SUMMATION OF CONTINUED FRACTIONS OF WHICH THE INDICES CONSTITUTE AN ARITHMETICAL PROGRESSION WHILE ALL THE NUMERATORS ARE UNITY, LIKEWISE THE RESOLUTION OF RICCATI EQUATIONS MAY BE RESOLVED BY FRACTIONS OF THIS KIND

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Opuscula analytica 2, 1785, p. 217-239

1. Since in the preceding dissertation I have set out a method, continued fractions being reduced to two integral formulas, which indeed succeeded happily in an infinite number of cases : but truly the cases, which may be considered the most simple, where all the numerators have been put equal to each other, has led to integral formulas of this kind, which until now in no manner may be permitted to be set out and to be compared among themselves, since still from this generally there may be had two continued fractions, the values of which can be shown conveniently enough:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\text{etc.}}}}=\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$$

and

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\text{etc.}}}}=\cot.\frac{2}{n},$$

of which indeed the one may be deduced easily from the other, if in place of n there may be written $n\sqrt{-1}$.

2. But when the indices follow some other arithmetic progression, I have reduced the summation of such continued fractions now some time ago, in Book XI of the old Commentaries of our Academy in a straight forwards manner, to a singular Riccati equation. But the method, by which I have established this, was set out there in an exceedingly succinct manner; whereby, since it may be seen there for the most part to be hidden, it will be worth the effort here to explain more fully ; particularly since not only may no-one have observed the strength of this method, but also I may be able to cover that more fully.

3. So that I may present the same investigation more clearly, I may begin from the general fraction, of which indeed all the numerators shall be unity, but the indices in general may be designated by the letters *a, b, c, d, e, f* etc. , thus so that the continued fraction itself may have the form:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

of which the value may be indicated by the letter S , for which requiring to be known at least approximately, the series of the following fractions may be formed in the customary manner from the indices a, b, c, d, e etc. :

$$\begin{matrix} a & b & c & d & e & \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} & \text{etc.,} \end{matrix}$$

of which both the numerators as well as the denominators may be determined in the following manner from the two preceding :

$$A=a,\ B=Ab+1,\ C=Bc+A,\ D=Cd+B \text{ etc.}$$
$$\mathfrak{A}=1,\ \mathfrak{B}=\mathfrak{A}b,\quad \mathfrak{C}=\mathfrak{B}c+\mathfrak{A},\ \mathfrak{D}=\mathfrak{C}d+\mathfrak{B} \text{ etc.}$$

4. But now in place of the letters A, B, C, D ... and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$. we may introduce others into the calculation, which shall be

$$A'=\frac{A}{a},\ B'=\frac{B}{ab},\ C'=\frac{C}{abc},\ D'=\frac{D}{abcd},\ \text{etc.}$$
$$\mathfrak{A}'=\frac{\mathfrak{A}}{a},\ \mathfrak{B}'=\frac{\mathfrak{B}}{ab},\ \mathfrak{C}'=\frac{\mathfrak{C}}{abc},\ \mathfrak{D}'=\frac{\mathfrak{D}}{abcd},\ \text{etc.,}$$

and these new letters will be determined in the following manner by the indices and by the two preceding terms :

$$A'=1,\quad B'=A'+\frac{1}{ab},\ C'=B'+\frac{A'}{bc},\ D'=C'+\frac{B'}{cd},\ \text{etc.}$$
$$\mathfrak{A}'=\frac{1}{a},\ \mathfrak{B}'=\mathfrak{A}',\qquad \mathfrak{C}'=\mathfrak{B}'+\frac{\mathfrak{A}'}{bc},\ \mathfrak{D}'=\mathfrak{C}'+\frac{\mathfrak{B}'}{cd},\ \text{etc.}$$

Therefore with these values set out for any case, these fractions:

$$\frac{A'}{\mathfrak{A}'},\ \frac{B'}{\mathfrak{B}'},\ \frac{C'}{\mathfrak{C}'},\ \frac{D'}{\mathfrak{D}'},\ \frac{E'}{\mathfrak{E}'}\ \text{etc.}$$

continually approach closer to the value S of the continued fraction proposed, and continued to infinity will become exactly equal to that.

5. So that the forms of these letters may be understood better, we may set out these more simply by the indices, and indeed initially we will find the following formulas for the numerators :

$$A' = 1,$$
$$B' = 1 + \frac{1}{ab},$$
$$C' = 1 + \frac{1}{ab} + \frac{1}{bc},$$
$$D' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{abcd}$$
$$E' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde}$$
$$F' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{ef} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde}$$
$$+ \frac{1}{abef} + \frac{1}{bcef} + \frac{1}{cdef} + \frac{1}{abcdef}$$
etc.

Truly the following formulas will be produced for the denominators :

$$\mathfrak{A}' = \frac{1}{a}$$
$$\mathfrak{B}' = \frac{1}{a}$$
$$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$$
$$\mathfrak{D}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd}$$
$$\mathfrak{E}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{abcde}$$
$$\mathfrak{F}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{aef} + \frac{1}{abcde} + \frac{1}{abcef} + \frac{1}{acdef}$$
etc.

in which latter formulas the individual terms have the factor $\frac{1}{a}$, with which omitted the remaining factors may be defined in the same manner by the indices *b, c, d, e, f* etc., from which the preceding Latin letters have been determined by all the indices *a, b, c, d, e* etc.

6. Now we will apply these set out to the case, which has been proposed by us here, where we assume the indices *a, b, c, d, e* to proceed according to an arithmetical progression. Therefore we may put in place the difference, by which these indices continually increase $= \Delta$, and the indices after the first will be :

$$b = a + \Delta,\ c = a + 2\Delta,\ d = a + 3\Delta,\ e = a + 4\Delta \text{ etc.}$$

Indeed we will not substitute these values into the denominators of our formulas, but we will use these especially for contracting the formulas.

7. Therefore with this progression established we may set out initially the numerators of our fractions in the following manner :

$$A' = \frac{1}{a}$$
$$B' = 1 + \frac{1}{ab}$$
$$C' = 1 + \frac{1}{ab} + \frac{1}{bc} = 1 + \frac{2}{ac}$$
$$D' = 1 + \frac{2}{ac} + \frac{1}{cd} + \frac{1}{abcd} = 1 + \frac{3}{ac} + \frac{1}{abcd}$$
$$E' = 1 + \frac{3}{ad} + \frac{1}{de} + \frac{1}{abcd} + \frac{2}{acde} = 1 + \frac{4}{ae} + \frac{3}{abde}$$
$$F' = 1 + \frac{5}{af} + \frac{6}{abdf} + \frac{1}{abcdef}$$
$$G' = 1 + \frac{6}{ag} + \frac{10}{abfg} + \frac{4}{abcefg}$$
$$H' = 1 + \frac{7}{ah} + \frac{15}{abgh} + \frac{10}{abcfgh} + \frac{1}{abcdefgh}$$
$$I' = 1 + \frac{8}{ai} + \frac{21}{abhi} + \frac{10}{abcghi} + \frac{5}{abcdfghi}$$
$$K' = 1 + \frac{9}{ak} + \frac{28}{abik} + \frac{35}{abchik} + \frac{15}{abcdghik} + \frac{1}{abcdefghik}$$
etc.

8. Just as in these forms all the first terms are ones, thus the numerators of the second progress according to the natural numbers, of the third following trigonal numbers, of the fourth following the first tetrahedral, then the second, third etc. The order in the denominators equally is evident enough. Hence therefore in general we will be able to show that formula, which may correspond to the index i. Thus if that latter may be designated by the letter Z', then truly in the order of the letters a, b, c, d, e etc. there would be $z = a + i\Delta$, truly the antecedents

$$y = a + (i-1)\Delta,\ x = a + (i-2)\Delta,\ v = a + (i-3)\Delta \text{ etc.},$$

and we will have

$$Z' = 1 + \frac{i}{az} + \frac{(i-1)(i-2)}{1\cdot 2abyz} + \frac{(i-2)(i-3)(i-4)}{1\cdot 2\cdot 3abcxyz} + \frac{(i-3)(i-4)(i-5)(i-6)}{1\cdot 2\cdot 3\cdot 4abcdvxyz} + \text{ etc.},$$

9. Now truly with regard to the Germanic letters $\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{D}'$ etc., any of these may be formed from the preceding Latin letters, while the letters a, b, c, d, e may be moved forwards by one step, then truly the individual terms may be multiplied by $\frac{1}{a}$, thus so that there will be as follows :

$$\mathfrak{A}'=1$$
$$\mathfrak{B}'=\frac{1}{a}$$
$$\mathfrak{C}'=\frac{1}{a}+\frac{1}{abc}$$
$$\mathfrak{D}'=\frac{1}{a}+\frac{2}{abd}$$
$$\mathfrak{E}'=\frac{1}{a}+\frac{3}{abe}+\frac{1}{abcde}$$
$$\mathfrak{F}'=\frac{1}{a}+\frac{4}{abf}+\frac{3}{abcef}$$
$$\mathfrak{G}'=\frac{1}{a}+\frac{5}{abg}+\frac{6}{abcfg}+\frac{1}{abcdefg}$$
$$\mathfrak{H}'=\frac{1}{a}+\frac{6}{abh}+\frac{10}{abcgh}+\frac{4}{abcdfgh}$$
$$\mathfrak{I}'=\frac{1}{a}+\frac{7}{abi}+\frac{15}{abchi}+\frac{10}{abcdghi}+\frac{1}{abcdefghi}$$
etc.,

from which in general, if $\mathfrak{Z}'$ may correspond to the index i , there will be

$$\mathfrak{Z}'=\frac{1}{a}+\frac{(i-1)}{abz}+\frac{(i-2)(i-3)}{1\cdot 2\cdot abcyz}+\frac{(i-3)(i-4)(i-5)}{1\cdot 2\cdot 3\cdot abcdxyz}+\frac{(i-4)(i-5)(i-6)(i-7)}{1\cdot 2\cdot 3\cdot 4\cdot abcdevxyz}+\text{etc.}$$

