CONCERNING THE SERIES SUMMATION FOR RECIPROCAL POWERS BY A NEW AND EASIER METHOD

[E597]

Opuscula analytica 2, 1785, p. 257-274

1. Since first I was showing the sums of these series, I deduced those from this principle, so that for each sine and cosine innumerable circular arcs correspond, which all shall be the roots of an infinitude of equations, by which an arc is accustomed to be expressed by the sine or cosine. Hence from the coefficients of these same equations also not only the sums of the roots themselves, but also I have assigned the sums of any power of these. Truly later I have derived the same sums also from other principles, but which all were depending on the property of the circle mentioned. Now truly I have observed these sums from another much simpler principle and able to be deduced depending on analytical operations only, which more accurate method it will please to be set out here. [It appears that the current method is actually the same as used initially in E61; see the *O.O.* edition for details]

2. Moreover the integration of this formula

$$\int\left(\frac{z^{m-1}\pm z^{n-m-1}}{1\pm z^{n}}\right)dz$$

provided this principle for me, for the case, where after the integration there is set $z=1$. Indeed I have shown in Book XIX *Nov. Comment*. this integration can be expressed in the following manner by the customary rules of integration :

$$\int\left(\frac{z^{m-1}+z^{n-m-1}}{1+z^{n}}\right)dz=\frac{\pi}{n\sin.\frac{m\pi}{n}}$$

and

$$\int\left(\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}\right)dz=\frac{\pi}{n\text{tang}.\frac{m\pi}{n}}.$$

[See e.g. E59 & E60; E462; also Euler's *Integral Calculus*, Vol. I, §82 & §84.] But if truly the same formulas may be expanded by series , on putting $z=1$ there will become

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}-\frac{1}{n+m}+\frac{1}{2n+m}-\frac{1}{3n+m}+\frac{1}{4n+m}-\text{etc.}$$
$$\frac{1}{n-m}-\frac{1}{2n+m}+\frac{1}{3n-m}-\frac{1}{4n-m}+\text{etc.}$$

and

$$\frac{\pi}{n\text{tang.}\frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n+m} + \frac{1}{2n+m} + \frac{1}{3n+m} + \frac{1}{4n+m} + \text{etc.}$$
$$-\frac{1}{n-m} - \frac{1}{2n-m} - \frac{1}{3n-m} - \frac{1}{4n-m} - \text{etc.},$$

which two series therefore are worthy of greater attention, because clearly everything is present in these, which not only concerns the sums of powers, but also have been extended to similar summations.

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3. We may consider initially the first form

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}$$

and with the two similar terms contracted we will have

$$\frac{\pi}{n\sin.\frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} + \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm} + \text{etc.}$$

Now we may assume, so that the formulas may become simpler, $m = 1$ and there will be

$$\frac{\pi}{n\sin.\frac{m\pi}{n}} = 1 + \frac{2}{nn-1} - \frac{2}{4nn-1} + \frac{2}{9nn-1} - \frac{2}{16nn-1} + \text{etc.}$$

or

$$\frac{\pi}{2n\sin.\frac{m\pi}{n}} - \frac{1}{2} = \frac{1}{nn-1} - \frac{1}{4nn-1} + \frac{1}{9nn-1} - \frac{1}{16nn-1} + \text{etc.}$$

4. Now we may resolve these into infinite geometric series in the customary manner and there will become :

$$\frac{1}{nn-1} = \frac{1}{nn} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}} + \text{etc.},$$
$$-\frac{1}{4nn-1} = -\frac{1}{4nn} - \frac{1}{4^2n^4} - \frac{1}{4^3n^6} - \frac{1}{4^4n^8} - \frac{1}{4^5n^{10}} - \text{etc.},$$
$$\frac{1}{9nn-1} = \frac{1}{4nn} + \frac{1}{9^2n^4} + \frac{1}{9^3n^6} + \frac{1}{9^4n^8} + \frac{1}{9^5n^{10}} + \text{etc.},$$
$$-\frac{1}{16nn-1} = -\frac{1}{16nn} - \frac{1}{16^2n^4} - \frac{1}{16^3n^6} - \frac{1}{16^4n^8} - \frac{1}{16^5n^{10}} - \text{etc.}$$
$$\text{etc.}$$

Therefore the sum of all these infinite series will be

$$= \frac{\pi}{2n\sin.\frac{\pi}{n}} - \frac{1}{2}.$$

5. Therefore now we may gather these series following the vertical lines, which in the end we may put for the sake of brevity

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{etc.}=A\pi\pi,$$
$$1-\frac{1}{4^2}+\frac{1}{9^2}-\frac{1}{16^2}+\frac{1}{25^2}-\text{etc.}=B\pi^4,$$
$$1-\frac{1}{4^3}+\frac{1}{9^3}-\frac{1}{16^3}+\frac{1}{25^3}-\text{etc.}=C\pi^6,$$
$$1-\frac{1}{4^4}+\frac{1}{9^4}-\frac{1}{16^4}+\frac{1}{25^4}-\text{etc.}=D\pi^8$$
$$\text{etc.}$$

Hence therefore we will come upon the following equation :

$$\frac{\pi}{2n\sin.\frac{\pi}{n}}-\frac{1}{2}=\frac{A\pi\pi}{nn}+\frac{B\pi^4}{n^4}+\frac{C\pi^6}{n^6}+\frac{D\pi^8}{n^8}+\text{etc.}$$

6. Again for the sake of brevity, we may put $\frac{\pi}{n}=x$, so that the following equation may be produced :

$$\frac{x}{2\sin.x}-\frac{1}{2}=Axx+Bx^4+Cx^6+Dx^8+Ex^{10}+\text{etc.},$$

were now it is understood all the coefficients assumed *A, B, C* etc. can be defined by the due expansion; with which found we may obtain the sums of all the series contained in this form

$$1-\frac{1}{4^i}+\frac{1}{9^i}-\frac{1}{16^i}+\frac{1}{25^i}-\text{etc.}$$

or in this

$$1-\frac{1}{2^{2i}}+\frac{1}{3^{2i}}-\frac{1}{4^{2i}}+\frac{1}{5^{2i}}-\text{etc.}$$

with *i* denoting some whole number.

7. Now since by the most well-known series there shall be

$$\sin.x=x-\frac{x^3}{1\cdot2\cdot3}+\frac{x^5}{1\cdot2\cdots5}-\frac{x^7}{1\cdot2\cdots7}+\text{etc. },$$

for this simpler series we may write

$$\sin.x=\alpha x-\beta x^3+\gamma x^5-\delta x^7+\varepsilon x^9-\text{etc. },$$

so that there shall be

$$\alpha=1,\ \beta=\frac{\alpha}{2\cdot3},\ \gamma=\frac{\beta}{4\cdot5},\ \delta=\frac{\gamma}{6\cdot7},\ \varepsilon=\frac{\delta}{8\cdot9}\ \text{ etc.}$$

with which in place we may transfer the term $-\frac{1}{2}$ to the right side and we may multiply each side by this series equal to sin. x and there will become :

$$\begin{array}{llllllll}
\frac{x}{2}=\frac{1}{2}\alpha x & +\alpha Ax^3 & +\alpha Bx^5 & +\alpha Cx^7 & +\alpha Dx^9 & +\alpha Ex^{11} & +\alpha Fx^{13} & +\text{etc.}\\
 & -\frac{1}{2}\beta & -\beta A & -\beta B & -\beta C & -\beta D & -\beta E & \\
 & & +\frac{1}{2}\gamma & +\gamma A & +\gamma B & +\gamma C & +\gamma D & \\
 & & & -\frac{1}{2}\delta & -\delta A & -\delta B & -\delta C & \\
 & & & & +\frac{1}{2}\varepsilon & +\varepsilon A & +\varepsilon B & \\
 & & & & & -\frac{1}{2}\zeta & -\zeta A & \\
 & & & & & & +\frac{1}{2}\eta &
\end{array}$$

8. Because this equality must remain, whatever value may be attributed to the letter x , the individual powers of this must be able to cancel each other amongst themselves separately. Indeed in the first place the terms containing x on account of $\alpha=1$ at once remove each other, the remaining powers on account of $\alpha=1$ will give the following determinations

$$\begin{aligned}
A&=\tfrac{1}{2}\beta,\\
B&=\beta A-\tfrac{1}{2}\gamma,\\
C&=\beta B-\gamma A+\tfrac{1}{2}\delta,\\
D&=\beta C-\gamma B+\delta A-\tfrac{1}{2}\varepsilon,\\
E&=\beta D-\gamma C+\delta B-\varepsilon A+\tfrac{1}{2}\zeta\\
&\text{etc.}
\end{aligned}$$

Therefore with the aid of such formulas the sums of any of the other equal powers will be able to be assigned.

9. But with the sum of this series found

$$s=1-\frac{1}{2^{2i}}+\frac{1}{3^{2i}}-\frac{1}{4^{2i}}+\frac{1}{5^{2i}}-\text{etc.}$$

from that also the sums of the related series will be able to be defined :

$$t=1+\frac{1}{3^{2i}}+\frac{1}{5^{2i}}+\frac{1}{7^{2i}}+\frac{1}{9^{2i}}+\text{etc.}$$

and

$$u = 1 + \frac{1}{2^{2i}} + \frac{1}{3^{2i}} + \frac{1}{4^{2i}} + \frac{1}{5^{2i}} + \text{etc.}$$

Indeed since there shall be

$$t = u\left(1 - \frac{1}{2^{2i}}\right) = \frac{2^{2i}-1}{2^{2i}} u$$

and

$$s = u\left(1 - \frac{2}{2^{2i}}\right) = \frac{2^{2i}-2}{2^{2i}} u,$$

there will be

$$u = \frac{2^{2i} s}{2^{2i}-2}$$

and hence

$$t = \frac{2^{2i}-1}{2^{2i}-2} s;$$

but in the following the sums of these series also will be elicited at once from our general series.

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10. So that if here also the pairs of terms may be contracted, this series may arise

$$\frac{\pi}{n \text{tang.} \frac{m\pi}{n}} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm} - \text{etc.}$$

Here again we may put $m = 1$ and with the division made by 2 we will have

$$\frac{1}{nn-1} + \frac{1}{4nn-1} + \frac{1}{9nn-1} + \frac{1}{16nn-1} + \frac{1}{25nn-1} + \text{etc.} = \frac{1}{2} - \frac{\pi}{2n \text{tang.} \frac{\pi}{n}}$$

Now these individual fractions may be resolved into series as above and there will be :

$$\frac{1}{nn-1} = \frac{1}{nn} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}} + \frac{1}{n^{12}} + \text{etc.},$$
$$\frac{1}{4nn-1} = \frac{1}{4nn} + \frac{1}{4^2 n^4} + \frac{1}{4^3 n^6} + \frac{1}{4^4 n^8} + \frac{1}{4^5 n^{10}} + \frac{1}{4^6 n^{12}} + \text{etc.},$$
$$\frac{1}{9nn-1} = \frac{1}{9nn} + \frac{1}{9^2 n^4} + \frac{1}{9^3 n^6} + \frac{1}{9^4 n^8} + \frac{1}{9^5 n^{10}} + \frac{1}{9^6 n^{12}} + \text{etc.},$$
$$\frac{1}{16nn-1} = \frac{1}{16nn} + \frac{1}{16^2 n^4} + \frac{1}{16^3 n^6} + \frac{1}{16^4 n^8} + \frac{1}{16^5 n^{10}} + \frac{1}{16^6 n^{12}} + \text{etc.}$$

etc.

