CONCERNING A SIGNIFICANT ADVANCE

IN THE SCIENCE OF NUMBERS

Opuscula analytica, 2, 1785, p. 275-314
[E598]
[Shown to the St. Petersburg Academy on the 26th October 1775]

1. The advances are quite extraordinary, which the celebrated Lagrange has shown in the Commentaries of the Prussian Royal Academy for the year 1773, concerning the divisors of the most general formula $Btt + Ctu + Duu$, and they illuminate the science of numbers in the greatest light, which even now is surrounded by so much darkness. But on this account, since that treatment is especially general, those, who have not been trained well enough in considerations of this kind, encounter some difficulty, nor are they able to understand well enough the strength of such lofty demonstrations. On account of which it will not be in vain, to set out all the investigations more carefully, on which these demonstrations depend, and for more special formulas to be adopted, since in this way they will be able to be understood everything more easily. Concerning which I may explain more accurately, how much more support will be able to be brought to most of the theorems especially, the truth of which may be allowed to be understood by me indeed by induction alone, it may be able to advance, from which it will become much more apparent, how much hitherto a perfect demonstration of these may be desired.

LEMMA

2. If p and q were numbers prime between themselves, then clearly all the numbers can be understood to be in this general form $\alpha p \pm \beta q$, and that in infinitely many ways.
The demonstration of this lemma itself is easily found here and there .

PROBLEM 1

3. *If p and q shall be numbers prime between themselves, n may denote some given number, either positive or negative, to find all the divisors of this formula : $pp + nqq$.*

SOLUTION

D may denote some divisor contained in this form $pp + nqq$, and d shall be the quotient arising from this division, thus so that $Dd = pp + nqq$. Now it is evident here at once the number d must become prime to q ; indeed if q [and d] were to have a common divisor, p too must have the same, contrary to the hypothesis ; on account of which the number p will be able to be expressed thus by d and q, so that there shall become $p = \alpha d \pm \beta q$, with which value substituted there will become

$$Dd = \alpha\alpha dd \pm 2\alpha\beta dq + (\beta\beta + n)qq,$$

and thus the divisor

$$D = \alpha\alpha d \pm 2\alpha\beta q + (\tfrac{\beta\beta+n}{d})qq,$$

where $\frac{\beta\beta+n}{d}$ therefore will be a whole number, which shall be $= h$, and thus

$D = \alpha\alpha d \pm 2\alpha\beta q + hqq$ will be had,

for which form we may write

$$D = frr \pm gqr + hqq,$$

thus so that there shall become $f = d$, $r = \alpha$, $g = 2\beta$, and on account of $h = \frac{\beta\beta+n}{d}$, there will become $fh = \beta\beta + n$; and hence there will become $4fh - gg = 4n$. Hence therefore it is apparent all the divisors of the form $pp + nqq$ always to be contained in this form:

$$D = frr \pm gqr + hqq,$$

provided there were $4fh - gg = 4n$. And in turn, if there were

$$D = frr \pm gqr + hqq,$$

there will be

$$4Df = 4ffrr \pm 4fgqr + 4fhqq;$$

and thus on account of $4fh = 4n + gg$, there will be

$$4Df = (2fr \pm gq)^2 + 4nqq,$$

which form agrees with the proposed form, since it must be divided by 4.

COROLLARY 1

4. Therefore in general it will be allowed to deal with all the divisors of the proposed formula $pp + nqq$ in this most widely accessible form: $frr + grs + hss$, [replacing q by s], provided there were $4fh - gg = 4n$, or $fh - \frac{1}{4}gg = n$; from which it is apparent innumerable formulas of this kind can be shown, since the number g will be allowed to be taken as it pleases. Moreover from that the numbers f and h will be allowed to be defined, so that there may become

$$4fh = 4n + gg.$$

SCHOLIUM

5. Moreover, since innumerable formulas of this kind : $frr + grs + hss$ may be allowed to be shown, in which there shall be $4fh - gg = 4n$, hence little of gain will be seen to be added to our presentation. Indeed for some proposed number *D,* requiring to be judged, each shall be able to be a divisor of the form $pp + nqq$, all these innumerable formulas must be considered, whether perhaps that same number *D* may be contained in some of those. Therefore a particular one found, which we must refer to accept from the illustrious Lagrange, consists in this, so that an infinite multitude of formulas of that kind may lead in some case for a small number to be recalled, that which we may set out in the following problem.

PROBLEM 2

6. *The general form of the divisors found before,* $frr + grs + hss$, *in which there shall be* $4fh - gg = 4n$, *may be transformed into another of the same kind,* $f'tt + g'tu + h'uu$, *in which there shall be* $g' < f'$ *or* h', *with the property* $4f'h' - g'g' = 4n$ *remaining.*

SOLUTION

We may put $f < h$ and the number *g* to be somewhat greater than *f,* and we may put in place $r = t - \alpha s$, with which value substituted this formula will arise :

$$ftt + (g - 2\alpha)ts + (\alpha\alpha - \alpha g + h)ss\,;$$

where clearly α will be able to be assumed thus, so that there may become $g - 2\alpha f < f$, where indeed nothing matters whether $g - 2\alpha f$ may become positive or negative. Therefore we may put in place $g - 2\alpha f = \pm g'$, thus so that certainly there shall be $g' < f'$, then truly on account of the analogy we may write f' in place of f, and $\alpha\alpha f - \alpha g + h = h'$, and there will be

$$4f'h' - g'g' = 4fh - gg = 4n.$$

Therefore in this manner the proposed form is reduced to this :

$$f'tt \pm g'ts + h'ss\,,$$

in which certainly there becomes $g' < f'$. Because if now it may eventuate, so that hitherto g' were greater than *h',* then in a similar manner this same formula will be able to be transformed into the other, in which the middle coefficient will be less than either end one you please, from which it is apparent the proposed form

$$frr \pm grs + hss$$

always can be converted into another similar form

$$f'tt \pm g'tu + h'uu,$$

in which g' shall be less than f' and h', and in a similar manner there may become

$$4f'h' - g'g' = 4n.$$

COROLLARY 1

7. Therefore in this manner an infinite multitude of formulas $frr + grs + hss$, in which $4fh - gg = 4n$, and most can be reduced to a small enough number, while clearly all these formulas are to be excluded, in which the middle coefficient g is greater than either of the end ones.

COROLLARY 2

8. Therefore since there shall be both $f > g$ as well as $h > g$, there will become $4fh > 4gg$. Therefore there shall be $4fh = 4gg + \Delta$, and since there must become $4fh - gg = 4n$, there will be $3gg + \Delta = 4n$, and thus $3gg < 4n$, hence therefore $g < \sqrt{\frac{4n}{3}}$; on account of which in place of g it will suffice to have assumed successively only these values, which are less than $\sqrt{\frac{4n}{3}}$, from which the values of the individual letters f and h will be deduced easily from the equation $4fh = gg + 4n$, with which done plainly all the divisors of the form $pp + nqq$ certainly will be held in some of these simple formulas.

SCHOLIUM

9. Because the equation $4fh - gg = 4n$ cannot have a place, unless g shall be an even number, we may write $2g$ at once for g, so that the form d shall become $frr + 2grs + hss$, with there being $fh - gg = n$, which therefore can be reduced thus to a form, so that there shall be $2g < f$ or $< h$. But this reduction can be put in place most conveniently by a step, provided that in place of α in the above reduction there may be written unity. Thus so that if the divisor were

$$D = frr + 2grs + hss,$$

there will be also

$$D = f'rr + 2g'rs + h'ss,$$

with the twofold way present, or

$$f' = f,\ g' = f - g \text{ and } h' = f - 2g + h \ ,$$

or also

$$h' = h,\ g' = h - g \text{ and } f' = f - 2g + h,$$

since it is allowed to interchange the end terms between themselves. So that if here whether there were $2g' < f'$ or $2g' < h'$, that same operation as before must be continued, while there may become $2g < f$ or h; where it is required to be observed in these formulas the middle term 2grs can be taken both positive as well as negative, therefore so that the numbers r and s can denote all the integers, either positive or negative. Therefore from these premised we may investigate all the prime divisors of the numbers contained either in this form: $pp + nqq$, or in this : $pp - nqq$; if indeed composite divisors are composed from primes, thus so that with all the prime divisors known likewise all the composite numbers may be obtained.

PROBLEM 3

10. *To find all the prime divisors of the numbers contained in this form :* $pp + nqq$ *, with the numbers p and q prime between themselves, as well as with respect to the number n.*

SOLUTION

I. Indeed since here concerning the prime divisors there is need for a discussion only, for unless p also may be prime to n, the formula $pp + nqq$ also would admit all the divisors of the number n, which therefore require no discussion and are themselves produced at once. Therefore D shall be some divisor of the form $pp + nqq$, and we have seen always to be in the manner

$$D = frr + 2grs + hss,$$

with $fh - gg = n$ present, so that there shall be both $2g < f$ as well as $2g < h$; where hereby since hence there shall be $f > 2g$ and $h > 2g$, there will become $fh > 4gg$. There shall be therefore $fh = 4gg + \varDelta$, and because $fh - gg = n$, there will be $3gg + \varDelta = n$, and thus $gg < \frac{n}{3}$ and $g < \sqrt{\frac{n}{3}}$. Therefore with this condition a multitude of forms is reduced for the divisor D to that smaller number, for which the smaller would be the number n. Therefore since there shall be

$$D = frr + 2grs + hss,$$

there will become

$$Df = ffrr + 2fgrs + fhss$$

and thus on account of $fh = gg + n$ there will become

$$Df = (fr + gs)^2 + nss,$$

which is that same form proposed. In a similar manner with the letters f and h interchanged, there will be also

$$Dh = (hs + gr)^2 + nrr,$$

from which it is evident, if Df were a number of the form $pp + nqq$, then also the product Dh to become of this same form, thus so that it may suffice to be found in either way. Therefore for whatever case all the values of the letters t may be sought, which shall be f, f', f'', f''' etc., and all the prime divisors D will be prepared thus, so that either D, Df, Df', or Df'' etc. shall be numbers of the form $pp + nqq$. And these follow from the demonstrations of the illustrious Lagrange.

II. We may therefore be able to connect these with those, which at one time now I had commented on the forms of these divisors of primes, where I have shown all these divisors can be dealt with in an expression of this kind : $4ni + a$, while clearly a may denote certain numbers prime to $4n$ and likewise smaller than $4n$, there only half of such numbers occurs, with the rest hence clearly excluded. From which if α may denote these excluded numbers, it will be able to confirm no numbers contained in the form $4ni + \alpha$, which are able to be divisors of the form $pp + nqq$. But these forms agree especially with the preceding ones. If indeed there were $D = pp + nqq$, some or other of the numbers p and q must be odd, and thus q will be either even or odd. If in the first place q were even, and thus qq a number of the form $4i$, there will become $D = 4ni + pp$. From which it is apparent that letter α to include all the odd square numbers and prime to $4n$, or the remainders, which remain from the division of these squares made by $4n$. But if q were an odd number, and thus qq of the form $4i + 1$, hence there will become $D = 4ni + pp + n$. From which it is apparent the letter α also to include all the numbers of the form $pp + n$, which indeed shall be prime to $4n$, or the residues of these remaining from the division by $4n$. Truly these same numbers also result for α, if Df were a number of the form $pp + nqq$, that which will be able to be shown more easily in examples.

III. With these set out, since the form $4ni + a$ may include all the divisors of the form $pp + nqq$, but the other form $4ni + \alpha$ may involve no divisors within itself, from the first form $4ni + a$ all numbers will have to be excluded divisible by some number of the form $4ni + \alpha$. So that therefore if it may be able to be shown in this manner clearly all the numbers excluded from the formula $4ni + a$, which are unable to be divisors of the form $pp + nqq$, then evidently it will follow all the prime numbers of the form $4ni + a$ certainly to be divisors of the form $pp + nqq$, since we have excluded only composite numbers in this way. Therefore the whole matter corresponds to this, so that it may be shown the formula $4ni+\alpha$ plainly contains all the prime numbers, which are unable to be divisors of the form $pp+nqq$; which if it may be able to be shown, nothing further may be desired in this kind.

COROLLARY 1

11. Therefore for some number n all the numbers $4n$ smaller to some prime itself may be distributed into two classes, of which we have designated the one by the letter a, truly the other by the letter α, thus so that the formula $4ni+a$ may hold all the divisors of the form $pp+nqq$, truly the other formula $4ni+\alpha$ may exclude those divisors completely, nor any number of this form at any time shall be able to be a divisor of the formula $pp+nqq$. But the multitude of numbers of each class always is the same ; evidently if the multitude of all the numbers less than the number $4n$ and of the primes to that were $=2\lambda$ (indeed this same number is even always), the first form a contains the number λ, and the other form α will contain just as many.

COROLLARY 2

12. Concerning these formulas: $4ni+a$ and $4ni+\alpha$ now at one time I have shown, if the numbers a and a' may occur in the former class, then the product aa' to occur there also, that which is to be understood also concerning several numbers of the same class, which if they were a, a', a'', a''' etc., also the products both from two as well as from several of these numbers, and thus also all the powers of these will be found in the same class, evidently after division by $4n$ and the remainders were reduced smaller than $4n$. Thence also I have shown, if α were a number of the latter class, then in the same also will be found the numbers $a\alpha,\ a'\alpha,\ a''\alpha,\ a'''\alpha$ etc . From which it is clear the multitude of numbers of the latter class cannot be smaller than of the first class. But so that the multitude of each shall be exactly equal, that also can be shown easily. Then truly also this is certain, if $\alpha,\ \alpha',\ \alpha'',\ \alpha'''$ etc. were the numbers of the latter class, then both the squares of these, just as the products of these arising from the two terms in the former class, moreover again the products from three terms in turn will be found in the latter class.

COROLLARY 3

13. Therefore everything, which hitherto in this generally can be desired, correspond here, so that it may be shown the class $4ni+\alpha$ to contain all the prime numbers, which may be unable to be divisors of the form $pp+nqq$; then indeed all the prime numbers of the former form $4ni+a$ certainly will prevail to be divisors of some number of the form $pp+nqq$.

PROBLEM 4

14. *To find all the prime divisors of numbers contained in this form: $pp-nqq$, where indeed, as before, p and q not only shall be prime between themselves, but also prime to n.*

SOLUTION

I. The condition, so that p also shall be prime to n, thus only is added here, because all the other divisors of the number n also may come into consideration here, yet which we exclude here, as is shown by themselves. Therefore here it is apparent initially, if D were a prime divisor of the form $pp-nqq$, then also to become a divisor of the form , if indeed nqq were greater than pp. For if D were a divisor of the form $pp-nqq$, also it will be a divisor of the form $npp-nnqq$, which form, if in place of nq we may write r, will be changed into this : $npp-rr$. Then in the same manner as appeared before always to become

$$D = frr + 2grs + hss,$$

with $fh-gg=-n$ present, and that form always able to be reduced, so that there may become $2g<f$ and likewise $2g<h$, where indeed the signs of the numbers f and h are not to be considered, if perhaps either member may become negative; whereby since, on account of $f>2g$ and $h>2g$, there shall be $fh>4gg$, it is evident there cannot become

$$fh-gg=-n,$$

unless either f or h were negative, from which the form of the divisor thus must be constituted, so that there shall be

$$D = frr + 2grs - hss,$$

and there must become $-fh-gg=-n$, or $fh+gg=+n$. Therefore since $fh>4gg$, it is necessary, so that there shall be $5gg<n$, and thus $g<\sqrt{\frac{n}{5}}$, thus so that in this case fewer values may be left for g. But then there will become

$$Df = ffrr + 2fgrs - fhss,$$

or

$$Df = (fr+gs)^2 - nss,$$

which is the form itself proposed. Again moreover there will be

$$Dh = nrr - (gr - hs)^2 \ ,$$

which is our inverse form $npp - qq$. Hence therefore it is understood, if Df were a number of the form $pp - nqq$, then there the formula Dh itself to become a number of the form $npp - qq$.