10. Now we may increase the index z to infinity, so that the fraction $\frac{Z'}{\mathfrak{Z}'}$ may express the value of the continued fraction itself, which we have called S, thus so that there shall be $S=\frac{Z'}{\mathfrak{Z}'}$, and the reduction of the formulas found may be carried out in the following manner. Initially clearly for Z' there will be

$$\frac{i}{z}=\frac{i}{a+i\varDelta}=\frac{1}{\varDelta},\ \text{on account of}\ \ i=\infty;$$

for the third term there will be,

$$\frac{(i-1)(i-2)}{yz}=\frac{(i-1)(i-2)}{(a+(i-1)\varDelta)(a+i\varDelta)}=\frac{1}{\varDelta^2};$$

moreover again in a similar manner

$$\frac{(i-2)(i-3)(i-4)}{xyz}=\frac{(i-2)(i-3)(i-4)}{(a+(i-2)\varDelta)(a+(i-1)\varDelta)(a+i\varDelta)}=\frac{1}{\varDelta^3}$$

and

$$\frac{(i-3)(i-4)(i-5)(i-6)}{vxyz}=\frac{1}{\varDelta^4}\ \text{etc.}$$

By an equal manner for the formula $\mathfrak{Z}'$ there will be also

$$\frac{i-1}{z}=\frac{1}{\Delta},\quad \frac{(i-2)(i-3)}{yz}=\frac{1}{\Delta^2}$$

and thus henceforth ; on account of which for the case $i=\infty$ both out formulas thus are contracted conveniently, so that there may become

$$Z'=1+\frac{1}{a\Delta}+\frac{1}{1\cdot 2ab\Delta^2}+\frac{1}{1\cdot 2\cdot 3abc\Delta^3}+\frac{1}{1\cdot 2\cdot 3\cdot 4abcd\Delta^4}+\text{etc.}$$

and in a similar manner

$$\mathfrak{Z}'=\frac{1}{a}+\frac{1}{ab\Delta}+\frac{1}{1\cdot 2abc\Delta^2}+\frac{1}{1\cdot 2\cdot 3abcd\Delta^3}+\frac{1}{1\cdot 2\cdot 3\cdot 4abcde\Delta^4}+\text{etc.}$$

11. Therefore since the value sought shall be $S=\frac{Z'}{\mathfrak{Z}'}$, we may see, how we may be able to recall both series to finite expressions. In the end we may be able to return both these series more generally, while in place of the numerators, which are all equal to 1, we have substituted a certain geometric progression. Therefore we may put in place

$$p=1+\frac{x^{\Delta}}{a\Delta}+\frac{x^{2\Delta}}{1\cdot 2ab\Delta^2}+\frac{x^{3\Delta}}{1\cdot 2\cdot 3abc\Delta^3}+\text{etc.}$$

and

$$q=\frac{1}{a}+\frac{x^{\Delta}}{ab\Delta}+\frac{x^{2\Delta}}{1\cdot 2abc\Delta^2}+\frac{x^{3\Delta}}{1\cdot 2\cdot 3abcd\Delta^3}+\text{etc.}$$

So that if moreover hence we have elicited the values p and q in general, then truly we may put $x=1$, certainly there will become $S=\frac{p}{q}$. But here it is evident, both these series maintain a special relationship between each other, and by differentiation to be possible to convert the one into the other, as we may establish according to the following investigation.

12. In the first place therefore the former series differentiated gives simply :

$$\frac{xdp}{dx}=\frac{x^{\Delta}}{a}+\frac{x^{2\Delta}}{ab\Delta}+\frac{x^{3\Delta}}{1\cdot 2abc\Delta^2}+\frac{x^{4\Delta}}{1\cdot 2\cdot 3abcd\Delta^3}+\text{etc.},$$

which series compared with the other q evidently produces :

$$\frac{xdp}{dx}=x^{\Delta}q,$$

from which it is apparent, if only the sum of either of these two series were known, the other too can be assigned, since from the know value p there arises :

$$q=\frac{dp}{x^{\Delta-1}dx};$$

truly on the contrary from the known value q there becomes :

$$dp=x^{\Delta-1}qdx,$$

and thus

$$p=\int x^{\Delta-1}qdx,$$

which integral must be taken thus, so that on putting $x=0$ there may become $p=1$.

13. But before as we may differentiate the other series, we may multiply that by x^{a}, and on account of

$$a+\Delta=b,\ a+2\Delta=c,\ a+3\Delta=d \text{ etc.}$$

there will become:

$$x^{a}q=\frac{x^{a}}{a}+\frac{x^{b}}{ab\Delta}+\frac{x^{c}}{1\cdot 2abc\Delta^{2}}+\frac{x^{d}}{1\cdot 2\cdot 3abcd\Delta^{3}}+\text{etc.}$$

Now this equation differentiated and multiplied again by x will give :

$$xd\cdot\frac{x^{a}q}{dx}=x^{a}+\frac{x^{b}}{a\Delta}+\frac{x^{c}}{1\cdot 2ab\Delta^{2}}+\frac{x^{d}}{1\cdot 2\cdot 3abc\Delta^{3}}+\text{etc.},$$

but truly the first series p likewise multiplied by x^{a} produces

$$x^{a}p=x^{a}+\frac{x^{b}}{a\Delta}+\frac{x^{c}}{1\cdot 2ab\Delta^{2}}+\frac{x^{d}}{1\cdot 2\cdot 3abc\Delta^{3}}+\text{etc.}$$

which series since they shall be exactly equal, will be

$$\frac{x}{dx}d\cdot x^{a}q=x^{a}p,\ \text{and thus } d\cdot x^{a}q=px^{a-1}dx,$$

and thus two differential equations have arisen between p and q, from which the value of each will be able to be elicited.

14. Since from the first equation there shall be

$$q=\frac{dp}{x^{\Delta-1}dx}\ ,\ \text{there will become } x^{a}q=\frac{x^{a-\Delta+1}dp}{dx},$$

from which with the element dx assumed constant there will become

$$d\cdot x^{a}q=\frac{x^{a-\Delta+1}ddp+(a-\Delta+1)x^{a-\Delta}dpdx}{dx},$$

and thus with the quantity q replaced by the other p we obtain this differential equation of the second order :

$$px^{a-1}dx = \frac{x^{a-\Delta+1}ddp+(a-\Delta+1)x^{a-\Delta}dpdx}{dx},$$

which if it may be allowed to be resolved, the whole concern will be accomplished. Where it is required to be observed properly, since there shall be

$$p = 1 + \frac{x^{\Delta}}{a\Delta} + \frac{x^{2\Delta}}{1\cdot 2ab\Delta^2} + \text{etc.},$$

the integration thus must be established, so that on putting $x = 0$ there may become $p = 1$; then truly, since there is :

$$\frac{dp}{dx} = \frac{x^{\Delta-1}}{a} + \frac{x^{2\Delta-1}}{ab\Delta} + \text{etc.},$$

the other condition of the integration demands, so that on putting $x = 0$ there may become also $\frac{dp}{dx} = 0$, if indeed there were $\Delta > 1$; if indeed it may be $=1$, for the case $x = 0$ there must become $\frac{dp}{dx} = \frac{1}{a}$. But if $\Delta < 1$, there will have to become $\frac{dp}{dx} = \infty$.

15. Since the fraction put in place $x = 1$ may provide the value of our continued fraction *S*, in general we may put to be $\frac{p}{q} = z$, thus so that on putting $x = 1$ there may become $z = S$, from which it is apparent, for the case $x = 0$ there must become $z = a$. Therefore since there shall be $p = qz$,there will be $dp = qdz + zdq$; but from the first equation there was :

$$q = \frac{dp}{x^{\Delta-1}dx}$$

on account of which we will have :

$$q = \frac{qdz + zdq}{x^{\Delta-1}dx},$$

or $x^{\Delta-1}qdx = qdz + zdq$, from which there becomes

$$dq = \frac{x^{\Delta-1}qdx - qdz}{z};$$

but there was $dp = x^{\Delta-1}qdx$.

