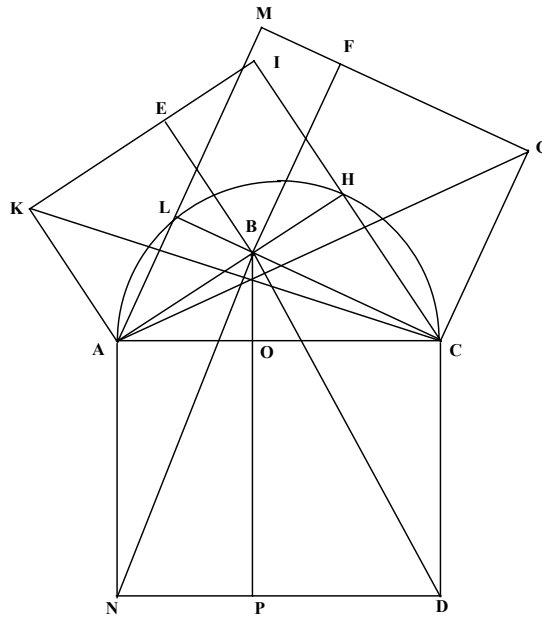
PROPOSITIO XLIV.

Esto ABC triangulum
 amblygonium , & laterum
 quadrata AD, AE, CF.
 ducatur ex C linea CH secans
 orthogonaliter in H & I, lineas AB,
 KE.

Dico AD quadratum, superare
 quadrata AE, CF, rectangulo ABH
 bis sumpto.

Demonstratio.

Ducatur ex A linea AM, secans in L & M
 orthogonaliter lineas CB GF, & super AC
 ut diametro describatur semicirculus
 AHC; ^a transibit is per H & L.
 Iunganturque ; puncta AG, CK, BD, BN,
 dein ex B, recta demittatur BO, secans
 orthogonaliter in O P, lineas AC, DN.
 Quoniam AM, CG lineae, per



Prop. 44. Fig.1

constructionem aequidistant, erit AGC
 triangulum aequale dimidio rectanguli GL : eodem modo triangulum CAK aequale est dimidio rectanguli
 AI; Rursum cum anguli ABE, CAN, per constructionem recti sint, addito communi angulo BAC, erunt
 anguli BAN, CAK inter se aequales ; sunt autem AB, AN latera,

[32]

aequalia duobus lateribus AC, AK; igitur triangulum NAB, aequale est triangulo CAK, & AI rectangulum,
 aequale rectangulo AP: eodem modo ostenditur rectangulum CM, aequari rectangulo CP. Unde AD
 quadratum, aequale est rectangulis AI, CM, simul sumptis. Sed AI rectangulum, superat quadratum AE,
 rectangulo BI, id est rectangulo ABH, & CM rectangulum, superat quadratum CF, rectangulo BM, id est
 rectangulo SBL : id est ABH rectangulo; igitur quadratum AD, superat quadrata AE, CF, rectangulo ABH
 bis sumpto. Q.e.d. ^a 31. Tertii.

Scholion.

*Propositionem hanc, uti & sequentem, licet demonstret Euclides Lib.2.prop.12 & 13. non iniucundum
 tamen fore putari, si utramque eo discursu quo Pythagorae 47. primi Element. hac loco demonstrarem.*

BOOK I.§2.

PROPOSITON 44.

Let ABC be an obtuse-angled triangle, and AD, AE, and CF the squares of the
 sides of the triangle. The line CH is drawn from C cutting the lines AB and KE
 in H and I at right-angles.

I assert that the square AB exceeds the sum of the squares AE and CF by the rectangle ABH taken twice.

Demonstration.

The line AM is drawn from A cutting the lines CB and GF orthogonally in L and M. On AC as diameter the semi-circle AHC is described, ^a that will cross these lines in H and L. The points AG, CK, BD, and BN are joined. The line BO is sent from B cutting the lines the lines AC and ND at right angles in O and P. Since the lines AM and CG are parallel from the construction, the triangle AGC is half the rect. GL. In the same manner the triangle CAK is equal to half the rect. AI. Again, as the angles ABE and CAN are right by construction, by the addition of the common angle BAC, the angles BAN and CAK are equal to each other. Also the sides AB and AN are equal to the two sides AC and AK. Therefore the triangle NAB is equal to the triangle CAK, and the rect. AI is equal to the rect. AP. In the same way the rect. CM is shown to be equal to the rect. CP. Hence the square AD is equal to the sum of the rectangles AI and CM. But the rect. AI exceeds the square AE by the rect. BI. i. e. rect. ABH, and the rect. CM exceeds the square CF by the rect. BM, i. e. by the rect. CBL, or by the rect. ABH. Therefore the square AD exceeds the squares AE and CF by twice the rect. ABH. Q.e.d. ^a 31. Tertii.

Note.

