CONCERNING ALGEBRAIC CURVES

ALL OF WHICH MAY BE MEASURED BY CIRCULAR ARCS

Shown to the Academy on the 20th August, 1781; [E783]

Mémoires de l'académie des sciences de St. Pétersbourg 11, 1830, p. 114-124

1. I had no doubt some years before to have offered this proposition itself as a significant theorem in print: *so that besides the circle no algebraic curve may be given, of which equal circular arcs may be able to be assigned for all its arcs.* Also I had been led astray by several plausible enough reasons, which confirmed me in this opinion, though hitherto I have considered these properly to be most distant from a perfect demonstration. But the particular reason for me was, after I had taken the greatest pains with this argument, because I was unable still to elicit a curve of this kind.

2. On account of which, since recently I was occupied investigating in general two algebraic curves in a similar argument, which satisfy a common rectification, and thence I could investigate infinitely many algebraic curves, the length of which it would be possible to measure by parabolic arcs, then truly also infinitely many algebraic curves satisfy the same rectification as the ellipse; I have been astonished especially, because, even if I may change the ellipse to a circle, the curves found nevertheless shall be different from the circle. Therefore here solemnly retracting my opinion, I may establish an easy method, with the aid of which innumerable algebraic curves can be found, of which all the circular arcs are equal.

3. Therefore with the circle with centre c proposed , Fig. 1 and 2, with radius $ca=1$, we may consider the description of the curve *AZ* thus requiring to be prepared, so that its indefinite arc *AZ* always shall be equal to the indefinite arc of this circle *az,* with which called $az=\omega$, and also the arc shall be $AZ=\omega$. Now I refer this curve to a certain fixed centre *C*, and I will investigate its nature by the equation between the distance $CZ=z$ and the angle $ACZ=\varphi$, so that the question may be satisfied. Therefore since this arc hence shall be

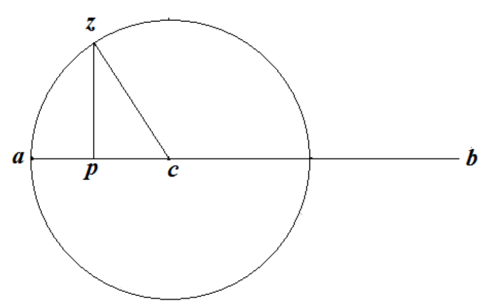

Fig. 1

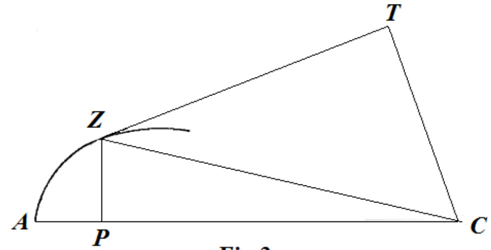

Fig.2

$AZ=\int\sqrt{\partial z^2+zz\partial\varphi^2}$, there must become $\partial\omega^2=\partial z^2+zz\partial\varphi^2$, from which there is deduced $\partial\varphi = \frac{\sqrt{\partial\omega^2 - \partial z^2}}{z}$, where therefore the whole problem is reduced to this, so that a relation of this kind may be sought between z and ω, by which the integral of this formula $\varphi = \int \frac{\sqrt{\partial\omega^2 - \partial z^2}}{z}$ may be expressed simpler by the circular arc.

4. But I have observed this can be established conveniently enough, if we may put the distance $CZ = b + \cos.\omega$; where at the limit [of the integration] I take the interval $cb = b$ and with the perpendicular zp sent from z there will become $cp = \cos.\omega$ and thus the distance CZ always must be taken equal to the interval bp. From which it is apparent for our initial A of the curve, the distance to become $CA = ba = b + 1$. Therefore since hence there shall become $\partial z = -\partial\omega \sin.\omega$, the differential formula given for $\partial\varphi$ on putting $z = b + \cos.\omega$ will adopt this neater form well enough

$$\partial\varphi = \frac{\partial\omega \cos.\omega}{b + \cos.\omega},$$

of which the integral therefore must be equal to the integral of the circular arc.

5. But this formula itself at once is separated into these parts $\partial\varphi = \partial\omega - \frac{b\partial\omega}{b + \cos.\omega}$, of which the first by itself is an element of the circle. For the other part we may put

$$\text{tang.}\tfrac{1}{2}\omega = t$$

and there will become

$$\partial\omega = \frac{2\partial t}{1 + tt};$$

then truly there becomes

$$\sin.\tfrac{1}{2}\omega = \frac{t}{\sqrt{1 + tt}} \text{ and } \cos.\tfrac{1}{2}\omega = \frac{1}{\sqrt{1 + tt}},$$

from which there is deduced

$$\cos.\omega = \cos^2.\tfrac{1}{2}\omega - \sin^2.\tfrac{1}{2}\omega = \frac{1 - tt}{1 + tt}.$$

Therefore there will become

$$b + \cos.\omega = \frac{b + 1 + (b - 1)tt}{1 + tt}$$

and thus there will become

$$\frac{b\partial\omega}{b + \cos.\omega} = \frac{2b\partial t}{(b + 1) + (b - 1)tt},$$

the integration of which always is reduced to the arc of a circle, provided there were $b > 1$.

6. For this integral requiring to be found it may be observed in general :

$$\int \frac{\partial t}{f+gtt} = \frac{1}{\sqrt{fg}} \text{Atang.} \frac{t\sqrt{g}}{\sqrt{f}},$$

from which in our case the angle will be

$$\varphi = \omega - \frac{2b}{\sqrt{bb-1}} \text{Atang.} t\sqrt{\frac{b-1}{b+1}}.$$

But truly so that the difference of those angles may be able to be assigned geometrically, it is necessary, so that the coefficient $\frac{2b}{\sqrt{bb-1}}$ shall be a rational number; and thus now it is clear, how often this has happened, always to be producing an algebraic curve *AZ* having an arc equal to the proposed circle.

7. Since there shall be $z = b + \cos.\omega$, several outstanding properties of this curve present themselves, which it will be agreed to be observed properly; for if the tangent *ZT* may be drawn to *Z* and there the angle may be called $CZT = \psi$, there will be

$$\sin.\psi = \frac{z\partial\varphi}{\partial\omega};$$

therefore on account of $\partial\varphi = \frac{\partial\omega\cos.\omega}{b+\cos.\omega}$ there will be $\sin.\psi = \cos.\omega$, thus so that the angle *CZT* always will be equal to $90° - \omega$ and thus on account of $AZ = \omega$ there will be always

$$\psi = \frac{\pi}{2} - \omega$$

with $\frac{\pi}{2}$ denoting a right angle. Hence, if from *C* the perpendicular *CT* may be sent to the tangent, there will be

$$CT = z\sin.\psi = z\cos.\omega = (b + \cos.\omega)\cos.\omega.$$

But with the perpendicular put $CT = p$ the radius of oscillation of the curve always to be agreed $= \frac{z\partial z}{\partial p}$. Therefore since there shall be

$$z\partial z = -\partial\omega\sin.\omega(b + \cos.\omega) \text{ and } \partial p = -\partial\omega\sin.\omega(\text{b} + 2\cos.\omega),$$

