CHAPTER IV

CONCERNING HOMOGENEOUS FLUIDS

CAPABLE OF ROTATIONAL MOTION WITHOUT COMPRESSION

PROBLEM 32

75. *Thus if a fluid* (Fig. 26) *may be rotating about the fixed axis OC, in order that the motion of the individual elements shall be uniform, truly the speed will be a function of whatever distance proportional to the distance from the same axis, to investigate whether each such motion may be able to be sustained?*

SOLUTION

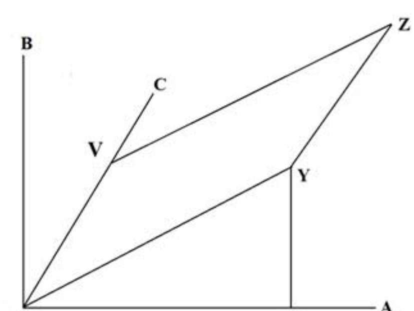


Fig. 26

An element of the fluid defined at *Z*, with the coordinates $OX = x,\ XY = y$ and $YZ = z$, of which therefore the distance from the *OC* axis is $ZV = \sqrt{(xx+yy)}$, which therefore is not changed in the motion. Therefore because the element *Z* is rotating about the point V normal in a plane normal to the axis *AC* ,its speed w will vanish, whereby the speeds *u* and *v* are prepared thus, in order that, while in the increment of the time *dt* the coordinates *x* and *y* take the increments *udt* and *vdt,* the distance $\sqrt{(xx+yy)}$ will not be changed, from which there is deduced to become $ux+vy=0$. Therefore there may be put $u = Ty$ and $v = -Tx$, the total speed will be $T = \sqrt{(xx+yy)}$, and thus by hypothesis *T* will be a function of $\sqrt{(xx+yy)}$; whereby we may put $T = \Gamma : \frac{xx+yy}{2}$, so that there shall become

$$\left(\frac{dT}{dx}\right) = x\Gamma' : \frac{xx+yy}{2} \text{ and } \left(\frac{dT}{dy}\right) = y\Gamma' : \frac{xx+yy}{2},$$

then truly $\left(\frac{dT}{dx}\right) = 0$ and $\left(\frac{dT}{dy}\right) = 0.$

With this circular motion established it is required to be seen, whether that may be consistent with the principles of fluid motion. And indeed in the first place it may demand on account of $w=0$, so that there shall be

$$\left(\frac{du}{dx}\right) + \left(\frac{dv}{dy}\right) = 0;$$

which happens incredibly well, since there shall become :

$$\left(\frac{du}{dx}\right) = y\left(\frac{dT}{dx}\right) = xy\Gamma' : \frac{xx+yy}{2} \text{ and } \left(\frac{dv}{dy}\right) = -\left(\frac{dT}{dy}\right) = -xy\Gamma' : \frac{xx+yy}{2},$$

Moreover again we will have for the other equation, for the sake of example by writing L for $\Gamma' : \frac{xx+yy}{2}$, so that there may become $dT = Lxdx + Lydy$,

$$\left(\tfrac{du}{dx}\right) = Lxy, \quad \left(\tfrac{du}{dy}\right) = T + Lyy, \quad \left(\tfrac{dv}{dx}\right) = -T - Lxx, \quad \left(\tfrac{dv}{dy}\right) = -Lxy,$$

from which we conclude:

$$U = TyLxy - Tx \cdot (T + Lyy) - TTx,$$
$$V = -Ty(T + Lxx) + Tx \cdot Lxy = -TTy$$

and $W = 0.$ Whereby since the formula $Udx+Vdy = -TT\left(xdx+ydy\right)$ certainly may admit integration, because T is a function of $xx+yy$, the other equation for the integrable motion found becomes integrable, to the extent that it postulates a possible motion, and with the action of the forces put in place:

$$\int (Pdx+Qdy+Rdz) = S$$

the pressure is defined thus, so that there shall become [Recall that 2*g* is taken as a distance numerically equal to the acceleration of gravity: hence variables such as pressure defined in terms of mass can be converted into similar variables in terms of weight.]

$$\tfrac{2gp}{b} = 2gS + \int TT(xdx+ydy) + f : t.$$

Therefore generally the motion present in the fluid can be described: which therefore will be worth the effort of a more careful examination. The distance from the axis shall be $ZV = \sqrt{(xx+yy)} = s,$ and there may be put

$$\int TT(xdx+ydy) = \int TTsds = \Gamma : s,$$

for the integral, and there will become $TT = \frac{\Gamma' : s}{s},$ from which the speed, with which the element Z is rotating about the axis OC at the distance $ZV = s$, will be

$$= Ts = \sqrt{s}\Gamma' : s$$

[We surmise T to be a generalized angular speed] and the pressure likewise is found :

$$p = bS + \tfrac{b}{2g}\Gamma : s + f : t.$$

COROLLARY 1

76. Therefore if the speed of the element *Z*, of which the distance from the axis of rotation is $ZV = s$, the speed may be put $= \Delta s$, hence there will become

$$\Gamma' : s = \frac{(\Delta : s)^2}{s}$$

and the pressure is found :

$$p = bS + \frac{b}{2g}\int \frac{ds}{s}(\Delta : s)^2 + f : t.$$

Whereby, if that speed shall become $= \alpha s^n$,

$$p = bS + \frac{\alpha\alpha b s^{2n}}{4ng} + f : t;$$

where in the case of a constant angular speed $= \alpha$ on account of $n = 0$ the pressure to become:

$$p = bS + \frac{\alpha\alpha b}{2g} ls + f : t.$$ [Following L'Hôspital's Rule.]

COROLLARY 2

77. An indefinite function of the time thus is introduced into the pressure *p*, since the pressure at some time may be increased or decreased by external forces as it pleases. But whatever may be put into effect by these forces, it is necessary, it is necessary that it may be contained in the general solution. Therefore if such a change in the external forces is not allowed, that function of the time will have to be omitted.

COROLLARY 3

78. Moreover motion of this kind may form filled with whirlpools, in whatever manner the forces *P*, *Q*, *R*, also shall be prepared, but by no means may it be able to sustain that motion, while these will be less able to be sustained, provided the formula

$$Pdx + Qdy + Rdz = dS$$

may be permitted to be integrated, evidently such a motion will able to be obtained by calling in help from external forces.

COROLLARY 4

79. Because the whole mass of the fluid is rotating about the fixed axis *OC*, from which the individual elements maintain the same distance, the whole mass of the vessel of which the axis shall be *CO*, can be considered to be included turning or rotating. But nothing with regard to the motion, refers to what kind of a figure may be attributed to that, provided all of its sections were circles normal to the axis *OC*.

SCHOLIUM 1

80. So that we may set this out more clearly, we may establish that the fluid be subjected to gravity only, the direction of which shall be *CO*, or rather the axis of gyration *OC* may be placed normally, so that there shall become $S=-z$; from which we will have for the pressure

$$p=b\left(h-z\right)+\tfrac{b}{2g}\int\tfrac{ds}{s}\left(\Delta:s\right)^{2}+f:t,$$

with the speed being $=\Delta:s$ for the distance from the axis $=s$. Again we may assume no change to arise from external forces, so that $f:t$ may become zero. Therefore the figure *EEFF* (Fig. 27) will represent a vertical section of the vessel made through the axis *OC*, in which *GHG* shall be the upper surface of the fluid, through which the pressure may vanish, from which with the distance taken from the axis $OP=s$, the height arising $PM=h+\tfrac{1}{2g}\int\tfrac{ds}{s}\left(\Delta:s\right)^{2}$, from which equation the upper surface of the fluid *GHG* is expressed. Therefore with the speed of rotation taken for the distance *s* to be $=\alpha s^{n}$, there will become

$$PM=h+\tfrac{\alpha\alpha}{4gn}s^{2n},$$

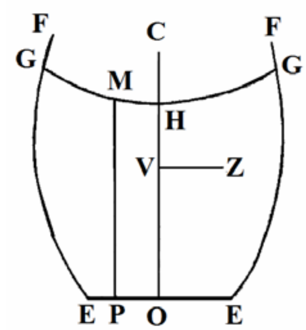


Fig.27

from which *GHG* will be hollowed out around *H* and with the minimum height there to be $OH=h$, if indeed *n* were a positive number. But truly if *n* shall be a negative number, in the middle *H* thus will be depressed indefinitely and a cavity will be left around the axis. Indeed in the case, where $n=1$ and the whole mass is rotating in the same time, this curve will be a parabola described about the axis *HC*, the parameter of which is $\tfrac{4g}{\alpha\alpha}$: and if the time of a single rotation, which is $\tfrac{2\pi}{\alpha}$ sec., may be put $=\theta''$:

the parameter will become $=\tfrac{g}{\pi\pi}$, and the pressure at $Z=b(h-z)+\tfrac{\pi\pi b}{\theta\theta}\cdot\tfrac{ss}{g}$,

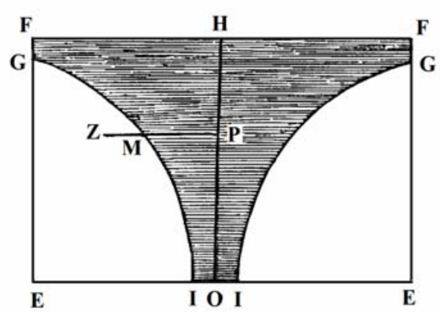


Fig. 28

from which likewise the pressure becomes known at the side of the vessel, which thus itself is had, so that, where that were greater, thus it shall become greater for the same height. But if truly it shall be $n=0$ or thus a negative number, that inconvenience arises, so that the elements close to the axis can will complete revolutions in an infinitely small time ; truly this inconvenience is removed at once, since the fluid shall leave an empty space around the axis, so that we may represent a motion of this kind, let there become $n=-\tfrac{1}{2}$, so that at the distance from the axis $=s$ the speed shall become $=\tfrac{\alpha}{\sqrt{s}}$ and the pressure $p=b(h-z)-\tfrac{\alpha\alpha b}{2gs}$, which for some height *z* will vanish at the distance $s=\frac{\alpha\alpha}{2g(h-z)}$. Therefore the space (Fig.28) around the axis $OH=h$ is left empty *FFGGII* is terminated by a hyperbola, with there being $HP.PM=\frac{\alpha\alpha}{2g}$, and thus the whirlpool is shown, thus so that the pressure everywhere along *GI* shall be zero: beyond this whirlpool at *Z*, where fluid is found at *Z*, the pressure will become:

$$p=b\cdot HP-\frac{\alpha\alpha b}{2g\cdot PZ} \text{ or } p=\frac{\alpha\alpha b}{2g}\left(\frac{1}{PM}-\frac{1}{PZ}\right)=\frac{\alpha\alpha b\cdot MZ}{2g\cdot PM\cdot PZ}.$$

Therefore whirlpools or circular channels of this nature can arise, whenever the rotational speed around the axis either is constant, or decreases with greater distances, and without doubt channels in the sea are required to be ascribed to this cause.

SCHOLIUM 2

81. With the fluid acted on by gravity alone, we will consider the axis (Fig. 29), around which the fluid is rotating, to be horizontal and gravity to be directed along the right line *OA*, so that there shall be $P=1$, $Q=0$ et $R=0$, and thus $S=x$. Again, at the distance $OP=s$ from the axis, the speed shall be $=\Delta:s$, and the pressure will become

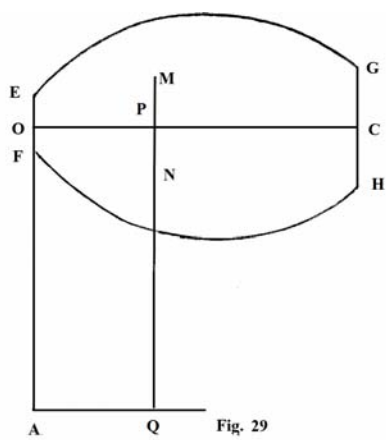


Fig. 29

$$p=bx+\frac{b}{2g}\int\frac{ds}{s}(\Delta:s)^2$$

with *f: t* being omitted, while we remove all the variation due to external forces. At first we will consider the case where $\Delta:s=\alpha s$ and therefore $p=b(x-h)+\frac{\alpha\alpha b}{2g}ss$, which situation may be shown in figure 29, where the axis *OC* of the vessel *EFGH* is considered to be horizontal, truly with the right line *OA* vertical. *N* may be taken in the vertical plane *AOC*, the distance of which from the axis $PN=s$, and the pressure will become:

$$p=b(\tfrac{\alpha\alpha}{2g}PN^2-QN),$$

but for the point *M*, where after *N* is carried through a half revolution, there will become:

$$p=b(\tfrac{\alpha\alpha}{2g}PM^2-QM),$$

and thus during the motion the same element experiences diverse pressures, which lest any may become negative, the constant $OA=h$ may be able to be taken negative. But in the case shown in the figure, lest the pressure may emerge negative anywhere: it will be required to insert a solid cylinder into the vessel, of which the radius $=k$ shall be so great, in order that there may become

$$\tfrac{\alpha\alpha}{2g}kk-h-k=0,$$

thus so outside this cylinder the pressure everywhere shall be positive and the fluid is rotating around this cylinder. Moreover for the case $n=\frac{1}{2}$ there becomes

$$p=b(x-h)-\tfrac{\alpha\alpha b}{2gs},$$

where therefore *h* will be agreed to be taken negative, so that the right line *OA* shall be agreed to be inclined upwards; then therefore

the pressure at *N* will be $=b(QN-\frac{\alpha\alpha}{2g\cdot PN})$

and

the pressure at *M* will be $=b(QM-\frac{\alpha\alpha}{2g\cdot PM})$,

from which so wide as cylinder will require to be inserted, so that with its radius put $=k$, there may become

$$h-k-\tfrac{\alpha\alpha}{2gk}=0 \text{ or } hk-\tfrac{\alpha\alpha}{2g}=kk,$$

hence

$$k=\tfrac{1}{2}h-\sqrt{\left(\tfrac{1}{4}hh-\tfrac{\alpha\alpha}{2g}\right)},$$

from which it is clear that *hh* must be greater than $\frac{2\alpha\alpha}{g}$. Then truly, since at an exceedingly great distance from the axis the pressure again will become negative, also the radius of the vessel cannot exceed this value

$$\tfrac{1}{2}h+\sqrt{\left(\tfrac{1}{4}hh-\tfrac{\alpha\alpha}{2g}\right)}.$$

Finally it is required to be observed, while motion of this kind has once begun, that itself can henceforth be able to persist, and lest on account of the friction of the vessel, the vessel itself may be rotating around the axis as part of the motion.

SCHOLIUM 3

82. If the fluid clearly may not be disturbed by any forces, so that there shall become $S=0$, then with the distance from the axis put $=s$, with the speed $=\Delta:s$, the pressure at the same distance will be

$$p=bx+\tfrac{b}{2g}\int\tfrac{ds}{s}\left(\Delta:s\right),$$

with all the variation from the external forces removed. Hence on putting the speed $\Delta:s=\alpha s^n$ there is produced:

$$p=b\left(h+\tfrac{\alpha\alpha}{4ng}s^{2n}\right),$$

and in the case $n=0$ there becomes:

$$p=b\left(h+\tfrac{\alpha\alpha}{2g}ls\right).$$

But if $\Delta:s=\alpha s^{-m}$, there will be

$$p = b\left(h - \frac{4ng}{\alpha\alpha} s^{-2m}\right).$$

Which cases deserve to be presented separately.

I. If $\Delta : s = \alpha s^n$, then h can be either $> 0,\ = 0,$ or < 0. And in the first place, if $h > 0$, the pressure everywhere will be positive, and indeed a minimum on the axis itself, by receding from which it will increase continually : and thus the whole cylinder of fluid will be able to rotate in this manner. Then likewise it happens also, if $h = 0$, only with this distinction, that the pressure vanishes at the axis itself. In the third place with h taken negative there becomes

$$p = b\left(\frac{\alpha\alpha}{2gn} s^{2n} - h\right),$$

from which since the pressure may be produced negative, provided $s^{2n} < \frac{2gh}{\alpha\alpha} n,$ here an empty space may be left around the axis and the fluid will be rotating around this hollow cylinder. From these cases the smallest body immersed in fluid will be pressed towards the axis by the force $\frac{\alpha\alpha b}{2g} s^{2n-1}$.

II. If $\Delta : s = \alpha$ and $p = b\left(h + \frac{\alpha\alpha}{2g} ls\right)$, the pressure around the axis is negative as far as the distance $s = e^{\frac{-2gh}{\alpha\alpha}}$, where it vanishes; therefore so great an empty cylinder will remain around the axis, around which the fluid will rotate.

III. If $\Delta : s = \alpha s^{-m}$ and $p = b\left(h - \frac{\alpha\alpha}{2mg} s^{-2m}\right),$ it is evident the constant h must be taken positive by necessity and the cylinder left to go around the axis in an empty space, of which the radius $= \sqrt[2m]{\frac{\alpha\alpha}{4mgh}},$ then truly in the fluid beyond this axis the pressure will increase continually, but at an infinite distance to become finally $= bh$. For this fluid a small particle immersed at a distance from the axis $= s$ will be forced towards the axis by the force $\frac{\alpha\alpha b}{2g} s^{-2m-1},$ which shall be inversely as the square of the distance from the axis , if $m = +\frac{1}{2}$, and the speed $= \frac{\alpha}{\sqrt{s}}$, or inversely as the inverse square root of the distance.

PROBLEM 33

83. *If a fluid may be rotating around some axis or the three speeds u, v, w of any point in the fluid shall be proportional to these formulas*

$$\alpha y - \beta z,\ \gamma z - \alpha x,\ \beta x - \gamma y,$$

to investigate the conditions, by which such a motion may be able to exist, while the fluid may be acted on by some forces P, Q, R.

SOLUTION

Therefore we may put :

$$u=T\left(\alpha y-\beta z\right),\ v=\ T\left(\gamma z-\alpha x\right),\ w=T(\beta x-\gamma y)$$

and the differential formulas hence will arise:

$$\begin{array}{l|l|l}
\left(\frac{du}{dx}\right)=\left(\alpha y-\beta z\right)\left(\frac{dT}{dx}\right) & \left(\frac{dv}{dx}\right)=-\alpha T+\left(\gamma z-\alpha x\right)\left(\frac{dT}{dx}\right) & \left(\frac{dw}{dx}\right)=\beta T+\left(\beta x-\gamma y\right)\left(\frac{dT}{dx}\right)\\
\left(\frac{du}{dy}\right)=\alpha T+\left(\alpha y-\beta z\right)\left(\frac{dT}{dy}\right) & \left(\frac{dv}{dy}\right)=\left(\gamma z-\alpha x\right)\left(\frac{dT}{dy}\right) & \left(\frac{dw}{dy}\right)=-\gamma T+\left(\beta x-\gamma y\right)\left(\frac{dT}{dy}\right)\\
\left(\frac{du}{dz}\right)=-\beta T+\left(\alpha y-\beta z\right)\left(\frac{dT}{dz}\right) & \left(\frac{dv}{dz}\right)=\ \gamma T+\left(\gamma z-\alpha x\right)\left(\frac{dT}{dz}\right) & \left(\frac{dw}{dz}\right)=\left(\beta x-\gamma y\right)\left(\frac{dT}{dz}\right)\\
\left(\frac{du}{dt}\right)=\left(\alpha y-\beta z\right)\left(\frac{dT}{dt}\right) & \left(\frac{dv}{dt}\right)=\left(\gamma z-\alpha x\right)\left(\frac{dT}{dt}\right) & \left(\frac{dw}{dt}\right)=\left(\beta x-\gamma y\right)\left(\frac{dT}{dt}\right)
\end{array}$$

Hence the first equation $\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0$ adopts this form:

$$\left(\alpha y-\beta z\right)\left(\frac{dT}{dx}\right)+\left(\gamma z-\alpha x\right)\left(\frac{dT}{dy}\right)+\left(\beta x-\gamma y\right)\left(\frac{dT}{dz}\right)=0.$$

For this equation is satisfied, if *T* were some function of the two quantities $\gamma x+\beta y+\alpha z$ and $xx+yy+zz$; if we may put

$$dT=M\left(\gamma dx+\beta dy+\alpha dz\right)+N\left(xdx+ydy+zdz\right),$$

then there will become

$$\left(\frac{dT}{dx}\right)=M\gamma+Nx,\quad \left(\frac{dT}{dy}\right)=M\beta+N\gamma,\quad \left(\frac{dT}{dz}\right)=M\alpha+Nz.$$

Therefore we may progress to the other equation; and we may expand out

$$u\left(\frac{du}{dx}\right)+v\left(\frac{dv}{dy}\right)+w\left(\frac{dw}{dz}\right)=0\,,$$

which with the factors substituted will be changed into this :

$$TT(\alpha\gamma z+\beta\gamma y-\alpha\alpha x-\beta\beta x).$$

Whereby since *T* may not involve the time *t*, on account of $\left(\frac{dT}{dt}\right)=0$ there will be also $\left(\frac{du}{dt}\right)=0$, from which there becomes

$$U = TT(\alpha\gamma z + \beta\gamma y - \alpha\alpha x - \beta\beta x)$$
$$V = TT(\beta\gamma x + \alpha\beta z - \gamma\gamma y - \alpha\alpha y)$$
$$W = TT(\alpha\beta y + \alpha\gamma x - \beta\beta z - \gamma\gamma z)$$

and therefore

$$\begin{aligned} &Udx+Vdy+Wdz \\ &= TTd\cdot\left(\alpha\gamma xz+\beta\gamma xy+\alpha\beta yz-\tfrac{1}{2}(\alpha\alpha+\beta\beta)xx-\tfrac{1}{2}(\gamma\gamma+\alpha\alpha)yy-\tfrac{1}{2}(\beta\beta+\gamma\gamma)zz\right) \\ &= -\tfrac{1}{2}TTd\cdot\left((\alpha y-\beta z)^2+(\gamma z-\alpha x)^2+(\beta x-\gamma z)^2\right). \end{aligned}$$

Which expression since it must be integrable, it is necessary that TT and likewise T shall be a function of this kind

$$(\alpha y-\beta z)^2+(\gamma z-\alpha x)^2+(\beta x-\gamma z)^2,$$

which, since it is reduced to this :

$$(\alpha\alpha+\beta\beta+\gamma\gamma)(xx+yy+zz)-(\gamma x+\beta y+\alpha z)^2,$$

certainly it is contained in the form satisfying the first general condition. Whereby on putting

$$(\alpha y-\beta z)^2+(\gamma z-\alpha x)^2+(\beta x-\gamma z)^2=(\alpha\alpha+\beta\beta+\gamma\gamma)ss,$$

provided some function of *s* may be taken for *T*, the other equation supplies this equation for the pressure *p*:

$$\tfrac{2gp}{b}=2gS+(\alpha\alpha+\beta\beta+\gamma\gamma)\int TTsds+f:t$$

with *S* being present for the action of the force $\int(Pdx+Qdy+Rdz)$.

COROLLARY 1

84. Therefore the speed of each particle at *Z* is

$$=\sqrt{(uu+vv+ww)}=Ts\sqrt{(\alpha\alpha+\beta\beta+\gamma\gamma)},$$

from which since *T* shall be a function of *s*, also it will be a function of the true speed itself. Moreover it is required to note this quantity *s* to designate the distance of the point *Z* from the axis, about which the rotation is done.

COROLLARY 2

85. Therefore provided that the individual particles of the fluid may be rotating about some axis, thus so that its speed shall be proportional to a function of the distance, motion of this kind may be found in a fluid.

COROLLARY 3

86. Again, it is required for this actual motion, that the pressure p shall be given a positive value ; and if it may happen, so that its value may become negative anywhere, where an empty space is going to be established in the fluid, so that by requiring a solid body be located at that place.

SCHOLIUM

87. Indeed this problem appears no broader than the preceding, since this motion may be made about a fixed axis and likewise such a situation may be attributed to that ; but yet, since perhaps the forms assumed of a more general kind for the speeds u, v, w may be deceptive, the establishment of these will be seen to be aided significantly for investigations of this kind, by bringing forwards others to be investigated. Now since the whole theory of the motion of fluids at this point is reduced to the resolution of equations, so that forms of this kind may be thought out for u, v, w , from which the first formula $\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)$ may vanish, then truly this equation $Udz+Vdy+Wdz$ may emerge integrable. But up to this point I have approached this work thus, so that in the first place I shall have satisfied the first condition, which thus may be allowed to be established more generally, then truly it will be required to elicit cases of this kind, for which the second condition also will be satisfied, which investigation will be seen in the cases established to be most illuminating and helpful. But if we may wish to begin in the reverse order with the latter condition, the work may be seen to become much more difficult and harder, thus so that here it may scarcely be allowed to expect a general solution. Yet I have observed the most widely apparent case, where the latter condition may be fulfilled; which evidently happens, if the formula $udx+vdy+wdz$ were integrable, where this case, in which the motion of fluids through tubes is to be examined primarily, almost as if by definition this theory were present at this stage for this alone ; from which I judge it to be worth the effort for this same case requiring to be established and to be resolved in the following chapter, and that thus even more so, since it may be allowed to extend that to more general fluids.

