CHAPTER V

CONCERNING THE MINIMAL DISTURBANCES OF AIR

IN TUBES HAVING A HYPERBOLIC CONOIDAL FIGURE

[*i.e.* Steady-State Conditions.]

PROBLEM 86

101. *If the figure of the tube* (Fig. 91) *may arise from the rotation of the equilateral hyperbolic arc AB made around the asymptote IL and the air contained in that initially will have been disturbed in some manner from the equilibrium state, to investigate the following disturbances of this at some time.*

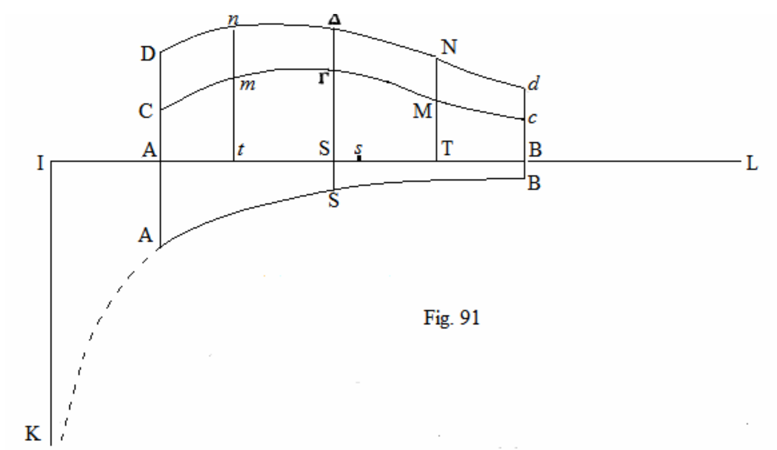

Fig. 91

SOLUTION

I shall be the intersection of the asymptotes, and with the distance taken $IS = S$, the amplitude of the tube there may be put to be $\Omega = \frac{C}{SS}$, moreover the initial disturbance of the air induced thus itself may be had, so that at the position S in the tube the density will have become $= Q$ and the speed along $IB = \Upsilon$. Now after some passage of the time t, from the solution found above (§.82 and §.84), where on account of $\alpha = 1,\ \ \beta = 0$, we may neglect the effect of gravity, air will be moved from S through the small distance $Ss = \upsilon$, so that there shall become [Recall that B is the equilibrium density of the air] :

$$\upsilon = SS\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + S\Gamma:(S+ct) + S\Delta:(S-ct)$$

on putting $\sqrt{\frac{2ga}{b}} = c$, then truly with the change from the constant density

$$q = Q\Big(1 - l\,\tfrac{Q}{B} + \Gamma:(S+ct) + \Delta:(S-ct) - S\Gamma':(S+ct) - S\Delta':(S-ct)\Big)$$

and the speed

$$\mathfrak{T} = cS\left(\varGamma' : (S+ct) - \varDelta' : (S - ct)\right).$$

But now the question is reduced to this, so that it may be adapted to the initial nature of the functions $\varGamma$ and $\varDelta$: therefore on putting $t = 0$, there must become $\upsilon = 0,\ q = Q$ and $\mathfrak{T} = \varUpsilon$, from which we deduce these equations:

$$\text{I. } SS\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + S\varGamma : S + S\varDelta : S = 0$$

$$\text{II. } -l\,\tfrac{Q}{B} + S\varGamma : S + S\varDelta : S - S\varGamma' : S - S\varDelta' : S = 0$$

$$\text{III. } \varUpsilon = cS\left(\varGamma' : S - \varDelta' : S\right),$$

of which indeed the two first equations agree with each other, so that it is understood both from the nature of the matter as well as from the differentiation of the first. For the sake of brevity we may put $\varGamma : S + \varDelta : S = x$ and $\varGamma : S - \varDelta : S = y$ so that the equations may be produced

$$\text{II. } -l\,\tfrac{Q}{B} + x - \tfrac{Sdx}{dS} = 0$$

and

$$\text{III. } \varUpsilon = \tfrac{cSdy}{dS},$$

and

$$\text{I. } SS\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + S : x = 0,$$

which certainly includes that circumstance itself . Hence therefore there is $x = -S\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B},$ thence truly $y = \tfrac{1}{c}\int \tfrac{\varUpsilon S}{S},$ thus moreover so that we may obtain:

$$\varGamma : S = -\tfrac{1}{2} S\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + \tfrac{1}{2c}\int \tfrac{\varUpsilon dS}{S}$$

and

$$\varDelta : S = -\tfrac{1}{2} S\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} - \tfrac{1}{2c}\int \tfrac{\varUpsilon dS}{S}.$$

from which the [differentials] become:

$$\varGamma' : S = -\tfrac{1}{2}\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} - \tfrac{1}{2S}\, l\,\tfrac{Q}{B} + \tfrac{\varUpsilon}{2cS}$$

and

$$\varDelta' : S = -\tfrac{1}{2}\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} - \tfrac{1}{2S}\, l\,\tfrac{Q}{B} - \tfrac{\varUpsilon}{2cS}.$$

Therefore since from the initial state for the initial point S of the axis the quantities may be given $l\,\tfrac{Q}{B} = \tfrac{Q-B}{B}$ approximately and $\varUpsilon$, hence the two curves $C\varGamma c$ and $D\varDelta d$ may be constructed , so that the applied lines of these shall be, but which curves may not be extended beyond the length of the tube *AB* , and then from the quadratures of these the curves arise :

$$\Gamma : S = ACS\Gamma \text{ and } \Delta : S = ADS\Delta.$$

In the time elapsed $=t$ from the construction of these curves , the intervals $ST = St = ct$ may be taken on the axis in each direction from the point *S*, then there will become :

the translation $Ss = SS\int \frac{dS}{SS} l\frac{Q}{B} + S\left(ACTM + ADtn\right)$

the density $q = Q\left(1 - l\frac{Q}{B} + ACTM + ADtn - S{\cdot}TM - S{\cdot}tn\right)$

the speed $\mathfrak{T} = cS\left(TM - tn\right),$

which determinations prevail, as long as the points *T* and *t* do not fall outside the tube *AB*.

COROLLARY 1

102. If (Fig. 92) above the axis *IL* within the extension of the tube *AB*, at the location of which

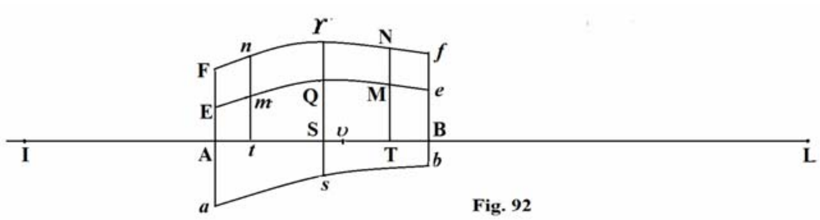

Fig. 92

S, with the interval put $IS = S$, the amplitude is $\Omega = \frac{ff}{SS}$, the two other curved lines are constructed *EQe* and $F\Upsilon f$ [concerning the density and speed respect.], the applied lines of which shall be Υ

$$SQ = \int \frac{dS}{SS} l\frac{Q}{B} + \frac{1}{S} l\frac{Q}{B} = \int \frac{dQ}{SQ} \text{ and } S\Upsilon = \frac{\Upsilon}{cS},$$

these functions are determined thus, so that there shall be

$$\Gamma' : S = -\tfrac{1}{2}SQ + \tfrac{1}{2}S\Upsilon \text{ and } \Delta' : S = -\tfrac{1}{2}SQ - \tfrac{1}{2}S\Upsilon$$

and hence again

$$\Gamma : S = -\tfrac{1}{2}AESQ + \tfrac{1}{2}AFS\Upsilon \text{ and } \Delta : S = -\tfrac{1}{2}AESQ - \tfrac{1}{2}AFS\Upsilon.$$

COROLLARY 2

103. Hence moreover in the elapsed time *t* and with the intervals $ST = St = ct$ taken from some point S from the intervals of the air situated around S, the density is found :

$$q = \mathrm{Q}\left(1 - l\tfrac{Q}{B} - \tfrac{1}{2}AETM - \tfrac{1}{2}AEtm + \tfrac{1}{2}AFTN - \tfrac{1}{2}AFtn + \tfrac{1}{2}S\left(TM + tm - TN + tn\right)\right),$$

the speed

$$\mathfrak{T} = \tfrac{1}{2}cS\left(-TM + TN + tm + tn\right)$$

and with the distance moved:

$$S\upsilon = SS\int \tfrac{dS}{SS}\,l\tfrac{Q}{B} - \tfrac{1}{2}S(AETM - AFTN + AEtm + AFtn),$$

from which since on putting $t = 0$ there may become $S\upsilon = 0$, it is evident to become

$$SS\int \tfrac{dS}{SS}\,l\tfrac{Q}{B} = S\cdot AESQ,$$

thus so that there shall be:

$$S\upsilon = \tfrac{1}{2}S(2AESQ - AETM - AEtm + AFTN - AFtn).$$

SCHOLIUM 1

104. The construction given in the corollaries, even if more terms may be taken than before, yet for the calculation offered in any case much more is required to be adapted to be presented, since in these the disturbances are presented separately, which arise either on account of a disturbance in the original density, or because of an impressed motion. Moreover this distinction is especially necessary, if we may wish to explore the continuation of each extended scale EQe and $F\Upsilon f$; which indeed is necessary generally for a perfect understanding of motion. But these expressions can be shown more conveniently, so that they may not depend on the end *A* of the tube ; indeed since from that construction itself it may be apparent:

$$\int \tfrac{dS}{SS}\,l\tfrac{Q}{B} = \tfrac{AESQ}{S},$$

and from the position

$$SQ = \int \tfrac{dS}{SS}\,l\tfrac{Q}{B} + \tfrac{1}{S}\,l\tfrac{Q}{B}$$

we may deduce

$$l\tfrac{Q}{B} = S\cdot SQ - AESQ,$$

from which values introduced we will obtain these determinations:

$$\frac{q}{Q} = 1 + \tfrac{1}{2}S(TM + tm - 2SQ) - \tfrac{1}{2}S\left(TN - tn\right) + \tfrac{1}{2}(SQtm - SQTM) + \tfrac{1}{2}TNtn$$
$$\mathfrak{T} = \tfrac{1}{2}cS(tm - TM + TN + tn)$$
$$S\upsilon = \tfrac{1}{2}S\left(SQtm - SQTM\right) + \tfrac{1}{2}S\cdot TNtn,$$

which formulas thus are more adapted to the calculation, since they depend on neither end of the tube. But this will be agreed to be observed especially, the distance of each point *S* from the start of the asymptote *I*, evidently $IS = S$, to enter into the computation.

SCHOLIUM 2

105. But before it may be allowed to progress further here, it is agreed to be considered I use more accurate scales, for the motion requiring to be defined. Clearly each must be constructed from the state, by which the air in the tube was induced initially, moreover this state I assume to be determined thus, so that the density of the air present at *S* were $= Q$, with the natural density being $= B$, indeed the speed along the direction $AB = \Upsilon$. With these in place the construction of the scale $F\Upsilon$ can be produced without any difficulty, since its applied line shall be $S\Upsilon = \frac{\Upsilon}{cS}$, truly the other scale *EQe*,

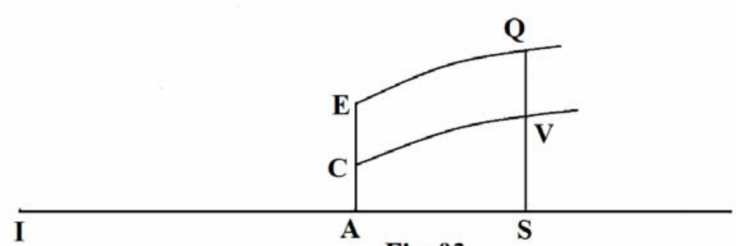

Fig. 93

which involves an integral formula, demands a certain explanation on account of the constant involved. Therefore in the first place (Fig. 93) since for the individual point *S* the density *Q* is given, thus the curve *CV* may be constructed, so that for the interval called $CSV = S$, its applied line shall be $SV = \frac{1}{SS} l \frac{Q}{B}$, clearly with some right line taken for unity. Then truly with this curve described with the scale *EQe*, for which there is a need, must be constructed thus, so that its applied line may be taken everywhere, $SQ = S \cdot SV + ACSV$, and thus the individual applied lines of this scale, by which the tube is extended, will be able to be assigned for the given curve *CV*. Then also since the area of the scale *AESQ* may enter into our formulas, it will be agreed to observe

$AESQ = S\cdot SQ - SS \cdot SV$ and thus $AESQ = S \cdot ACSV$.

