CONCERNING THE TRANSFORMATION OF SERIES

INTO CONTINUED FRACTIONS

WHERE LIKEWISE THIS THEORY SHALL BE GREATLY ENLARGED

[E593]
Opuscula analytica 2, 1785, p. 138-177

1. We may consider some continued fraction, which shall be

$$s = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

and in the first place we may seek the simple fractions , which continually will approach closer to the value of *s*, which we may form thus, so that there shall be :

$$\frac{A}{\mathfrak{A}} = a,\ \frac{B}{\mathfrak{B}} = a + \tfrac{1}{b},\ \frac{C}{\mathfrak{C}} = a + \cfrac{1}{b + \cfrac{1}{c}},\ \frac{D}{\mathfrak{D}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}} \quad \text{etc.}$$

Therefore the final value of this fraction will express the true value of the continued fraction proposed. Hence therefore to become apparent at once :

$$\frac{A}{\mathfrak{A}} = \frac{a}{1},\ \frac{B}{\mathfrak{B}} = \frac{ab+1}{b},\ \frac{C}{\mathfrak{C}} = \frac{abc+a+c}{bc+1}.$$

But just how these fractions may progress further, we may examine in the following manner.

2. Here it is evident the second factor to arise from the first, if in place of *a* there may be written

$$a + \frac{1}{b},$$

and in a similar manner the third to arise from the second, if in place of *b* there may be written

$$b + \frac{1}{c},$$

truly the fourth from the third, if in place of c there may be written

$$c+\frac{1}{d},$$

and thus henceforth. Hence therefore, if the indefinite fraction $\frac{P}{\mathfrak{P}}$ shall be formed from the indices *a, b, c, d, ... p* and the two sequences $\frac{Q}{\mathfrak{Q}}$ and $\frac{R}{\mathfrak{R}}$ may be put in place, which correspond to the indices *a, b, c, d, ... q* and *a, b, c, d, e, . . . r,* it is evident from the fraction $\frac{P}{\mathfrak{P}}$ the following $\frac{Q}{\mathfrak{Q}}$ to be found, if in place of p there may be written

$$p+\frac{1}{q},$$

that truly from $\frac{Q}{\mathfrak{Q}}$ the following $\frac{R}{\mathfrak{R}}$ to arise, if in place of q there may be written

$$q+\frac{1}{r}.$$

Now truly it is easy to see in the fraction $\frac{P}{\mathfrak{P}}$ both the numerator P as well as the denominator $\mathfrak{P}$ thus involve all the letters *a, b, c, d, ... p* , so that none of these may rise above the first dimension. Indeed if all the indices *a, b, c, d, e* etc. may be considered to be unequal, neither a square nor a higher power will be able to occur anywhere.

3. On account of which terms of two kinds occur both in P as well as in $\mathfrak{P}$, while the one kind plainly does not contain the index p, truly the other kind involves that as a factor; from which the numerator P will have a form of this kind $M+Np$ and in a similar manner $\mathfrak{P}$ will have this form $\mathfrak{M}+\mathfrak{N}p$, thus so that there shall be :

$$\frac{P}{\mathfrak{P}}=\frac{M+Np}{\mathfrak{M}+\mathfrak{N}p}.$$

Therefore in place of p we may write this form :

$$p+\frac{1}{q},$$

so that we may obtain the fraction $\frac{Q}{\mathfrak{Q}}$, which therefore, after we will have multiplied above and below by q, there will be

$$\frac{Q}{\mathfrak{Q}}=\frac{Mq+Npq+N}{\mathfrak{M}q+\mathfrak{N}pq+\mathfrak{N}}=\frac{(M+Np)q+N}{\mathfrak{N}+(\mathfrak{M}+\mathfrak{N}p)q}.$$