CHAPTER V

CONCERNED WITH THE MOTION OF FLUIDS IN THAT CASE WHERE THE INTEGRAL HAS THIS FORM $udx+vdy+wdz$

PROBLEM 34

88. *If the three speeds u, v, w of each fluid element may be prepared thus, so that the formula $udx+vdy+wdz$ may be allowed to be integrated, the equation will be produced by which the pressure of the fluid is expressed.*

SOLUTION

Since u, v, w shall be functions of the four variables x, y, z and t, because the formula $udx+vdy+wdz$ is put integrable, that is required to be understood, while the time t is assumed constant. Therefore I shall be its complete integral, which also, with the time t had for a variable, may provide the differential:

$$dI = udx+vdy+wdz+\Phi dt.$$

Hence therefore, so that at first we may elicit the acceleration:

$$U = u\left(\tfrac{du}{dx}\right)+v\left(\tfrac{dv}{dy}\right)+w\left(\tfrac{du}{dz}\right)+\left(\tfrac{du}{dt}\right),$$

these differential formulas will be most conducive according to our set up to be recalled in terms of the single element dx, which is done readily, while from that integrable form there becomes

$$\left(\tfrac{du}{dy}\right)=\left(\tfrac{dv}{dx}\right),\ \left(\tfrac{du}{dz}\right)=\left(\tfrac{dw}{dx}\right) \text{ and } \left(\tfrac{du}{dt}\right)=\left(\tfrac{d\Phi}{dx}\right),$$

from which we follow on with:

$$U = u\left(\tfrac{du}{dx}\right)+v\left(\tfrac{dv}{dx}\right)+w\left(\tfrac{dw}{dx}\right)+\left(\tfrac{d\Phi}{dt}\right),$$

and in a similar manner:

$$V = u\left(\tfrac{du}{dy}\right)+v\left(\tfrac{dv}{dy}\right)+w\left(\tfrac{dw}{dy}\right)+\left(\tfrac{d\Phi}{dy}\right)$$

$$W = u\left(\tfrac{du}{dz}\right)+v\left(\tfrac{dv}{dz}\right)+w\left(\tfrac{dw}{dz}\right)+\left(\tfrac{d\Phi}{dz}\right).$$

Since now with the action of the accelerative forces P, Q, R in place [*i.e.* the force per unit mass, effecting the acceleration]:

$$\int(Pdx+Qdy+Rdz)=S$$

and for an element of the fluid, which we may consider, with the pressure $=p$ and with the density $=q$, hence we will find the equation:

$$\tfrac{2gdp}{q}=2gdS-Udx-Vdy-Wdz,$$

in which the time *t* is assumed constant, so that now from this hypothesis there is found:

$$dx\left(\tfrac{d\Phi}{dx}\right)+dy\left(\tfrac{d\Phi}{dy}\right)+dz\left(\tfrac{d\Phi}{dz}\right)=d\Phi,$$

with this reduction observed in the remaining members there will be

$$Udx+Vdy+Wdz=udu+vdv+wdw+d\Phi,$$

which form since it shall be integrable, we will have :

$$2g\int\tfrac{dp}{q}=2gS-\tfrac{1}{2}(uu+vv+ww)-\Phi+f:t,$$

which equation is applicable, as often as *q* will have been a function of the pressure *p* only; but if in addition it may depend on other causes, where this equation may be found, it is necessary, that *q* shall be a function both of *p* as well as of this quantity

$$2gS-\tfrac{1}{2}(uu+vv+ww)-\Phi,$$

otherwise, this hypothesis is required to be excluded.

COROLLARY 1

89. Since the true speed of the element of the fluid shall be $=\sqrt{(uu+vv\ ww)}$, it is clear the pressure in this hypothesis that depends on this speed, so that, where the fluid is moving faster, there the pressure may be diminished more, and that in the squared ratio of the speed.

COROLLARY 2

90. So far the quantity Φ enters into this equation, in as much as the three speeds *u*, *v*, *w* also will depend on the time, thus so that at the same position of the volume the motion of the fluid may be varied with the time.

COROLLARY 3

91. But the function of the time above approaching only from the variation of the external forces, by which the whole mass of the fluid may be disturbed, it arises, since it shall be arbitrary, also it is required to adapt this function to the circumstances. But in that same motion there it disturbs nothing.

SCHOLIUM

92. This hypothesis appears so wide, so that almost all the motion of fluids, in which the authors at this point will have been occupied, may be included within it; from which it may be able to see that in short it may be necessary for the motion of fluids, unless now the cases in the preceding chapter may have occurred, in which this property is not accepted. Indeed in problem 32 we have seen motion able to exist with there being $u = Ty,\ v = -Tx$ and $w = 0$, provided T were a function of $xx+yy$ itself; nor truly does it satisfy this condition of our hypothesis, by which

$$udx + vdy = T\left(ydx - xdy\right),$$

except in the single case $T = \frac{1}{xx+yy}$, where truly the speed becomes

$$\sqrt{(uu+vv)} = \frac{1}{\sqrt{(xx+yy)}};$$

since yet the motion may be able to exist equally from the remaining cases. Thence if the density may depend on the location in some way, or the amount of heat [Euler and Latin could not distinguish between heat and temperature at the time] were greatly different in various regions, its variation without doubt would be able to be so irregular a quantity, so that in no way would it be able to be considered as a function of the quantity

$$2gS - \tfrac{1}{2}(uu+vv+ww) - \Phi\ ,$$

nor therefore shall it allow our equation to be integrated, which yet generally is necessary for real motion. Nor here, as above, is motion of this kind allowed to depart from equilibrium, as cases of motion of this kind cannot be given, since rather an equilibrium would be impossible on account of that, by necessity a motion would be considered to exist; therefore it is necessary that another entirely different motion may occur according to this hypotenuse, which it will be required to consider to extend only to certain types of movements. Then truly, since we have not adapted that to the first condition, that still needs a new restriction to be necessary, which we will investigate in the following chapter.

PROBLEM 35

93. *If the motion of fluids may be prepared thus, so that the formula* $udx+vdy+wdz$ *proves to be integrable, to determine these cases, in which the first condition for motion likewise will be fulfilled.*

SOLUTION

With q denoting the density of the fluid in that place, where the three speeds are u, v, w, the first condition of the motion, as the account of the density will have supplied the requirements, demands that there shall be

$$\left(\frac{d\cdot qu}{dx}\right)+\left(\frac{d\cdot qv}{dy}\right)+\left(\frac{d\cdot qw}{dz}\right)+\left(\frac{dq}{dt}\right)=0.$$

Now as before I shall be that function of x, y, z and t, from which there may become

$$dI = udx+vdy+wdz+\Phi dt,$$

and hence since there shall be

$$u=\left(\frac{dI}{dx}\right),\quad v=\left(\frac{dI}{dy}\right),\quad w=\left(\frac{dI}{dz}\right),$$

the equation established from that will lead to this:

$$q\left(\left(\frac{ddI}{dx^2}\right)+\left(\frac{ddI}{dy^2}\right)+\left(\frac{ddI}{dz^2}\right)\right)+\left(\frac{dq}{dx}\right)\left(\frac{dI}{dx}\right)+\left(\frac{dq}{dy}\right)\left(\frac{dI}{dy}\right)+\left(\frac{dq}{dz}\right)\left(\frac{dI}{dz}\right)+\left(\frac{dq}{dt}\right)=0.$$

Therefore the function I will be required to be assumed thus, so that it may satisfy this equation, which thus may be more difficult to put in place, if the density q may depend on the pressure p, since this must be defined finally by the other equation

$$\frac{2gdp}{q}=2gdS-udu-vdv-wdw-d\Phi\,;$$

therefore so that the equation will be resolved with the greatest difficulty. Meanwhile, in whatever manner it will have been allowed to set out these two equations, thence always a motion of this kind will be obtained, which on account of its nature in the fluid, which will have been assumed it will be able to be found. Therefore this hypothesis will scarcely be allowed to be called into use at any time, unless the density of the fluid everywhere and at all times were constant or $q=b$, for which case our equations will become :

$$\left(\frac{ddI}{dx^2}\right)+\left(\frac{ddI}{dy^2}\right)+\left(\frac{ddI}{dz^2}\right)=0$$

and

$$2gp = 2gbS - \tfrac{1}{2}b(uu+vv+ww+2\Phi)+f:t.$$

COROLLARY 1

94. Therefore with the density placed constant $q = b$, the investigation requires for the solution a function *I* of this kind, so that there shall be,

$$\left(\tfrac{ddI}{dx^2}\right)+\left(\tfrac{ddI}{dy^2}\right)+\left(\tfrac{ddI}{dz^2}\right)=0\,;$$

moreover with such a function found, then at last the speeds will become known:

$$u=\left(\tfrac{dI}{dx}\right),\ v=\left(\tfrac{dI}{dy}\right),\ \ w=d\left(\tfrac{dI}{dz}\right).$$

COROLLARY 2

95. If we may put $I=\Gamma:(\alpha x+\beta y+\gamma z)$, where indeed the time *t* can be introduced in some way, this relation arises between $\alpha,\ \beta,\ \gamma$, so that there must become $\alpha\alpha+\beta\beta+\gamma\gamma=0;$ which cannot happen, except with one variable allowed to be imaginary.

COROLLARY 3

96. But for this inconvenience it can happen in the account of the function, such as if there may be put

$$I = e^{\alpha x+\beta v}\left(A\sin z\sqrt{(\alpha\alpha+\beta\beta)}+B\cos z\sqrt{(\alpha\alpha+\beta\beta)}\right)$$

or

$$I = e^{z\sqrt{(\alpha\alpha+\beta\beta)}}\left(A\sin(\alpha x+\beta y)+B\cos(\alpha x+\beta y)\right),$$

where the constants *A* and *B* can involve the time in some manner.

SCHOLIUM 1

97. It is clear these values given for *I* to be especially specific; indeed the complete value will have to include two arbitrary functions, each of two variable quantities, while the value given by Corollary 2 is from a single function of a single variable. Meanwhile the letters $\alpha,\ \beta$ will be able to be taken as it pleases, from which innumerable values for *I* will be allowed to be shown easily. But however many values will have been found,

these added in turn always will bear a suitable value for *I*. Moreover infinitely many other values of this kind special values of this kind can be found, by taking some function of the quantity $\alpha x+\beta y+\gamma z$ with there being $\alpha\alpha+\beta\beta+\gamma\gamma=0$, with none had with respect to imaginary numbers, and since such functions may always be present in the form $M+N\sqrt{-1}$, thence always an infinite number of satisfying values for *I* will be allowed to be formed, functions of this kind are as follows:

$$\text{Ang.tang}\tfrac{M}{N},\ e^{\pm M}(A\cos N+B\sin N),\ e^{\pm N}(A\cos M+B\sin M).$$

Therefore since endless values are able to be assigned easily for *M* and *N*, hence endless real values will be allowed to be deduced for *I* ; which in addition by as many changes as it pleases, the quantities $\alpha,\ \beta,\ \gamma$, of which two real values always can be accepted, will be multiplied indefinitely. But a great deal is missing, whereby the sum of all the values of this kind is less than may be had for the general value of *I* .

SCHOLIUM 2

98. Moreover motion of this kind and more generally of this sort may involve an inconvenience, so that the like of these may scarcely find a place in actual use, therefore because the fluid is continually carried to a place where the pressure shall become negative and thus continuity is removed, then truly the vessel cannot be used for that continuity, except not only may it be moved likewise, but also its shape will be changed continually. Which it will suffice to be shown by a single example. Let $\beta=0$ and there may be taken $I=Ae^{\alpha x}\sin\alpha z,$ so that there may become

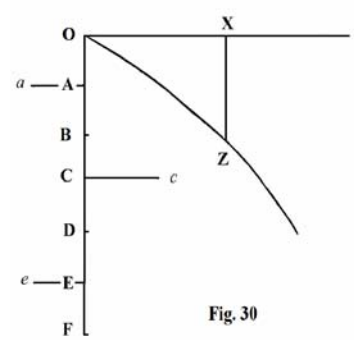


Fig. 30

$$u=A\alpha e^{\alpha x}\sin\alpha z,\ v=0 \text{ and } w=A\alpha e^{\alpha x}\cos\alpha z$$

and hence

$$uu+vv+ww=AA\alpha\alpha e^{2\alpha x}$$

The fluid may be acted on by gravity alone in the direction *ZY*, and there will become $S=h-z$, from which there arises for the pressure :

$$2gp=2gb(h-z)-\tfrac{1}{2}AA\alpha\alpha e^{2\alpha x},$$

if indeed we may put the same motion to endure in the same state. For the sake of brevity there shall be put $A\alpha=\ 2\sqrt{gh}$ and the pressure may vanish, when $z=h\left(1-e^{2\alpha x}\right)$.
Therefore with the right line (Fig. 30) *OX* taken horizontal, in which there shall be $OX=x$, and $XZ=-z$ inclined downwards, the curved line *OZ,* in which the pressure vanishes, will be logarithmic, and below that the pressures will follow in accordance with the depth, above that truly they will be negative, and therefore the continuity of the fluid

is destroyed [Recall that Euler had assumed initially the pressure at the upper surface to be zero, rather than atmospheric pressure, which thus indicates the fluid to be contained in a closed vessel with no air space present above.]. Yet meanwhile at *O*, where $x=0$ and $z=0$, there will $u=0$ and $w=2\sqrt{gh}$; thus so that here the fluid may move up. Again on the line to the vertical *OF* for the depth $OB=\frac{\pi}{\alpha}$ the motion will become downwards with the speed equal to $2\sqrt{gh}$; truly at *D* by taking $OD=\frac{2\pi}{\alpha}$ again it will be moving upwards with equal speed. Then truly with the depth taken $OA=\frac{\pi}{2\alpha}$ on account of $\alpha z=-\frac{\pi}{2}=-90^{o}$ will be carried along *Aa* by the horizontal motion $2\sqrt{gh}$ only, at *C* truly by taking $OC=3OA$ along *Oc*, at *E* again along *Ee*. The matter will be resolved with the remaining vertical right lines taken against X, except that the speeds will continually become greater. From which it is understood motion of this kind cannot be considered in any vessel, besides, since the continuation of the fluid in ascent beyond the curve *OZ* is resolved, then truly in addition the fluid released everywhere again mixes continually, evidently where it descends below the curve *OZ* .

SCHOLIUM 3

99. But motion of this kind is exceedingly difficult, to diagnose the fact that it may not be able to exist, and to be separated from our general theory. The reason for this inconvenience may be seen particularly in that situation, so that the speeds, with which the individual elements may be moved, we may have restricted to the point of the volume, on account of which the quantities *u*, *v* and *w* will always refer to the same point Z and for any time its particles, which may be moving at Z, will declare a motion, in which we can no longer consider the matter concerning the further progress of the same particles. Therefore since in many questions it shall be necessary to pursue the continual motion of each particle, just as if the motion required to be investigated may be adulatory and as if vibratory in nature, I will try to adapt these same principles of motion to this situation. With which agreed we will pursue that suitable, so that the special letters serving to designate the speeds may be removed from the calculation, and indeed in place of these other variables will be introduced, which themselves indicate the state of the fluid at a certain time. Accordingly I am going to undertake this investigation in the following chapter .

CHAPTER VI

CONCERNING THE MOTION OF FLUIDS BEING DEFINED FROM THEIR INITIAL STATE

PROBLEM 36

100. *To describe the variable quantities for a given initial state of a fluid, from which henceforth the state of the same fluid will be required to be defined with some elapse in the time.*

SOLUTION

in the initial state, for which we have assumed the time $t = 0$, we will consider some element of the fluid (Fig. 31), so that the point *Z* shall require to be determined by the three

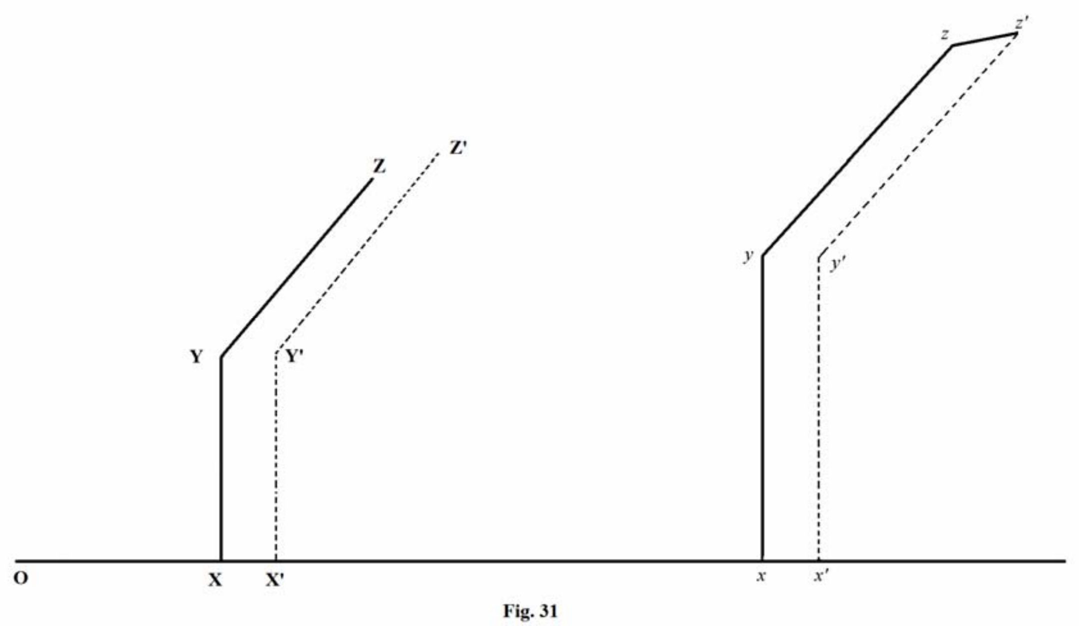


Fig. 31

coordinates $OX = X$, $XY = Y$ and $YZ = Z$, which therefore, as long as we may pursue the same element in motion, remain constants, but if nevertheless we may consider other elements of the fluid, we will be required to consider these as variable quantities. Now in the elapsed time *t* that same element shall be moved from *Z* to *z*, for which location we may call the coordinates $Ox = x,\ xy = y,\ yz = z,$ which therefore are required to be had for the four quantities *X*, *Y*, *Z* and for the *t*. Hence these coordinates $x, y, z,$ if serving for the same *X*, *Y*, *Z* only the *t* time may be changed, the whole path traversed successively by the element will be indicated, which initially was at Z . From which if its motion, while it passes through *z*, may be resolved following the coordinates, the speeds will become:

$$\text{along } Ox = \left(\tfrac{dx}{dt}\right),\ \text{along } xy = \left(\tfrac{dy}{dt}\right),\ \text{along } yz = \left(\tfrac{dz}{dt}\right),$$

and hence again the accelerations along these same directions will be had, which will be :

$$\text{along } Ox = \left(\tfrac{ddx}{dt^2}\right),\ \text{along } xy = \left(\tfrac{ddy}{dt^2}\right),\ \text{along } yz = \left(\tfrac{ddz}{dt^2}\right).$$

Also therefore it is evident these functions *x, y, z* must be endowed with this property, so that with the time *t* vanishing there may become

$$x = X,\ y = Y \ \text{ and } \ z = Z,$$

and by making $t = 0$, these formulas will exhibit only the initial speeds and accelerations of the same elements, while at that point it was at *Z.*

Again *q* will denote the density, which our element will have in the elapsed time $= t$ at *z*, and also the function *q* is a function of the four variables *X*, *Y*, *Z* and *t*, from which for any time each of the elements will be allowed to define the density.

Finally the pressure *p* agreeing for the element moving at *z* in the elapsed time *t*, and also this quantity will be a function of the four variables *X*, *Y*, *Z* and *t.*

With these in place the whole motion to be determined is reduced to this, so that, such functions of these five quantities *x, y, z, q* and *p,* the four variables *X, Y, Z* and *t,* will be investigated.

COROLLARY 1

101. Since *x* shall be a function of the four variables *X, Y, Z* and *t,* its differential shall be complete or arising from the variation of all :

$$dx = dX\left(\tfrac{dx}{dX}\right) + dY\left(\tfrac{dx}{dY}\right) + dZ\left(\tfrac{dx}{dZ}\right) + dt\left(\tfrac{dx}{dt}\right),$$

which also in a similar manner is required to be held by the remaining functions *y, z, p* and *q*.

COROLLARY 2

102. But if we may be able to go from z to z' in the time increment dt , on account of requiring to have *X, Y, Z* as constants, this place z' will have the coordinates being determined :

$$Ox' = x + dt\left(\tfrac{dx}{dt}\right),\ \ x'y' = y + dt\left(\tfrac{dy}{dt}\right),\ \ y'z' = z + dt\left(\tfrac{dz}{dt}\right),$$

then truly the density at $z' = q + dt\left(\frac{dq}{dt}\right)$ and at that same place the pressure $= p + dt\left(\frac{dp}{dt}\right)$.

COROLLARY 3

103. But if also it may be sought, where after the same time *t* the other element, which initially was at *Z'*, at what place nearby to *Z* it may be taken, and for *Z'* the coordinates shall become $X + dX$, $Y + dY$ and $Z + dZ$, if the place sought may be situated at *z'*, for that there will be

$$Ox' = x + dX\left(\frac{dx}{dX}\right) + dY\left(\frac{dx}{dY}\right) + dZ\left(\frac{dx}{dZ}\right),$$

which also is required to be understood both for the two remaining coordinates *x'y'* and *y'z'*, as well for the density and pressure.

SCHOLIUM

104. Now so that we may be able to define both the variation of the density as well as of the pressure, while the same element of the fluid progresses further in the incremental time *dt* , it is necessary that we may consider both the initial state for the two nearby elements *Z* and *Z'* as well as we may investigate the mutual situation of these after the time *t* . Indeed for this reason it will be required to judge, thence in the incremental time *dt* whether they may approach closer together or recede further apart, since in the one case the density increases, in the other it decreases. Truly here the approach or separation of the two elements themselves depends generally on their mutual situation, and thus it can happen, so that in the same fluid two infinitely small masses may approach each other, while others may recede. On account of which this judgment is required to be put in place in the same manner, as we have done above, while the movement of any two indefinitely small masses shall be considered, in which case only the two nearby elements must be considered at the same time, which also is required to be investigated for pressures, which finely I propose in the following problem.