SCHOLIUM 3

106. There will be a need for this curve (Fig. 94) *CV*, which can be called properly the scale of the density, accustomed to be used in place of this scale *EQe*. Therefore *CQc* shall be this same scale of the density; in which on putting the interval $IS = S$ the applied line shall become $SQ = \frac{1}{SS} l \frac{Q}{B}$; truly in the scale of the speeds $F\Upsilon f$, the applied line shall be $S\Upsilon = \frac{\Upsilon}{cS}$ as before. Therefore so that if we may wish to substitute that same scale *CQc* in place of the preceding one *EQe* , it will be required to write $S \cdot SQ + ACSQ$ in place of *SQ* in the formulas found , and $S \cdot ACSQ$ in place of the area *AESQ*. Now since there shall be $l \frac{Q}{B} = SS \cdot SQ,$ we will obtain from §.103, for whatever elapsed time interval taken, $ST = St = ct$

$$\text{density } q = Q \left\{ \begin{matrix} 1 - SS \cdot SQ - \frac{1}{2} S \cdot ACTM - \frac{1}{2} S \cdot ACtm + \frac{1}{2} TNtn + \frac{1}{2} SS(TM + tm) \\ -\frac{1}{2} S(TN - tn) + \frac{1}{2} S(ACTM + ACtm) \end{matrix} \right\},$$

which expression is contracted into this :

$$q = Q\left(1 + \tfrac{1}{2} SS(TM + tm - 2SQ) - \tfrac{1}{2} S(TN - tn) + \tfrac{1}{2} TNtn\right).$$

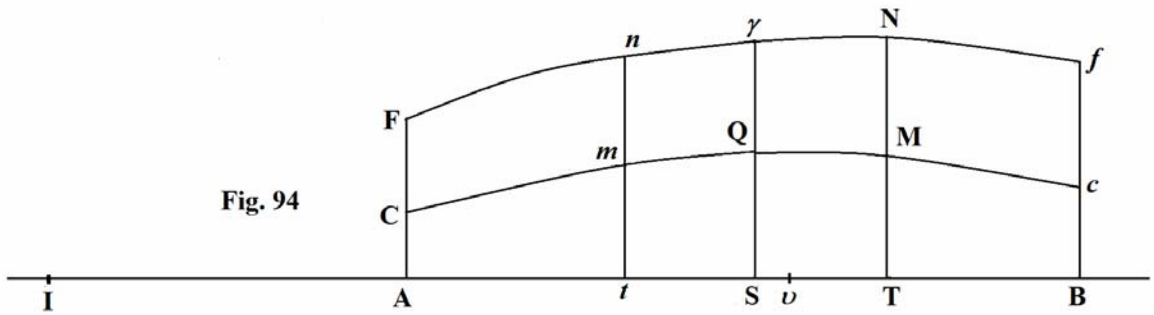

Fig. 94

Then the speed will be :

$$\mathfrak{T} = \tfrac{1}{2} cS\left(-S \cdot TM + S \cdot tm - ACTM + ACtm + TN + tn\right)$$

or

$$\mathfrak{T} = -\tfrac{1}{2} cSS\left(TM - tm\right) - \tfrac{1}{2} cS \cdot TMtm + \tfrac{1}{2} cS\left(TN + tn\right).$$

Truly finally the distance of the translation $S\upsilon$ is found :

$$S\upsilon = \tfrac{1}{2} S\left(2S \cdot ACSQ - S \cdot ACTM - S \cdot ACtm + TNtn\right)$$

or

$$S\upsilon = \tfrac{1}{2} SS\left(SQtm - SQTM\right) + \tfrac{1}{2} S \cdot TNtn.$$

Therefore in as much as I may use these formulas especially fitting for the calculation in the following sections.

PROBLEM 87

107. *If the tube were terminated at A, there either to be open or closed, and the scales of each density and speed have been put in place, just as in the preceding scholium* (106), *to investigate the continuation produced above the axis beyond A.*

SOLUTION

Let the distance (Fig. 95) $IA = a$, and initially we will consider the case, where the tube is open at A ; therefore there the density always will be equal to the natural density and thus $Q = B$, from which the scale of the density *AM* will pass through the point *A*, truly *FN* shall be the scale of the speed. Now since there must always be $q = Q = B$, in some elapsed time *t* and with the intervals $AT = At = ct$ taken on each side from the point *A* the formula, which then has been found to express the density at A, must provide $q = Q$. Whereby that formula, by taking the indefinite point *S* may be transferred to this point *A* itself : therefore there will become:

$$SQ = 0,\quad S\Upsilon = AF,\ \text{and}\ \ S = a;$$

from which the continuation if each scale is required to be prepared thus, so that there may become [see §.106]

$$SS\left(TM + tm\right) - S\left(TN - tn\right) + TNtn = 0$$

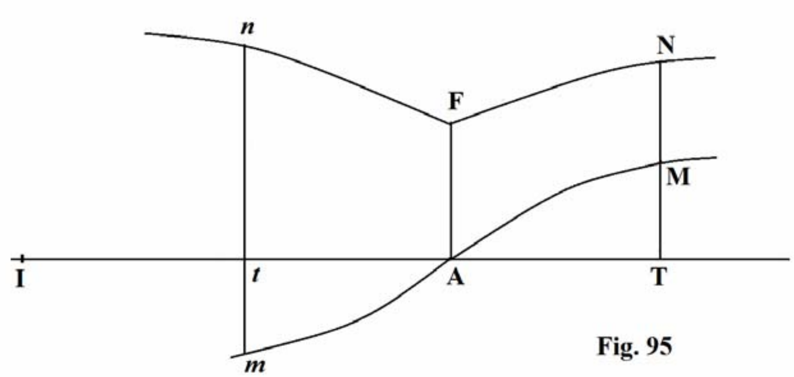

Fig. 95

and thus so that it may become, so that the continuation of neither may depend on the other. Therefore separately there must become $tm = -TM$, from which it is understood the scale of the density *AM* to be required to be continued thus, so that the continued part

Am shall be similar to the scale *AM* , but related to the opposite part of the axis. Truly for the continuation of the scale of the speed requiring to be found we may put in place

$$AT = At = x,\ TN = y \text{ and } tn = z,$$

and since there must be

$$TNtn - a(TN - tn) = 0,$$

there will become

$$\int ydx + \int zdx - ay + az = 0;$$

hence on being differentiated:

$$ydx + zdx - ady + adz = 0,$$

which multiplied by $e^{\frac{x}{a}}$ gives the integral

$$e^{\frac{x}{a}}az = \int e^{\frac{x}{a}}(ady - ydx) = e^{\frac{x}{a}}ay - 2\int e^{\frac{x}{a}}ydx$$

and thus $z = y - \frac{2}{a}e^{\frac{-x}{a}}\int e^{\frac{x}{a}}ydx$, thus with the integral taken thus, so that it may vanish on putting $x = 0$.

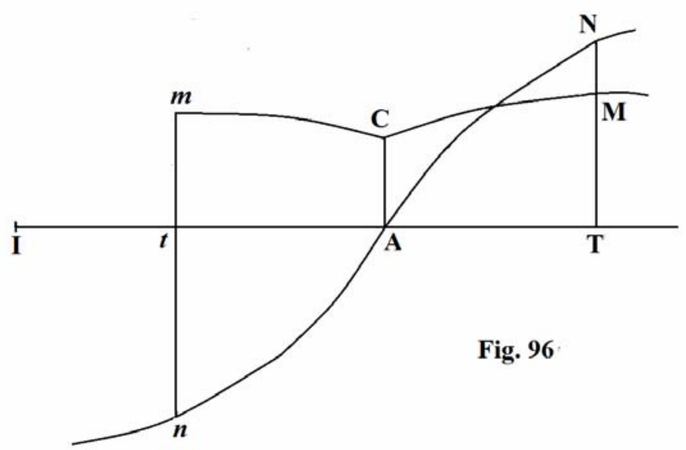

Fig. 96

Now the tube may be closed at *A* (Fig. 96) , and the speed there $\mathfrak{T}$ must be zero always, from which it is necessary that the scale of the speed *AN* may pass through the point. Therefore since in the elapsed time *t*, for which the intervals may be taken $AT = At = ct = x,$ the speed $\mathfrak{T}$ also may be allowed to become zero, whatever were the scale of the density *CM*, in the first place it is required to be $TN + tn = 0$, or the scales of the speed *AN* is similar to *An* itself continued placed on the opposite part of the axis, but for the continuation of the scale of the density there must become

$$-aa(TM - tm) - aTMtm = 0.$$

For this end we may put $AT = At = x,\ TM = y$ and $tm = z$

and this condition leads to that same equation:

$$a(y - z) + \int ydx + \int zdx = 0 \text{ or } ady - adz + ydx + zdx = 0,$$

of which the integral is :

$$e^{-\frac{x}{a}}az = \int e^{-\frac{x}{a}}(ady + ydx) = e^{-\frac{x}{a}}ay + 2\int e^{-\frac{x}{a}}ydx,$$

thus so that for this continuation we may have:

$$z = y + \tfrac{2}{a}e^{\frac{x}{a}}\int e^{-\frac{x}{a}}ydx,$$

COROLLARY

108. Therefore in the first case, where the tube is open at A , the continuation of the scale of the speed is prepared thus, so that there shall become

$$AFTN + AFtn = a(TN - tn)$$

and thus the area of the continued curve will be had from the found applied line tn

$$AFtn = IA\left(TN - tn\right) - AFTN.$$

COROLLARY 2

109. In a similar manner for the latter case, where the tube is closed at A, the continuation of the scale of the density CM is prepared thus, so that there shall become

$$ACTM + ACtm = a(tm - TM).$$

Whereby with its applied line tm found , the area of that thus itself will be had

$$ACtm = IA\left(tm - TM\right) - ACTM.$$

COROLLARY 3

110. If in the first case the air in the tube will have had no initial motion and the density alone were disturbed, then, since the scale of the speed falls on the axis, on account of $y = 0$ also the continuation of that falls on the axis. Truly with the latter case, if the density of the air initially were natural through the whole tube, then the scale of the density and likewise its continuation fall on the axis. But with these cases excepted, the continuations found from the principal scales disagree especially and the construction of these can be carried out only by quadratures, by calling in the aid of the exponential above.

PROBLEM 88

111. *If the tube were terminate at B, and there to be either open or closed, and the scales both of the density and of the speed follow the precepts of the form* (106) *to define the continuation upon the axis produced beyond B.*

SOLUTION

With the distance put (Fig. 97) $IB = b$, we may consider in the first case, where the tube is open at *B*, and thus both the density at the point *B* as well as initially always remains the same value *B*, from which the scale of the density *mB* will pass through the point *B* itself, while *nf* shall be the scale of the speed. The equal distances $Bt = BT = x$ may be taken on each side from *B* , with there being $x = ct$, [i.e. It is assumed throughout that changes in the speed and density of the air flow are transmitted at the speed of sound *c*] and as after the time *t* will be going to become the density at *B*, from paragraph 106

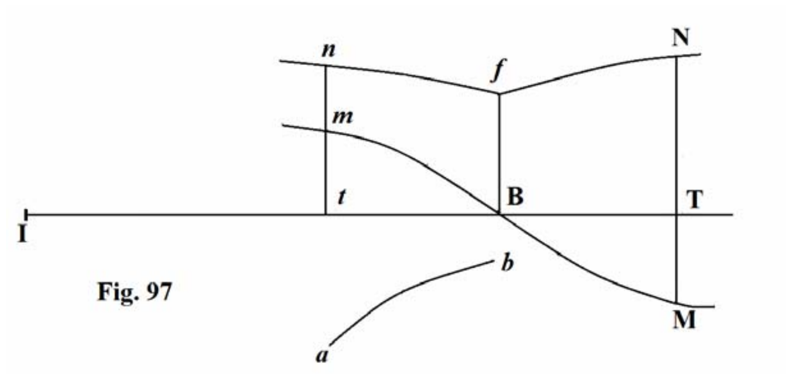

Fig. 97

$$q = Q(1+\tfrac{1}{2}bb(TM+tm)-\tfrac{1}{2}b(TN-tn)+\tfrac{1}{2}TNtn),$$

it is necessary, that in the first place there may become $TM + tm = 0$ and thus the scale of the density *Bm* shall be contained in a similar form *BM* requiring to describe the other part of the axis. Truly for the scale of the speed *nf* with the continuation *f N* requiring to be found, $tn = y$ and $TN = z$, and there must become

$$\int ydx + \int zdx = b(z-y),$$

from which it is deduced :

$$e^{-\frac{x}{b}}bz = \int e^{-\frac{x}{b}}\left(bdy + ydx\right) = e^{-\frac{x}{b}}by + 2\int e^{-\frac{x}{b}}ydx = -e^{-\frac{x}{b}}by + 2\int e^{-\frac{x}{b}}bdx$$

and thus

$$z = y + \tfrac{2}{b}e^{\frac{x}{b}}\int e^{-\frac{x}{b}}ydx = -y + 2e^{\frac{x}{b}}\int e^{-\frac{x}{b}}ydx.$$

Now the tube shall be closed at *B* (Fig. 98) and the other scale of the speed *nB* thus will pass through *B* , so that the speed there always shall become $\mathfrak{T} = 0$ and thus

$$0 = -bb\left(TM - tm\right) - b\cdot TMtm + b(TN + tn).$$

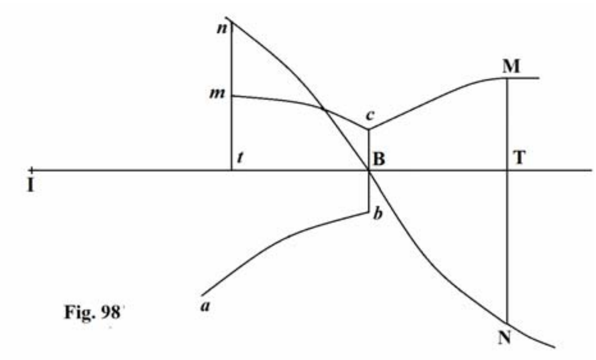

Fig. 98

Therefore initially there will be required to be $TN + tn = 0$, and the continuation of the scale *nB* will be *BN* equal to itself and placed contrarily ; indeed for the continued scale of the other density *cm* requiring to be found with the distances called $BT = Bt = x$, with $tm = y$ known and $TM = z$ unknown, there must become

$$\int ydx + \int zdx + b(z - y) = 0,$$

from which there is found

$$z = y - \tfrac{2}{b} e^{\frac{x}{b}} \int e^{-\frac{x}{b}} ydx = -y + 2e^{\frac{-x}{b}} \int e^{\frac{x}{b}} ydx.$$

COROLLARY 1

112. In the case (Fig. 97) , where the tube is open at B , after the individual applied lines *TN* were defined from the formulas found, there is no need for the area of this continuation itself to be sought, since the area shall be from the prescribed condition:

$$BfTN = IB\left(TN - tn\right) - Bftm.$$

COROLLARY 2

113. In a similar manner for the case (Fig. 98), where the tube is closed at *B*, the area of the continuation of the scale of the density *cM* is defined by this reckoning

$$BcTM = IB\left(tm - TM\right) - Bctm,$$

from which it suffices to have assigned its applied line TM .