the radius of osculation of the curve at *Z*, which we will call *r*,

$$= \frac{b+\cos.\omega}{b+2\cos.\omega},$$

which therefore at the beginning, where $\omega = 0$, will be $r = \frac{b+1}{b+2}$, and thus smaller than for the circle. But truly for the arc $\omega = \frac{\pi}{2}$ there will be $r = 1$ and thus equal to the radius of the circle. But by taking $w = \pi$ there will be $r = \frac{b-1}{b-2}$. From which it is apparent, unless there shall be $b > 2$, this radius of osculation to become negative or if it may be inclined in the opposite direction and thus meanwhile thus the turning point allowed to be opposite, which will arise, when $\cos.\omega = -\frac{b}{2}$, because therefore it will fall between $\omega = 90°$ and $\omega = 180°$. And at this place the radius of osculation will be infinitely great. Besides since there shall be $\partial\varphi = \frac{\partial\omega\cos.\omega}{b+\cos.\omega}$, it is evident the curve rises above the axis or the angle $ACZ = \varphi$ to increase from $\omega = 0^{\mathrm{o}}$ to $\omega = 90°$, but hence that angle itself again to decrease and thus to cut the axis AC, before there may become $\omega = 180°$, because then the angle φ will become negative. Indeed because on putting $\omega = 180°$ there becomes $t = \infty$ and thus $\text{Atang.}t\sqrt{\frac{b-1}{b+1}} = 90°$, and thus $\varphi = 180°\left(1 - \frac{b}{\sqrt{bb-1}}\right)$, where $\frac{b}{\sqrt{bb-1}} > 1$.

8. From the radius of osculation found $r = \frac{b+\cos.\omega}{b+2\cos.\omega}$ it will be able to assign conveniently the extent of the curve $AZ = \omega$. For if the extent γ were put in place, there will be

$$\partial\gamma = \frac{\partial\omega}{r} = \frac{\partial\omega(b+2\cos.\omega)}{b+\cos.\omega},$$

that is, there will become

$$\partial\gamma = \partial\omega + \frac{\partial\omega\cos.\omega}{b+\cos.\omega} = \partial\omega + \partial\varphi$$

and thus the extent γ may be equal always to the sum of the two angles ω and φ, as long as the angle φ falls above the axis. For if it may fall below, it must be taken negative. But since the height of the curve may continually increase, as long as the curve AZ is concave towards the same direction, but after it begins to incline to the contrary direction, which happens, when the point of opposite inflection is given (now we have observed such a point to occur, when $b + 2\cos.\omega = 0$ or where $\cos.\omega = -\frac{b}{2}$), then, since there shall be $z = b + \cos.\omega$, there will become $z = -\frac{b}{2}$, thus so that the opposite point of inflection always may fall at the distance $CZ = \frac{b}{2}$; from which we deduce the curve from the initial point A, where $z = b+1$, the concavity to be directed upwards from the axis, while the distance may become $z = \frac{b}{2}$, and as long as the distance were smaller than $\frac{b}{2}$, the concavity to be inclined in the opposite direction, because that cannot happen, unless there were $b < 2$, since $b-1$ is the minimum distance of the curve from the centre C; on account of which, if there were $b > 2$, the whole curve nowhere will have an opposite point of inflection.

9. But since our algebraic curve may never happen, unless this formula $\frac{b}{\sqrt{bb-1}}$ may be equal to a rational number, which we may put to be n, hence in turn it is gathered that $b = \frac{n}{\sqrt{nn-1}}$. Then therefore the angle ACZ will be

$$= \varphi = \omega - 2n\text{Atang.}t\sqrt{\frac{b-1}{b+1}},$$

where there is $t = \text{tang.}\frac{1}{2}\omega$. Therefore here there will be

$$\frac{b-1}{b+1}=\frac{n-\sqrt{nn-1}}{n+\sqrt{nn-1}}=\frac{1}{(n+\sqrt{nn-1})^2}$$

and thus there will be

$$t\sqrt{\frac{b-1}{b+1}}=\frac{t}{n+\sqrt{nn-1}}.$$

Therefore since by necessity there must be taken $n>1$, it is evident that tangent $t\sqrt{\frac{b-1}{b+1}}$ always to be less than t. Therefore for the sake of brevity we may put

$$t\sqrt{\frac{b-1}{b+1}}=u$$

and we may call the angle, of which the tangent is $u,=\theta$; we will have this formula

$$\varphi=\omega-2n\theta\,,$$

from which the following is deduced

GEOMETRICAL CONSTRUCTION OF THE CURVE SOUGHT

10. We will show therefore, how for any point of the circle *z*, the point corresponding to that *Z* on some curve may be able to be defined on the curve sought. Without doubt with some rational number taken for *n* greater than unity, there may be taken $b=\frac{n}{\sqrt{nn-1}}=cb$; then truly from the arc $az=\omega$ there will be had $t=\text{tang.}\frac{1}{2}\omega$ and hence also there will become known

$$u=t\sqrt{\frac{b-1}{b+1}}=\frac{t}{n+\sqrt{nn-1}}.$$

Now there may be a cut in the circular arc, the tangent of which is *u*, which may be put $=\theta$, and because *n* is a rational number, it will be assigned geometrically $=2n\theta$, with which done the angle *ACZ* may be constructed equal to the difference of the angles ω and $2n\theta$, so that clearly there may become $\varphi=\omega-2n\theta$, with which done the distance may be taken $CZ=b+\cos.\omega=bp$, and in this manner the corresponding points *Z* of the curve sought will be determined for the individual points of the circle *z*.

11. Hence it is apparent, when the arc $az=\omega$ vanishes, then the point *Z* to be incident at the same point A with there being $CA=ba$. But truly with the arc taken $az=180°=\pi$, since then there becomes $t=\text{tang.}\frac{1}{2}\pi=\infty$, there will be also $u=\infty$, from wich $\theta=90°$. Therefore for this case the angle will become $\varphi=180°-2n\cdot 90°=\pi(1-n)$. Whereby since there shall be $n>1$ always, the angle φ falls on the other side of the axis and here the angle will be $=\pi(n-1)$. Truly the distance

of the corresponding point from the centre C will be $b-1$, which is the minimum distance, to which our curve can approach towards the centre. But in this manner it will suffice for the extension of the curve to be described only from the maximum distance $b+1$ as far as to the minimum $b-1$, since therefore beyond these limits the curve is stretched out equally on both sides, from which it is understood both the maximum as well as the minimum distance shall become the diameters of the curve. And finally also it is apparent the length of the curve from the maximum distance to the following minimum distance to be equal to the semi periphery of the proposed circle. And because the angle between the maximum and minimum distance, which is $(n-1)\pi$, is commensurable with the periphery of the circle, it follows the number of the diameter must always to be finite.