PROBLEM 37

105. *In the elapsed time t if the density q and the pressure p* (Fig. 31) *of the element moving at z , which initially was at Z, for the same time to find the density and pressure of another element itself nearby at z'.*

SOLUTION

For the position of the proposed element at *z* the coordinates shall be $Ox = x,\ xy = y,\ yz = z;$ but for the element itself nearby at z' the coordinates shall be

$$Ox = x+\alpha,\ x'y' = y+\beta,\ y'z' = z+\gamma,$$

for the indefinitely small particles $\alpha,\ \beta,\ \gamma$. Now for the position *Z*, where the element *z* was present initially, with the coordinates put $OX = X$, $XY = Y$ and $YZ = Z$, Z' shall be the place, where the other element z' initially was held, and for which the coordinates $OX' = X + dX$, $X'Y' = Y + dY$, $Y'Z' = Z + dZ$, which differentials *dX*, *dY*, *dZ* will be required now to be defined by the particles given $\alpha,\ \beta,\ \gamma$. Moreover in turn from paragraph 104 we will have:

$$\alpha = dX\left(\tfrac{dx}{dX}\right)+dY\left(\tfrac{dx}{dY}\right)+dZ\left(\tfrac{dx}{dZ}\right)$$
$$\beta = dX\left(\tfrac{dy}{dX}\right)+dY\left(\tfrac{dy}{dY}\right)+dZ\left(\tfrac{dy}{dZ}\right)$$
$$\gamma = dX\left(\tfrac{dz}{dX}\right)+dY\left(\tfrac{dz}{dY}\right)+dZ\left(\tfrac{dz}{dZ}\right).$$

Hence therefore there becomes

$$\alpha\left(\tfrac{dy}{dZ}\right)-\beta\left(\tfrac{dx}{dZ}\right)=dX\left(\left(\tfrac{dx}{dX}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dy}{dX}\right)\left(\tfrac{dx}{dZ}\right)\right)+dY\left(\left(\tfrac{dx}{dY}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dy}{dY}\right)\left(\tfrac{dx}{dZ}\right)\right)$$
$$\beta\left(\tfrac{dz}{dZ}\right)-\gamma\left(\tfrac{dy}{dZ}\right)=dX\left(\left(\tfrac{dy}{dX}\right)\left(\tfrac{dz}{dZ}\right)-\left(\tfrac{dz}{dX}\right)\left(\tfrac{dy}{dZ}\right)\right)+dY\left(\left(\tfrac{dy}{dY}\right)\left(\tfrac{dz}{dZ}\right)-\left(\tfrac{dz}{dY}\right)\left(\tfrac{dy}{dZ}\right)\right).$$

From which for the sake of brevity we may put :

$$\left.\begin{aligned}&+\left(\tfrac{dx}{dX}\right)\left[\left(\tfrac{dy}{dY}\right)\left(\tfrac{dz}{dZ}\right)-\left(\tfrac{dz}{dY}\right)\left(\tfrac{dy}{dZ}\right)\right]\\&+\left(\tfrac{dx}{dY}\right)\left[\left(\tfrac{dz}{dX}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dy}{dX}\right)\left(\tfrac{dz}{dZ}\right)\right]\\&+\left(\tfrac{dx}{dZ}\right)\left[\left(\tfrac{dy}{dX}\right)\left(\tfrac{dz}{dY}\right)-\left(\tfrac{dz}{dX}\right)\left(\tfrac{dy}{dY}\right)\right]\end{aligned}\right\}=K,$$

there is deduced to become:

$$dX=\frac{1}{K}\left\{\begin{aligned}&+\alpha\left[\left(\tfrac{dy}{dY}\right)\left(\tfrac{dz}{dZ}\right)-\left(\tfrac{dz}{dY}\right)\left(\tfrac{dy}{dZ}\right)\right]\\&+\beta\left[-\left(\tfrac{dx}{dY}\right)\left(\tfrac{dz}{dZ}\right)+\left(\tfrac{dz}{dY}\right)\left(\tfrac{dx}{dZ}\right)\right]\\&+\gamma\left[\left(\tfrac{dx}{dY}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dy}{dY}\right)\left(\tfrac{dx}{dZ}\right)\right]\end{aligned}\right\}$$

$$dY = \frac{1}{K}\left\{\begin{array}{l} +\alpha\left[\left(\frac{dz}{dX}\right)\left(\frac{dy}{dZ}\right)-\left(\frac{dy}{dX}\right)\left(\frac{dz}{dZ}\right)\right] \\ +\beta\left[\left(\frac{dx}{dX}\right)\left(\frac{dz}{dZ}\right)-\left(\frac{dz}{dX}\right)\left(\frac{dx}{dZ}\right)\right] \\ +\gamma\left[-\left(\frac{dx}{dX}\right)\left(\frac{dy}{dZ}\right)+\left(\frac{dy}{dX}\right)\left(\frac{dx}{dZ}\right)\right] \end{array}\right\}$$

$$dZ = \frac{1}{K}\left\{\begin{array}{l} +\alpha\left[-\left(\frac{dy}{dY}\right)\left(\frac{dz}{dX}\right)+\left(\frac{dy}{dX}\right)\left(\frac{dz}{dY}\right)\right] \\ +\beta\left[\left(\frac{dz}{dX}\right)\left(\frac{dx}{dY}\right)-\left(\frac{dx}{dX}\right)\left(\frac{dz}{dY}\right)\right] \\ +\gamma\left[\left(\frac{dx}{dX}\right)\left(\frac{dy}{dY}\right)-\left(\frac{dy}{dX}\right)\left(\frac{dx}{dY}\right)\right] \end{array}\right\}.$$

Now with these differentials found we will have for z'

$$\text{the density } = q + dX\left(\frac{dq}{dX}\right) + dY\left(\frac{dq}{dY}\right) + dz\left(\frac{dq}{dZ}\right)$$

and

$$\text{the pressure } = p + dX\left(\frac{dp}{dX}\right) + dY\left(\frac{dp}{dY}\right) + dz\left(\frac{dp}{dZ}\right).$$

COROLLARY 1

106. That quantity K, which constitutes the denominator in these formulas, is expressed thus with the product set out :

$$\begin{aligned} K = &+\left(\frac{dx}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dz}{dZ}\right)+\left(\frac{dz}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dy}{dZ}\right)+\left(\frac{dy}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dx}{dZ}\right) \\ &-\left(\frac{dx}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dy}{dZ}\right)-\left(\frac{dz}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dx}{dZ}\right)-\left(\frac{dy}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dz}{dZ}\right), \end{aligned}$$

which expression is now freed from the order of the coordinates.

COROLLARY 2

107. If, in order to contract the calculation, we may put :

$$\begin{array}{lll} \left(\frac{dx}{dX}\right)=A, & \left(\frac{dx}{dY}\right)=D, & \left(\frac{dx}{dZ}\right)=G, \\ \left(\frac{dy}{dY}\right)=B, & \left(\frac{dy}{dZ}\right)=E, & \left(\frac{dy}{dX}\right)=H, \\ \left(\frac{dz}{dZ}\right)=C, & \left(\frac{dz}{dX}\right)=F, & \left(\frac{dz}{dY}\right)=I, \end{array}$$

this formula will appear more clearly :

$$K = ABC + DEF + GHI - AEI - BFG - CDH.$$

COROLLARY 3

108. Again from these with the required signs we will obtain

$$dX = \frac{\alpha(BC-El)+\beta(GI-CD)+\gamma(DE-BG)}{K}$$
$$dY = \frac{\alpha(EF-CH)+\beta(AC-FG)+\gamma(GH-AE)}{K}$$
$$dZ = \frac{\alpha(HI-BF)+\beta(DF-AI)+\gamma(AB-DH)}{K}.$$

COROLLARY 4

109. So that these formulas may be contracted further, whenever the use of these may be the fullest, again we may put :

$$BC-EI = \mathfrak{A},\ \ GI-CD = \mathfrak{D},\ \ DE-BG = \mathfrak{G},$$
$$AC-FG = \mathfrak{B},\ \ GH-AE = \mathfrak{E},\ \ EF-CH = \mathfrak{H},$$
$$AB-DH = \mathfrak{C},\ \ HI-BF = \mathfrak{F},\ \ DF-AI = \mathfrak{I},$$

so that we shall have

$$dX = \frac{\alpha\mathfrak{A}+\beta\mathfrak{D}+\gamma\mathfrak{G}}{K},\ dY = \frac{\alpha\mathfrak{H}+\beta\mathfrak{B}+\gamma\mathfrak{E}}{K},\ dZ = \frac{\alpha\mathfrak{F}+\beta\mathfrak{I}+\gamma\mathfrak{C}}{K}.$$

PROBLEM 38

110. *For the elapsed time t, if we may consider the element at z' close to the element at z , to show the motion of this element z' by the three speeds along the directions of the coordinates Ox, xy and yz.*

SOLUTION

The three speeds of the element at *z* shall be as above *u, v, w,* and we have seen here from the derivations to become

$$u = \left(\tfrac{dx}{dt}\right),\ \ v = \left(\tfrac{dy}{dt}\right),\ \ w = \left(\tfrac{dz}{dt}\right),$$

and these speeds equally may be regarded as functions of the four variables *X, Y, Z* and *t*. Now the coordinates of the point *z'* may remain as before $x+\alpha,\ y+\beta,\ z+\gamma$ and *Z'* its position in the beginning, and from that the differentials *dX, dY, dZ* will be determined from the given $\alpha,\ \beta,\ \gamma$ as by the preceding problem: with what done the speeds of the nearby point *z'* will be:

$$\text{along } Ox' = u + dX\left(\tfrac{du}{dX}\right) + dY\left(\tfrac{du}{dY}\right) + dz\left(\tfrac{du}{dZ}\right)$$
$$\text{along } x'y' = v + dX\left(\tfrac{dv}{dX}\right) + dY\left(\tfrac{dv}{dY}\right) + dz\left(\tfrac{dv}{dZ}\right)$$
$$\text{along } y'z' = w + dX\left(\tfrac{dw}{dX}\right) + dY\left(\tfrac{dw}{dY}\right) + dz\left(\tfrac{dw}{dZ}\right)$$

and with the letters *u, v, w* replaced thus the speeds will become:

$$\text{along } Ox', \ = \left(\tfrac{dx}{dt}\right) + dX\left(\tfrac{ddx}{dtdX}\right) + dY\left(\tfrac{ddx}{dtdY}\right) + dZ\left(\tfrac{ddx}{dtdZ}\right)$$
$$\text{along } x'y' \ = \left(\tfrac{dy}{dt}\right) + dX\left(\tfrac{ddy}{dtdX}\right) + dY\left(\tfrac{ddy}{dtdY}\right) + dZ\left(\tfrac{ddy}{dtdZ}\right)$$
$$\text{along } y'z' \ = \left(\tfrac{dz}{dt}\right) + dX\left(\tfrac{ddz}{dtdX}\right) + dY\left(\tfrac{ddz}{dtdY}\right) + dZ\left(\tfrac{ddz}{dtdZ}\right),$$

where in place of *dX*, *dY*, *dZ* it will be necessary to write the values $\alpha,\ \beta,\ \gamma$ found before.

SCHOLIUM

111. We may transfer these to the figure 23, where *Z* shall be that point, which we have considered in this way at *z* (Fig. 31), of which therefore the three speeds are :

$$u = \left(\tfrac{dx}{dt}\right),\quad v = \left(\tfrac{dy}{dt}\right),\quad w = \left(\tfrac{dz}{dt}\right),$$

but at the position of the point *z'* close to that, we will consider successively the three points *L, M, N,* for which there shall be $ZL = \alpha,\ ZM = \beta,\ ZN = \gamma$. Therefore for the point L in the values found above for the differentials *dX, dY, dZ* there must be put $\beta = 0,\ \gamma = 0$, for the point *M*, $\alpha = 0,\ \gamma = 0$ and for the point *N*, $\alpha = 0,\ \beta = 0$. Whereby the three speeds of the point *L* will be

	above there was
$\text{along } OX = \left(\tfrac{dx}{dt}\right) + \tfrac{\alpha\mathfrak{A}}{K}\left(\tfrac{ddx}{dtdX}\right) + \tfrac{\alpha\mathfrak{H}}{K}\left(\tfrac{ddx}{dtdY}\right) + \tfrac{\alpha\mathfrak{F}}{K}\left(\tfrac{ddx}{dtdZ}\right)$	$u+dx\left(\tfrac{du}{dx}\right)$
$\text{along } XY = \left(\tfrac{dy}{dt}\right) + \tfrac{\alpha\mathfrak{A}}{K}\left(\tfrac{ddy}{dtdX}\right) + \tfrac{\alpha\mathfrak{H}}{K}\left(\tfrac{ddy}{dtdY}\right) + \tfrac{\alpha\mathfrak{F}}{K}\left(\tfrac{ddy}{dtdZ}\right)$	$v+dx\left(\tfrac{dv}{dx}\right)$
$\text{along } YZ = \left(\tfrac{dz}{dt}\right) + \tfrac{\alpha\mathfrak{A}}{K}\left(\tfrac{ddz}{dtdX}\right) + \tfrac{\alpha\mathfrak{H}}{K}\left(\tfrac{ddz}{dtdY}\right) + \tfrac{\alpha\mathfrak{F}}{K}\left(\tfrac{ddz}{dtdZ}\right)$	$w+dx\left(\tfrac{dw}{dx}\right).$

Moreover the speeds of the point *M* will become:

before there was

$$\text{along } OX = \left(\tfrac{dx}{dt}\right) + \tfrac{\beta}{K}\left[\mathfrak{D}\left(\tfrac{ddx}{dtdX}\right) + \mathfrak{B}\left(\tfrac{ddx}{dtdY}\right) + \mathfrak{F}\left(\tfrac{ddx}{dtdZ}\right)\right] \qquad u+dy\left(\tfrac{du}{dy}\right)$$

$$\text{along } XY = \left(\tfrac{dy}{dt}\right) + \tfrac{\beta}{K}\left[\mathfrak{D}\left(\tfrac{ddy}{dtdX}\right) + \mathfrak{B}\left(\tfrac{ddy}{dtdY}\right) + \mathfrak{F}\left(\tfrac{ddy}{dtdZ}\right)\right] \qquad v+dy\left(\tfrac{dv}{dy}\right)$$

$$\text{along } YZ = \left(\tfrac{dz}{dt}\right) + \tfrac{\beta}{K}\left[\mathfrak{D}\left(\tfrac{ddz}{dtdX}\right) + \mathfrak{B}\left(\tfrac{ddz}{dtdY}\right) + \mathfrak{F}\left(\tfrac{ddz}{dtdZ}\right)\right] \qquad w+dy\left(\tfrac{dw}{dy}\right).$$

and finally the speeds of the point N :

before there was

$$\text{along } OX = \left(\tfrac{dx}{dt}\right) + \tfrac{\gamma}{K}\left[\mathfrak{G}\left(\tfrac{ddx}{dtdX}\right) + \mathfrak{E}\left(\tfrac{ddx}{dtdY}\right) + \mathfrak{C}\left(\tfrac{ddx}{dtdZ}\right)\right] \qquad u+dz\left(\tfrac{du}{dz}\right)$$

$$\text{along } XY = \left(\tfrac{dy}{dt}\right) + \tfrac{\gamma}{K}\left[\mathfrak{G}\left(\tfrac{ddy}{dtdX}\right) + \mathfrak{E}\left(\tfrac{ddy}{dtdY}\right) + \mathfrak{C}\left(\tfrac{ddy}{dtdZ}\right)\right] \qquad v+dz\left(\tfrac{dv}{dz}\right)$$

$$\text{along } YZ = \left(\tfrac{dz}{dt}\right) + \tfrac{\gamma}{K}\left[\mathfrak{G}\left(\tfrac{ddz}{dtdX}\right) + \mathfrak{E}\left(\tfrac{ddz}{dtdY}\right) + \mathfrak{C}\left(\tfrac{ddz}{dtdZ}\right)\right] \qquad w+dz\left(\tfrac{dw}{dz}\right).$$

With these formulas noted the following problem may be solved without any difficulty, if we may call in the aid of problem 19.

PROBLEM 39

112. *To investigate and to define the translation made at this point of time dt, and to define the increment of the density for the element of the fluid* (Fig. 23) *put in place, having the figure of a pyramid ZLMN.*

SOLUTION

This problem evidently agrees with the superior (12), from which the same solution will be had, only if the manner in which they have been changed in designation, will be observed properly. Evidently in the first place, the sides of the pyramid, which there were *dx, dy, dz,* here are α, β, γ *;* thence the speeds *u, v, w* here are designated by $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, $\left(\frac{dz}{dt}\right)$, and the differential formulas $\left(\frac{du}{dx}\right)$, $\left(\frac{dv}{dy}\right)$, $\left(\frac{dw}{dz}\right)$ are readily deduced from the preceding paragraph. Hence since the volume of this pyramid at Z shall be $\frac{1}{6}\alpha\beta\gamma$, after the translation its volume will be

$$\tfrac{1}{6}\alpha\beta\gamma + \tfrac{1}{6}\tfrac{\alpha\beta\gamma dt}{K}\begin{Bmatrix} \mathfrak{A}\left(\tfrac{ddx}{dtdX}\right) + \mathfrak{H}\left(\tfrac{ddx}{dtdY}\right) + \mathfrak{F}\left(\tfrac{ddx}{dtdZ}\right) \\ \mathfrak{D}\left(\tfrac{ddy}{dtdX}\right) + \mathfrak{B}\left(\tfrac{ddy}{dtdY}\right) + \mathfrak{J}\left(\tfrac{ddy}{dtdZ}\right) \\ \mathfrak{G}\left(\tfrac{ddz}{dtdX}\right) + \mathfrak{E}\left(\tfrac{ddz}{dtdY}\right) + \mathfrak{C}\left(\tfrac{ddz}{dtdZ}\right) \end{Bmatrix}.$$

But while this pyramid was at *Z*, its density was q, but after the increment of the time dt the density of the same particle by hypothesis this has become $q+dt\left(\frac{dq}{dt}\right)$. Whereby since each volume multiplied by its density ought to produce the same mass, hence the following equation arises determining the computation of the density:

$$\left.\begin{matrix} \mathfrak{A}\left(\frac{ddx}{dtdX}\right)+\mathfrak{H}\left(\frac{ddx}{dtdY}\right)+\mathfrak{F}\left(\frac{ddx}{dtdZ}\right) \\ \frac{K}{q}\left(\frac{dq}{dt}\right)+\mathfrak{D}\left(\frac{ddy}{dtdX}\right)+\mathfrak{B}\left(\frac{ddy}{dtdY}\right)+\mathfrak{J}\left(\frac{ddy}{dtdZ}\right) \\ \mathfrak{G}\left(\frac{ddz}{dtdX}\right)+\mathfrak{E}\left(\frac{ddz}{dtdY}\right)+\mathfrak{C}\left(\frac{ddz}{dtdZ}\right) \end{matrix}\right\} = 0,$$

where the values of the capital letters occurring here must be chosen from paragraphs 107 and 109. Therefore since thence there shall be

$$\left(\frac{ddx}{dtdX}\right)=\left(\frac{dA}{dt}\right),\quad \left(\frac{ddx}{dtdY}\right)=\left(\frac{dD}{dt}\right),\quad \left(\frac{ddx}{dtdZ}\right)=\left(\frac{dG}{dt}\right)\ \text{ etc.},$$

if in place of the Germanic letters the values may be written from paragraph 109, there will become

$$\left.\begin{matrix} +(BC-EI)\left(\frac{dA}{dt}\right)+(EF-CH)\left(\frac{dD}{dt}\right)+(HI-BF)\left(\frac{dG}{dt}\right) \\ +\frac{K}{q}\left(\frac{dq}{dt}\right)+(GI-CD)\left(\frac{dH}{dt}\right)+(AC-FG)\left(\frac{dB}{dt}\right)+(DF-AI)\left(\frac{dE}{dt}\right) \\ +(DE-BG)\left(\frac{dF}{dt}\right)+(GH-\mathrm{AE})\left(\frac{dI}{dt}\right)+(AB-DH)\left(\frac{dC}{dt}\right) \end{matrix}\right\} = 0,$$

which expression, if it may be prepared with the value of the letter *K*, which is

$$K = ABC+DEF+GHI-AEI-BFG-CDH,$$

it is readily seen the latter member of this to be reduced $\left(\frac{dK}{dt}\right)$, thus so that now the solution of the problem leads to this simple equation :

$$\frac{K}{q}\left(\frac{dq}{dt}\right)+\left(\frac{dK}{dt}\right)=0 \ \text{ or } \ K\left(\frac{dq}{dt}\right)+q\left(\frac{dK}{dt}\right)=0$$

or to this more neatly:

$$\left(\frac{d\cdot Kq}{dt}\right)=0.$$

From which we may understand *Kq* to be a function of this kind, of which the differential arising from the time *t* alone shall vanish, or which may remain the same for all time. Therefore it is evident that this cannot be the case, unless *Kq* shall be a function of the three variables *X, Y, Z* with the time excluded, unless the solution of the problem will be contained by this formula

$$Kq = f:(X,Y,Z).$$

COROLLARY 1

113. The quantity *K* is determined by these conditions, by which the coordinates *x, y, z* after the time *t* will depend on the principal values *X*, *Y*, *Z* in the initial state, just as the form of this shown in paragraph 106 declared. Therefore since the quantities *x, y, z* by necessity involve the time *t*, thus it has become necessary, so that from this form *Kq*, the account of the time has completely departed.

COROLLARY 2

114. Therefore so that if the density of the fluid *q* were a constant quantity, then the form *K* must be independent of the time. But if the density *q* were variable, its value will be able to be assigned for some time *t* , since there shall be

$$q = \frac{f:(X,Y,Z)}{K}.$$

COROLLARY 3

115. Here moreover in the first place it will be required to regard the quantity *q*, while the principal coordinates *X, Y, Z* remain the same, always to express the density of an element of the fluid ; therefore so that if the element may suffer no change in the density, the quantity *q* will remain constant, even if the remaining parts of the fluid may have different densities.

COROLLARY 4

116. Therefore if the fluid shall be heterogeneous or mixed from several fluids of diverse kinds, this account of the motion by requiring several to be defined takes precedence over the former ; since there the quantity *q* does not refer to a particle of the same fluid, but rather to the same place, thus so that it may express the density of all the particles, which pass successively through the same point.

SCHOLIUM 1

117. It is deserved with merit in the solution of this problem, so that finally by several routes it shall have led to the final simplicity ; and finally since by whatever case it has arrived at an integral differential equation, there is no doubt, why not another way may be

given, which at once shall lead to the same integral formula. But in whatever roundabout way I have thus arrived at these, so that I may wish to construct a solution in the same way as I have used above, since the account, by which we may consider this motion may indicate yet another way for the solution required to be come upon. Indeed we will consider at once a small part of the fluid in the initial state of the fluid under the pyramid figure *ZLMN* (Fig. 23) and the translation of this made in the finite time $=t$. Then therefore it may arrive at the final situation *zlmn*, which figure will be equally some irregular pyramid ; if indeed which may be in doubt, or after a finite time *t* may the faces of this pyramid be able to be considered safely as planes ? that first pyramid *ZLMN* at this stage may be taken as infinitely many smaller pyramids, and however many faces there were before, either convex or concave, now it must be recognised these be reduced infinitely closer to being planer and thus to be required to be had as planes. Therefore since at once we have assumed the principal pyramid *ZLMN* to be infinitely small, rightly too we will have the figure *zlmn* in the translated state truly for a pyramid. Therefore we will investigate the volume of this pyramid *zlmn*, so that, since we may put its density at *z* now $=q$ in the elapsed time $=t$, if it may be multiplied by q, it will produce the mass of this incremental portion, which since it remains the same always, shall be a function of this kind, which evidently shall not depend on the time ; or this same mass will be a function of the three quantities X, Y, Z with the time *t* excluded. Whereby since the preceding solution will have given finally, $Kq = f:(X,\ Y,\ Z)$, it is evident, if we may use the method indicated here, the volume of that incremental part itself must be found to be proportional to the quantity K itself.

SCHOLIUM 2

118. This consideration is quite worthwhile, as we may proceed more diligently. Therefore with the principal pyramid put in place $OX = X,\ XY = Y,\ YZ = Z$, then truly $ZL = dX,\ ZM = dY$ and $ZN = dZ$, so that its volume shall be $\frac{1}{6}dXdYdZ$; for the point *z* in the translated pyramid the coordinates shall be $Ox = x,\ xy = y$ and $yz = z.$ Now it will be considered, if the point may be defined in the initial state with these coordinates $X + dX,\ Y+dY,\ Z+dZ$, that in the time *t* is going to be translated to the point defined by these coordinates $x+\alpha,\ y+\beta,\ z+\gamma$, then there shall become :

$$\alpha = AdX + DdY + GdZ,\ \beta = HdX + BdY + EdZ,\ \gamma = FdX + IdY + CdZ.$$

Hence now from the four principal points of the pyramid, the four points translated may be defined, of which the coordinates will themselves be had thus:

$$\text{for } z\begin{Bmatrix} Ox=x \\ xy=y \\ yz=z \end{Bmatrix}, \qquad \text{for } l\begin{Bmatrix} Or=x+AdX \\ rp=y+HdX \\ pl=z+FdX \end{Bmatrix},$$

$$\text{for } m\begin{Bmatrix} Os=x+DdY \\ sq=y+Bdy \\ qm=z+IdY \end{Bmatrix}, \qquad \text{for } n\begin{Bmatrix} Ot=x+GdZ \\ to=y+EdZ \\ on=z+CdZ \end{Bmatrix}.$$

Now from these the sides of the translated pyramid are deduced :

$$\begin{aligned} zl^2 &= (AA+HH+FF)dX^2 \\ zm^2 &= (BB+II+DD)dY^2 \\ zn^2 &= (CC+GG+EE)dZ^2 \\ lm^2 &= (AdX-DdY)^2+(HdX-BdY)^2+(FdX-IdY)^2 \\ ln^2 &= (A\,dX-GdZ)^2+(HdX-EdZ)^2+(FdX-CdZ)^2 \\ mn^2 &= (DdY-GdZ)^2+(BdY-EdZ)^2+(IdY-CdZ)^2 \end{aligned}$$

Hence again following the precepts of paragraph 12 the cosines of the angles at z may be defined :

$$\begin{aligned} \cos lzm &= \nu = \tfrac{AD+BH+FI}{zl\cdot zm}dXdY \\ \cos lzn &= \mu = \tfrac{AG+EH+FI}{zm\cdot zn}dXdZ \\ \cos mzn &= \lambda = \tfrac{DG+BE+CI}{zm\cdot zn}dYdZ. \end{aligned}$$

With which values substituted the volume of the pyramid $zlmn$ is deduced

$$=\tfrac{1}{6}dXdYdZ\sqrt{\left\{\begin{aligned} &+(AA+HH+FF)(BB+II+DD)(CC+GG+EE) \\ &-(AD+BH+FI)^2(CC+GG+EE) \\ &-(AG+EH+CF)^2(BB+II+DD) \\ &-(DG+BE+CI)^2(AA+HH+FF) \\ &+2(AD+BH+FI)(AG+EH+CF)(DG+BE+CI) \end{aligned}\right.}$$

which form after the root sign if it may be expanded out, it taken to be precisely equal to the square of the quantity K : thus so that this volume shall be $=\tfrac{1}{6}KdXdYdZ$ and therefore its mass $=\frac{1}{6}KqdXdYdZ$, from which the quantity *Kq* at the time *t* by no means must depend on *t*.