16. These therefore follow from the first differential equation found $\frac{dp}{x^{\Delta-1}dx} = q$. Truly on the one hand, which is

$$d\cdot x^a q = px^{a-1}dx,$$

because

$$d\cdot x^{a}q = ax^{a-1}qdx + x^{a}dq$$

with *dq* put in place in the manner found it will produce

$$d\cdot x^{a}q = ax^{a-1}qdx + \tfrac{x^{a+\Delta-1}qdx - x^{a}qdz}{z},$$

we will have

$$px^{a-1}dx = ax^{a-1}qdx + \tfrac{x^{a+\Delta-1}qdx - x^{a}qdz}{z},$$

which equation, on account of $p = qz$, divided by *q* supplies that same differential equation of the first order:

$$x^{a-1}zdx = ax^{a-1}dx + \tfrac{x^{a+\Delta-1}dx - x^{a}dz}{z},$$

which multiplied by and divided by x^{a-1} provides

$$zzdx = azdx + x^{\Delta}dx - xdz,$$

of which the resolution therefore may be established, so that on putting $x = 0$ there may become $z = a$, then truly there may become $x = 1$, the value of z will give the value itself of the continued fraction which we seek.

17. Therefore the whole matter has led to the resolution of the differential equation of the first order :

$$zzdx = azdx + x^{\Delta}dx - xdz\,,$$

which evidently shall be contained in that celebrated Riccati equation. So that indeed it may be reduced to three terms only, there may be put $z = x^{a}y$, thus so that there shall be $y = \tfrac{z}{x^{a}}$, from which there becomes:

$$dz = ax^{\Delta-1}ydx + x^{a}dy,$$

with which values substituted our equation will adopt this form :

$$x^{a+1}dy + x^{2a}yydx = x^{\Delta}dx,$$

which therefore thus it will be required to be integrated, so that by putting $x = 0$, or indefinitely small, there may become $y = \tfrac{a}{x^{a}}$, that is $y = \infty$, with which made, if after the integration there may be put $x = 1$, the value of *y* will give the sum of the proposed continued fraction.

18. So that we may render this expression more simple, we may divide by x^{a+1}, so that we may have

$$dy + x^{a-1}yydx = x^{\Delta-a-1}dx;$$

now truly we may put in place $x^a = t$,thus so that in the case $x = 0$ there may become also $t = 0$, and in the case $x = 1$ also $t = 1$, from which if t may vanish, there must become $y = \frac{a}{t}$, or $y = \infty$. But with this value introduced, on account of $x = t^{\frac{1}{a}}$ and

$$dx = \frac{t^{\frac{1-a}{a}}dt}{a},$$

our equation will become

$$ady + yydt = t^{\frac{\Delta-2a}{a}}dt,$$

which is the most useful form of the Riccati equation.

19. Therefore when the proposed were of such a continued fraction :

$$a + \cfrac{1}{a+\Delta+\cfrac{1}{a+2\Delta+\cfrac{1}{a+3\Delta+\cfrac{1}{a+4\Delta+\text{etc.}}}}},$$

for its value being investigated, this same Riccati equation must be resolved :

$$ady + yydt = t^{\frac{\Delta-2a}{a}}dt;$$

where the integration thus will be required to be put in place, so that with t assumed infinitely small there may become $y = \frac{a}{t}$, with which done there may be established $t = 1$, and the value resulting for y will be the value of this continued fraction.

20. We may set out the simplest case, where for the right hand part the exponent of t becomes equal to zero, which therefore happens, if $\Delta = 2a$, and thus that same continued fraction

$$a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

for the sum of which we will have this differential equation :

$$ady + yydt = dt, \text{from which } dt = \frac{ady}{1-yy},$$

and on integrating

$$t=\frac{a}{2}l\frac{1+y}{1-y}+C,$$

which constant *C* thus is required to be taken, so that on putting $t=0$ there may become $y=\frac{a}{t}$, and thus $t=\frac{a}{y}$; from which it is apparent in this case to become $y=\infty$, from which it is understood at once, the integral equation thus to be constructed must become :

$$t=\frac{a}{2}l\frac{y+1}{y-1}+C,$$

or the integration thus to be established, so that by making $t=0$, y may become infinite. Now truly if *y* were infinite, there will become $l\frac{y+1}{y-1}=\frac{2}{y}$, whereby , since there must become $t=\frac{a}{y}$, there becomes $C=0$, thus so that the equation of the integral shall be just $t=\frac{a}{2}l\frac{y+1}{y-1}$, from which with the number *e* denoting the number, of which the hyperbolic logarithm shall be $=1$, there will become $e^{\frac{2t}{a}}=\frac{y+1}{y-1}$, and hence again $y=\frac{e^{\frac{2t}{a}}+1}{e^{\frac{2t}{a}}-1}$, from which on putting $t=1$ the sum of our continued fraction will be $\frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1}$, which is the same value, which I had found formerly.

21. Now we may consider also the remaining cases of the integrability of the Riccati equation, for which the exponent of t to the right-hand part is either -4, $-\frac{4}{3}$, $-\frac{8}{3}$, $-\frac{12}{5}$, $-\frac{12}{5}$, or $-\frac{12}{7}$, etc. Therefore initially there shall be $\frac{\Delta-2a}{a}=-4$, or $\Delta=-2a$, from which this continued fraction arises :

$$a+\cfrac{1}{-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\cfrac{1}{-7a+\text{etc.}}}}}$$

which evidently depends on the preceding. Indeed if we may put

$$-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\text{etc.}}}=s,$$

with the signs changed there will become

$$-s=a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}},$$

since the value of this now shall have been found, this case presents nothing new for us.

22. If there may be assumed

$$\frac{\varDelta-2a}{a}=-\frac{4}{3},\ \text{or}\ \varDelta=\frac{2}{3}a,$$

the following continued fraction thence arises :

$$a+\cfrac{1}{\frac{5}{3}a+\cfrac{1}{\frac{7}{3}a+\cfrac{1}{\frac{9}{3}a+\text{etc.}}}}$$

or if by setting $a=3\alpha$, the fraction will be

$$3\alpha+\cfrac{1}{5\alpha+\cfrac{1}{7\alpha+\cfrac{1}{9\alpha+\text{etc.}}}}$$

which is the first member from the above form itself truncated. The same arises in use, if there may be assumed

$$\frac{\varDelta-2a}{a}=-\frac{8}{3},\ \text{or}\ \varDelta=-\frac{2}{3}a,$$

from which this continued fraction arises, evidently on putting $a=3\alpha$:

$$3\alpha+\cfrac{1}{\alpha+\cfrac{1}{-\alpha+\cfrac{1}{-3\alpha+\cfrac{1}{-5\alpha+\text{etc.}}}}}$$

and thus we may reduce always to the principal form.

23. But the case of integrability in general may be held in this exponent : $-\frac{4i}{2i\pm1}$. Therefore on putting

$$\frac{\varDelta-2a}{a}=-\frac{4i}{2i\pm1},\ \text{there will become}\ \varDelta=\frac{\pm2a}{2i\pm1},$$

from which on putting $\frac{a}{2i\pm1}=\alpha$, there will become $\varDelta=\pm\,2\alpha$, therefore on account of $a=(2i\pm1)\alpha$, the continued fraction will be :

$$(2i\pm1)\alpha+\cfrac{1}{a\pm2\alpha+\cfrac{1}{a\pm4\alpha+\cfrac{1}{a\pm6\alpha+\text{etc.}}}}$$

where again it is evident all the odd numbers occur, thus so that thence the continued fraction arisen may be able always to be formed from our principal :

$$a+\cfrac{1}{a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}},$$

if either it truncated with some members, or it may be continued back to some upper member.

24. Therefore since all the cases of the integrable Riccati equations may lead to the same continued fraction, truly besides at this point no other cases will have been able to be established ; hence evidently it follows, if the indices of the continued fraction *a, b, c, d, e, f* etc. may constitutes some other arithmetical progression, then plainly in no manner can the sum be assigned, whenever that may depend on the case of an unresolvable Riccati equation. Thus so that if this continued fraction may be proposed :

$$a+\cfrac{1}{2a+\cfrac{1}{3a+\cfrac{1}{4a+\text{etc.}}}}$$

where $\Delta = a$, the summation will depend on this differential equation :

$$ady + yydt = \frac{dt}{t},$$

since the resolution of which may not be able to be extricated by any transcending quantities even now received in use, the value of this continued fraction depends neither on circular nor logarithmic quantities, nor may we seek it amongst all the quadratures of algebraic curves; from which it is little wonder, how the method used in the above dissertation for all such cases will have been without success.

25. Because still in the above dissertation we have given the sum of such continued fractions expressed by the two integral formulas, that expression itself will be able to be adapted also to the Riccati equation for the same cases, that which certainly may deserve the maximum attention, since at this stage besides in no way will this integrable case be able to be treated. On account of which it will be most worthwhile to compare the solutions among themselves, since hence help will be able to be sought without concern, and the Riccati equation being able to be treated successfully.

26. But I have shown in the above dissertation, if this continued fraction were proposed:

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

the value of that to be expressed by this fraction : $-\frac{A}{B}$, with

$$A=\int\frac{dx}{x^{2+\frac{b}{m}}\cdot e^{\frac{1+xx}{mx}}}$$

and

$$B=\int\frac{dx}{x^{1+\frac{b}{m}}\cdot e^{\frac{1+xx}{mx}}},$$

present, if indeed these two integrals may be extended from the limit $x=0$ as far as to the limit $x=\infty$.