11. Now therefore, so that we may make as above, we may collect the sum by the vertical columns, which in the end we may put in place

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}=\mathfrak{A}\pi^2,$$
$$1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\text{etc.}=\mathfrak{B}\pi^4,$$
$$1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\text{etc.}=\mathfrak{C}\pi^6,$$
$$1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\text{etc.}=\mathfrak{D}\pi^8$$
etc.

With which in place our equation will be

$$\frac{1}{2}-\frac{\pi}{2n\text{tang.}\frac{\pi}{n}}=\frac{\mathfrak{A}\pi^2}{n^2}+\frac{\mathfrak{B}\pi^4}{n^4}+\frac{\mathfrak{C}\pi^6}{n^6}+\frac{\mathfrak{D}\pi^8}{n^8}+\text{etc.}$$

12. Now we may make $\frac{\pi}{n}=x$, with which agreed on both the letters π and n likewise may be eliminated from the calculation, and there will be

$$\frac{1}{2}-\frac{x}{2\text{tang.}x}=\mathfrak{A}xx+\mathfrak{B}x^4+\mathfrak{C}x^6+\mathfrak{D}x^8+\mathfrak{E}x^{10}+\text{etc.},$$

where in place of this series for the sake of brevity we may write the letter s, so that there shall be

$$s=\frac{1}{2}-\frac{x}{2\text{tang.}x}=\frac{\sin.x-x\cos.x}{2\sin.x},$$

which equation multiplied by sin. x presents

$$s\sin.x=\frac{1}{2}\sin.x-\frac{1}{2}x\cos.x.$$

13. Now we may put in place as in the preceding expansion :

$$\sin.x=\alpha x-\beta x^3+\gamma x^5-\delta x^7+\varepsilon x^9-\text{etc.}\ ,$$

with there being

$$\alpha=1,\ \beta=\frac{1}{1\cdot2\cdot3},\ \gamma=\frac{1}{1 2\cdot3 4\cdot5},\ \delta=\frac{1}{1\cdot2\cdots7}\ \text{ etc.}$$

Now since there is

$$\cos.x=1-\frac{xx}{1\cdot2}+\frac{x^4}{1\cdot2\cdots4}-\frac{x^6}{1\cdot2\cdots6}+\frac{x^8}{1\cdot2\cdots8}-\text{etc.},$$

there will be

$$\cos.x=\alpha-3\beta xx+5\gamma x^4-7\delta x^6+9\varepsilon x^8-\text{etc.}$$

But then there will be

$$\tfrac{1}{2}\sin.x - \tfrac{1}{2}\cos.x = \beta x^3 - 2\gamma x^5 + 3\delta x^7 - 4\varepsilon x^9 + 5\zeta x^{11} - \text{etc.},$$

to which expression therefore the formula $s \sin. x$ must be equal.

14. Therefore in turn we may multiply the two series indicated by s and $\sin. x$ and the product will be found

$$\begin{array}{llllll} s\sin.x = \alpha\mathfrak{A}x^3 & +\alpha\mathfrak{B}x^5 & +\alpha\mathfrak{C}x^7 & +\alpha\mathfrak{D}x^9 & +\alpha\mathfrak{E}x^{11} & +\alpha\mathfrak{F}x^{13} + \text{etc.} \\ & -\beta\mathfrak{A} & -\beta\mathfrak{B} & -\beta\mathfrak{C} & -\beta\mathfrak{D} & -\beta\mathfrak{E} \\ & & +\gamma\mathfrak{A} & +\gamma\mathfrak{B} & +\gamma\mathfrak{C} & +\gamma\mathfrak{D} \\ & & & -\delta\mathfrak{A} & -\delta\mathfrak{B} & -\delta\mathfrak{C} \\ & & & & +\varepsilon\mathfrak{A} & +\varepsilon\mathfrak{B} \\ & & & & & -\zeta\mathfrak{A} \end{array}$$

which expression must be equal to the preceding one.

15. Therefore the individual powers of x separately amongst themselves may be equal and thence the following determinations may be formed

$$\begin{aligned} &\mathfrak{A} = \beta, \\ &\mathfrak{B} = \beta\mathfrak{A} - 2\gamma, \\ &\mathfrak{C} = \beta\mathfrak{B} - \gamma\mathfrak{A} + 3\delta, \\ &\mathfrak{D} = \beta\mathfrak{C} - \gamma\mathfrak{B} + \delta\mathfrak{A} - 4\varepsilon, \\ &\mathfrak{E} = \beta\mathfrak{D} - \gamma\mathfrak{C} + \delta\mathfrak{B} - \varepsilon\mathfrak{A} + 5\zeta \\ &\mathfrak{F} = \beta\mathfrak{E} - \gamma\mathfrak{D} + \delta\mathfrak{C} - \varepsilon\mathfrak{B} + \zeta\mathfrak{A} - 6\eta \\ &\qquad\qquad \text{etc.} \end{aligned}$$

16. Although with the aid of these formulas the determination of the coefficients $\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D}$ etc., as far as it will be allowed, can be continued, yet from the same principles other relations between these coefficients can be derived, by which the calculation will be raised somewhat [see E130]. Clearly we may resume the equation

$$\tfrac{1}{2} - \frac{x}{2\text{tang.}x} = s,$$

from which there becomes

$$\frac{x}{2\text{tang.}x} = s - \tfrac{1}{2}$$

and hence again

$$\frac{1-2s}{x} = \cot.x,$$

which cotangent may be established $= t$, so that there shall be

$$t = \frac{1-2s}{x};$$

therefore in place of s with the series substituted there will become

$$t = \frac{1}{x} - 2\mathfrak{A}x - 2\mathfrak{B}x^3 - 2\mathfrak{C}x^5 - 2\mathfrak{D}x^7 - \text{etc.}$$

17. Therefore since we will have put $\cot.x = t$ and thus

$$x = \text{Acot}.t,$$

there will be by differentiation

$$dx = -\frac{dt}{1+tt}$$

and hence

$$dt + dx(1+tt) = 0$$

or

$$\frac{dt}{dx} + 1 + tt = 0.$$

Truly there is

$$\frac{dt}{dx} = -\frac{1}{xx} - 2\mathfrak{A} - 6\mathfrak{B}xx - 10\mathfrak{C}x^4 - 14\mathfrak{D}x^6 - 18\mathfrak{E}x^8 - 22\mathfrak{F}x^{10} - \text{etc.};$$

truly besides there is found

$$\begin{aligned} 1 + tt = \frac{1}{xx} &- 4\mathfrak{A} - 4\mathfrak{B}xx - 4\mathfrak{C}x^4 - 4\mathfrak{D}x^6 - 4\mathfrak{E}x^8 - 4\mathfrak{F}x^{10} - \text{etc.} \\ &+1 \;+ 4\mathfrak{A}\mathfrak{A} \;+ 8\mathfrak{A}\mathfrak{B} \;+ 8\mathfrak{A}\mathfrak{C} + 8\mathfrak{A}\mathfrak{D} + 8\mathfrak{A}\mathfrak{E} \\ &\qquad\qquad\qquad\quad + 4\mathfrak{B}\mathfrak{B} + 8\mathfrak{B}\mathfrak{C} + 8\mathfrak{B}\mathfrak{D} \\ &\qquad\qquad\qquad\qquad\qquad\qquad + 4\mathfrak{C}\mathfrak{C} \end{aligned}$$

18. Therefore in the equality

$$\frac{dt}{dx} + 1 + tt = 0$$

the first members at once cancel out; but from the following the following determinations are gathered together :

$$\mathfrak{A} = \frac{1}{6},$$
$$\mathfrak{B} = \frac{2}{5}\mathfrak{A}\mathfrak{A},$$
$$\mathfrak{C} = \frac{2}{7}2\mathfrak{A}\mathfrak{B},$$
$$\mathfrak{D} = \frac{2}{9}(2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}),$$
$$\mathfrak{E} = \frac{2}{11}(2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}),$$
$$\mathfrak{F} = \frac{2}{13}(2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}),$$
$$\mathfrak{G} = \frac{2}{15}(2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D}),$$
$$\mathfrak{H} = \frac{2}{17}(2\mathfrak{A}\mathfrak{G} + 2\mathfrak{B}\mathfrak{F} + 2\mathfrak{C}\mathfrak{E} + \mathfrak{D}\mathfrak{D})$$

etc.

19. Now at one time in my *Introductione in analysin infinitorum* from these formulas I calculated long enough the values of these letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. etc., then truly I have continued to some longer terms, which values therefore I may append here:

$\mathfrak{A} = \frac{1}{1\cdot2\cdot3}\cdot\frac{1}{1} = \frac{1}{6}$ for the second power,

$\mathfrak{B} = \frac{2^2}{1\cdot2\cdots5}\cdot\frac{1}{3} = \frac{1}{90}$ . . . fourth,

$\mathfrak{C} = \frac{2^4}{1\cdot2\cdots7}\cdot\frac{1}{3} = \frac{1}{945}$ . . . sixth,

$\mathfrak{D} = \frac{2^6}{1\cdot2\cdots9}\cdot\frac{3}{5} = \frac{1}{9450}$ . . . eighth,

$\mathfrak{E} = \frac{2^8}{1\cdot2\cdots11}\cdot\frac{5}{3} = \frac{1}{93555}$ . . . tenth,

$\mathfrak{F} = \frac{2^{10}}{1\cdot2\cdots13}\cdot\frac{5}{3} = \frac{691}{105}$ . . . twelfeth,

$\mathfrak{G} = \frac{2^{12}}{1\cdot2\cdots15}\cdot\frac{35}{1}$ . . . fourteenth,

$\mathfrak{H} = \frac{2^{14}}{1\cdot2\cdots17}\cdot\frac{3617}{15}$ . . . sixteenth,

$\mathfrak{I} = \frac{2^{16}}{1\cdot2\cdots19}\cdot\frac{43867}{21}$ . . . eighteenth,

$\mathfrak{K} = \frac{2^{18}}{1\cdot2\cdots21}\cdot\frac{1222277}{55}$ . . . twentieth,

$\mathfrak{L} = \frac{2^{20}}{1\cdot2\cdots23}\cdot\frac{854513}{3}$ . . . twenty-second,

$\mathfrak{M} = \frac{2^{22}}{1\cdot2\cdots25}\cdot\frac{1181820455}{273}$ . . . twenty-fourth,

$\mathfrak{N} = \frac{2^{24}}{1\cdot2\cdots27}\cdot\frac{76977927}{1}$ . . . twenty-sixth,

$\mathfrak{O} = \frac{2^{26}}{1\cdot2\cdots29}\cdot\frac{23749461029}{15}$ . . . twenty-eighth,

$\mathfrak{P} = \frac{2^{28}}{1\cdot2\cdots31}\cdot\frac{8615841276005}{231}$ . . . thirtieth,

$\mathfrak{Q} = \frac{2^{30}}{1\cdot2\cdots33}\cdot\frac{84802531453387}{85}$ . . . thirty-second,

$\mathfrak{R} = \frac{2^{32}}{1\cdot2\cdots35}\cdot\frac{90219075042845}{3}$ . . . thirty-fourth.

