THE SOLUTION OF A QUESTION RELATING TO THE CALCULATION OF THE PROBABILITY OF HOW MUCH TWO SPOUSES SHOULD PAY, SO THAT AFTER THE DEATH OF EACH, A CERTAIN SUM OF MONEY MAY BE PAID TO THEIR HEIRS

[E599]

Opuscula analytica 2, 1785, p. 315-330

1. Here we assume a public treasury of this kind to be established, the resources of which shall be able to increase by a twentieth part each year, thus so that the sum of 100 Rubles may grow to 105 Rub. after a year ; whereby if for the sake of brevity we may put $\frac{105}{100}=\lambda$, the present sum of money $=C$ after n years will be estimated to become $\lambda^n C$. Moreover in turn a certain sum of money C requiring to be released after n years is considered at the present time to have the value $=\frac{C}{\lambda^n}$.

2. Now we may put the sum of money, which both spouses choose to accrue after the death of each, to be $=1000$ Rub., from which it is understood, if the time of this solution may be known, besides with the number of years lapsed being $=n$ its present value is going to become $=\frac{1000}{\lambda^n}$. Therefore these spouses at the present time are under obligation to bring together such an amount to the treasury. Truly since especially the time of the payment shall be uncertain, if indeed it must happen at last after the death of each, and truly the present value of this sum will be required to be determined following the rules of the calculus of probability, demanded from longevity observations. In the end I may use this table, which at one time I placed in a volume of the Memoirs of the Berlin Academy [Academy of Sciences of Berlin, Vol. 16, (1760), but not published until 1797, p.144-164], where, if a great number M of infants born at the same time may be considered, I have indicated the number of these still present after n years by the letter $(n)M$; from which it is understood such a letter (n) to designate a smaller fractions from these, for whom the number of years would be greater than the number n, and finally around 100 years to go to zero absolutely. Therefore here we may set out a table of these values for the individual years elapsed.

(1) = 0,804	(25) = 0,552	(49) = 0,370	(73) = 0,145
(2) = 0,768	(26) =0,544	(50) = 0,362	(74) = 0,135
(3) = 0,736	(27) = 0,535	(51) = 0,354	(75) = 0,125
(4) = 0,709	(28) = 0,525	(52) = 0,345	(76) = 0,114
(5) = 0,688	(29) = 0,516	(53) = 0,336	(77) = 0,104
(6) = 0,676	(30) = 0,507	(54) = 0,327	(78) = 0,093
(7) = 0,664	(31) = 0,499	(55) = 0,319	(79) = 0,082

(8) = 0,653	(32) = 0,490	(56) = 0,310	(80) = 0,072
(9) = 0,646	(33) = 0,482	(57) = 0,301	(81) = 0,063
(10) = 0,639	(34) = 0,475	(58) = 0,291	(82) = 0,054
(11) = 0,633	(35) = 0,468	(59) = 0,282	(83) = 0,046
(12) = 0,627	(36) = 0,461	(60) = 0,273	(84) = 0,039
(13) = 0,621	(37) = 0,454	(61) = 0,264	(85) = 0,032
(14) = 0,616	(38) = 0,446	(62) = 0,254	(86) = 0,026
(15) = 0,611	(39) = 0,439	(63) = 0,245	(87) = 0,020
(16) = 0,606	(40) = 0,432	(64) = 0,235	(88) = 0,015
(17) = 0,601	(41) = 0,426	(65) = 0,225	(89) = 0,011
(18) = 0,596	(42) = 0,420	(66) = 0,215	(90) = 0,008
(19) = 0,590	(43) = 0,413	(67) = 0,205	(91) = 0,006
(20) = 0,584	(44) = 0,406	(68) = 0,195	(92) = 0,004
(21) = 0,577	(45) = 0,400	(69) = 0,185	(93) = 0,003
(22) = 0,571	(46) = 0,393	(70) = 0,175	(94) = 0,002
(23) = 0,565	(47) = 0,386	(71) = 0,165	(95) = 0,001
(24) = 0,559	(48) = 0,378	(72) = 0,155	

3. Now for the present time we may put the age of the husband to be $= a$ years, of the wife truly $= b$ years, and so that the reasoning requiring to be put in place may be seen more clearly, we may imagine at the same time a large number of such couples, which shall be *N,* to be present of the same ages, who equally after the death of each may choose for their heirs to accrue the sum of 1000 Rub., from which, if the initial sum requiring to be paid may be put $= x$, the treasury may accept the sum *Nx* from all these .

4. But if it may be smiled at more, so that same amount of money *x* may not be paid at once from the initial total, but rather may be released distributed equally through a whole lifetime, we may apply our calculation to a two-fold solution, while by the first a sum $= x$ is released at once into the treasury, but the other above releases a certain sum $= z$ each year, evidently as long as not only both spouses were surviving, but also only one or the other was surviving. But for the complete solution, if anyone were to desire the whole sum of money to be paid at once initially, for this case it will be required to put $z = 0$ and the letter x will indicate the amount sought to be paid. But if anyone were to prefer the amount to be distributed equally through a whole lifetime, it would be required to put $x = z$ and *z* will be the sum requiring to be paid each year until the death of each spouse.

5. With these established, at once from the beginning for all these couples *N,* a sum will be paid $= Nx$. Now we may see, after *n* years had passed, how many marriages hitherto shall survive, both whole as well as dissolute, while evidently meanwhile one or the other had died ; then indeed for these individuals a sum $= z$ is to required to be paid into the treasury, the present value of which is estimated to be $= \frac{z}{\lambda^n}$. Besides truly for any current year it is required to find, how many marriages may have ended completely ;

indeed for as many as this it may eventuate, just as many of these with heirs must be paid that premium of 1000 Rub., of which therefore the present value will be $\frac{1000}{\lambda^n}$. Therefore in this way our calculation is required to be pursued as far as to the final limit of human life, and since both all the expenses as well as the returns were reduced to the present time, it may be agreed these to be equal to each other, from which it will be permitted to determine either x or z as it pleases.

6. From these premises we may begin from the first year, of which initially there may be put to be present N husbands, all of the same age $=a$, and just as many wives all of the same age $=b$, from which the treasury has taken the sum $=Nx$. Now therefore with the first year passing, following the table above, the number of husbands brought forwards hitherto surviving will be $\frac{(a+1)}{(a)}N$ and thus the number meanwhile deceased will be $\frac{(a)-(a+1)}{(a)}N$. In a similar manner the number of wives surviving hitherto will be $\frac{(b+1)}{(b)}N$, but the number of these, who meanwhile have died, will be $\frac{(b)-(b+1)}{(b)}N$. Therefore because any of these husbands surviving initially had a wife, this proportion may be established : so that the initial number of all the wives itself may be had to the number of these surviving , thus as the number of men surviving after one year to the number of these, the wives of whom hitherto will be surviving, which thus will be the number $\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}N$, by which each [couple] pays the sum $=z$ into the treasury ; of which the present value since it shall be $\frac{z}{\lambda}$, hence the value will arise

$$=\frac{(a+1)(b+1)}{(a)(b)}\cdot\frac{Nz}{\lambda}.$$

Then truly the number of these surviving husbands, who meanwhile have lost wives, will be

$$\frac{(a+1)}{(a)}\left(1-\frac{(b+1)}{(b)}\right)N;$$

who since likewise they pay the sum z into the treasury, that will be related to the initial sum

$$\frac{(a+1)}{(a)}\left(1-\frac{(b+1)}{(b)}\right)\frac{Nz}{\lambda},$$

from which hence it will be apparent the value taken together with the preceding to become

$$\frac{(a+1)}{(a)}\frac{Nz}{\lambda},$$

that which by itself is evident, because any surviving husband is held to pay thus sum z, evidently whether his wife lives or otherwise.

7. Now we will consider also these husbands, who have died within this year, of which the number is $\left(1-\frac{(a+1)}{(a)}\right)N;$ where two cases present themselves. The one case regards these cases the husbands, of whom hitherto the wives have survived, the number of which is found by the above analogy: so that the number of all the wives living initially itself is had to the number of these surviving after a year, thus as the number of men meanwhile dying to the number of these, of whom the wives still are living ; which number therefore will be

$$\frac{(b+1)}{(b)}\left(1-\frac{(a+1)}{(a)}\right)N;$$

from which individuals each also may bring together into the treasury a sum $=z$, of which the value relative to the initial value will be :

$$\frac{(b+1)}{(b)}\left(1-\frac{(a+1)}{(a)}\right)\frac{Nz}{\lambda},$$

from which all the returns from the first year passing flowing into the treasury will be :

$$\frac{Nz}{\lambda}\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}\right),$$

which quantity therefore consists of three parts. In the first place evidently the value $\frac{z}{\lambda}$ is multiplied by the number of husbands surviving, which is $\frac{(a+1)}{(a)}N$, then also by the number of wives surviving, which is $\frac{(b+1)}{(b)}N.$ Hence moreover the number of hitherto whole marriages must be taken away, since each does not provide z for two, but only z for one.

8. The next case considers these husbands [who have already died], of whom the wives are no longer alive. But from the preceding calculation it is apparent the number of these husbands passed away, of whom the wives too have passed away, to be

$$\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right)N.$$

Therefore just as many marriages have come to an end completely, of which therefore for the heirs for the treasury requiring to be paid will constitute the premium 1000 Rubles ; so that since it must be paid at once, no interest meanwhile has been able to be acquired, from which these expenses relative to the initial amount even now will prevail :

$$1000N\left(1-\frac{(a+1)}{(a)}-\frac{(b+1)}{(b)}+\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}\right).$$

[Note : in all there are four subsets into which the original set of N husband/wife pairs is divided:

1. Where both partners survive for the year, being $\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}N$;
2. Where the wife survives but not the husband: $\frac{(b+1)}{(b)}\left(1-\frac{(a+1)}{(a)}\right)N$;
3. Where the husband survives but not the wife: $\frac{(a+1)}{(a)}\left(1-\frac{(b+1)}{(b)}\right)N$;
4. Where neither the husband nor the wife survives : $\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right)N$.]

9. We may proceed now to the second year, of which initially the surviving husbands will be $\frac{(a+1)}{(a)}N,$ the wives truly $\frac{(b+1)}{(b)}N,$ among which the hitherto the whole marriages remain :

$$\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}N,$$

truly with the number broken

$$\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}\right)N,$$

thus so that the number of marriages completely finished shall be :

$$N\left(1-\frac{(a+1)}{(a)}-\frac{(b+1)}{(b)}+\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}\right)$$

or

$$N\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right).$$

But truly at the end of the second year the number of husbands hitherto surviving will be $\frac{(a+2)}{(a)}N,$ truly the number of these, who have died in these two years, $\left(1-\frac{(a+2)}{(a)}\right)N$. And in a similar manner the number of wives surviving hitherto will be $\frac{(b+2)}{(b)}N,$ truly of these who have died in the two years, $\left(1-\frac{(b+2)}{(b)}\right)N$, from which the number of marriages of which both partners have died in this two year period will be

$$\left(1-\frac{(a+2)}{(a)}\right)\left(1-\frac{(b+2)}{(b)}\right)N.$$

Whereby since the number of marriages expiring completely in the first year was :

$$\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right)N,$$

the number of these, which have concluded within this second year, will be

$$\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+2)}{(a)}-\frac{(b+2)}{(b)}-\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}+\frac{(a+2)}{(a)}\cdot\frac{(b+2)}{(b)}\right)N;$$

because, for which individuals, the sum of one thousand Rubles is paid, the total sum from interest related to the beginning will prevail :

$$\frac{1000N}{\lambda}\left(\frac{(a+1)-(a+2)}{(a)}+\frac{(b+1)-(b+2)}{(b)}-\frac{(a+1)(b+1)-(a+2)(b+2)}{(a)(b)}\right).$$

10. Because now at the beginning of the second year the number of marriages both whole as well as broken was

$$N\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}\right),$$

if hence we may take away the number of marriages expiring completely in this year, the number of these will remain, which will pay the sum z around the end of the second year, the number of which will be

$$N\left(\frac{(a+2)}{(a)}+\frac{(b+2)}{(b)}-\frac{(a+2)}{(a)}\cdot\frac{(b+2)}{(b)}\right).$$

Therefore as often as from these the sum z is brought into the treasury, from which the total value from the simple interest of two years will prevail from the beginning :

$$\frac{Nz}{\lambda^2}\left(\frac{(a+2)}{(a)}+\frac{(b+2)}{(b)}-\frac{(a+2)}{(a)}\cdot\frac{(b+2)}{(b)}\right).$$

11. Now from these established we will be able to progress for any year following. Therefore now we may put the number of years passed to be n and in this time the number of husbands surviving will be $\frac{(a+n)}{(a)}N,$ but before now the number of expired husbands will be $\left(1-\frac{(a+n)}{(a)}\right)N.$ In the same manner the number of wives hitherto

surviving will be $\frac{(b+n)}{(b)}N,$ of the deceased truly $\left(1-\frac{(b+n)}{(b)}\right)N$, from which the number of marriages both whole as well as broken at this time will be

$$\left(\frac{(a+n)}{(a)}+\frac{(b+n)}{(b)}-\frac{(a+n)}{(a)}\cdot\frac{(b+n)}{(b)}\right)N;$$

but truly the number of marriages completely finished in this whole time will be

$$\left(1-\frac{(a+n)}{(a)}\right)\left(1-\frac{(b+n)}{(b)}\right)N.$$

12. Now we may proceed to the end of this year and in a similar manner the number of marriages, either whole or broken, now will be

$$N\left(\frac{(a+n+1)}{(a)}+\frac{(b+n+1)}{(b)}-\frac{(a+n+1)}{(a)}\cdot\frac{(b+n+1)}{(b)}\right),$$

from which the individuals pay the sum z into the treasury, of which the value at the start of this transaction is z from which the whole sum around the end of this year paid into the treasury from the beginning will prevail :

$$\frac{Nz}{\lambda^{n+1}}\left(\frac{(a+n+1)}{(a)}+\frac{(b+n+1)}{(b)}-\frac{(a+n+1)}{(a)}\cdot\frac{(b+n+1)}{(b)}\right).$$

Then truly the number of all the marriages from this same beginning finished after a time of $n+1$ years will be

$$N\left(1-\frac{(a+n+1)}{(a)}\right)\left(1-\frac{(b+n+1)}{(b)}\right);$$

whereby , since as far as to the start of this year now marriages were to be finished

$$N\left(1-\frac{(a+n)}{(a)}\right)\left(1-\frac{(b+n)}{(b)}\right),$$

the number of these, which at last have ended this year, will be

$$N\left(\frac{(a+n)-(a+n+1)}{(a)}+\frac{(b+n)-(b+n+1)}{(b)}-\frac{(a+n)(b+n)-(a+n+1))(b+n+1)}{(a)(b)}\right).$$

Therefore because the sum of 1000 Rubles must be paid by these individuals, the value of these expenses related to the beginning will be :

$$\frac{1000N}{\lambda^n}\left(\frac{(a+n)-(a+n+1)}{(a)}+\frac{(b+n)-(b+n+1)}{(b)}-\frac{(a+n)(b+n)-(a+n+1))(b+n+1)}{(a)(b)}\right).$$