II. We may apply this also to that form of the divisor, which I have shown formerly [see E557]; and indeed initially if there were $D = pp - nqq$, for the cases in which q is an even number, and thus qq of the form $4i$, there will become $D = pp - 4ni$; from which if there may be put $D = 4ni + a$, on account of $pp > 4ni$, if there may be put $pp = 4nk + b$, such a form will be produced : $D = 4ni + b$, thus so that there shall be $a = b$, and thus may include all the square numbers prime to $4n$. But if q shall be an odd number, and thus qq of the form $4i + 1$, there will become $D = pp - n - 4ni$, and again on putting $pp = 4nk + b$ there will be produced $D = 4ni + b - n$, thus so that in this case there shall become $a = b - n$, where b can denote all the square numbers, or the remainders thence arising. In a similar manner if there were $D = npp - qq$, it is evident hence the values are going to be produced of the preceding for a to become negative, thus so that a may include all the square numbers, then also all the numbers of the form $pp - n$ taken, both positive as well as negative ; on account of which the form of all the divisors will be able to be shown thus, so that if there shall be $4ni \pm a$, moreover the form for the numbers from the excluded class of divisors will become $4ni \pm \alpha$, the number of which is equal to the former, evidently α will be allotted always just as many values as has the letter a.

III. Therefore also so that in this generally nothing further may be able to be desired, it remains only, so that the latter form $4ni \pm \alpha$ clearly may be shown to contain all the prime numbers, which under no circumstances may be able to be divisors of any number either of the form $pp - nqq$, or of $npp - qq$.

COROLLARY 1

15. With regard to these two formulas : $4ni \pm a$ and $4ni \pm \alpha$, of which the one includes all the divisors, truly the other one excludes all the divisors, the same prevail, which have been treated before. Evidently if a, a', a'' etc. may pertain to the first class, there also both all the powers, as well as all the products from two or more of these numbers will be found ; then truly if α shall be a number of the second class, in the same place also all the numbers $a\alpha,\ a'\alpha,\ a''\alpha$ etc. occur , thus so that the multitude of these numbers cannot be smaller than of the first class.

COROLLARY 2

16. Because the letter a includes all the squares, first of all its value will be 1, then truly also 9, 25 etc., unless the number n may have a divisor either 3, 5 etc. Indeed from these cases it will be required to exclude these cases, since in any case the form $4ni \pm a$ may not be able to be a prime number.

SCHOLIUM

17. Therefore with these general precepts put in place everything will emerge clearer, if we may set out particular cases ; here indeed several hitherto occur, which in general it will not be permitted to include. Moreover it will suffice that it may be shown by some examples, with which treated it will not be difficult to construct a table, which may show the forms of the prime divisors for all cases.

EXAMPLE 1

18. To find all the prime divisors of the numbers contained in the formula $pp + qq$, while clearly the numbers assumed for p and q to be prime between themselves.

SOLUTION

Initially with the divisor put

$$D = frr + 2grs + hss\,,$$

on account of $n = 1$ there must be $fh = gg + 1$, then truly $g < \sqrt{\tfrac{1}{3}}$; from which it is apparent no other value besides 0 to be assumed for g ; but then there will be $fh = 1$ and thus both $f = 1$ as well as $h = 1$, and thus all the divisors will be contained in this form $D = rr + ss$, thus so that the sum of two squares does not allow other divisors, unless which themselves shall be the sum of two squares. But the other form of divisors will be $4i + 1$, and all the numbers of the form $4n + 3$ or $4i - 1$ or may be excluded. So that therefore if it were possible to demonstrate the formula $4i - 1$ plainly to contain prime numbers, which may be unable to be divisors of the form $pp + qq$, then likewise it would be required to demonstrate also all the divisors of the form $4i + 1$ to become the sum of two squares. But this has been shown by me some time ago following Fermat [see E241].

EXAMPLE 2

[18a.] To find all the prime divisors of the form $pp - qq$.

SOLUTION

This example is referred to problem four, and there shall be $n = 1$, and since there must be $g < \sqrt{\frac{1}{5}}$, by necessity there will be required to become $g = 0$, and thus $fh = 1$, from which this form of divisor arises : $D = rr - ss$, which certainly clearly contains all the prime numbers except two. Although indeed this form has the factors $r + s$ and $r - s$, yet it contains all the primes, if there were $r - s = 1$, the account of which is unique. That other form of the divisor declares also, from which on account of $a = 1$, it becomes $4i \pm 1$, in which clearly all the odd numbers are contained, thus so that in this case nothing may be excluded, and the other form $4i \pm \alpha$ in this single case may not have a place. Moreover this case properly here is not concerned, because the divisors of the form $pp - qq$ agree amongst themselves.

EXAMPLE 3

19. To find all the prime divisors of the form $pp + 2qq$.

SOLUTION

This case pertains to the third problem, with $n = 2$, from which since there must be $g < \sqrt{\frac{2}{3}}$, there will be $g = 0$, and thus $fh = 2$, hence the form of the divisor will be $= rr + 2ss$. From which it is apparent numbers of the form $pp + 2qq$ do not allow other divisors, unless which shall be of the same form, which also now has been demonstrated some time ago. Moreover the other form $D = 8i + a$, on account of $a = pp$, or also $a = pp + 2$, give those values 1 and 3 for a, thus so that all the divisors of the form $pp + 2qq$ shall be either $8i + 1$ or $8i + 3$. Therefore the forms, which may be excluded from the class of the divisors, are $8i + 5$ and $8i + 7$, which therefore will require to be included under the form $8i + \alpha$. Therefore so that if it may be able to show only prime numbers of this form to be excluded from the class of divisors, likewise it may be required to show all the former prime numbers of the forms $8i + 1$ and $8i + 3$ to be contained in the formula $pp + 2qq$, indeed that which now has been shown. [E256] Moreover these two latter formulas also can be expressed thus: $8i - 1$ and $8i - 3$, thus so that the values of this α shall be the negative of this a, that which in general is required to be extended to the divisors of the form $pp + nqq$.

EXAMPLE 4

20. To find all the prime divisors of the form $pp-2qq$ or $2pp-qq$.

SOLUTION

From the fourth problem there is $n=2$, and thus, on account of $g<\sqrt{\frac{2}{5}}$, again there will be $g=0$ and $fh=2$, from which according to the divisors there will be $D=rr+2ss$, or also $D=rr-2ss$; from which it is apparent these forms admit no other divisors, unless which themselves shall be of the same form. But according to the form $D=8i+a$, because there is $a=pp$, or also $a=pp-2$, the values for a will be ± 1, therefore all the divisors will be contained in the form $8i\pm 1$; therefore all the numbers of the formula $8i\pm 3$ will be excluded. From which if only the prime numbers of the form $8i\pm 3$ may be excluded from the class of the divisors, it is necessary, that all the prime numbers of the form $8i\pm 1$ may be contained in the proposed form.

COROLLARY 1

21. Since in the third problem the reduction of the divisors to the form $pp+nqq$ may succeed generally only in a single way, in the case of the fourth problem such a reduction succeeds in an infinite number of ways ; indeed it is allowed thus always to assume the numbers p and q in a infinitude of ways, so that either the divisor D itself or Df of the formula $pp-nqq$ may be equal.

COROLLARY 2

22. But in the case of this example it is worthwhile to note, if there were $D=pp-2qq$, then also to become $D=2rr-ss$, because these two forms can become equal to each other, indeed with those equal there becomes

$$pp+ss=2(qq+rr)=(q+r)^2+(q-r)^2$$

thus so that there shall become $p=q+r$ and $s=q-r$.

EXAMPLE 5

23. To find the prime divisors of the form $pp+3qq$.

SOLUTION

Since here there is $n=3$, and thus $g<1$, there will be only $g=0$, and hence the divisor $D=rr+3ss$, thus so that also in this case all the prime divisors shall be of the form $pp+3qq$. But because the limit found for g may itself be equal to one, it need not be greater, we may set out also the case $g=1$, from which there becomes $fh=4$ and thus either $f=1$ and $h=4$, or $f=2$ and $h=2$. In the first case there becomes

$$D=rr+2rs+4ss=(r+s)^2+3ss,$$

which is the form proposed itself. In the other case there becomes :

$$D=2rr+2rs+2ss,$$

which form since it may have the factor 2, there must be put

$$D=rr+rs+ss,$$

but which equally is reduced to the proposed form. For if s is an even number, there may be put $s=2t$, there will become :

$$D=rr+2rt+4tt=(r+t)^2+3tt;$$

but if s is an odd number, r also must be odd, because otherwise it will return to the preceding case ; therefore $r+s$ will be an even number, from which on putting $r=2t-s$ there will become

$$D=4tt-2ts+ss=3tt+(t-s)^2,$$

from which it is apparent the above conclusion even now to prevail, and always to be $D=rr+3ss$. Then for the formula $12i+a$ on account of $a=pp$ there will be $a=1$, then truly the formula $a=pp+3$ gives $a=7$, from which all the divisors will be contained in either of the formulas : $12i+1$ or $12i+7$, which we may represent thus : $12i+1,\ 7$, or also in this manner $12i+1,-5$. If indeed all the values of a beyond $2n$ in general may be allowed to be suppressed, clearly by admitting negative numbers, then the other formula $12i\pm\alpha$, in which no divisor is present, will be $12i+5$ and $12i+11$, or $12i-1,+5$, from which in general it is apparent the values of α to be the negative of a.

EXAMPLE 6

[23a.] To find the prime divisors of the formula $pp-3qq$ or also of $3pp-qq$.

SOLUTION

By applying the fourth problem there will be here $n=3$, and thus $g<\sqrt{\frac{3}{5}}$, $g=0$ and $fh=3$ will follow, from which the divisor will be $D=rr-3ss$. Hence it is apparent these numbers admit no other divisors, unless which shall be of the same form. Then for the formula $12i\pm a$, on account of $a=pp$, or $a=3-pp$, no other values arise, besides $a=1$, thus so that all the divisors may be contained in this form : $12i\pm 1$. Therefore the formula excluding divisors will be $12i\pm 5$.

SCHOLIUM

24. Now at one time I have set out these formulas [see E164], and I have shown these other formulas not to be admitted, unless which shall be of the same form, because that does not always happen with greater numbers assumed for *n*. Moreover it will be agreed to exclude these cases, for which *n* is either a square number or divisible by a square. Indeed if there were $n=kmm$, then the formula $pp\pm kmmqq$ will agree with this : $pp\pm kqq$.

EXAMPLE 7

25. To find the prime divisors of the numbers $pp+5qq$.

SOLUTION

On account of $\sqrt{\frac{5}{3}}>g$ there will be either $g=0$ or $g=1$; in the first case there becomes $fh=5$, truly in the latter $fh=6$. The first case gives the divisor $D=rr+5ss$, which is the form proposed itself; truly the latter gives either

$$D=rr+2rs+6ss,$$
or
$$D=2rr+2rs+3ss,$$

of which forms that will be returned to the first by reduction, since there shall be

$$D=(r+s)^2+5ss;$$

truly this may disagree with that, since thence there may become

$$2D = 4rr + 4rs + 6ss = (2r + s)^2 + 5ss\,;$$

from which it is apparent all the divisors either these to be numbers of this form, or double of that, thus so that, if the divisor itself D were not of the form $pp + 5qq$, double $2D$ certainly shall be going to become of this form. Then for the form $20i + a$, on account of $a = pp$, its values hence produced will be 1 and 9, but from the other formula $= pp + 5$ the same values 1 and 9 are deduced. Truly since this is only about the divisors, for a also it will be possible to take $\frac{pp+5}{2}$, from which the values 3, 7 arise and thus the formula containing all the divisors will be $20i + 1,\ +3,\ +7,\ +9$, on the other hand truly the formula excluding the divisors will be $20i - 1,\ -3,\ -7,\ -9$. If now it may be possible to show that latter formula to contain all the prime numbers, which may be unable to be divisors of the form proposed, likewise it would be required to show all the prime numbers contained in the first form certainly to be divisors of some form of number, $pp + 5qq$ and thus either themselves or the double of these must have the same form. Moreover such numbers as far as to one hundred are :

1, 3, 7, 23, 29, 41, 43, 47, 61, 67, 83, 89.

EXAMPLE 8

26. To find all the prime divisors of numbers of the form $pp - 5qq$.

SOLUTION

From the fourth problem here there is $n = 5$, from which on account of $g < \sqrt{\frac{n}{5}}$ there can be taken $g = 0$, or also $g = 1$. Indeed nothing is harmed by assuming $g = 1$; yet it would only be superfluous to attribute a greater value to that. But $g = 0$ gives the divisor $rr - 5ss$, that is of the form proposed; truly the other value $g = 1$ gives $fh = 4$ and thus either

$$D = rr + 2rs - 4ss,$$

or

$$D = 2rr + 2rs - 2ss.$$

The former is reduced to $D = (r + s)^2 - 5ss$, that is to that proposed; the latter truly divided by 2 gives the divisor

$$D = rr + rs - ss,$$

which form also is reduced to that proposed, which I show thus. Either both the numbers r and s will be odd, or the one even, the other odd. For the latter case there shall be $s = 2t$ and there will become

$$D = rr + 2rt - 4tt, \text{ or } D = (r+t)^2 - 5tt.$$

But if both the numbers shall be odd, the sum of these $r + s$ will be even, for example $2t$, and thus $r - 2t - s$, from which there becomes

$$D = 4tt - 2ts - ss = 5tt - (t+s)^2.$$

Therefore it is apparent all the divisors of numbers of the proposed form also to be of the same form. Now according to the form $20i \pm a$ the value $a = pp$ gives 1 and 9, but the other value $a = 5 - pp$ gives likewise 1 et 9, thus so that all the divisors may be contained in these forms $20i \pm 1, \ \pm 9$. Moreover the other divisor forms excluded will be $20i \pm 3, \ \pm 7$.

SCHOLION

27. Since from these examples it may now be satisfied, how for smaller numbers n these individual operations may be required to be put in place, besides we may advance some examples concerning greater numbers .

EXAMPLE 9

28. To find all the divisors of prime numbers of the form $pp + 17qq$.

SOLUTION

Since there shall be $\sqrt{\frac{17}{3}} < 3$, for g we will have the three values 0, 1, 2. The first shall be $g = 0$, and thus $fh = 17$, hence the divisor arises $D = rr + 17ss$, [on putting $f = 1$] and thus of the proposed form itself. Secondly there may be assumed $g = 1$, there will become [recalling $fh - gg = n$] $fh = 18 = 1 \cdot 18 = 2 \cdot 9 = 3 \cdot 6$, from which these forms arise [from $D = frr + 2grs + hss$]:

$$1°.\ D = rr + 2rs + 18ss = (r+s)^2 + 17ss,$$
$$2°. D = 2rr + 2rs + 9ss,$$

from which there becomes

$$2D = 4rr + 4rs + 18ss = (2r+s)^2 + 17ss,$$

thus so that $2D$ shall be of the proposed form.

$$3°. \quad D = 3rr + 2rs + 6ss,$$

the threefold of which adopts the proposed form.