PREPARATION OF THE GENERAL FORMULAS FOR OTHER USES

20. Hitherto we have put $m = 1$, but now we may put $m = \frac{n-1}{2}$ and there will become

$$\frac{m\pi}{n} = \frac{(n-1)\pi}{2n} = \frac{1}{2}\pi - \frac{\pi}{2n},$$

from which there becomes

$$\sin.\frac{m\pi}{n} = \cos.\frac{\pi}{2n} \text{ and tang.}\frac{m\pi}{n} = \cot.\frac{\pi}{2n}.$$

Moreover these series themselves thus will be had :

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}} = \frac{1}{n-1} - \frac{1}{3n-1} + \frac{1}{5n-1} - \frac{1}{7n-1} + \frac{1}{9n-1} - \text{etc.}$$

$$+\frac{1}{n+1} - \frac{1}{3n+1} + \frac{1}{5n+1} - \frac{1}{7n+1} + \frac{1}{9n+1} - \text{etc.},$$

$$\frac{\pi}{2n\cot.\frac{\pi}{2n}} = \frac{1}{n-1} + \frac{1}{3n-1} + \frac{1}{5n-1} + \frac{1}{7n-1} + \frac{1}{9n-1} + \text{etc.}$$

$$-\frac{1}{n+1} - \frac{1}{3n+1} - \frac{1}{5n+1} - \frac{1}{7n+1} - \frac{1}{9n+1} - \text{etc.},$$

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21. Here also we may contract the two analogous terms and this series will be produced

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}} = \frac{2n}{nn-1} - \frac{6n}{9nn-1} + \frac{10n}{25nn-1} - \frac{14n}{49nn-1} + \text{etc.}$$

or

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}} = \frac{n}{nn-1} - \frac{3n}{9nn-1} + \frac{5n}{25nn-1} - \frac{7n}{49nn-1} + \text{etc.}$$

22. Therefore here all these fractions will be contained in this general form

$$\frac{in}{i^2n^2-1},$$

in which place i denotes all the odd numbers. Moreover this fraction applied to the general series provides

$$\frac{1}{in} + \frac{1}{i^3n^3} + \frac{1}{i^5n^5} + \frac{1}{i^7n^7} + \frac{1}{i^9n^9} + \text{etc.}$$

Hence therefore we may expand the individual fractions by the series

$$\frac{n}{nn-1} = \frac{1}{n} + \frac{1}{n^3} + \frac{1}{n^5} + \frac{1}{n^7} + \frac{1}{n^9} + \text{etc.},$$

$$-\frac{3n}{9nn-1} = -\frac{1}{3n} - \frac{1}{3^3n^3} - \frac{1}{3^5n^5} - \frac{1}{3^7n^7} - \frac{1}{3^9n^9} - \text{etc.},$$

$$\frac{5n}{25nn-1} = \frac{1}{5n} + \frac{1}{5^3n^3} + \frac{1}{5^5n^5} + \frac{1}{5^7n^7} + \frac{1}{5^9n^9} + \text{etc.},$$

$$-\frac{7n}{49nn-1} = -\frac{1}{7n} - \frac{1}{7^3n^3} - \frac{1}{7^5n^5} - \frac{1}{7^7n^7} - \frac{1}{7^9n^9} - \text{etc.},$$

etc.,

of which therefore the sum of all the series is

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}}.$$

23. Now also we may collect these series together by vertical columns and we may put in place

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.}=\mathfrak{a}\frac{\pi}{2},$$
$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}=\mathfrak{b}\frac{\pi^3}{2^3},$$
$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}=\mathfrak{c}\frac{\pi^5}{2^5},$$
$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\frac{1}{11^7}+\text{etc.}=\mathfrak{d}\frac{\pi^7}{2^7},$$
$$\text{etc.},$$

with which in place our equation will become

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}}=\frac{\mathfrak{a}\pi}{2n}+\mathfrak{b}\frac{\pi^3}{2^3n^3}+\mathfrak{c}\frac{\pi^5}{2^5n^5}+\mathfrak{d}\frac{\pi^7}{2^7n^7}+\text{etc.}$$

24. Now we may put $\frac{\pi}{2n}=x$ and our equation will adopt this form

$$\frac{x}{2\cos.x}=\mathfrak{a}x+\mathfrak{b}x^3+\mathfrak{c}x^5+\mathfrak{d}x^7+\mathfrak{e}x^9+\text{etc.}$$

Now therefore, if for the sake of brevity we may put

$$\cos.x=\alpha-\beta xx+\gamma x^4-\delta x^6+\varepsilon x^8-\text{etc.},$$

thus so that there shall be

$$\alpha=1,\ \beta=\frac{1}{1\cdot 2},\ \gamma=\frac{1}{1\cdot 2\cdot 3\cdot 4},\ \delta=\frac{1}{1\cdot 2\cdots 6}\ \text{etc.},$$

if we may multiply each side [of the above equation] by this series, this equation will arise

$$\begin{aligned}\tfrac{x}{2} = \alpha\mathfrak{a}x &+ \alpha\mathfrak{b}x^3 + \alpha\mathfrak{c}x^5 + \alpha\mathfrak{d}x^7 + \alpha\mathfrak{e}x^9 + \alpha\mathfrak{f}x^{11} + \alpha\mathfrak{g}x^{13} + \text{etc.}\\
&-\beta\mathfrak{a} \quad -\beta\mathfrak{b} \quad -\beta\mathfrak{c} \quad -\beta\mathfrak{d} \quad -\beta\mathfrak{e} \quad -\beta\mathfrak{f}\\
&\qquad +\gamma\mathfrak{a} \quad +\gamma\mathfrak{b} \quad +\gamma\mathfrak{c} \quad +\gamma\mathfrak{d} \quad +\gamma\mathfrak{e}\\
&\qquad\qquad -\delta\mathfrak{a} \quad -\delta\mathfrak{b} \quad -\delta\mathfrak{c} \quad -\delta\mathfrak{d}\\
&\qquad\qquad\qquad +\varepsilon\mathfrak{a} \quad +\varepsilon\mathfrak{b} \quad +\varepsilon\mathfrak{c}\\
&\qquad\qquad\qquad\qquad -\zeta\mathfrak{a} \quad -\varepsilon\mathfrak{b}\\
&\qquad\qquad\qquad\qquad\qquad +\eta\mathfrak{a}\end{aligned}$$

25. Therefore with the individual powers reduced to zero we will obtain the following determinations:

$$\begin{aligned}\mathfrak{a} &= \tfrac{1}{2},\\
\mathfrak{b} &= \beta\mathfrak{a},\\
\mathfrak{c} &= \beta\mathfrak{b} - \gamma\mathfrak{a},\\
\mathfrak{d} &= \beta\mathfrak{c} - \gamma\mathfrak{b} + \delta\mathfrak{a},\\
\mathfrak{e} &= \beta\mathfrak{d} - \gamma\mathfrak{c} + \delta\mathfrak{b} - \varepsilon\mathfrak{a},\\
\mathfrak{f} &= \beta\mathfrak{e} - \gamma\mathfrak{d} + \delta\mathfrak{c} - \varepsilon\mathfrak{b} + \zeta\mathfrak{a}\\
&\text{etc.}\end{aligned}$$

26. Now at one time I have shown the sums of these series with the aid of these formulas [See Euler's *Foundations of Differential Calculus*, § 122-125], from which the values for the letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc. presented thus will be found to be determined :

$\mathfrak{a} = \frac{1}{2}$ for the first powers,

$\mathfrak{b} = \frac{1}{1\cdot 2}\cdot\frac{1}{2}$ · · · third,

$\mathfrak{c} = \frac{5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{2}$ · · · fifth,

$\mathfrak{d} = \frac{61}{1\cdot 2\cdots 6}\cdot\frac{1}{2}$ · · · seventh,

$\mathfrak{e} = \frac{1385}{1\cdot 2\cdots 8}\cdot\frac{1}{2}$ · · · ninth,

$\mathfrak{f} = \frac{50521}{1\cdot 2\cdots 10}\cdot\frac{1}{2}$ · · · eleventh,

$\mathfrak{g} = \frac{2702765}{1\cdot 2\cdots 12}\cdot\frac{1}{2}$ · · · thirteenth,

$\mathfrak{h} = \frac{199360981}{1\cdot 2\cdots 14}\cdot\frac{1}{2}$ · · · fifteenth,

$\mathfrak{i} = \frac{19391512145}{1\cdot 2\cdots 16}\cdot\frac{1}{2}$ · · · seventeenth,

$\mathfrak{k} = \frac{2404879675441}{1\cdot 2\cdots 18}\cdot\frac{1}{2}$ · · · nineteenth,

27. Hence therefore we may put in place the sums of these series as far as to the twentieth power

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}=\frac{1}{1}\cdot\frac{\pi}{2^2},$$
$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.}=\frac{1}{1\cdot 2}\cdot\frac{\pi^3}{2^4},$$
$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=\frac{5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^5}{2^6},$$
$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=\frac{61}{1\cdot 2\cdots 6}\cdot\frac{\pi^7}{2^8},$$
$$1-\frac{1}{3^9}+\frac{1}{5^9}-\frac{1}{7^9}+\frac{1}{9^9}-\text{etc.}=\frac{1385}{1\cdot 2\cdots 8}\cdot\frac{\pi^9}{2^{10}},$$
$$1-\frac{1}{3^{11}}+\frac{1}{5^{11}}-\frac{1}{7^{11}}+\frac{1}{9^{11}}-\text{etc.}=\frac{50521}{1\cdot 2\cdots 10}\cdot\frac{\pi^{11}}{2^{12}},$$
$$1-\frac{1}{3^{13}}+\frac{1}{5^{13}}-\frac{1}{7^{13}}+\frac{1}{9^{13}}-\text{etc.}=\frac{2702765}{1\cdot 2\cdots 12}\cdot\frac{\pi^{13}}{2^{14}},$$
$$1-\frac{1}{3^{15}}+\frac{1}{5^{15}}-\frac{1}{7^{15}}+\frac{1}{9^{15}}-\text{etc.}=\frac{199360981}{1\cdot 2\cdots 14}\cdot\frac{\pi^{15}}{2^{16}},$$
$$1-\frac{1}{3^{17}}+\frac{1}{5^{17}}-\frac{1}{7^{17}}+\frac{1}{9^{17}}-\text{etc.}=\frac{19391512145}{1\cdot 2\cdots 16}\cdot\frac{\pi^{17}}{2^{18}},$$
$$1-\frac{1}{3^{19}}+\frac{1}{5^{19}}-\frac{1}{7^{19}}+\frac{1}{9^{19}}-\text{etc.}=\frac{2404879675441}{1\cdot 2\cdots 18}\cdot\frac{\pi^{19}}{2^{20}},$$
etc.