12. Hence it is understood also, how the equation may be able to be elicited between the coordinates $CP = x$ and $PZ = y$; since indeed there shall be

$$\text{tang.}\varphi = \tfrac{y}{x} \text{ and } \text{tang.}\tfrac{1}{2}\varphi = \sqrt{\tfrac{z-x}{z+x}},$$

for which equation there must be $\text{tang.}(\frac{1}{2}\omega - n\theta)$. Truly for which we may put $\text{tang.}\frac{1}{2}\omega = t$,and there will be $\cos.\omega = \frac{1-tt}{1+tt}$, from which on account of $z = b + \frac{1-tt}{1+tt}$ there is elicited $tt = \frac{b+1-z}{z-b+1}$ and hence

$$uu = \tfrac{b-1}{b+1}tt = \tfrac{bb-1-(b-1)z}{(b+1)z-bb+1}$$

and thus t and m may be expressed by functions of z and thus also $\text{tang.}n\theta$ by such a function, from which also the tangent of the angle $\frac{1}{2}\omega - n\theta$ may be defined by a function of z only. Hence with the square taken the formula $\frac{z-x}{z+x}$ is equal to some rational function of z, which equation finally on account of $z = \sqrt{xx+yy}$ with the square taken is reduced to some rational equation between x and y , which moreover generally rises to more dimensions, if indeed for the simplest case, where $n = 2$, has risen to the sixth order.

DESCRIPTION OF THE SIMPLEST CURVE WHERE $n = 2$.

13. Therefore here on account of $n = 2$ there will be $b = \frac{2}{\sqrt{3}} = \sec.30°$ and thus $b = 1,1547$ approximately. Therefore the maximum distance of the curve from the centre C (Fig. 3) or as if, of the greatest apse will be $CA = b+1 = 2,1547$, to which the curve is normal, and there the radius of osculation will be $r = \frac{b+1}{b-1} = 0,6830$. The minimum distance will be $b-1 = 0,1547$, which from the maximum will be different by the angle $-180°$

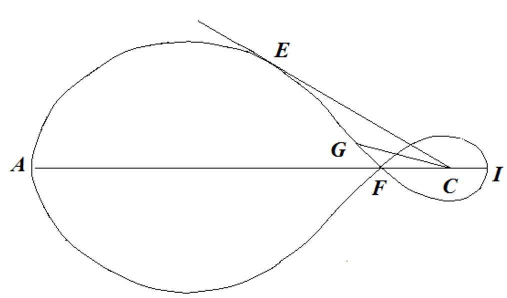

Fig.3.

and thus falls on the axis *AC* continued, which shall be *CI,* where the curve again will be normal to the axis. But truly the radius of osculation at *I* will be $\frac{b-1}{b-2} = -0,1830.$ But the length of the curve from the greatest apse A to the least I stretched out will be equal to the semi-periphery of the circle described with radius 1.

14. For other memorable points of the curve being defined with the arc taken $AZ = \omega$ the distance will be $CZ = b + \cos.\omega$. But for the angle $ACZ = \varphi$ we will have $\text{tang.}\frac{1}{2}\varphi = \text{tang.}(\frac{1}{2}\omega - 2\theta)$, where on putting $\text{tang.}\frac{1}{2}\omega = t$ there will become

$$\text{tang.}\theta = u = t\sqrt{\tfrac{b-1}{b+1}} = 0,2679t$$

and in turn

$$t = u\sqrt{\tfrac{b+1}{b-1}} = 3,7321u.$$

Therefore since there shall be $\text{tang.}\theta = u$, there will become $\text{tang.}2\theta = \frac{2u}{1-uu}$, from which there becomes

$$\text{tang.}(\tfrac{1}{2}\omega - 2\theta) = \tfrac{t(1-uu)-2u}{1-uu+2tu} = \text{tang.}\tfrac{1}{2}\varphi.$$

15. Now we may take the arc $AE = 90° = \frac{1}{2}\pi$ and the distance $CE = b$ and the angle $\varphi = 90° - \omega = 0,$ from which it is apparent the right line CE to be a tangent at *E* and there the radius of osculation to be $=1$. For the angle *ACE* requiring to be found we will have $t = 1$ and $u = 0,2679 = \text{tang.}\theta$. Therefore the angle $\theta = 15°\ 0'$ and thus $\frac{1}{2}\varphi = 15°\ 0'$ and in this manner the angle will become $ACE = 30°$.

16. Hence therefore the curve will approach towards the axis and since soon it will cut at *F,* where therefore, since there shall become $\varphi = 0$, there will be $t(1-uu) = 2u$ or $3,7321(1-uu) = 2$, from which there is found $uu = 0,4641$ and hence $t = 2,7321$. Therefore there will become $\frac{1}{2}\omega = 69°\ 54'$ and thus $\omega = 139°\ 48'$. From which it is apparent the curve to be inclined here under the axis at an angle 49° 48', truly the distance to become $CF = b - \sin.\ 49^{o}48' = 0,3909$. The radius of osculation at this place will be $= -1,0483$. Therefore here the curve is bent in the contrary direction and thus the contrary inflection point precedes the point *F.*

17. Therefore for this point, which shall be at *G,* now requiring to be found we have observed above that to happen, where the distance $CG = \frac{1}{2}b = 0,5773$, thus so that $\cos.\omega = -\frac{1}{2}b$ and thus $\omega = 125°16'$. Whereby at this place the curve is inclined under the right line *CG* by the angle 35° 16'. Because again there is $\frac{1}{2}\omega = 62°\ 38'$, there will be $t = 1,9319$ and hence again $u = 0,5176$, which is the tangent of the angle *θ,* which consequently will be 27° 22', therefore $\frac{1}{2}\varphi = 7^{o}54'$, consequently the angle

$FCG = 15° \ 48'$. Moreover from these principle points of the curve the extent of the curve will be able to be described exactly enough, from which, since the right line *AI* likewise shall be a diameter of the curve, the whole curve will have this figure (Fig. 3).

SUPPLEMENT

18. The solution of the following problem quite elegantly will enlarge on all the curves found by the preceding method much more easily and conveniently.

PROBLEM

To find the curve EZ (Fig. 4) *referred to the fixed point C, of which any arc EZ may hold the same ratio to the angle EZC everywhere.*

SOLUTION

19. Therefore here it is apparent at once the arc of the curve *EZ,* which is proportional to the angle EZC, to be equal to the arc of the circle measuring the same angle and thus, if these curves were algebraic, these are going to be satisfying our goal. For finding these we may put the angle $ECZ = \varphi$ and the distance $CZ = z$, so that we may have in place approximately $ZS = z\partial\varphi$ and $zS = \partial z$.

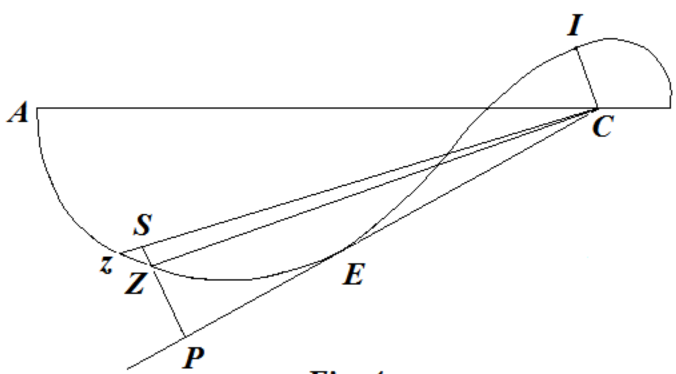

Fig. 4.