Thirdly there shall be $g = 2$ and thus $fh = 21 = 1 \cdot 21 = 3 \cdot 7$; from which there arises

$$1°. D = rr + 4rs + 21ss = (r + 2s)^2 + 17ss,$$
$$2°. D = 3rr + 4rs + 7ss,$$

of which the threefold again leads to the proposed form. On account of which all the divisors that will be prepared, so that either these themselves, or the double or treble of these will have the proposed form. Because thence it pertains to the form $68i + a$, the value $a = pp$ presents the numbers 1, 9, 25, 49, 13, 53, 33, 21; but the other value $a = pp + 17$ gives 21, 33 etc., which numbers agree with the preceding [See § 10, II & III]. But since here also the half and the third can occur, it is apparent initially the form $a = \frac{pp}{2}$ gives no suitable values ; but $a = \frac{pp}{3}$ provides the numbers : 3, 27, 7, 11, 39, 23, 31, 63. Then truly the formula $a = \frac{pp+17}{2}$ gives 9, 13, 21 etc., which now occur. Finally the formula $a = \frac{pp+17}{3}$ provides 7, 11, 27 etc., which likewise now are present. On account of which all the suitable values for a will be :

1, 3, 7, 9, 11, 13, 21, 23, 25, 27, 31, 33, 39, 49, 53, 63.

But these numbers can be found much more easily ; indeed we may find only some at once , because we know the products of these from two or more also must occur, but before everything the square numbers themselves occur, from which, because also 3 must occur, now clearly all are found. So that if now we may suppress all those numbers beyond the half of the number 68 , while we may appoint the affected complements to 68 of the greater half with a − sign, then the values of a will constitute the following series :

$$+1, \ +3, \ -5, \ +7, \ +9, \ +11, \ +13, \ -15, \ -19, \ +21,$$
$$+23, \ +25, \ +27, \ -29, \ +31, \ +33.$$

If now we may change the signs of all the numbers, we will obtain all the values of the letter α for the formula $68i + \alpha$, from which all the divisors have been excluded.

COROLLARY 3

29. Hence therefore it is evident also for all the other positive numbers assumed in place of n in the values of the letter a plainly all the odd numbers occur less than $2n$, and which likewise shall be prime to n , while all these have the +ve sign , the others have the −ve sign.

EXAMPLE 10

30. To find all the prime divisors of numbers contained in this : $pp-19qq$, also of $19pp-qq$.

SOLUTION

Here therefore on account of $n=19$ there will be $g<\sqrt{\frac{19}{5}}$, and thus $g<2$, from which we will have either $g=0$ or $g=1$. The first shall be $g=0$ and the divisor will be $D=rr-19ss$ on account of $fh=19$, and thus these divisors now are of the form proposed. Again there shall be $g=1$ and there becomes

$$fh=19-1=18=1\cdot18=2\cdot9=3\cdot6;$$

from which three cases are required to be set out :

1°. $D=rr+2rs-18ss=(r+s)^2-19ss$,

which now is contained in the proposed form.

2°. $D=2rr+2rs-9ss$,

the double of which may be returned to the form proposed.

3°. $D=3rr+2rs-6ss$,

the triple of which is contained in the form proposed. And thus all the divisors sought are held in either in the form proposed, or its double or treble.
Then for the form $4ni\pm a$, or $76i\pm a$ the values of a must be derived from the following formulas:

1°. $a=pp$ gives 1, 9, 25, 49, 5, 45, 17, 73, 61.

2°. $a=\frac{pp}{2}$ gives no suitable values, because all become even.

3°. $a=\frac{pp}{3}$, or $a=3tt$, provides these values : 3, 27, 75, 71, 15, 59, 51, 67, 31.

4°. $a=19-pp$ gives 15, 3 etc., which now occur,

5°. $a=\frac{19-pp}{2}$ gives 9, 5, 3 etc., the same which are present.

6°. $a=\frac{19-pp}{3}$ gives 5, 1, 15 etc., which also are present.

On which account all the suitable numbers for a will be required to be assumed, since both positive as well as negative can be assumed to be removed beyond 38 , while evidently the greater complements are adjoined to 76 :

1, 3, 5, 9, 15, 17, 25, 27, 31.

But for the other form $76i \pm \alpha$, in which no divisors are able to occur, the values of α are the following :

7, 11, 13, 21, 23, 29, 33, 35, 37.

SCHOLIUM

31. Thus far we have not assumed other values for n, besides the primes, on account of which we may add besides two examples concerning composite numbers.

EXAMPLE 11

32. To find all the prime divisors of the numbers contained in this form : $pp + 30qq$.

SOLUTION

Here on account of $n = 30$ and $g < \sqrt{10}$, in place of g it will be agreed to accept the four values, 0, 1, 2, 3, which individual values therefore we may run through :

I. $g = 0$ provides $fh = 30$, from which the following formulas arise for the divisor D :

$1^{\text{o}}.D = rr + 30ss,$

$2^{\text{o}}.D = 2rr + 15ss,$

$3^{\text{o}}.D = 3rr + 10ss,$

$4^{\text{o}}.D = 5rr + 6ss,$

the first of which agrees with the form proposed, while double of the second, three times the third and five times of the fourth ; where it is to be observed in place of the five fold also the six fold can be accepted, since if there were

$$5D = pp + 30qq,$$

then there will be also :

$$6D = pp + 30qq .$$

II. Now there shall be $g=1$, and there will be $fh=31$, from which the single form arises

$$D=rr+2rs+31ss=(r+s)^2+30ss,$$

which is that form proposed itself.

III. There shall be $g=2$, there will be $fh=34=1\cdot 34=2\cdot 17$; from which the two forms arise :

$1°.D=rr+4rs+34ss=(r+2s)^2+30ss,$
$2°.D=2rr+4rs+17ss,$

the second of which can be reduced to the proposed form.

IV. There shall be $g=3$ and there will be $fh=39=1\cdot 39=3\cdot 13$, from which again two forms arise :

$1°.D=rr+6rs+39ss=(r+3s)^2+30ss,$
$2°.D=3rr+6rs+13ss,$

the third of which adopts the proposed form. Therefore it follows all the divisors *D* to be prepared thus from these, so that either *D*, 2*D*, 3*D*, or 6*D* may be contained in the form proposed. Then truly for the form $4ni+a=120i+a$ before everything it may be observed the multitude of all the numbers smaller than 120 and likewise for 120 of the first to be 32, from which now certainly we may infer the number of the values both of the letter *a* as well as of α to be 16. Therefore since in the first place all the square numbers occur in *a*, the formula $a=pp$ will give only these numbers: 1 and 49; but truly the forms $\frac{pp}{2}$, $\frac{pp}{3}$, and $\frac{pp}{6}$ plainly give no numbers prime to 120. Truly the other form $a=pp+30$ give only these numbers : 31, 79. Hence moreover again $a=\frac{pp+30}{2}$, or these : $a=2tt+15$ give 17, 23, 47, 113. Again $a=\frac{pp+30}{3}$,or $a=3tt+10$ gives 13, 37. Therefore this form gives only two values. Finally $a=\frac{pp+30}{6}$, or $a=6tt+5$ gives these : 11, 29, 59, 101. But in this manner only 14 values will be produced for the letter *a,* thus so that two still may be desired. Truly here it depends in place of the more general formula $pp+30$ to be putting perhaps $pp+30qq$, from which on taking $p=3t$ and by dividing by 3 there can be put in place $a=3tt+10qq$. Now there shall be $q=2$, and there becomes $a=3tt+40$, from which the case $t=1$ gives $a=43$, but $t=3$ gives $a=67$; and in this way we have obtained all the 16 values of *a,* which proceed in order thus :

1, 11, 13, 17, 23, 29, 31, 37, 43, 47, 49, 59, 67, 79, 101, 113.

So that if now in place of the numbers greater than 60 the complements of these to 120 may be written with the $-$ ve sign, these numbers thus will be able to be set out :

$$+1,\ -7,\ +11,\ +13,\ +17,\ -19,\ +23,\ +29,\ +31,\ +37,$$
$$-41,\ +43,\ +47,\ +49,\ -53,\ +59,$$

where clearly all the odd numbers prime to 30 occur affected by the $+$ or $-$ sign, where if the signs may be changed, all the values of the letter α will be had for the formula $120i+\alpha$, of which all the numbers are excluded from the class of the divisors.

COROLLARY 1

33. Therefore all the divisors of numbers of the form $pp+30qq$ are distributed into four classes, the first of which contains all these, which are themselves of the form $pp+30qq$; the second class truly these, of which the equation doubled are of this form ; the third, of which they are the equation trebled, and finally the fourth those, of which the fifth or sixth multiple can be reduced to the form $pp+30qq$. Therefore these four classes can be shown in this manner, if we may designate the proposed form $pp+30qq$ by the letter F, and the divisors truly by the letter D :

$$\text{I. } D=F, \text{II. } 2D=F, \text{III. } 3D=F, \text{IV. } 5D=F;$$

where it will help to be observed, if there were $2D=F$, then also there becomes $15D=F$; and in a similar manner if there were $3D=F$, also there will be $10D=F$; but if there were $5D=F$, there will be also $6D=F$.

COROLLARY 2

34. When we say all the divisors of the numbers of the proposed form $pp+30qq$ to be contained in the form $120i+a$, it is not thus required to be understood, as though all the numbers contained in the form $120i+a$ shall be divisors, but thence all these must be excluded, which are divisible by some number of the form $120i+\alpha$. But with these taken away it may be seen to be maximally probable all the remaining numbers of the formula $120i+a$,and thus especially prime numbers, certainly to become divisors of some form of numbers of the form $pp+30qq$. But these prime numbers contained in the formula $120i+a$ can be assigned as it pleases thus far by an easy calculation, certainly which here are progressing in order as far as to 240 :

1, 11, 13, 17, 23, 29, 31, 37, 43, 47, 59, 67, 79, 101, 113,
131, 137, 149, 151, 157, 163, 167, 179, 199, 233.

COROLLARY 3

35. Since all the divisors are of the quadruple kind, then also the values of a may be agreed to be distributed into four classes, as thence the divisors arise either of the first, second, third, or fourth class, from which therefore we may write below the characters of each class 1, 2, 3, 6, in this manner :

1,	11,	13,	17,	23,	29,	31,	37,	43,	47,	49,	59,	67,	79,	101,	113
1,	6,	3,	2,	2,	6,	1,	3,	3,	2,	1,	6,	3,	1,	6,	2.

Here therefore it deserves to be observed four individual classes occur.

EXAMPLE 12

36. To find all the prime divisors of the numbers contained either in this form : $pp-30qq$, or in this : $30pp-qq$.

SOLUTION

Since here there shall be $\sqrt{\frac{30}{5}}<3$, for the letter g we will have only the three values 0, 1, 2. Hence since there shall be $fh=30-gg$, for the first case there will be $fh=30$, for the second $fh=29$ and for the third $fh=26$, which cases therefore we may set out.

I. Let there be $g=0$, and hence the following values arise :

1°. $D=rr-30ss$,
2°. $D=2rr-15ss$,
3°. $D=3rr-10ss$,
4°. $D=5rr-6ss$.

II. If $g=1$, the single form appears

$$D=rr+2rs-29ss=(r+s)^2-30ss,$$

which therefore is the form proposed itself.

III. If $g=2$, the two formulas arise

1°. $D=rr+4rs-26ss=(\mathrm{r}+2s)^2-30ss;$

again the proposed itself

2°. $D=2rr+4rs-13ss,$

the double of which becomes the number of the proposed form. Hence therefore the divisors of the multiple four kind arise, which are with the letter F for the proposed formula

$$\text{I. } D=F,\ \text{II. } 2D=F, \text{III. } 3D=F, \text{IV. } 6D=F.$$

Thence truly all the divisors for the formula containing $120i \pm a$ will be in the first place either $a=pp$, $a=\frac{pp}{2}$, $a=\frac{pp}{3}$, or $a=\frac{pp}{6}$, from which all the numbers prime to 30 are unable to arise except from the first form $a=pp$, and thus only two values arise: evidently 1 and 49. But the other form was $a=$ either $30-pp$, $a=\frac{30-pp}{2}$, $a=\frac{30-pp}{3}$, or $a=\frac{30-pp}{6}$, of which the first $a=30-pp$ provides these numbers: 29, 19, 91. But because we can put 30 in place of $30qq$, the formula $a=120-pp$ provides in addition these values : 119, 71. The second reduced to the form $a=2tt-15$ gives these numbers : 13, 7, 17, 83, 113, 107. To which formula truly $15pp-2qq$ will be equivalent , therefore on taking $p=3$ there will be also $a=135-2qq$, from which 13, 7, 103, 37 are produced ; and thus in addition the new number 103 appears. From the third form $a=3tt-10$ we obtain these new numbers : 7, 17, but the nearby form $a=10tt-3$ provides 37 in addition. From the final form $5tt-6$ we obtain these values : 1, 119; truly from the next form $a=6tt-5$ these : 1, 19, 49, 91. Hence it is to be noted especially these same numbers can arise from the diverse classes. Moreover thus far they will have been produced :

1, 7, 13, 17, 19, 29, 37, 49, 71, 83, 91, 103, 107, 113, 119,

of which the number of values is indeed only 15, since is must be 16; but since we know the complement of each number to 120 also must occur, this deficiency is readily supplied. Clearly 101 is missing as the complement of 19. But because the numbers a can be taken both positive as well as negative, it will be permitted to reject the complements, thus so that for a we will have the eight following values :

1, 7, 13, 17, 19, 29, 37, 49,

therefore the remainder provide the values of the letter α, which will be just as many

11, 23, 31, 41, 43, 47, 53, 59.

COROLLARY 1

37. Therefore in this case, with my theorem admitted, so that all the prime numbers in the form $4ni+a$ likewise shall be the divisors of the form $pp \pm nqq$, the prime numbers arising from our formula $120i \pm a$ as far as to 240 are the following :

1, 7, 13, 17, 19, 29, 37, 71, 83, 101, 103, 107, 113, 127, 137, 139, 149, 157, 191, 211, 223, 227, 233, 239.

COROLLARY 2

38. Because in this presentation we have seen the same numbers to be arising from diverse classes, it is evident not some four diverse classes to be put in place, but two of these can be merged into one. Indeed in the first place all the divisors of the four classes, according to which there was $5D = F$, or also $6D = F$, now in the first class there were found $D = F$, thus just as often as there were $5D = F$, also there shall become $D = F$. In a similar manner the divisors of the third class also may be contained in the second class. So that if indeed there were $3D = F$, always there will be also $2D = F$, on account of which all the divisors according to the proposed form $pp - 30qq$, or $30pp - qq$ can be recalled to two classes only : indeed there will be always either $D = F$ or $2D = F$.

COROLLARY 3

39. Therefore all the prime numbers arising from our form $120i \pm a$ will be of two kinds, while either these themselves or the duplicate form of these can be had, which it will be agreed to distinguish in a similar way as before, by writing below with the individual values the characters 1 or 2

1,	7,	13,	17,	19,	29,	37,	49
1,	2,	2,	2,	1,	1,	2,	1.

Where it may be observed both characters occur just as many times.

SCHOLIUM

40. So that if therefore for numbers of any form $pp \pm nqq$ all the prime divisors may be desired, it is allowed to assign these most easily from our general formula $4ni + a$; whereby on the other hand, if we may wish to use with the formulas shown by the illustrious Lagrange, there would be a need to elicit all the prime numbers with the maximum trouble from the individual forms $frr + 2grs + hss$; on account of which it is required to be chosen especially, so that a firm demonstration of this may be uncovered by my assertion, certainly so that at last that theory for the sum of perfect steps may be removed. But I believe such a demonstration perhaps can be hoped for soon, if the following considerations may be evaluated properly.

1). After some $pp \pm nqq$ for the proposed formula both my formulas $4ni + a$ and $4ni + \alpha$ will have been put in place, these likewise plainly include all the odd numbers for the proposed n primes; then truly all the divisors are referred to the first form $4ni + a$; but no numbers of the other form $4ni + \alpha$ can be proposed to be divisors, or all the numbers of the latter form are excluded completely from the class of divisors.