27. Now we may compare this continued fraction with the general, as we have treated here :

$$a+\cfrac{1}{a+\varDelta+\cfrac{1}{a+2\varDelta+\cfrac{1}{a+3\varDelta+\text{etc.}}}}$$

the value of which is held in this equation:

$$ady+yydt=t^{\frac{\varDelta-2a}{a}}dt\,,$$

with the integration evidently put in place thus, so that in the case $t=0$ there may become $y=\infty$; then truly there may be put $t=1$, from which value of y the sum of this continued fraction will be expressed .

28. Now with the comparison put in place there may become

$$m=\varDelta \text{ and } b=\varDelta-a\,,$$

with which values introduced these integral formulas will become

$$A=\int\frac{dx}{x^{3-\frac{a}{\varDelta}}\cdot e^{\frac{1+xx}{\varDelta x}}}$$

and

$$B=\int\frac{dx}{x^{2-\frac{a}{\varDelta}}\cdot e^{\frac{1+xx}{\varDelta x}}},$$

with the integrals again taken from $x=0$ to $x=\infty$; on account of which the value of y, which results from the equation

$$ady + yydt = t^{\frac{\Delta-2a}{a}}dt$$

for the case $t = 1$, will be equal to this fraction : $-\frac{A}{B}$, or if there were

$$y = \frac{\int dx : x^{3-\frac{a}{\Delta}} \cdot e^{\frac{1+xx}{\Delta x}}}{\int dx : x^{2-\frac{a}{\Delta}} \cdot e^{\frac{1+xx}{\Delta x}}}.$$

Moreover though this equality may be seen to be a special case, in which there becomes $t = 1$, here truly $x = \infty$, yet perhaps a relation of this kind between the two variables *t* and *x* will be able to be found, so that in general the quantity *y* will be equal to that formula.

29. Just as the consideration of our continued fraction has led us to the resolution of the Riccati equation, thus a direct method is given, by which this equation can be resolved by continued fractions. So that which may be shown more easily, the Riccati equation, which commonly is accustomed to be proposed in this form :

$$dy + ayydx = acx^{2m}dx,$$

may be transformed into another form more convenient for the present situation, by putting

$$y = x^m z \text{ and } x^{m+1} = t;$$

then indeed, if in place of $\frac{a}{m+1}$ there may be written *b*, this equation appears :

$$dz + \frac{mzdt}{(m+1)t} + bzzdt = bcdt.$$

30. Again we may put $\frac{m}{m+1} = -n$, so that there shall be $m = -\frac{n}{1+n}$ and this shall be the equation proposed by us :

$$dz - n\frac{zdt}{t} + bzzdt = bcdt ,$$

which, if we may use this substitution:

$$z = \frac{n+1}{bt} + \frac{c}{\mathrm{v}}$$

may be transformed into this form :

$$d\mathrm{v} - (n+2)\frac{\mathrm{v}dt}{t} + b\mathrm{vv}dt = bcdt ,$$

which only differs from that before, so that here the number *n* shall be made greater by two; whereby if again we may make

$$\mathrm{v}=\frac{n+3}{bt}+\frac{c}{u},$$

this equation will be produced :

$$du-(n+4)udt+buudt=bcdt\ ,$$

from which it is apparent, if the first equation may admit a resolution in the case $n=k$, then also its resolution is going to be empowered for the cases

$$n=k+2,\ n=k+4,\ n=k+6,\ \text{etc.}$$

and in general for the case $n=k+2i$, with *i* denoting some whole number.

31. So that if now successively we may substitute these due values in place of *v* and *u* and the values of the following letters themselves, the following continued fraction will be produced for the quantity *z* :

$$z=\frac{n+1}{bt}+\cfrac{c}{\frac{n+3}{bt}+\cfrac{c}{\frac{n+5}{bt}+\text{etc.}}}$$

which expression therefore shows the value of the quantity *z*, which agrees with the strength of this equation itself:

$$dz-n\frac{zdt}{t}+bzzdt=bcdt,$$

if clearly thus it may be integrated, so that on putting $t=0$ there may become $z=\infty$.

32. We may free this form from partial fractions, and we will obtain this form :

$$\frac{1}{z}=\cfrac{bt}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\cfrac{bbctt}{n+7+\text{etc.}}}}}$$

from which it is at once clear, on putting $t=0$ to become $\frac{1}{z}=0$ and thus $z=\infty.$ In a similar manner we are able to begin the resolution from the transformed equation, which was

$$d\mathrm{v}-(n+2)\frac{\mathrm{v}dt}{t}+b\mathrm{vv}dt=bcdt\ ,$$

which may be transformed in the first by putting $\mathrm{v}=\frac{bct}{-n-1+btz}$; indeed this will produce

$$dz - \frac{nzdt}{t} + bzzdt = bcdt\,,$$

in which equation the number *n* returned is smaller by two ; from which it is apparent, if the equation may admit a resolution in the case $n = k$, then also to be a resolution in the cases to follow

$$n = k-2,\ n = k-4,\ n = k-6$$

and in general $n = k-2i$. Whereby since the resolution may labour under no difficulty in the case $n = 0$, on taking $k = 0$ all the cases admitting a resolution will be contained in this formula : $n = \pm 2i$ and hence it will be become accustomed for the form

$$m = \frac{\mp 2i}{\pm 2i+1} = -\frac{2i}{2i\pm 1},$$

which contains the most noteworthy of the integrable cases.

33. We may put $n+2 = -v$, so that the equation, from which we may begin here, shall be

$$d\text{v} + \frac{v\text{v}dt}{t} + b\text{vv}dt = bcdt,$$

which therefore, by putting

$$\text{v} = \frac{bct}{v+1+btz}$$

is changed into this form:

$$dz + (v+2)\frac{zdt}{t} + bzzdt = bcdt,$$

in which now the number *v* is increased by two. Whereby if further we may put :

$$z = \frac{bct}{v+3+bty}$$

this equation will arise :

$$dy + (v+4)\frac{ydt}{t} + byydt = bcdt,$$

and thus by progressing further there will be come to $v+5,\ v+7$ etc.

34. Therefore we may substitute these values into the above equation, which is

$$d\text{v} + \frac{v\text{v}dt}{t} + b\text{vv}dt = bcdt,$$

and the following continued fraction will be produced for v will produce the following continued fraction :

$$\mathrm{v} = \cfrac{bct}{v+1\cfrac{+bbctt}{v+3+\cfrac{bbctt}{v+5+\cfrac{bbctt}{v+7+\text{etc.}}}}}$$

Therefore this expression has a place, if the differential equation thus may be integrated, so that on putting $t = 0$ there may become $\mathrm{v} = 0$.

35. Therefore from the Riccati equation we have elicited a pair of continued fractions, which so that we may be able to compare more easily between them, in place of *v* we may again write *n*, so that we may have these two differential equations:

$$\text{I.}\quad dz - n\tfrac{zdt}{t} + bzzdt = bcdt,$$

$$\text{II.}\ d\mathrm{v} + n\tfrac{\mathrm{v}dt}{t} + b\mathrm{vv}dt = bcdt,$$

and this continued fraction will arise from the former :

$$\frac{1}{z} = \cfrac{bt}{n+1+\cfrac{+bbctt}{n+3+\cfrac{bbctt}{n+5+\text{etc.}}}}$$

truly there arises from the latter:

$$\mathrm{v} = \cfrac{bct}{n+1\cfrac{+bbctt}{n+3+\cfrac{bbctt}{n+5+\text{etc.}}}}$$

which fractions agree completely between themselves, hence since there may become $\mathrm{v} = \frac{c}{z}$, certainly in whatever manner the one equation actually may be changed into the other, thus so that these two forms may be taken for one.

36. This resolution of the Riccati equation into a continued fraction therefore is more attention worthy, because this equation at this point in no manner will have been able to be resolved into a regular infinite series. Which series indeed I have shown formerly for its resolution [see Vol. II, § 940, Euler's *Intro. Integral Calculus* , as well as E269 and E267], they have been prepared thus, so that one infinite series divided by another other may give rise to the resolution of the Riccati equation ; but here from the discussion, a single simple series is put in place. Hence therefore the question arises, whether perhaps other differential equations may not be given, of which the resolution equally may be allowed to be presented by continued fractions.

SUMMATIO FRACTIONIS CONTINUAE CUIUS INDICES PROGRESSIONEM ARITHMETICAM CONSTITUUNT DUM NUMERATORES OMNES SUNT UNITATES UBI SIMUL RESOLUTIO AEQUATIONIS RICCATIANAE PER HUIUSMODI FRACTIONES DOCETUR

Conventui exhibita die 18. septembris 1775

Opuscula analytica 2, 1785, p. 217-239

l. Cum in praecedente dissertatione methodum exposuissem, fractiones continuas ad duas formulas integrales reducendi, ea quidem infinitis casibus feliciter successit: at vero casus, qui simplicissimus videtur, ubi omnes numeratores inter se ponuntur aequales, ad eiusmodi formulas integrales perduxit, quas nullo adhuc modo evolvere et inter se comparare licuit, cum tamen ex hoc genere binae fractiones continuae habeantur, quarum valores satis commode exhiberi possunt:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\text{etc.}}}}=\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$$

et

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\text{etc.}}}}=\cot.\frac{2}{n},$$

quarum quidem altera ex altera facile deducitur, si loco n scribatur $n\sqrt{-1}$.