Now so that hence we may obtain the following $\frac{R}{\mathfrak{R}}$, in place of q we may write

$$q+\frac{1}{r},$$

and after we will have multiplied above and below by r , there will arise

$$\frac{R}{\mathfrak{R}}=\frac{Nr+(M+Np)qr+M+Np}{\mathfrak{N}r+(\mathfrak{M}+\mathfrak{N}p)qr+\mathfrak{M}+\mathfrak{N}p},$$

or

$$\frac{R}{\mathfrak{R}}=\frac{M+Np+(N+Mq+Npq)r}{\mathfrak{M}+\mathfrak{N}p+(\mathfrak{N}+\mathfrak{M}q+\mathfrak{N}pq)r}.$$

Therefore since there shall be $P=M+Np$ and $Q=N+(M+Np)q$, there will be

$$R=P+Qr.$$

In a similar manner since there shall be $\mathfrak{P}=\mathfrak{M}+\mathfrak{N}p$ and $\mathfrak{Q}=\mathfrak{N}+(\mathfrak{M}+\mathfrak{N}p)q,$ there will be

$$\mathfrak{R}=\mathfrak{P}+\mathfrak{Q}r.$$

And thus it is apparent, how any of our simple fractions may be able to be formed easily from the two preceding ones.

4. Behold therefore a plain and clear enough demonstration of our rules for the conversion of continued fractions into simple fractions, where both the numerators as well as the denominators are formed following the same rule from the two preceding fractions. Therefore since for both the first fractions there shall be $A=a,\ \mathfrak{A}=1$, then truly $B=ab+1$ and $\mathfrak{B}=b$, from these two fractions it will be a simple matter to form all the following. So that which may appear clearer, for the individual indices *a, b, c, d, e* etc. we may write below the corresponding fractions for the order

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & \text{etc.} \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} & \frac{F}{\mathfrak{F}} & \frac{G}{\mathfrak{G}} & \text{etc.} \end{array}$$

and both the numerators as well as the denominators following the same rule will be determined from the two preceding in the following manner :

For the numerators	For the denominators
$A = a$	$\mathfrak{A} = 1$
$B = Ab + 1$	$\mathfrak{B} = b$
$C = Bc + A$	$\mathfrak{C} = \mathfrak{B}c + \mathfrak{A}$
$D = Cd + B$	$\mathfrak{D} = \mathfrak{C}d + \mathfrak{B}$
$E = De + C$	$\mathfrak{E} = \mathfrak{D}e + \mathfrak{C}$
$F = Ef + D$	$\mathfrak{F} = \mathfrak{E}f + \mathfrak{D}$
etc.	etc.

From which it is evident in the series of numerators in the first place the term before the first term from the law of the progression must be $=1$, but in the series of the denominators the term before the first term must be $=0$, thus so that the preceding first fraction shall be $\frac{1}{0}$.

5. Because by themselves it has been seen well enough these fractions

$$\frac{A}{\mathfrak{A}},\ \frac{B}{\mathfrak{B}},\ \frac{C}{\mathfrak{C}},\ \frac{D}{\mathfrak{D}}\ \text{etc.}$$

continually approach closer to the truth and finally end in the true value of the continued fraction, it is necessary, that the differences between pairs of fractions of these continue to become exhausted, on account of which we may establish these differences in order. Therefore initially we will have

$$\text{II} - \text{I} = \frac{B\mathfrak{A} - A\mathfrak{B}}{\mathfrak{A}\mathfrak{B}}.$$

now here in place of B and $\mathfrak{B}$ the values may be substituted from the table and the numerator will be produced $A\mathfrak{A}b + \mathfrak{A} - Ab$, which form on account of $\mathfrak{A} = 1$ will change into 1, thus so that there shall be

$$\frac{B}{\mathfrak{B}} - \frac{A}{\mathfrak{A}} = \frac{1}{\mathfrak{A}\mathfrak{B}}.$$