PROBLEM 40

119. *If a fluid may be acted on by some accelerative forces P, Q, R along the directions of the three coordinates, to investigate the equation, by which the pressure as the individual elements of the fluid may be determined.*

SOLUTION

In the elapsed time *t* some small particle of the fluid may be considered at *Z* (Fig. 25), to which for the sake of the calculation the figure of the parallelepiped *ZLMN zlmn* may be attributed, and for the point Z with the positive coordinates $OX = x,\ XY = y,\ YZ = z$ the sides of this parallelepiped shall be $ZL = \alpha,\ ZM = \beta$ and $Zz = \gamma$, so that its volume shall be $=\alpha\beta\gamma$ and its mass $=q\alpha\beta\gamma$. Now on putting the pressure at $z = p$, which is a function of the quantities *X, Y, Z* and of the time *t*, where *X, Y, Z* are the coordinates of this point, where the element was initially situated, which is now at *Z*. Therefore hence so that the pressure may be defined at *L*, of which the coordinates of the element are $x+\alpha,\ y,\ z,$ it will be required to be seen, where this element was initially, and from the preceding the positions of its coordinates were :

$$X+\frac{\alpha(BC-EI)}{K},\quad Y+\frac{\alpha(EF-CH)}{K},\quad Z+\frac{\alpha(HI-BF)}{K},$$

from which we conclude the pressure at *L* to become :

$$p+\frac{\alpha(BC-EI)}{K}\left(\frac{dp}{dX}\right)+\frac{\alpha(EF-CH)}{K}\left(\frac{dp}{dY}\right)+\frac{\alpha(HI-BF)}{K}\left(\frac{dp}{dY}\right),$$

the whole face of which *LN ln* is pushed along in the direction *AO* by the excess pressure above the pressure *p* at Z. Moreover the surface of this face is $=\beta y$, by which that excess multiplied gives the motive force, and this force divided by the mass $q\alpha\beta\gamma$ gives the size of the acceleration. Whereby since our small mass may be acted on in the direction *OA* by the accelerative force *P*, if from this that may be taken away, there will remain the true accelerative force along the direction *AO*. Therefore since the acceleration shall be $=\left(\frac{ddx}{dt^2}\right)$, this equation will be had:

$$\left(\frac{ddx}{dt^2}\right)=2gP-\frac{2g(BC-EI)}{Kq}\left(\frac{dp}{dX}\right)-\frac{2g(EF-CH)}{Kq}\left(\frac{dp}{dY}\right)-\frac{2g(HI-BF)}{Kq}\left(\frac{dp}{dZ}\right)$$

and in a similar manner for the two remaining directions there is found :

$$\left(\frac{ddy}{dt^2}\right)=2gQ-\frac{2g(GI-CD)}{Kq}\left(\frac{dp}{dX}\right)-\frac{2g(AC-FG)}{Kq}\left(\frac{dp}{dY}\right)-\frac{2g(DF-AI)}{Kq}\left(\frac{dp}{dZ}\right)$$
$$\left(\frac{ddz}{dt^2}\right)=2gR-\frac{2g(DE-BG)}{Kq}\left(\frac{dp}{dX}\right)-\frac{2g(GH-AE)}{Kq}\left(\frac{dp}{dY}\right)-\frac{2g(AB-DB)}{Kq}\left(\frac{dp}{dZ}\right).$$

Therefore with the Germanic letters introduced for the sake of brevity from paragraph 109, we come upon these three equations requiring to be defined for the pressure p :

$$\mathfrak{A}\left(\frac{dp}{dX}\right)+\mathfrak{H}\left(\frac{dp}{dY}\right)+\mathfrak{F}\left(\frac{dp}{dZ}\right)=KqP-\frac{Kq}{2g}\left(\frac{ddx}{dt^2}\right)$$
$$\mathfrak{D}\left(\frac{dp}{dX}\right)+\mathfrak{B}\left(\frac{dp}{dY}\right)+\mathfrak{I}\left(\frac{dp}{dZ}\right)=KqQ-\frac{Kq}{2g}\left(\frac{ddy}{dt^2}\right)$$
$$\mathfrak{G}\left(\frac{dp}{dX}\right)+\mathfrak{E}\left(\frac{dp}{dY}\right)+\mathfrak{C}\left(\frac{dp}{dZ}\right)=KqR-\frac{Kq}{2g}\left(\frac{ddz}{dt^2}\right).$$

So that hence we may define the formula, we will multiply the first by $\mathfrak{B}\mathfrak{C}-\mathfrak{E}\mathfrak{I}=K$, the second by $\mathfrak{E}\mathfrak{F}-\mathfrak{C}\mathfrak{H}=HK$, and the third by $\mathfrak{H}\mathfrak{I}-\mathfrak{B}\mathfrak{F}=FK$, on account of $A\mathfrak{A}+H\mathfrak{D}+F\mathfrak{C}=K$ on dividing by KK there will be found :

$$\left(\frac{dp}{dX}\right)=q(AP+HQ+FR)-\frac{q}{2g}\left(A\left(\frac{ddx}{dt^2}\right)+H\left(\frac{ddy}{dt^2}\right)+F\left(\frac{ddz}{dt^2}\right)\right)$$

and in a like manner there is elicited:

$$\left(\frac{dp}{dY}\right)=q(DP+BQ+IR)-\frac{q}{2g}\left(D\left(\frac{ddx}{dt^2}\right)+B\left(\frac{ddy}{dt^2}\right)+I\left(\frac{ddz}{dt^2}\right)\right)$$
$$\left(\frac{dp}{dZ}\right)=q(GP+EQ+CR)-\frac{q}{2g}\left(G\left(\frac{ddx}{dt^2}\right)+E\left(\frac{ddy}{dt^2}\right)+C\left(\frac{ddz}{dt^2}\right)\right).$$

Again the first may be multiplied by *dX*, the second by *dY*, the third by *dZ*, so that there may be obtained for the differential of the pressure p, if the time *t* may be kept constant, and since by the same hypothesis there shall be :

$$AdX+DdY+GdZ=dx,\ HdX+BdY+EdZ=dy \text{ and } FdX+IdY+CdZ=dz,$$

our three equations will coalesce into this one:

$$dp=q(Pdx+Qdy+Rdz)-\frac{q}{2g}\left(dx\left(\frac{ddx}{dt^2}\right)+dy\left(\frac{ddy}{dt^2}\right)+dz\left(\frac{ddz}{dt^2}\right)\right),$$

in the integration of which the time *t* is required to be had as constant.

COROLLARY 1

120. Since *x, y, z* shall be functions of *X, Y, Z* and *t*, if we may put the complete differential

$$dx = AdX + DdY + GdZ + Ldt,$$

there will become $\left(\frac{dx}{dt}\right) = L$ and thus $\left(\frac{ddx}{dt^2}\right) = \left(\frac{dL}{dt}\right)$, but in place of *dx* it will be required to write in this equation $AdX + DdY + GdZ$, since in that the time is assumed to be constant.

COROLLARY 2

121. But we have seen before, in whatever manner the density *q* may be variable, the quantity *Kq* not to involve the time *t*. But since the action S may depend on the location, at which the element of the fluid may be found after the time *t*, certainly there the time itself will be included.

SCHOLIUM

122. Because also in this solution we may accomplish a much simpler solution for the equation, than would be expected by roundabout calculations, there is no reason to doubt, why also a much easier and neater way may be found to accomplish the same solution. Nor truly will it be readily apparent, in what way the reasoning should be agreed on leading to that, as indeed it is evident the formula $Pdx+Qdy+Rdz$ expresses the differential of the action of the force on the element, the motion of which we consider, just as used in the above method. Truly the differential *dp* here has a completely different significance, as *p* here is a function of the variables *X, Y, Z* and *t*, and hence may be computed by taking *t* constant, whereas before *p* was a function of the quantities *x, y, z* and *t*, of which the differential *dp* was taken indeed with *t* likewise constant; truly here since these coordinates *x, y, z* now involve the time *t* , it is necessary for this differential to differ at once from the previous one. Then truly, even if the speeds $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, $\left(\frac{dz}{dt}\right)$ are expressed, which we have called *u, v, w* above, yet these formulas $\left(\frac{ddx}{dt^2}\right)$, $\left(\frac{ddy}{dt^2}\right)$, $\left(\frac{ddz}{dt^2}\right)$ will differ greatly from $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$, $\left(\frac{dw}{dt}\right)$; for they denote these accelerations , which we have designated by the letters *U, V* and *W* . But the reason for the discrepancy evidently is situated there, whereas here we have referred to the general calculation following the other four variables, and we have done before. From which indeed once we have established this to be suitable, so that the first equation found for the density found has been allowed to be integrated, on the other hand the other equation for the pressure will be seen to be more complicated.

PROBLEM 41

123. *With the initial state of some fluid given (Fig. 31)and with forces, the action of which it will support, to investigate the motion, by which it will be carried henceforth, and its state at some time.*

SOLUTION

We will consider some particle of the fluid at *Z* in the initial state, the position of which may be defined by the three coordinates $OX = X$, $XY = Y$ and $YZ = Z$: then truly the density of the same particle shall be $= Q$, truly the pressure $= P$. Besides moreover its motion shall be prepared thus, so that the speed produced may be resolved along the directions $OX = U$, $XY = V$ and $YZ = W$. Therefore since the initial state of the fluid shall be known, *Q, P, U, V, W* will be functions of the three given variables *X, Y, Z.* Now in the elapsed time *t*, the same particle, which was initially at *Z*, will arrive at *z*, the position of which may be defined by the similar coordinates $Ox = x$, $xy = y$ and $yz = z$, which therefore are to be regarded as functions of the four variables *X, Y, Z* and *t*, prepared thus, so that on putting the time $t = 0$, will be changed into the initial coordinates *X, Y* and *Z*, from which there follows in the same case $t = 0$ to become:

$$\left(\tfrac{dx}{dX}\right) = 1, \ \left(\tfrac{dy}{dX}\right) = 0, \ \left(\tfrac{dz}{dX}\right) = 0,$$
$$\left(\tfrac{dx}{dY}\right) = 0, \ \left(\tfrac{dy}{dY}\right) = 1, \ \left(\tfrac{dz}{dY}\right) = 0,$$
$$\left(\tfrac{dx}{dZ}\right) = 0, \ \left(\tfrac{dy}{dZ}\right) = 0, \ \left(\tfrac{dz}{dZ}\right) = 1.$$

Thence truly of the same particle, while after the time = *t* passes through the point *z* , its three speeds will be

$$\text{along } Ox = \left(\tfrac{dx}{dt}\right) = u, \ \text{along } xy = \left(\tfrac{dy}{dt}\right) = v, \ \text{along } yz = \left(\tfrac{dz}{dt}\right) = w,$$

from which with the time *t* vanishing, by necessity there shall become

$$\left(\tfrac{dx}{dt}\right) = U, \ \left(\tfrac{dy}{dt}\right) = V, \ \left(\tfrac{dz}{dt}\right) = W.$$

Again the density *q* and the pressure *p* of the particle now passing through *z* may be put in place, which two quantities likewise will be functions of the four variables *X, Y, Z* and *t*, prepared thus, so that on putting $t = 0$ there may become $q = Q$ and $p = P$.

Finally the accelerative forces, by which the particle will be acted on at *z*, are reduced to this :

$$\text{along } Ox = \mathfrak{P} \ \text{along } xy = \mathfrak{Q}, \ \text{along } yz = \mathfrak{R}.$$

With which in place it is clear a known motion to be returned from that, so that, whatever kind it may define, these five quantities *x, y, z, q* and *p* shall be functions of the four variables *X, Y, Z* and *t*, and this determination is demanded from the two following equations.

For a start this quantity is sought from the variables *x, y, z* :

$$K=\begin{cases}+\left(\frac{dx}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dz}{dZ}\right)+\left(\frac{dz}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dy}{dZ}\right)+\left(\frac{dy}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dx}{dZ}\right)\\-\left(\frac{dx}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dy}{dZ}\right)-\left(\frac{dz}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dx}{dZ}\right)-\left(\frac{dy}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dz}{dZ}\right)\end{cases}$$

from which as observed before, it is agreed on putting $t=0$ to become $K=1$. Therefore since we will have seen in problem 39, with the motion enduring for the same particle the quantity *Kq* to conserve the same value always, its value certainly must be equal to that, which it had initially on putting $t=0$, but then it becomes $K=1$ and $q=Q$. Whereby the first equation providing a determination of the motion will be $Kq=Q$ and thus $q=\frac{Q}{K}$.

The other equation we have deduced in the preceding problem, where the letter *g* is introduced designating the change in the height of a weight dropped [from rest; noting that 2*g* is a distance numerically equal to the acceleration of gravity in this scheme] in a time of one second, so that finally the time may be expressed in seconds and the speeds may be able to be expressed by the distances traversed in one second. Hence therefore the other equation containing a determination of the motion will be :

$$\frac{2gdp}{q}=2g\left(\mathfrak{P}dx+\mathfrak{Q}dy+\mathfrak{R}dz\right)-dx\left(\frac{ddx}{dt^2}\right)-dy\left(\frac{ddy}{dt^2}\right)-dz\left(\frac{ddz}{dt^2}\right),$$

in which it is required to be observed properly that the time *t* be assumed constant and only the initial coordinates *X*, *Y*, *Z* to be treated as variables. Whereby since *x*, *y*, *z* in addition involve the time *t* , the differentials of these *dx*, *dy*, *dz* are required to be taken conforming to this condition. But when the integral will have been found, in place of a constant some function of the time may be agreed to be added to that.

COROLLARY 1

124. Just as the latter equation has arisen from three quantities, this also it contains three determined quantities, by which it is effected, so that this may emerge integrable. Therefore in addition to be added in the first place from the nature of the relation between the density and pressure, generally five determinations will be had, and therefore however many, there is a need for just as many as five functions sought *x*, *y*, *z*, *q*, *p* requiring to be defined.

COROLLARY 2

125. Moreover, for the integral of the last equation found, if then the coordinates *X*, *Y*, *Z* may be considered as constants and only the time *t* may be taken as variable, the whole motion of this particle of the fluid will be had, which initially was at *Z*; and thence for

whatever time both its position and motion as well as the density and pressure will be able to be assigned.

COROLLARY 3

126. If that same particle, which initially was at *Z*, may allow no change in the density, there will be $q=Q$ always and thus form the first equation $K=1$. Hence therefore in place of *K* the above value assigned is defined by substituting a certain relation of the functions *x, y, z* , just as that must depend on the principal coordinates X, Y, Z .

SCHOLIUM 1

127. If we may consider the other equation more attentively, from its form it may be allowed to conjecture, how that may be required to be deduced from the theory of the forces acting. For initially two *Z* and *Z'* in place will be considered close to each other: the first will be determined by the coordinates $X, Y, Z,$ truly the other by these $X+dX,\ Y+dY,\ Z+dZ$. Now in the elapsed time $=t$ these two points will be transformed into *z* and *z'*, with the coordinates of the first being $x, y, z,$ truly of the second $x+dx,\ y+dy,\ z+dz$, where it may be observed these increments dx, dy, dz to be required to be taken properly from the differentiation if the functions *x, y, z*, while the time *t* is assumed constant, thus so that they may result from the variability of the principal coordinates X, Y, Z only. Now the interval may be called $zz'=ds$, through which extended the incremental fluid mass having a prismatic or cylindrical figure may be considered, of which the base shall be $=\delta\delta,$ and its volume $=\delta\delta ds$ and with the mass $=q\delta\delta ds$. Therefore since *p* is put to be the pressure at *z*, the pressure at $z'=p+dp$, with *dp* denoting that differential of the function *p*, which arises from the variability of the coordinates *X, Y, Z* only, with the time *t* assumed constant. Therefore this incremental mass [called a molecule by Euler, which might lead to confusion now] *zz'* will be acted on by the excess of the pressure on the base *z'* above the base *z* in the direction *z'z* by a motive force $=\delta\delta dp$, which divided by the mass $q\delta\delta ds$ gives the accelerative force $\frac{dp}{qds}$ along the same direction *z'z*. Therefore since the accelerative forces shall become $\mathfrak{P},\ \mathfrak{Q},\ \mathfrak{R}$ along the directions *Ox, xy, yz*, from these it is deduced the force along the direction *zz'*, which is found to be $\frac{\mathfrak{P}dx+\mathfrak{Q}dy+\mathfrak{R}dz}{ds}$, thus so that the total accelerative force along the direction *zz'* shall be

$$=\frac{\mathfrak{P}dx+\mathfrak{Q}dy+\mathfrak{R}dz}{ds}-\frac{dp}{qds}.$$

With this found we will consider the accelerations of the motion, which we have seen to be $\left(\frac{ddx}{dt^2}\right),\ \left(\frac{ddy}{dt^2}\right),\ \left(\frac{ddz}{dt^2}\right)$ along the directions *Ox, xy, yz* , and from these the acceleration along the direction *zz'* may be deduced, which produces :

$$\frac{dx}{ds}\left(\frac{ddx}{dt^2}\right)+\frac{dy}{ds}\left(\frac{ddy}{dt^2}\right)+\frac{dz}{ds}\left(\frac{ddz}{dt^2}\right),$$

and from the principles of the motion this acceleration will require to be put equal to the force of that acceleration multiplied by $2g$; and hence on multiplying by ds that equation itself arises containing the nature of the motion; which therefore will be allowed to be found without ambiguity. But this is done with the greatest certainty, for this almost unused equation to be of the greatest aid, in that the same equation may be found in more than one way, as hence may be illustrated by the nature of this new analysis.

SCHOLIUM 2

128. Since here only the principles of the motion to be treated has been established, perhaps I should demonstrate a short use of these formulas. Therefore in the first place for a progressive or parallel motion of the individual particles we may put :

$$x = X + L,\ y = Y + M,\ z = Z + N\ ,$$

with L, M, N being functions of the time t only, which shall vanish on making $t = 0$. Therefore since there shall be $\left(\frac{dx}{dX}\right)=1,\ \left(\frac{dy}{dY}\right)=1,\ \left(\frac{dz}{dZ}\right)=1,$ truly the reaming formulas shall all vanish, there will become $K = 1$ and $q = Q$, from which the density of each element remains the same ; or this hypothesis pertains to an almost incompressible fluid : yet meanwhile if it may consist of different kinds of matter, in the initial state Q can be regarded as a function of these X, Y and Z. Gravity may act only along the direction zy, so that there shall become $\mathfrak{P}=0,\ \ \mathfrak{Q}=0$ and $\mathfrak{R}=-1$, and there will be this other equation:

$$\frac{2gdp}{Q} = -2gdZ - dX\,\frac{ddL}{dt^2} - dY\,\frac{ddM}{dt^2} - dZ\,\frac{ddN}{dt^2}\,,$$

in order that it may be able to integrate which equation, the density Q must be the same everywhere and therefore $Q = b$, and thus the integral will be:

$$\frac{2g}{b}\,p = 2g\left(h - Z\right) - X\,\frac{ddL}{dt^2} - Y\cdot\frac{ddM}{dt^2} - Z\cdot\frac{ddN}{dt^2} + f : t;$$

therefore unless the motion shall be uniform, the above surface will not be horizontal.
We may consider that case, where the individual elements are rotating around the vertical axis in horizontal planes parallel to the horizontal. In this case the angle θ shall be some function of the time t and there may be established :

$$x = X\cos\theta - Y\sin\theta,\ \ y = Y\cos\theta + X\sin\theta,\ z = Z\ ,$$

hence on account of

$$\left(\frac{dx}{dX}\right)=\cos\theta,\ \ \left(\frac{dx}{dY}\right)=-\sin\theta,\ \ \left(\frac{dy}{dY}\right)=\cos\theta,\ \ \left(\frac{dy}{dX}\right)= X\sin\theta,\ \left(\frac{dz}{dZ}\right)=1$$

$K=\cos^2\theta+\sin^2\theta=1$ is deduced. Whereby so that the density may be put constant $q=Q=b$. Then there is found :

$$\left(\tfrac{dx}{dt}\right)=-\left(X\sin\theta+Y\cos\theta\right)\tfrac{d\theta}{dt},$$
$$\left(\tfrac{ddx}{dt^2}\right)=-\left(X\sin\theta+Y\cos\theta\right)\tfrac{dd\theta}{dt^2}+\left(Y\sin\theta-X\cos\theta\right)\tfrac{d\theta^2}{dt^2},$$
$$\left(\tfrac{dy}{dt}\right)=-\left(Y\sin\theta-X\cos\theta\right)\tfrac{d\theta}{dt},$$
$$\left(\tfrac{ddy}{dt^2}\right)=-\left(Y\sin\theta-X\cos\theta\right)\tfrac{dd\theta}{dt^2}-\left(X\sin\theta+Y\cos\theta\right)\tfrac{d\theta^2}{dt^2},$$

from which with the substitution made the other equation will become :

$$\tfrac{2g}{b}\,p=-2gdZ+\left(YdX+XdY\right)X\,\tfrac{dd\theta}{dt^2}+\left(XdX+YdY\right)\tfrac{d\theta^2}{dt^2},$$

where since t and θ shall be considered constants, on integrating there becomes

$$\tfrac{2gp}{b}=2g\left(h-Z\right)+XY\,\tfrac{dd\theta}{dt^2}+\tfrac{1}{2}\left(XX+YY\right)\tfrac{d\theta^2}{dt^2}+f:t.$$

Since here some function of the time may be allowed for θ, here the motion thus may be allowed to be much wider, than we have set out to be followed in the first method, from which this other method is required to be considered to be clearly more useful.
[The editor C.T. points out a slip in Euler's computation, in the *OO* edition, the value of the coefficient $\tfrac{dd\theta}{dt^2}$ in the penultimate equation $YdX+XdY$ to be $YdX-XdY$, unless $\tfrac{dd\theta}{dt^2}=0$.]

SCHOLIUM 3

129. I will not pursue the case advanced here further, since this same only idea shall be proposed to be shown concerning the application of this latter method, besides which a fuller setting out both of this method as well as of the preceding method would depart from the significant promotion of analysis, before anything may be allowed to hope for successfully. For since here the general analysis concerning functions of four variables may be considered, while that part, which consists of functions of two variables only, at this stage scarcely has began to be improved: certainly with so much hard work to be undertaken at once with trepidation. Therefore so that as if by ascending by steps we may strive towards the investigation of this motion of fluids in general, it may be seen by beginning with easier cases, where fewer variables occur. And here the likeness of the theory of the motion to geometry can most conveniently be divided into the three parts

the line, the plane and the solid; of which the two first parts, even if by abstraction have been formed in aid of the third, yet by no means are without appropriate use. And indeed several, which at this stage have been explored in the motion of fluids, refer to the flow through channels or tubes, which even if they may not be assumed to be the most straight, yet the fluid is not considered to be moving otherwise through these and if they shall be such, if indeed in the individual transverse sections no unequal motion is allowed. Thus deservedly the motion of fluids through tubes of this kind is allowed to be called linear. The second part is the plane or rather the surface, for which as if a moving fluid may be attributed only two dimensions, while clearly the third dimension may not consider any objectionable unequal motion. Therefore from these two parts we will be able at last to undertake the full treatment through all three dimensions with greater confidence.