2. Quando autem indices aliam quamcunque progressionem arithmeticam sequuntur, summationem talium fractionum continuarum iam olim in Tomo XI veterum Commentariorum nostrae Academiae singulari prorsus modo ad aequationem RICCATIANAM reduxi. Methodus autem, qua hoc praestiti, ibi nimis succincte est exposita; quare, cum ea plurimum in recessu habere videatur, eam hic operae pretium erit uberius explicare; praecipue cum non solum nemo vim illius methodi animadvertisse videatur, sed etiam ipse eius penitus essem oblitus.

3. Quo isthanc investigationem clarius ob oculos ponam, exordiar a fractione generali, cuius quidem omnes numeratores sint unitates, indices autem in genere litteris *a*, *b*, *c*, *d*, *e*, *f* etc. designentur, ita ut ipsa fractio continua hanc habeat formam:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

cuius valor littera *S* indicetur, ad quem proxime saltem cognoscendum, ex indicibus *a, b, c, d, e* etc. formetur more solito series sequentium fractionum:

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} \end{array} \text{ etc.,}$$

quarum tam numeratores quam denominatores sequenti modo ex binis praecedentibus determinantur:

$$A=a,\ B=Ab+1,\ C=Bc+A,\ D=Cd+B \text{ etc.}$$
$$\mathfrak{A}=1,\ \mathfrak{B}=\mathfrak{A}b,\quad \mathfrak{C}=\mathfrak{B}c+\mathfrak{A},\ \mathfrak{D}=\mathfrak{C}d+\mathfrak{B} \text{ etc.}$$

4. Nunc autem loco litterarum *A, B, C, D* ... et $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$.. alias in calculum introducamus, quae sint

$$A'=\frac{A}{a},\ B'=\frac{B}{ab},\ C'=\frac{C}{abc},\ D'=\frac{D}{abcd},\ \text{etc.}$$
$$\mathfrak{A}'=\frac{\mathfrak{A}}{a},\ \mathfrak{B}'=\frac{\mathfrak{B}}{ab},\ \mathfrak{C}'=\frac{\mathfrak{C}}{abc},\ \mathfrak{D}'=\frac{\mathfrak{D}}{abcd},\ \text{etc.,}$$

atque hae novae litterae sequenti modo per indices et binos antecedentes terminos determinabuntur:

$$A'=1,\quad B'=A'+\frac{1}{ab},\ C'=B'+\frac{A'}{bc},\ D'=C'+\frac{B'}{cd},\ \text{etc.}$$
$$\mathfrak{A}'=\frac{1}{a},\ \mathfrak{B}'=\mathfrak{A}',\qquad \mathfrak{C}'=\mathfrak{B}'+\frac{\mathfrak{A}'}{bc},\ \mathfrak{D}'=\mathfrak{C}'+\frac{\mathfrak{B}'}{cd},\ \text{etc.}$$

His igitur valoribus pro quovis casu evolutis, istae fractiones:

$$\frac{A'}{\mathfrak{A}'},\ \frac{B'}{\mathfrak{B}'},\ \frac{C'}{\mathfrak{C}'},\ \frac{D'}{\mathfrak{D}'},\ \frac{E'}{\mathfrak{E}'}\ \text{ etc.}$$

continuo propius ad valorem *S* fractionis continuae propositae accedent, et in infinitum continuatae ei prorsus aequabuntur.

5. Quo formae harum litterarum melius perspiciantur, eas simpliciter per indices evolvamus, ac primo quidem pro numeratoribus reperiemus sequentes formulas:

$$A' = 1,$$
$$B' = 1 + \frac{1}{ab},$$
$$C' = 1 + \frac{1}{ab} + \frac{1}{bc},$$
$$D' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{abcd}$$
$$E' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde}$$
$$F' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{ef} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde}$$
$$+ \frac{1}{abef} + \frac{1}{bcef} + \frac{1}{cdef} + \frac{1}{abcdef}$$
etc.

Pro denominatoribus vero prodibunt sequentes formulae:

$$\mathfrak{A}' = \frac{1}{a}$$
$$\mathfrak{B}' = \frac{1}{a}$$
$$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$$
$$\mathfrak{D}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd}$$
$$\mathfrak{E}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{abcde}$$
$$\mathfrak{F}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{aef} + \frac{1}{abcde} + \frac{1}{abcef} + \frac{1}{acdef}$$
etc.

in quibus posterioribus formulis singuli termini factorem habent $\frac{1}{a}$, quo omisso reliqui factores eodem modo per indices *b, c, d, e, f* etc. definientur, quo litterae latinae antecedentes per omnes indices *a, b, c, d, e* etc. sunt determinatae.

6. Accommodemus nunc istas evolutiones ad casum, qui hic nobis est propositus, ubi indices *a, b, c, d, e* secundum progressionem arithmeticam procedere assumimus. Statuamus igitur differentiam, qua hi indices continuo crescunt $= \varDelta$, eruntque indices post primum sequentes

$$b = a + \varDelta,\; c = a + 2\varDelta,\; d = a + 3\varDelta,\; e = a + 4\varDelta \text{ etc.}$$

Hos quidem valores in denominatoribus nostrarum formularum non substituemus, sed iis praecipue utemur ad formulas contrahendas.

7. Hac igitur progressione stabilita evolvamus primo numeratores nostrarum fractionum sequenti modo:

$$A' = \frac{1}{a}$$
$$B' = 1 + \frac{1}{ab}$$
$$C' = 1 + \frac{1}{ab} + \frac{1}{bc} = 1 + \frac{2}{ac}$$
$$D' = 1 + \frac{2}{ac} + \frac{1}{cd} + \frac{1}{abcd} = 1 + \frac{3}{ac} + \frac{1}{abcd}$$
$$E' = 1 + \frac{3}{ad} + \frac{1}{de} + \frac{1}{abcd} + \frac{2}{acde} = 1 + \frac{4}{ae} + \frac{3}{abde}$$
$$F' = 1 + \frac{5}{af} + \frac{6}{abdf} + \frac{1}{abcdef}$$
$$G' = 1 + \frac{6}{ag} + \frac{10}{abfg} + \frac{4}{abcefg}$$
$$H' = 1 + \frac{7}{ah} + \frac{15}{abgh} + \frac{10}{abcfgh} + \frac{1}{abcdefgh}$$
$$I' = 1 + \frac{8}{ai} + \frac{21}{abhi} + \frac{10}{abcghi} + \frac{5}{abcdfghi}$$
$$K' = 1 + \frac{9}{ak} + \frac{28}{abik} + \frac{35}{abchik} + \frac{15}{abcdghik} + \frac{1}{abcdefghik}$$

etc.

8. Quemadmodum in his formis omnes termini primi sunt unitates, ita numeratores secundorum secundum numeros naturales, tertiorum secundum trigonales, quartorum secundum pyramidales primos, tum secundos, tertios etc. progrediuntur. In denominatoribus ordo pariter satis est manifestus. Hinc igitur in genere eam formulam exhibere poterimus, quae indefinite respondeat indici *i*. Ita si ista expressio designetur littera *Z'*, tum vero in ordine litterarum *a, b, c, d, e* etc. fuerit $z = a + i\Delta$, antecedentes vero

$$y = a + (i-1)\Delta,\ x = a + (i-2)\Delta,\ v = a + (i-3)\Delta \text{ etc.},$$

habebimus

$$Z' = 1 + \frac{i}{az} + \frac{(i-1)(i-2)}{1\cdot 2abyz} + \frac{(i-2)(i-3)(i-4)}{1\cdot 2\cdot 3abcxyz} + \frac{(i-3)(i-4)(i-5)(i-6)}{1\cdot 2\cdot 3\cdot 4abcdvxyz} + \text{ etc.},$$

9. Quod iam ad litteras germanicas $\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{D}'$ etc. attinet, earum quaelibet ex antecedente latina formatur, dum litterae *a*, *b, c, d*, *e* uno gradu promoventur, tum vero singuli termini per $\frac{1}{a}$ multiplicentur, ita erit ut sequitur:

$$\mathfrak{A}'=1$$
$$\mathfrak{B}'=\frac{1}{a}$$
$$\mathfrak{C}'=\frac{1}{a}+\frac{1}{abc}$$
$$\mathfrak{D}'=\frac{1}{a}+\frac{2}{abd}$$
$$\mathfrak{E}'=\frac{1}{a}+\frac{3}{abe}+\frac{1}{abcde}$$
$$\mathfrak{F}'=\frac{1}{a}+\frac{4}{abf}+\frac{3}{abcef}$$
$$\mathfrak{G}'=\frac{1}{a}+\frac{5}{abg}+\frac{6}{abcfg}+\frac{1}{abcdefg}$$
$$\mathfrak{H}'=\frac{1}{a}+\frac{6}{abh}+\frac{10}{abcgh}+\frac{4}{abcdfgh}$$
$$\mathfrak{I}'=\frac{1}{a}+\frac{7}{abi}+\frac{15}{abchi}+\frac{10}{abcdghi}+\frac{1}{abcdefghi}$$
etc.,

unde in genere, si indici *i* respondeat $\mathfrak{Z}'$, erit

$$\mathfrak{Z}'=\frac{1}{a}+\frac{(i-1)}{abz}+\frac{(i-2)(i-3)}{1\cdot2\cdot abcyz}+\frac{(i-3)(i-4)(i-5)}{1\cdot2\cdot3\cdot abcdxyz}+\frac{(i-4)(i-5)(i-6)(i-7)}{1\cdot2\cdot3\cdot4 abcdevxyz}+\text{etc.}$$