2). It may be considered carefully in any case all the values of a to be grouped together according to an outstanding law, thus so that as if all jointly may constitute some complete entity, in which nothing may be lacking and nothing to be in excess, since all the products occur again in the same class from two or more of these numbers, thus so that, and likewise other suitable values will have been found for a, and from these all the remaining values are able to be found easily, especially since all the square numbers and the remainders of these with respect to the divisor $4n$ certainly will be advanced. From which if in this manner all the products and also the powers of all the numbers now found may be inserted, soon that same whole class will be filled, so that the multitude of all the numbers pertaining to this always shall be half plainly of all the numbers prime to $4n$ and smaller than that; truly the other half will provide the class of the numbers α, which in no manner can emerge as divisors.

3). Hence therefore it is apparent both these classes thus to be distinguished in turn from each other, and in the nature of the numbers themselves established to differ from each other maximally with the greatest discrimination, thus so that the numbers of either class evidently by its nature shall be different from the other class.

4). Because no numbers of the class $4ni+\alpha$ at any time are able to be divisors of any of the numbers of the form $pp\pm nqq$, this class must be regarded as the origin of all the numbers, the nature of which differs from the innate character of the divisor, which disagreement also must extend to all the numbers, which are divisible by any of the numbers of the class $4ni+\alpha$. If indeed such numbers may be divisors, also these same numbers of this class would be divisors, that which is denied by the nature of the matter.

5). But since the product from two numbers of the class $4ni+\alpha$ may pass into the class of divisors $4ni+a$, it is evident many must occur in the first class to be different from the nature of the divisors ; evidently all these, which are divisible by any number of the other class.

6). So that if now all these numbers in the class $4ni+a$ may be delineated or excluded, which may be changed by the nature of the divisor, it may be seen to be most probable all the remaining numbers to be predicted by the nature of the divisor. Since in this manner only composite numbers may be removed, it is evident clearly all the prime numbers contained in the form $4ni+a$ again to be the divisors of some numbers of the form $pp\pm nqq$. Therefore the whole situation returns to this, so that the strength of this perfect demonstration may be agreed for this same probability. But this truth, whatever it is, can thus be proposed more elegantly.

THEOREM TO BE DEMONSTRATED

41. *If a were a divisor of some number of the form* $pp+nqq$ *, thus so that there shall be* $aD=pp+nqq$ *, then as often as* $4ni+a$ *is a prime number, just as often also* $D(4ni+a)$ *will be a number of the form* $pp+nqq$ *.*

But here it is required to be observed : 1). The numbers p and q must be prime between themselves. 2). The divisor a also must be prime to n, because divisors of n itself hence are excluded. 3). So that if it may eventuate that the number $D(4ni+a)$ may not seem to be contained in the form $pp+nqq$, then always its square, or also its product by another square, certainly will be contained in that. Because therefore in this case there will be

$$D(4ni+a)=(\tfrac{p}{2})^2+n(\tfrac{q}{2})^2,$$

this resolution without exception is considered to deserve to be noted. Thus so that if there shall be $27=4^2+11\cdot 1^2$, there will be $a=27$ and $n=11$ and $D=1$, from which formula $4ni+a$ there emerges $44i+27$, which in the case $i=1$ provides 71, this is a prime number ; nor yet is it possible for $71=pp+11qq$ to be in integers. Truly there is

$$4\cdot 71=284=3^2+11\cdot 5^2,$$

and thus

$$71=(\tfrac{3}{2})^2+11\cdot(\tfrac{5}{2})^2.$$

But such cases rarely occur and thus they are not required to be removed, because numbers of the formula $4ni+\alpha$ thus are excluded from the class of divisors, so that, even if fractions may be taken for p and q, yet never are they able to be divisors.

SCHOLIUM

42. It would be superfluous to extend these investigations to the formulas of this kind: $mpp\pm nqq$, since all the divisors of numbers of the form $mpp\pm nqq$ always shall be divisors of numbers of the form $pp\pm mnqq$. Which therefore at one time in Book XIV of the old Comment. of the Academy I have commented on divisors of numbers of the form $mpp\pm nqq$ and the major part I have concluded from induction alone, now by the outstanding properties demonstrated by the Illust. Lagrange not only more are shown, but also the order has been led through with much greater certainty, thus so that now nothing further may be desired, except that a solid demonstration of the theorem brought forwards may be uncovered, which now indeed it will be permitted to expect soon. But my method enjoys this prerogative especially, because with its help clearly all the divisors of the formulas of this kind $mpp\pm nqq$ to be assigned, and they

are able to be continued as far as it pleases, that which I will declare as well in the following example.

EXAMPLE 13

43. To find all the divisors of the form $pp+39qq$.

Therefore in the first place, we may seek all the diverse forms of the divisors by the formulas of the Illust. Lagrange, and since there shall be $n=39$ and thus $\sqrt{\frac{39}{3}}<4$, it will suffice for g to assume these four values : 0, 1, 2, 3.

I. Therefore the value $g=0$ provides $fh=39$, from which these two forms arise :

$$1°.\ rr+39ss, \qquad 2°.\ 3rr+13ss,$$

the first of which gives the divisors $D=F$ and the other $3D=F$, with F denoting the form proposed.

II. The value $g=1$ gives $fh=40$, from which these forms arise :

1°. $D=rr+2rs+40s=(r+s)^2+39ss$, and thus $D=F$.

2°. $D=2rr+2rs+20ss$, but this form cannot be a prime number.

3°. $D=4rr+2rs+10ss$, which form likewise cannot give a prime number.

4°. $D=5rr+2rs+8ss$, from which there becomes $5D=F$, or also $8D=F$.

III. The case $g=2$ gives $fh=43$, from which the single form arises :

$D=rr+4rs+43ss=(r+2s)^2+39ss$, and thus $D=F$.

IV. Finally the case $g=3$ provides $fh=48$, from which the following forms containing prime numbers arise:

1°. $D=rr+6rs+48ss=(r+3s)^2+39ss$, and thus $D=F$.

2°. $D=3rr+6rs+16ss$
gives $3D=F$, or also $16D=F$. Hence therefore it appears generally to give three kinds of divisors:

$$1)\ D=F, \quad 2)\ 3D=F, \quad 3)\ 5D=F.$$

With which put in place we may set out the formula $4ni+a=156i+a$, where it may be observed initially of all the prime numbers to 156 itself, the smallest multitude to be 48, from which as far as to the half 78 there will be 24, of which the individual ones taken either positive or negative will provide values for the letter *a.* Therefore these numbers will be :

$$1, 5, 7, 11, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43,$$
$$47, 49, 53, 55, 59, 61, 67, 71, 73, 77,$$

where initially the squares have the + ve sign, which therefore will be

$$+1,\ +25,\ +49;$$

truly of the remaining numbers the squares on division by 156 may be removed beyond 78, from which there will become :

$$11^2=-35,\ 17^2=-23,\ 19^2=+49,\ 23^2=+61.$$

We may consider the form $pp+39$ for the remaining numbers, from which on accepting $p=1$, 40 will be provided, of which the numbers pertaining to the third kind, the divisor 5 has the sign +. Now because the preceding numbers are being referred to the first kind, the products of these by 5 also will have to be referred to the third kind, from which the following numbers arise :

$$+5,\ +41,\ -31,\ -19,\ -67,\ -7.$$

Now there shall be $p=2$ and there will be $4+39=43$, which is a divisor of the first class, from which also the numbers of this class now found multiplied by 43 will give the divisors of the first class, but which, since the number 43 shall be exceedingly large, will be found more easily from the following. Therefore there may be taken $p=3$ and there will be $pp+39=48$, of which the divisor 3 now has been excluded. Therefore there shall be $p=4$ and there will be $16+39=55$, of which we have now treated the divisor 5 ; truly the other divisor 11 also belongs to the third class; therefore the prime numbers of the first class multiplied by this will be :

$$+11,\ +59,\ -37,\ -73,\ +71,\ +47.$$

Also the numbers of the third class my be multiplied by 11 and the products removed, which are

$$+55,\ -17,\ -29,\ -53,\ +43,\ -77,$$

which may be returned to the first class. In this manner all our numbers with their due signs have been applied, which since they may be referred either to the first or to the third class, it is evident no divisors of the second class remain. Clearly all these numbers now may be held in the first class, on account of which all the values of a with their characters I or III written below thus will be had themselves :

$+1,$	$+5,$	$-7,$	$+11,$	$-17,$	$-19,$	$-23,$	$+25,$	$-29,$	$-31,$	$-35,$	$-37,$
I	III	III	III	I	III	I	I	I	III	I	III

$+41,$	$+43,$	$+47,$	$+49,$	$-53,$	$+55,$	$+59,$	$+61,$	-67	$+71,$	$-73,$	-77
III	I	III	I	I	I	III	I	III	III	III	I.

Nor truly is the second class completely useless: indeed prime numbers are given, which we may return to the first order, the resolution of which into integers cannot succeed and thus demands the square denominator 16 , a number of this kind is 61, which otherwise is unable to be returned to the first class, unless in this manner : $61=(\frac{25}{4})^2+39(\frac{3}{4})^2$. Truly there is $3\cdot 61=183=12^2+39\cdot 1^2$. Because if now negative values found for a may be converted into positive ones, by taking the complements to 156 the following values will be produced :

1,	5,	11,	25,	41,	43,	47,	49,	55,	59,	61,	71,	79,	83,	89,	103,
I	III	III	I	III	I	III	I	I	III	I	III	I	III	III	I

119,	121,	125,	127,	133,	137,	139,	149
III	I	III	I	I	III	I	III.

Therefore now all the prime numbers contained in the form $156i+a$ certainly will be divisors of some numbers of the form $pp+39qq$, and thus either these themselves, of the quintuples of these, or also the triples will be numbers of this form. Hence therefore all the prime divisors from 1 as far as to 312 will be the following:

1, 5, 11, 41, 43, 47, 59, 61, 71, 79, 83, 89, 103, 127, 137, 139, 149, 157,
167, 181, 197, 199, 211, 227, 239, 277, 281, 283, 293.

COROLLARY 1

44. So that it may be understood more easily, why in this case the second class may be taken back to the first, now above we have shown, if the divisor were

$$D=frr+2grs+hss,$$

with $fh=gg+n$ present, then not only Df, but also Dh can be reduced to the form $pp+nqq$. Moreover hence if there were more generally

$$k = ftt + 2gtu + huu\,,$$

then the product Dk also will be a number of the form $pp + nqq$; indeed with the calculation made there will be found :

$$Dk = (frt + g(ts + ru) + hsu)^2 + n\big(ts - ru\big)^2.$$

So that therefore if k were able to become a square, or divisible by a square, then this square can be omitted. For if $Dkll$ were a number of the form $pp + nqq$, then also with fractions admitted, Dk also will be of the same form. Thus in our case for the divisors of the second class there was

$$D = 3rr + 13ss,$$

and thus $k = 3tt + 13uu$, of which the value on assuming $t = 1$ and $u = 1$, will become $k = 16$, which since it shall be a square number, this form may be reduced to the first.

COROLLARY 2

45. Therefore now all the theorems, which I gave at one time in Book XIV of the old Commentaries have been taken to a much more certain level, after the forms of these divisors have been demonstrated by the celebrate Lagrange; and there is seen to be no doubt, why not soon, what may be desired in this hitherto generally, may be supplied by a perfect demonstration.

COROLLARY 3

46. Before I may abandon this argument completely, I may add at this point a memorable observation concerning the signs of the numbers *a,* while evidently all the values of this beyond $2n$ are removed. Indeed since the first and last of these numbers likewise may be taken to become $2n$, it is required to be seen clearly, whether each of these two numbers may have either the same or different signs, indeed in each case since any two of these numbers shall be equidistant from the ends, the sum of which therefore always is $2n$, also they will have either the same signs or opposite signs. Thus in our case, where there was $2n = 78$, the final 77 will have the sign $-$, while the first 1 always has the sign $+$, from which also the signs of two equidistant from the ends always will be opposite. But for the contrary in example 11**,** where there was $2n = 60$, the final number 59 will have the sign +, from which also any two others equidistant from the ends are taken to have the same sign in place, indeed the account of which phenomena will be able to be investigated without difficulty. But observations of this kind require much labour in the investigation of the divisors.

DE INSIGNI PROMOTIONE SCIENTIAE NUMERORUM

[E598]
Opuscula analytica, 2, 1785, p. 275-314
[Conventui exhibita die 26. octobris 1775]

1. Eximia omnino sunt, quae celeberrimus LA GRANGE in Comment. Academiae Regiae Borussicae pro Anno 1773 de divisoribus formulae generalissimae $Btt+Ctu+Duu$ demonstravit, et maximam lucem in scientia numerorum, quae etiamnunc tantis tenebris est involuta, accendunt. Ob hoc ipsum autem, quod ista tractatio maxime est generalis, ii, qui non satis sunt exercitati in huiusmodi speculationibus, non parum difficultatis offendunt neque vim talium sublimium demonstrationum satis perspicere valent. Quamobrem haud inutile erit omnia momenta, quibus hae demonstrationes innituntur, diligentius explicare atque ad formulas magis speciales accommodare, quandoquidem hoc modo omnia facilius intelligi poterunt. Deinde imprimis accuratius exponam, quantum firmamentum hinc plurimis theorematibus, quorum veritatem per solaro inductionem mihi quidem cognoscere licuit, afferri possit, unde multo clarius patebit, quantum adhuc ad eorum perfectam demonstrationem desideretur.

LEMMA

2. Si p et q fuerint numeri inter se primi, tum omnes plane numeri in hac forma generali $\alpha p\pm\beta q$ comprehendi possunt, idque infinitis modis.

Huius lemmatis demonstratio per se facilis passim invenitur.

PROBLEMA 1

3. *Si p et q sint numeri inter se primi, n vero denotet numerum quemcunque datum, sive positivum sive negativum, invenire omnes divisores huius formulae:* $pp+nqq$.

SOLUTIO

Denotet D divisorem quemcunque numeri in hac forma $pp+nqq$ contenti, sitque d quotus ex hac divisione ortus, ita ut sit $Dd=pp+nqq$. Hic iam statim evidens est numerum d ad q fore primum; si enim q haberet divisorem communem, eundem quoque p habere deberet, contra hypothesin; quamobrem numerus p per d et q ita exprimi poterit, ut sit $p=\alpha d\pm\beta q$ quo valore substituto fiet

$$Dd=\alpha\alpha dd\pm 2\alpha\beta dq+(\beta\beta+n)qq,$$

ideoque divisor

$$D=\alpha\alpha d\pm 2\alpha\beta q+(\tfrac{\beta\beta+n}{d})qq,$$

ubi ergo $\frac{\beta\beta+n}{d}$ erit numerus integer, qui sit $=h$, ita ut habeatur

$$D=\alpha\alpha d\pm 2\alpha\beta q+hqq\ ,$$

pro qua forma scribamus

$$D=frr\pm gqr+hqq\ ,$$

ita ut sit $f=d$, $r=\alpha$, $g=2\beta$, et ob $h=\frac{\beta\beta+n}{d}$ erit $fh=\beta\beta+n$; hincque fiet $4fh-gg=4n$. Hinc igitur patet omnes divisores formae $pp+nqq$ semper contineri in hac forma:

$$D=frr\pm gqr+hqq\ ,$$

dummodo fuerit $4fh-gg=4n$. Ac vicissim, si fuerit

$$D=frr\pm gqr+hqq\ ,$$

erit

$$4Df=4ffrr\pm 4fgqr+4fhqq\ ;$$

ideoque ob $4fh=4n+gg$ erit

$$4Df=(2fr\pm gq)^2+4nqq\ ,$$

quae forma a proposita non discrepat, si modo dividatur per 4.