Although the traditional method of this proposition and that following are used to demonstrate Prop. 12 and 13, Book 2, of Euclid's Elements, it is nevertheless not an unpleasant task to use the method set out in our own demonstration of Pythagoras's Theorem (Prop. 47 of the Elements).

[$\triangle AGC = \frac{1}{2}$ rect. GL (AM, GC par'el); $\triangle CAK = \frac{1}{2}$ rect. AI (AK, IC par'el); $\triangle NAB$ is congruent to $\triangle CAK$ (Two pairs of equal sides and included angle); \therefore rect. AI = rect. AP; Also, rect. CM = rect. CP;

\therefore Sq. AD = rect. AI + rect. CM;

But rect. AI = sq. AE + rect. AB.BE, and rect. CM = sq. CF + rect. CB.BL (= rect. AB.BE);

\therefore Sq. AD = rect. AI + rect. CM = sq. AE + sq. CF + 2.rect. AB.BE, as required.]

PROPOSITIO XLV.

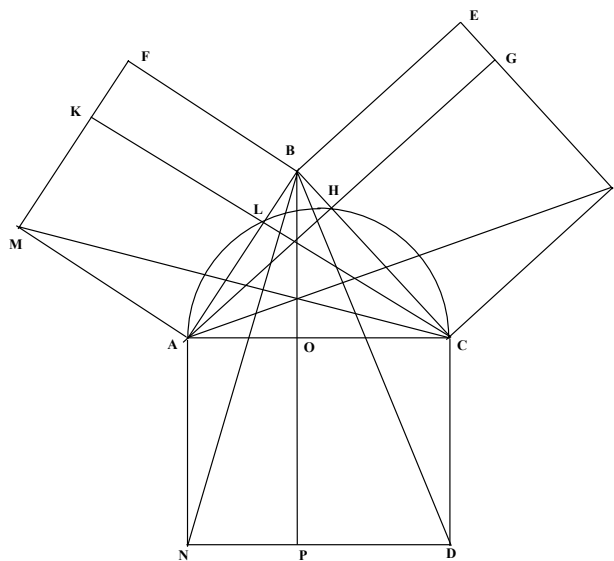
Esto ABC triangulum
 oxygonium, & laterum
 quadrata AD, CE, AF.

Ducaturque ex A linea AG, secans
 orthogonaliter in H & G, lineas
 CB, IE.

Dico AD quadratum deficere à
 quadrata CE, AF, rectangulo HBC
 bis sumpto.

Demonstratio.

Ducatur ex C linea, secans orthogonaliter
 in L & K lineas AB, FM, iunganturque
 puncta AE, CM, BD, BN. descripto dein
 super AC ut diametro semicirculo AHC;
 (qui transibit per ^a H & K) demittatur ex
 B, linea BO secans orthogonaliter in O &
 P, lineas AC, DN. Quoniam AM, LC



Prop. 45. Fig.1

lineae aequidistant, erit MAC triangulum aequale dimidio rectanguli AK : eodem modo triangulum AIC aequale dimidio rectanguli CG. Rursum cum anguli MAB, CAN, per constructionem sint aequales, addito communi angulo BAC, erunt anguli MAB, BAN quoque inter se aequales; sunt autem & latera AM, AC

equalia duobus lateribus AB, AN; igitur triangulum MAC, aequale triangulo BAN, & AK rectangulum, aequale esse rectangulo AP: duplo nimirum trianguli BAN. Eodem modo ostenditur rectangulum CG, aequari rectangulo CP. Quare AD quadratum, aequale est rectangulis CG, AK. Sed CG rectangulum, deficit à quadrato CE, rectangulo HE, id est rectangulo HBC, & AK rectangulum, deficit à quadrato AF, rectangulo LF, id est rectangulo ABL : id est rectangulo HBC; igitur quadratum AD, deficit à quadratis quadrata CE, AF, rectangulo HBC bis sumpto. Q.e.d. a 31. Tertii.

BOOK I. §2. PROPOSITON 45.

Let ABC be an acute-angled triangle, and AD, CE, and AF the squares of the sides of the triangle. The line AG is drawn from A cutting the lines CB and IE in H and I at right-angles. I assert that the square AD is less than the sum of the squares CE and AF by the rectangle ABC taken twice.

Demonstration.