EXPANSION OF THE LATTER SERIES § 20

28. With the analogous pairs of terms contracted this series will adopt this form

$$\frac{\pi}{4n\cot.\frac{\pi}{2n}}=\frac{1}{nn-1}+\frac{1}{9nn-1}+\frac{1}{25nn-1}+\frac{1}{49nn-1}+\text{etc.},$$

which fractions expanded in series will give :

$$\frac{1}{nn-1}=\frac{1}{nn}+\frac{1}{n^4}+\frac{1}{n^6}+\frac{1}{n^8}+\frac{1}{n^{10}}+\text{etc.},$$
$$\frac{1}{9nn-1}=\frac{1}{9nn}+\frac{1}{9^2n^4}+\frac{1}{9^3n^6}+\frac{1}{9^4n^8}+\frac{1}{9^5n^{10}}+\text{etc.},$$
$$\frac{1}{25nn-1}=\frac{1}{25n}+\frac{1}{25^2n^4}+\frac{1}{25^3n^6}+\frac{1}{25^4n^8}+\frac{1}{25^5n^{10}}+\text{etc.},$$
etc.,

of which the sum of all is

$$\frac{\pi}{4n}\text{tang.}\frac{\pi}{2n}.$$

29. So that now we may be able to gather these series vertically, we may establish

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{etc.}=\mathfrak{A}'\frac{\pi\pi}{2^2},$$
$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.}=\mathfrak{B}'\frac{\pi^4}{2^4},$$
$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}=\mathfrak{C}'\frac{\pi^6}{2^6},$$
etc.

From which our equation will become

$$\frac{\pi}{4n}\text{tang.}\frac{\pi}{2n}=\frac{\mathfrak{A}'\pi^2}{2^2n^2}+\frac{\mathfrak{B}'\pi^4}{2^4n^4}+\frac{\mathfrak{C}'\pi^6}{2^6n^6}+\frac{\mathfrak{D}'\pi^8}{2^8n^8}+\text{etc.}$$

30. Now we may put $\frac{\pi}{2n}=x$, so that our equation may become

$$\frac{x}{2}\text{tang.}x=\mathfrak{A}'xx+\mathfrak{B}'x^4+\mathfrak{C}'x^6+\text{etc.},$$

from which there will be

$$\text{tang.}x=2\mathfrak{A}'x+2\mathfrak{B}'x^3+2\mathfrak{C}'x^5+2\mathfrak{D}'x^7+\text{etc.};$$

in place of which series we may write the letter *t*, so that there shall be $\text{tang.}x=t$ and hence on differentiating

$$dx=\frac{dt}{1+tt},$$

and thus we will have

$$\frac{dt}{dx}=1+tt.$$

Truly there is

$$\frac{dt}{dx}=2\mathfrak{A}'+2\cdot3\mathfrak{B}'xx+2\cdot5\mathfrak{C}'x^4+2\cdot7\mathfrak{D}'x^6+\text{etc.}$$

31. With the expansion made in the same manner there will be

$$\begin{aligned}1+tt=1+4\mathfrak{A}'\mathfrak{A}'xx+8\mathfrak{A}'\mathfrak{B}'x^4&+8\mathfrak{A}'\mathfrak{C}'x^6+8\mathfrak{A}'\mathfrak{D}'x^8+\text{etc.}\\+4\mathfrak{B}'\mathfrak{B}'&\quad+8\mathfrak{B}'\mathfrak{C}'\end{aligned}$$

Hence the following determinations may be deduced :

$$\mathfrak{A}' = \tfrac{1}{2},$$
$$\mathfrak{B}' = \tfrac{2}{3}\cdot \mathfrak{A}'\mathfrak{A}',$$
$$\mathfrak{C}' = \tfrac{2}{5}\cdot 2\mathfrak{A}'\mathfrak{B}',$$
$$\mathfrak{D}' = \tfrac{2}{7}(2\mathfrak{A}'\mathfrak{C}' + \mathfrak{B}'\mathfrak{B}'),$$
$$\mathfrak{E}' = \tfrac{2}{9}(2\mathfrak{A}'\mathfrak{D}' + 2\mathfrak{B}'\mathfrak{C}'),$$
$$\mathfrak{F}' = \tfrac{2}{11}(2\mathfrak{A}'\mathfrak{E}' + 2\mathfrak{B}'\mathfrak{D}' + \mathfrak{C}'\mathfrak{C}'),$$
$$\mathfrak{G}' = \tfrac{2}{13}(2\mathfrak{A}'\mathfrak{F}' + 2\mathfrak{B}'\mathfrak{E}' + 2\mathfrak{C}'\mathfrak{D}')$$
etc.

32. These determinations agree almost completely with these, which we have found above in §18 for the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc.; indeed the whole distinction is found in the numerical coefficients. Truly by no means is there a need to compute these values separately, since these now may be able to be deduced from the above easily. Since indeed there shall be

$$\frac{\mathfrak{A}'}{2^2} = \frac{\mathfrak{A}(2^2-1)}{2^2},$$

there will be

$$\mathfrak{A}' = (2^2-1)\mathfrak{A}.$$

In a similar manner there will be

$$\mathfrak{B}' = (2^4-1)\mathfrak{B},\ \mathfrak{C}' = (2^6-1)\mathfrak{C},\ \mathfrak{D}' = (2^8-1)\mathfrak{D} \text{ etc.}$$

CONCLUSION

33. For the sake of these, who may have wished to express these completely numerical values by decimal fractions, we may attach the following table, in which all the powers of π have been expanded in decimal fractions, where in place of $\frac{\pi}{2}$ we have written q.

$$\frac{q}{1} = 1,57079\ 63267\ 94896\ 6192313216\ 916,$$

$$\frac{q^2}{1\cdot 2} = 1,23370\ 0550136169\ 82735\ 43113\ 750,$$

$$\frac{q^3}{1\cdot 2\cdot 3} = 0,64596\ 40975\ 06246\ 25365\ 57565\ 639,$$

$$\frac{q^4}{1\cdot 2\cdot 3\cdot 4} = 0,25366\ 95079\ 01048\ 01363\ 65633\ 664,$$

$$\frac{q^5}{1\cdot 2\cdots 5} = 0,07969\ 26262\ 46167\ 04512\ 05055\ 495,$$

$$\frac{q^6}{1\cdot 2\cdots 6} = 0,02086\ 34807\ 63352\ 96087\ 30516\ 372,$$

$$\frac{q^7}{1\cdot 2\cdots 7} = 0,004681754135318\ 68810\ 06854\ 639,$$

$$\frac{q^8}{1\cdot 2\cdots 8} = 0,000919260274839\ 42658\ 02417\ 162,$$

$$\frac{q^9}{1\cdot 2\cdots 9} = 0,000160441184787\ 3598218726\ 609,$$

$$\frac{q^{10}}{1\cdot 2\cdots 10} = 0,0000252020423730\ 6060\ 54810530,$$

$$\frac{q^{11}}{1\cdot 2\cdots 11} = 0,00000\ 35988\ 43235\ 21208\ 53404\ 685,$$

$$\frac{q^{12}}{1\cdot 2\cdots 12} = 0,0000004710\ 87477\ 8818171503670,$$

$$\frac{q^{13}}{1\cdot 2\cdots 13} = 0,00000\ 00569\ 21729\ 21967\ 92681178,$$

$$\frac{q^{14}}{1\cdot 2\cdots 14} = 0,00000\ 0006386603\ 0837918522\ 411,$$

$$\frac{q^{15}}{1\cdot 2\cdots 15} = 0,0000000006\ 68803\ 5109811467\ 232,$$

$$\frac{q^{16}}{1\cdot 2\cdots 16} = 0,0000000000\ 65659\ 63114\ 97947\ 236,$$

$$\frac{q^{17}}{1\cdot 2\cdots 17} = 0,00000\ 00000\ 06066\ 9357311061957,$$

$$\frac{q^{18}}{1\cdot 2\cdots 18} = 0,00000\ 00000\ 00529\ 44002\ 00734\ 624,$$

$$\frac{q^{19}}{1\cdot 2\cdots 19} = 0,00000\ 00000\ 00043\ 77065\ 46731374,$$

$$\frac{q^{20}}{1\cdot 2\cdots 20} = 0,00000\ 00000\ 00003\ 4377391790\ 986,$$

$$\frac{q^{21}}{1\cdot 2\cdots 21} = 0,00000\ 00000\ 00000\ 25714\ 22892\ 860,$$

$$\frac{q^{22}}{1\cdot 2\cdots 22} = 0,00000\ 00000\ 00000\ 0183599165\ 216,$$

$$\frac{q^{23}}{1\cdot 2\cdots 23} = 0,00000\ 00000\ 00000\ 0012538995\ 405,$$

$$\frac{q^{24}}{1\cdot 2\cdots 24} = 0,00000\,00000\,00000\,00008\,20675\,330,$$
$$\frac{q^{25}}{1\cdot 2\cdots 25} = 0,00000\,00000\,00000\,00000\,51564\,552,$$
$$\frac{q^{26}}{1\cdot 2\cdots 26} = 0,00000\,00000\,00000\,00000\,03115\,285,$$
$$\frac{q^{27}}{1\cdot 2\cdots 27} = 0,00000\,00000\,00000\,00000\,00181\,240,$$
$$\frac{q^{28}}{1\cdot 2\cdots 28} = 0,00000\,00000\,00000\,00000\,00010\,168,$$
$$\frac{q^{29}}{1\cdot 2\cdots 29} = 0,00000\,00000\,00000\,00000\,00000\,551,$$
$$\frac{q^{30}}{1\cdot 2\cdots 30} = 0,00000\,00000\,00000\,00000\,00000\,029,$$
$$\frac{q^{31}}{1\cdot 2\cdots 31} = 0,00000\,00000\,00000\,00000\,00000\,001.$$

Indeed these powers have been divided by certain numbers, but which generally are these themselves, by which the same powers of π occur divided in the previous formulas, from which the expansion in decimal fractions there is rendered easier.

DE SERIEBUS POTESTATUM RECIPROCIS

METHODO NOVA ET FACILLIMA SUMMANDIS

[E597]

Opuscula analytica 2, 1785, p. 257-274

1. Cum primum summas harum serierum docuissem, eas ex hoc principio deduxi, quod cuique sinui et cosinui innumerabiles arcus circulares respondent, qui omnes sint radices aequationum infinitarum, quibus arcus per sinum vel cosinum exprimi solent. Hinc enim ex coefficientibus istarum aequationum non solum summas ipsarum radicum, sed etiam earum potestatum quarumcunque assignavi. Postea vero easdem summas etiam ex aliis principiis derivavi, quae autem omnia memorata circuli proprietate innitebantur. Nunc vero observavi istas summas ex alio principio multo simpliciori et solis operationibus analyticis innixo deduci posse, quam methodum hic accuratius exposuisse iuvabit.