13. Now we may gather together both all the returns arising from the quantity z as well as the expenditures arising from the payment of these 1000 Rubles ; and indeed in the first place all the returns, which besides the principal sum Nx introduced into the treasury, have been found to be expressed by the three following series:

$$Nz\left\{\begin{array}{lllll} \frac{(a+1)}{\lambda(a)} & +\frac{(a+2)}{\lambda^2(a)} & +\frac{(a+3)}{\lambda^3(a)} & +\cdots+ & \frac{(a+n)}{\lambda^n(a)} \\ +\frac{(b+1)}{\lambda(b)} & +\frac{(b+2)}{\lambda^2(b)} & +\frac{(b+3)}{\lambda^3(b)} & +\cdots+ & \frac{(b+n)}{\lambda^n(b)} \\ -\frac{(a+1)(b+1)}{\lambda(a)(b)} & -\frac{(a+2)(b+2)}{\lambda^2(a)(b)} & -\frac{(a+3)(b+3)}{\lambda^3(a)(b)} & -\cdots- & \frac{(a+n)(b+n)}{\lambda^n(a)(b)} \end{array}\right\}.$$

So that therefore for the sake of brevity we may establish

$$\begin{aligned} P &= \frac{(a+1)}{\lambda}+\frac{(a+2)}{\lambda^2}+\frac{(a+3)}{\lambda^3}+\frac{(a+4)}{\lambda^4}+\cdots+\frac{(95)}{\lambda^{95-a}}, \\ Q &= \frac{(b+1)}{\lambda}+\frac{(b+2)}{\lambda^2}+\frac{(b+3)}{\lambda^3}+\frac{(b+4)}{\lambda^4}+\cdots+\frac{(95)}{\lambda^{95-b}}, \\ R &= \frac{(a+1)(b+1)}{\lambda}+\frac{(a+2)(b+2)}{\lambda^2}+\frac{(a+3)(b+3)}{\lambda^3}+\text{etc.} \end{aligned}$$

the whole sum will be returned

$$Nx+Nz\left(\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}\right).$$

In a similar manner we may gather together all the expenditures into one sum, which sum will be composed from the six following series :

$$1000N\left\{\begin{array}{lllll}
\frac{(a)}{(a)} & +\frac{(a+1)}{\lambda(a)} & +\frac{(a+2)}{\lambda^2(a)} & +\frac{(a+3)}{\lambda^3(a)} & +\text{etc.}\\
-\frac{(a+1)}{(a)} & -\frac{(a+2)}{\lambda(a)} & -\frac{(a+3)}{\lambda^2(a)} & -\frac{(a+4)}{\lambda^3(a)} & -\text{etc.}\\
+\frac{(b)}{(b)} & +\frac{(b+1)}{\lambda(b)} & +\frac{(b+2)}{\lambda^2(b)} & +\frac{(b+3)}{\lambda^3(b)} & +\text{etc.}\\
-\frac{(b+1)}{(b)} & -\frac{(b+2)}{\lambda(b)} & -\frac{(b+3)}{\lambda^2(b)} & -\frac{(b+4)}{\lambda^3(b)} & -\text{etc.}\\
-\frac{(a)(b)}{(a)(b)} & -\frac{(a+1)(b+1)}{\lambda(a)(b)} & -\frac{(a+2)(b+2)}{\lambda^2(a)(b)} & -\frac{(a+3)(b+3)}{\lambda^3(a)(b)} & -\text{etc.}\\
+\frac{(a+1)(b+1)}{(a)(b)} & +\frac{(a+2)(b+2)}{\lambda(a)(b)} & +\frac{(a+3)(b+3)}{\lambda^2(a)(b)} & +\frac{(a+4)(b+4)}{\lambda^3(a)(b)} & +\text{etc.}
\end{array}\right\}.$$

14. It is evident also here the sums of the three series put in place *P, Q, R* are able to be conveniently called in to help, and hence all the expenditures related to the initial expression are going to become of the following form

$$1000N\left(\frac{(a)+P}{(a)}-\frac{\lambda P}{(a)}+\frac{(b)+Q}{(b)}-\frac{\lambda Q}{(b)}-\frac{(a)(b)+R}{(a)(b)}+\frac{\lambda R}{(a)(b)}\right).$$

or

$$1000N\left(1+\frac{(1-\lambda)P}{(a)}+\frac{(1-\lambda)Q}{(b)}+\frac{(\lambda-1)R}{(a)(b)}\right);$$

consequently the equation for the solution of the equation proposed will be

$$x+z\left(\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}\right)=1000\left(1+\frac{(1-\lambda)P}{(a)}+\frac{(1-\lambda)Q}{(b)}+\frac{(\lambda-1)R}{(a)(b)}\right).$$

15. Now since there shall be

$$\lambda=\frac{105}{100},$$

there will be

$$\lambda-1=\frac{5}{100} \text{ and } 1000(\lambda-1)=50,$$

from which our equation will be

$$x+z\left(\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}\right)=1000-\frac{50P}{(a)}-\frac{50Q}{(b)}+\frac{50R}{(a)(b)}.$$

On account of which if the total premium may be paid at once from the beginning, thus so that there shall be $z=0$, this will be the premium :

$$x=1000-\frac{50P}{(a)}-\frac{50Q}{(b)}+\frac{50R}{(a)(b)}.$$

But if we may wish, so that the premium may be distributed equally through the whole interval of the time as far as to the death of each couple, there must be put $x=z$ and thus the following annual contribution will be produced :

$$z=\frac{1000-\frac{50P}{(a)}-\frac{50Q}{(b)}+\frac{50R}{(a)(b)}}{1+\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}},$$

and thus the whole matter returns to this, so that for any age of each spouse the values of the three series indicated by the letters *P, Q, R* may be investigated, which therefore we may set out in the following.

ESTABLISHMENT OF THE VALUES *P* AND *Q*

16. Because the series *Q* is defined in a similar manner from the age *b*, as the series *P* must be elicited from the age *a* , it will suffice to have expanded out only either for the individual ages.
Therefore since there shall be

$$P=\frac{(a+1)}{\lambda}+\frac{(a+2)}{\lambda^2}+\frac{(a+3)}{\lambda^3}+\cdots+\frac{(95)}{\lambda^{95-a}},$$

if all the terms of this series may be reduced to the same denominator λ^{95-a} and may be set out in the backwards order, there will become :

$$P=\frac{1}{\lambda^{95-a}}\Big((95)+(94)\lambda+(93)\lambda^2+\cdots+(a+1)\lambda^{94-a}\Big).$$

17. But the setting out of this series might be not a little tedious, if we may wish to resolve the same by the individual years. But because the values of the characters (*a*) and (*b*) thus are not certain, so that we may be allowed to acknowledge a little aberration, it will suffice to take together five terms following each other in turn, and to put in place of these five times the mean value of these ; thus so that for the first five terms there may be able to write $5(93)\lambda^2$, with which done the value of our letter *P* will be

$$P=\frac{5}{\lambda^{95-a}}\Big((93)\lambda^2+(88)\lambda^7+(83)\lambda^{12}+\cdots+(a+3)\lambda^{92-a}\Big);$$

whereby if we may designate this series by the letter p , so that there shall be

$$p=(93)\lambda^2+(88)\lambda^7+(83)\lambda^{12}+\cdots+(a+3)\lambda^{92-a},$$

with the value of the letter p found there will be

$$P=\frac{5p}{\lambda^{95-a}}$$

and hence

$$\frac{P}{(a)}=\frac{5p}{(a)\lambda^{95-a}}.$$

In the same manner, if there may be put

$$q=(93)\lambda^2+(88)\lambda^7+(83)\lambda^{12}+\cdots+(b+3)\lambda^{92-b},$$

there will be had:

$$Q=\frac{5q}{\lambda^{95-b}}$$

and

$$\frac{Q}{(b)}=\frac{5q}{(b)\lambda^{95-b}}.$$

ESTABLISHMENT OF THE THIRD VALUE R

18. The series, which we have designated by the letter *R*, was this :

$$R=\frac{(a+1)(b+1)}{\lambda}+\frac{(a+2)(b+2)}{\lambda\lambda}+\cdots+\frac{(a+n)(b+n)}{\lambda^n}+\cdots,$$

which series therefore it will be required to be continued so far, then the following terms vanish, which happens, if either of the numbers $(a+n)$ or $(b+n)$ will exceed 95, from which, at once it rises from the greater of these two numbers to that limit, and the series is considered to be terminated here.

19. But because both the numbers *a* and *b* enter into our calculation equally and nor does any difference thence arise, even if these two letters may be interchanged between themselves, thus so that *a* will denote the age of the wife and *b* the age of the husband, we will be able to assume the age *a* always to be greater than the age *b;* for if the wife were older than the husband, then *a* will designate the age of the wife, and *b* the age of the husband. Whereby since we will consider the age *b* as the lesser, we will

designate the difference by the letter *d*, thus so that there shall be $b = a - d$, where indeed the difference *d* will be nothing, if both spouses may have had the same age.

20. With this observed the final limit of our series will be there, where there becomes $a + n = 95$ and thus $n = 95 - a$, thus so that now our series shall become

$$R = \frac{(a+1)(b+1)}{\lambda} + \frac{(a+2)(b+2)}{\lambda\lambda} + \cdots + \frac{(95)(b+95-a)}{\lambda^{95-a}},$$

where therefore the final term is $\frac{(95)(95-d)}{\lambda^{95-a}}$. Now we may reduce, as before, all these fractions to the same denominator λ^{95-a} and we may put the whole series in reverse order, and we find

$$R = \frac{1}{\lambda^{95-a}} \left\{ \begin{matrix} (95)(95-d) + (94)(94-d)\lambda + (93)(93-d)\lambda^2 \\ +(92)(92-d)\lambda^3 + \cdots + (a+1)(a+1-d)\lambda^{94-a} \end{matrix} \right\},$$

of which therefore it will be required to compute the sum of the series according to the individual values of both the numbers *a* and *d*.

21. But so that this calculation may be rendered easier, again five terms as we have made before, we may contract into one, while evidently the sum of these we will estimate to be equal to five times the middle term among these, with which done we will have

$$R = \frac{5}{\lambda^{95-a}} \left((93)(93-d)\lambda^2 + (88)(88-d)\lambda^7 + \cdots + (a+3)(a+3-d)\lambda^{92-a} \right).$$

So that therefore if we may put

$$r = (93)(93-d)\lambda^2 + (88)(88-d)\lambda^7 + \cdots + (a+3)(a+3-d)\lambda^{92-a},$$

with the value of this series r found, the value itself which we seek,

$$R = \frac{5r}{\lambda^{95-a}};$$

and because for our calculation we need the value $\frac{R}{(a)(b)}$, there will be

$$\frac{R}{(a)(b)} = \frac{5r}{(a)(b)\lambda^{95-a}}$$

and with these values for the individual cases found our general equation will be

$$x+z(\tfrac{P}{(a)}+\tfrac{Q}{(b)}-\tfrac{R}{(a)(b)})=1000-\tfrac{50P}{(a)}-\tfrac{50Q}{(b)}+\tfrac{50R}{(a)(b)}.$$

22. Because here two numbers occur, *a* and *d,* this calculation demands much more labor than the preceding for the series *P* and *Q;* so that which we may ease, both the numbers *a* and *d* we may assume to increase or decrease by fives; on this account it will be required to set out all the cases for the various values of the differences *d,* as we may put to be successively 0, 5, 10, 15, 20 etc. From which we may set out these cases in the following manner.

CASE I

Where $d=0$ and thus $b=a$

Therefore here there will be:

$$r=(93)^2\lambda^2+(88)^2\lambda^7+(83)^3\lambda^{12}+\cdots+(a+3)^2\lambda^{92-a},$$

from which there becomes

$$R=\tfrac{5r}{\lambda^{95-a}}\ \text{ et }\ \tfrac{R}{(a)^2}=\tfrac{5r}{\lambda^{95-a}(a)^2}.$$

CASE II

Where $d=5$ and thus $b=a-5$

Then therefore there will be

$$r=(93)(88)\lambda^2+(88)(83)\lambda^7+(83)(78)\lambda^{12}+\cdots+(a+3)(a-2)\lambda^{92-a},$$

from which there becomes

$$R=\tfrac{5r}{\lambda^{95-a}}\ \text{ et }\ \tfrac{R}{(a)(a-5)}=\tfrac{5r}{\lambda^{95-a}(a)(a-5)}.$$

CASE III

Where $d=10$ and thus $b=a-10$

Here therefore there will be

$$r=(93)(83)\lambda^2+(88)(78)\lambda^7+(83)(73)\lambda^{12}+\cdots+(a+3)(a-7)\lambda^{92-a},,$$

from which there becomes

$$R=\frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-10)}=\frac{5r}{\lambda^{95-a}(a)(a-10)}.$$

CASE IV

Where $d=15$ and thus $b=a-15$

In this case there will be

$$r=(93)(78)\lambda^2+(88)(73)\lambda^7+(83)(68)\lambda^{12}+\cdots+(a+3)(a-12)\lambda^{92-a},,$$

from which there becomes

$$R=\frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-15)}=\frac{5r}{\lambda^{95-a}(a)(a-15)}.$$

CASE V

Where $d=20$ and thus $b=a-20$

Then therefore there will be

$$r=(93)(73)\lambda^2+(88)(68)\lambda^7+(83)(63)\lambda^{12}+\cdots+(a+3)(a-17)\lambda^{92-a},$$

from which there becomes

$$R=\frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-20)}=\frac{5r}{\lambda^{95-a}(a)(a-20)}.$$

CASE VI

Where $d=25$ and thus $b=a-25$

In this case there will be

$$r=(93)(68)\lambda^2+(88)(63)\lambda^7+(83)(58)\lambda^{12}+\cdots+(a+3)(a-22)\lambda^{92-a},$$

from which there becomes

$$R=\frac{5r}{\lambda^{95-a}} \quad \text{et} \quad \frac{R}{(a)(a-25)}=\frac{5r}{\lambda^{95-a}(a)(a-25)}.$$

CASE VII

Where $d=30$ and thus $b=a-30$

Then therefore there will be

$$r=(93)(63)\lambda^2+(88)(58)\lambda^7+(83)(53)\lambda^{12}+\cdots+(a+3)(a-27)\lambda^{27-a},$$

from which there becomes

$$R=\frac{5r}{\lambda^{95-a}} \quad \text{et} \quad \frac{R}{(a)(a-30)}=\frac{5r}{\lambda^{95-a}(a)(a-30)}.$$

THE SOLUTION OF SOME MORE DIFFICULT QUESTIONS

IN THE CALCULUS OF PROBABILITY

[E600]

Opuscula analytica 2**,** 1785, p. 331-346

1. A game commonly established everywhere [*i.e.* a lottery] gives the occasion for these questions, where from ninety tickets designated by the numbers 1, 2, 3, 4, ... 90 , at appointed times five of the allotted tickets are accustomed to be drawn. Hence questions of this kind therefore arise: evidently how great the probability shall be, so that, after a given number drawn had been carried out, either all ninety numbers will have been completed, or perhaps either 89 , 88 or fewer. Therefore these questions, as being the most difficult, are to be resolved here by the principles of the calculus of probability that I have put into use some time ago now. Nor let the objections of the most illustrious d'Alembert deter me, who has tried to render this calculation invalid. For after the great Geometer said goodbye to his mathematical studies, on these too he would appear to have proclaimed war, while generally he has attacked to overturn the most solid foundations. For whatever these objections of the maximum weight might be in the writings of the ignorant, yet thence it is not at all to be feared any detriment is going to be brought forwards from this knowledge.