COROLLARY 3

114. If the distance $IB = b$ were very great in comparison with the intervals $BT = Bt = x$, so that there shall become $e^{-\frac{x}{b}} = 1$, or if the fraction $\frac{a}{b}$ may be able to be considered as constant, there will become :

$$2e^{\frac{\pm x}{b}} \int e^{\frac{\mp x}{b}} ydx = 2y;$$

from which for each case there becomes $z = y$, and hence the continuation thence itself will be had, as if the tube were cylindrical.

PROBLEM 89

115. *If the aperture of the* (Fig. 99) *hyperbolic tube were at the end A and the air contained in that were disturbed in some manner from equilibrium through the small distance GH, to determine the propagation of this pulse.*

SOLUTION

Each scale of the density *GMH* and of the speed *GNH* may be described upon the interval GH, thus so that, if at *T*, with the distance put to be $IT = S$, initially the density were $= Q$ and the speed in the region were $AH = \Upsilon$, the applied lines shall be

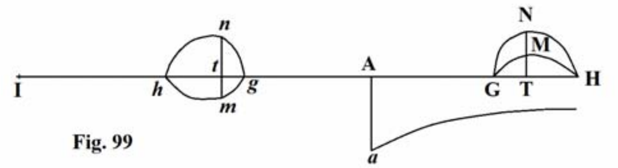

Fig. 99

$TM = \frac{1}{SS} l \frac{Q}{B}$ and $TN = \frac{\Upsilon}{cS}$.

With these scales put in place the continuation of these beyond *A* towards *I* thus must be put in place by the precepts established before: Clearly initially, since both the scales lie on the same axis *AG* itself from *A* to *G*, and each scale will fall equally on the axis *Ag* through just as great a distance $Ag = AG$; then truly, since the tube is open at *A*, the scale of the density will be continued to *gmh*, with the curve similar to *GMH* itself placed on the other side of the axis. Truly for the continuation of the scale of the speed with the axis taken $At = AT = x$ and with the applied line called $TM = y$, for the interval present $IA = a$, the quantity $e^{\frac{x}{a}}$, since *x* may be varied only through the minimum distance *GH*, so that it will be able to be regarded as constant, thus so that from paragraph 107 there shall be $z = y$ or $tn = TN$, and with the whole curve *gnh* similar to the scale *GNH* and placed on the same part of the axis. In this manner the whole continuation of each scale may be may be put in place, if the tube may be extended indefinitely to the other part ; but if the tube may be terminated somewhere at *B*, just as it may become open or closed there, thus new repetitions of each scale thence will arise, just as in cylindrical tubes. But here it is proposed to examine only the disturbance, which arises in that opening *Aa* itself, since that is shared with the free air itself and in that may be propagated further; but following which at this point repetitions may arise, here I will not pursue these as pure resonances, if indeed the account of these is clear enough from above. Therefore the air at *A* will be at rest, while a time $= \frac{AG}{c}$ will have passed ; and then likewise each pulse will arrive there. We may take each interval $AT = At = ct$, and since the distance of the point *A* from the centre of the hyperbola *I* is $IA = a$, in the formulas given in §106 there will become $S = a$ and $SQ = 0$, then from which it is deduced for the air at this place:

$$\text{the density } q = B\left(1 + \tfrac{1}{2}aa\left(TM - tm\right) - \tfrac{1}{2}a\left(TN - tn\right) - \text{area } GTN\right),$$

and since this area is required to be vanishing, there becomes $q = B$, as the nature of the matter demands for the open air. Ten truly for that same place the speed will be :

$$\mathfrak{T} = -\tfrac{1}{2}aac\left(TM + tm\right) + \tfrac{1}{2}ac(TN + tn)$$

or

$$\mathfrak{T} = -aac \cdot TM + ac \cdot TN = -\tfrac{aa}{SS} \cdot c \cdot l\,\tfrac{Q}{B} + \tfrac{a}{S}\Upsilon,$$

truly the distance of the movement $= -aa \cdot GTM + a \cdot GTN$, which indeed will be a minimum.

COROLLARY 1

116. From that same solution it is clear enough in tubes of this kind pulses in the same plane to be propagated with the same speed, as in cylindrical tubes ; from which it is permitted to conclude the figure of the tube plainly to impart nothing to the speed of the propagated pulses; even if we may not prevail to show that from theory for all the shapes of tubes .

COROLLARY 2

117. Then also it is observed everything, which we have shown above concerned with the propagation of pulses in equally wide tubes and with the repetition of these, also has a place here, since pulses to be excited in the smallest interval may show the same continuations of each scale ; therefore it would be superfluous to repeat here, what has been said before concerning resonances of sound of the same kind.

SCHOLIUM 1

118. Therefore I have brought up this problem, so that the stentorian effect produced to be explained may be attended to, since the shapes of these tubes (Fig. 100) may differ little from the conoidal hyperbola, which we discuss here. Therefore *kab* shall be an equilateral hyperbola described within the asymptotes *IB* and *IK*, the part of which *ab* rotated about the axis *AB* may produce the loud-sounding tube, in the opening of which *Bb* the tone may be reproduced in some manner. With this tone produced near *Bb* a density *Q* may be impressed on the air greater than the natural density and likewise the speed along $BA = \Upsilon$, therefore the direction of which is contrary to that which we have introduced into the solution of the problem. Then truly the distances may be put $IA = a$ and $IB = b$, which was *S* in the problem, and thence it is clear the speed of the air to be impressed in the wider opening *Aa* towards *I*, which shall become

$$= \tfrac{a}{b} \cdot \Upsilon + \tfrac{aac}{bb}\,l\,\tfrac{Q}{B};$$

which since the distance $IA = a$ shall be much smaller with the distance $IB = b$, also will result in a speed much smaller than Υ generated in the initial pulse, unless on account of the natural density, Q may barely be increased, therefore so that c may denote a distance around 1000 feet [per second]. Nor yet will it be allowed to attribute the effect to the speed of these pulses translated to Aa , but the particular cause is required to be sought in the amplitude of the opening Aa, through which that pulse has been endowed with an equal speed, also has been extended, and if so many tones may be produced there at the same time, just as many times as the opening may surpass the aperture of someone's mouth shouting out. For if the voice may be produced in the free air distant from the tube, wherever that may be turned it will be strongly diminished by spreading out over the distance BA; but now, while it may be held together in the tube and fill the whole distance Aa at a much greater speed, than would happen, if the tube were absent, it is no wonder that its effect will be so strong. Concerning the rest it is easily understood the figure of the tube itself hardly to confer much to this effect, while it may not be adverse to being used.

[Euler's opinion on this last point seems to be rather doubtful, as the expanded end of a trumpet or horn contributes greatly to the amount of sound emitted: the body of the instrument vibrates in unison with the sound generated within the tube, and especially the free end, to produce a louder sound.]

SCHOLIUM 2

119. Hence also an account is seen to be sought of the sounds, which trumpets, bugles, horns, and other similar instruments produce, which may disagree with flutes regarding this matter, because with these, as we have seen, all the air contained in the tube inflated by new air likewise is set in motion and hence the oscillatory motion follows, so that sound is produced. Truly in these instruments a certain sound, either by blowing from the mouth or, they are aroused into a vibratory motion by some elastic fiber at that end. But for each only a pulse of this kind is generated at that end, such as we have considered here, which thereafter are propagated along the whole length of the tube, and while they are prevented from spreading out to the sides, thus they acquire a greater strength, and thus the depth of the sound produced will depend chiefly on the pulses produced in succession by the opening, as on account of the cause also the sounds of these instruments are especially different from the sounds of flutes. Yet meanwhile also in these instruments the length and shape of the tube confer the most to the depth of the sound, thus so that here the cause of each sound may be seen to concur at the same time, since with the aid of these instruments not all, but only certain sounds are able to be produced on account of the depth of the notes, which generally maintain the account of the numbers 1, 2, 3, 4, 5 etc. progressing in the natural order. Which phenomenon we will have observed also with flutes, hence rather it is seen to be required to conclude, the cause of the sounds produced from trumpets, bugles, horns to be mixed, and partially first from the sound blown in, partially to be sought from the disturbance of all the air contained in these tubes. Truly since we have scarcely touched on the theory of motion of the air at this stage, truly most of the theory at this stage is lacking, we are unable to hope for a complete explanation of these sounds. But rather from these, which the first principles of this new science may grant to us, we must agree with, then at last to await an understanding, when that science will be able to be improved more.

PROBLEM 90

120. *If* (Fig. 101) *in a hyperbolic tube AaBb with the aperture open on both sides the air thus will be disturbed from the state of equilibrium, so that only its density may be varied without any motion, to define the motion of the oscillation, which henceforth will be generated in the tube.*

SOLUTION

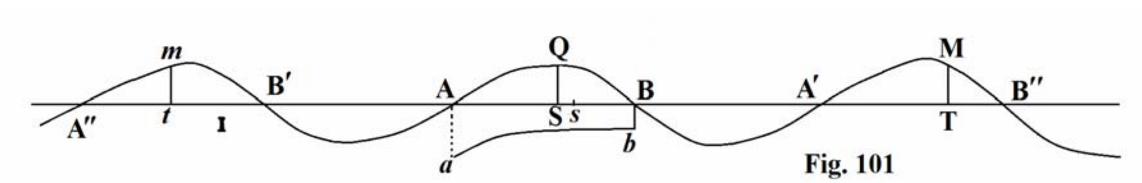

Fig. 101

I shall be the centre of the hyperbola, from which the tube is formed, and with the position of some point *S* assumed in the tube with the distance $IS = S$, the initial density of the air induced there shall be $= Q$, with the natural density $= B$, the applied line may be taken $SQ = \frac{1}{SS} l \frac{Q}{B}$ which will be the point *Q* in the scale of the density *AQB*, which, since the tube is open at each end, will be required to pass through both the terms *A* and *B*. But the scale of the speed lies on the axis by this hypothesis. Now by the precepts treated before this scale of the density may be continued easily on both sides, with the same being required to be described alternately for the opposite parts of the axis, so that the points A, B, A', B', A'', B'' etc. with the intervals $= AB$ planted at intervals themselves shall be the centres of the alternate equal arcs on both sides. Hence for some time from the start of the elapsed time *t* the state of the air, which initially was present at *S* , will be defined in the following manner: from some point *S* the distances $ST = St = ct$ may be parted on the axis on both sides, with *c* denoting the distance, through which the sound is propagated in one second, from the applied lines of the scale of the densities at these points *T* and *t* by §.106 there will become

I. Density $q = Q\left(1 + \frac{1}{2} SS\left(TM + tm - 2SQ\right)\right)$

II. Speed $\mathfrak{T} = -\frac{1}{2} cSS\left(TM - tm\right) - \frac{1}{2} cS \cdot TMtm$

III. Distance $Ss = \frac{1}{2} SS\left(SQtm - SQTM\right).$

From which it is apparent on taking $ST = St = 2AB$ or after the time $t = \frac{2AB}{c}$ to become $TM = tm = SQ$, and hence the density $q = Q$, the speed $\mathfrak{T} = 0$ and equally $Ss = 0,$ on account of $SQtm = 0$ and $SQTM = 0.$

COROLLARY 1

121. Therefore it is agreed the air in the tube to be performing two oscillations in the time $t = \frac{2AB}{c}$, from which since the times of the individual oscillations shall be $= \frac{AB}{c}$, $\frac{c}{AB}$ of the individual oscillations will be produced per second, which number likewise expresses the depth of the sound.

COROLLARY 2

122. Therefore the hyperbola clearly produces the same sound as the cylindrical tube of the same length, if indeed each shall be open at both sides and initially no motion were impressed on the air in the hyperbolic tube.

SCHOLIUM

123. Indeed we have found in the time $t = \frac{2AB}{c}$, when the air is restored perfectly to a pristine state, two oscillations to be performed, in the manner of vibrating strings, thus so that, if now the air may be moving in a maximum displacement, in this elapsed time it will have reverted to this state again. Hence indeed it does not follow in half the elapsed time $t = \frac{AB}{c}$ the air to pertain to another contrary excursion: then indeed only this may arise, if the curve *AQB* may correspond to two similar parts or the diameter may be had passing normally through the midpoint of the right line *AB*. So that unless that may happen, the individual oscillations thus themselves will follow, so that the alternate intervals shall be larger and smaller, from which it is necessary for a less pure sound to arise, than if all the intervals were equal. And this perhaps is the reason, why cylindrical flutes may produce purer sounds, than either convergent or diverging ones ; but here it agrees, that we here we have assume the first disturbance of the air with no adjoining motion, certainly with that acceding to a continued scale of the speeds for hyperbolic tubes it follows another law by far, than if the tube everywhere were uniformly great. But in whatever way the scale of the speed were prepared, from the formulas given in §.106 it is clear the disturbances to be determined thus follow from each scale, so that they may be composed from the effect of each arising separately themselves. Therefore since in this problem we have designated the effect arising from the scale of the density alone, now we may subject the scale of the speeds to be examined separately, thence so that each effect from the two disturbances taken together from whatever two scales, arising at the same time, may be able to be defined.