2. Hoc autem principium mihi suppeditavit integratio huius formulae

$$\int\left(\frac{z^{m-1}\pm z^{n-m-1}}{1\pm z^{n}}\right)dz$$

pro casu, quo post integrationem statuitur $z=1$. Ostendi enim in Tomo XIX Nov. Comment. per solitas integrationum operationes haec integralia sequenti modo exprimi

$$\int\left(\frac{z^{m-1}+z^{n-m-1}}{1+z^{n}}\right)dz=\frac{\pi}{n\sin.\frac{m\pi}{n}}$$

et

$$\int\left(\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}\right)dz=\frac{\pi}{n\operatorname{tang}.\frac{m\pi}{n}}.$$

Quodsi vero eaedem formulae per series infinitas evolvantur, posito $z=1$ erit

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}-\frac{1}{n+m}+\frac{1}{2n+m}-\frac{1}{3n+m}+\frac{1}{4n+m}-\text{etc.}$$
$$\frac{1}{n-m}-\frac{1}{2n+m}+\frac{1}{3n-m}-\frac{1}{4n-m}+\text{etc.}$$

et

$$\frac{\pi}{n\operatorname{tang}.\frac{m\pi}{n}}=\frac{1}{m}+\frac{1}{n+m}+\frac{1}{2n+m}+\frac{1}{3n+m}+\frac{1}{4n+m}+\text{etc.}$$
$$-\frac{1}{n-m}-\frac{1}{2n-m}-\frac{1}{3n-m}-\frac{1}{4n-m}-\text{etc.},$$

quae duae series eo maiori attentione sunt dignae, quod in iis omnia plane continentur, quae non solum circa summationes potestatum, sed etiam circa summationes similes sunt prolata.

EVOLUTIO PRIORIS SERIEl GENERALIS

3. Consideremus primo formam priorem

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}$$

ac binis terminis analogis contractis habebimus

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}+\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}+\frac{2m}{9nn-mm}-\frac{2m}{16nn-mm}+\text{etc.}$$

Sumatur nunc, quo formulae fiant simpliciores, $m = 1$ eritque

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=1+\frac{2}{nn-1}-\frac{2}{4nn-1}+\frac{2}{9nn-1}-\frac{2}{16nn-1}+\text{etc.}$$

sive

$$\frac{\pi}{2n\sin.\frac{m\pi}{n}}-\frac{1}{2}=\frac{1}{nn-1}-\frac{1}{4nn-1}+\frac{1}{9nn-1}-\frac{1}{16nn-1}+\text{etc.}$$

4. Nunc singulas has fractiones more solito in series infinitas geometricas resolvamus eritque

$$\frac{1}{nn-1}=\frac{1}{nn}+\frac{1}{n^4}+\frac{1}{n^6}+\frac{1}{n^8}+\frac{1}{n^{10}}+\text{etc.},$$
$$-\frac{1}{4nn-1}=-\frac{1}{4nn}-\frac{1}{4^2n^4}-\frac{1}{4^3n^6}-\frac{1}{4^4n^8}-\frac{1}{4^5n^{10}}-\text{etc.},$$
$$\frac{1}{9nn-1}=\frac{1}{4nn}+\frac{1}{9^2n^4}+\frac{1}{9^3n^6}+\frac{1}{9^4n^8}+\frac{1}{9^5n^{10}}+\text{etc.},$$
$$-\frac{1}{16nn-1}=-\frac{1}{16nn}-\frac{1}{16^2n^4}-\frac{1}{16^3n^6}-\frac{1}{16^4n^8}-\frac{1}{16^5n^{10}}-\text{etc.}$$

etc.

Harum igitur serierum infinitarum omnium summa erit

$$=\frac{\pi}{2n\sin.\frac{\pi}{n}}-\frac{1}{2}.$$

5. Nunc igitur has series secundum lineas verticales colligamus, quem in finem statuamus brevitatis gratia

$$1-\tfrac{1}{4}+\tfrac{1}{9}-\tfrac{1}{16}+\tfrac{1}{25}-\text{etc.}=A\pi\pi,$$
$$1-\tfrac{1}{4^2}+\tfrac{1}{9^2}-\tfrac{1}{16^2}+\tfrac{1}{25^2}-\text{etc.}=B\pi^4,$$
$$1-\tfrac{1}{4^3}+\tfrac{1}{9^3}-\tfrac{1}{16^3}+\tfrac{1}{25^3}-\text{etc.}=C\pi^6,$$
$$1-\tfrac{1}{4^4}+\tfrac{1}{9^4}-\tfrac{1}{16^4}+\tfrac{1}{25^4}-\text{etc.}=D\pi^8$$
$$\text{etc.}$$

Hinc igitur adipiscemur sequentem aequationem

$$\frac{\pi}{2n\sin.\frac{\pi}{n}}-\frac{1}{2}=\frac{A\pi\pi}{nn}+\frac{B\pi^4}{n^4}+\frac{C\pi^6}{n^6}+\frac{D\pi^8}{n^8}+\text{etc.}$$

6. Ponamus porro brevitatis gratia $\frac{\pi}{n}=x$, ut prodeat sequens aequatio

$$\frac{x}{2\sin.x}-\frac{1}{2}=Axx+Bx^4+Cx^6+Dx^8+Ex^{10}+\text{etc.},$$

ubi iam intelligitur per debitam evolutionem omnes coefficientes assumtos *A, B, C* etc. definiri posse; quibus inventis nanciscemur summas omnium serierum in hac forma contentarum

$$1-\frac{1}{4^i}+\frac{1}{9^i}-\frac{1}{16^i}+\frac{1}{25^i}-\text{etc.}$$

sive in hac

$$1-\frac{1}{2^{2i}}+\frac{1}{3^{2i}}-\frac{1}{4^{2i}}+\frac{1}{5^{2i}}-\text{etc.}$$

denotante *i* numerum integrum quemcunque.

7. Cum iam per seriem notissimam sit

$$\sin.x=x-\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdots 5}-\frac{x^7}{1\cdot 2\cdots 7}+\text{etc. },$$

pro hac serie simpliciter scribamus

$$\sin.x=\alpha x-\beta x^3+\gamma x^5-\delta x^7+\varepsilon x^9-\text{etc. },$$

ita ut sit

$$\alpha=1,\ \beta=\frac{\alpha}{2\cdot 3},\ \gamma=\frac{\beta}{4\cdot 5},\ \delta=\frac{\gamma}{6\cdot 7},\ \varepsilon=\frac{\delta}{8\cdot 9}\ \ \text{etc.}$$

quo posito membrum $-\frac{1}{2}$ ad dextram partem transferamus atque utrinque multiplicemus per hanc seriem ipsi sin. x aequalem fietque

$$\begin{array}{llllllll}
\frac{x}{2}=\frac{1}{2}\alpha x & +\alpha Ax^3 & +\alpha Bx^5 & +\alpha Cx^7 & +\alpha Dx^9 & +\alpha Ex^{11} & +\alpha Fx^{13} & +\text{etc.}\\
 & -\frac{1}{2}\beta & -\beta A & -\beta B & -\beta C & -\beta D & -\beta E & \\
 & & +\frac{1}{2}\gamma & +\gamma A & +\gamma B & +\gamma C & +\gamma D & \\
 & & & -\frac{1}{2}\delta & -\delta A & -\delta B & -\delta C & \\
 & & & & +\frac{1}{2}\varepsilon & +\varepsilon A & +\varepsilon B & \\
 & & & & & -\frac{1}{2}\zeta & -\zeta A & \\
 & & & & & & +\frac{1}{2}\eta &
\end{array}$$

8. Quoniam haec aequalitas subsistera debet, quicunque valor litterae x tribuatur, singulae eius potestates se mutuo seorsim destruere debent. Primo quidem termini ipsum x continentes ob $\alpha=1$ sponte se tollunt, reliquae potestates ob $\alpha=1$ sequentes dant determinationes

$$\begin{aligned}
&A=\tfrac{1}{2}\beta,\\
&B=\beta A-\tfrac{1}{2}\gamma,\\
&C=\beta B-\gamma A+\tfrac{1}{2}\delta,\\
&D=\beta C-\gamma B+\delta A-\tfrac{1}{2}\varepsilon,\\
&E=\beta D-\gamma C+\delta B-\varepsilon A+\tfrac{1}{2}\zeta\\
&\qquad\qquad\text{etc.}
\end{aligned}$$

Harum igitur formularum ope summae quantumvis altarum potestatum parium assignari poterunt.

9. Inventa autem summa huius seriei

$$s=1-\frac{1}{2^{2i}}+\frac{1}{3^{2i}}-\frac{1}{4^{2i}}+\frac{1}{5^{2i}}-\text{etc.}$$

ex ea quoque serierum agnatarum istarum summae definiri poterunt

$$t=1+\frac{1}{3^{2i}}+\frac{1}{5^{2i}}+\frac{1}{7^{2i}}+\frac{1}{9^{2i}}+\text{etc.}$$

et

$$u = 1 + \frac{1}{2^{2i}} + \frac{1}{3^{2i}} + \frac{1}{4^{2i}} + \frac{1}{5^{2i}} + \text{etc.}$$

Cum enim sit

$$t = u\left(1 - \frac{1}{2^{2i}}\right) = \frac{2^{2i}-1}{2^{2i}}u$$

et

$$s = u\left(1 - \frac{2}{2^{2i}}\right) = \frac{2^{2i}-2}{2^{2i}}u,$$

erit

$$u = \frac{2^{2i}s}{2^{2i}-2}$$

hincque

$$t = \frac{2^{2i}-1}{2^{2i}-2}s;$$

in sequentibus autem harum serierum summae etiam immediate ex nostris formulis generalibus elicientur.

EVOLUTIO SERIEl GENERALIS POSTERIORIS

10. Quodsi hic etiam bini termini analogi contrahantur, orietur ista series

$$\frac{\pi}{n\,\text{tang.}\frac{m\pi}{n}} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} - \frac{2m}{16nn-mm} - \text{etc.}$$

Ponamus hic iterum $m = 1$ et facta divisione per 2 habebimus

$$\frac{1}{nn-1} + \frac{1}{4nn-1} + \frac{1}{9nn-1} + \frac{1}{16nn-1} + \frac{1}{25nn-1} + \text{etc.} = \frac{1}{2} - \frac{\pi}{2n\,\text{tang.}\frac{\pi}{n}}$$

Nunc singulae istae fractiones in series resolvantur ut supra eritque

$$\frac{1}{nn-1} = \frac{1}{nn} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}} + \frac{1}{n^{12}} + \text{etc.},$$

$$\frac{1}{4nn-1} = \frac{1}{4nn} + \frac{1}{4^2n^4} + \frac{1}{4^3n^6} + \frac{1}{4^4n^8} + \frac{1}{4^5n^{10}} + \frac{1}{4^6n^{12}} + \text{etc.},$$

$$\frac{1}{9nn-1} = \frac{1}{9nn} + \frac{1}{9^2n^4} + \frac{1}{9^3n^6} + \frac{1}{9^4n^8} + \frac{1}{9^5n^{10}} + \frac{1}{9^6n^{12}} + \text{etc.},$$

$$\frac{1}{16nn-1} = \frac{1}{16nn} + \frac{1}{16^2n^4} + \frac{1}{16^3n^6} + \frac{1}{16^4n^8} + \frac{1}{16^5n^{10}} + \frac{1}{16^6n^{12}} + \text{etc.}$$

etc.