Now we may put the angle $EZC = \omega$,truly the arc $EZ = a\omega$,and because all the similar curves may equally satisfy the relation referred to the same point *C*, it will be allowed to take $a = 1$, so that there shall be the $EZ = \omega$ and therefore its element $Zz = \partial\omega$,and now the triangle *ZzS* presents at once these two equations

$$\partial z = \partial\omega \cos.\omega \text{ and } z\partial\varphi = \partial\omega \sin.\omega.$$

20. The first of these equations integrated gives at once $z = b + \sin.\omega$, from which from the other there becomes $\partial\varphi = \frac{\partial\omega \sin.\omega}{b+\sin.\omega}$. Hence it is evident at once at the point *E,* where the arc *EZ* vanishes, also the angle to become $\omega = 0$ and thus the distance $CE = b$ and that right line *CE* to become the tangent of the curve at the initial point *E* itself.

21. Therefore for the element of the angle $\partial\varphi$ we have

$$\partial\varphi = \partial\omega - \frac{b\partial\omega}{b+\sin.\omega} \text{ and thus } \varphi = \omega - \int \frac{b\partial\omega}{b+\sin.\omega},$$

for integrating which formula we may put $\operatorname{tang}.\frac{1}{2}\omega = t$, from which there becomes $\sin.\omega = \frac{2t}{1+tt}$ and $\partial\omega = \frac{2\partial t}{1+tt}$, from which the formula arises :

$$\frac{b\partial\omega}{b+\sin.\omega} = \frac{2b\partial t}{b(1+tt)+2t}.$$

We may put $\frac{1}{b} = \cos.\beta$, so that there may become

$$\frac{b\partial\omega}{b+\sin.\omega} = \frac{2\partial t}{1+tt+2t\cos.\beta},$$

of which formula the integral always expresses the arc of a circle, but only if there were $b > 1$ and by $\frac{1}{b}$ the cosine of some angle may be able to be referred to. But it is agreed the integral of this formula to become

$$= \frac{2}{\sin.\beta}\operatorname{Atang}.\frac{t\sin.\beta}{1+t\cos.\beta},$$

thus so that now likewise this equation has arisen

$$\varphi = \omega - \frac{2}{\sin.\beta}\operatorname{Atang}.\frac{t\sin.\beta}{1+t\cos.\beta};$$

from which it is apparent, as often as $\sin.\beta$ were a rational number, that angle itself can be assigned always geometrically and thus our curve to become algebraic, and because the angle β will be allowed to be taken in an infinite number of ways, likewise innumerable algebraic curves to be found satisfying our aim, certainly all of which may be measured by circular arcs. Moreover it is clear these curves with those, which we have found before, to agree perfectly, because here only another principle has been agreed at *E*.

22. Therefore since $\sin.\beta$ must be a rational number, we may put $\frac{1}{\sin.\beta} = n$, thus so that n shall be some number greater than unity either whole or a fraction, and on putting for the sake of brevity

$$\operatorname{Atang}.\frac{t\sin.\beta}{1+t\cos.\beta} = \theta$$

there will be

$$\varphi = \omega - 2n\theta,$$

which angle therefore at the beginning, where $\omega = 0$, also vanished. Therefore there will be $\frac{1}{2}\varphi = \frac{1}{2}\omega - n\theta$ and with the orthogonal coordinates put in place $CP = x$ and $PZ = y$ there will be

$$\text{tang.}\varphi = \tfrac{y}{x} \text{ and tang.}\tfrac{1}{2}\varphi = \tfrac{y}{z+x} = \sqrt{\tfrac{z-x}{z+x}}.$$

Again since there shall be

$$z = b + \sin.\omega = b + \tfrac{2t}{1+tt},$$

it is apparent also t to be equal to a function of z and hence also tang.θ, thus so that hence for any case the equation between the orthogonal coordinates x and y may be able to be elicited.

23. Now we will investigate especially the points of this curve and indeed initially we may take the arc $EA = 90° = \frac{\pi}{2}$ and the angle ω will be right and the distance CA to the curve will be normal and likewise the diameter of the curve , about which the curve is extended by being drawn out equally on each side. Therefore here there will be $\text{tang.}\frac{1}{2}\omega = t = 1$ and thus

$$\text{tang.}\theta = \tfrac{\sin.\beta}{1+\cos.\beta} = \text{tang.}\tfrac{1}{2}\beta,$$

thus so that there shall be $\theta = \frac{1}{2}\beta$, so that with this angle β found, of which the cosine is $\frac{1}{b}$, the angle ECA will be $= \frac{\pi}{2} - n\beta$. But that distance itself CA will be $b+1$, which will be the maximum, to which the curve is able to reach.

24. Now we will consider the part of this curve stretched backwards from the point E and we may take the arc EI equal to a quadrant, from which it will be required to put in place $\omega = -\frac{\pi}{2}$, and at this point I the distance will become $CI = b-1$, which is the minimum distance of all, to which the curve can fall, and here again CI will be normal to the curve and equally its diameter, from which it will suffice only the curve to be described from A by E as far as to I.

25. Therefore in this case on account of $t = -1$ there will be $\theta = \text{Atang.}\frac{-\sin.\beta}{1-\cos.\beta}$ and thus this angle θ will be negative and its tangent $\frac{\sin.\beta}{1-\cos.\beta}$, which expression is the cotangent of the angle $\frac{1}{2}\beta$, and thus there will become $-\theta = \frac{\pi}{2} - \frac{1}{2}\beta$, from which the angle is produced :

$$ECI = \varphi = -\tfrac{\pi}{2} + 2n(\tfrac{\pi}{2} - \tfrac{1}{2}\beta) = (n - \tfrac{1}{2})\pi - n\beta,$$

on account of which the angle intercepted between the maximum distance $CA = b+1$ and the minimum $CI = b-1$ will be $ACI = (n-1)\pi$, exactly as was found above.

26. Finally we will consider the case, where the arc is taken equal to the semi periphery *EZ* or $\omega=\pi$, therefore the distance where it will touch the curve again ; therefore then there will be $t=\infty$ and $\text{tang.}\theta=\text{tang.}\beta$ and thus $\theta=\beta$ and thus the angle will be $\varphi=\pi-2n\beta$, which is twice as great as the angle *ECA,* just as the nature of the diameter demands. Finally here it will help to have observed all the formulas found here can be reduced to the preceding, if in place of *t* there may be written $\frac{1-t}{1+t}$ and likewise the angle φ may be reduced to the angle $ECA=\frac{\pi}{2}-n\beta$.

DE CURVIS ALGEBRAICIS QUARUM OMNES ARCUS PER ARCUS CIRCULARES METIRI LICEAT

Convent. exhib. die 20 Augusti 1781
[E783]
Mémoires de l'académie des sciences de St. Pétersbourg 11, 1830, p. 114-124

1. Non dubitavi ante aliquot annos istam propositionem tanquam insigne theorema in medium proferre: *quod praeter circulum nulla detur curva algebraica, cuius arcubus omnibus aequales arcus circulares assignari queant.* Plures etiam adduxi rationes satis probabiles, quae me in hac opinione confirmabant, quanquam probe perspexi eas a perfecta demonstratione adhuc plurimum distare. Praecipua autem ratio mihi erat, quod, postquam in hoc argumento plurimum elaborassem, nullam tamen huiusmodi curvam elicere potuerim.