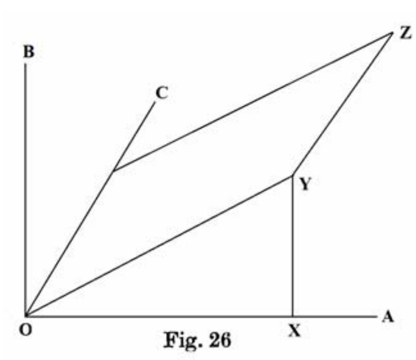
CAPUT IV

DE FLUIDORUM HOMOGENEORUM NULLIUS COMPRESSIONIS CAPACIUM MOTU GYRATORIO

PROBLEMA 32

75. *Si fluidum ita* (Fig. 26) *circa axem fixum OC gyretur, ut singulorum elementorum motus sit uniformis, celeritas vero functioni cuicunque distantiae ab eodem axe proportionalis, investigare, utrum talis motus subsistere possit?*

SOLUTIO



Consideretur fluidi elementum in *Z*, coordinatis $OX = x$, $XY = y$ et $YZ = z$ definitum, cuius ergo ab axe *OC* distantia est $ZV = \sqrt{(xx+yy)}$, quae ergo in motu non mutatur. Quoniam ergo elementum *Z* in plano Fig. 26 ad axem *AG* normali gyratur circa punctum *V*, eius celeritas *w* evanescet, celeritates vero *u* et *v* ita sunt comparatae, ut, dum coordinatae *x* et *y* tempusculo *dt* incrementa capiunt *udt* et *vdt*, distantia $\sqrt{(xx+yy)}$ non varietur, ex quo colligitur fore $ux+vy=0$. Statuatur ergo $u = Ty$ et $v\ Tx$, erit tota celeritas $T\sqrt{(xx+yy)}$, ideoque per hypothesin *T* functio ipsius $\sqrt{(xx+yy)}$; quare ponamus $T = \Gamma : \frac{xx+yy}{2}$, ut sit

$$\left(\tfrac{dT}{dx}\right) = x\Gamma' : \tfrac{xx+yy}{2} \quad \text{et} \quad \left(\tfrac{dT}{dy}\right) = y\Gamma' : \tfrac{xx+yy}{2},$$

tum vero $\left(\frac{dT}{dx}\right) = 0$ et $\left(\frac{dT}{dy}\right) = 0.$

His circa motum stabilitis videndum est, num is cum principiis motus fluidorum consistere possit. Ac prima quidem aequatio ob $w=0$ postulat, ut sit

$$\left(\tfrac{du}{dx}\right) + \left(\tfrac{dv}{dy}\right) = 0;$$

quod egregie evenit, cum sit

$$\left(\tfrac{du}{dx}\right) = y\left(\tfrac{dT}{dx}\right) = xy\Gamma' : \tfrac{xx+yy}{2} \quad \text{et} \quad \left(\tfrac{dv}{dy}\right) = -\left(\tfrac{dT}{dy}\right) = -xy\Gamma' : \tfrac{xx+yy}{2},$$

Porro autem pro altera aequatione habebimus, pro $\Gamma' : \frac{xx+yy}{2}$ brevitatis gratia scribendo *L*, ut sit $dT = Lxdx + Lydy,$

$$\left(\frac{du}{dx}\right) = Lxy,\ \left(\frac{du}{dy}\right) = T + Lyy,\ \left(\frac{dv}{dx}\right) = -T - Lxx,\ \left(\frac{dv}{dy}\right) = -Lxy,\ ,$$

unde concludimus:

$$U = TyLxy - Tx\cdot(T + Lyy) - TT\ x,$$
$$V = -Ty(T + Lxx) + Tx\cdot Lxy = -TTy$$

et $W = 0.$ Quare cum formula $Udx+Vdy = -TT\left(xdx\ ydy\right)$ utique integrationem admittat, quia T est functio ipsius $xx+yy$, altera aequatio pro motu inventa integrabilis evadit, quemadmodum motus possibilitas postulat, et posita virium actione

$$\int (Pdx+Qdy + Rdz) = S$$

pressio ita definitur, ut sit

$$\tfrac{2gp}{b} = 2gS + \int TT(xdx+ydy)\ +f\ :t.$$

Omnino ergo motus descriptus in fluido inesse potest: quem ergo accuratius scrutari operae erit pretium. Sit distantia ab axe $ZV = \sqrt{(xx+yy)} = s,$ ac ponatur integrale

$$\int TT(xdx+ydy) = \int TTsds = \Gamma : s,$$

eritque $TT = \frac{\Gamma':s}{s}$, unde celeritas, qua elementum Z in distantia $ZV = s$ circa axem OC gyratur, erit

$$= Ts = \sqrt{s}\Gamma' :\ s$$

et pressio ibidem invenitur

$$p = bS + \tfrac{b}{2g}\Gamma : s\ + f\ :t.$$

COROLLARIUM 1

76. Si ergo elementi Z, cuius distantia ab axe gyrationis est $ZV = s$, celeritas ponatur $= \Delta s$, fiet hinc

$$\Gamma' : s = \frac{(\Delta{:}s)^2}{s}$$

atque pressio reperitur

$$p = bS + \tfrac{b}{2g}\int \tfrac{ds}{s}\left(\Delta : s\right)^2 + f\ :t.$$

Quare, si illa celeritas sit $= \alpha s^n$, fit

$$p = bS + \tfrac{\alpha\alpha bs^{2n}}{4ng} + f:t;$$

ubi notandum casu celeritatis constantis $= \alpha$ ob $n = 0$ fieri

$$p = bS + \tfrac{\alpha\alpha b}{2g} ls + f:t.$$

COROLLARIUM 2

77. Functio temporis indefinita ideo in pressionem *p* ingreditur, quoniam per vires externas quovis tempore pressio pro lubitu vel augeri vel diminui potest. Quicquid autem per vires externas effici potest, necesse est, ut id in solutione generali contineatur. Si ergo talis mutatio in viribus externis non admittatur, eam temporis functionem omitti oportet.

COROLLARIUM 3

78. Quicunque autem fingatur huiusmodi motus vorticosus, vires sollicitantes *P*, *Q*, *R*, quomodocunque etiam sint comparatae, neutiquam impediunt, quominus is subsistere queat, dummodo formula

$$Pdx + Qdy + Rdz = dS$$

integrationem admittat, semper scilicet viribus externis in subsidium vocandis talis motus obtineri poterit.

COROLLARIUM 4

79. Quoniam tota fluidi massa circa axem fixum *OC* gyratur, a quo singula elementa eandem conservant distantiam, tota massa vasi rotundo seu tornato inclusa concipi potest, cuius axis sit *CO.* Ad motum autem nihil refert, qualis figura ipsi tribuatur, dummodo omnes eius sectiones ad axem *OC* normales fuerint circuli.

SCHOLION 1

80. Quo haec clarius exponamus, fluidum soli gravitati subiectum statuamus, cuius directio sit *CO*, seu potius axis gyrationis *OC* ponatur normalis, ut sit $S = -z$; unde pro pressione habebimus

$$p = b(h - z) + \tfrac{b}{2g}\int \tfrac{ds}{s}(\Delta:s)^2 + f:t,$$

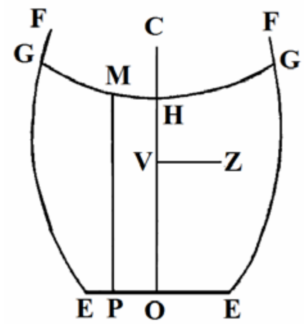


Fig.27

existente celeritate ad distantiam $= s$ ab axe $= \Delta:s$. Sumamus porro in viribus externis nullam evenire mutationem, ut *f* : *t* in nihilum abeat. Repraesentet (Fig. 27) ergo figura *EEFF* sectionem verticalem vasis per axem *OC* factam, in qua sit *GHG* superficies fluidi suprema, per quam pressio evanescat, unde sumta distantia ab axe $OP = s$, erit altitudo

$PM = h + \frac{1}{2g}\int \frac{ds}{s}\left(\Delta : s\right)^2,$ qua aequatione figura suprema superficiei *GHG* exprimitur.

Sumta ergo celeritate gyrationis pro distantia $s = \alpha s^n$, fiet

$$PM = h + \frac{\alpha\alpha}{4gn} s^{2n},$$

unde superficies *GHG* circa medium *H* erit excavata ibique minima altitudo $OH = h$, siquidem *n* sit numerus positivus. At vero si *n* sit numerus negativus, in medio *H* adeo in infinitum deprimetur et cavitatem circa axem relinquet. Casu quidem, quo $n = 1$ et totum fluidum eodem tempore revolvitur, haec curva erit parabola circa axem *HC* descripta, cuius parameter est $\frac{4g}{\alpha\alpha}$: ac si tempus unius revolutionis, quod est $\frac{2\pi}{\alpha}$ min.sec., ponatur $= \theta''$, erit

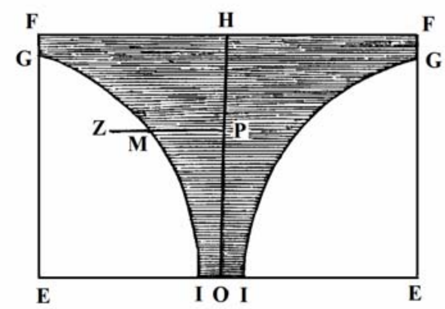


Fig. 28

parameter $= \frac{g}{\pi\pi}$ et pressio in $Z \; = b(h - z) + \frac{\pi\pi b}{\theta\theta} \cdot \frac{ss}{g}$, unde simul pressio in latera vasis innotescit, quae ita se habet, ut, quo id fuerit amplius, eo futura sit maior pro eadem altitudine. Quodsi vero sit $n = 0$ vel adeo numerus negativus, id incommodum nascitur, quod elementa axi proxima revolutiones suas tempore infinite parvo conficerent; verum hoc incommodum sponte tollitur, cum fluidum spatium vacuum circa axem relinquat, cuiusmodi motum ut repraesentemus, sit $n = -\frac{1}{2}$, ut in distantia ab axe $= s$ sit celeritas $= \frac{\alpha}{\sqrt{s}}$ et pressio $p = b(h - z) - \frac{\alpha\alpha b}{2gs}$, quae pro quavis altitudine *z* evanescit in distantia $s = \frac{\alpha\alpha}{2g(h - z)}$. Spatium ergo (Fig.28) circa axem $OH = h$ vacuum relictum *FFGGII* hyperbolis terminatur, existente $HP.PM = \frac{\alpha\alpha}{2g}$, sicque voraginem exhibet, ita ut per *GI* pressio ubique sit nulla: extra hanc voraginem in *Z*, ubi fluidum reperitur in *Z*, erit pressio

$$p = b \cdot HP - \frac{\alpha\alpha b}{2g \cdot PZ} \text{ seu } p = \frac{\alpha\alpha b}{2g}\left(\frac{1}{PM} - \frac{1}{PZ}\right) = \frac{\alpha\alpha b \cdot MZ}{2g \cdot PM \cdot PZ}.$$

Huiusmodi ergo voragines seu euripi oriuntur, quoties celeritas gyratoria circa axem vel est constans, vel in maioribus distantiis decrescit, huicque causae sine dubio euripi in mari sunt adscribendi.

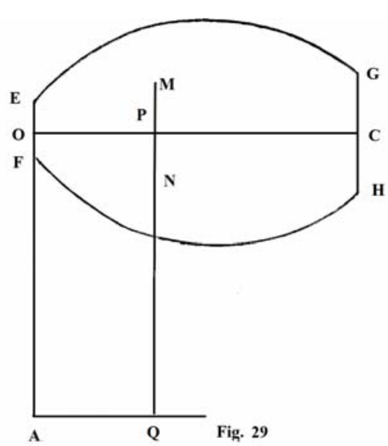


Fig. 29

SCHOLION 2

81. Manente fluido soli gravitati exposito, consideremus (Fig. 29) axem, circa quem fluidum gyratur, esse horizontalem et gravitatem dirigi secundum rectam *OA*,

ut sit $P=1,\ Q=0$ et $R=0,$ ideoque $S=x$. Sit porro in distantia $=s$ ab axe celeritas $=\Delta:s$,
ac pressio fiet

$$p=bx+\tfrac{b}{2g}\int\tfrac{ds}{s}\left(\Delta:s\right)^2$$

omittendo *f: t*, dum a viribus externis omnem variationem removemus. Contemplemur primo casum quo $\Delta:s=\alpha s$ et propterea $p=b(x-h)+\frac{\alpha\alpha b}{2g}ss$, qui status in figura 29 repraesentetur, ubi vasis *EFGH* axis *OC* intelligendus est horizontalis, recta vero $OA=h$ verticalis. Sumatur *N* in plano verticali *AOC*, cuius distantia ab axe $PN=s$, erit pressio

$$p=b(\tfrac{\alpha\alpha}{2g}PN^2-QN),$$

pro puncto autem *M*, quo illud *N* post semirevolutionem defertur, erit

$$p=b(\tfrac{\alpha\alpha}{2g}PM^2-QM),$$

sicque durante motu idem elementum diversas pressiones patitur, quae ne unquam fiant negativae, constans $OA=h$ negative accipi posset. Casu autem in figura repraesentato, ne usquam pressio evadat negativa: in vas inseri oportet cylindrum solidum, cuius semidiameter $=k$ tantus sit, ut fiat

$$\tfrac{\alpha\alpha}{2g}kk-h-k=0,$$

ita ut extra hunc cylindrum pressio ubique fiat positiva et fluidum circa hunc cylindrum revolvatur. Pro casu autem $n=\frac{1}{2}$ fit

$$p=b(x-h)-\tfrac{\alpha\alpha b}{2gs},$$

ubi ergo *h* negativum capi convenit, ut recta *OA* sursum vergere sit censenda; tum ergo erit

$$\text{pressio in } M=b(QN-\tfrac{\alpha\alpha}{2g\cdot PN})$$

et

$$\text{pressio in } M=b(QM-\tfrac{\alpha\alpha}{2g\cdot PM})$$

ex quo cylindrum tam crassum inseri oportet, ut eius semidiametro posito $=k$ fiat

$$h-k-\tfrac{\alpha\alpha}{2gk}=0 \text{ seu } hk-\tfrac{\alpha\alpha}{2g}=kk,$$

hinc

$$k=\tfrac{1}{2}h-\sqrt{\left(\tfrac{1}{4}hh-\tfrac{\alpha\alpha}{2g}\right)},$$

unde patet *hh* maius esse debere quam $\frac{2\alpha\alpha}{g}$. Deinde vero, quia in nimis magna distantia ab axe pressio iterum fieret negativa, etiam semidiameter vasis excedere non debet hunc valorem

$$\tfrac{1}{2}h+\sqrt{\left(\tfrac{1}{4}hh-\tfrac{\alpha\alpha}{2g}\right)}.$$

Ceterum notandum est, dum huiusmodi motus semel inceperit, eum deinceps subsistere posse, ac ne ob attritum vasis debilitetur, ipsum vas pari motu circa axem revolvatur.

SCHOLION 3

82. Si fluidum a nullis plane viribus sollicitetur, ut sit $S=0$, tum posita in distantia ab axe $=s$ celeritate $=\Delta:s$, erit pressio in eadem distantia

$$p=bx+\tfrac{b}{2g}\int\tfrac{ds}{s}\left(\Delta:s\right),$$

sublata omni variatione in viribus externis. Hinc posita celeritate $\Delta:s=\alpha s^{n}$ prodit

$$p=b\left(h+\tfrac{\alpha\alpha}{4ng}s^{2n}\right)$$

et casu $n=0$ fit

$$p=b\left(h+\tfrac{\alpha\alpha}{2g}ls\right).$$

At $\Delta:s=\alpha s^{-m}$ erit

$$p=b\left(h-\tfrac{4ng}{\alpha\alpha}s^{-2m}\right).$$

Qui casus seorsim evolvi merentur.

I. Si $\Delta:s=\alpha s^{n}$, tum h esse potest vel >0 vel $=0$ vel <0. Ac primo, si $h>0$, pressio ubique erit positiva, et minima quidem in ipso axe, a quo recedendo continuo crescit: sicque integer fluidi cylindrus hoc modo gyrari poterit. Idem deinde evenit etiam, si $h=0$, hoc tantum discrimine, quod in ipso axe pressio evanescit. Tertio sumto h negativo fit

$$p=b\left(\tfrac{\alpha\alpha}{2}s^{2n}-h\right),$$

unde cum pressio prodeat negativa, quamdiu $s^{2n}<\tfrac{2gh}{\alpha\alpha}n,$ hoc spatium circa axem vacuum relinquetur et fluidum circa hunc cylindrum cavum gyrabitur. His casibus corpusculum minimum fluido immersum axem versus urgebitur vi $\tfrac{\alpha\alpha b}{2g}s^{2n-1}$.

II. Si $\Delta s=\alpha$ et $p=b\left(h+\tfrac{\alpha\alpha}{2g}ls\right)$, pressio circa axem est negativa usque ad distantiam $s=e^{\frac{-2gh}{\alpha\alpha}}$, ubi evanescit; tam amplus ergo cylindrus circa axem vacuus relinquetur, circa quem fluidum gyrabitur.

III. Si $\Delta:s=\alpha s^{-m}$ et $p=b\left(h-\tfrac{\alpha\alpha}{2mg}s^{-2m}\right),$ patet constantem h necessario positivam sumi debere et cylindrum circa axem vacuum relictum iri, cuius radius $=\sqrt[2m]{\tfrac{\alpha\alpha}{4mgh}},$ tum vero in fluido extra hunc axem pressionem crescere continuo, at in distantia infinita demum fieri $=bh$. Corpusculum huic fluido in distantia ab axe $=s$ immersum ad axem pelletur vi $\frac{\alpha\alpha b}{2g}s^{-2m-1}$, quae quadrato distantiae ab axe reciproce fit proportionalis, si $m+\frac{1}{2}$, et celeritas $=\frac{\alpha}{\sqrt{s}}$ seu reciproce in subduplicata ratione distantiae.

PROBLEMA 33

83. *Si fluidum gyretur circa axem quemcunque seu ternae cuiusque puncti celeritates u, v, w proportionales sint his formulis*

$$\alpha y-\beta z,\ \gamma z-\alpha x,\ \beta x-\gamma y,$$

conditiones explorare, quibus talis motus in fluido existere queat, dum fluidum a viribus quibuscunque P, Q, R sollicitatur.

SOLUTIO

Ponamus ergo:

$$u=T\left(\alpha y-\beta z\right),\ v=\ T\left(\gamma z-\alpha x\right),\ w=T(\beta x-\gamma y)$$

et formulae differentiales hinc natae erunt:

$$\left(\tfrac{du}{dx}\right)=\left(\alpha y-\beta z\right)\left(\tfrac{dT}{dx}\right) \quad \left(\tfrac{dv}{dx}\right)=-\alpha T+\left(\gamma z-\alpha x\right)\left(\tfrac{dT}{dx}\right) \quad \left(\tfrac{dw}{dx}\right)=\beta T+\left(\beta x-\gamma y\right)\left(\tfrac{dT}{dx}\right)$$

$$\left(\tfrac{du}{dy}\right)=\alpha T+\left(\alpha y-\beta z\right)\left(\tfrac{dT}{dy}\right) \quad \left(\tfrac{dv}{dy}\right)=\left(\gamma z-\alpha x\right)\left(\tfrac{dT}{dy}\right) \quad \left(\tfrac{dw}{dy}\right)=-\gamma T+\left(\beta x-\gamma y\right)\left(\tfrac{dT}{dy}\right)$$

$$\left(\tfrac{du}{dz}\right)=-\beta T+\left(\alpha y-\beta z\right)\left(\tfrac{dT}{dz}\right) \quad \left(\tfrac{dv}{dz}\right)=\gamma T+\left(\gamma z-\alpha x\right)\left(\tfrac{dT}{dz}\right) \quad \left(\tfrac{dw}{dz}\right)=\left(\beta x-\gamma y\right)\left(\tfrac{dT}{dz}\right)$$

$$\left(\tfrac{du}{dt}\right)=\left(\alpha y-\beta z\right)\left(\tfrac{dT}{dt}\right) \quad \left(\tfrac{dv}{dt}\right)=\left(\gamma z-\alpha x\right)\left(\tfrac{dT}{dt}\right) \quad \left(\tfrac{dw}{dt}\right)=\left(\beta x-\gamma y\right)\left(\tfrac{dT}{dt}\right)$$

Hinc prima equatio $\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0$ induit hanc formam:

$$\left(\alpha y-\beta z\right)\left(\tfrac{dT}{dx}\right)+\left(\gamma z-\alpha x\right)\left(\tfrac{dT}{dy}\right)+\left(\beta x-\gamma y\right)\left(\tfrac{dT}{dz}\right)=0.$$

Cui aequationi satisfit, si T fuerit functio quaecunque harum duarum quantitatum $\gamma x+\beta y+\alpha z$ et $xx+yy+zz$; nam si ponamus

$$dT=M\left(\gamma dx+\beta dy+\alpha dz\right)+N\left(xdx+ydy+zdz\right),$$

erit

$$\left(\tfrac{dT}{dx}\right)=M\gamma+Nx,\ \ \left(\tfrac{dT}{dy}\right)=M\beta+N\gamma,\ \ \left(\tfrac{dT}{dz}\right)=M\alpha+Nz.$$

Ad alteram ergo aequationem progrediamur; ac primo formulam

$$u\left(\tfrac{du}{dx}\right)+v\left(\tfrac{dv}{dy}\right)+w\left(\tfrac{dw}{dz}\right)=0$$

evolvamus, quae factis substitutionibus abit in

$$TT(\alpha\gamma z+\beta\gamma y-\alpha\alpha x-\beta\beta x).$$

Quare cum T non involvat tempus t, ob $\left(\tfrac{dT}{dt}\right)=0$ erit etiam $\left(\tfrac{du}{dt}\right)=0$,
unde fit

$$U=TT(\alpha\gamma z+\beta\gamma y-\alpha\alpha x-\beta\beta x)$$
$$V=TT(\beta\gamma x+\alpha\beta z-\gamma\gamma y-\alpha\alpha y)$$
$$W=TT(\alpha\beta y+\alpha\gamma x-\beta\beta z-\gamma\gamma z)$$

ac propterea

$$\begin{aligned}&Udx+Vdy+Wdz\\&=TTd\cdot\left(\alpha\gamma xz+\beta\gamma xy+\alpha\beta yz-\tfrac{1}{2}(\alpha\alpha+\beta\beta)xx-\tfrac{1}{2}(\gamma\gamma+\alpha\alpha)yy-\tfrac{1}{2}(\beta\beta+\gamma\gamma)zz\right)\\&=-\tfrac{1}{2}TTd\cdot\left((\alpha y-\beta z)^2+(\gamma z-\alpha x)^2+(\beta x-\gamma z)^2\right).\end{aligned}$$

Quae expressio cum debeat esse integrabilis, necesse est, ut TT ideoque et T sit functio huius quantitatis

$$(\alpha y-\beta z)^2+(\gamma z-\alpha x)^2+(\beta x-\gamma z)^2,$$

quae, quia reducitur ad hanc:

$$(\alpha\alpha+\beta\beta+\gamma\gamma)(xx+yy+zz)-(\gamma x+\beta y+\alpha z)^2,$$

utique in forma generali primae conditioni satisfaciente continetur. Quare ponendo

$$(\alpha y-\beta z)^2+(\gamma z-\alpha x)^2+(\beta x-\gamma z)^2=(\alpha\alpha+\beta\beta+\gamma\gamma)ss,$$

dummodo pro T sumatur functio quaecunque ipsius s, altera aequatio hanc pro pressione p suppeditat aequationem

$$\tfrac{2gp}{b}=2gS+(\alpha\alpha+\beta\beta+\gamma\gamma)\int TTsds+f:t$$

existente S actione virium $\int(Pdx+Qdy+Rdz)$.

COROLLARIUM 1

84. Celeritas vero cuiusque particulae in Z est

$$=\sqrt{(uu+vv+ww)}=Ts\sqrt{(\alpha\alpha+\beta\beta+\gamma\gamma)},$$

unde cum *T* sit functio ipsius *s*, erit etiam functio ipsius celeritatis verae. Notandum autem est hanc quantitatem *s* designare distantiam puncti *Z* ab axe, circa quem fit gyratio.

COROLLARIUM 2

85. Dummodo ergo singulae fluidi particulae uniformiter circa axem quemcunque revolvantur, ita ut cuiusque celeritas sit functioni distantiae proportionalis, huiusmodi motus in fluido locum habere potest.

COROLLARIUM 3

86. Ad motus realitatem autem porro requiritur, ut pressio *p* valorem obtineat positivum; ac si eveniat, ut eius valor usquam sit negativus, ibi spatium a fluido vacuum est statuendum, quod fit corpus solidum in eum locum collocando.

SCHOLION

87. Problema quidem hoc non latius patet quam praecedens, cum et hic motus fiat circa axem fixum perindeque sit, qualis situs ipsi tribuatur; verumtamen, quia formae pro celeritatibus *u*, *v*, *w* assumtae speciem saltem mentiuntur generaliorem, earum evolutio non parum adiumenti ad huiusmodi investigationes alias suscipiendas afferre videtur. Quandoquidem nunc universa motus fluidorum Theoria ad resolutionem huiusmodi aequationum est perducta totumque negotium huc redit, ut pro u, *v*, *w* eiusmodi formae excogitentur, quibus primo formula $\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)$ evanescat, tum vero haec $Udz+Vdy+Wdz$ integrabilis evadat. Hactenus autem hoc opus ita sum aggressus, ut primum conditioni priori satisfecerim, quod adeo generalissime praestare licuit, tum vero hinc eiusmodi casus elicere oportebat, quibus etiam posteriori conditioni satisfieret, quae investigatio casibus evolutis non mediocriter illustrari et adiuvari videtur. Quodsi inverso ordine a conditione posteriori exordiri velimus, opus multo magis difficile et arduum videatur, ita ut hic solutionem generalem vix expectare liceat. Casum tamen vehementer late patentem observavi, quo posterior conditio impletur; quod scilicet fit, si formula $udx+vdy+wdz$ fuerit integrabilis, ubi imprimis notandum est hunc casum in motu fluidorum per tubos, in quo fere solo definiendo Theoria adhuc fuit occupata, locum habere; unde operae pretium esse arbitror isti casui evolvendo sequens Caput destinare, idque eo magis, quod eum ad omnis generis fluida extendere licet.