11. Nunc igitur, ut supra fecimus, per columna verticales summam colligamus, quem in finem statuamus

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}=\mathfrak{A}\pi^2,$$
$$1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\text{etc.}=\mathfrak{B}\pi^4,$$
$$1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\text{etc.}=\mathfrak{C}\pi^6,$$
$$1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\text{etc.}=\mathfrak{D}\pi^8$$
etc.

Quibus positis aequatio nostra erit

$$\frac{1}{2}-\frac{\pi}{2n\,\text{tang.}\frac{\pi}{n}}=\frac{\mathfrak{A}\pi^2}{nn}+\frac{\mathfrak{B}\pi^4}{n^4}+\frac{\mathfrak{C}\pi^6}{n^6}+\frac{\mathfrak{D}\pi^8}{n^8}+\text{etc.}$$

12. Faciamus nunc $\frac{\pi}{n}=x$, quo pacto ambae litterae π et n simul ex calculo elidentur, eritque

$$\frac{1}{2}-\frac{x}{2\,\text{tang.}x}=\mathfrak{A}xx+\mathfrak{B}x^4+\mathfrak{C}x^6+\mathfrak{D}x^8+\mathfrak{E}x^{10}+\text{etc.},$$

ubi loco huius seriei brevitatis gratia scribamus litteram s, ut sit

$$s=\frac{1}{2}-\frac{x}{2\,\text{tang.}x}=\frac{\sin.x-x\cos.x}{2\sin.x},$$

quae aequatio per sin. x multiplicata praebet

$$s\sin.x=\tfrac{1}{2}\sin.x-\tfrac{1}{2}x\cos.x.$$

13. Statuamus nunc ut in praecedente evolutione

$$\sin.x=\alpha x-\beta x^3+\gamma x^5-\delta x^7+\varepsilon x^9-\text{etc.},$$

existente

$$\alpha=1,\ \beta=\frac{1}{1\cdot2\cdot3},\ \gamma=\frac{1}{1\cdot2\cdot3\cdot4\cdot5},\ \delta=\frac{1}{1\cdot2\cdots7}\ \text{ etc.}$$

Quia nunc est

$$\cos.x=1-\frac{xx}{1\cdot2}+\frac{x^4}{1\cdot2\cdots4}-\frac{x^6}{1\cdot2\cdots6}+\frac{x^8}{1\cdot2\cdots8}-\text{etc.},$$

erit

$$\cos.x=\alpha-3\beta xx+5\gamma x^4-7\delta x^6+9\varepsilon x^8-\text{etc.}$$

Tum autem erit

$$\tfrac{1}{2}\sin.x - \tfrac{1}{2}\cos.x = \beta x^3 - 2\gamma x^5 + 3\delta x^7 - 4\varepsilon x^9 + 5\zeta x^{11} - \text{etc.},$$

cui ergo expressioni formula s sin. x debet esse aequalis.

14. Binas igitur series per s et sin. x indicatas invicem multiplicemus et productum reperietur

$$\begin{array}{lllllll} s\sin.x = \alpha\mathfrak{A}x^3 & +\alpha\mathfrak{B}x^5 & +\alpha\mathfrak{C}x^7 & +\alpha\mathfrak{D}x^9 & +\alpha\mathfrak{E}x^{11} & +\alpha\mathfrak{F}x^{13} & +\text{etc.} \\ & -\beta\mathfrak{A} & -\beta\mathfrak{B} & -\beta\mathfrak{C} & -\beta\mathfrak{D} & -\beta\mathfrak{E} & \\ & & +\gamma\mathfrak{A} & +\gamma\mathfrak{B} & +\gamma\mathfrak{C} & +\gamma\mathfrak{D} & \\ & & & -\delta\mathfrak{A} & -\delta\mathfrak{B} & -\delta\mathfrak{C} & \\ & & & & +\varepsilon\mathfrak{A} & +\varepsilon\mathfrak{B} & \\ & & & & & -\zeta\mathfrak{A} & \end{array}$$

quae expressio praecedenti debet esse aequalis.

15. Singulae igitur potestates ipsius x seorsim inter se aequentur indeque formentur sequentes determinationes

$$\begin{aligned} &\mathfrak{A} = \beta, \\ &\mathfrak{B} = \beta\mathfrak{A} - 2\gamma, \\ &\mathfrak{C} = \beta\mathfrak{B} - \gamma\mathfrak{A} + 3\delta, \\ &\mathfrak{D} = \beta\mathfrak{C} - \gamma\mathfrak{B} + \delta\mathfrak{A} - 4\varepsilon, \\ &\mathfrak{E} = \beta\mathfrak{D} - \gamma\mathfrak{C} + \delta\mathfrak{B} - \varepsilon\mathfrak{A} + 5\zeta \\ &\mathfrak{F} = \beta\mathfrak{E} - \gamma\mathfrak{D} + \delta\mathfrak{C} - \varepsilon\mathfrak{B} + \zeta\mathfrak{A} - 6\eta \\ &\qquad\qquad \text{etc.} \end{aligned}$$

16. Quanquam ope harum formularum determinatio coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., quousque libuerit, continuari potest, tamen ex iisdem principiis aliae relationes inter hos coefficientes derivari possunt, quibus calculus haud mediocriter sublevabitur. Resumamus scilicet aequationem

$$\tfrac{1}{2} - \frac{x}{2\tan g.x} = s,$$

unde fit

$$\frac{x}{2\tan g.x} = s - \tfrac{1}{2}$$

hincque porro

$$\frac{1-2s}{x} = \cot.x,$$

quae cotangens statuatur $= t$, ut sit

$$t = \tfrac{1-2s}{x};$$

ergo loco *s* serie substituta fiet

$$t = \tfrac{1}{x} - 2\mathfrak{A}x - 2\mathfrak{B}x^3 - 2\mathfrak{C}x^5 - 2\mathfrak{D}x^7 - \text{etc.}$$

17. Cum igitur posuerimus $\cot.x = t$ ideoque

$$x = \mathrm{A}\cot.t,$$

erit differentiando

$$dx = -\tfrac{dt}{1+tt}$$

hincque

$$dt + dx(1+tt) = 0$$

sive

$$\tfrac{dt}{dx} + 1 + tt = 0.$$

Est vero

$$\tfrac{dt}{dx} = -\tfrac{1}{xx} - 2\mathfrak{A} - 6\mathfrak{B}xx - 10\mathfrak{C}x^4 - 14\mathfrak{D}x^6 - 18\mathfrak{E}x^8 - 22\mathfrak{F}x^{10} - \text{etc.};$$

praeterea vero reperitur

$$\begin{aligned} 1 + tt = \tfrac{1}{xx} &- 4\mathfrak{A} - 4\mathfrak{B}xx - 4\mathfrak{C}x^4 - 4\mathfrak{D}x^6 - 4\mathfrak{E}x^8 - 4\mathfrak{F}x^{10} - \text{etc.} \\ &+1 + 4\mathfrak{A}\mathfrak{A} + 8\mathfrak{A}\mathfrak{B} + 8\mathfrak{A}\mathfrak{C} + 8\mathfrak{A}\mathfrak{D} + 8\mathfrak{A}\mathfrak{E} \\ &\qquad + 4\mathfrak{B}\mathfrak{B} + 8\mathfrak{B}\mathfrak{C} + 8\mathfrak{B}\mathfrak{D} \\ &\qquad\qquad + 4\mathfrak{C}\mathfrak{C} \end{aligned}$$

18. In aequalitate igitur

$$\tfrac{dt}{dx} + 1 + tt = 0$$

prima membra sponte se tollunt; ex sequentibus autem colliguntur sequentes determinationes

$$\mathfrak{A} = \frac{1}{6},$$
$$\mathfrak{B} = \frac{2}{5}\mathfrak{A}\mathfrak{A},$$
$$\mathfrak{C} = \frac{2}{7}2\mathfrak{A}\mathfrak{B},$$
$$\mathfrak{D} = \frac{2}{9}(2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}),$$
$$\mathfrak{E} = \frac{2}{11}(2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}),$$
$$\mathfrak{F} = \frac{2}{13}(2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}),$$
$$\mathfrak{G} = \frac{2}{15}(2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D}),$$
$$\mathfrak{H} = \frac{2}{17}(2\mathfrak{A}\mathfrak{G} + 2\mathfrak{B}\mathfrak{F} + 2\mathfrak{C}\mathfrak{E} + \mathfrak{D}\mathfrak{D})$$

etc.

19. Ex his formulis iam olim in *Introductione* mea *in analysin infinitorum* valores istarum litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. etc. satis longe computavi, deinceps vero ad aliquot terminos longius continuavi, quos valores igitur hic apponam:

$\mathfrak{A} = \frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{1} = \frac{1}{6}$ pro potestatibus secundis,

$\mathfrak{B} = \frac{2^2}{1\cdot 2\cdots 5}\cdot\frac{1}{3} = \frac{1}{90}$ . . . . quartis,

$\mathfrak{C} = \frac{2^4}{1\cdot 2\cdots 7}\cdot\frac{1}{3} = \frac{1}{945}$ . . . . sextis,

$\mathfrak{D} = \frac{2^6}{1\cdot 2\cdots 9}\cdot\frac{3}{5} = \frac{1}{9450}$ . . . . octavis,

$\mathfrak{E} = \frac{2^8}{1\cdot 2\cdots 11}\cdot\frac{5}{3} = \frac{1}{93555}$ . . . . decimis,

$\mathfrak{F} = \frac{2^{10}}{1\cdot 2\cdots 13}\cdot\frac{5}{3} = \frac{691}{105}$ . . . . duodecimis,

$\mathfrak{G} = \frac{2^{12}}{1\cdot 2\cdots 15}\cdot\frac{35}{1}$ . . . . decimis quartis,

$\mathfrak{H} = \frac{2^{14}}{1\cdot 2\cdots 17}\cdot\frac{3617}{15}$ . . . . decimis sextis,

$\mathfrak{I} = \frac{2^{16}}{1\cdot 2\cdots 19}\cdot\frac{43867}{21}$ . . . . decimis octavis,

$\mathfrak{K} = \frac{2^{18}}{1\cdot 2\cdots 21}\cdot\frac{1222277}{55}$ . . . . vigesimis,

$\mathfrak{L} = \frac{2^{20}}{1\cdot 2\cdots 23}\cdot\frac{854513}{3}$ . . . . vigesimis secundis,