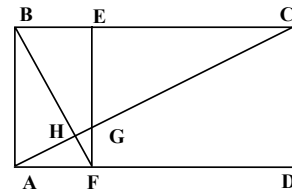
A line is drawn from C cutting the lines AB and FM orthogonally in L and K. The points AI, CM, BD, and BN are joined. On AC as diameter the semi-circle AHC is described that crosses these lines in H and L. The line BO is sent from B cutting the lines AC and DN at right angles in O and P. Since the lines AM and LC are parallel from the construction, the triangle MAC is half the rect. AK. In the same manner the triangle AIC is equal to half the rect. CG. Again, as the angles MAB and CAN are equal by construction, by the addition of the common angle BAC, the angles MAC and BAN are equal to each other. Also the sides AM and AC are equal to the two sides AB and AN. Therefore the triangle MAC is equal to the triangle BAN, and hence the rect. AK is equal to the rect. AP, clearly double the triangle BAN. In the same way the rect. CG is shown to be equal to the rect. CP. Hence the square AD is equal to the sum of the rectangles CG and AK. But the rect. CG is less than the square CE by the rect. HE. i. e. rect. HBC, and the rect. AK is less than the square AF by the rect. LF, i. e. by the rect. ABL, or by the rect. HBC. Therefore the square AD is less than the squares CE and AF by twice the rect. HBC. Q.e.d. a 31. Tertii.

[$\Delta AIC = \frac{1}{2}$ rect. GC (AG, IC par'l); $\Delta CAM = \frac{1}{2}$ rect. AK (AM, KC par'l); ΔNAB is congruent to ΔCAM (Two pairs of equal sides and included angle); \therefore rect. AK = rect. AP; Also, rect. CG = rect. CP;
 \therefore Sq. AD = rect. AK + rect. CG;
 But rect. AK = sq.. AF - rect. HB.BC, and rect. CG = sq.. CF - rect. HB.BC ;
 \therefore Sq. AD = rect. AK + rect. CG = sq.. AF + sq.. CF - 2.rect. HB.BC, as required.]

[33]

PROPOSITIO XLVI.

Parallelogrammi ABC, diametrum AC, secet EF; aequidistatis ipsi AB in G. ponatur insuper BF. Dico GH, HA, HC tres esse in continua proportione.



Prop. 46. Fig.1

Demonstratio.

Similia namque sunt triangula, AHF, BHC, quemmadmodum & triangula GHF, AHG: quare ut CH ad HA, hoc est BH ad HF, ita HA ad HG. Q.e.d.

BOOK I. §2. PROPOSITON 46.

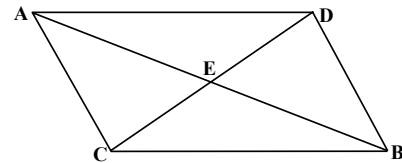
The diameter AC of the parallelogram ABC is cut by the line EF passing through G, EF is to be equidistant from AB. In addition the line BF is put in place.
 I assert that the three lines GH, HA, and HC are in continued proportion.

Demonstration.

For the triangles AHF and BHC are similar, as are the triangles GHF and AHB [see Fig. 46.1]. Whereby as CH is to HA, that is BH is to HF, and thus HA to HG. Q.e.d.

PROPOSITIO XLVII.

Sint AB CD cuiuscumque parallelogrammi; diametri AB.
 Dico diametrorum quadrata simul sumpta, aequalia esse quadratis laterum figurae.



Prop. 47. Fig.1

Demonstratio.

Per trigesimam nonam huius, quadrata AC, AD aequantur quadratis EC, EA bis sumptis : & per iandem quadrata CB, BD aequalia sunt iisdem ; scilicet quadratis EC, EB bis sumptis, sed quadratum CD^a aequatur EC quadrato quater sumpto, & AB quadratum , quadrat ES quater sumpto : patet ergo veritas propositions. Q.e.d. ^a 4. Secundii.

BOOK I. §2. PROPOSITON 47.

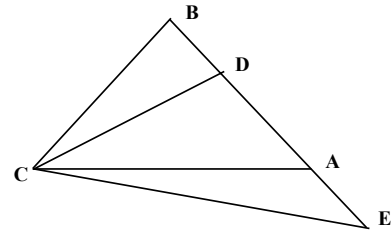
Let AB and CD be the sides of any parallelogram with diagonal AB.
 I assert that the sum of the squares of the diagonals is equal to the sum of the squares of the sides of the figure.

Demonstration.

By the 39th Proposition of this Book, the sum of the squares of the sides AC and AD is equal to twice the sum of the squares of EC and EA: and by the same proposition, the sum of the squares of the sides CB and BD is equal, of course, to twice the sum of the squares EC and EB. But the square CD^a is equal to four times the square EC, and the square AB to four times the square EA: from which the truth of the proposition is apparent. Q.e.d. ^a 4. Secundii.

PROPOSITIO XLVIII.

Esto ABC triangulum isosceles, ductaque ex C, alterutro angulorum aequalium, utcunque linea CD, quae AB lateri, occurrat in D, fiat ipsi CD aequalis DE, iunganturque EC. Dico angulum BCD, duplum esse anguli ACE.



Prop. 48. Fig.1

Demonstratio.