10. Augeamus nunc indicem *z* in infinitum, ut fractio $\frac{Z'}{\mathfrak{Z}'}$ exprimat ipsum valorem fractionis continuae, quem vocavimus *S*, ita ut futurum sit $S=\frac{Z'}{\mathfrak{Z}'}$, atque reductio formularum inventarum sequenti modo peragetur. Primo scilicet pro *Z'* erit

$$\frac{i}{z}=\frac{i}{a+i\Delta}=\frac{1}{\Delta},\ \text{ob } i=\infty;$$

pro termino tertio erit,

$$\frac{(i-1)(i-2)}{yz}=\frac{(i-1)(i-2)}{(a+(i-1)\Delta)(a+i\Delta)}=\frac{1}{\Delta^2};$$

porro autem simili modo

$$\frac{(i-2)(i-3)(i-4)}{xyz}=\frac{(i-2)(i-3)(i-4)}{(a+(i-2)\Delta)(a+(i-1)\Delta)(a+i\Delta)}=\frac{1}{\Delta^3}$$

et

$$\frac{(i-3)(i-4)(i-5)(i-6)}{vxyz}=\frac{1}{\Delta^4}\ \text{etc.}$$

Pari modo pro formula $\mathfrak{Z}'$ erit etiam

$$\frac{i-1}{z}=\frac{1}{\Delta},\quad \frac{(i-2)(i-3)}{yz}=\frac{1}{\Delta^2}$$

et ita porro; quamobrem pro casu $i = \infty$ ambae nostrae formulae ita commode contrahuntur, ut fiat

$$Z' = 1 + \frac{1}{a\Delta} + \frac{1}{1\cdot 2ab\Delta^2} + \frac{1}{1\cdot 2\cdot 3abc\Delta^3} + \frac{1}{1\cdot 2\cdot 3\cdot 4abcd\Delta^4} + \text{etc.}$$

similique modo

$$\mathfrak{Z}' = \frac{1}{a} + \frac{1}{ab\Delta} + \frac{1}{1\cdot 2abc\Delta^2} + \frac{1}{1\cdot 2\cdot 3abcd\Delta^3} + \frac{1}{1\cdot 2\cdot 3\cdot 4abcde\Delta^4} + \text{etc.}$$

11. Cum igitur sit valor quaesitus $S = \frac{Z'}{\mathfrak{Z}'}$, videamus, quomodo ambas series infinitas inventas ad expressiones finitas revocare queamus. Hunc in finem ambas series generaliores reddamus, dum loco numeratorum, qui omnes sunt 1, progressionem quandam geometricam substituimus. Statuamus igitur

$$p = 1 + \frac{x^{\Delta}}{a\Delta} + \frac{x^{2\Delta}}{1\cdot 2ab\Delta^2} + \frac{x^{3\Delta}}{1\cdot 2\cdot 3abc\Delta^3} + \text{etc.}$$

et

$$q = \frac{1}{a} + \frac{x^{\Delta}}{ab\Delta} + \frac{x^{2\Delta}}{1\cdot 2abc\Delta^2} + \frac{x^{3\Delta}}{1\cdot 2\cdot 3abcd\Delta^3} + \text{etc.}$$

Quod si autem hinc valores *p* et *q* eruerimus in genere, tum vero ponamus $x = 1$, utique proveniet $S = \frac{p}{q}$. Hic autem manifestum est, ambas has series *q* egregiam inter se tenere affinitatem, ac per differentiationem unam in alteram converti posse, quam investigationem sequenti modo instituamus.

12. Primo igitur series prior simpliciter differentiata dat

$$\frac{xdp}{dx} = \frac{x^{\Delta}}{a} + \frac{x^{2\Delta}}{ab\Delta} + \frac{x^{3\Delta}}{1\cdot 2abc\Delta^2} + \frac{x^{4\Delta}}{1\cdot 2\cdot 3abcd\Delta^3} + \text{etc.},$$

quae series cum altera *q* comparata manifesto praebet

$$\frac{xdp}{dx} = x^{\Delta}q,$$

unde patet, si modo summa alterius harum duarum serierum esset cognita, alteram quoque assignari posse, quandoquidem ex cognito valore *p* prodit

$$q = \frac{dp}{x^{\Delta-1}dx};$$

contra vero ex cognito valore *q* fit

$$dp = x^{\Delta-1}qdx,$$

ideoque

$$p=\int x^{\Delta-1}qdx,$$

quod integrale ita sumi debet, ut posito $x=0$ fiat $p=1$.

13. Ante autem quam alteram seriem differentiemus, eam multiplicemus per x^a, atque ob

$$a+\Delta=b,\ a+2\Delta=c,\ a+3\Delta=d \text{ etc.}$$

erit

$$x^a q=\frac{x^a}{a}+\frac{x^b}{ab\Delta}+\frac{x^c}{1\cdot 2abc\Delta^2}+\frac{x^d}{1\cdot 2\cdot 3abcd\Delta^3}+\text{etc.}$$

Nunc ista aequatio differentiata iterumque per x multiplicata dabit

$$xd\cdot\frac{x^a q}{dx}=x^a+\frac{x^b}{a\Delta}+\frac{x^c}{1\cdot 2ab\Delta^2}+\frac{x^d}{1\cdot 2\cdot 3abc\Delta^3}+\text{etc.},$$

at vero prior series p itidem per x^a multiplicata praebet

$$x^a p=x^a+\frac{x^b}{a\Delta}+\frac{x^c}{1\cdot 2ab\Delta^2}+\frac{x^d}{1\cdot 2\cdot 3abc\Delta^3}+\text{etc.}$$

quae series cum sint perfecte aequales, erit

$$\frac{x}{dx}d\cdot x^a q=x^a p,\ \text{ideoque } d\cdot x^a q=px^{a-1}dx,$$

sicque duas nacti sumus aequationes differentiales inter p et q, ex quibus valorem utriusque eruere licebit.

14. Cum ex priore aequatione sit

$$q=\frac{dp}{x^{\Delta-1}dx}\ \text{, erit } x^a q=\frac{x^{a-\Delta+1}dp}{dx},$$

unde sumto elemento dx constante fiet

$$d\cdot x^a q=\frac{x^{a-\Delta+1}ddp+(a-\Delta+1)x^{a-\Delta}dpdx}{dx},$$

sicque elisa quantitate q pro altera p nanciscimur hanc aequationem differentialem secundi gradus:

$$px^{a-1}dx=\frac{x^{a-\Delta+1}ddp+(a-\Delta+1)x^{a-\Delta}dpdx}{dx},$$

quam si resolvere licuerit, totum negotium erit confectum. Ubi probe notandum, cum sit

$$p = 1 + \frac{x^{\Delta}}{a\Delta} + \frac{x^{2\Delta}}{1 \cdot 2ab\Delta^2} + \text{etc.},$$

integrationem ita institui debere, ut posito $x = 0$ fiat $p = 1$; tum vero, quia est

$$\frac{dp}{dx} = \frac{x^{\Delta-1}}{a} + \frac{x^{2\Delta-1}}{ab\Delta} + \text{etc.},$$

altera conditio integrationis postulat, ut posito $x = 0$ etiam fiat $\frac{dp}{dx} = 0$, si quidem fuerit $\Delta > 1$; si enim esset $= 1$, casu $x = 0$ fieri debebit $\frac{dp}{dx} = \frac{1}{a}$. At si $\Delta < 1$, fieri debebit $\frac{dp}{dx} = \infty$.

15. Cum fractio posito $x = 1$ praebeat valorem nostrae fractionis continuae *S*, ponamus in genere esse $\frac{p}{q} = z$, ita ut posito $x = 1$ fiat $z = S$, unde patet, casu $x = 0$ fieri debere $z = a$. Cum igitur sit $p = qz$, erit $dp = qdz + zdq$; erat autem ex prima aequatione

$$q = \frac{dp}{x^{\Delta-1}dx}$$

quamobrem habebimus

$$q = \frac{qdz + zdq}{x^{\Delta-1}dx},$$

sive $x^{\Delta-1}qdx = qdz + zdq$, unde fit

$$dq = \frac{x^{\Delta-1}qdx - qdz}{z};$$

erat autem $dp = x^{\Delta-1}qdx$.