2. Quamobrem, cum nuper in simili argumento occupatus in genere binas curvas algebraicas investigassem, quae communi rectificatione gauderent, indeque infinitas curvas algebraicas investigassem, quarum longitudo per arcus parabolicos metiri liceret, tum vero etiam infinitas curvas algebraicas cum ellipsi eadem rectificatione gaudentes, maxime obstupui, quod, etiamsi ellipsin in circulum converterem, nihilominus curvae inventae a circulo essent diversae. Sententiam igitur meam hic solenniter retractans methodum facilem exponam, cuius ope innumerabiles curvae algebraicae inveniri possunt, quarum omnes arcus circularibus sunt aequales.

3. Proposito igitur circulo centro c, Fig. 1 et 2, radio ca, descripto concipiamus curvam AZ ita comparatam, ut eius arcus indefinitus AZ semper aequalis sit arcui indefinito illius circuli az, quo vocato $az=\omega$ sit quoque arcus $AZ=\omega$. Hanc iam curvam ad centrum quoddam fixum C refero eiusque naturam per aequationem inter distantiam $CZ=z$ et angulum $ACZ=\varphi$ investigabo, ut quaesito satisfiat. Cum igitur hinc sit arcus $AZ=\int\sqrt{\partial z^2+zz\partial\varphi^2}$, fieri debet $\partial\omega^2=\partial z^2+zz\partial\varphi^2$, unde deducitur $\partial\varphi=\frac{\sqrt{\partial\omega^2-\partial z^2}}{z}$, ubi ergo totum negotium huc redit, ut eiusmodi relatio inter z et ω exquiratur, quae integrale huius formulae

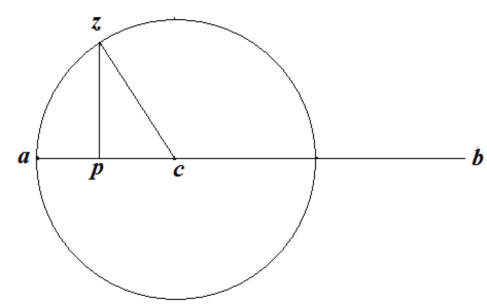

Fig. 1

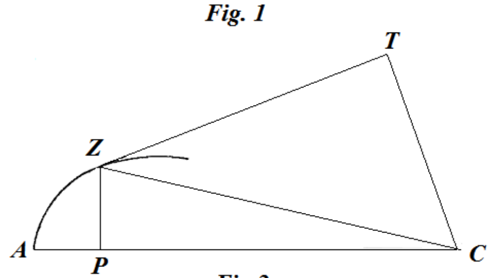

Fig.2

$\varphi = \int \frac{\sqrt{\partial\omega^2 - \partial z^2}}{z}$ per arcum circularem simpliciter exprimat.

4. Observavi autem hoc satis commode praestari posse, si statuamus distantiam $CZ = b + \cos.\omega$; quem in finem sumo intervallum $cb = b$ ac demisso ex z perpendiculo zp fiet $cp = \cos.\omega$ sicque distantia CZ semper aequalis capi debet intervallo bp. Unde patet pro initio A nostrae curvae fore distantiam $CA = ba = b + 1$. Cum igitur hinc fiat $\partial z = -\partial\omega \sin.\omega$, formula differentialis pro $\partial\varphi$ data posito $z = b + \cos.\omega$ induet hanc formam satis concinnam

$$\partial\varphi = \frac{\partial\omega \cos.\omega}{b + \cos.\omega},$$

cuius ergo integrale arcui circulari aequale esse debet.

5. Ista autem formula sponte in has partes discerpitur $\partial\varphi = \partial\omega - \frac{b\partial\omega}{b + \cos.\omega}$, quarum prima per se est elementum circuli. Pro altera parte ponamus

$$\text{tang.}\tfrac{1}{2}\omega = t$$

fietque

$$\partial\omega = \frac{2\partial t}{1+tt};$$

tum vero fit

$$\sin.\tfrac{1}{2}\omega = \frac{t}{\sqrt{1+tt}} \text{ et } \cos.\tfrac{1}{2}\omega = \frac{1}{\sqrt{1+tt}},$$

unde colligitur

$$\cos.\omega = \cos^2.\tfrac{1}{2}\omega - \sin^2.\tfrac{1}{2}\omega = \frac{1-tt}{1+tt}.$$

Erit ergo

$$b + \cos.\omega = \frac{b+1+(b-1)tt}{1+tt} \text{ sicque erit}$$

$$\frac{b\partial\omega}{b+\cos.\omega} = \frac{2b\partial t}{(b+1)+(b-1)tt},$$

cuius integratio semper ad arcum circularem reducitur, dummodo fuerit $b > 1$.

6. Ad hoc integrale inveniendum notetur esse in genere

$$\int \frac{\partial t}{f+gtt} = \frac{1}{\sqrt{fg}} \text{Atang.} \frac{t\sqrt{g}}{\sqrt{f}},$$

unde pro nostro casu erit angulus

$$\varphi = \omega - \frac{2b}{\sqrt{bb-1}} \text{Atang.} t\sqrt{\frac{b-1}{b+1}}.$$

At vero ut horum angulorum differentia geometrice assignari queat, necesse est, ut coefficiens $\frac{2b}{\sqrt{bb-1}}$ sit numerus rationalis; atque adeo iam evidens est, quoties hoc contigerit, semper prodituram esse curvam algebraicam AZ cum circulo proposito arcus aequales habentem.

7. Cum sit, $z = b+\cos.\omega$ plures egregiae proprietates huius curvae se offerunt, quas probe notari conveniet; namque si ad Z ducatur tangens ZT et vocetur angulus $CZT = \psi$, erit

$$\sin.\psi = \tfrac{z\partial\varphi}{\partial\omega};$$

ergo $\partial\varphi = \frac{\partial\omega\cos.\omega}{b+\cos.\omega}$ erit $\sin.\psi = \cos.\omega$, ita ut angulus CZT semper aequetur $90°-\omega$ ideoque ob $AZ=\omega$ semper erit

$$\psi = \tfrac{\pi}{2} - \omega$$

denotante $\frac{\pi}{2}$ angulum rectum. Hinc, si ex C in tangentem demittatur perpendiculum CT, erit

$$CT = z\sin.\psi = z\cos.\omega = (b+\cos.\omega)\cos.\omega.$$

Posito autem hoc perpendiculo $CT = p$ constat semper esse radium osculi curvae $= \frac{z\partial z}{\partial p}$. Cum igitur sit

$$z\partial z = -\partial\omega\sin.\omega(b+\cos.\omega) \text{ et } \partial p = -\partial\omega\sin.\omega(b+2\cos.\omega),$$