2. Those who would take pains in investigations of this kind, easily observe the resolution of questions of this kind to demand an especially complex calculation, which the aid of a certain symbol has been able to overcome for me, which now I have used with the greatest success a number of times. Evidently a symbol of this kind

$$\left(\frac{p}{q}\right),$$

which may represent a fraction enclosed in brackets, specifies this product for me :

$$\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-3}{3}\cdot\frac{p-3}{4}\cdots\frac{p-q+1}{q},$$

of which the value can be shown easily for any case. Moreover concerning this character it will help to have observed the following.

1°. Always there shall be

$$\left(\frac{p}{q}\right)=\left(\frac{p}{p-q}\right).$$

2°. If $q = 0$, there is always

$$\left(\frac{p}{0}\right) = 1.$$

3°. If *q* shall be either a negative number or a number greater than *p,* the value of $\left(\frac{p}{q}\right)$ itself always shall be $= 0$.

4°. Thence if p shall be a negative number, then this formula

$$\left(\frac{-p}{q}\right)$$

can be reduced to this :

$$\pm\left(\frac{p-q+1}{q}\right),$$

where the $+$ prevails, if the number *q* were even, − truly, if odd; from which it is apparent this same form also can be changed into the form $\pm\left(\frac{p-q+1}{p-1}\right)$.

3. From these premises, I am going to treat most generally the questions arising from the game mentioned. Clearly I will indicate the number of tickets by the letter *m,* which I assume to be designated by the individual different letters *a, b, c, d* etc., lest the use of absolute numbers may produce confusion. Then by some draw I may suppose to withdraw *i* tickets from these, from which the number of all the variations, which can pertain to these treatments, will be $=\left(\frac{m}{i}\right)$. Besides if a number of these draws were put in place successively $= n$, it is apparent from the principles of combinations the number of all these variations, to which they may be able to pertain, to be $\left(\frac{m}{i}\right)^n$. Therefore I may run through the following problems in this manner.

PROBLEM 1

If the number of tickets designated by the letters a, b, c, d etc. shall be m, and subsequently any i of the tickets in the draw may be removed, and now the number of draws performed were n, there is sought, how great the probability would be, so that all the m letters a, b, c, d etc. may have emerged.

SOLUTION

4. Here in the first place it is required to be observed, since in n draws the number of tickets extracted is in, all the letters cannot be brought out, unless there were $in > m$ and thus

$$n > \tfrac{m}{i}$$

or at least not less. Now the number $\varDelta$ shall indicate the number of all the variations, which can eventuate in these n draws, and there will be, as we have indicated now,

$$\varDelta = \left(\tfrac{m}{i}\right)^n;$$

which since it shall be the number of all the possible cases, for our question hence all the cases can be excluded, which contain fewer than m letters.

Therefore in the first place, if the number of letters shall be only $= m-1$, because it can happen in m ways, the number of cases, which contain only $m-1$ or fewer letters, will be

$$m\left(\tfrac{m-1}{i}\right)^n,$$

which number we may put $= A$.

In a similar manner, if two letters may be excluded, because this can happen in $\left(\tfrac{m}{2}\right)$ ways, the number of cases containing only $m-2$ or fewer letters will be

$$\left(\tfrac{m}{2}\right)\cdot\left(\tfrac{m-2}{i}\right)^n,$$

which number we will indicate by the letter B .

Again C shall be the number of cases, which contain only $m-3$ or fewer letters, and there will be

$$C = \left(\tfrac{m}{3}\right)\cdot\left(\tfrac{m-3}{i}\right)^n.$$

And in the same manner there becomes

$$D = \left(\tfrac{m}{4}\right)\cdot\left(\tfrac{m-4}{i}\right)^n,\ \ E = \left(\tfrac{m}{5}\right)\cdot\left(\tfrac{m-5}{i}\right)^n \text{ etc.}$$

And with these elements in place the number of all the cases , which may contain all the m letters, to be

$$\varDelta - A + B - C + D - \text{etc.},$$

which number we will indicate by the letter Σ.

5. It is evident this number Σ is able to be determined by the theory of combinations alone, and thus certainly to be liable to no doubt, thus just as a geometrical truth may be able to be shown. Hence moreover following the principles of probability the number of favorable cases divided by the number of all the possible cases will provide the probability sought ; which therefore if it may be put $=\Pi$, there will become

$$\Pi = \frac{\Sigma}{\Delta}.$$

Whereby since there shall be

$$\Sigma = \Delta - A + B - C + D - \text{etc.},$$

with the values assigned for the letters Δ, A, B, C etc. being substituted, there will become :

$$\Sigma = \left(\frac{m}{i}\right)^n - \left(\frac{m}{1}\right)\left(\frac{m-1}{i}\right)^n + \left(\frac{m}{2}\right)\left(\frac{m-2}{i}\right)^n - \left(\frac{m}{3}\right)\left(\frac{m-3}{i}\right)^n + \text{etc.},$$

This form divided by $\left(\frac{m}{i}\right)^n$ will give the probability, so that after n draws all m letters will have appeared, from which by necessity this expression Σ must be equal to zero always, as often as there were $n < \frac{m}{i}$, so that also the calculation for the simpler cases requiring to be put in place to happen actually will be apparent. Just as if there were $m = 7,\ n = 3,\ i = 2,$ there will be

$$\left(\frac{m}{i}\right)^n = 21^3 = 9261,$$

$$\left(\frac{m}{1}\right)\left(\frac{m-1}{i}\right)^n = 7\cdot 15^3 = 32625,$$

$$\left(\frac{m}{2}\right)\left(\frac{m-2}{i}\right)^n = 21\cdot 10^3 = 21000,$$

$$\left(\frac{m}{3}\right)\left(\frac{m-3}{i}\right)^n = 35\cdot 6^3 = 7560,$$

$$\left(\frac{m}{4}\right)\left(\frac{m-4}{i}\right)^n = 35\cdot 3^3 = 945,$$

$$\left(\frac{m}{5}\right)\left(\frac{m-5}{i}\right)^n = 21\cdot 1^3 = 21,$$

$$\left(\frac{m}{6}\right)\left(\frac{m-6}{i}\right)^n = 0,$$

from which there is produced $\Sigma = 0$.

6. Therefore provided that *n* shall not be less than $\frac{m}{i}$, there will be always $\Sigma = 0$; but if there were $n = \frac{m}{i}$ or $m = in$, this case is especially remarkable; for then our formula found for Σ can be reduced to the product from the separate constant factors. For there will be

$$\Sigma = \left(\tfrac{m}{i}\right)\left(\tfrac{m-i}{i}\right)\left(\tfrac{m-2i}{i}\right)\left(\tfrac{m-3i}{i}\right)\cdots\left(\tfrac{i}{i}\right);$$

which so that we may show by an example, we assume, as before, $n = 3$ and $i = 2$, truly there shall be $m = 6$; and the former form given for Σ provides $\Sigma = 90$, truly the other gives $\Sigma = 15 \cdot 6 \cdot 1 = 90$.

7. But nevertheless these formulas, if greater numbers may be taken for *n*, become very large, yet in any case by logarithms it will be able to assign the value of the probability Π without difficulty. Indeed since there shall be

$$\frac{A}{\Delta} = \left[\frac{m\left(\frac{m-1}{i}\right)^n}{\left(\frac{m}{i}\right)^n} = m \cdot \left(\frac{(m-1)!}{i!(m-i-1)!} \times \frac{i!(m-i)!}{(m)!} \right)^n = \right] m \cdot \frac{(m-i)^n}{m^n},$$

$$\frac{B}{A} = \frac{m-1}{2} \cdot \frac{(m-1-i)^n}{(m-1)^n},$$

$$\frac{C}{B} = \frac{m-2}{3} \cdot \frac{(m-2-i)^n}{(m-2)^n}$$

etc.,

hence with logarithms taken there will be

$$l\frac{A}{\Delta} = lm - nl\frac{m}{m-i},$$

$$l\frac{B}{A} = l\frac{m-1}{2} - nl\frac{m-1}{m-1-i},$$

$$l\frac{C}{B} = l\frac{m-2}{3} - nl\frac{m-2}{m-2-i}$$

etc.,

from which there is deduced

$$l\tfrac{A}{\Delta} = lm - nl\tfrac{m}{m-i},$$
$$l\tfrac{B}{\Delta} = l\tfrac{A}{\Delta} + l\tfrac{m-1}{2} - nl\tfrac{m-1}{m-1-i},$$
$$l\tfrac{C}{\Delta} = l\tfrac{B}{\Delta} + l\tfrac{m-2}{3} - nl\tfrac{m-2}{m-2-i}$$
etc.,

From which therefore the values $\frac{A}{\Delta}$, $\frac{B}{\Delta}$, $\frac{C}{\Delta}$ etc. are found easily, with which found the probability sought will be

$$\Pi = 1 - \tfrac{A}{\Delta} + \tfrac{B}{\Delta} - \tfrac{C}{\Delta} + \tfrac{D}{\Delta} - \text{etc.}$$

8. We may apply these to the case of the game mentioned initially, where there is $m = 90$ and $i = 5$, and there will be, as follows:

$$l\tfrac{A}{\Delta} = \quad l90 - \; l\tfrac{90}{85} = \quad 1{,}9542425 - n\cdot 0{,}0248236,$$
$$l\tfrac{B}{\Delta} = l\tfrac{A}{\Delta} + l\tfrac{89}{2} - nl\tfrac{89}{84} = l\tfrac{A}{\Delta} + 1{,}6483600 - n\cdot 0{,}0251107,$$
$$l\tfrac{C}{\Delta} = l\tfrac{B}{\Delta} + l\tfrac{88}{3} - nl\tfrac{88}{83} = l\tfrac{B}{\Delta} + 1{,}4673614 - n\cdot 0{,}0254046,$$
$$l\tfrac{D}{\Delta} = l\tfrac{C}{\Delta} + l\tfrac{87}{4} - nl\tfrac{87}{82} = l\tfrac{C}{\Delta} + 1{,}3374593 - n\cdot 0{,}0257054,$$
$$l\tfrac{E}{\Delta} = l\tfrac{D}{\Delta} + l\tfrac{86}{5} - nl\tfrac{86}{81} = l\tfrac{D}{\Delta} + 1{,}2355285 - n\cdot 0{,}0260135,$$
$$l\tfrac{F}{\Delta} = l\tfrac{E}{\Delta} + l\tfrac{85}{6} - nl\tfrac{85}{80} = l\tfrac{E}{\Delta} + 1{,}1512676 - n\cdot 0{,}0263289,$$
$$l\tfrac{G}{\Delta} = l\tfrac{F}{\Delta} + l\tfrac{84}{7} - nl\tfrac{84}{79} = l\tfrac{F}{\Delta} + 1{,}0791813 - n\cdot 0{,}026522$$
etc.

9. Here it is evident, where a greater number of draws *n* may be taken*,* there that same progression converges more quickly, thus so that, if *n* may denote an exceedingly large number, there shall always be going to be produced $\Pi = 1$; then evidently it will be especially probable all the numbers *m* will come forth at once. But on the other hand, if the number *n* may scarcely surpass the minimum value $\frac{m}{i} = 18$, the expansion of these terms may become especially tedious, since there shall be a need for more terms, before the expansion may become vanishing.

We may suppose $n = 100$, so that we may answer this equation, how great the probability shall be, so that after a hundred draws all ninety numbers will have emerged. Here therefore there will be :

$$l\tfrac{A}{\Delta}=9{,}47188,\text{ therefore }\tfrac{A}{\Delta}=0{,}2964,$$
$$l\tfrac{B}{\Delta}=8{,}60917,\text{ therefore }\tfrac{B}{\Delta}=0{,}0407,$$
$$l\tfrac{C}{\Delta}=7{,}53607,\text{ therefore }\tfrac{C}{\Delta}=0{,}0034,$$
$$l\tfrac{D}{\Delta}=6{,}30299,\text{ therefore }\tfrac{D}{\Delta}=0{,}0002,$$
$$l\tfrac{E}{\Delta}=4{,}93719,\text{ therefore }\tfrac{E}{\Delta}=0{,}0000,$$

therefore [the probability of all the favourable cases]

$$\Pi=0{,}7411.$$

10. There shall be $n=200$ and for this case there will be

$$l\tfrac{A}{\Delta}=6{,}98952,\text{ therefore }\tfrac{A}{\Delta}=0{,}00098,$$
$$l\tfrac{B}{\Delta}=3{,}61574,\text{ therefore }\tfrac{B}{\Delta}=0{,}00000,$$

from which the probability is deduced, so that after 200 draws all the numbers will have emerged,

$$\Pi=0{,}99902,$$

which probability certainly is every close to the certitude of everything emerging.

PROBLEM 2

With everything in place, which have been established in the preceding problem, there is sought, how great the probability shall become, so that at least $m-1$ letters will have emerged after n draws.

SOLUTION

11. Here therefore the number of the draws containing all *m* letters is not excluded, from which it is apparent the number of draws to become greater in our present case. But for the calculation to succeed, if the number of these cases may be put to be Σ', I have found to become

$$\Sigma'=\Delta-B+2C-3D+4E-5F+\text{etc.},$$

from which the probability, so that after *n* draws at least $m-1$ of the letters will have emerged, will be

$$\Pi'=\tfrac{\Sigma'}{\Delta}$$

and thus

$$\Pi' = 1 - \tfrac{B}{\Delta} + 2\tfrac{C}{\Delta} - 3\tfrac{D}{\Delta} + 4\tfrac{E}{\Delta} - \text{etc.}$$

12. Therefore in this case there will be

$$\Sigma' = \left(\tfrac{m}{i}\right)^n - \left(\tfrac{m}{2}\right)\cdot\left(\tfrac{m-2}{i}\right)^n + 2\left(\tfrac{m}{3}\right)\cdot\left(\tfrac{m-3}{i}\right)^n - 3\left(\tfrac{m}{4}\right)\cdot\left(\tfrac{m-4}{i}\right)^n + 4\left(\tfrac{m}{5}\right)\cdot\left(\tfrac{m-5}{i}\right)^n - \text{etc.},$$

from which, if an application may be made to the game mentioned, since the letters Δ, A, B, C, D etc. may retain the same values, the calculation put in place from the values found $\tfrac{B}{\Delta}, \tfrac{C}{\Delta}, \tfrac{D}{\Delta}$ etc. will be performed easily. Thus, if after 100 draws the probability is required, so that at least 89 of the numbers will have emerged, on account of

$$\tfrac{B}{\Delta} = 0{,}0407, \ \tfrac{C}{\Delta} = 0{,}0034, \ \tfrac{D}{\Delta} = 0{,}0002$$

this will be the probability

$$\Pi' = 0{,}9655.$$

From which it follows the probability, that only fewer numbers will have emerged, to become 0,0345.