16. Haec igitur sequuntur ex priori aequatione differentiali inventa $\frac{dp}{x^{\Delta-1}dx} = q$. Altera vero, quae est

$$d \cdot x^a q = px^{a-1}dx,$$

quia

$$d \cdot x^a q = ax^{a-1}qdx + x^a dq$$

loco *dq* posito valore modo invento prodit

$$d \cdot x^a q = ax^{a-1}qdx + \frac{x^{a+\Delta-1}qdx - x^a qdz}{z},$$

habebimus

$$px^{a-1}dx = ax^{a-1}qdx + \frac{x^{a+\Delta-1}qdx - x^a qdz}{z},$$

quae aequatio, ob $p = qz$, per q divisa suppeditat istam aequationem differentialem primi gradus:

$$x^{a-1}zdx = ax^{a-1}dx + \frac{x^{a+\Delta-1}dx - x^a dz}{z},$$

quae per z multiplicata et per x^{a-1} divisa praebet

$$zzdx = azdx + x^{\Delta}dx - xdz,$$

cuius ergo resolutio si ita instituatur, ut posito $x = 0$ fiat $z = a$, tum vero fiat $x = 1$, valor ipsius z dabit ipsum valorem fractionis continuae quem quaerimus.

17. Totum igitur negotium perductum est ad resolutionem aequationis differentialis primi gradus

$$zzdx = azdx + x^{\Delta}dx - xdz,$$

quae manifesto in celeberrima illa aequatione RICCATIANA continetur. Ut enim ad tres tantum terminos reducatur, ponatur $z = x^a y$, ita ut sit $y = \frac{z}{x^a}$, unde fit

$$dz = ax^{\Delta-1}ydx + x^a dy,$$

quibus valoribus substitutis nostra aequatio hanc induet formam:

$$x^{a+1}dy + x^{2a}yydx = x^{\Delta}dx,$$

quam ergo ita integrari oportet, ut posito $x = 0$, sive infinite parvo, fiat $y = \frac{a}{x^a}$, hoc est $y = \infty$, quo facto, si post integrationem statuatur $x = 1$, valor ipsius y dabit summam fractionis continuae propositae.

18. Quo hanc expressionem simpliciorem reddamus, dividamus per x^{a+1}, ut habeamus

$$dy + x^{a-1}yydx = x^{\Delta-a-1}dx;$$

nunc vero statuamus $x^a = t$, ita ut casu $x = 0$ fiat quoque $t = 0$, et casu $x = 1$ etiam $t = 1$, unde si t evanescat, fieri debet $y = \frac{a}{t}$, sive $y = \infty$. Hoc autem valore introducto, ob $x = t^{\frac{1}{a}}$ et

$$dx = \frac{t^{\frac{1-a}{a}}dt}{a},$$

aequatio nostra fiet

$$ady + yydt = t^{\frac{\Delta-2a}{a}}dt,$$

quae est forma maxime usitata aequationis RICCATIANAE.

19. Quando ergo proposita fuerit talis fractio continua :

$$a + \cfrac{1}{a+\Delta+\cfrac{1}{a+2\Delta+\cfrac{1}{a+3\Delta+\cfrac{1}{a+4\Delta+\text{etc.}}}}}$$

pro eius valore investigando resolvi debet ista aequatio RICCATIANA:

$$ady + yydt = t^{\frac{\Delta-2a}{a}}dt;$$

ubi integrationem ita institui oportet, ut sumta *t* infinite parva fiat $y = \frac{a}{t}$ quo facto statuatur $t = 1$, et valor pro y resultans erit valor huius fractionis continuae.

20. Evolvamus casum simplicissimum, quo ad dextram partem exponens ipsius *t* fit nihilo aequalis, quod ergo evenit, si $\Delta = 2a$, ideoque ipsa fractio continua

$$a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

pro cuius summa habebimus hanc aequationem differentialem:

$$ady + yydt = dt, \text{unde } dt = \frac{ady}{1-yy},$$

et integrando

$$t = \frac{a}{2} l \frac{1+y}{1-y} + C,$$

quae constans *C* ita est capienda, ut posito $t = 0$ fiat $y = \frac{a}{t}$, ideoque $t = \frac{a}{y};$ unde patet hoc casu fieri $y = \infty$, ex quo statim intelligitur, aequationem integralem ita instrui debere:

$$t = \frac{a}{2} l \frac{1+y}{1-y} + C,$$

sive integrationem ita institui, ut facto $t = 0$ fiat *y* infinitum. Nunc vero si *y* infinitum, erit $l \frac{y+1}{y-1} = \frac{2}{y},$ quare, cum fieri debeat $t = \frac{a}{y}$, fit $C = 0$, ita ut iusta aequatio integralis sit $t = \frac{a}{2} l \frac{y+1}{y-1},$ unde denotante *e* numerum, cuius logarithmus hyperbolicus $= 1$, fiet

$e^{\frac{2t}{a}} = \frac{y+1}{y-1}$, hincque porro $y = \frac{e^{\frac{2t}{a}}+1}{e^{\frac{2t}{a}}-1}$, unde posito $t=1$ summa nostrae fractionis continuae erit $\frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1}$, qui est idem valor, quem iam dudum inveneram.

21. Contemplemur nunc etiam reliquos casus integrabilitatis aequationis RICCATIANAE, quibus exponens ipsius t ad partem dextram est vel -4, vel $-\frac{4}{3}$, vel $-\frac{8}{3}$, vel $-\frac{12}{5}$, vel $-\frac{12}{5}$, vel $-\frac{12}{7}$, etc. Sit igitur primo $\frac{\Delta-2a}{a}=-4$, vel $\Delta=-2a$, unde nascitur haec fractio continua :

$$a+\cfrac{1}{-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\cfrac{1}{-7a+\text{etc.}}}}}$$

quae manifesto a praecedente pendet. Si enim ponamus

$$-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\text{etc.}}} = s,$$

mutatis signis erit

$$-s=a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

cuius valor cum iam sit inventus, iste casus nihil novi nobis offert.

22. Si sumatur

$$\frac{\Delta-2a}{a}=-\frac{4}{3}, \text{ sive } \Delta=\frac{2}{3}a,$$

inde sequens nascitur fractio continua:

$$a+\cfrac{1}{\frac{5}{3}a+\cfrac{1}{\frac{7}{3}a+\cfrac{1}{\frac{9}{3}a+\text{etc.}}}}$$

sive statuendo $a=3\alpha$, erit fractio

$$3\alpha+\cfrac{1}{5\alpha+\cfrac{1}{7\alpha+\cfrac{1}{9\alpha+\text{etc.}}}}$$

quae est ipsa forma superior primo membro truncata. Idem usu venit, si sumatur

$$\frac{\Delta-2a}{a}=-\frac{8}{3},\ \text{sive}\ \ \Delta=-\frac{2}{3}a,$$

unde oritur haec fractio continua, ponendo scilicet $a=3\alpha$:

$$3\alpha+\cfrac{1}{\alpha+\cfrac{1}{-\alpha+\cfrac{1}{-3\alpha+\cfrac{1}{-5\alpha+\text{etc.}}}}}$$

sicque semper ad principalem formam reducimur.

23. Casus autem integrabilitatis in genere hoc continentur exponente: $-\frac{4i}{2i\pm1}$. Posito igitur

$$\frac{\Delta-2a}{a}=-\frac{4i}{2i\pm1},\ \text{fiet}\ \Delta=\frac{\pm2a}{2i\pm1},$$

unde posito $\frac{a}{2i\pm1}=\alpha$, erit $\Delta=\pm\,2\alpha$, ergo ob $a=(2i\pm1)\alpha$ fractio continua erit

$$(2i\pm1)\alpha+\cfrac{1}{a\pm2\alpha+\cfrac{1}{a\pm4\alpha+\cfrac{1}{a\pm6\alpha+\text{etc.}}}}$$

ubi manifesto iterum omnes numeri impares occurrunt, ita ut fractio continua inde nata semper formari possit ex nostra principali

$$a+\cfrac{1}{a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}}$$

si vel aliquot membris truncetur, vel retro ad aliquot membra superne continuetur.

24. Cum igitur omnes casus integrabilitatis aequationis RICCATIANAE ad eandem fractionem continuam perducant, praeterea vero nulli adhuc alii casus evolvi potuerint; hinc manifesto sequitur, si indices fractionis continuae *a*, *b*, *c*, *d*, *e*, *f* etc. quamcunque aliam progressionem arithmeticam constituant, tum summam nullo plane modo assignari posse, quandoquidem ea pendet a casu irresolubili aequationis RICCATIANAE. Ita si proponatur haec fractio continua:

$$a+\cfrac{1}{2a+\cfrac{1}{3a+\cfrac{1}{4a+\text{etc.}}}}$$

ubi est $\Delta=a$, summatio pendebit ab ista aequatione differentiali:

$$ady + yydt = \frac{dt}{t},$$

cuius resolutio cum per nullas quantitates transcendentes etiam nunc usu receptas expediri possit, valorem huius fractionis continuae frustra inter quantitates a circulo vel a logarithmis pendentes, vel adeo inter omnes quadraturas curvarum algebraicarum quaerimus; unde mirum non est, quod methodus in superiori dissertatione usitata pro talibus casibus omni successu caruerit.

25. Quoniam tamen in dissertatione superiore summam talium fractionum continuarum per binas formulas integrales expressam dedimus, illa ipsa expressio etiam ad aequationem RICCITIANAM pro iisdem casibus accommodari poterit, id quod utique maximam attentionem meretur, cum nullo adhuc modo ista aequatio praeter casus integrabiles tractari potuerit. Quamobrem maxime operae erit pretium solutiones inter se comparare, quandoquidem hinc haud contemnendum subsidium, aequationem RICCITIANAM feliciori successu tractandi, expectari poterit.