erit radius osculi curvae in Z, quem vocemus r,

$$= \tfrac{b+\cos.\omega}{b+2\cos.\omega},$$

qui ergo in initio, ubi $\omega = 0$, erit $r = \frac{b+1}{b+2}$ ideoque minor quam in circulo. At vero pro arcu $\omega = \frac{\pi}{2}$ erit $r = 1$ ideoque radio circuli aequalis. Sumto autem $w = \pi$ erit $r = \frac{b-1}{b-2}$. Unde patet, nisi sit $b > 2$, hunc radium osculi fieri negativum sive in plagam contrariam vergere ideoque interea curvam punctum flexus contrarii esse passam, quod eveniet, ubi $\cos.\omega = -\frac{b}{2}$, quod ergo inter $\omega = 90°$ et $\omega = 180°$ cadet. Hocque loco radius osculi erit infinite magnum.
Praeterea cum sit $\partial\varphi = \frac{\partial\omega\cos.\omega}{b+\cos.\omega}$, manifestum est curvam supra axem ascendere sive angulum $ACZ = \varphi$ augeri ab $\omega = 0^{\circ}$ ad $\omega = 90°$, hinc autem istum angulum iterum decrescere atque adeo curvam axem AC secare, antequam fiat $\omega = 180°$, quia tum angulus φ fiet negativus. Quia enim posito $\omega = 180°$ fit $t = \infty$ ideoque $\text{Atang.}t\sqrt{\frac{b-1}{b+1}} = 90°$ ideoque $\varphi = 180°\left(1-\frac{b}{\sqrt{bb-1}}\right)$, ubi $\frac{b}{\sqrt{bb-1}} > 1$.

8. Ex radio osculi invento $r = \frac{b+\cos.\omega}{b+2\cos.\omega}$ etiam commode assignari potest amplitudo curvae $AZ = \omega$. Si enim amplitudo ponatur γ, erit

$$\partial\gamma = \frac{\partial\omega}{r} = \frac{\partial\omega(b+2\cos.\omega)}{b+\cos.\omega},$$

hoc est, erit

$$\partial\gamma = \partial\omega + \frac{\partial\omega\cos.\omega}{b+\cos.\omega} = \partial\omega + \partial\varphi$$

sicque amplitudo γ semper aequatur summae angulorum ω et φ, quamdiu scilicet angulus φ supra axem cadit. Si enim infra axem cadat, negative accipi debet. Cum autem amplitudo curvae continuo augeatur, quamdiu curva *AZ* versus eandem partem est concava, postquam autem coepit in partem contrariam vergere, quod evenit, ubi punctum flexus contrarii datur (iam notavimus tale punctum occurrere, ubi $b+2\cos.\omega = 0$ seu ubi $\cos.\omega = -\frac{b}{2}$), tum, cum sit $z = b+\cos.\omega$, fiet $z = -\frac{b}{2}$, ita ut punctum flexus contrarii semper incidat in distantiam $CZ = \frac{b}{2}$; unde colligimus curvam ab initio *A*, ubi $z = b+1$, concavitatem axi obvertere, donec fiat distantia $z = \frac{b}{2}$, et quamdiu distantia minor fuerit quam $\frac{b}{2}$, concavitatem in partem contrariam vergi, id quod evenire nequit, nisi fuerit $b < 2$, quia $b-1$ minima distantia curvae a centro *C;* quamobrem, si fuerit $b > 2$, tota curva nusquam habebit punctum flexus contrarii.

9. Cum autem nostrae curvae algebraicae fieri nequeant, nisi haec formula $\frac{b}{\sqrt{bb-1}}$ aequetur numero rationali, quem ponamus *n,* hinc vicissim colligitur $b = \frac{n}{\sqrt{nn-1}}$. Tum igitur erit angulus *ACZ*

$$= \varphi = \omega - 2n\text{Atang.}t\sqrt{\frac{b-1}{b+1}},$$

ubi est $t = \text{tang.}\frac{1}{2}\omega$. Hic igitur erit

$$\frac{b-1}{b+1} = \frac{n-\sqrt{nn-1}}{n+\sqrt{nn-1}} = \frac{1}{(n+\sqrt{nn-1})^2}$$

sicque erit

$$t\sqrt{\frac{b-1}{b+1}} = \frac{t}{n+\sqrt{nn-1}}.$$

Quia igitur necessario sumi debet $n > 1$, manifestum est istam tangentem $t\sqrt{\frac{b-1}{b+1}}$ semper minorem esse quam *t.* Ponamus ergo brevitatis gratia

$$t\sqrt{\frac{b-1}{b+1}} = u$$

et vocemus angulum, cuius tangens est $u, = \theta$; habebimus hanc formulam

$$\varphi = \omega - 2n\theta,$$

unde deducitur sequens

CONSTRUCTIO GEOMETRICA CURVARUM QUAESITARUM

10. Monstrabimus igitur, quomodo pro quovis circuli puncto z punctum ei respondens Z in qualibet curva quaesita definiri queat. Sumto nimirum pro n numero quocunque rationali unitate maiore capiatur $b = \frac{n}{\sqrt{nn-1}} = cb$; tum vero ex arcu $az = \omega$ habebitur $t = \text{tang.}\frac{1}{2}\omega$ hincque etiam innotescet

$$u = t\sqrt{\frac{b-1}{b+1}} = \frac{t}{n+\sqrt{nn-1}}.$$

Nunc abscindatur in circulo arcus, cuius tangens est u, qui ponatur $= \theta$, et quia n est numerus rationalis, geometrice assignabitur $= 2n\theta$, quo facto construatur angulus ACZ aequalis differentiae angulorum ω et $2n\theta$, ut scilicet fiat $\varphi = \omega - 2n\theta$, quo facto sumatur distantia $CZ = b + \cos.\omega = bp$, hocque modo pro singulis circuli punctis z determinabuntur puncta correspondentia Z curvae quaesitae.

11. Hinc patet, quando arcus $az = \omega$ evanescit, tum punctum Z incidere in ipsum punctum A existente $CA = ba$. At vero sumto arcu $az = 180° = \pi$, quia tum fit $t = \text{tang.}\frac{1}{2}\pi = \infty$, erit etiam $u = \infty$, unde $\theta = 90°$. Pro hoc ergo casu fiet angulus $\varphi = 180° - 2n \cdot 90° = \pi(1-n)$. Quare cum semper sit $n > 1$, angulus φ ad alteram axis partem cadet eritque hic angulus $= \pi(n-1)$. Distantia vero puncti respondentis a centro C erit $b-1$, quae est minima distantia, ad quam nostra curva versus centrum accedere potest. Sufficiet autem hoc modo tractum curvae tantum a distantia maxima $b+1$ usque ad minimam $b-1$ descripsisse, propterea quod ultra hos terminos curva utrinque aequaliter porrigitur, unde intelligitur tam distantiam maximam quam minimam fore curvae diametros. Denique etiam ultro patet longitudinem curvae a distantia maxima ad sequentem minimam semiperipheriae circuli propositi aequari. Et quia angulus inter maximam et minimam distantiam, qui est $(n-1)\pi$, cum peripheria circuli est commensurabilis, sequitur numerum diametrorum semper esse debere finitum.