PROBLEM 91

124. *If in the first place the disturbance of the equilibrium may be agreed to be only in the motion, with the natural density remaining everywhere, or if the scale of the speed may be given, from that to define the following disturbances of the air in a hyperbolic tube AaBb* (Fig. 102) *with both ends open.*

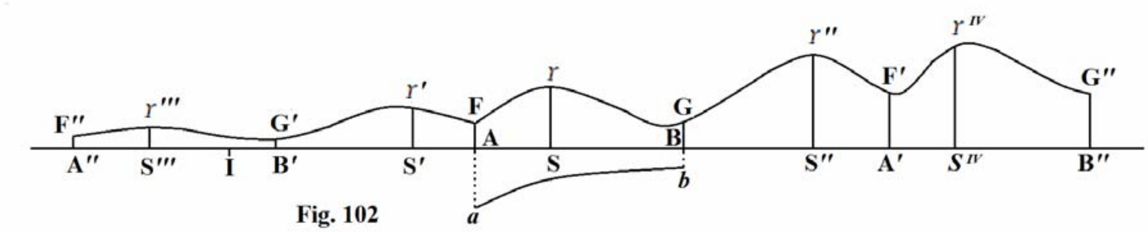

Fig. 102

SOLUTION

Therefore $F\Upsilon G$ shall be the given scale of the speeds, the applied lines of which are defined thus from the speeds impressed initially, so that, if at *S*, the distance of which from the centre of the hyperbola shall be $IS = S$, the speed in the region *AB* will have been Υ, the length $S\Upsilon$ may become $= \frac{\Upsilon}{cS}$. Therefore this curve will be required to be continued on both sides following the precepts treated in problems in §.87 and §.88. To this end the distances may be put in place $IA = a$ and $IB = b$, and for the continuance of each beyond *A* requiring to be found there may be taken $AS' = AS = x$, so that there shall $x = S - a$, and the applied line may be put $S\Upsilon = y$, and the applied line at *S′* requiring to be put in place

$$S'\Upsilon' = z = y - \tfrac{2}{a} e^{\frac{-x}{a}} \int e^{\frac{x}{a}} y dx,$$

with this integral taken thus, so that it may vanish on putting $x = 0$. Therefore on account of $x = S - a$ there will be

$$S'\Upsilon' = S\Upsilon = y - \tfrac{2}{a} e^{\frac{-S}{a}} \int e^{\frac{S}{a}} dS \cdot S\Upsilon$$

and the area

$$AFS'\Upsilon' = a\left(S\Upsilon - S'\Upsilon'\right) - AFS\Upsilon.$$

Now we may progress beyond *B*, and on assuming $BS'' = BS = x = b - S$, with $S\Upsilon = y$ remaining, the applied line at *S″* arising from problem 88 is found:

$$z = y + \tfrac{2}{b} e^{\frac{x}{b}} \int e^{\frac{-x}{b}} y dx$$

or

$$S''\Upsilon'' = S\Upsilon = y - \tfrac{2}{b} e^{\frac{-S}{b}} \int e^{\frac{S}{b}} dS \cdot S\Upsilon$$

and the area

$$BGS''\Upsilon'' = b\left(S''\Upsilon'' - S\Upsilon\right) - BGS\Upsilon.$$

Now we may return to the aperture *A* and with the distance taken

$$AS''' = AS'' = x = 2b - a - S\,,$$

there shall become $S''\Upsilon'' = y$, and with the applied line requiring to be raised at S''' there will become:

$$z = y - \tfrac{2}{a} e^{\frac{-x}{a}} \int e^{\frac{x}{a}} y dx$$

or

$$S'''\Upsilon''' = S''\Upsilon'' + \tfrac{2}{a} e^{\frac{S}{a}} \int e^{\frac{-S}{a}} dS \cdot S''\Upsilon'',$$

which with the reduction made changed into this form:

$$S'''\Upsilon''' = S''\Upsilon'' + \tfrac{2(b-a)}{a(a+b)} e^{\frac{S}{a}} \int e^{\frac{-S}{a}} dS \cdot S\Upsilon + \tfrac{2(b-a)}{a(a+b)} e^{\frac{-S}{b}} \int e^{\frac{S}{b}} dS \cdot S\Upsilon.$$

We may return to the aperture *Bb* and we may take

$$BS^{IV} = BS' = x = b - 2a + S,$$

and by putting $S'\Upsilon' = y$ the applied line at S^{IV} will be required to be put in place :

$$z = y + \tfrac{2}{b} e^{\frac{x}{b}} \int e^{\frac{-x}{b}} y dx$$

or

$$S^{IV}\Upsilon^{IV} = S'\Upsilon' + \tfrac{2}{b} e^{\frac{S}{b}} \int e^{\frac{-S}{b}} dS \cdot S'\Upsilon',$$

which is reduced to this form by substituting the value found above in place of $S'\Upsilon'$:

$$S^{IV}\Upsilon^{IV} = S\Upsilon - \tfrac{2(b-a)}{a(a+b)} e^{\frac{-S}{a}} \int e^{\frac{S}{a}} dS \cdot S\Upsilon - \tfrac{2(b-a)}{b(a+b)} e^{\frac{S}{b}} \int e^{\frac{-S}{b}} dS \cdot S\Upsilon$$

and thus it will be permitted to progress further, as far as it pleases.

Truly here we see chiefly, in what state the air shall be going to become, which was initially at *S*, after the time $t = \frac{2AB}{c}$, in which case the abscissas on both sides lie in the abscissas at S^{IV} and S''', thus so that in the general solution of §.106 there may become

$$TN = S^{IV}\Upsilon^{IV} \text{ and } tn = S'''\Upsilon''',$$

therefore then there may be deduced:

$$\text{density} \quad q = Q\left(1-\tfrac{1}{2}S\left(S^{IV}\Upsilon^{IV} - S'''\Upsilon'''\right)+\tfrac{1}{2}S'''\Upsilon'''\, S^{IV}\Upsilon^{IV}\right)$$

$$\text{speed} \quad \mathfrak{T} = \tfrac{1}{2}cS\left(S^{IV}\Upsilon^{IV} + S'''\Upsilon'''\right)$$

$$\text{translation} \quad Ss = \tfrac{1}{2}S\cdot S'''\Upsilon'''\, S^{IV}\Upsilon^{IV}.$$

Moreover this arcs $S'''\Upsilon'''\, S^{IV}\Upsilon^{IV}$ thus can be expressed by more applied lines; since from the nature of the continuation there shall become:

$$\text{I.} \quad S\Upsilon S'\Upsilon' = a\left(S\Upsilon - S'\Upsilon'\right)$$

$$\text{II.} \quad S''\Upsilon''\, S'''\Upsilon''' = a\left(S''\Upsilon'' - S'''\Upsilon'''\right)$$

$$\text{III.} \quad S\Upsilon S''\Upsilon'' = b\left(S''\Upsilon'' - S\Upsilon\right)$$

$$\text{IV.} \quad S'\Upsilon'\, S^{IV}\Upsilon^{IV} = b\left(S^{IV}\Upsilon^{IV} - S'\Upsilon'\right),$$

on being combined there will become:

$$\text{IV} - \text{I.} \quad S\Upsilon\, S^{IV}\Upsilon^{IV} = b\left(S^{IV}\Upsilon^{IV} - S'\Upsilon'\right) - a\left(S\Upsilon - S'\Upsilon'\right)$$

$$\text{II} - \text{III.} \quad S\Upsilon\, S'''\Upsilon''' = a\left(S''\Upsilon'' - S'''\Upsilon'''\right) - b\left(S''\Upsilon'' - S\Upsilon\right),$$

from which we deduce the area :

$$S'''\Upsilon''' S^{IV}\Upsilon^{IV} = (b-a)\left(S\Upsilon - S'\Upsilon' - S''\Upsilon''\right) + bS^{IV}\Upsilon^{IV} - aS'''\Upsilon'''.$$

In which if we may substitute the values found for these applied lines, this same area is found by putting $S\Upsilon = y$:

$$\tfrac{2(b-a)}{a(a+b)}\left(e^{\frac{-S}{a}}\int e^{\frac{S}{a}}ydS - e^{\frac{S}{a}}\int e^{\frac{-S}{a}}ydS + e^{\frac{-S}{b}}\int e^{\frac{S}{b}}ydS - e^{\frac{S}{b}}\int e^{\frac{-S}{b}}ydS\right),$$

where the integrals thus must be taken, so that these, which involve *a*, vanish on putting $S = a$, truly the others, on putting $S = b$. Therefore this expression for the area found introduced in $\tfrac{1}{2}S$ provides the distance of the translation *Ss*. Thereupon we will have for the rest of the elements:

$$S^{IV}\Upsilon^{IV} - S'''\Upsilon''' =$$

$$= \tfrac{-2(b-a)}{a(a+b)}\left(\tfrac{1}{a}e^{\frac{-S}{a}}\int e^{\frac{S}{a}}ydS + \tfrac{1}{a}e^{\frac{S}{a}}\int e^{\frac{-S}{a}}ydS + \tfrac{1}{b}e^{\frac{-S}{b}}\int e^{\frac{S}{b}}ydS + \tfrac{1}{b}e^{\frac{S}{b}}\int e^{\frac{-S}{b}}ydS\right)$$

and

$$S^{IV}\Upsilon^{IV}+S'''\Upsilon'''=$$
$$=2y-\frac{2(b-a)}{a(a+b)}\left(\tfrac{1}{a}e^{\frac{-S}{a}}\int e^{\frac{S}{a}}ydS-\tfrac{1}{a}e^{\frac{S}{a}}\int e^{\frac{-S}{a}}ydS+\tfrac{1}{b}e^{\frac{-S}{b}}\int e^{\frac{S}{b}}ydS-\tfrac{1}{b}e^{\frac{S}{b}}\int e^{\frac{-S}{b}}ydS\right).$$

COROLLARY 1

125. Hence therefore it is apparent on account of the scale of the speed after the time $=\frac{2AB}{c}$ the state of the air in the tube can differ a great deal from the initial state, so that the difference there may emerge smaller, where the fraction $\frac{2(b-a)}{a(a+b)}$ were smaller and where the principal scale will approach closer to the axis $F\Upsilon G$.

COROLLARY 2

126. Therefore so that if likewise a motion were impressed on the air in a hyperbolic tube, after the time $\frac{2AB}{c}$ the air in the tube will not be considered to have completed two oscillations; much less will be able to regain the time for some other oscillations, but rather the sound perceived thence will be very coarse and lacking in harmony.

SCHOLIUM

127. Therefore flutes formed according to a hyperbolic shape will suffer from this significant defect, that by no means will they produce sounds pure and suitable for harmony, therefore so that the motion will not be allowed to be resolved into distinct oscillations. And this defect thus will be greater, where a flute of this kind may be blown into stronger, since then much less oscillations will be able to be distinguished, but a bearable sound will be produced by a most gentle inflation. But this flaw in hyperbolic flutes is easily understood not to be a special property, but for that to be extended to all other forms, where they may differ more from equal width. Hence therefore the reasoning is understood, why all flutes of the kind, which are accustomed to be put in place in wind organs, may have either a cylindrical or prismatic shape, so that the width everywhere shall be the same, and it is observed only these shapes to be adapted for music; so that it is allowed to be done in two ways, while these openings on top are either open or closed. But if we may wish to close hyperbolic flutes, much more coarse sound will be produced, since then neither scale considered separately will prevail to produce a regular oscillatory motion ; from which the case will not at all be worth the effort to establish, where the hyperbolic tube may be open at one end, and truly closed in the other.