PROBLEM 3

With the same in place as hitherto, it is sought, how great the probability shall be, so that at least $m-2$ *letters may have been extracted after n draws.*

SOLUTION

13. The number of all the cases, which may contain at least $m-2$ letters, thus is defined by the letters established before Δ, A, B, C, D etc., so that there shall be

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.},$$

which expression itself is obtained with the values restored in this manner:

$$\Sigma'' = \left(\tfrac{m}{i}\right)^n - \left(\tfrac{2}{2}\right)\left(\tfrac{m}{3}\right)\cdot\left(\tfrac{m-3}{i}\right)^n + \left(\tfrac{3}{2}\right)\left(\tfrac{m}{4}\right)\cdot\left(\tfrac{m-4}{i}\right)^n - \left(\tfrac{4}{2}\right)\left(\tfrac{m}{5}\right)\cdot\left(\tfrac{m-5}{i}\right)^n + \text{etc.},$$

and hence the probability will be :

$$\Pi = 1 - \tfrac{C}{\Delta} + 3\tfrac{D}{\Delta} - 6\tfrac{E}{\Delta} + 10\tfrac{F}{\Delta} - \text{etc.}$$

Therefore, for the game mentioned before, if the probability may be sought, so that after 100 draws at least 88 numbers will have emerged, there will be found

$$\Pi'' = 0{,}9972,$$

from which the probability, so that the opposite happens, will be $= 0{,}0028$.

GENERAL PROBLEM

With the same in place as before, it is enquired, how great the probability shall be, so that after n draws at least $m-\lambda$ *letters will have emerged.*

SOLUTION

14. The number of cases containing so many letters for the minimum is expressed conveniently by our symbols, so that there shall be

$$\left(\tfrac{m}{i}\right)^n - \left(\tfrac{\lambda}{\lambda}\right)\left(\tfrac{m}{\lambda+1}\right)\cdot\left(\tfrac{m-\lambda-1}{i}\right)^n + \left(\tfrac{\lambda+1}{\lambda}\right)\left(\tfrac{m}{\lambda+2}\right)\cdot\left(\tfrac{m-\lambda-2}{i}\right)^n$$
$$- \left(\tfrac{\lambda+2}{\lambda}\right)\left(\tfrac{m}{\lambda+3}\right)\cdot\left(\tfrac{m-\lambda-3}{i}\right)^n + \text{etc.},$$

which formula divided by the first term $\left(\tfrac{m}{i}\right)^n$ will provide the probability sought.

15. In these probabilities being estimated generally it is assumed all the letters to be drawn with equal ease, but which the illustrious d'Alembert says cannot be assumed. For it is considered likewise to be required to be with respect to all the draws now performed before ; for if certain letters frequently will have been drawn too much, then these are going to be emerging more rarely in the following draws; truly the opposite to happen, if certain letters emerge exceedingly rarely. This reasoning if it may be valid, also shall be going to prevail, if the following draws were made finally after a year or thus as far as a whole century, also why may they not be established somewhere else ; and by the same reasoning, all the draws also must be considered, which were carried out at one time to such an extent in some parts of the world, for which indeed scarcely anything more absurd can be devised.

DEMONSTRATION OF THE PRECEDING SOLUTION

16. Since the number of all the letters *a, b, c, d* etc., for which we have assumed the designated individual tickets, shall be $=m$, this collection of all the letters I will call the principal system, from which other derived systems, which may contain fewer letters, it will be convenient to have formed, which thus I separate into orders, so that the first order may contain all the systems, which may contain only $m-1$, of which the number therefore will be $=m$.
Truly I may refer all systems to the second order, in which the number of letters is $m-2$, the number of which will be

$$\frac{m}{1}\cdot\frac{m-1}{2}=\left(\frac{m}{2}\right).$$

Moreover the third order will have all the systems, where the number of letters is $m-3$, the number of which is

$$\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}=\left(\frac{m}{3}\right).$$

In the same manner the number of systems of the fourth order containing only $m-4$ will be

$$\left(\frac{m}{4}\right);$$

but of the fifth order, where only $m-5$ are present, the number of systems will be

$$\left(\frac{m}{5}\right)$$

and thus henceforth.

17. So that which may become clearer, we may consider the principle system with these six letters present:

$$a\ b\ c\ d\ e\ f\ ,$$

from which therefore the following drawn sequences of each order will result, which we will show with this table:

I.	II.	III.	IV.	V.
abcde	*abcd*	*abc*	*ab*	*a*
abcdf	*abce*	*abd*	*ac*	*b*
abcef	*abcf*	*abe*	*ad*	*c*
abdef	*abde*	*abf*	*ae*	*d*
acdef	*abdf*	*acd*	*af*	*e*
bedef	*abef*	*ace*	*bc*	*f*
	acde	*acf*	*bd*	
	acdf	*ade*	*be*	
	acef	*adf*	*bf*	
	adef	*aef*	*cd*	
	bcde	*bcd*	*ce*	
	bcdf	*bce*	*cf*	
	bcef	*bcf*	*de*	
	bdef	*bde*	*df*	
	cdef	*bdf*	*ef*	
		bef		
		cde		
		edf		
		cef		
		def		

where therefore the number of systems of the first order is $6=\left(\frac{6}{1}\right)$, of the second order $15=\left(\frac{6}{2}\right)$, of the third order $20=\left(\frac{6}{3}\right)$, fourth $=20=\left(\frac{6}{3}\right)$, of the fifth $=6=\left(\frac{6}{5}\right)$.

18. Now it is evident the individual systems of each lesser order to be contained in all the greater orders, as often as it may happen more to be observed lying between. Thus for the case $m=6$ the system of the first order *abcde* occurs in this system once. But the system of the second order *abcd* occurs in the first order twice, in the second it occurs once. Thence the system of the third order *abc* occurs in the first order three times, in the second three times, but is found once in the third. The system of the fourth order *ab* in the first order four times, in the second six times, in the fourth, in the fourth is present once. Finally a system of the fifth order occurs five times in the first order, ten times in the second, ten times in the third, five times in the fourth, and once in the fifth. From which it is evident these numbers agree with the binomial coefficients raised to powers, if indeed all the systems may be contained once in the principal system itself.

19. Hence therefore in general for any system of whatever lesser order it can be assigned easily, in how many it may occur in any greater order, that which will be declared in the following table, where I will denote the principal system by the letter *O,*

moreover the systems of the first, second, third, fourth, etc. orders by the Roman numerals I, II, III, IV, V, VI etc. .

	O	I	II	III	IV	V	VI
m	1						
$m-1$	1	1					
$m-2$	1	2	1				
$m-3$	1	3	3	1			
$m-4$	1	4	6	4	1		
$m-5$	1	5	10	10	5	1	
$m-6$	1	6	15	20	15	5	1
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
$m-\lambda$	$\left(\frac{\lambda}{0}\right)$	$\left(\frac{\lambda}{1}\right)$	$\left(\frac{\lambda}{2}\right)$	$\left(\frac{\lambda}{3}\right)$	$\left(\frac{\lambda}{4}\right)$	$\left(\frac{\lambda}{5}\right)$	$\left(\frac{\lambda}{6}\right)$

20. We will now consider the number of tickets, which may actually be extracted both by some draw from the principal system, as well as can be considered drawn from some derived systems, which indeed will be able to be deduced easily from the principal. So that if a single letter may be extracted by some draw, the number of different draws from the principal system will be

$$=\left(\frac{m}{1}\right);$$

but if two letters may be extracted at the same time, the number of all the different draws will be

$$\frac{m}{1}\cdot\frac{m-1}{2}=\left(\frac{m}{2}\right).$$

If any three letters may be extracted in some draw, the number of different draws will be

$$\left(\frac{m}{3}\right)$$

and in general, if i letters may be extracted in some draw, the number of all the different draws will be

$$\left(\frac{m}{i}\right).$$

But if such extractions also may be considered to happen from the derived systems, for any number drawn from the system of the first order of divisors will be $\left(\frac{m-1}{i}\right)$, of the second order $\left(\frac{m-2}{i}\right)$, of the third order $\left(\frac{m-3}{i}\right)$ and thus henceforth.

21. But now if these extractions may be repeated twice, because any draw for the principal system not only are all the remaining extractions able to follow, but also the number of diverse cases will be $\left(\frac{m}{i}\right)^2$. If three extractions successively may be put in place, the number of all the cases will be $\left(\frac{m}{i}\right)^3$; and in general, if n extractions themselves may follow, the number of all possible cases will be $\left(\frac{m}{i}\right)^n$, which we have designated by the letter $\varDelta$ above, thus so that there shall be

$$\varDelta = \left(\frac{m}{i}\right)^n .$$

22. In a similar manner the number of all the cases, which are able to happen in any system of the first order, is $\left(\frac{m-1}{i}\right)^n$; whereby since the number of these systems shall be $\left(\frac{m}{i}\right)$, the number of all the cases which the first order provides, will be $\left(\frac{m}{1}\right)\left(\frac{m-1}{i}\right)^n$ and which we will designate by the letter A, thus so that there shall be

$$A = \left(\frac{m}{1}\right)\left(\frac{m-1}{i}\right)^n .$$

It is easily understood in the same manner the number of all the cases, which are able to arise from the individual systems, to be for the second order

$$B = \left(\frac{m}{2}\right)\left(\frac{m-2}{i}\right)^n ,$$

for the third order

$$C = \left(\frac{m}{3}\right)\left(\frac{m-3}{i}\right)^n ,$$

for the fourth order

$$D = \left(\frac{m}{4}\right)\left(\frac{m-4}{i}\right)^n ,$$

and thus so on. From these premises the solutions of the preceding individual problems will be allowed to be set out easily.

FOR THE FIRST PROBLEM

23. Since in this problem from all the cases possible, of which the number is $\varDelta$, these must be enumerated, which involve all the m, thence we may exclude in the first place all the cases, which contain only $m-1$ or fewer letters, which will happen, if we may take

away all the cases possible of the first order, of which the number is A. For in this manner the cases, which contain $m-1$ letters, will be removed from within. But truly the cases, which contain $m-2$ letters, in this manner will be taken away twice ; from which they will be deficient once in the formula $\varDelta - A$, thus so that the number of these $1-2 = -1$. But for the cases containing $m-3$ letters, the number which occur in the formula $\varDelta - A$ will be $1-3=-2$. In a similar manner for $m-4$ we will have $1-4=-3$ and so on thus, which case deficiencies again therefore will have to be restored.

24. Moreover the case of the form $m-2$, the deficiencies will be restored if B may be added once to the formula $\varDelta - A$. But in this manner terms of the form $m-3$ are added three times, since yet they shall have been deficient by twice the amount ; therefore now they will be in excess by times one, or the sign will be $+1$. But to the form $m-4$ six times the amount is added, since it was lacking only by three times, and thus the index number will be $+3$. In a similar manner for terms of the form $m-5$ the index will be $10-4=+6$ and so on thus.

25. Therefore since we may remove these excessive cases again, we may subtract all the cases of the third order, $=C$. Indeed in this manner terms of the form $m-3$ will be removed completely, but for the remainder frequently too much will be taken away, evidently the index of the order $m-4$ will be -1, the index for the order $m-5$ will be -4 etc.

26. Because the form $m-4$ is deficient by one, restitution will be had by adding the letter D. But the lesser terms now will be in excess according to the indices 1, 5, 15 etc., from which on subtracting E these will be removed; because the subtraction has been too much, it will be restored by addition of the letter F and so on thus.

27. Hence it has been shown well enough all the other cases containing letters fewer than m to be taken from the form $\varDelta$; of which the number of the remaining will be

$$\varDelta - A + B - C + D - E + F - \text{etc.},$$

which we have indicated by $\varSigma$; and thus the solution of the first problem has been demonstrated rigorously.

[*i.e.* $\varSigma = \left(\frac{m}{i}\right)^n - \left(\frac{m}{1}\right)\left(\frac{m-1}{i}\right)^n + \left(\frac{m}{2}\right)\left(\frac{m-2}{i}\right)^n - \left(\frac{m}{3}\right)\left(\frac{m-3}{i}\right)^n + \text{etc.}$; from which $\varPi = \frac{\varSigma}{\varDelta}$.]

FOR THE SECOND PROBLEM

28. It is evident the following is the same, by which we have used here, the reasoning from the preceding demonstration can be set out for the solution of the second problem. Nor will there be a need to set out everything at so much length. Indeed since from the

number of cases possible *Δ* these shall be required to be enumerated, which contain only $m-1$, hence it is apparent at once to be by excluding all the cases containing $m-2$, which may come about, if from the number *Δ* the number *B* may be taken away. But the above table given in § 19 declares by this manner terms of the form $m-3$ to be going to be removed three times, since they were required still to be taken away only once, and thus for the remaining forms. For these requiring to be put in place the number 2*C* may be added, where deficient numbers of the form $m-3$ will be removed completely; but numbers of the form $m-4$ with the index 3 will be in excess, and the following more in excess. Where the former may be removed, again the number 3*D* must be subtracted, from which the removed terms of the form $m-4$ will be excluded. The deficient numbers of the form $m-5$ and of the following again will be restored by the addition of the number 4E and thus so forth ; with which operations performed the number of the remaining will be

$$\Sigma' = \Delta - B + 2C - 3D + 4E - 5F + \text{etc.};$$

and thus the solution of the second problem has been demonstrated.

FOR THE THIRD PROBLEM

29. Here the number *C* must be subtracted from the number *Δ* , so that the cases $m-3$ of the tickets may be excluded; and because with this agreed the number is subtracted four times *D*, since it will only be in excess once, again the number 3*D* must be added, by which that and the following deficiencies may be restored. Because it exceeds it will be removed by subtraction of the number 6*E*, truly the deficiencies are restored by the addition of the number 10*F* and thus so forth; from which the number of cases containing the $m-2$ letters will be

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.},$$

just as I have asserted in the solution of the third problem. Therefore in this manner this solution also has been firmly demonstrated.

SOLUTIO QUAESTIONIS AD CALCULUM PROBABILITATIS PERTINENTIS QUANTUM DUO CONIUGES PERSOLVERE DEBEANT UT SUIS HAEREDIBUS POST UTRIUSQUE MORTEM CERTA ARGENTI SUMMA PERSOLVATUR

[E599]

Opuscula analytica 2, 1785, p. 315-330

1. Assumimus hic eiusmodi aerarium publicum esse constitutum, cuius facultates quotannis vicesima sui parte augeri queant, ita ut summa 100 Rubellonum post annum ad 105 Rub. excrescat; quare si brevitatis gratia ponamus $\frac{105}{100}=\lambda$, praesens pecuniae summa $=C$ post *n* annos aestimanda erit $\lambda^n C$. Vicissim autem quaevis pecuniae summa *C* post *n* annos solvenda praesenti tempore valorem habere censenda est $=\frac{C}{\lambda^n}$.

2. Ponamus nunc argenti summam, quam ambo coniuges post utriusque mortem acquirere optant, esse $=1000$ Rub., unde intelligitur, si tempus huius solutionis esset cognitum, annorum praeterlapsorum numero existente $=n$ eius valorem praesentem futurum esse $=\frac{1000}{\lambda^n}$ Tantum igitur illi coniuges praesenti tempore in aerarium conferre tenerentur. Verum cum tempus solutionis maxime sit incertum, siquidem demum post utriusque mortem fieri debet, verum et praesentem valorem huius summae secundum regulas calculi probabilium ex longaevis mortalitatis observationibus petitas determinari oportet. Hunc in finem utar tabula, quam olim in Tomo Memor. Berol. pro Anno 1760 inserui, ubi, si praemagnus numerus *M* infantum simul natorum consideretur, eorum numerum post *n* annos adhuc superstitum indicavi charactere (*n*)*M;* ex quo intelligitur talem characterem (*n*) designare fractiones eo minores, quo maior fuerit annorum numerus *n,* ac tandem circa 100 annos prorsus in nihilum abire. Tabulam igitur horum valorum pro singulis annis elapsis hic exponamus.