12. Hinc etiam intelligitur, quomodo aequationem inter coordinatas $CP = x$ et $PZ = y$ erui oporteat; cum enim sit

$$\text{tang.}\varphi = \frac{y}{x} \quad \text{et} \quad \text{tang.}\frac{1}{2}\varphi = \sqrt{\frac{z-x}{z+x}},$$

cui aequari debet $\text{tang.}(\frac{1}{2}\omega - n\theta)$. Quia vero posuimus $\text{tang.}\frac{1}{2}\omega = t$, erit $\cos.\omega = \frac{1-tt}{1+tt}$, unde ob $z = b + \frac{1-tt}{1+tt}$ elicitur $tt = \frac{b+1-z}{z-b+1}$ hincque

$$uu = \frac{b-1}{b+1}tt = \frac{bb-1-(b-1)z}{(b+1)z-bb+1}$$

sicque *t* et *m* per functiones ipsius *z* ideoque etiam $\text{tang.}n\theta$ per talem functionem exprimetur, unde etiam tangens anguli $\frac{1}{2}\omega - n\theta$ per functionem solius *z* definietur. Hinc sumtis quadratis formula $\frac{z-x}{z+x}$ aequatur functioni rationali ipsius *z*, quae aequatio denique ob $z = \sqrt{xx+yy}$ sumendis quadratis ad aequationem rationalem inter *x* et *y* reducitur, quae autem plerumque ad plurimas dimensiones assurgit, si quidem pro casu simplicissimo, quo $n = 2$, ad sextum ordinem ascendit.

DESCRIPTIO CURVAE SIMPLICISSIMAE QUO $n = 2$

13. Hic ergo ob $n = 2$ erit $b = \frac{2}{\sqrt{3}} = \sec.30°$ ideoque proxime $b = 1{,}1547$. Maxima igitur curvae distantia a centro *C* (Fig. 3) seu quasi absis summa erit $CA = b+1 = 2{,}1547$, ad quam curva est normalis, ibique radius osculi erit $r = \frac{b+1}{b-1} = 0{,}6830$. Minima distantia erit $b-1 = 0{,}1547$, quae a maxima distabit angulo $-180°$ ideoque in axem *AC* continuatum cadet, quae sit *CI*, ubi curva iterum ad axem erit normalis. At vero radius osculi in *I* erit $\frac{b-1}{b-2} = -0{,}1830$. Longitudo autem curvae ab abside summa *A* ad imam *I* protensae aequabitur semi-peripheriae circuli radio 1 descripti.

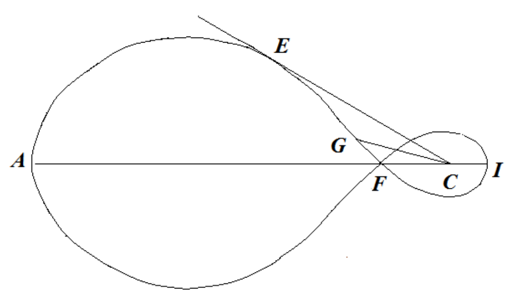

Fig.3.

14. Pro aliis curvae punctis memorabilibus definiendis sumto arcu $AZ = \omega$ erit distantia $CZ = b + \cos.\omega$. Pro angulo autem $ACZ = \varphi$ habebimus $\text{tang.}\frac{1}{2}\varphi = \text{tang.}(\frac{1}{2}\omega - 2\theta)$, ubi posito $\text{tang.}\frac{1}{2}\omega = t$ erit

$$\text{tang.}\theta = u = t\sqrt{\frac{b-1}{b+1}} = 0{,}2679t$$

et vicissim

$$t = u\sqrt{\frac{b+1}{b-1}} = 3{,}7321u.$$

Cum igitur sit $\text{tang.}\theta = u$, erit $\text{tang.}2\theta = \frac{2u}{1-uu}$, unde fit

$$\text{tang.}(\tfrac{1}{2}\omega - 2\theta) = \frac{t(1-uu)-2u}{1-uu+2tu} = \text{tang.}\tfrac{1}{2}\varphi.$$

15. Sumamus nunc arcum $AE = 90° = \frac{1}{2}\pi$ eritque distantia $CE = b$ et angulus $\varphi = 90° - \omega = 0,$ unde patet rectam *CE* curvam *E* tangere ibique radium osculi fore $= 1$. Pro angulo *ACE* investigando habemus $t = 1$ et $u = 0,2679 = \text{tang.}\theta$. Erit ergo angulus $\theta = 15°\ 0'$ ideoque $\frac{1}{2}\varphi = 15°\ 0'$ hocque modo erit angulus *ACE* 30°.

16. Hinc igitur curva ad axem appropinquabit cumque mox secabit in *F*, ubi ergo, cum fiat $\varphi = 0$, erit $t(1-uu) = 2u$ sive $3,7321(1-uu) = 2$, unde reperitur $uu = 0,4641$ hincque $\text{t} = 2,7321$. Erit ergo $\frac{1}{2}\omega = 69°\ 54'$ ideoque $\omega = 139°\ 48'$. Unde patet curvam hic ad axem sub angulo 49° 48' esse inclinatam, distantiam vero fore $CF = b - \sin.\ 49°48' = 0,3909$. Radius osculi hoc loco erit $= -1,0483$. Hic ergo curva iam in contrariam partem est inflexa ideoque punctum flexus contrarii praecessit punctum *F*.

17. Ad hoc ergo punctum, quod sit in *G*, inveniendum iam supra notavimus id incidere, ubi distantia $CG = \frac{1}{2}b = 0,5773$, ita ut $\cos.\omega = -\frac{1}{2}b$ ideoque $\omega = 125°16'$. Quare hoc loco curva ad rectam *CG* inclinatur sub angulo 35° 16'. Quia porro est $\frac{1}{2}\omega = 62°\ 38'$, erit $t = 1,9319$ hincque porro $u = 0,5176$, quae est tangens anguli θ, qui consequenter erit 27° 22', ergo $\frac{1}{2}\varphi = 7°54'$, consequenter angulus $FCG = 15°\ 48'$. Ex his autem principalibus curvae punctis tractus curvae facile satis exacte describi poterit, unde, cum recta *AI* simul curvae sit diameter, tota curva habet hanc figuram (Fig. 3).

SUPPLEMENTUM

18. Solutio sequentis problematis non parum elegantis omnes curvas methodo praecedente inventas multo facilius et commodius largietur.

PROBLEMA

Invenire curvam EZ (Fig. 4) *ad punctum fixum C relatam, cuius quilibet arcus EZ ad angulum EZC ubique eandem teneat rationem.*

SOLUTIO

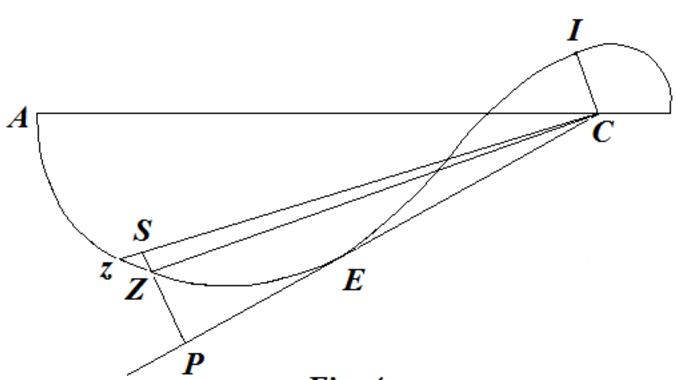

Fig. 4.