26. In superiori autem dissertatione ostendi, si haec proposita fuerit fractio continua:

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

eius valorem exprimi per hanc fractionem: $-\frac{A}{B}$, existente

$$A = \int \frac{dx}{x^{2+\frac{b}{m}} \cdot e^{\frac{1+xx}{mx}}}$$

et

$$B = \int \frac{dx}{x^{1+\frac{b}{m}} \cdot e^{\frac{1+xx}{mx}}},$$

siquidem haec duo integralia a termino $x = 0$ usque ad terminum $x = \infty$ extendantur.

27. Comparemus nunc hanc fractionem continuam cum generali, quam hic tractavimus:

$$a+\cfrac{1}{a+\Delta+\cfrac{1}{a+2\Delta+\cfrac{1}{a+3\Delta+\text{etc.}}}}$$

cuius valor continetur in hac aequatione:

$$ady + yydt = t^{\frac{\Delta-2a}{a}} dt,$$

integratione scilicet ita instituta, ut casu $t = 0$ fiat $y = \infty$; tum vero statuatur $t = 1$, unde valor ipsius y summam istius fractionis continuae exprimet.

28. Comparatione igitur instituta fiet

$$m = \Delta \text{ et } b = \Delta - a,$$

quibus valoribus introductis formulae illae integrales erunt

$$A = \int \frac{dx}{x^{3-\frac{a}{\Delta}} \cdot e^{\frac{1+xx}{\Delta x}}}$$

et

$$B = \int \frac{dx}{x^{2-\frac{a}{\Delta}} \cdot e^{\frac{1+xx}{\Delta x}}},$$

integralibus iterum sumtis ab $x = 0$ ad $x = \infty$; quamobrem valor ipsius *y*, qui ex aequatione

$$ady + yydt = t^{\frac{\Delta - 2a}{a}} dt$$

pro casu $t = 1$ resultat, aequalis erit isti fractioni: $-\frac{A}{B}$, sive erit

$$y = \frac{\int dx : x^{3-\frac{a}{\Delta}} \cdot e^{\frac{1+xx}{\Delta x}}}{\int dx : x^{2-\frac{a}{\Delta}} \cdot e^{\frac{1+xx}{\Delta x}}}.$$

Quanquam autem haec aequalitas tantum casus speciales, quibus ibi fit $t = 1$, hic vero $x = \infty$, spectat, tamen fortasse eiusmodi relatio inter binas variabiles *t* et *x* inveniri poterit, ut in genere quantitas *y* illi formulae aequetur.

29. Quemadmodum consideratio fractionis nostrae continuae nos ad resolutionem aequationis RICCATIANAE perduxit, ita vicissim datur methodus directa, qua ista aequatio per fractiones continuas resolvi potest. Quod quo facilius ostendatur, aequatio RICCATIANA, quae vulgo hac forma proponi solet:

$$dy + ayydx = acx^{2m}dx,$$

in aliam formam ad praesens institutum magis accommodatam transfundatur, ponendo

$$y = x^m z \text{ et } x^{m+1} = t;$$

tum enim, si loco $\frac{a}{m+1}$ scribatur *b*, prodit ista aequatio:

$$dz + \frac{mzdt}{(m+1)t} + bzzdt = bcdt.$$

30. Ponamus porro $\frac{m}{m+1} = -n$, ut sit $m = -\frac{n}{1+n}$ et nobis proposita sit ista aequatio:

$$dz - n\frac{zdt}{t} + bzzdt = bcdt \text{ ,}$$

quae, si adhibeamus hanc substitutionem:

$$z = \frac{n+1}{bt} + \frac{c}{v}$$

transmutatur in hanc formam:

$$dv - (n+2)\frac{vdt}{t} + bvvdt = bcdt \text{ ,}$$

quae a priore hoc tantum differt, ut hic numerus *n* binario maior sit factus; quare si porro faciamus

$$v = \frac{n+3}{bt} + \frac{c}{u},$$

prodibit haec aequatio:

$$du - (n+4)udt + buudt = bcdt \text{ ,}$$

unde patet, si prima aequatio resolutionem admittat casu $n = k$, tum etiam eius resolutionem in potestate fore casibus

$$n = k+2,\ n = k+4,\ n = k+6, \text{ etc.}$$

et in genere casu $n = k + 2i$, denotante *i* numerum integrum quemcunque.

31. Quod si iam successive loco *v* et *u* et sequentium litterarum valores istos debitos substituamus, pro quantitate *z* sequens prodibit fractio continua:

$$z = \frac{n+1}{bt} + \cfrac{c}{\frac{n+3}{bt} + \cfrac{c}{\frac{n+5}{bt} + \text{etc.}}}$$

quae ergo expressio exhibet valorem quantitatis *z*, qui ipsi convenit vi huius aequationis:

$$dz - n\frac{zdt}{t} + bzzdt = bcdt,$$

si scilicet ita integretur, ut posito $t = 0$ fiat $z = \infty$.

32. Liberemus hanc formam a fractionibus partialibus, et obtinebimus hanc formam:

$$\frac{1}{z}=\cfrac{bt}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\cfrac{bbctt}{n+7+\text{etc.}}}}}$$

unde statim patet, posito $t=0$ fore $\frac{1}{z}=0$ ideoque $z=\infty.$ Simili modo resolutionem ab aequatione transformata exordiri possumus, quae erat

$$dv-(n+2)\frac{vdt}{t}+bvvdt=bcdt\,,$$

quae in primam transformatur ponendo $v=\frac{bct}{-n-1+btz}$; prodibit enim

$$dz-\frac{nzdt}{t}+bzzdt=bcdt\,,$$

in qua aequatione numerus *n* binario redditus est minor; unde patet, si aequatio resolutionem admittat casu $n=k$, tum etiam resolutionem esse successuram casibus

$$n=k-2,\ n=k-4,\ n=k-6$$

et in genere $n=k-2i$. Quare cum resolutio nulla laboret difficultate casu $n=0$, sumto $k=0$ omnes casus resolutionem admittentes continebuntur in hac formula: $n=\pm 2i$ hincque pro forma consueta fiet

$$m=\frac{\mp 2i}{\pm 2i+1}=-\frac{2i}{2i\pm 1},$$

quae continet casus notissimos integrabilitatis.

33. Ponamus $n+2\ =-v$, ut aequatio, a qua hic inchoamus, sit

$$dv+\frac{vvdt}{t}+bvvdt=bcdt,$$

quae ergo, posito

$$v=\frac{bct}{v+1+btz}$$

transmutatur in hanc formam:

$$dz+(v+2)\frac{zdt}{t}+bzzdt=bcdt,$$

in qua nunc numerus v binario augetur. Quare si ulterius ponamus

$$z=\frac{bct}{v+3+bty}$$

orietur haec aequatio:

$$dy+(v+4)\tfrac{ydt}{t}+byydt=bcdt,$$

sicque ulterius progrediendo pervenietur ad $v+5,\ v+7$ etc.

34. Substituamus ergo istos valores in superiore aequatione, quae est

$$dv+\tfrac{vvdt}{t}+bvvdt=bcdt,$$

ac pro *v* prodibit sequens fractio continua:

$$v=\cfrac{bct}{v+1\cfrac{+bbctt}{v+3+\cfrac{bbctt}{v+5+\cfrac{bbctt}{v+7+\text{etc.}}}}}$$

Haec igitur expressio locum habet, si aequatio differentialis ita integretur, ut posito $t=0$ fiat $v=0$.

35. Geminas igitur ex aequatione RICCATIANA elicuimus fractiones continuas, quas quo facilius inter se comparare queamus, loco *v* scribamus iterum *n*, ut habeamus has duas aequationes differentiales:

$$\text{I.}\quad dz-n\tfrac{zdt}{t}+bzzdt=bcdt,$$
$$\text{II.}\ dv+n\tfrac{vdt}{t}+bvvdt=bcdt,$$

atque ex priore nascetur ista fractio continua:

$$\frac{1}{z}=\cfrac{bt}{n+1+\cfrac{+bbctt}{n+3+\cfrac{bbctt}{n+5+\text{etc.}}}}$$

ex altera vero oritur

$$v=\cfrac{bct}{v+1\cfrac{+bbctt}{v+3+\cfrac{bbctt}{v+5+\text{etc.}}}}$$

quae ergo fractiones prorsus inter se conveniunt, oum hinc fiat $v=\frac{c}{z}$, quippe quo modo altera aequatio in alteram actu convertitur, ita ut hae duae formae pro unica sint habendae.

36. Ista resolutio aequationis RICCATIANAE in fractionem continuam eo magis est attentione digna, quod haec aequatio nullo adhuc modo in seriem infinitam regularem resolvi potuerit. Quas enim series olim pro eius resolutione exhibui, ita sunt comparatae, ut una series infinita per aliam divisa resolutionem aequationis RICCATIANAE

suppeditet; hic autem de unica serie simplici sermo instituitur. Hinc igitur nascitur quaestio, num forte non etiam aliae aequationes differentiales dentur, quarum resolutionem pariter per fractiones continuas expedire liceat.