COROLLARIUM 1

4. In genere igitur omnes divisores formulae propositae $pp+nqq$ comprehendere licebit in ista formula latissime patente: $frr+grs+hss$, dummodo fuerit $4fh-gg=4n$, sive $fh-\frac{1}{4}gg=n$; unde patet innumeras huiusmodi formulas exhiberi posse, quoniam numerum g pro lubitu accipere licet. Ex eo autem numeros f et h ita definiri oportet, ut fiat

$$4fh=4n+gg\ .$$

SCHOLION

5. Quoniam autem innumerabiles huiusmodi formulas: $frr+grs+hss$ exhibere licet, in quibus sit $4fh-gg=4n$, parum hinc lucri ad nostrum institutum afferri videtur. Proposito enim quocunque numero *D,* ad diiudicandum, utrum esse possit divisor formae $pp+nqq$, omnes illae innumerabiles formulae considerari deberent, num forte iste

numerus D in quapiam illarum contineatur. Praecipuum igitur inventum, quod Illustri LA GRANGE acceptum referre debemus, in hoc consistit, quod infinitam illam huiusmodi formularum multitudinem ad exiguum numerum pro quovis casu revocare docuit, id quod in sequente problemate exponamus.

PROBLEMA 2

6. *Formam generalem divisorum ante inventam,* $frr+grs+hss$*, in qua sit* $4fh-gg=4n$*, in aliam eiusdem formae,* $f'tt+g'tu+h'uu$*, transmutare, in qua sit* $g'<f'$ *vel* h'*, manente proprietate* $4f'h'-g'g'=4n$.

SOLUTIO

Ponamus esse $f<h$ et numerum g quantumvis esse maiorem quam f, ac statuamus $r=t-\alpha s$, quo valore substituto orietur ista forma:

$$ftt+(g-2\alpha)ts+(\alpha\alpha-\alpha g+h)ss\,;$$

ubi manifesto α ita assumi poterit, ut fiat $g-2\alpha f<f$, ubi quidem animadvertendum est nihil referre, utrum $g-2\alpha f$ prodeat positivum an negativum. Statuatur igitur $g-2\alpha f=g'$, ita ut certe sit $g'<f'$, tum vero ob analogiam loco f scribatur f' et $\alpha\alpha f-\alpha g+h=h'$ eritque

$$4f'h'-g'g'=4fh-gg=4n.$$

Hoc igitur modo forma proposita reducta est ad hanc:

$$f'tt\pm g'ts+h'ss$$

in qua certe est $g'<f'$. Quod si iam eveniat, ut g' adhuc maius fuerit quam h', tum simili modo ista formula in aliam transformari poterit, in qua coëfficiens medius utrovis extremo sit minor, unde patet formam propositam

$$frr\pm grs+hss$$

semper in aliam similis formae

$$f'tt\pm g'tu+h'uu$$

converti posse, in qua g' minus sit quam f' et h', simulque etiam fiat

$$4f'h'-g'g'=4n\,.$$

COROLLARIUM 1

7. Hoc igitur modo infinita multitudo formularum $frr+grs+hss$, in qua $4fh-gg=4n$, plerumque ad satis exiguum numerum reduci potest, dum scilicet omnes illae formulae excludi possunt, in quibus coëfficiens medius g maior est alterutro extremorum.

COROLLARIUM 2

8. Cum igitur sit tam $f>g$ quam $h>g$, erit $4fh>4gg$. Sit igitur $4fh=4gg+\Delta$, et cum esse debeat $4fh-gg=4n$, erit $3gg+\Delta=4n$, ideoque $3gg<4n$, hinc ergo $g<\sqrt{\frac{4n}{3}}$; quamobrem loco g successive eos tantum valores assumsisse sufficiet, qui sunt minores quam $\sqrt{\frac{4n}{3}}$, ex quibus singulis facile colligentur valores litterarum f et h ex aequatione $4fh=gg+4n$, quo facto omnes plane divisores formae $pp+nqq$ certe continebuntur in quapiam harum formularum simpliciorum.

SCHOLION

9. Quoniam aequatio $4fh-gg=4n$ locum habere nequit, nisi g sit numerus par, pro g statim scribamus *2g,* ut forma d sit $frr+2grs+hss$, existente $fh-gg=n$, quae ergo forma semper ita reduci potest, ut sit $2g<f$ vel $<h$. Haec autem reductio commodissime per gradus institui potest, dum loco α in superiori reductione scribitur unitas. Ita si fuerit divisor

$$D=frr+2grs+hss,$$

erit quoque

$$D=f'rr+2g'rs+h'ss,$$

existente duplici modo vel

$$f'=f,\ g'=f-g \text{ et } h'=f-2g+h,$$

vel etiam

$$h'=h,\ g'=h-g \text{ et } f'=f-2g+h,$$

quoniam membra extrema inter se commutare licet. Quod si hic nondum fuerit $2g'<f'$ seu $2g'<h'$, ista operatio tam diu continuari debet, donec fiat $2g<f$ vel h; ubi notandum in his formulis terminum medium *2grs* tam positivum quam negativum accipi

posse, propterea quod numeri r et s denotare possunt omnes numeros integros sive positivos sive negativos. His igitur praemissis investigemus omnes divisores primos numerorum vel in hac forma: $pp+nqq$, vel in hac: $pp-nqq$ contentorum; si quidem divisores compositi ex primis componuntur, ita ut cognitis omnibus divisoribus primis simul omnes compositi habeantur.

PROBLEMA 3

10. *Invenire omnes divisores primos numerorum in hac forma:* $pp+nqq$ *contentorum, existentibus numeris p et q tam inter se primis quam respectu numeri n.*

SOLUTIO

I. Quia enim hic de divisoribus primis tantum sermo est, nisi p esset quoque primus ad n, formula $pp+nqq$ etiam admitteret omnes divisores numeri n, qui propterea nullam investigationem requirunt et sponte se produnt. Sit ergo D divisor quicunque formae $pp+nqq$, ac modo vidimus semper fore

$$D = frr+2grs+hss,$$

existente $fh-gg=n$, ita ut sit tam $2g<f$ quam $2g<h$; quare cum hinc sit $f>2g$ et $h>2g$, erit $fh>4gg$. Sit igitur $fh=4gg+\Delta$, et quia $fh-gg=n$, erit $3gg+\Delta=n$, ideoque $gg<\frac{n}{3}$ et $g<\sqrt{\frac{n}{3}}$. Hac igitur conditione multitudo formarum pro divisore D ad eo minorem numerum reducitur, quo minor fuerit numerus n. Cum igitur sit

$$D = frr+2grs+hss,$$

erit

$$Df = ffrr+2fgrs+fhss$$

ideoque ob $fh=gg+n$ fiet

$$Df = (fr+gs)^2+nss,$$

quae est ipsa forma proposita. Simili modo permutatis litteris f et h erit quoque

$$Dh = (hs+gr)^2+nrr,$$

unde patet, si fuerit *Df* numerus formae $pp+nqq$, tum etiam productum *Dh* fore eiusdem formae, ita ut sufficiat alterutram invenisse. Pro quovis ergo casu quaerantur omnes valores litterae *t,* qui sint f, f', f'', f''' etc., atque omnes divisores primi *D* ita erunt comparati, ut vel *D*, vel *Df*, vel *Df',* vel *Df''* etc. sint numeri formae $pp+nqq$. Haecque sequuntur ex demonstrationibus Illustris LA GRANGE.

II. Haec igitur coniungamus cum iis, quae iam olim de formis horum divisorum primorum sum commentatus, ubi ostendi omnes hos divisores comprehendi posse in huiusmodi expressione: $4ni+a$, dum scilicet *a* denotat certos numeros primos ad 4*n* simulque minores quam 4*n,* ubi tantum semissis talium numerorum occurrit, reliquis hinc prorsus exclusis.
Unde si α denotet hos numeros exclusos, affirmari poterit nullos numeros, in forma $4ni+\alpha$ contentos, esse posse divisores formae $pp+nqq$. Istae autem formae egregie conveniunt cum praecedentibus. Si enim fuerit $D=pp+nqq$, alteruter numerorum *p* et *q* debet esse impar, ideoque *q* vel par vel impar. Sit primo *q* par, ideoque *qq* numerus formae 4*i,* fiet $D=4ni+pp$. Unde patet litteram illam α complecti omnes numeros quadratos impares et primos ad 4*n,* sive residua, quae ex divisione horum quadratorum, per 4*n* facta, remanent. Sin autem fuerit *q* numerus impar, ideoque *qq* formae $4i+1$, hinc fiet $D=4ni+pp+n$. Unde patet litteram α etiam complecti omnes numeros formae $pp+n$, qui quidem ad 4*n* sint primi, vel eorum residua ex divisione per 4*n* remanentia. Iidem vero etiam numeri pro α resultant, si fuerit *Df* numerus formae $pp+nqq$, id quod in exemplis facilius ostendi poterit.

III. His expositis, cum forma $4ni+a$ complectatur omnes divisores formae $pp+nqq$, altera autem forma $4ni+\alpha$ nullos divisores in se involvat, ex priori forma $4ni+\alpha$ excludi debebunt omnes numeri divisibiles per quempiam numerum formae $4ni+a$. Quod si igitur demonstrari posset hoc modo ex formula $4ni+\alpha$ omnes plane numeros excludi, qui nequeunt esse divisores formae $pp+nqq$, tum manifesto sequeretur omnes numeros primos formae $4ni+a$ certe esse divisores formae $pp+nqq$, quandoquidem tantum numeros compositos hoc modo exclusimus. Totum ergo negotium huc redit, ut demonstretur formulam $4ni+\alpha$ omnes plane continere numeros primos, qui nequeunt esse divisores formae $pp+nqq$; quod si demonstrari posset, nihil amplius in hoc genere desideraretur.

COROLLARIUM 1

11. Pro quovis ergo numero *n* omnes numeri ipso 4*n* minores ad eumque primi in duas classes distribuentur, quarum alteram littera *a,* alteram vero littera α designavimus, ita ut formula $4ni+a$ contineat omnes divisores formae $pp+nqq$, altera vero formula $4ni+\alpha$ divisores illos penitus excludat, neque ullus numerus istius formae unquam esse possit divisor formulae $pp+nqq$. Multitudo autem numerorum utriusque classis semper est eadem; scilicet si multitudo omnium numerorum minorum quam numerus 4*n* ad

eumque primorum fuerit $=2\lambda$ (semper enim iste numerus est par), prior forma a continet λ numeros, totidemque etiam continebit altera forma α.

COROLLARIUM 2

12. Circa has formulas: $4ni+a$ et $4ni+\alpha$ iam olim demonstravi, si numeri a et a' in priore classe occurrant, tum ibi quoque occurrere productum aa', id quod etiam de pluribus numeris huius classis est intelligendum, qui si fuerint a, a', a'', a''' etc., etiam producta tam ex binis quam pluribus horum numerorum, atque adeo etiam omnes eorum potestates in eadem classe reperientur, postquam scilicet per $4n$ divisi ad residua minora quam $4n$ fuerint reducti. Deinde etiam demonstravi, si α fuerit numerus posterioris classis, tum in eadem quoque reperiri debere numeros $a\alpha,\ a'\alpha,\ a''\alpha,\ a'''\alpha$ etc . Unde patet multitudinem numerorum posterioris classis minorem esse non posse quam primae classis. Quod autem multitudo utrinque sit prorsus aequalis, id etiam facile demonstrari potest. Tum vero etiam hoc certum est, si $\alpha,\ \alpha',\ \alpha'',\ \alpha'''$ etc. fuerint numeri posterioris classis, tum tam eorum quadrata quam eorum producta ex binis in priorem classem ingredi, producta autem ex ternis iterum in classe posteriore reperiri.

COROLLARIUM 3

13. Omnia igitur, quae adhuc in hoc genere desiderari possunt, huc redeunt, ut demonstretur classem $4ni+\alpha$ omnes continere numeros primos, qui nequeant esse divisores formae $pp+nqq$; tum enim evictum erit omnes numeros primos formae prioris $4ni+a$ certe esse divisores cuiuspiam numeri formae $pp+nqq$.

PROBLEMA 4

14: *Invenire omnes divisores primos numerorum in hac forma: $pp-nqq$ contentorum, ubi quidem, ut ante, p et q non solum sint primi inter se, sed etiam primi ad n.*

SOLUTIO

I. Conditio, quod p sit etiam primus ad n, ideo tantum hic adiicitur, quia alias etiam omnes divisores numeri n hic in censum venirent, quos tamen hic excludimus, utpote per se manifestos. Hic igitur primo patet, si fuerit D divisor primus formae $pp-nqq$, tum etiam fore divisorem formae $nqq-pp$, siquidem fuerit nqq maius quam pp. Nam si fuerit D divisor formae $pp-nqq$, erit quoque divisor formae $npp-nnqq$, quae forma, si loco nq scribamus r, abit in hanc: $npp-rr$. Deinde eodem modo ut ante patet semper fore

$$D=frr+2grs+hss,$$

existente $fh-gg=-n$, hancque formam semper ita reduci posse, ut fiat $2g<f$ simulque $2g<h$, ubi quidem signa numerorum f et h non respiciuntur, si forte

alterutrum membrum fiat negativum; quare cum, ob $f > 2g$ et $h > 2g$, sit $fh > 4gg$, evidens est fieri non posse

$$fh - gg = -n,$$

nisi vel f vel h fuerit negativum, unde forma divisoris ita debet constitui, ut sit

$$D = frr + 2grs - hss,$$

fierique debet $-fh - gg = -n$, sive $fh + gg = +n$. Quoniam igitur $fh > 4gg$, necesse est, ut sit $5gg < n$, ideoque $g < \sqrt{\frac{n}{5}}$, ita ut hoc casu pauciores valores pro g relinquantur. Tum autem erit

$$Df = ffrr + 2fgrs - fhss,$$

sive

$$Df = (fr + gs)^2 - nss,$$

quae est forma ipsa proposita. Porro autem erit

$$Dh = nrr - (gr - hs)^2,$$

quae est forma nostra inversa $npp - qq$. Hinc igitur intelligitur, si fuerit Df numerus formae $pp - nqq$, tum eo ipso formulam Dh fore numerum formae $npp - qq$.

II. Accommodemus haec etiam ad eam formam divisorum, quam olim exhibui; ac primo quidem si fuerit $D = pp - nqq$, pro casibus, quibus q est numerus par, ideoque qq formae $4i$, fiet $D = pp - 4ni$; unde si ponatur $D = 4ni + a$, ob $pp > 4ni$, si ponatur $pp = 4nk + b$, prodibit talis forma: $D = 4ni + b$, ita ut sit $a = b$, ideoque omnes numeros quadratos ad $4n$ primos in se complectatur. Sin autem sit q numerus impar, ideoque qq formae $4i + 1$, fiet $D = pp - n - 4ni$, positoque iterum $pp = 4nk + b$ prodit $D = 4ni + b - n$, ita ut hoc casu sit $a = b - n$, ubi b denotare potest omnes numeros quadratos, vel residua inde orta. Simili modo si fuerit $D = npp - qq$, evidens est valores pro a hinc prodituros praecedentium fore negativos, ita ut a comprehendat omnes numeros quadratos, deinde etiam omnes numeros formae $pp - n$, tam positive quam negative sumtos; quamobrem forma omnium divisorum ita exhiberi poterit, ut sit $4ni \pm a$, forma autem pro numeris ex classe divisorum exclusis erit $4ni \pm \alpha$, quorum multitudo aequalis est priori, scilicet α semper totidem sortietur valores, quot habet littera a.