$\mathfrak{M} = \frac{2^{22}}{1\cdot 2\cdots 25}\cdot\frac{1181820455}{273}$ . . . . vigesimis quartis,

$\mathfrak{N} = \frac{2^{24}}{1\cdot 2\cdots 27}\cdot\frac{76977927}{1}$ . . . . vigesimis sextis,

$\mathfrak{O} = \frac{2^{26}}{1\cdot 2\cdots 29}\cdot\frac{23749461029}{15}$ . . . . vigesimis octavis,

$\mathfrak{P} = \frac{2^{28}}{1\cdot 2\cdots 31}\cdot\frac{8615841276005}{231}$ . . . . trigesimis,

$\mathfrak{Q} = \frac{2^{30}}{1\cdot 2\cdots 33}\cdot\frac{84802531453387}{85}$ . . . . trigesimis secundis,

$\mathfrak{R} = \frac{2^{32}}{1\cdot 2\cdots 35}\cdot\frac{90219075042845}{3}$ . . . . trigesimis quartis,

PRAEPARATIO FORMULARUM GENERALIUM AD ALIOS USUS

20. Hactenus posuimus $m = 1$, nunc autem statuamus $m = \frac{n-1}{2}$ eritque

$$\frac{m\pi}{n} = \frac{(n-1)\pi}{2n} = \frac{1}{2}\pi - \frac{\pi}{2n},$$

unde fit

$$\sin.\frac{m\pi}{n} = \cos.\frac{\pi}{2n} \text{ et tang.}\frac{m\pi}{n} = \cot.\frac{\pi}{2n}.$$

Ipsae autem series ita se habebunt:

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}}=\frac{1}{n-1}-\frac{1}{3n-1}+\frac{1}{5n-1}-\frac{1}{7n-1}+\frac{1}{9n-1}-\text{etc.}$$
$$+\frac{1}{n+1}-\frac{1}{3n+1}+\frac{1}{5n+1}-\frac{1}{7n+1}+\frac{1}{9n+1}-\text{etc,}$$
$$\frac{\pi}{2n\cot.\frac{\pi}{2n}}=\frac{1}{n-1}+\frac{1}{3n-1}+\frac{1}{5n-1}+\frac{1}{7n-1}+\frac{1}{9n-1}+\text{etc.}$$
$$-\frac{1}{n+1}-\frac{1}{3n+1}-\frac{1}{5n+1}-\frac{1}{7n+1}-\frac{1}{9n+1}-\text{etc,}$$

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21. Contrahantur hic etiam bini termini analogi ac prodibit haec series

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}}=\frac{2n}{nn-1}-\frac{6n}{9nn-1}+\frac{10n}{25nn-1}-\frac{14n}{49nn-1}+\text{etc.}$$

sive

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}}=\frac{n}{nn-1}-\frac{3n}{9nn-1}+\frac{5n}{25nn-1}-\frac{7n}{49nn-1}+\text{etc.}$$

22. Hic igitur omnes istae fractiones continentur in hac forma generali

$$\frac{in}{i^2n^2-1},$$

in ubi i denotat omnes numeros impares. Haec autem fractio in seriem infinitam conversa praebet

$$\frac{1}{in}+\frac{1}{i^3n^3}+\frac{1}{i^5n^5}+\frac{1}{i^7n^7}+\frac{1}{i^9n^9}+\text{etc.}$$

Hinc igitur singulas fractiones per series evolvamus

$$\frac{n}{nn-1}=\frac{1}{n}+\frac{1}{n^3}+\frac{1}{n^5}+\frac{1}{n^7}+\frac{1}{n^9}+\text{etc.,}$$
$$-\frac{3n}{9nn-1}=-\frac{1}{3n}-\frac{1}{3^3n^3}-\frac{1}{3^5n^5}-\frac{1}{3^7n^7}-\frac{1}{3^9n^9}-\text{etc.,}$$
$$\frac{5n}{25nn-1}=\frac{1}{5n}+\frac{1}{5^3n^3}+\frac{1}{5^5n^5}+\frac{1}{5^7n^7}+\frac{1}{5^9n^9}+\text{etc.,}$$
$$-\frac{7n}{49nn-1}=-\frac{1}{7n}-\frac{1}{7^3n^3}-\frac{1}{7^5n^5}-\frac{1}{7^7n^7}-\frac{1}{7^9n^9}-\text{etc.,}$$
etc.,

quarum igitur serierum omnium summa est

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}}.$$

23. Nunc etiam has series per columnas verticales colligamus ac statuamus

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.}=\mathfrak{a}\frac{\pi}{2},$$
$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}=\mathfrak{b}\frac{\pi^3}{2^3},$$
$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}=\mathfrak{c}\frac{\pi^5}{2^5},$$
$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\frac{1}{11^7}+\text{etc.}=\mathfrak{d}\frac{\pi^7}{2^7},$$
$$\text{etc.},$$

quibus positis aequatio nostra erit

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}}=\frac{\mathfrak{a}\pi}{2n}+\mathfrak{b}\frac{\pi^3}{2^3n^3}+\mathfrak{c}\frac{\pi^5}{2^5n^5}+\mathfrak{d}\frac{\pi^7}{2^7n^7}+\text{etc.}$$

24. Ponamus nunc $\frac{\pi}{2n}=x$ et equatio nostra hanc induet formam

$$\frac{x}{2\cos.x}=\mathfrak{a}x+\mathfrak{b}x^3+\mathfrak{c}x^5+\mathfrak{d}x^7+\mathfrak{e}x^9+\text{etc.}$$

Nunc igitur, si brevitatis gratia ponamus

$$\cos.x=\alpha-\beta xx+\gamma x^4-\delta x^6+\varepsilon x^8-\text{etc.},$$

ita ut sit

$$\alpha=1,\ \beta=\frac{1}{1\cdot 2},\ \gamma=\frac{1}{1\cdot 2\cdot 3\cdot 4},\ \delta=\frac{1}{1\cdot 2\cdots 6}\ \text{etc.},$$

si per hanc seriem utrinque multiplicemus, orietur ista aequatio

$$\begin{array}{llllllll}
\frac{x}{2}=\alpha\mathfrak{a}x & +\alpha\mathfrak{b}x^3 & +\alpha\mathfrak{c}x^5 & +\alpha\mathfrak{d}x^7 & +\alpha\mathfrak{e}x^9 & +\alpha\mathfrak{f}x^{11} & +\alpha\mathfrak{g}x^{13} & +\text{etc.}\\
 & -\beta\mathfrak{a} & -\beta\mathfrak{b} & -\beta\mathfrak{c} & -\beta\mathfrak{d} & -\beta\mathfrak{e} & -\beta\mathfrak{f} & \\
 & & +\gamma\mathfrak{a} & +\gamma\mathfrak{b} & +\gamma\mathfrak{c} & +\gamma\mathfrak{d} & +\gamma\mathfrak{e} & \\
 & & & -\delta\mathfrak{a} & -\delta\mathfrak{b} & -\delta\mathfrak{c} & -\delta\mathfrak{d} & \\
 & & & & +\varepsilon\mathfrak{a} & +\varepsilon\mathfrak{b} & +\varepsilon\mathfrak{c} & \\
 & & & & & -\zeta\mathfrak{a} & -\varepsilon\mathfrak{b} & \\
 & & & & & & +\eta\mathfrak{a} &
\end{array}$$

25. Singulis igitur potestatibus ad nihilum reductis nanciscemur sequentes determinationes:

$$\mathfrak{a} = \tfrac{1}{2},$$
$$\mathfrak{b} = \beta\mathfrak{a},$$
$$\mathfrak{c} = \beta\mathfrak{b} - \gamma\mathfrak{a},$$
$$\mathfrak{d} = \beta\mathfrak{c} - \gamma\mathfrak{b} + \delta\mathfrak{a},$$
$$\mathfrak{e} = \beta\mathfrak{d} - \gamma\mathfrak{c} + \delta\mathfrak{b} - \varepsilon\mathfrak{a},$$
$$\mathfrak{f} = \beta\mathfrak{e} - \gamma\mathfrak{d} + \delta\mathfrak{c} - \varepsilon\mathfrak{b} + \zeta\mathfrak{a}$$
etc.

26. Ope harum formularum iam olim summas istarum serierum exhibui, unde valores pro praesentibus litteris $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc. ita reperientur determinati:

$\mathfrak{a} = \frac{1}{2}$ pro potestatibus primis,

$\mathfrak{b} = \frac{1}{1\cdot 2}\cdot\frac{1}{2}$ . . . . . . tertiis,

$\mathfrak{c} = \frac{5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{2}$ . . . . . . quintis,

$\mathfrak{d} = \frac{61}{1\cdot 2\cdots 6}\cdot\frac{1}{2}$ . . . . . . septimis,

$\mathfrak{e} = \frac{1385}{1\cdot 2\cdots 8}\cdot\frac{1}{2}$ . . . . . . nonis,

$\mathfrak{f} = \frac{50521}{1\cdot 2\cdots 10}\cdot\frac{1}{2}$ . . . . . . undecimis,

$\mathfrak{g} = \frac{2702765}{1\cdot 2\cdots 12}\cdot\frac{1}{2}$ . . . . . . decimis tertiis,

$\mathfrak{h} = \frac{199360981}{1\cdot 2\cdots 14}\cdot\frac{1}{2}$ . . . . . . decimis quintis,

$\mathfrak{i} = \frac{19391512145}{1\cdot 2\cdots 16}\cdot\frac{1}{2}$ . . . . . . decimis septimis,

$\mathfrak{k} = \frac{2404879675441}{1\cdot 2\cdots 18}\cdot\frac{1}{2}$ . . . . . . decimis nonis,

27. Hinc igitur summas istarum serierum usque ad potestatem vigesimam apponamus

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}=\frac{1}{1}\cdot\frac{\pi}{2^2},$$
$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.}=\frac{1}{1\cdot 2}\cdot\frac{\pi^3}{2^4},$$
$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=\frac{5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^5}{2^6},$$
$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=\frac{61}{1\cdot 2\cdots 6}\cdot\frac{\pi^7}{2^8},$$
$$1-\frac{1}{3^9}+\frac{1}{5^9}-\frac{1}{7^9}+\frac{1}{9^9}-\text{etc.}=\frac{1385}{1\cdot 2\cdots 8}\cdot\frac{\pi^9}{2^{10}},$$
$$1-\frac{1}{3^{11}}+\frac{1}{5^{11}}-\frac{1}{7^{11}}+\frac{1}{9^{11}}-\text{etc.}=\frac{50521}{1\cdot 2\cdots 10}\cdot\frac{\pi^{11}}{2^{12}},$$
$$1-\frac{1}{3^{13}}+\frac{1}{5^{13}}-\frac{1}{7^{13}}+\frac{1}{9^{13}}-\text{etc.}=\frac{2702765}{1\cdot 2\cdots 12}\cdot\frac{\pi^{13}}{2^{14}},$$
$$1-\frac{1}{3^{15}}+\frac{1}{5^{15}}-\frac{1}{7^{15}}+\frac{1}{9^{15}}-\text{etc.}=\frac{199360981}{1\cdot 2\cdots 14}\cdot\frac{\pi^{15}}{2^{16}},$$
$$1-\frac{1}{3^{17}}+\frac{1}{5^{17}}-\frac{1}{7^{17}}+\frac{1}{9^{17}}-\text{etc.}=\frac{19391512145}{1\cdot 2\cdots 16}\cdot\frac{\pi^{17}}{2^{18}},$$
$$1-\frac{1}{3^{19}}+\frac{1}{5^{19}}-\frac{1}{7^{19}}+\frac{1}{9^{19}}-\text{etc.}=\frac{2404879675441}{1\cdot 2\cdots 18}\cdot\frac{\pi^{19}}{2^{20}},$$
etc.