(1) = 0,804	(25) = 0,552	(49) = 0,370	(73) = 0,145
(2) = 0,768	(26) =0,544	(50) = 0,362	(74) = 0,135
(3) = 0,736	(27) = 0,535	(51) = 0,354	(75) = 0,125
(4) = 0,709	(28) = 0,525	(52) = 0,345	(76) = 0,114
(5) = 0,688	(29) = 0,516	(53) = 0,336	(77) = 0,104
(6) = 0,676	(30) = 0,507	(54) = 0,327	(78) = 0,093
(7) = 0,664	(31) = 0,499	(55) = 0,319	(79) = 0,082
(8) = 0,653	(32) = 0,490	(56) = 0,310	(80) = 0,072
(9) = 0,646	(33) = 0,482	(57) = 0,301	(81) = 0,063
(10) = 0,639	(34) = 0,475	(58) = 0,291	(82) = 0,054
(11) = 0,633	(35) = 0,468	(59) = 0,282	(83) = 0,046

(12) = 0,627	(36) = 0,461	(60) = 0,273	(84) = 0,039
(13) = 0,621	(37) = 0,454	(61) = 0,264	(85) = 0,032
(14) = 0,616	(38) = 0,446	(62) = 0,254	(86) = 0,026
(15) = 0,611	(39) = 0,439	(63) = 0,245	(87) = 0,020
(16) = 0,606	(40) = 0,432	(64) = 0,235	(88) = 0,015
(17) = 0,601	(41) = 0,426	(65) = 0,225	(89) = 0,011
(18) = 0,596	(42) = 0,420	(66) = 0,215	(90) = 0,008
(19) = 0,590	(43) = 0,413	(67) = 0,205	(91) = 0,006
(20) = 0,584	(44) = 0,406	(68) = 0,195	(92) = 0,004
(21) = 0,577	(45) = 0,400	(69) = 0,185	(93) = 0,003
(22) = 0,571	(46) = 0,393	(70) = 0,175	(94) = 0,002
(23) = 0,565	(47) = 0,386	(71) = 0,165	(95) = 0,001
(24) = 0,559	(48) = 0,378	(72) = 0,155	

3. Ponamus nunc praesenti tempore aetatem mariti esse $=a$ annorum, uxoris vero $=b$ annorum, et quo ratiocinia instituenda clarius percipi queant, fingamus simul ingentem numerum talium coniugum, qui sit *N,* eiusdem aetatis adesse, qui pariter suis haeredibus post utriusque mortem summam 1000 Rub. acquirere optent, unde, si summa initio persolvenda statuatur $=x$, aerarium ab his omnibus accipiet summam *Nx.*

4. Sin autem magis arrideat, ut istud pretium x non statim ab initio totum, sed potius per totam vitam aequaliter distributum solvatur, calculum nostrum ad duplicem solutionem accommodemus, dum altera statim ab initio summa $=x$ in aerarium solvitur, altera autem quotannis insuper quaepiam summa $=z$ solvitur, quamdiu scilicet non solum ambo coniuges, sed etiam alteruter tantum superstites fuerint. Solutione autem hoc modo absoluta si quis voluerit totum pretium statim ab initio persolvere, pro hoc casu poni oportebit $z=0$ et littera x quaesitum pretium indicabit. Sin autem quis maluerit hoc pretium per totam vitam aequaliter distribui, poni oportebit $x=z$ eritque z summa singulis annis solvenda usque ad mortem utriusque coniugis.

5. His constitutis statim ab initio ab omnibus illis N coniugiis solvetur summa $=Nx$. Nunc videamus, postquam elapsi fuerint n anni, quot coniugia adhuc tam integra quam dissoluta, dum scilicet interea alteruter fuerit mortuus, sint superfutura; tum enim a singulis istis in aerarium solvetur summa $=z$, cuius valor praesens aestimandus est $=\frac{z}{\lambda^n}$. Praeterea vero pro quovis anno currente inquirendum est, quot coniugia penitus extinguantur; quoties enim hoc evenit, toties eorum haeredibus praemium illud 1000 Rub. persolvi debet, cuius ergo valor praesens erit $\frac{1000}{\lambda^n}$. Hoc igitur modo calculum nostrum prosequi oportet usque ad extremum vitae humanae terminum, et cum omnes tam expensae quam reditus fuerint ad praesens tempus reducti, eos inter se aequari conveniet, unde pro lubitu sive x sive z determinare licebit.

6. His praemissis incipiamus ab anno primo, cuius initio adesse ponuntur N mariti, omnes eiusdem aetatis $=a$, totidemque uxores eiusdem aetatis $=b$, a quibus aerarium accepit summam $=Nx$. Nunc igitur elapso anno primo secundum tabulam supra allatam numerus maritorum adhuc superstitum erit $\frac{(a+1)}{(a)}N$ ideoque numerus interea defunctorum $=\frac{(a)-(a+1)}{(a)}N.$ Simuli modo numerus uxorum adhuc superstitum erit $\frac{(b+1)}{(b)}N$, earum autem, quae interea sunt mortuae, numerus $\frac{(b)-(b+1)}{(b)}N.$ Quia igitur quilibet horum maritorum superstitum initio habuit coniugem, instituatur haec proportio: uti numerus omnium uxorum initio se habet ad earum numerum superstitum, ita numerus virorum elapso anno superstitum ad numerum eorum, quorum uxores adhuc erunt superstites qui ergo numerus erit $\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}N$, a quibus singulis in aerarium solvitur summa $=z$; cuius valor praesens cum sit $\frac{z}{\lambda}$ hinc orietur valor

$$=\frac{(a+1)(b+1)}{(a)(b)}\cdot\frac{Nz}{\lambda}.$$

Tum vero numerus eorum maritorum, qui interea uxores amiserint, erit

$$\frac{(a+1)}{(a)}\left(1-\frac{(b+1)}{(b)}\right)N;$$

qui cum itidem in aerarium solvant summam z, ea ad initium relata erit

$$\frac{(a+1)}{(a)}\left(1-\frac{(b+1)}{(b)}\right)\frac{Nz}{\lambda},$$

unde patet hunc valorem cum praecedente coniunctum fore

$$\frac{(a+1)}{(a)}\frac{Nz}{\lambda},$$

id quod per se est manifestum, quia quilibet maritus superstes hanc summam z solvere tenetur, sive eius uxor adhuc vivat sive secus.

7. Consideremus nunc etiam eos maritos, qui intra hunc annum erunt mortui, quorum numerus est $\left(1-\frac{(a+1)}{(a)}\right)N;$ ubi duo casus se offerunt. Alter casus eos spectat maritos, quorum uxores adhuc sunt superstites, quorum numerus per superiorem analogiam invenitur: uti se habet numerus omnium uxorum initio viventium ad earum numerum post annum superstitum, ita numerus virorum interea defunctorum ad eorum numerum, quorum uxores adhuc sunt superstites; qui ergo numerus erit

$$\tfrac{(b+1)}{(b)}\left(1-\tfrac{(a+1)}{(a)}\right)N;$$

quae cum singulae etiam in aerarium conferant summam $=z$,eius valor ad initium relatus erit

$$\tfrac{(b+1)}{(b)}\left(1-\tfrac{(a+1)}{(a)}\right)\tfrac{Nz}{\lambda},$$

unde omnes reditus primo anno elapso aerarium influentes erunt

$$\tfrac{Nz}{\lambda}\left(\tfrac{(a+1)}{(a)}+\tfrac{(b+1)}{(b)}-\tfrac{(a+1)}{(a)}\cdot\tfrac{(b+1)}{(b)}\right),$$

quae ergo quantitas tribus constat partibus. Primo scilicet valor $\tfrac{z}{\lambda}$ multiplicatur per numerum maritorum superstitum, qui est $\tfrac{(a+1)}{(a)}N$, deinde etiam per numerum uxorum superstitum, qui est $\tfrac{(b+1)}{(b)}N.$ Hinc autem auferri debet numerus coniugiorum adhuc integrorum, quia singula non duo $z,$ sed tantum unum z expendunt.

8. Alter casus eos spectat maritos, quorum uxores non amplius sunt superstites. Ex praecedente autem calculo apparet numerum eorum maritorum mortuorum, quorum uxores interea quoque sunt defunctae, esse

$$\left(1-\tfrac{(a+1)}{(a)}\right)\left(1-\tfrac{(b+1)}{(b)}\right)N.$$

Tot ergo coniugia penitus sunt extincta, quorum igitur haeredibus ex aerario solvendum erit praemium constitutum 1000 Rub.; quod cum statim persolvi debeat, nullam usuram lucrari interea potuit, unde istae expensae ad initium relatae etiamnunc valebunt

$$1000N\left(1-\tfrac{(a+1)}{(a)}-\tfrac{(b+1)}{(b)}+\tfrac{(a+1)}{(a)}\cdot\tfrac{(b+1)}{(b)}\right).$$

9. Progrediamur nunc ad annum secundum, cuius initio superstites erant mari $\tfrac{(a+1)}{(a)}N,$ uxores vero $\tfrac{(b+1)}{(b)}N,$ inter quos subsistent adhuc coniugia integra

$$\tfrac{(a+1)}{(a)}\cdot\tfrac{(b+1)}{(b)}N,$$

soluta vero

$$\left(\tfrac{(a+1)}{(a)}+\tfrac{(b+1)}{(b)}-\tfrac{(a+1)}{(a)}\cdot\tfrac{(b+1)}{(b)}\right)N,$$

ita ut numerus coniugiorum penitus extinctorum sit

$$N\left(1-\frac{(a+1)}{(a)}-\frac{(b+1)}{(b)}+\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}\right)$$

sive

$$N\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right).$$

At vero in fine anni secundi numerus maritorum superstitum adhuc erit $\frac{(a+2)}{(a)}N,$ numerus vero eorum, qui hoc biennio sunt mortui, $\left(1-\frac{(a+2)}{(a)}\right)N$. Similique modo numerus uxorum adhuc superstitum erit $\frac{(b+2)}{(b)}N,$ earum vero, quae biennio sunt mortuae, $\left(1-\frac{(b+2)}{(b)}\right)N$, unde numerus coniugiorum hoc biennio extinctorum erit

$$\left(1-\frac{(a+2)}{(a)}\right)\left(1-\frac{(b+2)}{(b)}\right)N.$$

Quare cum numerus coniugiorum primo anno extinctorum fuerit

$$\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right)N,$$

numerus eorum, quae intra hunc secundum annum sunt extincta, erit

$$\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+2)}{(a)}-\frac{(b+2)}{(b)}-\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}+\frac{(a+2)}{(a)}\cdot\frac{(b+2)}{(b)}\right)N;$$

pro quibus singulis quia persolvitur summa mille Rub., tota summa ob usuram ad initium relata valebit

$$\frac{1000N}{\lambda}\left(\frac{(a+1)-(a+2)}{(a)}+\frac{(b+1)-(b+2)}{(b)}-\frac{(a+1)(b+1)-(a+2)(b+2)}{(a)(b)}\right).$$

10. Quia nunc initio secundi anni numerus coniugiorum tam integrorum quam solutorum erat

$$N\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+1)}{(a)}\cdot\frac{(b+1)}{(b)}\right),$$

si hinc auferamus numerum coniugiorum hoc anno extinctorum, remanebit numerus eorum, qui circa finem secundi anni singuli solvent summam z, quorum ergo numerus erit

$$N\left(\frac{(a+2)}{(a)}+\frac{(b+2)}{(b)}-\frac{(a+2)}{(a)}\cdot\frac{(b+2)}{(b)}\right).$$

Toties igitur ab his summa z in aerarium infertur, unde totus valor ob usuram duorum annorum minutus primo initio valebit

$$\frac{Nz}{\lambda^2}\left(\frac{(a+2)}{(a)}+\frac{(b+2)}{(b)}-\frac{(a+2)}{(a)}\cdot\frac{(b+2)}{(b)}\right).$$

11. His expositis iam ad annum quemcunque sequentum progredi poterimus. Ponamus igitur iam elapsos esse n annos hocque tempore numerus maritorum superstitum erit $\frac{(a+n)}{(a)}N,$ ante autem iam defunctorum $\left(1-\frac{(a+n)}{(a)}\right)N.$ Eodemque modo numerus uxorum adhuc superstitum est $\frac{(b+n)}{(b)}N,$ demortuarum vero $\left(1-\frac{(b+n)}{(b)}\right)N$, unde numerus coniugiorum tam integrorum quam solutorum hoc tempore erit

$$\left(\frac{(a+n)}{(a)}+\frac{(b+n)}{(b)}-\frac{(a+n)}{(a)}\cdot\frac{(b+n)}{(b)}\right)N;$$

at vero numerus coniugiorum toto hoc tempore penitus extinctorum erit

$$\left(1-\frac{(a+n)}{(a)}\right)\left(1-\frac{(b+n)}{(b)}\right)N.$$

12. Iam procedamus ad finem istius anni ac simili modo numerus coniugiorum, sive integrorum sive solutorum, nunc erit

$$N\left(\frac{(a+n+1)}{(a)}+\frac{(b+n+1)}{(b)}-\frac{(a+n+1)}{(a)}\cdot\frac{(b+n+1)}{(b)}\right),$$

a quibus singulis in aerarium persolvitur summa $z,$ cuius valor ad initium translatus est z unde tota summa circa finem huius anni in aerarium soluta pro initio valebit

$$\frac{Nz}{\lambda^{n+1}}\left(\frac{(a+n+1)}{(a)}+\frac{(b+n+1)}{(b)}-\frac{(a+n+1)}{(a)}\cdot\frac{(b+n+1)}{(b)}\right).$$

Tum vero numerus omnium coniugiorum ab ipso initio usque ad tempus $n+1$ annorum extinctorum erit

$$N\left(1-\frac{(a+n+1)}{(a)}\right)\left(1-\frac{(b+n+1)}{(b)}\right);$$

quare, cum usque ad initium huius anni iam extincta fuissent

$$N\left(1-\frac{(a+n)}{(a)}\right)\left(1-\frac{(b+n)}{(b)}\right)$$

coniugia, numerus eorum, quae hoc demum anno sunt extincta, erit

$$N\left(\frac{(a+n)-(a+n+1)}{(a)}+\frac{(b+n)-(b+n+1)}{(b)}-\frac{(a+n)(b+n)-(a+n+1))(b+n+1)}{(a)(b)}\right).$$

Quoniam igitur pro his singulis expendi debet summa 1000 Rub., valor harum expensarum ad initium relatus erit

$$\frac{1000N}{\lambda^n}\left(\frac{(a+n)-(a+n+1)}{(a)}+\frac{(b+n)-(b+n+1)}{(b)}-\frac{(a+n)(b+n)-(a+n+1))(b+n+1)}{(a)(b)}\right).$$

13. Colligamus nunc omnes tam reditus ex quantitate z oriundos quam expensas ex solutione illorum 1000 Rub. ortas; ac primo quidem omnes reditus, qui praeter summam principalem Nx in aerarium inferuntur, per ternas sequentes series expressi inveniuntur:

$$Nz\left\{\begin{array}{l}\frac{(a+1)}{\lambda(a)}+\frac{(a+2)}{\lambda^2(a)}+\frac{(a+3)}{\lambda^3(a)}+\cdots+\frac{(a+n)}{\lambda^n(a)}\\+\frac{(b+1)}{\lambda(b)}+\frac{(b+2)}{\lambda^2(b)}+\frac{(b+3)}{\lambda^3(b)}+\cdots+\frac{(b+n)}{\lambda^n(b)}\\-\frac{(a+1)(b+1)}{\lambda(a)(b)}-\frac{(a+2)(b+2)}{\lambda^2(a)(b)}-\frac{(a+3)(b+3)}{\lambda^3(a)(b)}-\cdots-\frac{(a+n)(b+n)}{\lambda^n(a)(b)}\end{array}\right\}.$$

.