CAPUT V

DE MINIMIS AERIS AGITATIONIBUS IN TUBIS FIGURAM CONOIDICAM HYPERBOLICAM HABENTIBUS

PROBLEMA 86

101. *Si* (Fig. 91) *tubi figura oriatur ex conversione arcus hyperbolae aequilaterae AB circa asymtotam IL facta et aër in eo contentus initio quomodocunque de statu aequilibrii fuerit deturbatus, agitationes eius sequentes ad quodvis tempus investigare.*

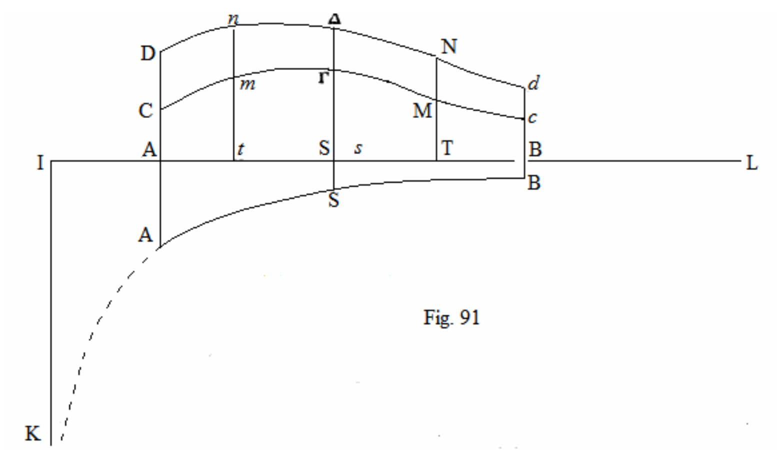

Fig. 91

SOLUTIO

Sit *I* intersectio asymtotarum, sumtoque spatio $IS = S$ amplitudo tubi ibi ponatur $\Omega = \frac{C}{SS}$, agitatio autem aëri primum inducta ita se habeat, ut in tubi loco *S* fuerit densitas $= Q$ et celeritas secundum $IB = \Upsilon$. Elapso iam tempore quocunque *t*, ex solutione supra (§. 82 et §. 84) exhibita, ubi ob $\alpha = 1$, $\beta = 0$ negligamus gravitatis effectum, aër ex *S* per spatiolum $Ss = \upsilon$ erit translatus, ut sit

$$\upsilon = SS\int \tfrac{dS}{SS} l \tfrac{Q}{B} + S\Gamma:(S+ct) + S\Delta:(S-ct)$$

posito $\sqrt{\frac{2ga}{b}} = c$, tum vero erit mutata constante densitas

$$q = Q\left(1 - l\tfrac{Q}{B} + \Gamma:(S+ct) + \Delta:(S-ct) - S\Gamma':(S+ct) - S\Delta':(S-ct)\right)$$

et celeritas

$$\mathfrak{T} = cS\left(\Gamma':(S+ct) - \Delta':(S-ct)\right).$$

Quaestio autem nunc huc redit, ut ad statum initialem natura functionum Γ et Δ accommodetur: posito ergo $t=0$, fieri debet $\upsilon=0$, $q=Q$ et $\mathfrak{T}=\Upsilon$, unde colligimus has aequationes:

$$\text{I. } SS\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + S\Gamma : S + S\Delta : S = 0$$
$$\text{II. } -l\,\tfrac{Q}{B} + S\Gamma : S + S\Delta : S - S\Gamma' : S - S\Delta' : S = 0$$
$$\text{III. } \Upsilon = cS\left(\Gamma' : S - \Delta' : S\right),$$

quarum quidem duae priores inter se conveniunt, ut tam ex rei natura quam differentiatione prioris intelligitur. Ponamus brevitatis gratia $\Gamma : S + \Delta : S = x$ et $\Gamma : S - \Delta : S = y$ ut prodeat

$$\text{II. aequatio } -l\,\tfrac{Q}{B} + x - \tfrac{Sdx}{dS} = 0$$

et

$$\text{III. aequatio } \Upsilon = \tfrac{cSdy}{dS},$$

atque

$$\text{I. } SS\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + S : x = 0,$$

quae quidem illam in se complectitur. Hinc ergo est $x = -S\int \frac{dS}{SS}\, l\,\frac{Q}{B}$, inde vero $y = \frac{1}{c}\int \frac{\Upsilon S}{S}$, ita aut obtineamus:

$$\Gamma : S = -\tfrac{1}{2}S\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + \tfrac{1}{2c}\int \tfrac{\Upsilon dS}{S}$$

et

$$\Delta : S = -\tfrac{1}{2}S\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} - \tfrac{1}{2c}\int \tfrac{\Upsilon dS}{S}.$$

unde fit

$$\Gamma' : S = -\tfrac{1}{2}\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} - \tfrac{1}{2S}\, l\,\tfrac{Q}{B} + \tfrac{\Upsilon}{2cS}$$

et

$$\Delta' : S = -\tfrac{1}{2}\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} - \tfrac{1}{2S}\, l\,\tfrac{Q}{B} - \tfrac{\Upsilon}{2cS}.$$

Cum igitur pro singulis axis punctis *S* ex statu initiali dentur quantitates $l\,\frac{Q}{B} = \frac{Q-B}{B}$ proxime et Υ, hinc construantur duae curvae $C\Gamma c$ et $D\Delta d$, ut sint earum applicatae , quae curvae autem non ultra tubi longitudinem *AB* extendantur, eritque tum per quadraturas harum curvarum

$$\Gamma : S = ACS\Gamma \text{ et } \Delta : S = ADS\Delta.$$

His curvis constructis elapso tempore $=t$, in axe a puncto S utrinque capiantur intervalla $ST=St=ct$ eritque tum

$$\text{translatio}\quad Ss=SS\int\tfrac{dS}{SS}l\tfrac{Q}{B}+S\left(ACTM+ADtn\right)$$

$$\text{densitas}\quad q=Q\left(1-l\tfrac{Q}{B}+ACTM+ADtn-S\cdot TM-S\cdot tn\right)$$

$$\text{celeritas}\quad \mathfrak{T}=cS\left(TM-tn\right),$$

quae determinationes valent, quamdiu puncta T et t non extra tubum AB cadunt.

COROLLARIUM 1

102. Si (Fig. 92) super axe IL intra tubi extensionem AB, in cuius loco

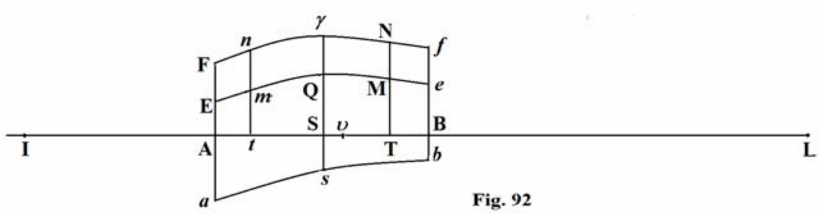

Fig. 92

S, posito intervallo $IS=S$, est amplitudo $\varOmega=\frac{ff}{SS}$, aliae duae construantur lineae curvae EQe et $F\Upsilon f$, quarum applicatae sint

$$SQ=\int\tfrac{dS}{SS}l\tfrac{Q}{B}+\tfrac{1}{S}l\tfrac{Q}{B}=\int\tfrac{dQ}{SQ}\quad\text{et}\quad S\Upsilon=\tfrac{\Upsilon}{cS},$$

illae functiones hinc ita determinantur, ut sit

$$\varGamma':S=-\tfrac{1}{2}SQ+\tfrac{1}{2}S\Upsilon\quad\text{et}\quad\varDelta':S=-\tfrac{1}{2}SQ-\tfrac{1}{2}S\Upsilon$$

hincque porro

$$\varGamma:S=-\tfrac{1}{2}AESQ+\tfrac{1}{2}AFS\Upsilon\quad\text{et}\quad\varDelta:S=-\tfrac{1}{2}AESQ-\tfrac{1}{2}AFS\Upsilon.$$

COROLLARIUM 2

103. Hinc autem elapso tempore t sumtisque a puncto S intervallis $ST=St=ct$ reperitur aëris, qui initio ad S versabatur:
densitas

$$q=\mathrm{Q}\left(1-l\tfrac{Q}{B}-\tfrac{1}{2}AETM-\tfrac{1}{2}AEtm+\tfrac{1}{2}AFTN-\tfrac{1}{2}AFtn+\tfrac{1}{2}S\left(TM+tm-TN+tn\right)\right),$$

celeritas

$$\mathfrak{T} = \tfrac{1}{2}cS\left(-TM + TN + tm + tn\right)$$

et translatio

$$S\upsilon = SS\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} - \tfrac{1}{2}S(AETM - AFTN + AEtm + AFtn),$$

unde cum posito $t = 0$ fiat $S\upsilon = 0$, patet esse

$$SS\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} = S\cdot AESQ,$$

ita ut sit

$$S\upsilon = \tfrac{1}{2}S(2AESQ - AETM - AEtm + AFTN - AFtn).$$

SCHOLION 1

104. Constructio in corollariis data, etsi plures terminos comprehendit quam prior, tamen ad calculum quovis casu oblato evolvendum multo magis est accommodata, quoniam in ea agitationes, quae vel ob turbatam initio densitatem vel ob motum impressum oriuntur, seorsim exhibentur. Haec autem distinctio maxime est necessaria, si continuationem utriusque scalae extructae *EQe* et $F\Upsilon f$ explorare velimus; quod quidem ad perfectam motus cognitionem omnino est necessarium. Commodius autem hae expressiones exhiberi possunt, ut non pendeant a tubi termino *A;* cum enim ex ipsa constructione pateat esse:

$$\int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} = \tfrac{AESQ}{S},$$

ex positione

$$SQ = \int \tfrac{dS}{SS}\, l\,\tfrac{Q}{B} + \tfrac{1}{S}\, l\,\tfrac{Q}{B}$$

colligimus

$$l\,\tfrac{Q}{B} = S\cdot SQ - AESQ,$$

quibus valoribus indroductis obtinebimus has determinationes:

$$\tfrac{q}{Q} = 1 + \tfrac{1}{2}S(TM + tm - 2SQ) - \tfrac{1}{2}S\left(TN - tn\right) + \tfrac{1}{2}(SQtm - SQTM) + \tfrac{1}{2}TNtn$$

$$\mathfrak{T} = \tfrac{1}{2}cS(tm - TM + TN + tn)$$

$$S\upsilon = \tfrac{1}{2}S\left(SQtm - SQTM\right) + \tfrac{1}{2}S\cdot TNtn,$$

quae formulae ideo ad calculum magis sunt accommodatae, quod a neutra tubi termina pendent. Id autem hic imprimis notari convenit distantiam cuiusque puncti *S* ab asymtotae initia *I* scilicet $IS = S$ in computum venire.

SCHOLION 2

105. Antequam autem hic ulterius progredi liceat, accuratius scalas, quibus ad motum definiendum utor, perpendi convenit. Utraque scilicet ex statu, qui aëri in tubo contenta initia fuerit inductus, construi debet, hunc autem statum ita determinari assumo, ut aëris ad *S* versantis densitas fuerit $=Q$, naturali existente $=B$, celeritas vero secundum directionem $AB=\Upsilon$. His positis constructio scalae $F\Upsilon$ nulla laborat difficultate, cum eius sit applicata $S\Upsilon=\frac{\Upsilon}{cS}$, altera vero scala *EQe*,

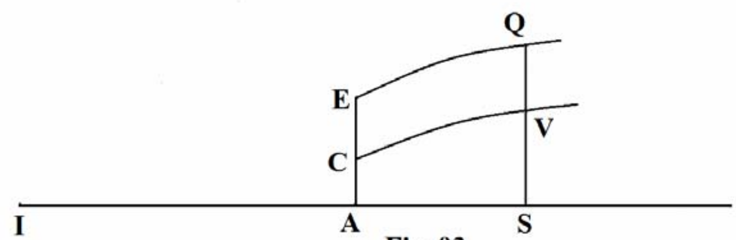

Fig. 93

quae formulam integralem involvit, quandam dilucidationem ob constantem implexam postulat. Primo igitur (Fig. 93) cum in singulis punctis *S* densitas datur *Q*, ita construatur curva *CV*, ut vocato intervallo $CSV=S$ sit eius applicata $SV=\frac{1}{SS}l\frac{Q}{B}$, sumta scilicet recta quacunque pro unitate. Tum vero hac curva descripta scala *EQe*, qua opus est, ita construi debet, ut capiatur ubique eius applicata $SQ=S\cdot SV+ACSV$, sicque huius scalae singulae applicatae, qua tubus extenditur, ex curva data *CV* assignari poterunt. Cum deinde etiam area scalae *AESQ* in nostras formulas ingrediatur, notari convenit esse

$$AESQ=S\cdot SQ-SS\cdot SV \quad \text{ideoque} \quad AESQ=S\cdot ACSV.$$

SCHOLION 3

106. Operae pretium erit (Fig. 94) hanc curvam *CV*, quae proprie scala densitatum vocari potest, loco illius scalae *EQe* ad usum adhibere. Sit igitur *CQc* ista scala densitatum; in qua posito intervallo $IS=S$ sit applicata $SQ=\frac{1}{SS}l\frac{Q}{B}$, in scala celeritatum vero $F\Upsilon f$ sit ut ante applicata $S\Upsilon=\frac{\Upsilon}{cS}$. Quodsi igitur istam scalam *CQc* loco praecedentis *EQe* substituere velimus, in formulis inventis loco *SQ* scribi oportet
$S\cdot SQ+ACSQ$ et $S\cdot ACSQ$ loco illius areae *AESQ*. Cum nunc sit $l\frac{Q}{B}=SS\cdot SQ,$ ex paragrapho 103 obtinebimus pro tempore quocunque elapso *t* sumtis intervallis
$ST=St=ct$

$$\text{densitatem } q=Q\begin{Bmatrix}1-SS\cdot SQ\ -\frac{1}{2}S\cdot ACTM-\frac{1}{2}S\cdot ACtm+\frac{1}{2}TNtn+\frac{1}{2}SS\left(TM+tm\right)\\ -\frac{1}{2}S\left(TN-tn\right)+\frac{1}{2}S(ACTM+ACtm)\end{Bmatrix},$$

quae expressio contrahitur in hanc:

$$q = Q\left(1+\tfrac{1}{2}SS\left(TM+tm-2SQ\right)-\tfrac{1}{2}S\left(TN-tn\right)+\tfrac{1}{2}TNtn\right).$$

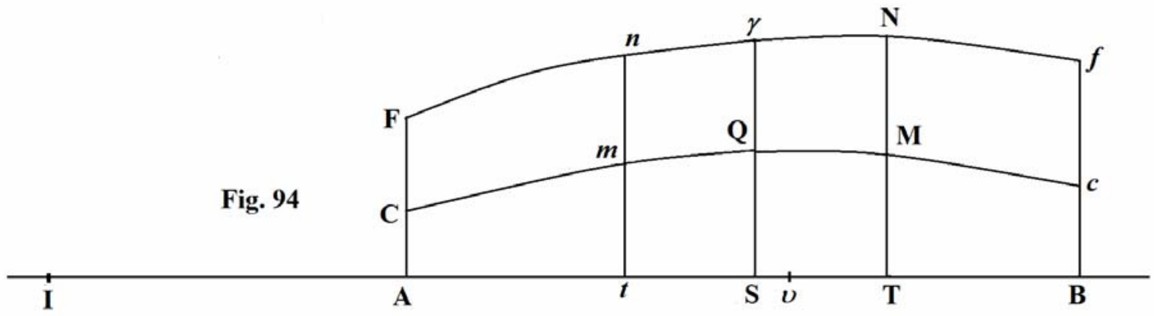

Fig. 94

Deinde erit celeritas

$$\mathfrak{T} = \tfrac{1}{2}cS\left(-S\cdot TM + S\cdot tm - ACTM + ACtm + TN + tn\right)$$

seu

$$\mathfrak{T} = -\tfrac{1}{2}cSS\left(TM - tm\right) - \tfrac{1}{2}cS\cdot TMtm + \tfrac{1}{2}cS\left(TN + tn\right).$$

Denique vero spatium translationis *Sv* reperitur:

$$S\upsilon = \tfrac{1}{2}S\left(2S\cdot ACSQ - S\cdot ACTM - S\cdot ACtm + TNtn\right)$$

seu

$$S\upsilon = \tfrac{1}{2}SS\left(SQtm - SQTM\right) + \tfrac{1}{2}S\cdot TNtn.$$

His ergo formulis utpote ad calculum maxime accommodatis in sequentibus utar.