III. Quo igitur etiam in hoc genere nihil amplius desiderari queat, id tantum superest, ut demonstretur formam posteriorem $4ni \pm \alpha$ omnes plane continere numeros primos, qui nunquam esse queant divisores ullius numeri vel formae $pp - nqq$, vel $npp - qq$.

COROLLARIUM 1

15. De his binis formulis: $4ni \pm a$ et $4ni \pm \alpha$, quarum illa omnes divisores involvit, haec vero excludit, eadem valent, quae ante sunt tradita. Scilicet si a, a', a'' etc. ad priorem classem pertineant, ibidem quoque reperientur tam omnes potestates quam producta ex binis pluribusve horum numerorum; tum vero si α sit numerus posterioris classis, ibidem quoque occurrent omnes numeri $a\alpha$, $a'\alpha$, $a''\alpha$ etc., ita ut multitudo horum numerorum minor esse nequeat quam prioris classis.

COROLLARIUM 2

16. Quoniam littera a complectitur omnia quadrata, ante omnia eius valor erit 1, tum vero etiam 9, 25 etc., nisi numerus n habeat divisorem vel 3, vel 5 etc. His enim casibus ista quadrata excludi oportet, quia alioquin forma $4ni \pm a$ numerus primus fieri non posset.

SCHOLION

17. His igitur generalibus praeceptis expositis omnia clariora evadent, si casus particulares evolvamus; hic enim plura adhuc occurrent, quae in genere attingere non licuit. Sufficiet autem id in aliquibus exemplis ostendisse, quibus pertractatis non difficile erit tabulam construere, quae pro omnibus casibus formas divisorum primorum exhibeat.

EXEMPLUM 1

18. Invenire omnes divisores primos numerorum in formula $pp + qq$ contentorum, dum scilicet pro p et q assumantur numeri inter se primi.

SOLUTIO

Posito divisore primo

$$D = frr + 2grs + hss,$$

ob $n = 1$ debet esse $fh = gg + 1$, tum vero $g < \sqrt{\tfrac{1}{3}}$; unde patet pro g alium valorem assumi non posse praeter 0 ; tum autem erit $fh = 1$ ideoque tam $f = 1$ quam $h = 1$, sicque omnes divisores in hac forma $D = rr + ss$ continebuntur, ita ut summa duorum quadratorum alios divisores non admittat, nisi qui ipsi sint summae duorum quadratorum. Altera autem forma divisorum erit $4i + 1$, et excludentur omnes numeri formae $4n + 3$ sive $4i - 1$. Quod si ergo demonstrari posset formulam $4i - 1$ omnes plane continere numeros primos, qui nequeunt esse divisores formae $pp + qq$, tum simul demonstratum esset etiam omnes divisores primos formae $4i + 1$ fore summam duorum quadratorum. Hoc autem iam dudum a me post FERMATIUM est demonstratum.

EXEMPLUM 2

[18a.] Invenire omnes divisores primos formae $pp-qq$.

SOLUTIO

Hoc exemplum ad problema quartum refertur, estque $n=1$, et quia debet esse $g<\sqrt{\frac{1}{5}}$, necessario fieri oportet $g=0$, ideoque $fh=1$, unde oritur haec forma divisorum: $D=rr-ss$, quae utique continet omnes plane numeros primos excepto binario. Quamquam enim haec forma habet factores $r+s$ et $r-s$, tamen continet omnes primos, si fuerit $r-s=1$, cuius ratio est peculiaris. Id etiam altera divisorum forma declarat, qua, ob $a=1$, fit $4i\pm 1$, in qua omnes plane numeri impares continentur, ita ut hoc casu nulli excludantur, alteraque forma $4i\pm\alpha$ hoc solo casu nullum locum habeat. Ceterum hic casus proprie huc non pertinet, quia divisores formae $pp-qq$ per se constant.

EXEMPLUM 3

19. Invenire omnes divisores primos formae $pp+2qq$.

SOLUTIO

Hic casus pertinet ad problema tertium, existente $n=2$, unde cum debeat esse $g<\sqrt{\frac{2}{3}}$, erit $g=0$, ideoque $fh=2$, hinc forma divisorum erit $=rr+2ss$. Unde patet numeros formae $pp+2qq$ alios non admittere divisores, nisi qui sint eiusdem formae, quod quidem etiam iam dudum est demonstratum. Altera autem forma $D=8i+a$, ob $a=pp$, vel etiam $a=pp+2$, pro a hos dat valores: 1 et 3, ita ut omnes divisores formae $pp+2qq$ sint vel $8i+1$ vel $8i+3$. Formae ergo, quae ex classe divisorum excluduntur, sunt $8i+5$ et $8i+7$, quas igitur sub forma $8i+\alpha$ complecti oportet. Quod si ergo demonstrari posset solos numeros primos harum formarum ex classe divisorum excludi, simul demonstratum esset omnes numeros primos priorum formarum $8i+1$ et $8i+3$ contineri in formula $pp+2qq$, id quod quidem iam est ostensum. Ceterum binae posteriores formulae etiam ita exprimi possunt: $8i-1$ et $8i-3$, ita ut valores ipsius α sint negativi ipsius a, id quod in genere de divisoribus formae $pp+nqq$ est tenendum.

EXEMPLUM 4

20. Invenire omnes divisores primos formae $pp-2qq$ sive $2pp-qq$.

SOLUTIO

Ex problemate quarto est $n=2$, ideoque, ob $g<\sqrt{\frac{2}{5}}$, erit iterum $g=0$ et $fh=2$, unde pro divisoribus erit $D=rr+2ss$, vel etiam $D=rr-2ss$; unde patet has formas nullos alios divisores admittere, nisi qui ipsi sint eiusdem formae. Pro forma autem $D=8i+a$, quia est $a=pp$, vel etiam $a=pp-2$, valores pro a erunt ± 1, ergo omnes divisores continebuntur in forma $8i\pm 1$; excluduntur ergo omnes numeri formae $8i\pm 3$. Unde si soli numeri primi formae $8i\pm 3$ ex classe divisorum excludantur, necesse est, ut omnes numeri primi formae $8i\pm 1$ in forma proposita contineantur.

COROLLARIUM 1

21. Cum in problemate tertio reductio divisorum ad formam $pp+nqq$ plerumque unico tantum modo succedat, in casu problematis quarti talis reductio semper infinitis modis succedit; semper enim numeros p et q infinitis modis ita assumere licet, ut vel ipse divisor D vel Df formulae $pp-nqq$ aequetur.

COROLLARIUM 2

22. Casu autem huius exempli notari meretur, si fuerit $D=pp-2qq$, tum etiam fore $D=2rr-ss$, quoniam hae duae formae inter se aequales fieri possunt, iis enim aequatis fit

$$pp+ss=2(qq+rr)=(q+r)^2+(q-r)^2$$

ita ut sit $p=q+r$ et $s=q-r$.

EXEMPLUM 5

23. Invenire divisores primos formae $pp+3qq$.

SOLUTIO

Quia hic est $n=3$, ideoque $g<1$, tantum erit $g=0$, hincque divisor $D=rr+3ss$, ita ut etiam hoc casu omnes divisores primi sint formae $pp+3qq$. Quia autem limes pro g inventus ipsi unitati aequatur, quam superare non debet, evolvamus etiam casum $g=1$, unde fit $fh=4$ ideoque vel $f=1$ et $h=4$, vel $f=2$ et $h=2$. Priori casu fit

$$D=rr+2rs+4ss=\left(r+s\right)^2+3ss,$$

quae est ipsa forma proposita. Altero casu fit

$$D = 2rr + 2rs + 2ss\,,$$

quae forma cum factorem habeat *2,* statui debet

$$D = rr + rs + ss\,,$$

quae autem pariter ad propositam reducitur. Nam si s est numerus par, puta $s = 2t$, erit

$$D = rr + 2rt + 4tt = (r+t)^2 + 3tt\,;$$

sin autem s est numerus impar, etiam r debebit esse impar, quia alioquin ad casum praecedentem revolveremur; erit ergo $r+s$ numerus par, unde posito $r = 2t - s$ fiet

$$D = 4tt - 2ts + ss = 3tt + (t-s)^2\,,$$

unde patet superiorem conclusionem etiamnunc valere, semperque esse $D = rr + 3ss$. Deinde pro formula $12i + a$ ob $a = pp$ erit $a = 1$, tum vero formula $a = pp + 3$ dat $a = 7$, unde omnes divisores continebuntur in alterutra harum formularum: $12i + 1$ vel $12i + 7$, quas coniunctim ita repraesentemus: $12i + 1,\ 7$, vel etiam hoc modo $12i + 1, -5$. Si enim omnes valores ipsius a infra $2n$ in genere deprimere liceat, admittendis scilicet numeris negativis, tum altera formula $12i \pm \alpha$, in qua nullus divisor continetur, erit $12i + 5$ et $12i + 11$, vel $12i - 1, +5$, unde patet in genere valores ipsius α negativos esse ipsius *a.*

EXEMPLUM 6

[23a.] Invenire divisores primos formulae $pp - 3qq$ sive etiam $3pp - qq$.

SOLUTIO

Applicando hic problema quartum erit $n = 3$, ideoque $g < \sqrt{\frac{3}{5}}$, consequentur $g = 0$ et $fh = 3$, unde divisor erit $D = rr - 3ss$. Hinc patet hos numeros nullos alios divisores admittere, nisi qui sint eiusdem formae. Deinde pro formula $12i \pm a$, ob $a = pp$, vel $a = 3 - pp$, alii valores non prodeunt, praeter $a = 1$, ita ut omnes divisores contineantur in hac forma: $12i \pm 1$. Formula igitur divisores excludens erit $12i \pm 5$.

SCHOLION

24. Istas formulas iam olim expedivi, et demonstravi eas alios divisores non admittere, nisi qui sint eiusdem formae, id quod in maioribus numeris pro n assumtis non semper contingit. Conveniet autem eos casus excludere, quibus n est vel numerus quadratus, vel per quadratum divisibilis. Si enim foret $n = kmm$, tum formula $pp \pm kmmqq$ conveniret cum hac: $pp \pm kqq$.

EXEMPLUM 7

25. Invenire divisores primos numerorum $pp + 5qq$.

SOLUTIO

Ob $\sqrt{\frac{5}{3}} > g$ erit vel $g = 0$ vel $g = 1$**;** priori casu fit $fh = 5$, posteriori vero $fh = 6$. Prior casus dat divisorem $D = rr + 5ss$, quae est ipsa forma proposita; posterior vero dat vel

$$D = rr + 2rs + 6ss, \text{vel}$$
$$D = 2rr + 2rs + 3ss,$$

quarum formarum illa per reductionem ad primam redit, cum sit

$$D = (r + s)^2 + 5ss \;;$$

haec vero ab illa discrepat, cum inde fiat

$$2D = 4rr + 4rs + 6ss = (2r + s)^2 + 5ss\,;$$

unde patet omnes divisores vel ipsos esse numeros huius formae, vel eorum dupla, ita ut, si ipse divisor D non fuerit formae $pp + 5qq$, eius duplum $2D$ certe futurum sit huius formae. Deinde pro forma $20i + a$, ob $a = pp$, eius valores hinc nati erunt 1 et 9, ex altera autem formula $= pp + 5$ colliguntur iidem valores 1 et 9. Quia vero hic tantum de divisoribus agitur, pro a etiam sumi poterit $\frac{pp+5}{2}$, unde oriuntur valores 3, 7, sicque formula omnes divisores continens erit $20i + 1, \ +3, \ +7, \ +9$, contra vero formula divisores excludens erit $20i - 1, \ -3, \ -7, \ -9$. Si iam demonstrari posset istam postremam formulam continere omnes numeros primos, qui nequeunt esse divisores formae propositae, simul demonstratum foret omnes numeros primos in priore forma contentos certo esse divisores cuiuspiam numeri formae, $pp + 5qq$ ideoque vel ipsos vel eorum dupla eandem formam habere debere. Tales autem numeri usque ad centum sunt 1, 3, 7, 23, 29, 41, 43, 47, 61, 67, 83, 89.

EXEMPLUM 8

26. Invenire divisores primos numerorum formae $pp-5qq$.

SOLUTIO

Hic ex problemate quarto est $n=5$, unde ob $g<\sqrt{\frac{n}{5}}$ sumi poterit $g=0$, vel etiam $g=1$. Nihil enim nocet sumere $g=1$; superfluum tantum foret ipsi maiorem valorem tribuere. At $g=0$ dat divisorem $rr-5ss$, hoc est formae propositae; alter vero valor $g=1$ dat $fh=4$ ideoque vel

$$D=rr+2rs-4ss, \text{vel}$$
$$D=2rr+2rs-2ss.$$

Prior reducitur ad $D=(r+s)^2-5ss$, hoc est ad propositam; posterior vero per 2 divisa dat divisorem

$$D=rr+rs-ss,$$

quae forma etiam ad propositam reducitur, quod ita ostendo. Vel ambo numeri r et s erunt impares, vel alter par, alter impar. Pro casu posteriore sit $s=2t$ eritque

$$D=rr+2rt-4tt, \text{ sive } D=(r+t)^2-5tt.$$

Sin autem ambo numeri sint impares, erit eorum summa $r+s$ par, puta $2t$, ideoque $r-2t-s$, unde fit

$$D=4tt-2ts-ss=5tt-(t+s)^2.$$

Patet igitur omnes divisores numerorum formae propositae quoque eiusdem esse formae. Iam pro forma $20i\pm a$ valor $a=pp$ praebet 1 et 9, alter autem valor $a=5-pp$ praebet itidem 1 et 9, ita ut omnes divisores contineantur in hac forma $20i\pm 1,\ \pm 9$. Altera autem forma divisores excludens erit $20i\pm 3,\ \pm 7$.

SCHOLION

27. Quoniam ex his exemplis iam satisfiquet, quomodo pro minoribus numeris n singulas has operationes institui oporteat, aliquot exempla circa numeros maiores adhuc afferamus.

EXEMPLUM 9

28. Invenire omnes divisores primos numerorum formae $pp+17qq$.

SOLUTIO

Cum sit $\sqrt{\frac{17}{3}} < 3$, pro g habebimus tres valores 0, 1, 2. Primo sit $g = 0$, ideoque $fh = 17$, hinc divisor oritur $D = rr + 17ss$, ideoque ipsius formae propositae. Secundo sumatur $g = 1$, erit $fh = 18 = 1 \cdot 18 = 2 \cdot 9 = 3 \cdot 6$, unde nascuntur hae formae:

$$1°.\ D = rr + 2rs + 18ss = (r+s)^2 + 17ss,$$
$$2°. D = 2rr + 2rs + 9ss, \text{unde fit}$$

$$2D = 4rr + 4rs + 18ss = (2r+s)^2 + 17ss,$$

ita ut *2D* sit formae propositae.

3°. $D = 3rr + 2rs + 6ss$,

cuius triplum induit formam propositam.

Tertio sit $g = 2$ ideoque $fh = 21 = 1 \cdot 21 = 3 \cdot 7$; unde oritur

$$1°. D = rr + 4rs + 21ss = (r+2s)^2 + 17ss,$$
$$2°. D = 3rr + 4rs + 7ss,$$

cuius triplum iterum formam propositam induit. Quamobrem omnes divisores ita erunt comparati, ut vel ipsi, vel eorum dupla, vel eorum tripla habeant formam propositam. Quod deinde ad formam $68i + a$ attinet, valor $a = pp$ praebet numeros 1, 9, 25, 49, 13, 53, 33, 21; alter autem valor $a = pp + 17$ dat 21, 33 etc., qui numeri cum praecedentibus conveniunt. Quia autem hic etiam subdupla et subtripla occurrere possunt, primo patet formam $a = \frac{pp}{2}$ nullos dare valores idoneos; at $a = \frac{pp}{3}$ sequentes praebet numeros: 3, 27, 7, 11, 39, 23, 31, 63. Deinde vero formula $a = \frac{pp+17}{2}$ dat 9, [13,] 21 etc., qui numeri iam occurrunt. Denique formula $a = \frac{pp+17}{3}$ praebet 7, 11, 27 etc., qui itidem iam adsunt. Quamobrem omnes valores idonei pro *a* erunt

1, 3, 7, 9, 11, 13, 21, 23, 25, 27, 31, 33, 39, 49, 53, 63.