Quodsi ergo brevitatis gratia statuamus

$$\begin{aligned}P&=\frac{(a+1)}{\lambda}+\frac{(a+2)}{\lambda^2}+\frac{(a+3)}{\lambda^3}+\frac{(a+4)}{\lambda^4}+\cdots+\frac{(95)}{\lambda^{95-a}},\\Q&=\frac{(b+1)}{\lambda}+\frac{(b+2)}{\lambda^2}+\frac{(b+3)}{\lambda^3}+\frac{(b+4)}{\lambda^4}+\cdots+\frac{(95)}{\lambda^{95-b}},\\R&=\frac{(a+1)(b+1)}{\lambda}+\frac{(a+2)(b+2)}{\lambda^2}+\frac{(a+3)(b+3)}{\lambda^3}+\text{etc.}\end{aligned}$$

erit tota summa redituum

$$Nx+Nz\left(\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}\right).$$

Colligamus simili modo omnes expensas in unam summam, quae summa ex sex sequentibus seriebus erit composita:

$$1000N\left\{\begin{array}{l}
\frac{(a)}{(a)}+\frac{(a+1)}{\lambda(a)}+\frac{(a+2)}{\lambda^2(a)}+\frac{(a+3)}{\lambda^3(a)}+\text{etc.}\\
-\frac{(a+1)}{(a)}-\frac{(a+2)}{\lambda(a)}-\frac{(a+3)}{\lambda^2(a)}-\frac{(a+4)}{\lambda^3(a)}-\text{etc.}\\
+\frac{(b)}{(b)}+\frac{(b+1)}{\lambda(b)}+\frac{(b+2)}{\lambda^2(b)}+\frac{(b+3)}{\lambda^3(b)}+\text{etc.}\\
-\frac{(b+1)}{(b)}-\frac{(b+2)}{\lambda(b)}-\frac{(b+3)}{\lambda^2(b)}-\frac{(b+4)}{\lambda^3(b)}-\text{etc.}\\
-\frac{(a)(b)}{(a)(b)}-\frac{(a+1)(b+1)}{\lambda(a)(b)}-\frac{(a+2)(b+2)}{\lambda^2(a)(b)}-\frac{(a+3)(b+3)}{\lambda^3(a)(b)}-\text{etc.}\\
+\frac{(a+1)(b+1)}{(a)(b)}+\frac{(a+2)(b+2)}{\lambda(a)(b)}+\frac{(a+3)(b+3)}{\lambda^2(a)(b)}+\frac{(a+4)(b+4)}{\lambda^3(a)(b)}+\text{etc.}
\end{array}\right\}.$$

14. Perspicuum est etiam hic summas trium serierum constitutas *P, Q, R* commode in subsidium vocari posse hincque omnes expensas ad initium relatas expressum iri per sequentem formam

$$1000N\left(\frac{(a)+P}{(a)}-\frac{\lambda P}{(a)}+\frac{(b)+Q}{(b)}-\frac{\lambda Q}{(b)}-\frac{(a)(b)+R}{(a)(b)}+\frac{\lambda R}{(a)(b)}\right).$$

seu

$$1000N\left(1+\frac{(1-\lambda)P}{(a)}+\frac{(1-\lambda)Q}{(b)}+\frac{(\lambda-1)R}{(a)(b)}\right);$$

consequenter aequatio pro solutione quaestionis propositae erit

$$x+z\left(\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}\right)=1000\left(1+\frac{(1-\lambda)P}{(a)}+\frac{(1-\lambda)Q}{(b)}+\frac{(\lambda-1)R}{(a)(b)}\right).$$

15. Cum iam sit

$$\lambda=\frac{105}{100},$$

erit

$$\lambda-1=\frac{5}{100}\ \text{ et }\ 1000(\lambda-1)=50,$$

unde nostra aequatio erit

$$x+z\left(\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}\right)=1000-\frac{50P}{(a)}-\frac{50Q}{(b)}+\frac{50R}{(a)(b)}.$$

Quantobrem si totum pretium statim ab initio persolvi debeat, ita ut sit $z=0$, erit hoc pretium

$$x=1000-\tfrac{50P}{(a)}-\tfrac{50Q}{(b)}+\tfrac{50R}{(a)(b)}.$$

Sin autem velimus, ut pretium per totum temporis intervallum usque ad mortem utriusque coniugis aequaliter distribuatur, poni debet $x=z$ atque contributio annua prodibit sequens

$$z=\frac{1000-\frac{50P}{(a)}-\frac{50Q}{(b)}+\frac{50R}{(a)(b)}}{1+\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}},$$

sicque totum negotium huc redit, ut pro qualibet aetate utriusque coniugis valores ternarum serierum litteris *P, Q, R* insignitarum investigentur, quos ergo in sequentibus evolvamus.

EVOLUTIO VALORUM *P* ET *Q*

16. Quoniam series *Q* simili modo ex aetate *b* definitur, quo series *P* ex aetate *a* erui debet, sufficiet alterutram tantum pro singulis aetatibus evolvisse.
Cum igitur sit

$$P=\frac{(a+1)}{\lambda}+\frac{(a+2)}{\lambda^2}+\frac{(a+3)}{\lambda^3}+\cdots+\frac{(95)}{\lambda^{95-a}},$$

si omnes termini huius seriei ad eandem denominationem λ^{95-a} reducantur atque ordine retrograda disponantur, fiet

$$P=\tfrac{1}{\lambda^{95-a}}\Big((95)+(94)\lambda+(93)\lambda^2+\cdots+(a+1)\lambda^{94-a}\Big).$$

17. Evolutio autem huius seriei non parum foret taediosa, si per singulos annos eam absolvere vellemus. Quia autem valores characterum (*a*) et (*b*) non adeo sunt certi, ut non aliquam aberrationem agnoscere debeamus, sufficiet quinos terminos se insequentes invicem coniungere eorumque summam quintuplo termini medii aequalem statuere, ita ut pro quinis prioribus terminis scribi queat $5(93)\lambda^2$, quo facto valor nostrae litterae *P* erit

$$P=\tfrac{5}{\lambda^{95-a}}\Big((93)\lambda^2+(88)\lambda^7+(83)\lambda^{12}+\cdots+(a+3)\lambda^{92-a}\Big);$$

quare si hanc seriem littera *p* designerons, ut sit

$$p=(93)\lambda^2+(88)\lambda^7+(83)\lambda^{12}+\cdots+(a+3)\lambda^{92-a},$$

invento valore litterae p erit

$$P=\frac{5p}{\lambda^{95-a}}$$

hincque

$$\frac{P}{(a)}=\frac{5p}{(a)\lambda^{95-a}}.$$

Eodem modo, si ponatur

$$q=(93)\lambda^2+(88)\lambda^7+(83)\lambda^{12}+\cdots+(b+3)\lambda^{92-b},$$

habebitur

$$Q=\frac{5q}{\lambda^{95-b}}$$

et

$$\frac{Q}{(b)}=\frac{5q}{(b)\lambda^{95-b}}.$$

EVOLUTIO TERTII VALORIS R

18. Series, quam littera R designavimus, erat haec

$$R=\frac{(a+1)(b+1)}{\lambda}+\frac{(a+2)(b+2)}{\lambda\lambda}+\cdots+\frac{(a+n)(b+n)}{\lambda^n}+\cdots,$$

quam seriem eo usque continuari oportet, donec termini sequentes evanescant, quod fit, si alteruter numerorum $(a+n)$ vel $(b+n)$ superet 95, unde, statim ac maior horum duorum numerorum ad istum terminum exsurgit, series hic terminata est censenda.

19. Quia autem ambo numeri a et b in nostrum calculum aequaliter ingrediuntur neque ullum discrimen inde nascitur, etiamsi hae duae litterae inter se permutentur, ita ut a denotet aetatem uxoris et b aetatem mariti, assumere poterimus aetatem a semper esse maiorem quam b; si enim uxor natu maior fuerit quam maritus, tum a designabit aetatem uxoris, at b mariti. Quare cum aetatem b tanquam minorem spectemus, discrimen littera d designemus, ita ut sit $b=a-d$, ubi quidem differentia d nulla erit, si ambo coniuges eandem habuerint aetatem.

20. Hoc observato ultimus nostrae seriei terminus ibi erit, ubi fit $a+n=95$ ideoque $n=95-a$, ita ut iam nostra series futura sit

$$R=\frac{(a+1)(b+1)}{\lambda}+\frac{(a+2)(b+2)}{\lambda\lambda}+\cdots+\frac{(95)(b+95-a)}{\lambda^{95-a}},$$

ubi ergo ultimus terminus est $\frac{(95)(95-d)}{\lambda^{95-a}}$, Reducamus nunc, ut ante, omnes has fractiones ad eandem denominationem λ^{95-a} ac totam seriem ordine retrograda disponamus reperimusque

$$R=\frac{1}{\lambda^{95-a}}\left\{\begin{array}{l}(95)(95-d)+(94)(94-d)\lambda+(93)(93-d)\lambda^{2}\\+(92)(92-d)\lambda^{3}+\cdots+(a+1)(a+1-d)\lambda^{94-a}\end{array}\right\},$$

cuius ergo seriei summam pro singulis valoribus amborum numerorum *a* et *d* computari oportet.

21. Quo autem iste calculus facilior reddatur, iterum quinos terminos, ut ante fecimus, in unum contrahamus, dum scilicet eorum summam quintuplo medii inter eos aequalem aestimabimus, quo facto habebimus

$$R=\frac{5}{\lambda^{95-a}}\Big((93)(93-d)\lambda^{2}+(88)(88-d)\lambda^{7}+\cdots+(a+3)(a+3-d)\lambda^{92-a}\Big).$$

Quodsi ergo ponamus

$$r=(93)(93-d)\lambda^{2}+(88)(88-d)\lambda^{7}+\cdots+(a+3)(a+3-d)\lambda^{92-a},$$

invento valore huius seriei *r* erit ipse valor, quem quaerimus,

$$R=\frac{5r}{\lambda^{95-a}};$$

et quoniam pro nostro calculo indigemus valore $\frac{R}{(a)(b)}$, erit

$$\frac{R}{(a)(b)}=\frac{5r}{(a)(b)\lambda^{95-a}}$$

hisque valoribus pro singulis casibus inventis aequatio nostra generalis erit

$$x+z\Big(\frac{P}{(a)}+\frac{Q}{(b)}-\frac{R}{(a)(b)}\Big)=1000-\frac{50P}{(a)}-\frac{50Q}{(b)}+\frac{50R}{(a)(b)}.$$

22. Quoniam hic duo numeri occurrunt, *a* et *d,* iste calculus multo maiorem laborem postulat quam praecedens pro seriebus *P* et *Q;* quem ut sublevemus, ambos numeros *a* et *d* per quinarium vel crescere vel decrescere assumerons; hanc ob rem plures casus evolvi oportebit pro variis valoribus differentiae *d,* quam successive statuemus 0, 5, 10, 15, 20 etc. Unde hos casus sequenti modo ordine referantus.

CASUS I

QUO $d=0$ IDEOQUE $b=a$

Hic ergo erit

$$r=(93)^2\lambda^2+(88)^2\lambda^7+(83)^3\lambda^{12}+\cdots+(a+3)^2\lambda^{92-a},$$

unde fit

$$R=\tfrac{5r}{\lambda^{95-a}} \quad \text{et} \quad \tfrac{R}{(a)^2}=\tfrac{5r}{\lambda^{95-a}(a)^2}.$$

CASUS II

QUO $d=5$ IDEOQUE $b=a-5$

Tum ergo erit

$$r=(93)(88)\lambda^2+(88)(83)\lambda^7+(83)(78)\lambda^{12}+\cdots+(a+3)(a-2)\lambda^{92-a},$$

unde fit

$$R=\tfrac{5r}{\lambda^{95-a}} \quad \text{et} \quad \tfrac{R}{(a)(a-5)}=\tfrac{5r}{\lambda^{95-a}(a)(a-5)}.$$

CASUS III

QUO $d=10$ IDEOQUE $b=a-10$

Hic ergo erit

$$r=(93)(83)\lambda^2+(88)(78)\lambda^7+(83)(73)\lambda^{12}+\cdots+(a+3)(a-7)\lambda^{92-a},,$$

unde fit

$$R=\tfrac{5r}{\lambda^{95-a}} \quad \text{et} \quad \tfrac{R}{(a)(a-10)}=\tfrac{5r}{\lambda^{95-a}(a)(a-10)}.$$

CASUS IV

QUO $d=15$ IDEOQUE $b=a-15$

Hoc casu erit

$$r=(93)(78)\lambda^2+(88)(73)\lambda^7+(83)(68)\lambda^{12}+\cdots+(a+3)(a-12)\lambda^{92-a},,$$

unde fit

$$R=\tfrac{5r}{\lambda^{95-a}} \quad \text{et} \quad \tfrac{R}{(a)(a-15)}=\tfrac{5r}{\lambda^{95-a}(a)(a-15)}.$$

CASUS V

QUO $d=20$ IDEOQUE $b=a-20$

Tum ergo erit

$$r=(93)(73)\lambda^2+(88)(68)\lambda^7+(83)(63)\lambda^{12}+\cdots+(a+3)(a-17)\lambda^{92-a},$$

unde fit

$$R=\tfrac{5r}{\lambda^{95-a}} \quad \text{et} \quad \tfrac{R}{(a)(a-20)}=\tfrac{5r}{\lambda^{95-a}(a)(a-20)}.$$

CASUS VI

QUO $d=25$ IDEOQUE $b=a-25$

Hoc casu erit

$$r=(93)(68)\lambda^2+(88)(63)\lambda^7+(83)(58)\lambda^{12}+\cdots+(a+3)(a-22)\lambda^{92-a},$$

unde fit

$$R=\tfrac{5r}{\lambda^{95-a}} \quad \text{et} \quad \tfrac{R}{(a)(a-25)}=\tfrac{5r}{\lambda^{95-a}(a)(a-25)}.$$

CASUS VII

QUO $d=30$ IDEOQUE $b=a-30$

Tum ergo erit

$$r=(93)(63)\lambda^2+(88)(58)\lambda^7+(83)(53)\lambda^{12}+\cdots+(a+3)(a-27)\lambda^{27-a},$$

unde fit

$$R=\tfrac{5r}{\lambda^{95-a}} \quad \text{et} \quad \tfrac{R}{(a)(a-30)}=\tfrac{5r}{\lambda^{95-a}(a)(a-30)}.$$

SOLUTIO
QUARUNDAM QUAESTIONUM DIFFICILIORUM
IN CALCULO PROBABILIUM

[E600]

Opuscula analytica 2**,** 1785, p. 331-346

1. His quaestionibus occasionem dedit ludus passim publice institutus, quo ex nonaginta schedulis, numeris 1, 2, 3, 4, ... 90 signatis, statis temporibus quinae schedulae sorte extrahi solent. Hinc ergo huiusmodi quaestiones oriuntur: quanta scilicet sit probabilitas, ut, postquam datus extractionum numerus fuerit peractus, vel omnes nonaginta numeri exierint vel saltem 89 vel 88 vel pauciores. Has igitur quaestiones, utpote difficillimas, hic ex principiis calculi probabilium iam pridem usu receptis resolvere constitui. Neque me deterrent obiectiones Illustris D'ALEMBERT, qui hunc calculum suspectum reddere est conatus. Postquam enim summus Geometra studiis mathematicis valedixit, iis etiam bellum indixisse videtur, dum pleraque fundamenta solidissime stabilita evertere est aggressus. Quamvis enim hae obiectiones apud ignaros maximi ponderis esse debeant, haud tamen metuendum est inde ipsi scientiae ullum detrimentum allatum iri.