PROBLEMA 87

107. *Si tubus in A fuerit terminatus ibique sive apertus sive clausus utriusque scalae densitatum et celeritatum, prouti in scholio praecedente* (106) *sunt constitutae, continuationem super axe ultra A producto investigare.*

SOLUTIO

Sit (Fig. 95) distantia $IA = a$, ac primo consideremus casum, quo tubus in *A* est apertus; ibi ergo densitas semper naturali erit aequalis ideoque $Q = B$, unde scala densitatum *AM* per punctum *A* transibit, scala vero celeritatum sit *FN*. Cum iam perpetuo esse debeat $q = Q = B$, elapso tempore quocunque *t* sumtisque

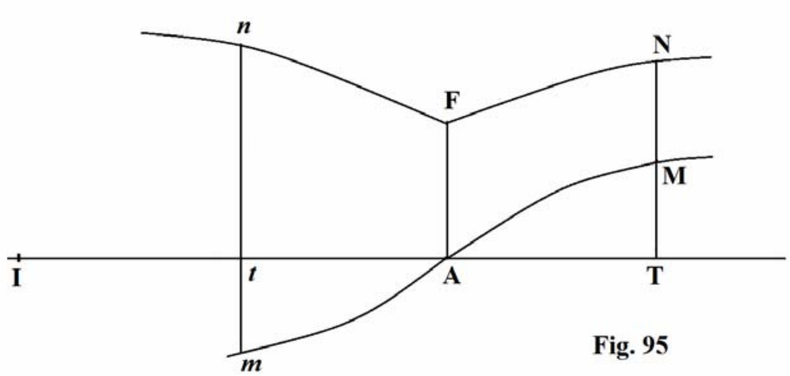

Fig. 95

a puncto *A* utrinque intervallis *AT* = *At* = *ct* formula, quae tum densitatem in *A* exprimere est inventa, praebere debet $q = Q$.
Quare illa formula sumendo punctum indefinitum *S* in ipso puncto *A* huc transferatur: fiet ergo

$$SQ = 0,\quad S\Upsilon = AF,\quad \text{et}\quad S = a;$$

unde continuationem utriusque scalae ita comparatam esse oportet, ut fiat

$$SS\left(TM + tm\right) - S\left(TN - tn\right) + TNtn = 0$$

idque ita, ut neutrius continuatio ab altera pendeat. Seorsim ergo esse debet $tm = -TM$, ex quo intelligitur scalam densitatum *AM* ita continuari oportere, ut pars continuata *Am* ipsi scalae *AM* similis sit, sed ad axis partem contrariam relata. Pro continuatione vero scalae celeritatum invenienda statuamus

$$AT = At = x,\ TN = y\ \text{ et }\ tn = z\ ,$$

et cum esse debeat

$$TNtn - a\left(TN - tn\right) = 0,$$

fiet

$$\int ydx + \int zdx - ay + az = 0;$$

hinc differentiando

$$ydx + zdx - ady + adz = 0,$$

quae per $e^{\frac{x}{a}}$ multiplicata dat integrale

$$e^{\frac{x}{a}}az = \int e^{\frac{x}{a}}\left(ady - ydx\right) = e^{\frac{x}{a}}ay - 2\int e^{\frac{x}{a}}ydx$$

ideoque $z = y - \frac{2}{a}e^{-\frac{x}{a}}\int e^{\frac{x}{a}}ydx,$
integrali hoc ita capto, ut evanescat posito $x = 0$.

Sit iam (Fig. 96) tubus in *A* clausus, ibique celeritas $\mathfrak{T}$

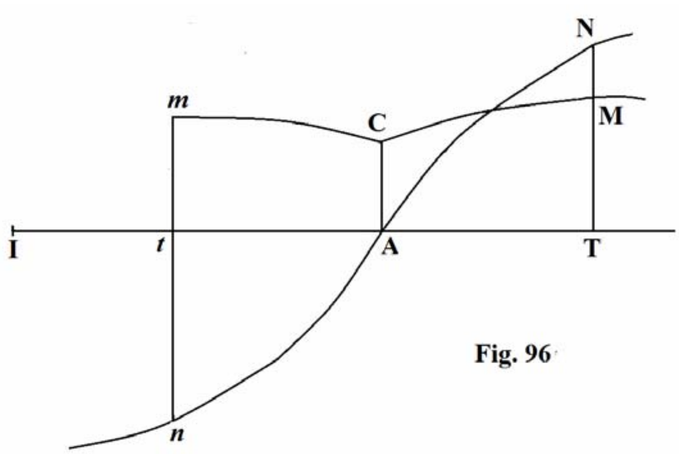

Fig. 96

semper esse debet nulla, unde scala celeritatum *AN* per punctum *A* transeat necesse est. Cum igitur elapso tempore *t*, pro quo sumantur intervalla $AT = At = ct = x$, celeritas $\mathfrak{T}$ quoque nulla prodire debeat, quaecunque fuerit scala densitatum *CM*, primo esse oportet $TN + tn = 0$, seu scalae celeritatum *AN* continuatio *An* ipsi est similis in contrariam axis partem disposita, pro continuatione autem scalae densitatum esse debet

$$-aa(TM - tm) - aTMtm = 0.$$

Ponamus in hunc finem

$$AT = At = x,\ TM = y \text{ et } tm = z$$

haecque conditio ad istam deducit aequationem:

$$a(y-z)+\int ydx+\int zdx=0 \text{ seu } ady-adz+ydx+zdx=0,$$

cuius integrale est

$$e^{-\frac{x}{a}}az = \int e^{-\frac{x}{a}}(ady+ydx)=e^{-\frac{x}{a}}ay+2\int e^{-\frac{x}{a}}ydx,$$

ita ut pro hac continuatione habeamus:

$$z = y+\tfrac{2}{a}e^{\frac{x}{a}}\int e^{-\frac{x}{a}}ydx,$$

COROLLARIUM

108. Casu ergo priori, quo tubus in *A* est apertus, continuatio scalae celeritatum ita est comparata, ut sit

$$AFTN + AFtn = a(TN-tn)$$

ideoque inventa applicata *tn* habebitur area curvae continuatae

$$AFtn = IA(TN-tn) - AFTN.$$

COROLLARIUM 2

109. Simili modo pro casu posteriori, quo tubus in *A* est clausus, continuatio scalae densitatum *CM* ita est comparata, ut sit

$$ACTM + ACtm = a(tm-TM)..$$

Quare inventa eius applicata *tm*, eius area ita se habebit

$$ACtm = IA(tm-TM) - ACTM.$$

COROLLARIUM 3

110. Si priori casu aër in tubo nullum habuerit motum initialem solaque densitas fuerit perturbata, tum, quia scala celeritatum in axem incidit, ob $y = 0$ etiam eius continuatio in axem incidet. Posteriori vero casu, si aëris densitas initio per totum tubum fuerit naturalis, tum scala densitatum simulque eius continuatio in axem incidit. His autem casibus exceptis continuationes inventae maxime discrepant a scalis principalibus earumque constructio nonnisi per quadraturas, quantitatem exponentialem insuper in subsidium vocando, perfici potest.

PROBLEMA 88

111. *Si tubus in B fuerit terminatus ibique vel apertus vel clausus, utriusque scalae densitatum et celeritatum secundum praecepta* (106) *formatae continuationem super axe ultra B producto definire.*

SOLUTIO

Posita (Fig. 97) distantia $IB = b$, examinemus primo casum, quo tubus in *B* est apertus, ideoque densitas in *B* tam initio quam semper eadem *B*, unde scala densitatum *mB* per ipsum punctum *B* transibit, *nf* vero sit scala celeritatum. Capiantur utrinque a *B* spatia aequalia $Bt = BT = x$, existente $x = ct$, atque cum post tempus *t* futura sit densitas in *B*

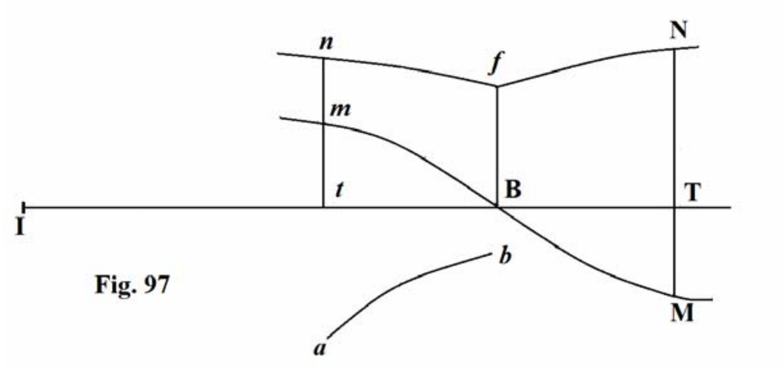

Fig. 97

ex paragrapho 106

$$q = Q(1+\tfrac{1}{2}bb(TM+tm)-\tfrac{1}{2}b(TN-tn)+\tfrac{1}{2}TNtn),$$

necesse est, ut primo fiat $TM + tm = 0$ ideoque scala densitatum *Bm* in formam similem *BM* ad alteram axis partem describendam continuetur. Pro scalae vero celeritatum *nf* continuatione *f N* invenienda vocentur $tn = y$ et $TN = z$, esseque debebit

$$\int ydx + \int zdx = b(z-y),$$

unde colligitur:

$$e^{-\frac{x}{b}}bz = \int e^{-\frac{x}{b}}\left(bdy + ydx\right) = e^{-\frac{x}{b}}by + 2\int e^{-\frac{x}{b}}ydx = -e^{-\frac{x}{b}}by + 2\int e^{-\frac{x}{b}}bdx$$

ideoque

$$z = y + \tfrac{2}{b}e^{\frac{x}{b}}\int e^{-\frac{x}{b}}ydx = -y + 2e^{\frac{x}{b}}\int e^{-\frac{x}{b}}ydx.$$

Sit nunc (Fig. 98) tubus in *B* clausus et altera scala celeritatum *nB* ita per *B* transibit, ut semper ibi fiat celeritas $\mathfrak{T} = 0$ ideoque

$$0 = -bb\left(TM - tm\right) - b \cdot TMtm + b(TN + tn).$$

Primo ergo esse oportet $TN + tn = 0$, et scalae *nB* continuatio erit *BN* ipsi aequalis et contrarie sita; pro alterius vero scalae densitatum *cm* continuatione inveniendaunde reperitur vocatis spatiis $BT = Bt = x$, applicata cognita $tm = y$ et incognita $TM = z$, fieri debet

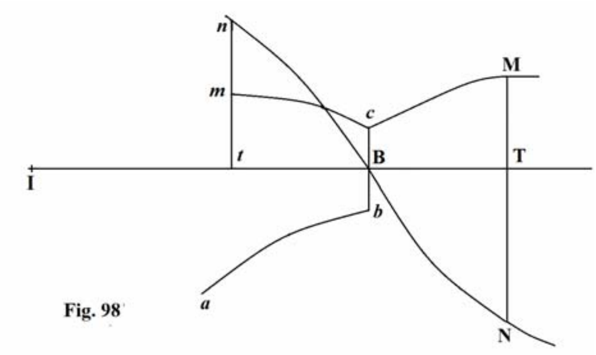

Fig. 98

$$\int ydx + \int zdx + b(z - y) = 0,$$

unde reperitur

$$z = y - \tfrac{2}{b} e^{\frac{x}{b}} \int e^{-\frac{x}{b}} ydx = -y + 2e^{\frac{-x}{b}} \int e^{\frac{x}{b}} ydx.$$

COROLLARIUM 1

112. Casu (Fig. 97) , quo tubus in *B* est apertus, postquam ex formula inventa singulae applicatae *TN* fuerint definitae, non opus est aream huius continuationis seorsim quaeri, cum ex conditione praescripta sit area

$$BfTN = IB\left(TN - tn\right) - Bftm.$$

COROLLARIUM 2

113. Simili modo pro casu (Fig. 98), quo tubus in *B* est clausus, continuationis scalae densitatum *cM* area hac ratione definitur

$$BcTM = IB\left(tm - TM\right) - Bctm,$$

ex quo sufficit eius applicatas *TM* assignavisse.

COROLLARIUM 3

114. Si distantia $IB = b$ prae intervallis $BT = Bt = x$ fuerit valde magna, ut sit $e^{-\frac{x}{b}} = 1$, vel si fractio $\frac{a}{b}$ ut constans spectari possit, erit

$$2e^{\frac{\pm x}{b}} \int e^{\frac{\mp x}{b}} ydx = 2y;$$

ex quo pro utroque casu fit $z = y$, hincque continuatio perinde se habebit, ac si tubus esset cylindricus.