CAPUT V

DE MOTU FLUIDORUM EO CASU QUO INTEGRABILIS EST HAEC FORMA $udx+vdy+wdz$

PROBLEMA 34

88. *Si cuiusque fluidi elementi ternae celeritates u, v, w ita sint comparatae, ut formula* $udx+vdy+wdz$ *integrationem admittat, aequationem, qua pressio fluidi exprimitur, evolvere.*

SOLUTIO

Cum *u, v, w* sint functiones quatuor variabilium *x, y, z* et *t*, quoniam formula $udx+vdy+wdz$ integrabilis ponitur, id intelligendum est, dum tempus *t* constans assumitur. Sit ergo *I* eius integrale completum, quod etiam tempore *t* pro variabili habito differentiatum praebeat:

$$dI=udx+vdy+wdz+\Phi dt.$$

Hinc igitur, ut primo accelerationem:

$$U=u\left(\tfrac{du}{dx}\right)+v\left(\tfrac{dv}{dy}\right)+w\left(\tfrac{du}{dz}\right)+\left(\tfrac{du}{dt}\right)$$

eliciamus, ad nostrum institutum plurimum conducet has formulas differentiales ad solum elementum *dx* revocare, quod facile fit, dum ex illius formae integrabilitate est

$$\left(\tfrac{du}{dy}\right)=\left(\tfrac{dv}{dx}\right),\ \left(\tfrac{du}{dz}\right)=\left(\tfrac{dw}{dx}\right)\ \text{ et }\ \left(\tfrac{du}{dt}\right)=\left(\tfrac{d\Phi}{dx}\right),$$

unde consequimur:

$$U=u\left(\tfrac{du}{dx}\right)+v\left(\tfrac{dv}{dx}\right)+w\left(\tfrac{dw}{dx}\right)+\left(\tfrac{d\Phi}{dt}\right)$$

similique modo

$$V=u\left(\tfrac{du}{dy}\right)+v\left(\tfrac{dv}{dy}\right)+w\left(\tfrac{dw}{dy}\right)+\left(\tfrac{d\Phi}{dy}\right)$$

$$W=u\left(\tfrac{du}{dz}\right)+v\left(\tfrac{dv}{dz}\right)+w\left(\tfrac{dw}{dz}\right)+\left(\tfrac{d\Phi}{dz}\right).$$

Cum iam posita virium acceleratricium *P, Q, R* actione:

$$\int(Pdx+Qdy+Rdz)=S$$

et in fluidi elemento, quod consideramus, pressione $=p$ et densitate $=q$, hanc invenerimus aequationem:

$$\tfrac{2gdp}{q}=2gdS-Udx-Vdy-Wdz,$$

in qua tempus *t* constans assumitur, quoniam in hac hypothesi est:

$$dx\left(\tfrac{d\Phi}{dx}\right)+dy\left(\tfrac{d\Phi}{dy}\right)+dz\left(\tfrac{d\Phi}{dz}\right)=d\Phi,$$

erit hac reductione in reliquis membris observata

$$Udx+Vdy+Wdz=udu+vdv+wdw+d\Phi,$$

quae forma cum sit integrabilis, habebimus:

$$2g\int\tfrac{dp}{q}=2gS-\tfrac{1}{2}(uu+vv+ww)-\Phi+f:t,$$

quae aequatio locum habet, quoties q fuerit functio solius pressionis p; sin autem insuper ab alia causa pendeat, quo haec aequatio locum habere possit, necesse est, ut q sit functio cum ipsius p tum huius quantitatis

$$2gS-\tfrac{1}{2}(uu+vv+ww)-\Phi,$$

alioquin haec hypothesis est excludenda.

COROLLARIUM 1

89. Cum vera elementi fluidi celeritas sit $=\sqrt{(uu+vv\ ww)}$, patet in hac hypothesi pressionem ita ab hac celeritate pendere, ut, quo celerius fluidum movetur, eo pressio magis diminuatur, idque in ratione duplicata celeritatis.

COROLLARIUM 2

90. Quantitas Φ eatenus in hanc aequationem ingreditur, quatenus ternae celeritates u, v, w etiam a tempore pendent, ita ut in eodem spatii loco motus fluidi Cum tempore varietur.

COROLLARIUM 3

91. Functio autem temporis insuper accedens a sola variatione virium externarum, quibus tota fluidi massa sollicitatur, provenit, quae cum sit arbitraria, etiam hanc functionem ad circumstantias accommodari oportet. In ipso autem motu ea nihil turbat.

SCHOLION

92. Hypothesis haec tam late patet, ut fere omnes fluidorum motus, in quibus definiendis Auctores adhuc fuerunt occupati, in se complectatur; ex quo videri posset eam ad motum fluidorum prorsus esse necessariam, nisi iam casus in praecedente capite occurrissent, in quibus haec proprietas non deprehenditur. In problemate 32 enim vidimus motum subsistere posse existente $u=Ty$, $v=-Tx$ et $w=0$, dummodo T fuerit functio ipsius $xx+yy$; neque vero hic conditio nostrae hypotheseos, qua

$$udx+vdy=T\left(ydx-xdy\right),$$

locum habet, nisi solo casu $T=\frac{1}{xx+yy}$, quo fit celeritas vera

$$\sqrt{(uu+vv)}=\frac{1}{\sqrt{(xx+yy)}};$$

cum tamen reliquis casibus motus aeque subsistere possit. Deinde si densitas utcunque a loco pendeat, seu calor fuerit maxime diversus in variis regionibus, eius varietas sine dubio tam esse poterit irregularis, ut nullo modo tanquam functio quantitatis

$$2gS-\tfrac{1}{2}(uu+vv+ww)-\Phi$$

spectari queat, neque propterea nostra aequatio integrationem admittat, quod tamen ad motus realitatem omnino est necessarium. Neque hic ut supra de aequilibrio regerere licet motum huiusmodi casibus dari non posse, cum potius ob id ipsum, quod aequilibrium sit impossibile, necessario motus existere debeat; motus igitur omnino alius ac secundum hanc hypothesin eveniat necesse est, ex quo ea nonnisi ad certas motus species patere est censenda. Tum vero, quia, eam ad primam conditionem nondum accommodavimus, ea adhuc nova limitatione indiget, quam in sequente problemate investigabimus.

PROBLEMA 35

93. *Si motus fluidorum ita sit comparatus, ut formula* $udx+vdy+wdz$ *integrabilis existit, eos casus determinare, quibus simul prima conditio ad motum requisita adimpletur.*

SOLUTIO

Cum denotante q densitatem fluidi in eo loco, ubi ternae celeritates sunt u, v, w, prima motus conditio, quam densitatis ratio suppeditaverat, postulat,
ut sit

$$\left(\frac{d\cdot qu}{dx}\right)+\left(\frac{d\cdot qv}{dy}\right)+\left(\frac{d\cdot qw}{dz}\right)+\left(\frac{dq}{dt}\right)=0.$$

Sit nunc ut ante I ea functio ipsarum x, y, z et t, ex qua fiat

$$dI=udx+vdy+wdz+\Phi dt,$$

et quia hinc est

$$u=\left(\frac{dI}{dx}\right),\quad v=\left(\frac{dI}{dy}\right),\quad w=\left(\frac{dI}{dz}\right),$$

aequatio illa evoluta perducetur ad hanc:

$$q\left(\left(\frac{ddI}{dx^2}\right)+\left(\frac{ddI}{dy^2}\right)+\left(\frac{ddI}{dz^2}\right)\right)+\left(\frac{dq}{dx}\right)\left(\frac{dI}{dx}\right)+\left(\frac{dq}{dy}\right)\left(\frac{dI}{dy}\right)+\left(\frac{dq}{dz}\right)\left(\frac{dI}{dz}\right)+\left(\frac{dq}{dt}\right)=0.$$

Functionem ergo *I* necessario ita assumi oportet, ut huic aequationi satisfiat, quod eo difficilius praestatur, si densitas *q* a pressione *p* pendeat, quia haec demum per alteram aequationem

$$\frac{2gdp}{q}=2gdS-udu-vdv-wdw-d\Phi$$

definiri debet; quo ergo casu illa aequatio difficillime resolvetur. Interim, quocunque modo has duas aequationes simul expedire licuerit, semper inde eiusmodi motus obtinetur, qui in fluido eius ratione densitatis indolis, quae fuerit assumta, locum habere poterit. Hanc ergo hypothesin vix unquam ad usum revocare licebit, nisi densitas fluidi ubique et semper fuerit constans seu $q=b$, pro quo casu aequationes nostrae evadent:

$$\left(\frac{ddI}{dx^2}\right)+\left(\frac{ddI}{dy^2}\right)+\left(\frac{ddI}{dz^2}\right)=0$$

et

$$2gp=2gbS-\tfrac{1}{2}b(uu+vv+ww+2\Phi)+f:t.$$

COROLLARIUM 1

94. Posita ergo densitate constante $q=b$, ad solutionem requiritur investigatio eiusmodi functionum *1*, ut sit

$$\left(\frac{ddI}{dx^2}\right)+\left(\frac{ddI}{dy^2}\right)+\left(\frac{ddI}{dz^2}\right)=0\,;$$

tali autem functione inventa tum demum celeritates innotescent

$$u=\left(\frac{dI}{dx}\right),\ v=\left(\frac{dI}{dy}\right),\ \ w=d\left(\frac{dI}{dz}\right).$$

COROLLARIUM 2

95. Si ponamus $I=\Gamma:(\alpha x+\beta y+\gamma z),$, ubi quidem tempus *t* utcunque immisceri potest, haec relatio inter $\alpha,\ \beta,\ \gamma$ oritur, ut esse debeat $\alpha\alpha+\beta\beta+\gamma\gamma=0;$ quod, nisi una imaginaria admittatur, fieri nequit.

COROLLARIUM 3

96. Huic autem incommodo ratione functionis occurri potest, veluti si ponatur

$$I=\ e^{\alpha x+\beta v}\left(A\sin z\sqrt{(\alpha\alpha+\beta\beta)}+B\cos z\sqrt{(\alpha\alpha+\beta\beta)}\right)$$

vel

$$I=\ e^{z\sqrt{(\alpha\alpha+\beta\beta)}}\left(A\sin(\alpha x+\beta y)+B\cos(\alpha x+\beta y)\right),$$

ubi constantes *A* et *B* tempus utcunque involvere possunt.

SCHOLION 1

97. Evidens est hos valores pro *I* datos maxime esse speciales; completus enim valor complecti deberet duas functiones arbitrarias, utramque duarum quantitatum variabilium, dum valor corollario 2 datus est unica functio unicae variabilis. Interim litterae $\alpha,\ \beta$ pro lubitu accipi possunt, unde facile innumerabiles valores pro *I* exhibere licet. Quotcunque autem valores fuerint inventi, ii invicem additi idoneum semper valorem pro *I* praebent. Infiniti autem alli valores huiusmodi speciales inveniri possunt, sumendo functionem quamcunque quantitatis $\alpha x+\beta y+\gamma z$ existente $\alpha\alpha+\beta\beta+\gamma\gamma=0$, nullo respectu ad imaginaria habito, et cum tales functiones semper in forma $M+N\sqrt{-1}$ contineantur, inde semper infinitos valores pro *I* satisfacientes formare licet, cuiusmodi sunt:

$$\text{Ang.tang}\,\tfrac{M}{N},\ e^{\pm M}(A\cos N+B\sin N),\ e^{\pm N}(A\cos M+B\sin M).$$

Quoniam ergo pro *M* et *N* facile infiniti valores assignari possunt, hinc infinities infinitos valores reales pro *I* colligere licebit; quae multitudo insuper mutando pro lubitu quantitates $\alpha,\ \beta,\ \gamma$ quarum semper binae reales accipi possunt, in immensum multiplicabitur. Plurimum autem abest, quominus summa omnium huiusmodi valorum pro valore generali ipsius *I* haberi queat.

SCHOLION 2

98. Huiusmodi autem motus plerumque eiusmodi incommoda implicant, ut eorum similitudo in mundo vix locum inveniat, propterea quod fluidum continuo in loca fertur, ubi pressio fit negativa ideoque continuitas tollitur, tum vero vas ad id continendum adhiberi nequit, nisi non tantum simul moveatur, sed etiam eius figura continuo mutetur. Quod unico exemplo ostendisse sufficiet. Sit $\beta=0$ capiaturque $I=Ae^{\alpha x}\sin\alpha z,$ ut fiat

$$u=A\alpha e^{\alpha x}\sin\alpha z,\ v=0\ \text{ et }\ w=A\alpha e^{\alpha x}\cos\alpha z$$

hincque

$$uu+vv+ww=AA\alpha\alpha e^{2\alpha x}$$

Urgeatur fluidum a sola gravitate in directione *ZY*, eritque $S=h-z$, ex quo pro pressione prodit

$$2gp=2gb(h-z)-\tfrac{1}{2}AA\alpha\alpha e^{2\alpha x},$$

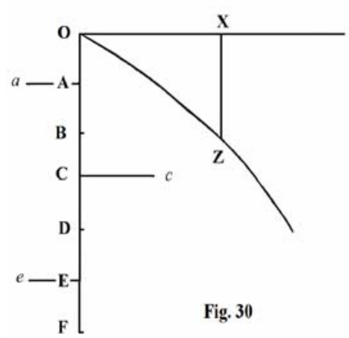


Fig. 30

siquidem eundem motus statum perpetuo durare ponamus. Sit brevitatis gratia $A\alpha=2\sqrt{gh}$ et pressio evanescet, ubi $z=h\left(1-e^{2\alpha x}\right)$. Sumta ergo (Fig. 30) *OX* recta horizontali, in qua sit $OX = x$, et verticalis deorsum vergens $XZ = -z$, linea curva *OZ*, in qua pressio evanescit, erit logarithmica, et infra eam pressiones sequentur rationem profunditatum, supra eam vero erunt negativae, ibique ergo fluidi continuitas tollitur. Interim tamen in *O*, ubi $x = 0$ et $z = 0$, fit $u = 0$ et $w = 2\sqrt{gh}$; ita ut hic fluidum sursum moveatur. In linea porro verticali *OF* ad profunditatem $OB = \frac{\pi}{\alpha}$ motus fiet deorsum pari celeritate $2\sqrt{gh}$; in *D* vero sumta $OD = \frac{2\pi}{\alpha}$ iterum aeque celeriter sursum movebitur. Tum vero sumta profunditate $OA = \frac{\pi}{2\alpha}$ ob $\alpha z = -\frac{\pi}{2} = -90^{\circ}$ solo motu horizontali $2\sqrt{gh}$ secundum *Aa* feretur, in *C* vero sumto $OC = 3OA$ secundum *Oc*, in *E* vero iterum secundum *Ee*. Simili modo res se habebit in reliquis rectis verticalibus versus *X* sumtis, nisi quod celeritates continuo fiunt maiores. Ex quo intelligitur eiusmodi motum in nullo vase concipi posse, praeterquam, quod fluidi continuitas in ascensu ultra curvam *OZ* solvitur, tum vero insuper passim fluidum solutum se iterum continuo admiscet, ubi scilicet infra curvam *OZ* descendit.

SCHOLION 3

99. Vehementer autem difficile est huiusmodi motus, qui re ipsa existere nequeunt, dignoscere et ab aequationibus nostris generatim separare. Cuius incommodi causa praecipue in eo posita videtur, quod celeritates, quibus singula fluidi elementa moventur, ad spatii puncta restrinximus, quandoquidem quantitates *u*, *v* et *w* perpetuo ad idem punctum *Z* referuntur et quovis tempore eius particulae, quae in *Z* versatur, motum declarant, in quo negotio ad ulteriorem progressum eiusdem particulae non amplius respicimus. Cum igitur in pluribus quaestionibus necesse sit cuiusque particulae motum continuo prosequi, veluti si motus undulatorius et quasi vibratorius est investigandus, eadem motus principia ad hoc institutum accommodare conabor. Quo pacto id commodi assequemur, ut litterae peculiares celeritatibus designandis inservientes ex calculo elidantur, earum vero loco aliae variabiles erunt introducendae, quae fluidi statum ad certum tempus in se complectantur. In sequente itaque capite hanc investigationem sum suscepturus.

CAPUT VI

DE MOTU FLUIDORUM EX STATU INITIALI DEFINIENDO

PROBLEMA 36

100. *Dato fluidi statu initiali quantitates variabiles describere, ex quibus deinceps eiusdem fluidi statum elapso tempore quocunque definire oportet.*

SOLUTIO

In statu initiali, pro quo sumimus tempus $t = 0$, consideremus (Fig. 31) fluidi elementum quodcunque, quod sit in puncto *Z* per ternas coordinatas

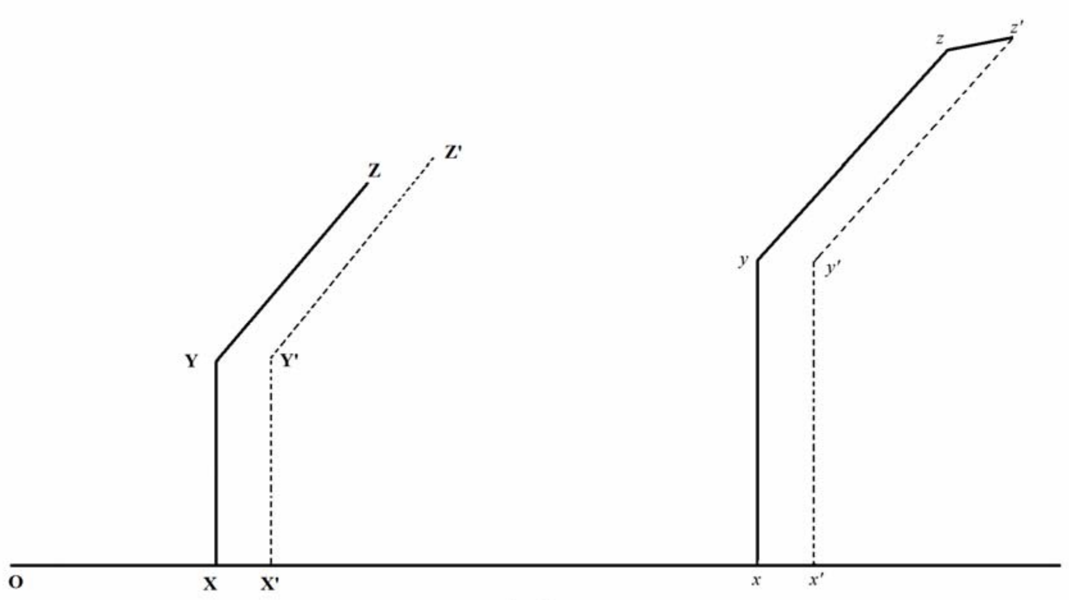


Fig. 31

$OX = X$, $XY = Y$ et $YZ = Z$ determinato, quae ergo, quamdiu idem elementum in motu prosequimur, manent constantes, sin autem ad alia fluidi elementa respicimus, ut quantitates variabiles erunt spectandae. Iam elapso tempore *t* translatum sit istud elementum ex *Z* in *z*, pro quo loco vocemus coordinatas $Ox = x,\ xy = y,\ yz = z,$ quae ergo pro functionibus quatuor quantitatum *X*, *Y*, *Z* et temporis *t* sunt habendae. Hinc istae coordinatae *x*, *y*, *z*, si servatis *X*, *Y*, *Z* iisdem solum tempus *t* varietur, totam viam ab elemento, quod initio erat in *Z,* successive percursam indicabunt. Ex quo si eius motus, dum per *z* transit, secundum coordinatas resolvatur, erunt celeritates:

$$\text{secundum } Ox = \left(\tfrac{dx}{dt}\right),\ \text{secundum } xy = \left(\tfrac{dy}{dt}\right),\ \text{secundum } yz = \left(\tfrac{dz}{dt}\right),$$

atque hinc porro accelerationes secundum easdem directiones habebuntur, quae erunt:

$$\text{secundum } Ox = \left(\tfrac{ddx}{dt^2}\right),\ \text{secundum } xy = \left(\tfrac{ddy}{dt^2}\right),\ \text{secundum } yz = \left(\tfrac{ddz}{dt^2}\right).$$

Praeterea etiam perspicuum est illas functiones *x*, *y*, *z* hac praeditas esse debere proprietate, ut evanescente tempore *t* fiat

$$x = X,\ y = Y \ \text{ et }\ z = Z,$$

ac facto $t = 0$, formulae modo exhibitae dabunt celeritates et accelerationes initiales eiusdem elementi, dum adhuc erat in *Z.*

Denotet porro *q* densitatem, quam nostrum elementum elapso tempore $= t$ in *z* habebit, eritque etiam *q* functio quatuor variabilium *X, Y, Z* et *t,* unde ad quodvis tempus cuiusque clementi densitatem definire licebit.

Sit denique *p* pressio conveniens elemento in *z* versanti elapso tempore *t*, atque etiam haec quantitas erit functio quatuor variabilium *X, Y, Z* et *t*.

His positis tota motus determinatio huc redit, ut, quales sint functiones istae quinque quantitates *x, y, z, q* et *p* quatuor variabilium *X, Y, Z* et *t*, investigetur.

COROLLARIUM 1

101. Cum *x* sit functio quatuor variabilium *X, Y, Z* et *t*, erit eius differentiale completum seu ex variatione omnium natum

$$dx = dX\left(\tfrac{dx}{dX}\right)+dY\left(\tfrac{dx}{dY}\right)+dZ\left(\tfrac{dx}{dZ}\right)+dt\left(\tfrac{dx}{dt}\right),$$

quod idem simili modo de reliquis functionibus *y, z, p* et *q* est tenendum.

COROLLARIUM 2

102. Quodsi idem elementum tempusculo *dt* ex *z* in *z′* pervenire ponamus, ob *X, Y, Z* pro constantibus habendas erunt coordinatae hunc locum *z′* determinantes:

$$Ox' = x + dt\left(\tfrac{dx}{dt}\right),\ \ x'y' = y+dt\left(\tfrac{dy}{dt}\right),\ \ y'z' = z+dt\left(\tfrac{dz}{dt}\right),$$

tum vero densitas in $z' = q+dt\left(\tfrac{dq}{dt}\right)$ et pressio ibidem $= p+dt\left(\tfrac{dp}{dt}\right)$.

COROLLARIUM 3

103. Sin autem quaeratur, ubi post idem tempus *t* aliud elementum, quod initio erat in *Z′*, loco ipsi *Z* proximo deprehendatur, ac pro *Z′* coordinatae sint $X+dX$, $Y+dY$ et $Z+dZ$, si locus quaesitus statuatur in *z′*, erit pro eo

$$Ox' = x+dX\left(\tfrac{dx}{dX}\right)+dY\left(\tfrac{dx}{dY}\right)+dZ\left(\tfrac{dx}{dZ}\right),$$

quod etiam tum de binis reliquis coordinatis *x′y′* et *y′z′* quam pro densitate et pressione est intelligendum.

SCHOLION

104. Ut nunc tam densitatis quam pressionis variationem, dum idem elementum fluidi tempusculo *dt* ulterius progreditur, definire queamus, necesse est, ut simul in statu initiali duo elementa proxima in *Z* et *Z′* contemplemur eorumque situm mutuum post tempus *t* investigemus. Hac enim ratione diiudicare licebit, quanto deinceps tempusculo *dt* vel propius ad se invicem accedant vel longius recedant, quia illo casu densitas crescit, hoc vero decrescit. Verum hic ipse binorum elementorum accessus vel recessus plurimum ab eorum situ mutuo pendent, fierique adeo potest, ut in eadem fluidi massa infinite parva

bina ad se invicem accedant, dum alia recedant. Quamobrem hoc iudicium eodem modo est instituendum, uti supra fecimus, dum translatio cuiusdam massae infinite parvae consideratur, in quo negotio tamen bina elementa proxima simul perpendi debent, quod idem quoque ad pressionis investigationem requiritur, quem in finem sequens problema propono.