19. Hic igitur statim patet arcum curvae *EZ*, quia angulo *EZC* est proportionalis, aequalem fore arcui circulari eundem angulum metientis ideoque, si hae curvae fuerint algebraicae, eas scopo nostro esse satisfacturas. Ad eas inveniendas ponamus angulum $ECZ = \varphi$ et

distantiam $CZ = z$, ut habeamus pro situ proxima $ZS = z\partial\varphi$ et $zS = \partial z$.
Ponamus nunc angulum $EZC = \omega$, arcum vero $EZ = a\omega$, et quia omnes curvae similes ad idem punctum *C* relatae aeque satisfaciunt, sumere licebit $a = 1$, ut sit arcus $EZ = \omega$ eiusque ergo elementum $Zz = \partial\omega$, et nunc triangulum *ZzS* statim praebet has duas aequationes

$$\partial z = \partial\omega\cos.\omega \text{ et } z\partial\varphi = \partial\omega\sin.\omega.$$

20. Prior harum aequationum integrata statim dat $z = b + \sin.\omega$, unde ex altera fit $\partial\varphi = \frac{\partial\omega\sin.\omega}{b+\sin.\omega}$. Hinc statim manifestum est in puncto *E*, ubi arcus *EZ* evanescit, fore etiam angulum $\omega = 0$ ideoque distantiam $CE = b$ et hanc rectam *CE* fore curvae tangentem in ipso initio *E*.

21. Pro elemento ergo angulari $\partial\varphi$ habemus

$$\partial\varphi = \partial\omega - \frac{b\partial\omega}{b+\sin.\omega} \text{ ideoque } \varphi = \omega - \int\frac{b\partial\omega}{b+\sin.\omega},$$

ad quam formulam integrandam ponamus $\tang.\frac{1}{2}\omega = t$, unde fit $\sin.\omega = \frac{2t}{1+tt}$ et $\partial\omega = \frac{2\partial t}{1+tt}$, unde oritur formula

$$\frac{b\partial\omega}{b+\sin.\omega} = \frac{2b\partial t}{b(1+tt)+2t}.$$

Ponamus $\frac{1}{b} = \cos.\beta$, ut oriatur

$$\frac{b\partial\omega}{b+\sin.\omega} = \frac{2\partial t}{1+tt+2t\cos.\beta},$$

cuius formulae integrale semper exprimet arcum circuli, si modo fuerit $b > 1$ et per $\frac{1}{b}$ cosinum cuiuspiam anguli referri queat. Constat autem huius formulae integrale fore

$$= \frac{2}{\sin.\beta}\text{Atang.}\frac{t\sin.\beta}{1+t\cos.\beta},$$

ita ut iam nacti simus hanc aequationem

$$\varphi = \omega - \frac{2}{\sin.\beta}\text{Atang.}\frac{t\sin.\beta}{1+t\cos.\beta};$$

unde patet, quoties $\sin.\beta$ fuerit numerus rationalis, istum angulum semper geometrice assignari posse ideoque curvam nostram fore algebraicam, et quia angulum β infinitis modis accipere licet, simul reperiri innumerabiles curvas algebraicas scopo nostro satisfacientes, quippe quarum omnes arcus per arcus circulares mensurantur. Evidens autem est has curvas cum iis, quas ante invenimus, perfecte convenire, quia hic tantum aliud principium est assumtum in *E*.

22. Quoniam igitur $\sin.\beta$ debet esse numerus rationalis, ponamus $\frac{1}{\sin.\beta}=n$, ita ut n sit numerus quicunque unitate maior sive integer sive fractus, ac posito brevitatis gratia

$$\text{Atang.}\frac{t\sin.\beta}{1+t\cos.\beta}=\theta$$

erit

$$\varphi=\omega-2n\theta\,,$$

qui ergo angulus in principio, ubi $\omega=0$, etiam evanescit. Erit igitur $\frac{1}{2}\varphi=\frac{1}{2}\omega-n\theta$ ac positis coordinatis orthogonalibus $CP=x$ et $PZ=y$ erit

$$\text{tang.}\varphi=\frac{y}{x}\text{ et tang.}\tfrac{1}{2}\varphi=\frac{y}{z+x}=\sqrt{\frac{z-x}{z+x}}.$$

Cum porro sit

$$z=b+\sin.\omega=b+\frac{2t}{1+tt},$$

patet etiam t aequari functioni ipsius z hincque etiam tang.θ, ita ut hinc pro quovis casu aequatio inter coordinatas orthogonales x et y erui queat.

23. Investigemus nunc praecipua puncta huius curvae ac primo quidem capiamus arcum $EA=90^\circ=\frac{\pi}{2}$ eritque angulus ω rectus et distantia CA ad curvam erit normalis simulque erit curvae diameter, circa quam curva utrinque pari tractu protenditur. Hic igitur erit $\text{tang.}\frac{1}{2}\omega=t=1$ ideoque

$$\text{tang.}\theta=\frac{\sin.\beta}{1+\cos.\beta}=\text{tang.}\tfrac{1}{2}\beta,$$

ita ut $\theta=\frac{1}{2}\beta$, unde invento hoc angulo β, cuius cosinus est $\frac{1}{b}$, erit angulus $ECA=\frac{\pi}{2}-n\beta$. Ipsa autem distantia CA erit $b+1$, quae erit maxima, ad quam curva pertingere potest.

24. Consideremus nunc portionem huius curvae a puncto E retro protensam ac sumamus arcum EI quadranti aequalem, unde statui oportebit $\omega=-\frac{\pi}{2}$, atque in hoc puncto I erit distantia $CI=b-1$, quae est omnium minima, ad quam curva descendere potest, hicque iterum erit CI ad curvam normalis pariterque eius diameter, unde sufficiet curvam tantum ab A per E usque ad I descripsisse.

25. Hoc igitur casu ob $t=-1$ erit . $\theta=\text{Atang.}\frac{-\sin.\beta}{1-\cos.\beta}$ sicque iste angulus θ erit negativus eiusque tangens $\frac{\sin.\beta}{1-\cos.\beta}$, quae expressio est cotangens anguli $\frac{1}{2}\beta$, sicque erit $-\theta=\frac{\pi}{2}-\frac{1}{2}\beta$, unde prodit angulus

$$ECI = \varphi = -\tfrac{\pi}{2} + 2n(\tfrac{\pi}{2} - \tfrac{1}{2}\beta) = (n - \tfrac{1}{2})\pi - n\beta,$$

quamobrem angulus inter distantiam maximam $CA = b + 1$ et minimam $CI = b - 1$ interceptus erit $ACI = (n-1)\pi$, prorsus uti supra est inventus.

26. Consideremus denique casum, quo arcus *EZ* semiperipheriae aequalis accipitur sive $\omega = \pi$, ubi ergo distantia curvam iterum tanget; tum igitur erit $t = \infty$ et tang.$\theta =$ tang.β ideoque $\theta = \beta$ sicque erit angulus $\varphi = \pi - 2n\beta$, qui est duplo maior quam angulus *ECA*, prorsus ut indoles diametri postulat. Ceterum hic notasse iuvabit omnes formulas hic inventas ad praecedentes reduci posse, si loco *t* scribatur $\frac{1-t}{1+t}$ simulque angulus φ minuatur angulo $ECA = \frac{\pi}{2} - n\beta$.