PROBLEMA 89

115. *Si* (Fig. 99) *tubus hyperbolicus in termina A fuerit apertus et aër in eo contentus per spatium minimum GH de aequilibrio utcunque turbetur, huius pulsus propagationem determinare.*

SOLUTIO

Super intervallo *GH* describatur utraque scala densitatum *GMH* et celeritatum *GNH*, ita ut, si in *T*, posita distantia $IT = S$, initio fuerit densitas $= Q$ et celeritas in plagam $AH = \Upsilon$, sint applicatae

$TM = \frac{1}{SS} l \frac{Q}{B}$ et $TN = \frac{\Upsilon}{cS}$.

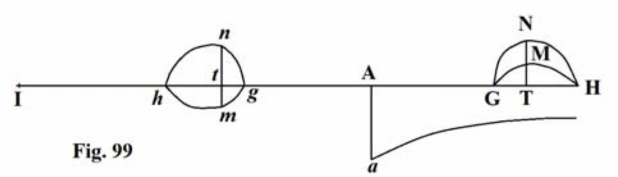

Fig. 99

His scalis constitutis per praecepta ante tradita continuatio earum ultra *A* versus *1* ita institui debet: Primo scilicet, quia ambae scalae ab *A* ad *G* in ipsum axem *AG* incidunt, per tantumdem spatium $Ag = AG$ utraque scala pariter in axem *Ag* cadet; tum vero, quia tubus ad *A* est apertus, scala densitatum continuabitur in *gmh*, curva simili ipsi *GMH* ad contrariam axis partem sita. Pro continuatione vero scalae celeritatum sumta abscissa $At = AT = x$ et vocata applicata $TM = y$, existente intervallo $IA = a$, quantitas $e^{\frac{x}{a}}$, quia *x* tantum per spatiolum minimum *GH* variatur, ut constans spectari poterit, ita ut sit ex paragrapho 107 $z = y$ seu $tn = TN$, totaque curva *gnh* similis scalae *GNH* et ad eandem axis partem disposita. Hoc modo tota utriusque scalae continuatio haberetur, si tubus ad alteram partem in infinitum esset extensus; sin autem alicubi in *B* terminaretur, prout ibi foret apertus vel clausus, novae repetitiones utriusque scalae inde orirentur, prorsus ut in tubis cylindricis. Hic autem tantum agitationem, quae in ipso orificio *Aa* gignetur, perscrutari est propositum, quoniam ea cum aëre libero communicatur in eoque ulterius propagatur; quae autem deinceps adhuc nascerentur repetitiones, eas hic utpote meras resonantias non persequar, siquidem earum ratio ex superioribus satis est manifesta. In *A* ergo aër erit tranquillus, donec effluxerit tempus $= \frac{AG}{c}$; ac tum simul uterque pulsus eo appellet. Sumamus utrinque intervallum $AT = At = ct$, et quia puncti *A* distantia a centro hyperbolae *1* est $IA = a$, in formulis paragrapho 106 datis erit $S = a$ et $SQ = 0$, unde tum colligitur pro aëre in hoc loco:

$$\text{densitas }\ q = B\left(1+\tfrac{1}{2}aa\left(TM-tm\right)-\tfrac{1}{2}a\left(TN-tn\right)-\text{area }GTN\right),$$

et quia haec area pro evanescente est habenda, fit $q=B$, uti rei natura pro aëre aperto postulat. Deinde vero ibidem erit

$$\text{celeritas }\ \mathfrak{T} = -\tfrac{1}{2}aac\left(TM+tm\right)+\tfrac{1}{2}ac(TN+tn)$$

seu

$$\mathfrak{T} = -aac\cdot TM + ac\cdot TN = -\tfrac{aa}{SS}\cdot c\cdot l\tfrac{Q}{B}+\tfrac{a}{S}\Upsilon,$$

spatium vero translationis $=-aa\cdot GTM + a\cdot GTN$, quod quidem erit minimum.

COROLLARIUM 1

116. Ex ipsa solutione satis liquet in huiusmodi tubis pulsus eadem plane celeritate propagari, atque in tubis cylindricis; ex quo concludere licet figuram tubi nihil plane conferre ad celeritatem propagationis pulsuum; etiamsi id ex Theoria pro omnibus tubi figuris ostendere non valeamus.

COROLLARIUM 2

117. Deinde etiam perspicitur omnia, quae supra de pulsuum propagatione in tubis aequaliter amplis eorumque repetitione demonstravimus, etiam hic locum habere, cum pulsus in intervallo minima excitati easdem exhibeant utriusque scalae continuationes; superfluum ergo foret, quae ante de eiusdem soni resonantia sunt dicta, hic repetere.

SCHOLION 1

118. Problema hoc ideo attuli, ut explicationi effectus a tubis stentoreis editi inserviret, quandoquidem harum tubarum figura (Fig. 100) parum discrepat a conoidica hyperbolica, quam hic tractamus. Sit ergo *kab* hyperbola aequilatera intra asymtotas *IB* et *IK* descripta, cuius portio *ab* circa axem *AB* gyrata generet tubam stentoream, in cuius orificio *Bb* vox quaecunque edatur. Hac voce aëri ad *Bb* proxima densitas imprimatur *Q* maior naturali simulque celeritas secundum $BA=\Upsilon$, cuius ergo directio illi, quam in solutionem problematis introduximus, est contraria. Tum vero ponatur distantia $IA=a$ et $IB=b$, quae in problemate erat *S*, atque inde patet aëri in orificio ampliori *Aa* imprimi celeritatem versus *I*, quae sit

$$=\tfrac{a}{b}\cdot\Upsilon+\tfrac{aac}{bb}l\tfrac{Q}{B};$$

quae cum distantia $IA=a$ multo sit minor distantia $IB=b$, etiam multo foret minor celeritate Υ in primo pulsu genita, nisi ob densitatem naturali *B* maiorem *Q* haud mediocriter augeretur, propterea quod *c* denotat distantiam quasi 1000 pedum. Neque tamen huic pulsus in *Aa* translati celeritati effectum tribuere licet, sed praecipua causa

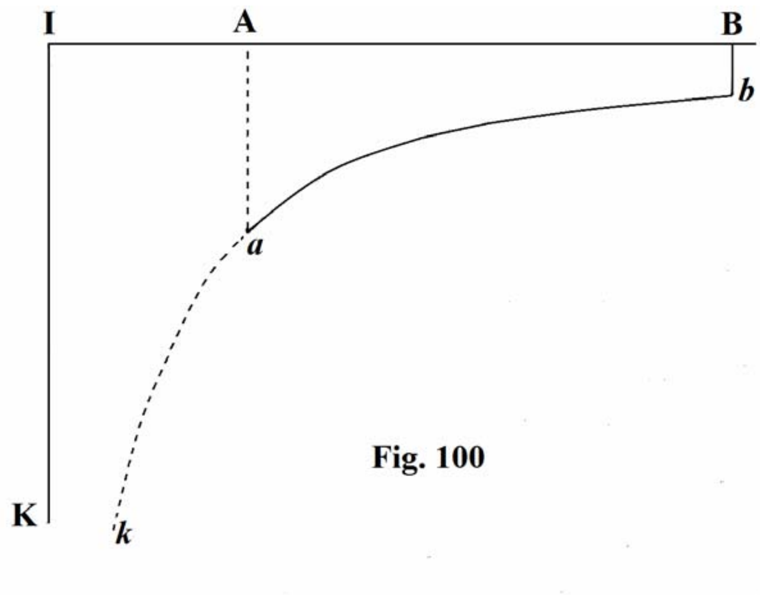

Fig. 100

& quaerenda est in amplitudine orificii *Aa*, per quod is pulsus pari velocitate praeditus est extensus idemque, praestat, ac si tot voces ibi simul ederentur, quoties haec apertura superat aperturam oris clamantis. Si enim remota tuba vox in liberum aërem ederetur, ea quaqua versus diffusa in distantia *BA* vehementer iminueretur; nunc autem, dum in tuba cohibetur totumque spatium *Aa* multo maiore celeritate implet, quam fieret, si tubus abesset, mirum non est, quod eius effectus tam sit fortis. De cetero facile intelligitur figuram ipsam tubi haud multum ad hunc effectum conferre, dum ab usitata non admodum abhorreat.

SCHOLION 2

119. Hinc etiam ratio petenda videtur eorum sonorum, quos tubae, buccina, cornua aliaque instrumenta similia edunt, quae in hoc a tibiis discrepant, quod in his, uti vidimus, totus aër in tubo contentus per novi aëris inflationem simul concitatur hincque motum oscillatorium consequitur, quo sonus producitur. In illis vero instrumentis iam sonus quidam per orificium inflatur, vel ex ore inflantis vel, dum in eo termino elastrum quodpiam ad motum vibratorium excitantur. Utroque autem modo in illo termino tantum eiusmodi pulsus excitatur, quales hic sumus contemplati, qui deinceps per totam tubi longitudinem propagantur, et dum a diffusione ad latera coërcentur, eo maiorem vim acquirunt, sicque gravitas soni editi potissimum a pulsibus successive in orificio productis pendet, quam ob causam etiam soni horum instrumentorum a sonis tibiarum maxime differunt. Interim tamen etiam in istis instrumentis longitudo et figura tubi plurimum ad soni gravitatem conferunt, ita ut hic utraque soni causa simul concurrere videatur, quandoquidem horum instrumentorum ope non omnes, sed tantum certi soni ratione gravitatis edi possunt, qui plerumque rationem numerorum 1, 2, 3, 4, 5 etc. naturali ordine progredientium inter se tenent. Quod phaenomenon cum etiam in tibiis observaverimus, hinc potius concludendum videtur sonorum a tubis, buccinis, cornibus etc. editorum causam esse mixtam, et partim in sonitu primum inflato, partim in agitatione totius aëris in his tubis contenti quaeri debere. Verum quia Theoriam motus aëris vix adhuc libavimus, plurimum adhuc abest, quominus perfectam horum sonorum explicationem sperare queamus. Quin potius in iis, quae prima huius novae scientiae principia nobis largiuntur, acquiescere debemus, uberiorem cognitionem tum demum expectaturi, quando eam scientiam magis excolere licuerit.

PROBLEMA 90

120. *Si* (Fig. 101) *in huiusmodi tubo hyperbolico AaBb utrinque aperto aër ita de statu aequilibrii deturbetur, ut eius sola densitas sine ullo motu varietur, motum oscillatorium, qui deinceps in tubo generabitur, definire.*

SOLUTIO

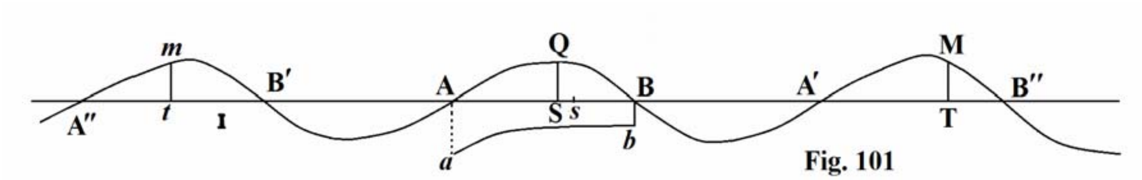

Fig. 101

Sit *I* centrum hyperbolae, ex qua tubus est formatus, et posita puncti in tubo utcunque assumti *S* distantia $IS = S$, inducta sit ibi aëri initio densitas $= Q$, naturali existente $= B$, capiaturque

$$\text{applicata } SQ = \tfrac{1}{SS} l \tfrac{Q}{B},$$

erit punctum *Q* in scala densitatum *A Q B*, quam, quia tubus utrinque est apertus, per ambos terminos *A* et *B* transire oportet. Scala autem celeritatum hac hypothesi in axem incidit. Iam per praecepta ante tradita facile haec scala densitatum utrinque continuatur, eandem alternatim ad contrarias axis partes describendo, ut puncta *A*, *B*, *A'*, *B'*, *A''*, *B''* etc. intervallis $= AB$ inter se dissita sint centra arcuum utrinque alternatim aequalium. Hinc ad quodvis tempus ab initio elapsum *t* status aëris, qui initio circa *S* versabatur, sequenti modo definietur: a puncto *S* utrinque in axe abscindantur spatia $ST = St = ct$, denotante *c* spatium, per quod sonus uno minuto secundo propagatur, ex applicatis scalae densitatum in his punctis *T* et *t* per paragraphum 106 erit

I. Densitas $q = Q\left(1+\tfrac{1}{2}SS\left(TM + tm - 2SQ\right)\right)$

II. Celeritas $\mathfrak{T} = -\tfrac{1}{2}cSS\left(TM - tm\right) - \tfrac{1}{2}cS\cdot TMtm$

III. Spatiolum $Ss = \tfrac{1}{2}SS\left(SQtm - SQTM\right).$

Unde patet sumto $ST = St = 2AB$ seu post tempus $t = \tfrac{2AB}{c}$ fieri $TM = tm = SQ$, hincque densitatem $q = Q$, celeritatem $\mathfrak{T} = 0$ et pariter $Ss = 0$, ob $SQtm = 0$ et $SQTM = 0$.

COROLLARIUM 1

121. Aër igitur in tubo termpore $t = \frac{2AB}{c}$ duas oscilliationes peregisse est censendus, unde cum tempora singularum oscillationum sint $= \frac{AB}{c}$, singulis minutis secundis edentur $\frac{c}{AB}$ oscillationes, qui numerus simul soni gravitatem exprimit.