Hi autem numeri multo facilius inveniri possunt; statim enim atque aliquos tantum reperimus, quoniam novimus eorum producta ex binis pluribusve etiam occurrere debere, ante omnia autem omnes numeri quadrati per se occurrunt, ex quibus, quia etiam 3 occurrit, iam omnes plane reperiuntur. Quod si iam omnes hos numeros infra semissem numeri 68 deprimamus, dum maiorum complementa ad 68 signo − affecta apponimus, tum valores ipsius *a* sequentem seriem constituent:

$$+1,\ +3,\ -5,\ +7,\ +9,\ +11,\ +13,\ -15,\ -19,\ +21,$$
$$+23,\ +25,\ +27,\ -29,\ +31,\ +33.$$

Si iam omnium horum numerorum signa mutemus, obtinebimus omnes valores litterae α pro formula $68i+\alpha$, ex qua omnes divisores sunt exclusi.

COROLLARIUM 3

29. Hinc igitur perspicuum est etiam pro omnibus aliis numeris positivis loco n assumtis in valoribus litterae a omnes plane occurrere numeros impares minores quam $2n$, et qui simul ad n sint primi, dum alli signo +, alii signo − sunt affecti.

EXEMPLUM 10

30. Invenire omnes divisores primos numerorum in hac formula: $pp-19qq$, vel etiam $19pp-qq$ contentorum.

SOLUTIO

Hic igitur ob $n=19$ erit $g<\sqrt{\frac{19}{5}}$, ideoque $g<2$, unde habebimus vel $g=0$ vel $g=1$. Sit primo $g=0$ eritque divisor $D=rr-19ss$ ob $fh=19$, ideoque hi divisores iam sunt ipsius formae propositae. Sit porro $g=1$ fietque $fh=19-1=18=1\cdot 18=2\cdot 9=3\cdot 6$; unde tres casus sunt evolvendi:

1°. $D=rr+2rs-18ss=(r+s)^2-19ss$,
quae forma iam in proposita continetur.

2°. $D=2rr+2rs-9ss$,

cuius duplum ad formam propositam redit.

3°. $D=3rr+2rs-6ss$,

cuius triplum in forma proposita continetur. Sicque omnes divisores quaesiti vel ipsi, vel eorum dupla, vel eorum tripla in forma proposita continentur.
Deinde pro forma $4ni\pm a$, sive $76i\pm a$ valores ipsius a ex sequentibus formulis derivari debent:

1°. $a=pp$ dat 1, 9, 25, 49, 5, 45, 17, 73, 61.

2°. $a=\frac{pp}{2}$ dat nullos valores idoneos, quia omnes forent pares.

3°. $a=\frac{pp}{3}$, sive $a=3tt$, praebet hos valores: 3, 27, 75, 71, 15, 59, 51, 67, 31.

4°. $a=19-pp$ dat 15, 3 etc.,
qui iam occurrunt,

5°. $a = \frac{19-pp}{2}$ dat 9, 5, 3 etc.,
qui iidem iam adsunt.
6°. $a = \frac{19-pp}{3}$ dat 5, 1, 15 etc.,

qui etiam adsunt. Quamobrem omnes numeri idonei pro a assumendi, quoniam tam positive quam negative accipi possunt, infra 38 deprimi possunt, dum scilicet maiorum complementa ad 76 apponuntur:

1, 3, 5, 9, 15, 17, 25, 27, 31.

Pro altera autem forma $76i \pm \alpha$, in qua nulli divisores occurrere possunt, valores ipsius α sunt sequentes:

7, 11, 13, 21, 23, 29, 33, 35, 37.

SCHOLION

31. Hactenus alios numeros pro n non assumsimus, praeter primos, quamobrem etiam adhuc adiungamus duo exempla circa numeros compositos.

EXEMPLUM 11

32. Invenire omnes divisores primos numerorum in hac forma contentorum: $pp + 30qq$.

SOLUTIO

Hic ob $n = 30$ et $g < \sqrt{10}$, loco g quatuor valores assumi conveniet, 0, 1, 2, 3, quos ergo singulos percurramus :

I. $g = 0$ praebet $fh = 30$, unde pro divisore D sequentes formulae nascuntur:

$1^o. D = rr + 30ss,$

$2^o. D = 2rr + 15ss,$

$3^o. D = 3rr + 10ss,$

$4^o. D = 5rr + 6ss,$

quarum prima cum forma proposita congruit, tum vero secundae duplum, tertiae triplum et quartae quintuplum; ubi notetur loco quintupli etiam sextuplum sumi posse, quandoquidem si fuerit

$$5D = pp + 30qq,$$

tum etiam erit

$$6D = pp + 30qq.$$

II. Sit iam $g = 1,$ erit $fh = 31$, unde unica forma nascitur

$$D = rr + 2rs + 31ss = (r+s)^2 + 30ss,$$

quae est ipsa forma proposita.

III. Sit $g = 2,$ erit $fh = 34 = 1\cdot 34 = 2\cdot 17$; unde duae formae nascuntur:

$1°. D = rr + 4rs + 34ss = (r+2s)^2 + 30ss,$
$2°. D = 2rr + 4rs + 17ss,$

cuius duplum ad formam propositam reducetur.

IV. Sit $g = 3$ eritque $fh = 39 = 1\cdot 39 = 3\cdot 13$, unde iterum duae nascuntur formae:

$1°. D = rr + 6rs + 39ss = (r+3s)^2 + 30ss,$
$2°. D = 3rr + 6rs + 13ss,$

cuius triplum induit formam propositam. Ex his igitur sequitur omnes divisores *D* ita esse comparatos, ut vel *D,* vel 2D, vel *3D,* vel *6D* in forma proposita contineantur. Deinde vero pro forma $4ni + a = 120i + a$ ante omnia notetur multitudinem omnium numerorum minorum quam 120 simulque ad 120 primorum esse 32, unde iam certo inferre possumus numerum valorum tam litterae *a* quam α esse 16. Cum igitur primo in *a* omnes numeri quadrati occurrant, formula $a = pp$ dabit hos tantum numeros: 1 et 49; at vero formae $\frac{pp}{2}$, $\frac{pp}{3}$ et $\frac{pp}{6}$ nullos plane praebent numeros ad 120 primos. Altera vero forma

$a = pp + 30$ praebet hos tantum numeros: 31, 79. Hinc autem porro $a = \frac{pp+30}{2}$, sive haec: $a = 2tt + 15$ praebet 17, 23, 47, 113. Porro $a = \frac{pp+30}{3}$, sive $a = 3tt + 10$ praebet 13, 37. Haec igitur forma tantum dat duos valores. Denique $a = \frac{pp+30}{6}$, sive $a = 6tt + 5$ dat hosce: 11, 29, 59, 101. Hoc autem modo tantum 14 prodierunt valores pro littera *a,* ita ut duo adhuc desiderentur. Verum hic perpendendum est loco formulae $pp + 30$ generalius poni potuisse $pp + 30qq$, unde sumendo $p = 3t$ et per 3 dividendo statui poterit $a = 3tt + 10qq$. Sit nunc $q = 2$, fietque $a = 3tt + 40$, unde casus $t = 1$ praebet $a = 43$, at $t = 3$ dat $a = 67$; hocque modo nacti sumus omnes 16 valores ipsius *a,* qui ordine ita procedunt:

1, 11, 13, 17, 23, 29, 31, 37, 43, 47, 49, 59, 67, 79, 101, 113.

Quod si iam loco numerorum maiorum quam 60 eorum complementa ad 120 cum signo − scribantur, isti numeri ita disponi poterunt:

$$+1,\ -7,\ +11,\ +13,\ +17,\ -19,\ +23,\ +29,\ +31,\ +37,$$
$$-41,\ +43,\ +47,\ +49,\ -53,\ +59,$$

ubi omnes plane numeri impares ad 30 primi occurrunt vel signo + vel − affecti, ubi si signa mutentur, habebuntur omnes valores litterae α pro formula $120i + \alpha$, cuius omnes numeri ex classe divisorum excluduntur.

COROLLARIUM 1

33. Omnes ergo divisores numerorum formae $pp + 30qq$ in quatuor classes distribuuntur, quarum prima continet eos, qui ipsi sunt formae $pp + 30qq$; secunda classis vero eos, quorum dupla sunt eius formae; tertia, quorum tripla, et quarta denique eos, quorum quintupla vel etiam sextupla ad formam $pp + 30qq$ reduci possunt. Hae igitur quatuor classes, si formam propositam $pp + 30qq$ littera *F,* divisores vero littera *D* designemus, hoc modo repraesentari possunt:

$$\text{I. } D = F, \text{II. } 2D = F, \text{III. } 3D = F, \text{IV. } 5D = F;$$

ubi notasse iuvabit, si fuerit $2D = F$, tum etiam fore $15D = F$; similique modo si fuerit $3D = F$, erit etiam $10D = F$; at si fuerit $5D = F$, erit etiam $6D = F$.

COROLLARIUM 2

34. Quando dicimus omnes divisores numerorum formae propositae $pp + 30qq$ in forma $120i + a$ contineri, id non ita est intelligendum, quasi omnes numeri in formula $120i + a$ contenti essent divisores, sed inde excludi debent omnes illi, qui per quempiam

numerum formae $120i+\alpha$ sunt divisibiles. His autem sublatis maxime probabile videtur omnes reliquos numeros formulae $120i+a$, ideoque imprimis numeros primos, certe fore divisores cuiuspiam numeri formae $pp+30qq$. Isti autem numeri primi in formula $120i+a$ contenti facili negotio quousque libuerit assignari possunt, quippe qui hoc ordine usque ad 240 progrediuntur:

1, 11, 13, 17, 23, 29, 31, 37, 43, 47, 59, 67, 79, 101, 113,
131, 137, 149, 151, 157, 163, 167, 179, 199, 233.

COROLLARIUM 3

35. Quoniam omnes divisores sunt quadruplicis generis, inde etiam valores ipsius a in quatuor classes distribui conveniet, prouti inde oriuntur divisores vel primae, vel secundae, vel tertiae, vel quartae classis, quibus ergo subscribamus characteres cuiusque classis 1, 2, 3, 6, hoc modo:

1,	11,	13,	17,	23,	29,	31,	37,	43,	47,	49,	59,	67,	79,	101,	113
1,	6,	3,	2,	2,	6,	1,	3,	3,	2,	1,	6,	3,	1,	6,	2.

Hic igitur notari meretur singulas classes quater occurrere .

EXEMPLUM 12

36. Invenire omnes divisores primos numerorum in hac forma: $pp-30qq$, sive in hac: $30pp-qq$ contentorum.

SOLUTIO

Cum hic sit $\sqrt{\frac{30}{5}}<3$, pro littera g habemus tantum tres valores 0, 1, 2. Hinc cum sit $fh=30-gg$, pro primo casu erit $fh=30$, pro secundo $fh=29$ et pro tertio $fh=26$, quos igitur casus evolvamus.

I. Sit $g=0$ et hinc nascuntur sequentes valores:
1°. $D=rr-30ss$,
2°. $D=2rr-15ss$,
3°. $D=3rr-10ss$,
4°. $D=5rr-6ss$.

II. Si $g=1$, unica forma nascitur

$$D=rr+2rs-29ss=(r+s)^2-30ss,$$

quae ergo est ipsa forma proposita.

III. Si $g = 2$, oriuntur duae formulae

$$1°.\ D = rr + 4rs - 26ss = (r + 2s)^2 - 30ss;$$

iterum ipsa proposita

$$2°.\ D = 2rr + 4rs - 13ss,$$

cuius duplum fit numerus formae propositae. Hinc ergo nascuntur quadruplicis generis divisores, qui posita littera F pro formula proposita sunt

$$\text{I. } D = F,\ \text{II. } 2D = F, \text{III. } 3D = F, \text{IV. } 6D = F.$$

Deinde vero pro formula omnes divisores continente $120i \pm a$ erit primo vel $a = pp$, vel $a = \frac{pp}{2}$, vel $a = \frac{pp}{3}$, vel $a = \frac{pp}{6}$, unde alii numeri ad 30 primi oriri nequeunt nisi ex prima forma $a = pp$, ideoque duo tantum valores hinc nascuntur: scilicet 1 et 49. Altera autem forma erat $a =$ vel $30 - pp$, vel $a = \frac{30-pp}{2}$, vel $a = \frac{30-pp}{3}$, vel $a = \frac{30-pp}{6}$ quarum prima $a = 30 - pp$ praebet hos numeros: 29, 19, 91. Quia autem loco 30 ponere possumus *30qq,* formula $a = 120 - pp$ praebet insuper hos valores: 119, 71. Secunda ad formam $a = 2tt - 15$ reducta dat hosce numeros: 13, 7, 17, 83, 113, 107. Huic vero formulae aequivalet $15pp - 2qq$, ergo sumto $p = 3$ erit quoque $a = 135 - 2qq$, unde prodeunt 13, 7, 103, 37 ; ideoque insuper novus 103 accedit. Ex tertia forma $a = 3tt - 10$ hosce novos numeros nanciscimur: 7, 17, forma autem affinis $a = 10tt - 3$ praebet insuper 37. Ex ultima forma $5tt - 6$ nascuntur hi valores: 1, 119; ex forma vero affini $a = 6tt - 5$ isti: 1, 19, 49, 91. Hinc imprimis notandum est eosdem numeros ex diversis classibus oriri posse. Prodierunt autem hactenus:

1, 7, 13, 17, 19, 29, 37, 49, 71, 83, 91, 103, 107, 113, 119,

quorum valorum numerus quidem tantum est 15, cum esse deberet 16; quia autem novimus cuiusque numeri complementum ad 120 etiam occurrere debere, iste defectus facile suppletur. Deerat scilicet 101 tanquam complementum ipsius 19. Quia autem numeri *a* tam positive quam negative accipi possunt, complementa reiicere licet, ita ut pro *a* habeamus octo sequentes valores:

1, 7, 13, 17, 19, 29, 37, 49,

reliqui igitur numeri praebent valores litterae α, qui erunt totidem

11, 23, 31, 41, 43, 47, 53, 59.

COROLLARIUM 1

37. Hoc igitur casu, admisso meo theoremate, quod omnes numeri primi in forma $4ni+a$ contenti simul sint divisores numerorum formae $pp \pm nqq$, numeri primi ex nostra formula $120i \pm a$ orti usque ad 240 sunt sequentes:

1, 7, 13, 17, 19, 29, 37, 71, 83, 101, 103, 107, 113, 127, 137,
139, 149, 157, 191, 211, 223, 227, 233, 239.

COROLLARIUM 2

38. Quoniam in hac evolutione vidimus eosdem numeros ex diversis classibus ortos esse, manifestum est nequicquam quatuor classes diversas esse constitutas, sed binas earum in unam coalescere posse. Primo enim omnes divisores quartae classis, pro quibus erat $5D = F$, sive etiam $6D = F$, iam in prima classe $D = F$ reperiuntur, ita ut semper, quoties fuerit $5D = F$, etiam futurum sit $D = F$. Simili modo divisores tertiae classis etiam continentur in classe secunda. Quod si enim fuerit $3D = F$, semper etiam erit $2D = F$, quamobrem omnes divisores pro forma proposita $pp - 30qq$, vel $30pp - qq$ ad duas tantum classes priores revocari possunt: semper enim erit vel $D = F$ vel $2D = F$.