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28. Binis terminis analogis contractis haec series hanc induet formam

$$\frac{\pi}{4n\cot.\frac{\pi}{2n}}=\frac{1}{nn-1}+\frac{1}{9nn-1}+\frac{1}{25nn-1}+\frac{1}{49nn-1}+\text{etc.},$$

quae fractiones in series evolutae dabunt

$$\frac{1}{nn-1}=\frac{1}{nn}+\frac{1}{n^4}+\frac{1}{n^6}+\frac{1}{n^8}+\frac{1}{n^{10}}+\text{etc.},$$
$$\frac{1}{9nn-1}=\frac{1}{9\ nn}+\frac{1}{9^2n^4}+\frac{1}{9^3n^6}+\frac{1}{9^4n^8}+\frac{1}{9^5n^{10}}+\text{etc.},$$
$$\frac{1}{25nn-1}=\frac{1}{25n}+\frac{1}{25^2n^4}+\frac{1}{25^3n^6}+\frac{1}{25^4n^8}+\frac{1}{25^5n^{10}}+\text{etc.},$$
etc.,

quarum igitur omnium summa est

$$\frac{\pi}{4n}\text{tang.}\frac{\pi}{2n}.$$

29. Quo nunc has series verticaliter colligere queamus, statuamus

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{etc.}=\mathfrak{A}'\frac{\pi\pi}{2^2},$$
$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.}=\mathfrak{B}'\frac{\pi^4}{2^4},$$
$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}=\mathfrak{C}'\frac{\pi^6}{2^6},$$
etc.

Unde aequatio nostra fiet

$$\frac{\pi}{4n}\,\text{tang.}\frac{\pi}{2n}=\frac{\mathfrak{A}'\pi^2}{2^2n^2}+\frac{\mathfrak{B}'\pi^4}{2^4n^4}+\frac{\mathfrak{C}'\pi^6}{2^6n^6}+\frac{\mathfrak{D}'\pi^8}{2^8n^8}+\text{etc.}$$

30. Ponamus nunc $\frac{\pi}{2n}=x$, ut aequatio nostra fiat

$$\frac{x}{2}\,\text{tang.}x=\mathfrak{A}'xx+\mathfrak{B}'x^4+\mathfrak{C}'x^6+\text{etc.},$$

unde erit

$$\text{tang.}x=2\mathfrak{A}'x+2\mathfrak{B}'x^3+2\mathfrak{C}'x^5+2\mathfrak{D}'x^7+\text{etc.};$$

cuius seriei loco scribamus litteram *t,* ut sit $\text{tang.}x=t$ hincque differentiando

$$dx=\frac{dt}{1+tt},$$

ideoque habebimus

$$\frac{dt}{dx}=1+tt.$$

Est vero

$$\frac{dt}{dx}=2\mathfrak{A}'+2\cdot 3\mathfrak{B}'xx+2\cdot 5\mathfrak{C}'x^4+2\cdot 7\mathfrak{D}'x^6+\text{etc.}$$

31. Eodem modo facta evolutione erit

$$\begin{aligned}1+tt=1+4\mathfrak{A}'\mathfrak{A}'xx+8\mathfrak{A}'\mathfrak{B}'x^4&+8\mathfrak{A}'\mathfrak{C}'x^6+8\mathfrak{A}'\mathfrak{D}'x^8+\text{etc.}\\ +4\mathfrak{B}'\mathfrak{B}'&\quad+8\mathfrak{B}'\mathfrak{C}'\end{aligned}$$

Hine igitur sequentes deducuntur determinationes:

$$\mathfrak{A}' = \tfrac{1}{2},$$
$$\mathfrak{B}' = \tfrac{2}{3}\cdot \mathfrak{A}'\mathfrak{A}',$$
$$\mathfrak{C}' = \tfrac{2}{5}\cdot 2\mathfrak{A}'\mathfrak{B}',$$
$$\mathfrak{D}' = \tfrac{2}{7}(2\mathfrak{A}'\mathfrak{C}' + \mathfrak{B}'\mathfrak{B}'),$$
$$\mathfrak{E}' = \tfrac{2}{9}(2\mathfrak{A}'\mathfrak{D}' + 2\mathfrak{B}'\mathfrak{C}'),$$
$$\mathfrak{F}' = \tfrac{2}{11}(2\mathfrak{A}'\mathfrak{E}' + 2\mathfrak{B}'\mathfrak{D}' + \mathfrak{C}'\mathfrak{C}'),$$
$$\mathfrak{G}' = \tfrac{2}{13}(2\mathfrak{A}'\mathfrak{F}' + 2\mathfrak{B}'\mathfrak{E}' + 2\mathfrak{C}'\mathfrak{D}')$$
etc.

32. Istae determinationes fere prorsus conveniunt cum iis, quas supra § 18 pro litteris $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc., invenimus; totum enim discrimen reperitur in coefficientibus numericis. Neutiquam vero opus est istos valores seorsim computare, cum ii iam ex superioribus facillime deduci queant. Cum enim sit

$$\frac{\mathfrak{A}'}{2^2} = \frac{\mathfrak{A}(2^2-1)}{2^2},$$

erit

$$\mathfrak{A}' = (2^2-1)\mathfrak{A}.$$

Simili modo erit

$$\mathfrak{B}' = (2^4-1)\mathfrak{B},\ \mathfrak{C}' = (2^6-1)\mathfrak{C},\ \mathfrak{D}' = (2^8-1)\mathfrak{D} \text{ etc.}$$

CONCLUSIO

33. In gratiam eorum, qui hos valores penitus numerice per fractiones decimales exprimere voluerint, subiungamus sequentem tabulam, in qua omnes potestates ipsius π per fractiones decimales sunt evolutae, ubi loco $\frac{\pi}{2}$ scripsimus q.

$$\frac{q}{1} = 1,57079\ 63267\ 94896\ 6192313216\ 916,$$

$$\frac{q^2}{1\cdot 2} = 1,23370\ 0550136169\ 82735\ 43113\ 750,$$

$$\frac{q^3}{1\cdot 2\cdot 3} = 0,64596\ 40975\ 06246\ 25365\ 57565\ 639,$$

$$\frac{q^4}{1\cdot 2\cdot 3\cdot 4} = 0,25366\ 95079\ 01048\ 01363\ 65633\ 664,$$

$$\frac{q^5}{1\cdot 2\cdots 5} = 0,07969\ 26262\ 46167\ 04512\ 05055\ 495,$$

$$\frac{q^6}{1\cdot 2\cdots 6} = 0,02086\ 34807\ 63352\ 96087\ 30516\ 372,$$

$$\frac{q^7}{1\cdot 2\cdots 7} = 0,004681754135318\ 68810\ 06854\ 639,$$

$$\frac{q^8}{1\cdot 2\cdots 8} = 0,000919260274839\ 42658\ 02417\ 162,$$

$$\frac{q^9}{1\cdot 2\cdots 9} = 0,000160441184787\ 3598218726\ 609,$$

$$\frac{q^{10}}{1\cdot 2\cdots 10} = 0,0000252020423730606054810530,$$

$$\frac{q^{11}}{1\cdot 2\cdots 11} = 0,00000\ 35988\ 43235\ 21208\ 53404\ 685,$$

$$\frac{q^{12}}{1\cdot 2\cdots 12} = 0,0000004710\ 87477\ 8818171503670,$$

$$\frac{q^{13}}{1\cdot 2\cdots 13} = 0,00000\ 00569\ 21729\ 21967\ 92681178,$$

$$\frac{q^{14}}{1\cdot 2\cdots 14} = 0,00000\ 00063866030837918522\ 411,$$

$$\frac{q^{15}}{1\cdot 2\cdots 15} = 0,0000000006\ 68803\ 5109811467\ 232,$$

$$\frac{q^{16}}{1\cdot 2\cdots 16} = 0,0000000000\ 65659\ 63114\ 97947\ 236,$$

$$\frac{q^{17}}{1\cdot 2\cdots 17} = 0,00000\ 00000\ 06066\ 9357311061957,$$

$$\frac{q^{18}}{1\cdot 2\cdots 18} = 0,00000\ 00000\ 00529\ 44002\ 00734\ 624,$$

$$\frac{q^{19}}{1\cdot 2\cdots 19} = 0,00000\ 00000\ 00043\ 77065\ 46731374,$$

$$\frac{q^{20}}{1\cdot 2\cdots 20} = 0,00000\ 00000\ 00003\ 43773\ 91790\ 986,$$

$$\frac{q^{21}}{1\cdot 2\cdots 21} = 0,00000\ 00000\ 00000\ 25714\ 22892\ 860,$$

$$\frac{q^{22}}{1\cdot 2\cdots 22} = 0,00000\ 00000\ 00000\ 01835\ 99165\ 216,$$

$$\frac{q^{23}}{1\cdot 2\cdots 23} = 0,00000\ 00000\ 00000\ 00125\ 38995\ 405,$$

$$\frac{q^{24}}{1\cdot2\cdots24} = 0,00000\,00000\,00000\,00008\,20675\,330,$$
$$\frac{q^{25}}{1\cdot2\cdots25} = 0,00000\,00000\,00000\,00000\,51564\,552,$$
$$\frac{q^{26}}{1\cdot2\cdots26} = 0,00000\,00000\,00000\,00000\,03115\,285,$$
$$\frac{q^{27}}{1\cdot2\cdots27} = 0,00000\,00000\,00000\,00000\,00181\,240,$$
$$\frac{q^{28}}{1\cdot2\cdots28} = 0,00000\,00000\,00000\,00000\,00010\,168,$$
$$\frac{q^{29}}{1\cdot2\cdots29} = 0,00000\,00000\,00000\,00000\,00000\,551,$$
$$\frac{q^{30}}{1\cdot2\cdots30} = 0,00000\,00000\,00000\,00000\,00000\,029,$$
$$\frac{q^{31}}{1\cdot2\cdots31} = 0,00000\,00000\,00000\,00000\,00000\,001.$$

Hae quidem potestates divisae sunt per certos numeros, qui autem plerumque sunt ii ipsi, per quos eaedem potestates ipsius π in superioribus formulis divisi occurrunt, unde evolutio in fractiones decimales eo facilior redditur.