COROLLARIUM 2

122. Tubus igitur hyperbolicus eundem plane edet sonum ac tubus cylindricus eiusdem longitudinis, siquidem uterque utrinque sit apertus et in hyperbolico aëri primum nullus motus fuerit impressus.

SCHOLION

123. Invenimus quidem tempore $t = \frac{2AB}{c}$, quo aër perfecte in pristinum statum restituitur, duas oscillationes peragi, ad similitudinem cordarum vibrantium, ita ut, si nunc aër in excursione maxima versetur, elapso hoc tempore iterum in eandem revertatur. Verum hinc non sequitur elapso tempore dimidio $t = \frac{AB}{c}$ aërem ad alteram excursionem contrariam pertingere: tum enim tantum hoc eveniret, si curva *AQB* duabus constaret partibus similibus seu diametrum haberet per medium punctum rectae *AB* normaliter transeuntem. Quod nisi eveniat, singulae oscillationes ita se invicem excipient, ut alternatim intervalla sint maiora et minora, unde sonum minus purum oriri necesse est, quam si omnia intervalla essent aequalia. Atque haec fortasse praecipua est causa, quod tibiae cylindricae puriores sonos edant, quam vel convergentes vel divergentes; huc autem accedit, quod hic primam aëris agitationem cum nullo motu coniunctam assumsimus, quippe quo accedente continuatio scalae celeritatum pro tubis hyperbolicis longe aliam sequitur legem, ac si tubus ubique esset aequaliter amplus. Quomodocunque autem scala celeritatum fuerit comparata, ex formulis paragrapho 106 datis liquet agitationes sequentes ita ex utraque scala determinari, ut componantur ex effectu utriusque seorsim producto. Cum igitur in hoc problemate effectum ex sola scala densitatum oriundum assignaverimus, nunc scalam celeritatum seorsim examini subiiciamus, ut deinceps utrumque effectum coniungendo agitationes ex duabus quibuscunque scalis simul oriundae definiri queant.

PROBLEMA 91

124. *Si prima aequilibrii perturbatio in solo motu constet, densitate ubique naturali relicta, seu si detur scala celeritatum, ex ea agitationes aëris sequentes in tubo hyperbolico AaBb* (Fig. 102) *utrinque aperto definire.*

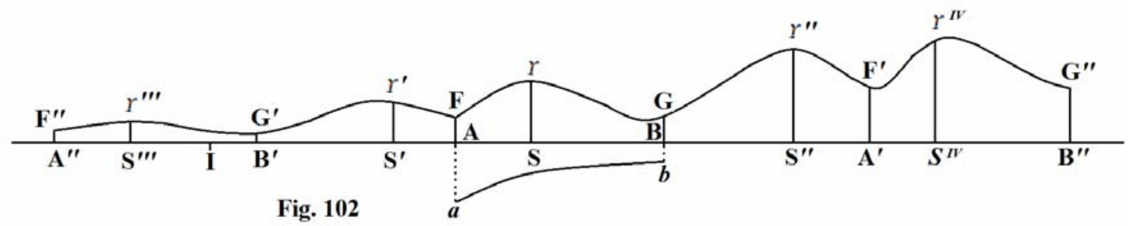

Fig. 102

SOLUTIO

Sit ergo $F\Upsilon G$ scala celeritatum data, cuius applicatae ex celeritatibus initio impressis ita definiuntur, ut, si in *S*, cuius distantia a centro hyperbolae sit $IS = S$, celeritas in plagam *AB* fuerit Υ, fiat $S\Upsilon = \frac{\Upsilon}{cS}$. Hanc igitur curvam utrinque continuari oportet secundum praecepta in problematibus 87 et 88 tradita. Statuantur in hunc finem distantiae $IA = a$ et $IB = b$, et pro continuatione ultra *A* invenienda capiatur $AS' = AS = x$, ut sit $x = S - a$, et ponatur applicata $ST = y$, eritque applicata in *S'* constituenda

$$S'\Upsilon' = z = y - \tfrac{2}{a} e^{\frac{-x}{a}} \int e^{\frac{x}{a}} y dx,$$

integrali hoc ita sumto, ut evanescat posito $x = 0$. Ergo ob $x = S - a$ erit

$$S'\Upsilon' = S\Upsilon' = y - \tfrac{2}{a} e^{\frac{-S}{a}} \int e^{\frac{S}{a}} dS \cdot S\Upsilon$$

et area

$$AFS'\Upsilon' = a\left(S\Upsilon - S'\Upsilon'\right) - AFS\Upsilon.$$

Iam ultra *B* progrediamur, sumtisque $BS'' = BS = x = b - S$, manente $S\Upsilon = y$, applicata in *S''* erigenda ex problemate 88 reperitur

$$z = y + \tfrac{2}{b} e^{\frac{x}{b}} \int e^{\frac{-x}{b}} y dx$$

seu

$$S''\Upsilon'' = S\Upsilon = y - \tfrac{2}{b} e^{\frac{-S}{b}} \int e^{\frac{S}{b}} dS \cdot S\Upsilon$$

et area

$$BGS''\Upsilon'' = b\left(S''\Upsilon'' - S\Upsilon\right) - BGS\Upsilon.$$

Nunc ad aperturam *A* revertamur et sumto spatio

$$AS''' = AS'' = x = 2b - a - S,$$

sit $S''\Upsilon'' = y$, et applicata in *S'''* erigenda erit

$$z = y - \tfrac{2}{a} e^{\frac{-x}{a}} \int e^{\frac{x}{a}} y dx$$

seu *S* –*S*

$$S'''\Upsilon''' = S''\Upsilon'' + \tfrac{2}{a}e^{\frac{S}{a}}\int e^{\frac{-S}{a}}dS\cdot S''\Upsilon'',$$

quae facta reductione transit in hanc formam:

$$S'''\Upsilon''' = S''\Upsilon'' + \tfrac{2(b-a)}{a(a+b)}e^{\frac{S}{a}}\int e^{\frac{-S}{a}}dS\cdot S\Upsilon + \tfrac{2(b-a)}{a(a+b)}e^{\frac{-S}{b}}\int e^{\frac{S}{b}}dS\cdot S\Upsilon.$$

Redeamus ad aperturam *Bb* sumamusque

$$BS^{IV} = BS' = x = b-2a+S,$$

et posito $S'\Upsilon' = y$ applicata in S^{IV} statuenda erit

$$z = y + \tfrac{2}{b}e^{\frac{x}{b}}\int e^{\frac{-x}{b}}ydx$$

seu

$$S^{IV}\Upsilon^{IV} = S'\Upsilon' + \tfrac{2}{b}e^{\frac{S}{b}}\int e^{\frac{-S}{b}}dS\cdot S'\Upsilon',$$

quae loco $S'\Upsilon'$ supra inventum valorem substituendo reducitur ad hanc formam:

$$S^{IV}\Upsilon^{IV} = S\Upsilon - \tfrac{2(b-a)}{a(a+b)}e^{\frac{-S}{a}}\int e^{\frac{S}{a}}dS\cdot S\Upsilon - \tfrac{2(b-a)}{b(a+b)}e^{\frac{S}{b}}\int e^{\frac{-S}{b}}dS\cdot S\Upsilon$$

sicque ulterius progredi licet, quousque libuerit.

Verum hic potissimum videamus, in quo statu futurus sit aër, qui initio erat ad *S*, post tempus $t = \frac{2AB}{c}$, quo casu abscissae utrinque abscindendae cadent in S^{IV} et S''', ita ut in solutione generali paragraphi 106 fiat

$$TN = S^{IV}\Upsilon^{IV} \quad \text{et} \quad tn = S'''\Upsilon''',$$

tum igitur colligetur

densitas $\quad q = Q\left(1-\tfrac{1}{2}S\left(S^{IV}\Upsilon^{IV} - S'''\Upsilon'''\right) + \tfrac{1}{2}S'''\Upsilon'''\, S^{IV}\Upsilon^{IV}\right)$

celeritas $\quad \mathfrak{T} = \tfrac{1}{2}cS\left(S^{IV}\Upsilon^{IV} + S'''\Upsilon'''\right)$

translatio $\quad Ss = \tfrac{1}{2}S\cdot S'''\Upsilon'''\, S^{IV}\Upsilon^{IV}.$

Area autem haec $S'''\Upsilon'''\, S^{IV}\Upsilon^{IV}$ ita per meras applicatas exprimi potest; cum ex continuationis indole sit

$$\text{I.}\quad S\Upsilon S'\Upsilon' = a\left(S\Upsilon - S'\Upsilon'\right)$$
$$\text{II.}\quad S''\Upsilon''\ S'''\Upsilon''' = a\left(S''\Upsilon'' - S'''\Upsilon'''\right)$$
$$\text{III.}\quad S\Upsilon S''\Upsilon'' = b\left(S''\Upsilon'' - S\Upsilon\right)$$
$$\text{IV.}\quad S'\Upsilon'\ S^{IV}\Upsilon^{IV} = b\left(S^{IV}\Upsilon^{IV} - S'\Upsilon'\right),$$

erit combinando:

$$\text{IV} - \text{I.}\quad S\Upsilon\ S^{IV}\Upsilon^{IV} = b\left(S^{IV}\Upsilon^{IV} - S'\Upsilon'\right) - a\left(S\Upsilon - S'\Upsilon'\right)$$
$$\text{II} - \text{III.}\quad S\Upsilon\ S'''\Upsilon''' = a\left(S''\Upsilon'' - S'''\Upsilon'''\right) - b\left(S''\Upsilon'' - S\Upsilon\right),$$

unde colligimus aream

$$S'''\Upsilon'''S^{IV}\Upsilon^{IV} = (b-a)\left(S\Upsilon - S'\Upsilon' - S''\Upsilon''\right) + bS^{IV}\Upsilon^{IV} - aS'''\Upsilon'''.$$

In qua si valores pro his applicatis inventos substituamus, reperitur ista area ponendo $S\Upsilon = y$:

$$\frac{2(b-a)}{a(a+b)}\left(e^{\frac{-S}{a}}\int e^{\frac{S}{a}}ydS - e^{\frac{S}{a}}\int e^{\frac{-S}{a}}ydS + e^{\frac{-S}{b}}\int e^{\frac{S}{b}}ydS - e^{\frac{S}{b}}\int e^{\frac{-S}{b}}ydS\right),$$

ubi integralia ita capi debent, ut ea, quae involvunt *a*, evanescant posito $S = a$, altera vero posito $S = b$. Haec ergo expressio pro area inventa in $\frac{1}{2}S$ ducta praebet spatium translationis *Ss.* Deinde pro reliquis elementis habebimus:

$$S^{IV}\Upsilon^{IV} - S'''\Upsilon''' =$$
$$= \frac{-2(b-a)}{a(a+b)}\left(\frac{1}{a}e^{\frac{-S}{a}}\int e^{\frac{S}{a}}ydS + \frac{1}{a}e^{\frac{S}{a}}\int e^{\frac{-S}{a}}ydS + \frac{1}{b}e^{\frac{-S}{b}}\int e^{\frac{S}{b}}ydS + \frac{1}{b}e^{\frac{S}{b}}\int e^{\frac{-S}{b}}ydS\right)$$

et

$$S^{IV}\Upsilon^{IV} + S'''\Upsilon''' =$$
$$= 2y - \frac{2(b-a)}{a(a+b)}\left(\frac{1}{a}e^{\frac{-S}{a}}\int e^{\frac{S}{a}}ydS - \frac{1}{a}e^{\frac{S}{a}}\int e^{\frac{-S}{a}}ydS + \frac{1}{b}e^{\frac{-S}{b}}\int e^{\frac{S}{b}}ydS - \frac{1}{b}e^{\frac{S}{b}}\int e^{\frac{-S}{b}}ydS\right)$$

COROLLARIUM 1

125. Hinc ergo patet ob scalam celeritatum post tempus $= \frac{2AB}{c}$ statum aëris in tubo multum ab initiali discrepare posse, quod discrimen eo minus evadet, quo minor fuerit fractio $\frac{2(b-a)}{a(a+b)}$ et quo propius scala principalis $F\Upsilon G$ ad axem accesserit.

COROLLARIUM 2

126. Quodsi ergo aëri in tubo hyperbolico simul motus fuerit impressus, post tempus $\frac{2AB}{c}$ aër in tubo non duas oscillationes perfecisse censebitur; multo minus ad aliud quodpiam tempus oscillationes revocari poterunt, sed potius sonus inde perceptus valde erit rudis et ad harmoniam ineptus.

SCHOLION

127. Tibiae ergo ad figuram hyperbolicam formatae hoc insigni vitio laborabunt, ut sonos neutiquam puros atque ad harmoniam idoneos edant, propterea quod motum non in oscillationes distinctas resolvere licet. Hocque vitium eo erit maius, quo fortius huiusmodi tibia inflatur, quoniam tum multo minus oscillationes distingui poterunt, inflatione autem lenissima sonus etiamnunc tolerabilis edetur. Facile autem intelligitur hoc vitium tibiis hyperbolicis non esse proprium, sed ad omnes alias formas eo magis extendi, quo magis ab amplitudine aequabili differant. Ratio igitur hinc perspicitur, cur omnis generis tibiae, quae Organis pneumaticis inseri solent, figuram habeant vel cylindricam vel prismaticam, ut amplitudo ubique sit eadem, haecque sola figura ad Musicam accommoda videtur; quod quidem duplici modo fieri licet, dum eae superne vel apertae sunt vel clauduntur. At si tibias hyperbolicas claudere velimus, soni multo rudiores edentur, quia tum neutra scala seorsim considerata motum oscillatorium regularem producere valet; ex quo operae pretium haud erit casum, quo tubus hyperbolicus in altero termino apertus, in altero vero clausus sumeretur, evolvi.