COROLLARIUM 3

39. Omnes igitur numeri primi ex nostra forma $120i \pm a$ oriundi duplicis erunt generis, dum vel ipsi vel eorum dupla formam propositam haberi possunt, quos simili modo ut ante distingui conveniet, subscribendo singulis valoribus characteres vel 1 vel 2

1,	7,	13,	17,	19,	29,	37,	49
1,	2,	2,	2,	1,	1,	2,	1.

Ubi notetur ambos characteres totidem occurrere.

SCHOLION

40. Quod si ergo pro numeris cuiuscunque formae $pp \pm nqq$ omnes divisores primi desiderentur, eos facillime ex nostra forma generali $4ni + a$ assignare licet; dum contra, si formulis ab illustri LA GRANGE exhibitis uti vellemus, opus foret maxime molestum ex singulis formis $frr + 2grs + hss$ omnes numeros primos elicere; quamobrem maxime est optandum, ut demonstratio firma illius mei asserti detegatur, quippe quo demum ista Theoria ad summum perfectionis gradum evehetur. Arbitror autem talem demonstrationem mox fortasse sperari posse, si sequentia momenta probe perpendantur.

1). Postquam pro formula proposita quacunque $pp \pm nqq$ ambae meae formulae $4ni + a$ et $4ni + \alpha$ fuerint constitutae, eae simul omnes plane numeros impares ad propositum n primos complectuntur; tum vero omnes divisores ad formam priorem $4ni + a$ referuntur; nulli autem numeri alterius formae $4ni + \alpha$ possunt esse divisores propositae, sive omnes numeri posterioris formae ex classe divisorum penitus excluduntur.

2). Probe perpendatur quovis casu omnes valores ipsius a egregia lege inter se cohaerere, ita ut omnes coniunctim quasi ambitum quendam completum constituant, in quo nihil deficiat nihilque abundet, quandoquidem omnia producta ex binis pluribusve horum numerorum iterum in eadem classe occurrunt, ita ut, simul atque aliqui valores idonei pro a fuerint inventi, ex iis reliqui omnes facile definiri queant, praecipue quoniam omnes numeri quadrati eorumve residua respectu divisoris $4n$ certe ingrediuntur. Unde si hoc modo omnia producta atque etiam potestates numerorum iam inventorum inserantur, mox tota ista classis ita adimplebitur, ut multitudo omnium numerorum huc pertinentium semper sit semissis omnium plane numerorum ad $4n$ primorum eoque minorum; altera vero semissis praebebit classem numerorum α, qui nullo modo divisores evadere possunt.

3). Hinc igitur patet ambas istas classes discrimine maxime memorabili et in natura ipsa numerorum fundato a se invicem discrepare atque adeo essentialiter a se invicem distingui, ita ut numeri alterius classis natura sua ab altera classe prorsus sint diversi.

4). Quoniam nulli numeri classis $4ni+\alpha$ unquam esse possunt divisores ullius numeri formae $pp\pm nqq$, ista classis tanquam origo spectari debet omnium numerorum, quorum natura ab indole divisorum abhorret, quae repugnantia quoque ad omnes numeros extendi debet, qui divisibiles sunt per ullum numerum classis $4ni+\alpha$. Si enim tales numeri possent esse divisores, etiam isti huius classis numeri forent divisores, id quod naturae rei repugnat.

5). Cum autem producta ex binis numeris classis $4ni+\alpha$ in classem divisorum $4ni+a$ transeant, manifestum est in prima classe plurimos occurrere debere ab indole divisorum alienas; omnes scilicet eos, qui per ullum numerum alterius classis sunt divisibiles.

6). Quod si iam omnes isti numeri in classe $4ni+a$ deleantur sive excludantur, qui natura divisorum refragantur, maxime probabile videtur reliquos numeros omnes indole divisorum fore praeditos. Cum hoc modo tantum numeri compositi expungantur, evidens est omnes plane numeros primos in forma $4ni+a$ contentos revera fore divisores cuiuspiam numeri formae $pp\pm nqq$. Totum ergo negotium huc redit, ut isti probabilitati vis perfectae demonstrationis concilietur. Haec autem veritas, siqua est, elegantius ita proponi potest.

THEOREMA DEMONSTRANDUM

41. *Si fuerit a divisor cuiuspiam numeri formae $pp+nqq$, ita ut sit $aD=pp+nqq$, tum quoties $4ni+a$ est numerus primus, toties quoque erit $D(4ni+a)$ numerus formae $pp+nqq$.*

Hic autem sequentia notari oportet: 1). Numeros p et q inter se esse debere primos. 2). Divisorem a etiam primum esse debere ad n, quoniam divisores ipsius n hinc excluduntur. 3). Quod si forte eveniat, ut numerus $D(4ni+a)$ non videatur in forma $pp+nqq$ contineri, tum semper eius quadruplum, vel etiam eius productum per aliud quadratum, certe in ea contineri. Quoniam igitur hoc casu erit

$$D(4ni+a)=(\tfrac{p}{2})^2+n(\tfrac{q}{2})^2,$$

haeo resolutio nullam exceptionem mereri est censenda. Ita cum sit $27=4^2+11\cdot 1^2$, erit $a=27$ et $n=11$ et $D=1$, unde formula $4ni+a$ evadit $44i+27$, quae casu $i=1$ praebet 71, hoc est numerum primum; neque tamen in integris esse potest $71=pp+11qq$. Est vero

$$4\cdot 71=284=3^2+11\cdot 5^2,$$

ideoque

$$71=(\tfrac{3}{2})^2+11\cdot(\tfrac{5}{2})^2.$$

Tales autem casus raro occurrunt et ideo non sunt excipiendi, quia numeri formulae $4ni+\alpha$ ita ex classe divisorum excluduntur, ut, etiamsi pro p et q numeri fracti accipiantur, tamen nunquam divisores esse queant.

SCHOLlON

42. Superfluum foret has investigationes ad huiusmodi formulas: $mpp\pm nqq$ extendere, cum omnes divisores numerorum formae $mpp\pm nqq$ semper sint etiam divisores numerorum formae $pp\pm mnqq$. Quae igitur olim in Tomo XIV Comment. Vet. Academiae de divisoribus numerorum formae $mpp\pm nqq$ sum commentatus et magnam partem ex sola inductione conclusi, nunc per egregias proprietates ab Illustri LA GRANGE demonstratas non solum plurimum illustrantur, sed etiam ad multo maiorem certitudinis gradum perducuntur, ita ut iam nihil amplius desideretur, nisi ut solida demonstratio theorematis allati detegatur, quam nunc quidem mox exspectare licebit. Mea autem methodus imprimis hac gaudet praerogativa, quod eius ope omnes plane divisores huiusmodi formularum $mpp\pm nqq$ assignari et, quousque libuerit, continuari possunt, id quod insuper sequenti exemplo declarabo.

EXEMPLUM 13

43. Invenire omnes divisores formae $pp+39qq$.

Primo igitur per formulas Illustris LA GRANGE quaeramus omnes diversas formas horum divisorum, et cum sit $n=39$ ideoque $\sqrt{\frac{39}{3}}<4$, sufficiet pro *g* assumere hos quatuor valores: 0, 1, 2, 3.

I. Valor igitur $g=0$ praebet $fh=39$, unde hae duae formae nascuntur:

$$1^\circ.\ rr+39ss, \qquad 2^\circ.\ 3rr+13ss,$$

quarum prior dat divisores $D=F$ et altera $3D=F$, denotante *F* formam propositam.

II. Valor $g=1$ dat $fh=40$, unde nascuntur istae formae:

1°. $D=rr+2rs+40s=(r+s)^2+39ss$,
ideoque $D=F$.

2°. $D=2rr+2rs+20ss$,
quae forma autem numerus primus esse nequit.

3°. $D=4rr+2rs+10ss$,
quae forma itidem non dat numeros primos.

4°. $D=5rr+2rs+8ss$,
unde fit $5D=F$, vel etiam $8D=F$.

III. Casus $g=2$ dat $fh=43$, unde unica forma oritur:
$D=rr+4rs+43ss=(r+2s)^2+39ss$,
ideoque $D=F$.

IV. Casus denique $g=3$ praebet $fh=48$, unde sequentes formae numeros primos continentes oriuntur:
1°. $D=rr+6rs+48ss=(r+3s)^2+39ss$,
ideoque $D=F$.

2°. $D=3rr+6rs+16ss$
dat $3D=F$, vel etiam $16D=F$. Hinc igitur patet omnino dari tria genera divisorum:

$$1)\ D=F, \quad 2)\ 3D=F, \quad 3)\ 5D=F.$$

Quibus constitutis evolvamus formulam $4ni+a=156i+a$, ubi primo notetur omnium numerorum ad 156 primorum ipsoque minorum multitudinem esse 48, unde usque ad semissem 78 erunt 24, quorum singuli vel positive vel negative sumti praebent valores pro littera *a*. Isti ergo numeri erunt:

1, 5, 7, 11, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43,
47, 49, 53, 55, 59, 61, 67, 71, 73, 77,

ubi primo quadrata habent signum +, qui ergo sunt

$$+1,\ +25,\ +49;$$

reliquorum vero numerorum quadrata divisione per 156 deprimantur infra 78, unde fiet

$$11^2=-35,\ 17^2=-23,\ 19^2=+49,\ 23^2=+61.$$

Pro reliquis numeris consideremus formam $pp+39$, unde sumto $p=1$ prodit 40, cuius numeri ad genus tertium pertinentis divisor 5 habet signum +. Iam quia praecedentes numeri ad genus primum sunt referendi, eorum producta per 5 etiam ad genus tertium referri debebunt, unde nascentur sequentes numeri:

$$+5,\ +41,\ -31,\ -19,\ -67,\ -7.$$

Sit nunc $p=2$ eritque $4+39=43$, qui est divisor primae classis, unde etiam numeri huius classis iam inventi per 43 multiplicati dabunt divisores primae classis, qui autem, cum numerus 43 sit nimis magnus, facilius ex sequentibus reperientur. Sumatur igitur $p=3$ eritque $pp+39=48$, cuius divisor 3 iam est exclusus. Sit ergo $p=4$ eritque $16+39=55$, cuius divisorem 5 iam tractavimus; alter vero divisor 11 etiam ad tertiam classem pertinet; per hunc ergo numeri primae classis multiplicati erunt:

$$+11,\ +59,\ -37,\ -73,\ +71,\ +47.$$

Multiplicentur etiam numeri tertiae classis per 11 et producta depressa, qui sunt

$$+55,\ -17,\ -29,\ -53,\ +43,\ -77,$$

revertentur ad classem primam. Hoc modo omnes nostri numeri signa sua debita sunt adepti, qui cum vel ad primam vel ad tertiam classem referantur, manifestum est nullos divisores secundae classis relinqui. Omnes scilicet hi numeri iam in prima classe continentur, quamobrem omnes valores ipsius *a* cum characteribus suis I vel III subscriptis ita se habebunt:

$+1,$	$+5,$	$-7,$	$+11,$	$-17,$	$-19,$	$-23,$	$+25,$	$-29,$	$-31,$	$-35,$	$-37,$
I	III	III	III	I	III	I	I	I	III	I	III

$+41,$	$+43,$	$+47,$	$+49,$	$-53,$	$+55,$	$+59,$	$+61,$	-67	$+71,$	$-73,$	-77
III	I	III	I	I	I	III	I	III	III	III	I.

Neque vero classis secunda prorsus est inutilis: dantur enim numeri primi, quos ad primam classem retulimus, quorum resolutio in integris non succedit atque adeo denominatorem quadratum 16 postulat, cuiusmodi numerus est 61, qui aliter ad primam classem redigi nequit, nisi hoc modo: $61=(\frac{25}{4})^2+39(\frac{3}{4})^2$. Est vero $3\cdot 61=183=12^2+39\cdot 1^2$. Quod si iam valores negativi pro *a* inventi in positivos convertantur, sumendis complementis ad 156 sequentes valores prodibunt:

1,	5,	11,	25,	41,	43,	47,	49,	55,	59,	61,	71,	79,	83,	89,	103,
I	III	III	I	III	I	III	I	I	III	I	III	I	III	III	I

119,	121,	125,	127,	133,	137,	139,	149
III	I	III	I	I	III	I	III.

Nunc igitur omnes numeri primi in forma $156i+a$ contenti certe erunt divisores cuiuspiam numeri formae $pp+39qq$, atque adeo vel ipsi, vel eorum quintupla, vel etiam tripla erunt numeri huius formae. Hinc ergo omnes divisores primi ab 1 usque ad 312 erunt sequentes:

1, 5, 11, 41, 43, 47, 59, 61, 71, 79, 83, 89, 103, 127, 137, 139, 149, 157,
167, 181, 197, 199, 211, 227, 239, 277, 281, 283, 293.

COROLLARIUM 1

44. Quo facilius intelligatur, cur hoc casu classis secunda ad primam sit revoluta, iam supra ostendimus, si divisor fuerit

$$D=frr+2grs+hss,$$

existente $fh=gg+n$, tum non solum *Df,* sed etiam *Dh* ad formam $pp+nqq$ reduci posse. Hinc autem generalius si fuerit

$$k=ftt+2gtu+huu,$$

tum productum *Dk* etiam erit numerus formae $pp+nqq$; facto enim calculo reperitur

$$Dk=(frt+g(ts+ru)+hsu)^2+n\left(ts-ru\right)^2.$$

Quod si ergo k fieri queat quadratum, vel divisibile per quadratum, tum hoc quadratum omitti poterit. Nam si fuerit $Dkll$ numerus formae $pp+nqq$,
tum, etiam admissis fractionibus, erit quoque Dk eiusdem formae. Ita nostro casu pro divisoribus secundae classis erat

$$D=3rr+13ss,$$

ideoque $k=3tt+13uu$, cuius valor sumto $t=1$ et $u=1$ fiet $k=16$, qui cum sit numerus quadratus, haec forma ad primam reducitur.

COROLLARIUM 2

45. Nunc igitur omnia theoremata, quae circa huiusmodi divisores olim in Comment. veter. Tomo XIV dederam, multo maiorem gradum certitudinis sunt adepta, postquam a celeb. LA GRANGE formae istorum divisorum sunt demonstratae; atque nullum dubium esse videtur, quin mox, quod in hoc genere adhuc desideratur, perfecta demonstratione muniatur.

COROLLARIUM 3

46. Antequam hoc argumentum penitus deseram, memorabilem adhuc observationem adiungam circa signa numerorum a, dum scilicet omnes eius valores infra $2n$ deprimuntur. Cum enim horum numerorum primus et ultimus simul sumti fiant $2n$, dispiciendum est, utrum hi duo numeri habeant vel paria signa vel disparia, utroque enim casu bini quicunque horum numerorum ab extremis aequidistantes, quorum ergo summa semper est $2n$, etiam habebunt sive eadem signa sive contraria. Ita nostro casu, quo erat $2n=78$, ultimus 77 habebat signum −, dum primus 1 semper habet signum +, unde etiam signa binorum ab extremis aeque distantium perpetuo erunt contraria. E contrario autem in exemplo 11, ubi erat $2n=60$, ultimus numerus 59 habebat signum +, unde etiam bini quicunque alii ab extremis aequidistantes eodem signo affecti deprehenduntur, cuius quidem phaenomeni ratio haud difficulter investigari poterit. Huiusmodi autem observationes laborem investigationis divisorum non mediocriter sublevant.