Angulus DAC aequalis b est duabus angulis DEC, ACE, id est per constructionem, angulo DCE una cum angulo ACE, id est angulo DCA una cum angulo ACE bis sumpto. Sed angulo DAC aequatur angulus BCA, cum ABC sit isosceles; igitur angulus BCA, aequatur angulo DCA, una cum angulo ACE bis sumpto, dempto igitur communi angulo DCA, manet angulus BCD aequalis angulo ACE bis sumpto. Q.e.d. b 16. Primi.

BOOK I. §2.

PROPOSITIO 48.

Let ABC be an isosceles triangle and from one of the equal angles C is drawn some line CD that crosses the side AB in D, and CD is made equal to DE, and EC is joined. I assert the angle BCD to be the double of the angle ACE

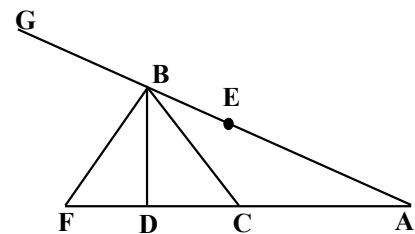
Demonstration.

The angle DAC is equal to the sum of the angles DEC and ACE^b, that is equal by construction, to the angles DCE and the angle ACE, that is to the angle DCA together with the angle ACE taken twice. But the angle DAC is equal to the angle BCA as ABC is isosceles. Therefore the angle BCA is equal to the angle DCA together with the angle ACE taken twice. Therefore with the common angle DCA taken away, it appears that the angle BCD is equal to twice the angle ACE. Q.e.d. b 16. Primi.

[34]

PROPOSITIO XLIX.

Esto ABC triangulum obtusangulum, oporteat AB latus subtensum angulo obtuso ita secare in E, ut AE, EB quadrata, aequalia sint quadratis AC, CB.



Prop. 49. Fig.1

Constructio & demonstratio.

Producta AC in F, ut AC, CF lineae sint inter se aequales, iunguntur FB; producanturque AB in G ut GB, BF lineae sint aequales; tum AG bifariam secetur in E : Dico factum esse quod petitur : Quoniam AC, CF lineae sunt inter se aequales, erunt ABC, CBF angula aequalia, & quia angulus ACB obtusus est, erit AB latus, subtensum angulo ACB, maius latere BF, id est per constructionem, latere GB. quare punctum E cadet inter A & B. igitur cum AG linea, divisa sit bifariam in E, & non bifariam in B, erunt AB, BG quadrata dupla^a quadratorum AE, EB : sed & AB, BG quadrata, id est AB, BF, dupla b sunt quadratorum AC, CB, igitur quadrata AE, EB, aequalia sunt quadratis AC, CB. divisimus igitur latus AB, &c. . Q.e.f. a 9. Secundi; b 39 Huius.

Aliter.

Demittatur ex B linea BD, normalis ad rectam AC, seceturque AB in E ut AEB rectangulum aequetur rectangulo ACD (quod fieri posse ex eo constat, quod AB linea maior sit linea AD.) Dico factum esse quod petitur. Quadratum AB aequale est quadratis AC, CB & ACD ^c rectangulo bis sumpto, sed & AB quadratum aequale ^d est quadratis AE, EB & AEB rectangulo bis sumpto, igitur quadratae AC, CB, una cum rectangulo ACD aequalia sunt quadratis AE, EB; una cum rectangulo AEB: sunt autem ACD, AEB rectangula inter se per constructionem aequalia, igitur & quadrata AC, CB, aequalia sunt quadratis AE, EB. divisimus igitur, &c. Q.e.d. c. 12 Secundi; d. 6 Secundi.

BOOK I. §2.

PROPOSITON 49.

Let ABC be an obtuse-angled triangle, the side AB is required to be subtended by an obtuse angle to be cut thus in E, in order that the sum of the squares on AE and EB are equal to the sum of the squares AC and CB.

Construction and demonstration.

The line AC is produced to F in order that the lines AC and CF are equal to each other, and the line FB is joined. The line AB is extended to G in order that GB and BF are equal, then AG is bisected in E. I assert that what was sought has been accomplished. Since the lines AC and CF are equal to each other, the triangles ABC and CBF [have an equal side]. Since the angle ACB is obtuse, the line AB subtended by the angle ACB is greater than the line BF, that is equal to the line GB by construction. Whereby the point E lies between A and B, [as GB < BA]. Therefore, as the line AG is bisected in E, and not bisected in B, the sum of the squares AB and BG is double the sum of the squares ^a AE and EB. And also, the sum of the squares AB and BG, that is AB and BF ^b is double the sum of the squares AC and CB. Therefore the sum of the squares AE and EB is equal to the sum of the squares AC and BC. Therefore we divide the line AB, etc. Q.e.f.

^a 9. Secundi; ^b 39 Huius.

[$AB^2 + BC^2 = AB^2 + BF^2 = 2.(AC^2 + BC^2)$; also, $AB^2 + BC^2 = (AE + EB)^2 + (AE - EB)^2$, from which the result $AC^2 + BC^2 = AE^2 + EB^2$ follows.]