2. Qui in huiusmodi investigationibus elaborarunt, facile perspicient resolutionem harum quaestionum calculus maxime intricatos postulare, quos autem mihi beneficia certorum characterum, quibus iam aliquoties optima successu sum usus, superare licuit. Huiusmodi scilicet character

$$\left(\frac{p}{q}\right),$$

quo fractio uncinulis inclusa repraesentatur, mihi denotat istud productum

$$\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-3}{3}\cdot\frac{p-3}{4}\cdots\frac{p-q+1}{q},$$

cuius ergo valor quovis casu facile exhiberi potest. Circa Hunc characterem autem sequentia notasse iuvabit.

1°. Semper est

$$\left(\frac{p}{q}\right)=\left(\frac{p}{p-q}\right).$$

2°. Si $q=0$, semper est

$$\left(\frac{p}{0}\right)=1.$$

3°. Si q sit vel numerus negativus vel maior quam p, valor ipsius $\left(\frac{p}{q}\right)$ semper est $=0$.

4°. Deinde si p numerus negativus, tum istam formulam

$$\left(\frac{-p}{q}\right)$$

reducere licet ad hanc

$$\pm\left(\frac{p-q+1}{q}\right),$$

ubi signum + valet, si q numerus par, − vero, si impar; unde patet istam formam etiam in hanc mutari posse $\pm\left(\frac{p-q+1}{p-1}\right)$.

3. His praemissis quaestiones ex ludo memorato natas generalissime sum tractaturus. Numerum scilicet schedularum denotabo littera m, quas singulas litteris diversis a, b, c, d etc. signatas assumo, ne usus numerorum absolutorum confusionem pariat. Deinde quovis tractu ex his schedulis i schedulas extrahi supponam, unde numerus omnium variationum, quae in his tractibus contingere possunt, erit $=\left(\frac{m}{i}\right)$. Praeterea si numerus tractuum successive institutorum fuerit $=n$, ex principiis combinationum patet numerum omnium variationum, quae contingere queant, esse $\left(\frac{m}{i}\right)^n$. Hoc ergo modo sequentia problemata percurram.

PROBLEMA 1

Si numerus schedularum litteris a, b, c, d etc. signatarum sit m indeque quolibet tractu extrahantur i schedulae atque iam numerus tractuum peractorum fuerit $=n$, quaeritur, quanta sit probabilitas, ut omnes m litterae a, b, c, d etc. exierint.

SOLUTIO

4. Hic primo observandum est, quoniam in n tractibus numerus schedularum extractarum est in, omnes litteras exire non posse, nisi fuerit $in > m$ ideoque

$$n > \frac{m}{i}$$

vel saltem non minus. Denotet iam $\varDelta$ numerum omnium variationum, quae in his n tractibus evenire possunt, eritque, ut iam indicavimus,

$$\varDelta = \left(\frac{m}{i}\right)^n;$$

qui cum sit numerus omnium casuum possibilium, pro nostra quaestione hinc omnes casus excludi debent, qui pauciores quam m litteras continent.

Primo ergo, si numerus litterarum tantum esset $= m-1$, quod m modis fieri potest, numerus casuum, qui tantum $m-1$ litteras vel pauciores continent, erit

$$m\left(\frac{m-1}{i}\right)^n,$$

quem numerum ponamus $= A$.

Simili modo, si binae litterae excludantur, quod $\left(\frac{m}{2}\right)$ modis fieri potest, numerus casuum tantum $m-2$ vel pauciores litteras continentium erit

$$\left(\frac{m}{2}\right)\cdot\left(\frac{m-2}{i}\right)^n,$$

quem numerum littera B indicemus.

Porro sit C numerus omnium casuum, qui tantum $m-3$ litteras vel pauciores continent, eritque

$$C = \left(\frac{m}{3}\right)\cdot\left(\frac{m-3}{i}\right)^n.$$

Eodemque modo fit

$$D = \left(\frac{m}{4}\right)\cdot\left(\frac{m-4}{i}\right)^n,\quad E = \left(\frac{m}{5}\right)\cdot\left(\frac{m-5}{i}\right)^n \text{ etc.}$$

Atque his elementis constitutis inveni numerum ommum casuum, qui omnes m litteras contineant, esse

$$\varDelta - A + B - C + D - \text{etc.},$$

quem numerum indicemus littera $\varSigma$.

5. Evidens est hunc numerum $\varSigma$ per solam theoriam combinationum determinari posse ideoque nulli prorsus dubio esse obnoxium, ita ut tanquam veritas geometrica spectari possit. Hinc autem secundum principia probabilium numerus casuum favorabilium per numerum omnium casuum possibilium divisus praebebit probabilitatem quaesitam; quae ergo si ponatur $= \varPi$ *,*
erit

$$\varPi = \frac{\varSigma}{\varDelta}.$$

Quare cum sit

$$\Sigma = \Delta - A + B - C + D - \text{etc.},$$

pro litteris *Δ, A, B, C* etc. valores assignatos substituendo erit

$$\Sigma = \left(\tfrac{m}{i}\right)^n - \left(\tfrac{m}{1}\right)\left(\tfrac{m-1}{i}\right)^n + \left(\tfrac{m}{2}\right)\left(\tfrac{m-2}{i}\right)^n - \left(\tfrac{m}{3}\right)\left(\tfrac{m-3}{i}\right)^n + \text{etc.},$$

Haec forma per $\left(\tfrac{m}{i}\right)^n$ divisa dabit probabilitatem, quod post *n* tractus omnes *m* litterae exierint, unde necessario ista expressio Σ semper nihilo aequalis esse debet, quoties fuerit $n < \tfrac{m}{i}$, quod etiam calculum pro casibus simplicioribus instituenti revera evenire patebit. Veluti si fuerit $m = 7,\ n = 3,\ i = 2,$ erit

$$\left(\tfrac{m}{i}\right)^n = 21^3 = 9261,$$
$$\left(\tfrac{m}{1}\right)\left(\tfrac{m-1}{i}\right)^n = 7\cdot 15^3 = 32625,$$
$$\left(\tfrac{m}{2}\right)\left(\tfrac{m-2}{i}\right)^n = 21\cdot 10^3 = 21000,$$
$$\left(\tfrac{m}{3}\right)\left(\tfrac{m-3}{i}\right)^n = 35\cdot 6^3 = 7560,$$
$$\left(\tfrac{m}{4}\right)\left(\tfrac{m-4}{i}\right)^n = 35\cdot 3^3 = 945,$$
$$\left(\tfrac{m}{5}\right)\left(\tfrac{m-5}{i}\right)^n = 21\cdot 1^3 = 21,$$
$$\left(\tfrac{m}{6}\right)\left(\tfrac{m-6}{i}\right)^n = 0,$$

unde prodit $\Sigma = 0$.

6. Dummodo ergo *n* non sit minus quam $\tfrac{m}{i}$, semper erit $\Sigma = 0$; at si fuerit $n = \tfrac{m}{i}$ sive $m = in$, hic casus maxime est memorabilis; tum enim formula nostra pro Σ inventa reduci potest ad productum ex meris factoribus constans. Erit enim

$$\Sigma = \left(\tfrac{m}{i}\right)\left(\tfrac{m-i}{i}\right)\left(\tfrac{m-2i}{i}\right)\left(\tfrac{m-3i}{i}\right)\cdots\left(\tfrac{i}{i}\right);$$

quod ut exemplo illustremus, sumamus, ut ante, $n = 3$ et $i = 2$, sit vero $m = 6$; atque forma prior pro Σ data praebet $\Sigma = 90$, altera vero dat $\Sigma = 15\cdot 6\cdot 1 = 90$.

7. Quanquam autem hae formulae, si pro *n* maiores numeri accipiantur, valde fiunt prolixae, tamen per logarithmos haud difficile erit quovis casu valorem probabilitatis Π assignare. Cum enim sit

$$\frac{A}{\varDelta} = m\cdot\frac{(m-i)^n}{m^n},$$

$$\frac{B}{A} = \frac{m-1}{2}\cdot\frac{(m-1-i)^n}{(m-1)^n},$$

$$\frac{C}{B} = \frac{m-2}{3}\cdot\frac{(m-2-i)^n}{(m-2)^n}$$

etc.,

hence with logarithms taken there will be

$$l\frac{A}{\varDelta} = lm - nl\frac{m}{m-i},$$

$$l\frac{B}{A} = l\frac{m-1}{2} - nl\frac{m-1}{m-1-i},$$

$$l\frac{C}{B} = l\frac{m-2}{3} - nl\frac{m-2}{m-2-i}$$

etc.,

ex quibus colligitur

$$l\frac{A}{\varDelta} = lm - nl\frac{m}{m-i},$$

$$l\frac{B}{\varDelta} = l\frac{A}{\varDelta} + l\frac{m-1}{2} - nl\frac{m-1}{m-1-i},$$

$$l\frac{C}{\varDelta} = l\frac{B}{\varDelta} + l\frac{m-2}{3} - nl\frac{m-2}{m-2-i}$$

etc.,

Unde ergo facile inveniuntur valores $\frac{A}{\varDelta}$, $\frac{B}{\varDelta}$, $\frac{C}{\varDelta}$ etc., quibus inventis probabilitas quaesita erit

$$\varPi = 1 - \frac{A}{\varDelta} + \frac{B}{\varDelta} - \frac{C}{\varDelta} + \frac{D}{\varDelta} - \text{etc.}$$

8. Applicemus haec ad casum ludi initio memorati, quo est $m = 90$ et $i = 5$, eritque, ut sequitur:

$$l\tfrac{A}{\Delta} = \quad l90 - l\tfrac{90}{85} = \quad 1,9542425 - n\cdot 0,0248236,$$
$$l\tfrac{B}{\Delta} = l\tfrac{A}{\Delta} + l\tfrac{89}{2} - nl\tfrac{89}{84} = l\tfrac{A}{\Delta} + 1,6483600 - n\cdot 0,0251107,$$
$$l\tfrac{C}{\Delta} = l\tfrac{B}{\Delta} + l\tfrac{88}{3} - nl\tfrac{88}{83} = l\tfrac{B}{\Delta} + 1,4673614 - n\cdot 0,0254046,$$
$$l\tfrac{D}{\Delta} = l\tfrac{C}{\Delta} + l\tfrac{87}{4} - nl\tfrac{87}{82} = l\tfrac{C}{\Delta} + 1,3374593 - n\cdot 0,0257054,$$
$$l\tfrac{E}{\Delta} = l\tfrac{D}{\Delta} + l\tfrac{86}{5} - nl\tfrac{86}{81} = l\tfrac{D}{\Delta} + 1,2355285 - n\cdot 0,0260135,$$
$$l\tfrac{F}{\Delta} = l\tfrac{E}{\Delta} + l\tfrac{85}{6} - nl\tfrac{85}{80} = l\tfrac{E}{\Delta} + 1,1512676 - n\cdot 0,0263289,$$
$$l\tfrac{G}{\Delta} = l\tfrac{F}{\Delta} + l\tfrac{84}{7} - nl\tfrac{84}{79} = l\tfrac{F}{\Delta} + 1,0791813 - n\cdot 0,026522$$

etc.

9. Perspicuum hic est, quo maior accipiatur numerus tractuum n, eo promptius istam progressionem convergere, ita ut, si n denotet numerum vehementer magnum, semper proxime proditurum sit $\Pi = 1$; tum scilicet maxime erit probabile omnes prorsus m numeros exiisse. Contra autem, si numerus n parum superet minimum valorem $\frac{m}{i} = 18$, evolutio horum terminorum maxime fiet operosa, cum pluribus terminis sit opus, antequam ad evanescentes perveniatur.

Sumamus $n = 100$, ut huic quaestioni respondeamus, quanta sit probabilitas, ut post centum tractus omnes nonaginta numeri exierint. Hic ergo erit

$$l\tfrac{A}{\Delta} = 9,47188, \text{ ergo } \tfrac{A}{\Delta} = 0,2964,$$
$$l\tfrac{B}{\Delta} = 8,60917, \text{ ergo } \tfrac{B}{\Delta} = 0,0407,$$
$$l\tfrac{C}{\Delta} = 7,53607, \text{ ergo } \tfrac{C}{\Delta} = 0,0034,$$
$$l\tfrac{D}{\Delta} = 6,30299, \text{ ergo } \tfrac{D}{\Delta} = 0,0002,$$
$$l\tfrac{E}{\Delta} = 4,93719, \text{ ergo } \tfrac{E}{\Delta} = 0,0000,$$

ergo

$$\Pi = 0,7411.$$

10. Sit $n = 200$ et pro hoc casu erit

$$l\tfrac{A}{\Delta} = 6,98952, \text{ ergo } \tfrac{A}{\Delta} = 0,00098,$$
$$l\tfrac{B}{\Delta} = 3,61574, \text{ ergo } \tfrac{B}{\Delta} = 0,00000,$$

unde colligatur probabilitas, quod post 200 extractiones omnes numeri exierint,

$$\Pi = 0{,}99902,$$

quae probabilitas certitudini omnes exiisse valde est propinqua.

PROBLEM 2

Positis, quae in problemate praecedente sunt constituta, quaeritur, quanta futura sit probabilitas, ut saltem $m-1$ *litterae post n tractus exierint.*

SOLUTIO

11. Hic ergo numerus tractuum omnes *m* litteras continentium non excluditur, unde patet tractuum numerum nostro praesenti casu fore maiorem. Calculo autem subducto, si numerus horum casuum ponatur Σ', inveni fore

$$\Sigma' = \Delta - B + 2C - 3D + 4E - 5F + \text{etc.},$$

unde probabilitas, quod post *n* tractus saltem $m-1$ litterae exierint, erit

$$\Pi' = \frac{\Sigma'}{\Delta}$$

ideoque

$$\Pi' = 1 - \frac{B}{\Delta} + 2\frac{C}{\Delta} - 3\frac{D}{\Delta} + 4\frac{E}{\Delta} - \text{etc.}$$

12. Hoc ergo casu erit

$$\Sigma' = \left(\frac{m}{i}\right)^n - \left(\frac{m}{2}\right)\cdot\left(\frac{m-2}{i}\right)^n + 2\left(\frac{m}{3}\right)\cdot\left(\frac{m-3}{i}\right)^n - 3\left(\frac{m}{4}\right)\cdot\left(\frac{m-4}{i}\right)^n + 4\left(\frac{m}{5}\right)\cdot\left(\frac{m-5}{i}\right)^n - \text{etc.},$$

unde, si applicatio fiat ad ludum memoratum, cum litterae Δ, A, B, C, D etc. eosdem retineant valores, calculus per logarithmos institutus ex inventis valoribus $\frac{B}{\Delta}$, $\frac{C}{\Delta}$, $\frac{D}{\Delta}$ etc. facile perficietur. Ita, si post 100 tractus requiratur probabilitas, quod saltem 89 numeri exierint, ob

$$\frac{B}{\Delta} = 0{,}0407,\ \frac{C}{\Delta} = 0{,}0034,\ \frac{D}{\Delta} = 0{,}0002$$

erit ista probabilitas

$$\Pi' = 0{,}9655.$$

Unde sequitur probabilitatem, quod tantum pauciores numeri exierint, fore 0,0345.