PROBLEMA 37

105. *Elapso tempore t si detur densitas q et pressio p* (Fig. 31) *elementi in z versantis, quod initio fuerat in Z, pro eodem tempore invenire densitatem et pressionem alius elementi ipsi proximi in z'.*

SOLUTIO

Pro loco elementi propositi in z sint coordinatae $Ox = x,\ xy = y,\ yz = z$;
pro elemento autem ipsi proximo in z' sint

$$Ox = x+\alpha,\ x'y' = y+\beta,\ y'z' = z+\gamma,$$

existentibus particulis $\alpha,\ \beta, \gamma$ infinite parvis. Iam pro loco Z, ubi elementum z initio fuerat, positis coordinatis $OX = X,\ XY = Y$ et $YZ = Z$, sit Z' locus, ubi alterum elementum z' initio haeserat, pro eoque coordinatae
$OX' = X + dX,\ X'Y' = Y + dY,\ Y'Z' = Z + dZ,$ quae differentialia iam dX, dY, dZ per illas particulas datas $\alpha,\ \beta,\ \gamma$ definiri oportet. Vicissim autem ex paragrapho 104 habemus:

$$\alpha = dX\left(\tfrac{dx}{dX}\right)+dY\left(\tfrac{dx}{dY}\right)+dZ\left(\tfrac{dx}{dZ}\right)$$
$$\beta = dX\left(\tfrac{dy}{dX}\right)+dY\left(\tfrac{dy}{dY}\right)+dZ\left(\tfrac{dy}{dZ}\right)$$
$$\gamma = dX\left(\tfrac{dz}{dX}\right)+dY\left(\tfrac{dz}{dY}\right)+dZ\left(\tfrac{dz}{dZ}\right).$$

Hinc ergo fit

$$\alpha\left(\tfrac{dy}{dZ}\right)-\beta\left(\tfrac{dx}{dZ}\right) = dX\left(\left(\tfrac{dx}{dX}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dy}{dX}\right)\left(\tfrac{dx}{dZ}\right)\right)+dY\left(\left(\tfrac{dx}{dY}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dy}{dY}\right)\left(\tfrac{dx}{dZ}\right)\right)$$
$$\beta\left(\tfrac{dz}{dZ}\right)-\gamma\left(\tfrac{dy}{dZ}\right) = dX\left(\left(\tfrac{dy}{dX}\right)\left(\tfrac{dz}{dZ}\right)-\left(\tfrac{dz}{dX}\right)\left(\tfrac{dy}{dZ}\right)\right)+dY\left(\left(\tfrac{dy}{dY}\right)\left(\tfrac{dz}{dZ}\right)-\left(\tfrac{dz}{dY}\right)\left(\tfrac{dy}{dZ}\right)\right).$$

Unde si brevitatis gratia ponamus:

$$\left.\begin{aligned}
&+\left(\tfrac{dx}{dX}\right)\left[\left(\tfrac{dy}{dY}\right)\left(\tfrac{dz}{dZ}\right)-\left(\tfrac{dz}{dY}\right)\left(\tfrac{dy}{dZ}\right)\right]\\
&+\left(\tfrac{dx}{dY}\right)\left[\left(\tfrac{dz}{dX}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dy}{dX}\right)\left(\tfrac{dz}{dZ}\right)\right]\\
&+\left(\tfrac{dx}{dZ}\right)\left[\left(\tfrac{dy}{dX}\right)\left(\tfrac{dz}{dY}\right)-\left(\tfrac{dz}{dX}\right)\left(\tfrac{dy}{dY}\right)\right]
\end{aligned}\right\} = K,$$

colligitur fore

$$dX = \frac{1}{K}\left\{\begin{array}{l} +\alpha\left[\left(\frac{dy}{dY}\right)\left(\frac{dz}{dZ}\right)-\left(\frac{dz}{dY}\right)\left(\frac{dy}{dZ}\right)\right] \\ +\beta\left[-\left(\frac{dx}{dY}\right)\left(\frac{dz}{dZ}\right)+\left(\frac{dz}{dY}\right)\left(\frac{dx}{dZ}\right)\right] \\ +\gamma\left[\left(\frac{dx}{dY}\right)\left(\frac{dy}{dZ}\right)-\left(\frac{dy}{dY}\right)\left(\frac{dx}{dZ}\right)\right] \end{array}\right\}$$

$$dY = \frac{1}{K}\left\{\begin{array}{l} +\alpha\left[\left(\frac{dz}{dX}\right)\left(\frac{dy}{dZ}\right)-\left(\frac{dy}{dX}\right)\left(\frac{dz}{dZ}\right)\right] \\ +\beta\left[\left(\frac{dx}{dX}\right)\left(\frac{dz}{dZ}\right)-\left(\frac{dz}{dX}\right)\left(\frac{dx}{dZ}\right)\right] \\ +\gamma\left[-\left(\frac{dx}{dX}\right)\left(\frac{dy}{dZ}\right)+\left(\frac{dy}{dX}\right)\left(\frac{dx}{dZ}\right)\right] \end{array}\right\}$$

$$dZ = \frac{1}{K}\left\{\begin{array}{l} +\alpha\left[-\left(\frac{dy}{dY}\right)\left(\frac{dz}{dX}\right)+\left(\frac{dy}{dX}\right)\left(\frac{dz}{dY}\right)\right] \\ +\beta\left[\left(\frac{dz}{dX}\right)\left(\frac{dx}{dY}\right)-\left(\frac{dx}{dX}\right)\left(\frac{dz}{dY}\right)\right] \\ +\gamma\left[\left(\frac{dx}{dX}\right)\left(\frac{dy}{dY}\right)-\left(\frac{dy}{dX}\right)\left(\frac{dx}{dY}\right)\right] \end{array}\right\}.$$

Inventis nunc his differentialibus pro loco z' habebimus

$$\text{densitatem} = q + dX\left(\frac{dq}{dX}\right) + dY\left(\frac{dq}{dY}\right) + dz\left(\frac{dq}{dZ}\right)$$

et

$$\text{pressionem} = p + dX\left(\frac{dp}{dX}\right) + dY\left(\frac{dp}{dY}\right) + dz\left(\frac{dp}{dZ}\right).$$

COROLLARIUM 1

106. Quantitas illa *K*, quae in his formulis denominatorem constituit, facta evolutione ita exprimitur:

$$\begin{aligned} K = &+\left(\frac{dx}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dz}{dZ}\right)+\left(\frac{dz}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dy}{dZ}\right)+\left(\frac{dy}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dx}{dZ}\right) \\ &-\left(\frac{dx}{dX}\right)\left(\frac{dz}{dY}\right)\left(\frac{dy}{dZ}\right)-\left(\frac{dz}{dX}\right)\left(\frac{dy}{dY}\right)\left(\frac{dx}{dZ}\right)-\left(\frac{dy}{dX}\right)\left(\frac{dx}{dY}\right)\left(\frac{dz}{dZ}\right), \end{aligned}$$

quae expressio iam immunis est ab ordine coordinatarum.

COROLLARIUM 2

107. Si ad calculum contrahendum ponamus:

$$\left(\frac{dx}{dX}\right)=A,\quad \left(\frac{dx}{dY}\right)=D,\quad \left(\frac{dx}{dZ}\right)=G,$$
$$\left(\frac{dy}{dY}\right)=B,\quad \left(\frac{dy}{dZ}\right)=E,\quad \left(\frac{dy}{dX}\right)=H,$$
$$\left(\frac{dz}{dZ}\right)=C,\quad \left(\frac{dz}{dX}\right)=F,\quad \left(\frac{dz}{dY}\right)=I,$$

magis perspicua evadet haec forma:

$$K=ABC+DEF+GHI-AEI-BFG-CDH.$$

COROLLARIUM 3

108. His porro iisdem adhibendis signis obtinebimus

$$dX=\frac{\alpha(BC-El)+\beta(GI-CD)+\gamma(DE-BG)}{K}$$
$$dY=\frac{\alpha(EF-CH)+\beta(AC-FG)+\gamma(GH-AE)}{K}$$
$$dZ=\frac{\alpha(HI-BF)+\beta(DF-AI)+\gamma(AB-DH)}{K}.$$

COROLLARIUM 4

109. Quo hae formulae magis contrahantur, quandoquidem earum amplissimus erit usus, ponamus porro:

$$BC-EI=\mathfrak{A},\quad GI-CD=\mathfrak{D},\quad DE-BG=\mathfrak{G},$$
$$AC-FG=\mathfrak{B},\quad GH-AE=\mathfrak{E},\quad EF-CH=\mathfrak{H},$$
$$AB-DH=\mathfrak{C},\quad HI-BF=\mathfrak{F},\quad DF-AI=\mathfrak{I},$$

ut habeamus

$$dX=\frac{\alpha\mathfrak{A}+\beta\mathfrak{D}+\gamma\mathfrak{G}}{K},\; dY=\frac{\alpha\mathfrak{H}+\beta\mathfrak{B}+\gamma\mathfrak{E}}{K},\; dZ=\frac{\alpha\mathfrak{F}+\beta\mathfrak{I}+\gamma\mathfrak{C}}{K}.$$

PROBLEMA 38

110. *Elapso tempore t, si elementum in z' ipsi elemento in z proximum concipiatur, huius elementi z' motum per ternas celeritates secundum coordinatarum Ox, xy et yz directiones exhibere.*

SOLUTIO

Sint elementi in *z* ternae celeritates ut supra *u, v, w*, ac vidimus ex receptis hic denominationibus fore

$$u=\left(\tfrac{dx}{dt}\right),\quad v=\left(\tfrac{dy}{dt}\right),\quad w=\left(\tfrac{dz}{dt}\right),$$

eruntque hae celeritates pariter tanquam functiones quatuor variabilium *X, Y, Z* et *t* spectandae. Maneant iam ut ante puncti z' coordinatae $x+\alpha,\ y+\beta,\ z+\gamma$ sitque Z' eius locus in principio, et pro eo differentialia *dX, dY, dZ* ex datis $\alpha,\ \beta,\ \gamma$ per problema praecedens determinentur: quo facto erunt puncti proximi z' celeritates

$$\text{along } Ox'=u+dX\left(\tfrac{du}{dX}\right)+dY\left(\tfrac{du}{dY}\right)+dz\left(\tfrac{du}{dZ}\right)$$
$$\text{along } x'y'=v+dX\left(\tfrac{dv}{dX}\right)+dY\left(\tfrac{dv}{dY}\right)+dz\left(\tfrac{dv}{dZ}\right)$$
$$\text{along } y'z'=w+dX\left(\tfrac{dw}{dX}\right)+dY\left(\tfrac{dw}{dY}\right)+dz\left(\tfrac{dw}{dZ}\right)$$

atque elisis litteris *u*, *v*, *w* eae ita se habebunt

$$\text{secundum } Ox',\ =\left(\tfrac{dx}{dt}\right)+dX\left(\tfrac{ddx}{dtdX}\right)+dY\left(\tfrac{ddx}{dtdY}\right)+dZ\left(\tfrac{ddx}{dtdZ}\right)$$
$$\text{secundum } x'y'\ =\left(\tfrac{dy}{dt}\right)+dX\left(\tfrac{ddy}{dtdX}\right)+dY\left(\tfrac{ddy}{dtdY}\right)+dZ\left(\tfrac{ddy}{dtdZ}\right)$$
$$\text{secundum } y'z'\ =\left(\tfrac{dz}{dt}\right)+dX\left(\tfrac{ddz}{dtdX}\right)+dY\left(\tfrac{ddz}{dtdY}\right)+dZ\left(\tfrac{ddz}{dtdZ}\right),$$

ubi loco *dX, dY, dZ* valores ante inventos per $\alpha,\ \beta,\ \gamma$ scribi oportet.

SCHOLION

111. Transferamus haec ad figuram 23, ubi *Z* sit id punctum, quod modo in *z* (Fig. 31) consideravimus, cuius ergo ternae celeritates sunt

$$u=\left(\tfrac{dx}{dt}\right),\quad v=\left(\tfrac{dy}{dt}\right),\quad w=\left(\tfrac{dz}{dt}\right),$$

loco puncti autem z' illi proximi successive consideremus tria puncta *L, M, N,* pro quibus sit $ZL=\alpha,\ ZM=\beta,\ ZN=\gamma.$ Pro puncto ergo L in valoribus supra pro differentialibus *dX, dY, dZ* inventis poni debet $\beta=0,\ \gamma=0$, pro puncto *M,* $\alpha=0,\ \gamma=0$ et pro puncto *N,* $\alpha=0,\ \beta=0$. Quare puncti *L* ternae celeritates erunt

supra erat

$$\text{sec. } OX = \left(\frac{dx}{dt}\right) + \frac{\alpha\mathfrak{A}}{K}\left(\frac{ddx}{dtdX}\right) + \frac{\alpha\mathfrak{H}}{K}\left(\frac{ddx}{dtdY}\right) + \frac{\alpha\mathfrak{F}}{K}\left(\frac{ddx}{dtdZ}\right) \quad u + dx\left(\frac{du}{dx}\right)$$

$$\text{sec. } XY = \left(\frac{dy}{dt}\right) + \frac{\alpha\mathfrak{A}}{K}\left(\frac{ddy}{dtdX}\right) + \frac{\alpha\mathfrak{H}}{K}\left(\frac{ddy}{dtdY}\right) + \frac{\alpha\mathfrak{F}}{K}\left(\frac{ddy}{dtdZ}\right) \quad v + dx\left(\frac{dv}{dx}\right)$$

$$\text{sec. } YZ = \left(\frac{dz}{dt}\right) + \frac{\alpha\mathfrak{A}}{K}\left(\frac{ddz}{dtdX}\right) + \frac{\alpha\mathfrak{H}}{K}\left(\frac{ddz}{dtdY}\right) + \frac{\alpha\mathfrak{F}}{K}\left(\frac{ddz}{dtdZ}\right) \quad w + dx\left(\frac{dw}{dx}\right).$$

Puncti autem *M* celeritates erunt supra erat

$$\text{sec. } OX = \left(\frac{dx}{dt}\right) + \frac{\beta}{K}\left[\mathfrak{D}\left(\frac{ddx}{dtdX}\right) + \mathfrak{B}\left(\frac{ddx}{dtdY}\right) + \mathfrak{F}\left(\frac{ddx}{dtdZ}\right)\right] \quad u + dy\left(\frac{du}{dy}\right)$$

$$\text{sec. } XY = \left(\frac{dy}{dt}\right) + \frac{\beta}{K}\left[\mathfrak{D}\left(\frac{ddy}{dtdX}\right) + \mathfrak{B}\left(\frac{ddy}{dtdY}\right) + \mathfrak{F}\left(\frac{ddy}{dtdZ}\right)\right] \quad v + dy\left(\frac{dv}{dy}\right)$$

$$\text{sec. } YZ = \left(\frac{dz}{dt}\right) + \frac{\beta}{K}\left[\mathfrak{D}\left(\frac{ddz}{dtdX}\right) + \mathfrak{B}\left(\frac{ddz}{dtdY}\right) + \mathfrak{F}\left(\frac{ddz}{dtdZ}\right)\right] \quad w + dy\left(\frac{dw}{dy}\right).$$

ac denique puncti *N* celeritates supra erat

$$\text{sec. } OX = \left(\frac{dx}{dt}\right) + \frac{\gamma}{K}\left[\mathfrak{G}\left(\frac{ddx}{dtdX}\right) + \mathfrak{E}\left(\frac{ddx}{dtdY}\right) + \mathfrak{C}\left(\frac{ddx}{dtdZ}\right)\right] \quad u + dz\left(\frac{du}{dz}\right)$$

$$\text{sec. } XY = \left(\frac{dy}{dt}\right) + \frac{\gamma}{K}\left[\mathfrak{G}\left(\frac{ddy}{dtdX}\right) + \mathfrak{E}\left(\frac{ddy}{dtdY}\right) + \mathfrak{C}\left(\frac{ddy}{dtdZ}\right)\right] \quad v + dz\left(\frac{dv}{dz}\right)$$

$$\text{sec. } YZ = \left(\frac{dz}{dt}\right) + \frac{\gamma}{K}\left[\mathfrak{G}\left(\frac{ddz}{dtdX}\right) + \mathfrak{E}\left(\frac{ddz}{dtdY}\right) + \mathfrak{C}\left(\frac{ddz}{dtdZ}\right)\right] \quad w + dz\left(\frac{dw}{dz}\right).$$

His autem formulis notatis sequens problema haud difficulter solvetur, si problema 19 in subsidium vocemus.

PROBLEMA 39

112. *Positis quae hactenus sunt explicata, elementi fluidi* (Fig. 23) *figuram pyramidalem habentis ZLMN, translationem tempusculo dt factam investigare et densitatis incrementum definire.*

SOLUTIO

Problema hoc prorsus convenit cum superiori (12), unde eandem quoque solutionem habebit, simodo, quae hic in designatione sunt mutata, probe observentur. Primo scilicet pyramidis latera, quae ibi erant *dx, dy, dz,* hic sunt α, β, γ ; deinde celeritates *u*, *v*, *w* hic designantur per $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, $\left(\frac{dz}{dt}\right)$, et formulae differentiales $\left(\frac{du}{dx}\right)$, $\left(\frac{dv}{dy}\right)$, $\left(\frac{dw}{dz}\right)$ ex paragrapho praecedente facile decerpuntur. Hinc cum istius pyramidis volumen in *Z* esset $\frac{1}{6}\alpha\beta\gamma$, post translationem eius volumen erit

$$\frac{1}{6}\alpha\beta\gamma+\frac{1}{6}\frac{\alpha\beta\gamma dt}{K}\left\{\begin{matrix}\mathfrak{A}\left(\frac{ddx}{dtdX}\right)+\mathfrak{H}\left(\frac{ddx}{dtdY}\right)+\mathfrak{F}\left(\frac{ddx}{dtdZ}\right)\\ \mathfrak{D}\left(\frac{ddy}{dtdX}\right)+\mathfrak{B}\left(\frac{ddy}{dtdY}\right)+\mathfrak{J}\left(\frac{ddy}{dtdZ}\right)\\ \mathfrak{G}\left(\frac{ddz}{dtdX}\right)+\mathfrak{E}\left(\frac{ddz}{dtdY}\right)+\mathfrak{C}\left(\frac{ddz}{dtdZ}\right)\end{matrix}\right\}.$$

Dum autem haec pyramis erat in Z, eius densitas erat q, post tempusculum autem dt eiusdem particulae densitas per hypotheses hic factas est $q+dt\left(\frac{dq}{dt}\right)$. Quare cum utrumque volumen per suam densitatem multiplicatum eandem massam praebere debeat, sequens hinc nascitur aequatio densitatis rationem determinans:

$$\left.\begin{matrix} & \mathfrak{A}\left(\frac{ddx}{dtdX}\right)+\mathfrak{H}\left(\frac{ddx}{dtdY}\right)+\mathfrak{F}\left(\frac{ddx}{dtdZ}\right)\\ \frac{K}{q}\left(\frac{dq}{dt}\right)+ & \mathfrak{D}\left(\frac{ddy}{dtdX}\right)+\mathfrak{B}\left(\frac{ddy}{dtdY}\right)+\mathfrak{J}\left(\frac{ddy}{dtdZ}\right)\\ & \mathfrak{G}\left(\frac{ddz}{dtdX}\right)+\mathfrak{E}\left(\frac{ddz}{dtdY}\right)+\mathfrak{C}\left(\frac{ddz}{dtdZ}\right)\end{matrix}\right\}=0,$$

ubi litterarum maiuscularum hic occurrentium valores ex paragraphis 107 et 109 desumi debent. Cum igitur inde sit

$$\left(\frac{ddx}{dtdX}\right)=\left(\frac{dA}{dt}\right),\quad \left(\frac{ddx}{dtdY}\right)=\left(\frac{dD}{dt}\right),\quad \left(\frac{ddx}{dtdZ}\right)=\left(\frac{dG}{dt}\right)\ \text{ etc.},$$

si loco litterarum germanicarum valores ex paragrapho 109 scribantur, erit

$$\left.\begin{matrix} & +(BC-EI)\left(\frac{dA}{dt}\right)+(EF-CH)\left(\frac{dD}{dt}\right)+(HI-BF)\left(\frac{dG}{dt}\right)\\ \frac{K}{q}\left(\frac{dq}{dt}\right) & +(GI-CD)\left(\frac{dH}{dt}\right)+(AC-FG)\left(\frac{dB}{dt}\right)+(DF-AI)\left(\frac{dE}{dt}\right)\\ & +(DE-BG)\left(\frac{dF}{dt}\right)+(GH-AE)\left(\frac{dI}{dt}\right)+(AB-DH)\left(\frac{dC}{dt}\right)\end{matrix}\right\}=0,$$

quae expressio si comparetur cum valore litterae K, qui est

$$K=ABC+DEF+GHI-AEI-BFG-CDH,$$

facile perspicitur illius membrum posterius reduci ad $\left(\frac{dK}{dt}\right)$, ita ut iam solutio problematis perducatur ad hanc simplicem aequationem

$$\frac{K}{q}\left(\frac{dq}{dt}\right)+\left(\frac{dK}{dt}\right)=0\ \text{ seu }\ K\left(\frac{dq}{dt}\right)+q\left(\frac{dK}{dt}\right)=0$$

vel ad hanc concinniorem

$$\left(\frac{d\cdot Kq}{dt}\right)=0.$$

Ex quo intelligimus *Kq* eiusmodi esse functionem, cuius differentiale ex sola variabilitate temporis *t* ortum evanescat, seu quae omni tempore maneat eadem. Manifestum ergo est hoc fieri non posse, nisi *Kq* sit functio tantum harum trium variabilium *X, Y, Z* tempore excluso, unde problematis solutio continebitur hac formula

$$Kq = f:(X,Y,Z).$$

COROLLARIUM 1

113. Quantitas *K* determinatur per conditiones, quibus coordinatae *x, y, z* post tempus *t* a principalibus *X, Y, Z* in statu initiali pendent, quemadmodum eius forma paragrapho 106 exhibita declarat. Cum igitur quantitates *x, y, z* necessario tempus *t* involvant, id ita fieri necesse est, ut ex forma *Kq* temporis ratio penitus egrediatur.

COROLLARIUM 2

114. Quodsi ergo densitas fluidi *q* fuerit quantitas constans, tum ipsa forma *K* a tempore debet esse immunis. Sin autem densitas *q* fuerit variabilis, eius quantitas ad quodvis tempus t assignari poterit, cum sit

$$q = \frac{f:(X,Y,Z)}{K}.$$

COROLLARIUM 3

115. Hic autem imprimis notari oportet quantitatem *q*, dum coordinatae principales *X, Y, Z* manent eaedem, perpetuo eiusdem fluidi elementi densitatem exprimere; quod ergo elementum si nullam mutationem in densitate patiatur, manebit *q* quantitas constans, etiamsi reliquae fluidi partes diversas habeant densitates.

COROLLARIUM 4

116. Si ergo fluidum sit heterogeneum seu ex fluidis pluribus diversae naturae mixtum, haec ratio motum definiendi plurimum praestat praecedenti; quoniam ibi quantitas q non ad eandem fluidi particulam, sed ad eundem locum refertur, ita ut omnium particularum, quae successive per idem punctum transeunt, densitates exprimat.

SCHOLION 1

117. In solutione huius problematis merito desideratur, quod demum per plures ambages ad postremam simplicitatem sit perducta; et quia tandem quasi casu ad aequationem differentialem integrabilem est perventum, nullum est dubium, quin alia detur via, quae immediate ad istam formulam integralem perducat. In ambages autem illas ideo incidi, quod solutionem eodem modo, quo supra sum usus, adstruere volui, cum tamen ratio, qua hic motum intuemur, aliam viam commonstret ad solutionem perveniendi. Consideretur enim statim in statu initiali molecula fluidi (Fig. 23) sub figura pyramidali

ZLMN et quaeratur eius translatio tempore finito $=t$ facta. Tum igitur perveniat in situm *zlmn*, quae figura erit pariter pyramis utcunque irregularis; si enim quis dubitet, an post tempus finitum *t* hedrae huius pyramidis etiamnunc pro planis tuto haberi queant? is priorem pyramidem *ZLMN* adhuc infinities minorem accipiat, et quantumvis ante hedrae fuerint convexae seu concavae, nunc agnoscere debebit eas infinite propius ad planitiem reduci atque adeo pro planis haberi oportere. Quoniam igitur statim pyramidem principalem *ZLMN* infinite parvam assumimus, recte quoque in statu translato figuram *zlmn* pro vero pyramide habebimus. Istius ergo pyramidis *zlmn* investigetur volumen, quod, cum eius densitas iam in *z* elapso tempore $=t$ statuatur $=q$, si per *q* multiplicetur, prodibit massa istius moleculae, quae quia perpetuo manet eadem, eiusmodi sit functio necesse est, quae a tempore plane non pendeat; seu ista massa erit functio trium quantitatum *X*, *Y*, *Z* tantum excluso tempore *t*. Quare cum solutio praecedens tandem dederit $Kq=f:(X,\ Y,\ Z)$, perspicuum est, si methodo hic indicata utamur, volumen illius moleculae ipsi quantitati *K* proportionale inveniri debere.

SCHOLION 2

118. Haec consideratio omnino est digna, quam diligentius prosequamur. Posito ergo pro pyramide principali $OX=X$, $XY=Y$, $YZ=Z$, tum vero $ZL=dX$, $ZM=dY$ et $ZN=dZ$, ut eius volumen sit $\frac{1}{6}dXdYdZ$; sint pro puncto *z* in pyramide translata coordinatae $Ox=x$, $xy=y$ et $yz=z$. Nunc consideretur, si punctum in statu initiali his coordinatis $X+dX$, $Y+dY$, $Z+dZ$ definiatur, id tempore *t* translatum iri in punctum his coordinatis $x+\alpha$, $y+\beta$, $z+\gamma$ definitum, dum sit:

$$\alpha = AdX+DdY+GdZ,\ \beta = HdX+BdY+EdZ,\ \gamma = FdX+IdY+CdZ.$$

Hinc iam ex quaternis punctis pyramidis principalis, quaterna puncta translatae definiantur, quorum coordinatae ita se habebunt:

$$pro\ z\begin{Bmatrix} Ox=x \\ xy=y \\ yz=z \end{Bmatrix},\qquad pro\ l\begin{Bmatrix} Or=x+AdX \\ rp=y+HdX \\ pl=z+FdX \end{Bmatrix},$$

$$pro\ m\begin{Bmatrix} Os=x+DdY \\ sq=y+Bdy \\ qm=z+IdY \end{Bmatrix},\qquad pro\ n\begin{Bmatrix} Ot=x+GdZ \\ to=y+EdZ \\ on=z+CdZ \end{Bmatrix}.$$

Ex his iam colliguntur latera pyramidis translatae:

$$zl^2 = (AA+HH+FF)dX^2$$
$$zm^2 = (BB+II+DD)dY^2$$
$$zn^2 = (CC+GG+EE)dZ^2$$
$$lm^2 = (AdX-DdY)^2+(HdX-BdY)^2+(FdX-IdY)^2$$
$$ln^2 = (A\,dX-GdZ)^2+(HdX-EdZ)^2+(FdX-CdZ)^2$$
$$mn^2 = (DdY-GdZ)^2+(BdY-EdZ)^2+(IdY-CdZ)^2$$

Hinc porro secundum praecepta paragraphi 12 definiantur angulorum ad z cosinus:

$$\cos lzm = \nu = \tfrac{AD+BH+FI}{zl\cdot zm}dXdY$$
$$\cos lzn = \mu = \tfrac{AG+EH+FI}{zm\cdot zn}dXdZ$$
$$\cos mzn = \lambda = \tfrac{DG+BE+CI}{zm\cdot zn}dYdZ.$$

Quibus valoribus substitutis volumen pyramidis *zlmn* deducitur

$$=\tfrac{1}{6}dXdYdZ\sqrt{\begin{cases}+(AA+HH+FF)(BB+II+DD)(CC+GG+EE)\\-(AD+BH+FI)^2(CC+GG+EE)\\-(AG+EH+CF)^2(BB+II+DD)\\-(DG+BE+CI)^2(AA+HH+FF)\\+2(AD+BH+FI)(AG+EH+CF)(DG+BE+CI)\end{cases}}$$

quae forma post signum radicale si evolvatur, praecise quadrato quantitatis K aequalis deprehenditur: ita ut hoc volumen sit $=\tfrac{1}{6}KdXdYdZ$ eiusque propterea massa $=\tfrac{1}{6}KqdXdYdZ$, unde quantitas Kq a tempore t neutiquam pendere debet.