Otherwise...

The line BD is drawn from B normal to the line AC. AB is divided in E in order that the rect. AE.EB is equal to the rect. AC.CD (which is possible since it agrees with the fact that the line AB is greater than the line AD.) I assert that what was sought has been accomplished. The square AB is equal ^d to the sum of the squares ^c AC and CB and twice the rect. AC.CD, and as the square AB is equal to the sum of the squares AE and EB and twice the rect. AE.ED. Hence the sum of the squares AC and CB together with [twice] the rect. AC.CD is equal to the sum of the squares AE and EB together with [twice] the rect. AE.ED: but the rectangles AC.CD and AE.EB are equal by construction. Therefore the sum of the squares AC and CB is equal to the sum of the squares AE and EB. Therefore we may divide, etc. Q.e.d. ^c 12 Secundi; ^d 6 Secundi.

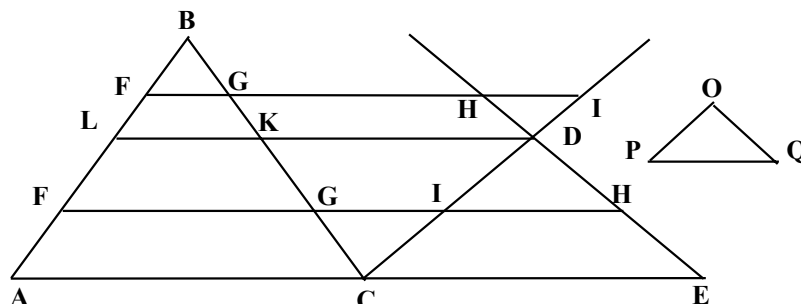
PROPOSITIO L.

Datis duobus triangulis ABC, CDE inaequalis altitudinis, super eadem, vel aequali basi in directum constitutis; lineam ducere, parallelam basi AC, quae auferat triangula in data ratione M ad N.

Constructio & demonstratio.

Ducatur ex D linea DK, parallela basi AC, occurrens ABC trianguli lateribus, in K & L: dein fiat ut M ad N, sic LBK triangulum, ad triangulum OPQ, quod simile sit triangulo CDE. tum quedam ducatur FG, parallela basi AC,

occurrens ABC trianguli lateribus in F & G, & CDE triangulo lateribus in H & I, ut^e FG sit ad HI, sic ut LK ad PQ. Dico factum esse quod peritur. Quoniam FG est ad HI, ut LK ad PQ, erit permutando invertendo ut LK ad FG, sic PQ ad HI :



Prop. 50. Fig.1

sed LBK triangulum, est ad triangulum FBG, in duplicata^f ratione eius, quam habet LK linea, ad lineam FG. & POI triangulum est ad triangulum CDE in duplicata^g ratione lineae PQ

[35]

ad HI, quia POQ, HDI triangula sunt similia; igitur ut triangulum LBK ad triangulum FBG, ita est triangulum POQ, ad triangulum HDI. & permutando ut LBK triangulum, ad triangulum POQ, sic FBG triangulum, est ad triangulum HDI: sed LBK triangulum, per constructionem est ad triangulum POQ. ut M ad N, igitur ut M ad N sic FBG triangulum est ad triangulum HDI; Duximus igitur lineam, &c. Q.e.f. ^e 16. Huius; ^f 19 Sexti; ^g ibid.

BOOK I.§2.

PROPOSITON 50.

With two triangles ABC and CDE given of unequal altitude, either on the same or on another equal base along the agreed direction, to draw a line parallel to the base AC by means of which the areas of the triangles can be removed in the given ratio M to N.

Construction and demonstration.

A line DK is drawn from D, parallel to the base AC, cutting the sides of the triangle ABC in K and L. The triangles LBK and OPQ shall be constructed with areas in the given ratio M to N, and triangle OPQ shall be similar to triangle CDE. Then indeed FG is drawn parallel to the base AC, cutting the sides of triangle ABC in F and G, and the sides of triangle CDE in H and I^e, and as FG is to HI, thus LK to PQ. I assert that what was sought has been accomplished.

Since FG is to HI as LK to PQ, the ratio will on inverting and swapping: as LK to FG, thus PQ to HI. But triangle LBK is to triangle FBG in the ratio of the square^f of this ratio that the line LK has to the line FG. Triangle POQ is to triangle HDI [text has CDE] as the^g ratio of the squares of the line PQ to the line HI,

[35]

since the triangles POQ and HDI are similar. Therefore as triangle LBK is to triangle FBG, thus triangle POQ is to triangle HDI & by swapping the ratio, as triangle LBK to triangle POQ, thus triangle FBG is to triangle HDI. But triangle LBK is by construction to triangle POQ as M to N: therefore as M to N thus

triangle FBG is to triangle HDI. We have been led therefore to a line, etc. Q.e.f. ^e 16. Huius; ^f 19 Sexti; ^g ibid.