PROBLEMA 3

Iisdem positis, ut hactenus, quaeritum, quanta sit probabilitas, ut saltem $m-2$ *litterae post n tractus fuerint extractae.*

SOLUTIO

13. Numerus omnium casuum, qui saltem $m-2$ litteras contineant, per litteras ante stabilitas Δ, A, B, C, D etc. ita definitur, ut sit

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.},$$

quae expressio restitutis valoribus hoc modo se habet

$$\Sigma'' = \left(\tfrac{m}{i}\right)^n - \left(\tfrac{2}{2}\right)\left(\tfrac{m}{3}\right)\cdot\left(\tfrac{m-3}{i}\right)^n + \left(\tfrac{3}{2}\right)\left(\tfrac{m}{4}\right)\cdot\left(\tfrac{m-4}{i}\right)^n - \left(\tfrac{4}{2}\right)\left(\tfrac{m}{5}\right)\cdot\left(\tfrac{m-5}{i}\right)^n + \text{etc.},$$

atque hinc probabilitas erit

$$\Pi = 1 - \tfrac{C}{\Delta} + 3\tfrac{D}{\Delta} - 6\tfrac{E}{\Delta} + 10\tfrac{F}{\Delta} - \text{etc.}$$

Pro ludo igitur ante memorato si quaeratur probabilitas, ut post 100 tractus saltem 88 numeri exierint, ea reperietur

$$\Pi'' = 0{,}9972,$$

unde probabilitas, quod contrarium evenit, erit = 0,0028.

PROBLEMA GENERALE

Iisdem positis, ut ante, quaeritur, quanta sit probabilitas, ut post n tractus saltem $m-\lambda$ *litterae exierint.*

SOLUTIO

14. Numerus casuum ad minimum tot litteras continentium per nostros characteres ita commode exprimitur, ut sit

$$\left(\tfrac{m}{i}\right)^n - \left(\tfrac{\lambda}{\lambda}\right)\left(\tfrac{m}{\lambda+1}\right)\cdot\left(\tfrac{m-\lambda-1}{i}\right)^n + \left(\tfrac{\lambda+1}{\lambda}\right)\left(\tfrac{m}{\lambda+2}\right)\cdot\left(\tfrac{m-\lambda-2}{i}\right)^n$$
$$- \left(\tfrac{\lambda+2}{\lambda}\right)\left(\tfrac{m}{\lambda+3}\right)\cdot\left(\tfrac{m-\lambda-3}{i}\right)^n + \text{etc.},$$

quae formula per terminum primum $\left(\frac{m}{i}\right)^n$ divisa praebebit probabilitatem quaesitam.

15. In his probabilitatibus aestimandis utique assumitur omnes litteras ad extrahendum aeque esse proclives, quod autem Ill. D'ALEMBERT negat assumi posse. Arbitratur enim simul ad omnes tractus iam ante peractos respici oportere; si enim quaepiam litterae nimis crebro fuerint extractae, tum eas in sequentibus tractibus rarius exituras; contrarium vero evenire, si quaepiam litterae nimis raro exierint. Haec ratio si valeret, etiam valitura esset, si sequentes tractus demum post annum vel adeo integrum saeculum, quin etiam si in alio quocunque loco instituerentur; atque ob eandem rationem etiam ratio haberi deberet omnium tractuum, qui tam olim in quibuscunque terrae locis fuerint peracti, quo certe vix quicquam absurdius excogitari potest.

DEMONSTRATIO SOLUTIONUM PRAECEDENTIUM

16. Cum numerus omnium litterarum *a, b, c, d* etc., quibus singulas schedulas signatas assumimus, sit $= m$, Hunc litterarum complexum vocabo systema principale, unde alia systemata derivata, quae pauciores litteras contineant, formari conveniet, quae ita in ordines dispesco, ut ordo primus complectatur omnia systemata, quae tantum $m-1$ litteras contineant, quorum ergo numerus erit $= m$.
Ad ordinem vero secundum referam omnia systemata, in quibus litterarum numerus est $m-2$, quorum numerus erit

$$\frac{m}{1}\cdot\frac{m-1}{2}=\left(\frac{m}{2}\right).$$

Ordo autem tertius habebit omnia systemata, ubi numerus litterarum est $m-3$, quorum numerus est

$$\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}=\left(\frac{m}{3}\right).$$

Eodem modo numerus systematum quarti ordinis tantum $m-4$ litteras continentium erit

$$\left(\frac{m}{4}\right);$$

quinti autem ordinis, ubi tantum $m-5$ litterae insunt, numerus systematum erit

$$\left(\frac{m}{5}\right)$$

et ita porro.

17. Quae quo fiant clariora, systema principale his sex litteris

$$a\ b\ c\ d\ e\ f$$

constans contemplemur, ex quo ergo sequentia derivata cuiusque ordinis resultabunt, quae hac tabula exhibemus:

I.	II.	III.	IV.	V.
abcde	*abcd*	*abc*	*ab*	*a*
abcdf	*abce*	*abd*	*ac*	*b*
abcef	*abcf*	*abe*	*ad*	*c*
abdef	*abde*	*abf*	*ae*	*d*
acdef	*abdf*	*acd*	*af*	*e*
bedef	*abef*	*ace*	*bc*	*f*
	acde	*acf*	*bd*	
	acdf	*ade*	*be*	
	acef	*adf*	*bf*	
	adef	*aef*	*cd*	
	bcde	*bcd*	*ce*	
	bcdf	*bce*	*cf*	
	bcef	*bcf*	*de*	
	bdef	*bde*	*df*	
	cdef	*bdf*	*ef*	
		bef		
		cde		
		edf		
		cef		
		def		

ubi ergo numerus systematum ordinis primi est $6=\left(\frac{6}{1}\right)$, ordinis secundi $15=\left(\frac{6}{2}\right)$, ordinis tertii $20=\left(\frac{6}{3}\right)$, quarti $=20=\left(\frac{6}{3}\right)$, quinti $=6=\left(\frac{6}{5}\right)$.

18. Nunc evidens est singula systemata cuiusque ordinis inferioris in omnibus superioribus contineri, quod quoties eveniat plurimum interest observasse. Ita pro casu $m=6$ systema primi ordinis *abcde* in ordine hoc semel occurrit. At systema secundi ordinis *abcd* in primo ordine bis, in secundo semel occurrit. Deinde systema tertii ordinis *abc* in primo ordine ter, in secundo ter, at in tertio semel reperitur. Systema quarti ordinis *ab* in primo ordine quater, in secundo sexies, in tertio quater, in quarto semel inest. Denique systema quinti ordinis in primo ordine quinquies occurrit, in secundo decies, in tertio decies, in quarto quinquies, in quinto semel. Ex quo manifestum est hos numeros

convenire cum coëfficientibus binomii ad potestates elevati, siquidem omnia systemata in ipso principali semel continentur.

19. Hinc ergo in genere pro quovis systemate cuiuspiam ordinis inferioris facile assignari potest, quot modis in quolibet ordine superiore occurrat, id quod sequens tabula manifesto declarabit, ubi systema principale littera *O,* systemata autem primi, secundi, tertii, quarti etc. ordinis notis romanis I, II, III, IV, V, VI etc. denotabo.

	O	I	II	III	IV	V	VI
m	1						
$m-1$	1	1					
$m-2$	1	2	1				
$m-3$	1	3	3	1			
$m-4$	1	4	6	4	1		
$m-5$	1	5	10	10	5	1	
$m-6$	1	6	15	20	15	5	1
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
$m-\lambda$	$\left(\frac{\lambda}{0}\right)$	$\left(\frac{\lambda}{1}\right)$	$\left(\frac{\lambda}{2}\right)$	$\left(\frac{\lambda}{3}\right)$	$\left(\frac{\lambda}{4}\right)$	$\left(\frac{\lambda}{5}\right)$	$\left(\frac{\lambda}{6}\right)$

20. Consideremus nunc numerum schedularum, quae quovis tractu tam ex systemate principali actu extrahuntur, quam ex systematibus derivatis extrahi concipi possunt, quae quidem ex principali facillime deduci poterunt. Quodsi quovis tractu unica littera extrahatur, pro systemate principali numerus tractuum diversorum erit

$$=\left(\frac{m}{1}\right);$$

sin autem binae litterae simul extrahantur, numerus omnium tractuum diversorum erit

$$\frac{m}{1}\cdot\frac{m-1}{2}=\left(\frac{m}{2}\right).$$

Si ternae litterae quovis tractu extrahantur, numerus tractuum diversorum erit

$$\left(\frac{m}{3}\right)$$

atque in genere, si *i* litterae quovis tractu extrahantur, numerus omnium tractuum diversorum erit

$$\left(\frac{m}{i}\right).$$

Sin autem tales extractiones etiam ex systematibus derivatis fieri concipiantur, pro quolibet systemate ordinis primi numerus tractuum diversorum erit $\left(\frac{m-1}{i}\right)$, ordinis secundi $\left(\frac{m-2}{i}\right)$, ordinis tertii $\left(\frac{m-3}{i}\right)$ et ita porro.

21. Quodsi iam hae extractiones bis repetantur, quoniam pro systemate principali quemlibet tractum non solum reliquae omnes sequi possunt, sed etiam ipsae, numerus diversorum casuum erit $\left(\frac{m}{i}\right)^2$. Si tres extractiones successive instituantur, omnium casuum numerus erit $\left(\frac{m}{i}\right)^3$; atque in genere, si n extractiones sibi succedant, numerus omnium casuum possibilium erit $\left(\frac{m}{i}\right)^n$, quem numerum littera Δ supra designavimus, ita ut sit $\Delta = \left(\frac{m}{i}\right)^n$.

22. Simili modo numerus omnium casuum, qui in quolibet systemate primi ordinis locum habere possunt, est $\left(\frac{m-1}{i}\right)^n$; quare cum horum systematum numerus sit $\left(\frac{m}{i}\right)$, numerus omnium casuum, quem primus ordo praebet, erit $\left(\frac{m}{1}\right)\left(\frac{m-1}{i}\right)^n$ quemque littera A designemus, ita ut sit

$$A = \left(\frac{m}{1}\right)\left(\frac{m-1}{i}\right)^n .$$

Eodem modo facile intelligitur numerum omnium casuum, qui ex singulis systematibus oriri possunt, esse pro ordine secundo

$$B = \left(\frac{m}{2}\right)\left(\frac{m-2}{i}\right)^n$$

pro ordine tertio

$$C = \left(\frac{m}{3}\right)\left(\frac{m-3}{i}\right)^n$$

pro ordine quarto

$$D = \left(\frac{m}{4}\right)\left(\frac{m-4}{i}\right)^n ,$$

et ita porro. His iam praemissis solutiones singulorum problematum praecedentium facile expedire licebit.

PRO PROBLEMATE PRIMO

23. Cum in hoc problemate ex omnibus casibus possibilibus, quorum numerus est Δ, ii enumerari debeant, qui omnes *m* litteras involvunt, inde excludamus primo omnes casus, qui tantum $m-1$ litteras vel pauciores continent, quod fiet, si omnes casus possibiles primi ordinis, quorum numerus est *A,* auferamus. Hoc enim modo casus, qui $m-1$ litteras continent, e medio tollentur. At vero casus, qui $m-2$ litteras continent, bis auferentur hoc modo; unde in formula $\Delta - A$ semel deficient, ita ut eorum numerus $1-2 = -1$. At pro casibus $m-3$ litteras continentibus numerus, quo in formula $\Delta - A$ occurrent, erit $1-3=-2$. Simili modo pro $m-4$ habebimus $1-4=-3$ et ita porro, qui ergo casus deficientes iterum restitui debebunt.

24. Casus autem formae $m-2$ semel deficientes restituentur, si ad formulam $\Delta - A$ addatur *B*. Hoc autem modo terminas formae $m-3$ ter adduntur, cum tamen bis tantum defecissent; ergo nunc semel abundabunt, sive index erit $+1$. At $m-4$ forma sexies adiicitur, cum tantum ter defuisset, ideoque index erit $+3$. Simili modo pro terminis formae $m-5$ index erit $10-4=+6$ et ita porro.

25. Ut igitur hos casus iam abundantes iterum tollamus, subtrahamus omnes casus ordinis tertii, $=C$. Hoc enim modo termini formae $m-3$ penitus tollentur, reliqui autem nimis crebro auferentur, scilicet pro ordine $m-4$ index erit -1, pro ordine $m-5$ index erit -4 etc.

26. Quia forma $m-4$ semel deficit, restitutio fiet addendo litteram *D.* Inferiores autem nunc redundabunt secundum indices 1, 5, 15 etc., unde *E* subtrahendo hi tollentur; quod nimis subtractum est, additione litterae *F* restituetur et ita porro.

27. Hinc iam satis manifestum est ex forma Δ sublatos esse omnes casus pauciores quam *m* litteras continentes; quorum ergo restantium numerus erit

$$\Delta - A + B - C + D - E + F - \text{etc.},$$

quem indicavimus per Σ ; sicque solutio primi problematis firmiter est demonstrata.

PRO PROBLEMATE SECUNDO

28. Manifestum est secundum idem, quo hic usi sumus, ratiocinium procedendo demonstrationem. pro secundi problematis solutione adornari posse. Nec opus erit omnia tam prolixe exponere. Cum enim ex numero casuum possibilium Δ ii sint enumerandi, qui tantum $m-1$ litteras continent, statim patet hinc excludendos esse omnes casus $m-2$ litteras continentes, quod fiet, si a numero *d* numerus *B* subtrahatur. At tabula supra § 19 data declarat hoc modo terminos formae $m-3$ ter ablaturos esse, cum tamen semel tantum subtrahi debuissent, et ita de reliquis formis. Ad eos restituendos addatur numerus

$2C$, quo numeri deficientes formae $m-3$ penitus tollentur; redundabunt autem numeri formae $m-4$ indice 3, ac magis superfluunt sequentes. Quo priores tollantur, iterum subtrahi debet numerus $3D$, quo sublato termini formae $m-4$ exclusi erunt. Deficientes numeri formae $m-5$ et sequentium iterum additione numeri 4E erunt restituendi et ita porro; quibus operationibus peractis numerus restantium erit

$$\Sigma' = \Delta - B + 2C - 3D + 4E - 5F + \text{etc.};$$

sicque solutio secundi problematis est demonstrata .

PRO PROBLEMATE TERTIO

29. Hic a numero Δ subtrahi debet numerus C, quo casus $m-3$ schedularum excludantur; et quia hoc pacto numerus D quater subtrahitur, cum tantum semel redundabat, iterum addi debet numerus $3D$, quo ille et sequeutes deficientes restituantur. Quod excedit subtractione numeri $6E$ tolletur, deficientes vero additione numeri $10F$ restituendi sunt et ita porro; unde numerus casuum $m-2$ litteras continentium erit

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.},$$

quemadmodum in solutione tertii problematis asseveravi. Hoc modo igitur haec etiam solutio firmiter est demonstrata.