PROBLEMA 40

119. *Si fluidum a viribus quibuscunque acceleratricibus P, Q, R secundum directiones ternarum coordinatarum sollicitetur, aequationem investigare, qua pressio in singulis fluidi elementis determinatur.*

SOLUTIO

Elapso tempore t consideretur (Fig. 25) molecula fluidi quaecunque in Z, cui calculi gratia figura parallelepipedi *ZLMN zlmn* tribuatur, ac pro puncto Z positis coordinatis $OX = x$, $XY = y$, $YZ = z$ sint latera huius parallelepipedi $ZL = \alpha$, $ZM = \beta$ et $Zz = \gamma$, ut eius volumen sit $=\alpha\beta\gamma$ et massa $=q\alpha\beta\gamma$. Iam posita pressione in z = p, quae est functio quantitatum *X*, *Y*, *Z* et temporis *t*, ubi *X*, *Y*, *Z* sunt coordinatae eius puncti, ubi elementum, quod iam est in *Z*, initio erat situm. Quo igitur hinc pressio in *L* definiatur, cuius elementi coordinatae sunt $x+\alpha,\ y,\ z,$ videndum est, ubi hoc elementum initio fuerat, et ex praecedentibus eius loci coordinatae erant

$$X+\tfrac{\alpha(BC-EI)}{K},\quad Y+\tfrac{\alpha(EF-CH)}{K},\quad Z+\tfrac{\alpha(HI-BF)}{K},$$

unde concludimus pressionem in *L* fore:

$$p+\tfrac{\alpha(BC-EI)}{K}\left(\tfrac{dp}{dX}\right)+\tfrac{\alpha(EF-CH)}{K}\left(\tfrac{dp}{dY}\right)+\tfrac{\alpha(HI-BF)}{K}\left(\tfrac{dp}{dY}\right),$$

cuius excessu supra pressionem *p* in Z tota hedra *LN ln* secundum directionem *AO* urgetur. Istius autem hedrae superficies est $=\beta y$, per quam ille excessus multiplicatus dat vim motricem, haecque per massam $q\alpha\beta\gamma$ divisa vim acceleratricem. Quare cum nostra molecula in directione *OA* sollicitetur vi acceleratrice P, si ab hac illa auferatur, remanebit vera vis acceleratrix secundum directionem *AO*. Cum ergo acceleratio sit $=\left(\tfrac{ddx}{dt^2}\right)$, habebitur haec aequatio

$$\left(\tfrac{ddx}{dt^2}\right)=2gP-\tfrac{2g(BC-EI)}{Kq}\left(\tfrac{dp}{dX}\right)-\tfrac{2g(EF-CH)}{Kq}\left(\tfrac{dp}{dY}\right)-\tfrac{2g(HI-BF)}{Kq}\left(\tfrac{dp}{dZ}\right)$$

similique modo pro duabus reliquis directionibus reperitur:

$$\left(\tfrac{ddy}{dt^2}\right)=2\mathrm{g}Q-\tfrac{2g(GI-CD)}{Kq}\left(\tfrac{dp}{dX}\right)-\tfrac{2g(AC-FG)}{Kq}\left(\tfrac{dp}{dY}\right)-\tfrac{2g(DF-AI)}{Kq}\left(\tfrac{dp}{dZ}\right)$$
$$\left(\tfrac{ddz}{dt^2}\right)=2gR-\tfrac{2g(DE-BG)}{Kq}\left(\tfrac{dp}{dX}\right)-\tfrac{2g(GH-AE)}{Kq}\left(\tfrac{dp}{dY}\right)-\tfrac{2g(AB-DB)}{Kq}\left(\tfrac{dp}{dZ}\right).$$

Introducendis ergo brevitatis gratia litteris germanicis ex paragrapho 109 adipiscimur has tres aequationes pro pressione *p* definienda:

$$\mathfrak{A}\left(\tfrac{dp}{dX}\right)+\mathfrak{H}\left(\tfrac{dp}{dY}\right)+\mathfrak{F}\left(\tfrac{dp}{dZ}\right)=KqP-\tfrac{Kq}{2g}\left(\tfrac{ddx}{dt^2}\right)$$
$$\mathfrak{D}\left(\tfrac{dp}{dX}\right)+\mathfrak{B}\left(\tfrac{dp}{dY}\right)+\mathfrak{I}\left(\tfrac{dp}{dZ}\right)=KqQ-\tfrac{Kq}{2g}\left(\tfrac{ddy}{dt^2}\right)$$
$$\mathfrak{G}\left(\tfrac{dp}{dX}\right)+\mathfrak{E}\left(\tfrac{dp}{dY}\right)+\mathfrak{C}\left(\tfrac{dp}{dZ}\right)=KqR-\tfrac{Kq}{2g}\left(\tfrac{ddz}{dt^2}\right).$$

Ut hinc formulam definiamus, multiplicemus primam per $\mathfrak{B}\mathfrak{C}-\mathfrak{E}\mathfrak{I}=K,$ secundam per $\mathfrak{E}\mathfrak{F}-\mathfrak{C}\mathfrak{H}=HK,$ et tertiam per $\mathfrak{H}\mathfrak{I}-\mathfrak{B}\mathfrak{F}=FK,$, ob $A\mathfrak{A}+H\mathfrak{D}+F\mathfrak{C}=K$ reperietur per *KK* dividendo:

$$\left(\frac{dp}{dX}\right)=q\left(AP+HQ+FR\right)-\frac{q}{2g}\left(A\left(\frac{ddx}{dt^2}\right)+H\left(\frac{ddy}{dt^2}\right)+F\left(\frac{ddz}{dt^2}\right)\right)$$

similique modo elicitur:

$$\left(\frac{dp}{dY}\right)=q\left(DP+BQ+IR\right)-\frac{q}{2g}\left(D\left(\frac{ddx}{dt^2}\right)+B\left(\frac{ddy}{dt^2}\right)+I\left(\frac{ddz}{dt^2}\right)\right)$$

$$\left(\frac{dp}{dZ}\right)=q\left(GP+EQ+CR\right)-\frac{q}{2g}\left(G\left(\frac{ddx}{dt^2}\right)+E\left(\frac{ddy}{dt^2}\right)+C\left(\frac{ddz}{dt^2}\right)\right).$$

Multiplicetur porro prima per dX, secunda per dY, tertia per dZ, ut obtineatur differentiale pressionis p, si tempus t constans statuatur, et cum in eadem hypothesi sit:

$$AdX+DdY+GdZ=dx,\ HdX+BdY+EdZ=dy \text{ et } FdX+IdY+CdZ=dz,$$

nostrae tres aequationes in hanc unam coalescent:

$$dp=q\left(Pdx+Qdy+Rdz\right)-\frac{q}{2g}\left(dx\left(\frac{ddx}{dt^2}\right)+dy\left(\frac{ddy}{dt^2}\right)+dz\left(\frac{ddz}{dt^2}\right)\right),$$

in cuius integratione tempus t pro constante est habendum.

COROLLARIUM l

120. Cum x, y, z sint functiones ipsarum X, Y, Z et t, si ponamus differentiale completum

$$dx=AdX+DdY+GdZ+Ldt,$$

erit $\left(\frac{dx}{dt}\right)=L$ ideoque $\left(\frac{ddx}{dt^2}\right)=\left(\frac{dL}{dt}\right)$, loco dx autem in hac aequatione scribi oportet $AdX+DdY+GdZ$ $AdX + DdY + GdZ$, quia in ea tempus constans assumitur.

COROLLARIUM 2

121. Ante autem vidimus, quomodocunque densitas q sit variabilis, quantitatem Kq tempus t non involvere. Cum autem actio S a loco, in quo elementum fluidi post tempus t reperitur, pendeat, ea utique tempus in se includet.

SCHOLION

122. Quoniam etiam in hac solutione ad aequationem multo simpliciorem pertigimus, quam per calculi ambages expectare licebat, nullum est dubium, quin etiam via faciliori et concinniori ad eandem solutionem pertingere liceat. Neque vero facile patet, quomodo ratiocinium eo perducens dirigi conveniat, id quidem perspicuum est formulam $Pdx+Qdy+Rdz$ exprimere differentiale actionis virium in elementum, cuius motum consideramus, prorsus uti in methodo superiori. Verum differentiale dp hic prorsus diversam habet significationem, dum p hic est functio variabilium X, Y, Z et t, hincque

sumendo *t* constans computatur, dum ante *p* fuerat functio quantitatum *x, y,z* et *t*, ex cuius differentiatione sumto quidem *t* item constante, differentiale *dp* capiebatur, quia vero hic ipsae coordinatae *x, y, z* iam tempus *t* involvunt, hoc differentiale ab illo prorsus discrepet necesse est. Tum vero, etsi $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, $\left(\frac{dz}{dt}\right)$
celeritates, quas supra u, v, w vocavimus, exprimunt, tamen hae formulae $\left(\frac{ddx}{dt^2}\right)$, $\left(\frac{ddy}{dt^2}\right)$, $\left(\frac{ddz}{dt^2}\right)$ plurimum discrepant ab $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$, $\left(\frac{dw}{dt}\right)$; denotant enim ipsas accelerationes, quas supra litteris *U, V* et *W* designavimus. Ratio autem discrepantiae manifesto in eo est sita, quod hic universum calculum ad longe alias quaternas variabiles referimus, atque ante fecimus. Unde quidem statim hoc commodi sumus nacti, ut prior aequatio pro densitate inventa integrationem admiserit, contra vero altera pro pressione magis complicata videtur.

PROBLEMA 41

123. *Dato (Fig. 31) fluidi cuiuscunque statu initiali et viribus, quarum actionem sustinet, investigare motum, quo deinceps feretur, eiusque statum ad quodvis tempus.*

SOLUTIO

In statu initiali consideremus fluidi particulam quamcunque in Z, cuius locus definiatur ternis coordinatis $OX = X$, $XY = Y$ et $YZ = Z$: tum vero eiusdem particulae sit densitas $= Q$, pressio vero $= P$. Praeterea autem eius motus ita sit comparatus, ut resolutus praebeat celeritates secundum directiones $OX = U$, $XY = V$ et $YZ = W$. Cum igitur status initialis sit cognitus, erunt *Q, P, U, V, W* functiones datae ternarum variabilium *X, Y, Z.* Elapso iam tempore *t*, eadem particula, quae initio erat in *Z*, pervenerit in *z*, cuius locus similibus coordinatis $Ox = x,\ xy = y$ et $yz = z$ definiatur, quae ergo spectandae sunt ut functiones quatuor variabilium *X, Y, Z* et *t*, ita comparatae, ut posito tempore $t = 0$, abeant in coordinatas initiales *X, Y* et *Z*, ex quo sequitur eodem casu $t = 0$ fore:

$$\left(\frac{dx}{dX}\right)=1,\ \left(\frac{dy}{dX}\right)=0,\ \left(\frac{dz}{dX}\right)=0,$$
$$\left(\frac{dx}{dY}\right)=0,\ \left(\frac{dy}{dY}\right)=1,\ \left(\frac{dz}{dY}\right)=0,$$
$$\left(\frac{dx}{dZ}\right)=0,\ \left(\frac{dy}{dZ}\right)=0,\ \left(\frac{dz}{dZ}\right)=1.$$

Deinde vero eiusdem particulae, dum post tempus = *t* per punctum *z* transit, eius ternae celeritates erunt

$$\text{secundum } Ox=\left(\frac{dx}{dt}\right)=u,\ \text{secundum } xy=\left(\frac{dy}{dt}\right)=v,\ \text{secundum } yz=\left(\frac{dz}{dt}\right)=w,$$

unde evanescente tempore *t* fiat necesse est

$$\left(\tfrac{dx}{dt}\right)=U,\ \left(\tfrac{dy}{dt}\right)=V,\ \left(\tfrac{dz}{dt}\right)=W.$$

Statuatur porro particulae iam per *z* transeuntis densitas *q* et pressio *p*, quae duae quantitates itidem erunt functiones quatuor variabilium *X, Y, Z* et *t*, ita comparatae, ut posito $t=0$ fiat $q=Q$ et $p=P$.

Vires denique acceleratrices, quibus particula in *z* urgetur, reducantur ad has

$$\text{secundum } Ox=\mathfrak{P}\ \text{secundum } xy=\mathfrak{Q},\ \text{secundum } yz=\mathfrak{R}.$$

Quibus positis evidens est cognitionem motus eo redire, ut, quales hae quinque quantitates *x, y, z, q* et *p* sint functiones quatuor variabilium *X, Y, Z* et *t*, definiatur, haecque determinatio ex sequentibus duabus aequationibus est petenda.

Pro priori quaeratur ex variabilibus x, y, z haec quantitas:

$$K=\begin{array}{l}+\left(\tfrac{dx}{dX}\right)\left(\tfrac{dy}{dY}\right)\left(\tfrac{dz}{dZ}\right)+\left(\tfrac{dz}{dX}\right)\left(\tfrac{dx}{dY}\right)\left(\tfrac{dy}{dZ}\right)+\left(\tfrac{dy}{dX}\right)\left(\tfrac{dz}{dY}\right)\left(\tfrac{dx}{dZ}\right)\\-\left(\tfrac{dx}{dX}\right)\left(\tfrac{dz}{dY}\right)\left(\tfrac{dy}{dZ}\right)-\left(\tfrac{dz}{dX}\right)\left(\tfrac{dy}{dY}\right)\left(\tfrac{dx}{dZ}\right)-\left(\tfrac{dy}{dX}\right)\left(\tfrac{dx}{dY}\right)\left(\tfrac{dz}{dZ}\right)\end{array}$$

unde ex ante notatis constat posito $t=0$ fore $K=1$. Cum igitur viderimus in problemate 39 durante motu pro eadem particula quantitatem *Kq* perpetuo eundem valorem conservare, eius valor utique illi aequalis esse debet, quem habebat initio posito $t=0$, tum autem fit $K=1$ et $q=Q$. Quocirca prior aequatio motus determinationem continens erit $Kq=Q$ ideoque $q=\frac{Q}{K}$.

Alteram aequationem in problemate praecedente elicuimus, ubi introducitur littera *g* altitudinem lapsus gravium tempore unius minuti secundi designans, eum in finem, ut tempus *t* in minutis secundis et celeritates per spatia uno minuto secundo percursa exprimi queant. Hinc igitur altera aequatio motus determinationem continens erit:

$$\tfrac{2gdp}{q}=2g\left(\mathfrak{P}dx+\mathfrak{Q}dy+\mathfrak{R}dz\right)-dx\left(\tfrac{ddx}{dt^2}\right)-dy\left(\tfrac{ddy}{dt^2}\right)-dz\left(\tfrac{ddz}{dt^2}\right),$$

in qua aequatione differentiali probe observandum est tempus *t* constans assumi solasque coordinatas initiales *X, Y, Z* ut variabiles tractari. Quare cum *x, y, z* insuper tempus *t* involvant, earum differentialia *dx, dy, dz* huic conditioni conformiter sunt capienda. Cum autem integrale fuerit inventum, loco constantis ei quamcunque temporis functionem adiici conveniet.

COROLLARIUM 1

124. Quemadmodum posterior aequatio ex tribus est nata, ita etiam tres continet determinationes, quibus efficiendum est, ut ea integrabilis evadat. Adiuncta ergo priori

insuperque ex natura fluidi relatione inter densitatem et pressionem, omnino quinque habentur determinationes, ideoque tot, quot opus est ad quinque functiones quaesitas x, y, z, q, p definiendas.

COROLLARIUM 2

125. Integrali autem posterioris aequationis invento, si tum coordinatae X, Y, Z ut constantes spectentur et solum tempus t variabile accipiatur, habebitur totus motus eius particulae fluidi, quae initio erat in Z; indeque ad quodvis tempus tam eius locus et motus quam densitas et pressio assignari poterit.

COROLLARIUM 3

126. Si ista particula, quae initio erat in Z, nullam densitatis mutationem admittat, perpetuo erit $q = Q$ ideoque ex aequatione priori $K = 1$. Hinc ergo loco ipsius K valorem supra assignatum substituendo certa relatio functionum x, y, z definitur, quemadmodum ea a coordinatis principalibus X, Y, Z pendere debent.

SCHOLION 1

127. Si alteram aequationem attentius contemplemur, ex eius forma coniicere licet, quomodo ea ex theoria virium sollicitantium sit deducenda. In statu enim initiali considerentur duo puncta sibi proxima Z et Z', quorum illud coordinatis X, Y, Z, hoc vero istis $X+dX$, $Y+dY$, $Z+dZ$ determinetur. Iam elapso tempore $=t$ haec duo puncta transferantur in z et z', illius coordinatis existentibus x, y, z, huius vero $x+dx$, $y+dy$, $z+dz$, ubi probe notetur haec incrementa dx, dy, dz ex differentiatione functionum x, y, z, dum tempus t constans assumitur, esse capienda, ita ut ex sola variabilitate coordinatarum principalium X, Y, Z resultent. Vocetur nunc intervallum $zz' = ds$, per quod extensa concipiatur molecula fluida figuram habens prismaticam seu cylindricam, cuius basis sit $=\delta\delta$, eritque eius volumen $=\delta\delta ds$ et massa $=q\delta\delta ds$. Quoniam igitur pressio in z ponitur p, erit pressio in $z' = p+dp$, denotante dp id differentiale functionis p, quod ex variabilitate solarum coordinatarum X, Y, Z nascitur tempore t constante assumto. Haec ergo molecula zz' ab excessu pressionis in basi z' supra basin z in directione $z'z$ urgetur vi motrice $=\delta\delta dp$, quae per massam $q\delta\delta ds$ divisa dat vim acceleratricem $\frac{dp}{qds}$ secundum eandem directionem $z'z$. Cum vero adsint vires acceleratrices $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}$ secundum directiones Ox, xy, yz, ex his colligatur vis secundum directionem zz', quae reperitur $\frac{\mathfrak{P}dx+\mathfrak{Q}dy+\mathfrak{R}dz}{ds}$, ita ut iam tota vis acceleratrix secundum directionem zz' sit

$$=\frac{\mathfrak{P}dx+\mathfrak{Q}dy+\mathfrak{R}dz}{ds}-\frac{dp}{qds}.$$

Hac inventa considerentur accelerationes motus, quas secundum directiones

Ox, xy, yz vidimus esse $\left(\frac{ddx}{dt^2}\right)$, $\left(\frac{ddy}{dt^2}\right)$, $\left(\frac{ddz}{dt^2}\right)$, ex iisque colligatur accereratio secundum directionem *zz'*, quae prodit:

$$\frac{dx}{ds}\left(\frac{ddx}{dt^2}\right)+\frac{dy}{ds}\left(\frac{ddy}{dt^2}\right)+\frac{dz}{ds}\left(\frac{ddz}{dt^2}\right),$$

atque ex motus principiis hanc accelerationem aequalem esse oportet vi acceleratrici illi per 2*g* multiplicatae; hincque per *ds* multiplicando ipsa aequatio altera motus naturam continens oritur; quam ergo statim sine tantis ambagibus invenire licuisset. In hoc autem fere inusitato calculi genere maximi certe est momenti eandem aequationem plus uno modo elicuisse, cum hinc natura istius novae analyseos non mediocriter illustretur.

SCHOLION 2

128. Quia hic motus tantum principia tradere constitui, brevibus saltem usum harum formularum ostendam. Primo igitur pro motu progressivo seu parallelo singularum particularum ponamus:

$$x = X + L,\ y = Y + M,\ z = Z + N,$$

existentibus *L, M, N* eiusmodi functionibus ipsius temporis *t* tantum, quae facto $t = 0$ evanescant. Cum igitur sit $\left(\frac{dx}{dX}\right)=1$, $\left(\frac{dy}{dY}\right)=1$, $\left(\frac{dz}{dZ}\right)=1$, reliquae vero formulae differentiales omnes evanescant, erit $K = 1$ et $q = Q$, unde densitas cuiusque elementi manet eadem; seu haec hypothesis ad fluidum pertinet nullius compressionis capax: interim tamen si ex materiis heterogeneis constet, in statu initiali *Q* spectari poterit ut functio ipsarum *X, Y* et *Z*. Agat sola gravitas secundum directionem *zy*, ut sit $\mathfrak{P} = 0$, $\mathfrak{Q} = 0$ et $\mathfrak{R} = -1$, eritque altera aequatio:

$$\frac{2gdp}{Q} = -2gdZ - dX\,\frac{ddL}{dt^2} - dY\,\frac{ddM}{dt^2} - dZ - \frac{ddN}{dt^2},$$

quae aequatio ut possit integrari, densitas *Q* ubique debet esse eadem ideoque $Q = b$, atque integrale erit:

$$\frac{2gdp}{Q} = -2gdZ - dX\,\frac{ddL}{dt^2} - dY\,\frac{ddM}{dt^2} - dZ\,\frac{ddN}{dt^2},$$

nisi ergo motus sit uniformis, suprema superficies horizontalis non erit.

Deinde casum perpendamus, quo singula elementa circa axem verticalem in planis horizonti parallelis revolvuntur. In hunc finem sit angulus θ functio quaecunque temporis *t* et statuatur:

$$x = X\cos\theta - Y\sin\theta,\ \ y = Y\cos\theta + X\sin\theta,\ z = Z\ ,$$

hinc ob

$$\left(\frac{dx}{dX}\right)=\cos\theta,\ \ \left(\frac{dx}{dY}\right)=-\sin\theta,\ \ \left(\frac{dy}{dY}\right)=\cos\theta,\ \ \left(\frac{dy}{dX}\right)=X\sin\theta,\ \left(\frac{dz}{dZ}\right)=1$$

colligitur $K = \cos^2\theta + \sin^2\theta = 1$. Quare ut ante densitas statuatur constans $q = Q = b$. Deinde reperitur:

$$\left(\frac{dx}{dt}\right)=-\left(X\sin\theta+Y\cos\theta\right)\frac{d\theta}{dt},$$
$$\left(\frac{ddx}{dt^2}\right)=-\left(X\sin\theta+Y\cos\theta\right)\frac{dd\theta}{dt^2}+\left(Y\sin\theta-X\cos\theta\right)\frac{d\theta^2}{dt^2},$$
$$\left(\frac{dy}{dt}\right)=-\left(Y\sin\theta-X\cos\theta\right)\frac{d\theta}{dt},$$
$$\left(\frac{ddy}{dt^2}\right)=-\left(Y\sin\theta-X\cos\theta\right)\frac{dd\theta}{dt^2}-\left(X\sin\theta+Y\cos\theta\right)\frac{d\theta^2}{dt^2},$$

unde facta substitutione fit altera aequatio

$$\frac{2g}{b}p=-2gdZ+\left(YdX+XdY\right)X\,\frac{dd\theta}{dt^2}+\left(XdX+YdY\right)\frac{d\theta^2}{dt^2},$$

ubi cum t et θ pro constantibus sint habenda, prodit integrando

$$\frac{2gp}{b}=2g\left(h-Z\right)+XY\,\frac{dd\theta}{dt^2}+\frac{1}{2}\left(XX+YY\right)\frac{d\theta^2}{dt^2}+f:t\,/$$

Cum hic pro θ functionem quamcunque temporis accipere liceat, hic motus multo latius patet eo, quem supra priorem methodum secuti evolvimus, ex quo haec altera methodus insigni usu praedita est censenda.

SCHOLION 3

129. Casus hic allatos fusius non prosequor, cum hic ideam tantum circa applicationem huius posterioris methodi exhibere sit propositum, praeterquam quod uberior evolutio tam istius quam praecedentis methodi insignem analyseos promotionem exigit, antequam quicquam cum successu sperare liceat. Cum enim haec universa analysis circa functiones quatuor variabilium versetur, dum ea pars, quae in functionibus duarum tantum variabilium consistit, vix adhuc excoli est coepta: temere certe tam arduum negotium subito susciperetur. Quod igitur quasi per gradus ascendendo ad hanc motus fluidorum investigationem in genere tendamus, a casibus facilioribus, ubi pauciores variabiles occurrunt, inchoandum videtur. Atque hic ad geometriae similitudinem Theoriam motus fluidorum in tres partes linearem, planam et solidam commodissime partimur; quarum binae priores, etsi per abstractionem in subsidium tertiae sunt formatae, tamen proprio usu neutiquam destituuntur. Pleraque enim, quae adhuc de motu fluidorum sunt explorata, ad fluxum per canales seu tubos referuntur, qui etsi non angustissimi assumi soient, tamen fluidum non aliter per eos moveri concipitur, ac si tales essent, siquidem in singulis sectionibus transversis nulla motus inaequalitas admittitur. Ita merito motum fluidi per huiusmodi tubos motum linearem appellare licet. Secunda pars est plana vel potius superficialis, qua fluido moto quasi duae tantum dimensiones tribuuntur, dum scilicet tertia dimensio nulli motus inaequalitati obnoxia consideratur. His ergo duabus demum partibus accuratius evolutis tractationem plenam per omnes tres dimensiones maiori fiducia adgredi poterimus.