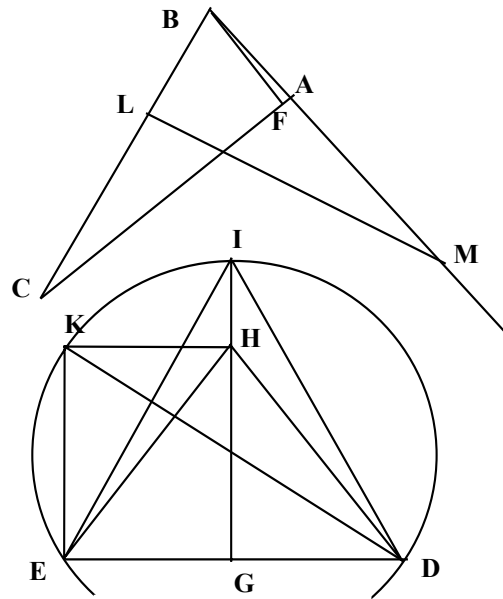
[Since $FG/HI = LK/PQ$ then $LK/FG = PQ/HI$. But $\Delta LBK/\Delta FBG = (LK/FG)^2$.
 $\Delta POQ/\Delta HDI = (PQ/HI)^2$ as ΔPOQ is similar to ΔHDI . Therefore $\Delta LBK/\Delta POQ = \Delta FBG/\Delta HDI$.
 Hence $\Delta LBK/\Delta POQ = M/N = \Delta FBG/\Delta HDI$.]

PROPOSITIO LI.

Dato triangulo ABC, linea DE, & angulo ABC. Oporteat in eodem angulo ABC rectam subtendere LM, datae DE aequalem, quae auferat triangulum, dato ABC triangulo aequale. Oportet autem ABC triangulum, non esse maius isosceli, quod super DE poni potest, habens ad verticem angulum, dato ABC aequalem.

Constructio & demonstratio.

Super DE linea segmentum fiat circuli, continens angulum, ABC aequalem. Divisaque DE bifariam in G, erigatur recta GI, normalis ad lineam AC, iunganturque puncta DI, IE. Dein ex B demittatur linea BF, normalis ad basim AC : fiatque ut DE ad AC, ita BF ad HG, erunt ABC, DHE ^a triangula inter se aequalia, & H punctum non cadet supra I, quia ABC triangulum per constructionem non est maius triangulo DIE: tum recta ducatur HK, parallela basi DE, occurrens circulo in K. Iunctisque punctis DK, EK, fiat BM aequalis ipsi DK, & BL aequalis ipsi KE, iunganturque LM. Dico LBM triangulum satisfacere petitioni: Quoniam HK, DE lineae sunt parallelae, erit EKD triangulum aequale triangulo DHE, id est per constructionem triangulo ABC; est autem angulus ABC, aequalos angulo DIE, id est angulo ^b DKE, & DK, DE latera per constructionem aequalia lateribus LB, BM. igitur triangulum LBM aequale est triangulo DKE, id est triangulo ABC, & ML latus aequale lateri DE. igitur angulo ABC rectam subtendimus, &c. Q.e.f. ^a Clau. ad 23 sexti; ^b 21 Terti.



Prop. 51. Fig.1

BOOK I.§2.

PROPOSITON 51.

For the given triangle ABC, the line DE and the angle ABC: it is required to subtend a line LM in the angle ABC equal to the line DE, which gives a triangle equal in area to the given triangle ABC. But it is necessary that the triangle ABC shall not be greater than the isosceles triangle that can be placed on DE, having a vertical angle equal to the given angle ABC.

Construction and demonstration.

The segment of a circle containing an angle [DIE] equal to the angle ABC is constructed on the line DE. The line DE is then divided in two equal parts by G, and the line GI erected normal to the line ED [text has AC], and the points DI and IE are joined. Then from B the line BF is sent, normal to the base AC: the ratio DE to AC shall be made as BF to HG, and triangles ABC and DHE ^a are equal to each other in area. The

point H will not fall on I, since triangle ABC by construction is not greater than triangle DIE: then the line HK is drawn parallel to the base DE, crossing the circle in K. With the points DK and EK joined, BM is made equal to DK and BL equal to KE, and LM is joined. I say that triangle LBM satisfies the requirements.

Since the lines HK and DE are parallel, triangle EKD has the same area as triangle DHE, that is in turn by construction equal to triangle ABC; but the angle ABC is equal to angle DIE, that is in turn equal to the angle DKE^b, and the sides DK and DE by construction are equal to the sides LB and BM. Therefore the triangle LBM is equal to the triangle DKE, that is to triangle ABC, and the side ML is equal to the side DE. Therefore we subtend the side to the angle, etc. Q.e.f. ^a Clau. ad 23 sexti; ^b 21 Terti.

[$\Delta ABC / \Delta DHE = (\frac{1}{2} AC.BF) / (\frac{1}{2} ED.HG) = (AC/ED).(BF/HG) = 1$, as the altitudes are in the inverse ratio to the bases of the triangles. The rest of the proof then follows easily.]

PROPOSITIO LII.

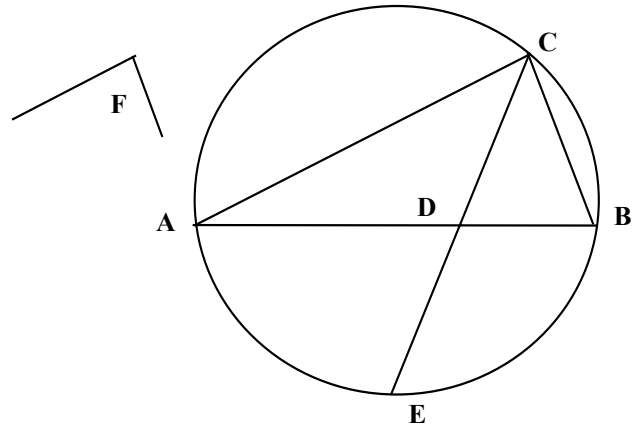
Data recta AB, utcunque divisa in D; super AB triangulum constituere ACB, habens ad C verticem, angulum dato F, aequalem; quem recta ex C per D acta, dividat bifariam.

Constructio & demonstratio.

Super AB recta segmentum describitur circuli, continens angulum BCA aequalem dato F; perfectoque circulo ACB, secetur arcus AEB bifariam in E, & ex E per D linea agatur EC, occurrens circuli peripheriae in C: iunganturque puncta AC, CB. Dico factum esse quod peritur, cum enim arcus AEB per

[36]

constructionem bifariam in E sit divisus, erunt ACE, BCE anguli inter se aequales, adeoque angulus ACB bifariam dividus, sed angulus ACB per constructionem est aequalis angulo dato F; igitur super AB recta triangulum constituimus, &c. Q.e.f. ^a 27 Tertii.



Prop. 52. Fig.1

BOOK I.§2.

PROPOSITON 52.

For the given line AB divided in some manner in D; to set up a triangle ACB on AB having the angle at C equal to the given angle F, which the line drawn from C passing through D shall bisect.

Construction and demonstration.

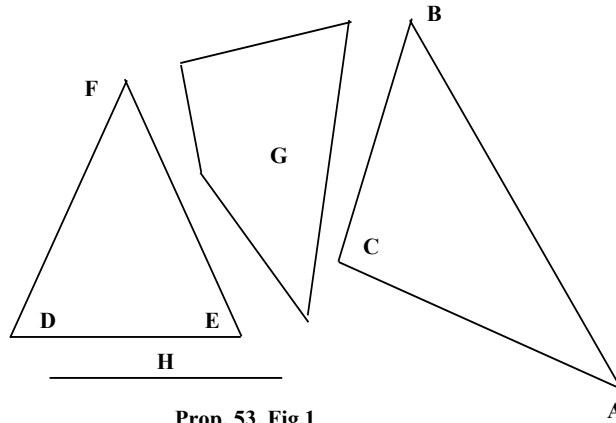
The segment of a circle is described on the line AB, containing the angle BCA equal to the angle F [The diagram Fig. 52.1 shown has B and C transposed from the original diagram]; and with a complete circle ACB the arc AEB is bisected in E, and the line EC is sent from E through D, cutting the circumference of the circle in C: the points AC and CB are joined. I assert that what is required has been done. Since the arc AEB

[36]

is indeed bisected by construction, the angles ACE and BCE are equal to each other, and thus the angle ACB is bisected. But the angle ACB is equal to the angle F by construction, hence upon the line AB we have set up a triangle, etc. Q.e.f. ^a 27 Terti.

PROPOSITIO LIII.

Data angulo ABC,
 rectam subtendere,
 quae ABC triangulum
 aequale constituat, figurae
 rectilineae G; sic ut AB, BC
 laterum , differentia sit aequalis
 datae rectae H.



Prop. 53. Fig.1

Constructio & demonstratio.

Fiat EFD triangulum isosceles aequale
 figurae rectilineae G habens angulum
 ad verticem aequalem angulo ABC:

habita dein media DF, & excessu extrematum H, ^b inveniantur extremae AB, BC; iungunturque puncta AC. Dico factum esse quod petitur. Quoniam angulo ABC, per constructionem aequalis est angulus EFD, sit autem ut AB ad DF, sic DF ad BC per constructionem, id est EF ad BC, erit ABC triangulum, aequale triangulo DFE: id est per constructionem, figurae rectilineae G: est autem H differentia laterum AB, BC: igitur angulo ABC rectam subtendimus, &c.
 Q.e.f. ^b Per A....ium & alios; ^c 15 Sexti.

BOOK I.§2.

PROPOSITON 53.

For the given angle ABC to subtend a line by which the triangle ABC shall have the same area as the rectilinear figure G; thus, as the difference of the sides AB and BC shall be equal to the given line H.

Construction and demonstration.

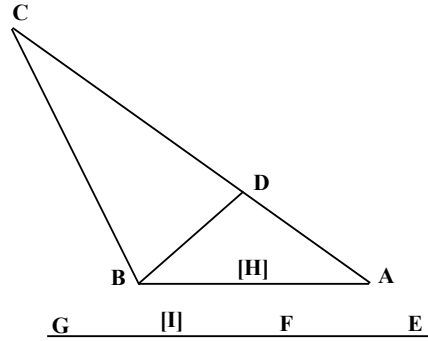
Let the given isosceles triangle EFD to made equal in area to the rectilinear figure G, having an angle to the vertical equal to the angle ABC : then DF is the mean proportional of the lengths AB and BC, and with H the difference of AB and BC, the lengths AB and BC can then be found, and the points AC joined. I assert that what is sought has been found.

Since the angle ABC by construction is equal to the angle EFD, moreover AB shall be to DF, thus as DF to BC by construction, that is in turn as EF to BC, and the triangle ABC will be equal in area to the triangle DFE: that is by construction equal to the rectilineat figure G: moreover H is the difference of the sides AB and BC. Therefore we subtend the line to the angle ABC, etc. Q.e.f. ^b Per A....ium & alios; ^c 15 Sexti.

[We are given no details about G, and must accept a numerical value for the area G and nothing else. G is equal to $\frac{1}{2} FD^2 \sin F = \frac{1}{2} AB \cdot BC \sin B$. Hence $AB - BC = H$ and $AB \cdot BC = FD^2 = 2G/\sin B$, from which a quadratic equation can be used to find the lengths of AB and BC.]

PROPOSITIO LIV.

Data angulo ABC, rectam subtendere AC, ut ex B in AC, demissa linea BD, angulum producat ADB, aequalem angulo ABC, abscindatque; rectam CD, ad quam AB linea, datam habeat rationem H ad I.



Prop. 54. Fig.1

Constructio & demonstratio.

Fiat ut H ad I, sic AB linea ad lineam FG. dataque AB media, & FG extremarum differentia, inveniuntur extremae EG, EF: erit EG maior recta AB. Ducatur igitur ex A linea AC aequalis rectae EG, occurrens BC lateri in C. sumptaque CD aequali ipsi FG, iungantur puncta. Dico factum esse quod petitur. Quoniam tam AC, EG lineae, quam DC, FG sunt inter se aequales, erit AD reliqua, aequalis reliquae EF. Quare AD est ad AB, ut AB ad AC; est autem angulus CAB, communis triangulis ADB, ACB; igitur triangula ADB, ACB sunt^d inter se similia; angulusque ADB aequalis angulo ABC. Rursum cum DC per constructionem sit aequalis ipsi FG, erit ut AB ad FG, sic AB ad DC, sed AB est ad FG per constructionem ut H ad I. Igitur ut H ad I, sic AB est ad DC; unde angulo ABC rectam subtendimus, &c.
 Q.e.f.^d 23 Sexti.

BOOK I.§2.

PROPOSITON 54.

For the given angle ABC, to subtend a line AC, in order that a line BD sent from B to AC gives the angle ADB equal to ABC, and the line CD cuts the line AC so that CD to AB is in a given ratio H to I.

Construction and demonstration.

Let the ratio of line AB to the line FG thus be made as H to I. Given the mean AB [i. e. $AB^2 = EF.EG$] and the difference FG of the largest and smallest lengths EG and EF, the extremes EG and EF can be found, where EG is larger than AB. The line AC is therefore drawn from A equal to the line EG, cutting the line BC in C, and CD is taken equal to FG, and the points are joined. I assert that the required result has been done.

Since the line pairs AC, EG and DC, FG are equal to each other, the remainder AD is equal to the remainder EF. Whereby AD is to AB as AB is to AC; But the angle CAB is common to the triangles ADB and ACB: therefore the triangles ADB and ACB are similar to each other^d. Also the angle ADB is equal to the angle ABC. Again since by construction DC is equal to FG, AB is to FG thus as AB to DC; but AB is to FG by construction as H is to I. Therefore as H to I, thus AB is to DC; hence we can subtend the line to the angle ABC, etc. Q.e.f.^b 23 Sexti.

[Given $AB/FG = H/I$ or $EG - EF = (I/H).AB$, and $AB^2 = EG.EF$, then the lengths EG and EF can be found: this involves solving a quadratic equation (?)

Δ ADB is similar to Δ ABC; hence $AB/AD = AC/AB$ and $AB^2 = AC.AD$, i. e. AB is the mean proportional of AC and AD