Again there will be

$$\text{III} - \text{II} = \frac{C\mathfrak{B} - B\mathfrak{C}}{\mathfrak{B}\mathfrak{C}},$$

of which the numerator, if in place of C and $\mathfrak{C}$ the values assigned may be written, produces

$$\mathfrak{B}(Bc + A) - B(\mathfrak{B}c + \mathfrak{A}) = A\mathfrak{B} - B\mathfrak{A}.$$

But just as we have seen to be $A\mathfrak{B} - B\mathfrak{A} = 1$, from which this numerator itself will be -1 and thus

$$\frac{C}{\mathfrak{C}}-\frac{B}{\mathfrak{B}}=-\frac{1}{\mathfrak{BC}}.$$

Again there is

$$\text{IV}-\text{III}=\frac{D\mathfrak{C}-C\mathfrak{D}}{\mathfrak{CD}},$$

where, if in place of D and $\mathfrak{D}$ the assigned values may be written, there will be

$$\mathfrak{C}D-C\mathfrak{D}=\mathfrak{C}(CD+B)-C(\mathfrak{C}D+\mathfrak{B})=B\mathfrak{C}-C\mathfrak{B}.$$

But just as we have seen to be $C\mathfrak{B}-B\mathfrak{C}=-1$, from which it is concluded

$$\frac{D}{\mathfrak{D}}-\frac{C}{\mathfrak{C}}=+\frac{1}{\mathfrak{CD}}.$$

In a similar manner for the following there will be found :

$$\frac{E}{\mathfrak{E}}-\frac{D}{\mathfrak{D}}=-\frac{1}{\mathfrak{DE}},\ \frac{F}{\mathfrak{F}}-\frac{E}{\mathfrak{E}}=+\frac{1}{\mathfrak{EF}}\ \text{etc.}$$

6. Hence therefore we will be able to define our individual fractions from the first alone $\frac{A}{\mathfrak{A}}=a$ and from the fractions involving the Germanic letters alone, since we will have

$$\frac{B}{\mathfrak{B}}=a+\frac{1}{\mathfrak{AB}},$$
$$\frac{C}{\mathfrak{C}}=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}},$$
$$\frac{D}{\mathfrak{D}}=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}},$$
$$\frac{E}{\mathfrak{E}}=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}}-\frac{1}{\mathfrak{DE}},$$
$$\frac{F}{\mathfrak{F}}=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}}-\frac{1}{\mathfrak{DE}}+\frac{1}{\mathfrak{EF}}$$

etc.

7. Therefore since the final or the infinitesimal true value of the proposed continued fraction may be shown, which we may designate by the letter *s*, there will be

$$s=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}}-\frac{1}{\mathfrak{DE}}+\frac{1}{\mathfrak{EF}}-\frac{1}{\mathfrak{FG}}+\text{etc.};$$

and thus we have reduced the continued fraction to an infinite series of fractions, all the numerators of which are in turn $+1$ and -1, truly the denominators may be determined by the Germanic letters alone, thus so that there shall be no need to establish the values of the letters *A, B, C* etc. , but it may suffice to have extricated the following formulas :

$$\mathfrak{A}=1,\ \mathfrak{B}=b,\ \mathfrak{C}=\mathfrak{B}c+\mathfrak{A},\ \mathfrak{D}=\mathfrak{C}d+\mathfrak{B},\ \mathfrak{E}=\mathfrak{D}e+\mathfrak{C}\ \text{etc.}$$

8. Therefore since each expression may begin from the quantity *a*, that will be removed from that calculation at once, because the Germanic letters clearly do not depend on that ; so that, from what we have found so far, they may be returned here, so that the proposed continued fraction

$$s=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}$$

if the Germanic letters may be defined from its indices *b, c, d, e* etc., where indeed at once there shall be $\mathfrak{A}=1$, there shall become always

$$s=\tfrac{1}{\mathfrak{AB}}-\tfrac{1}{\mathfrak{BC}}+\tfrac{1}{\mathfrak{CD}}-\tfrac{1}{\mathfrak{DE}}+\text{etc.},$$

which progression may proceed indefinitely, if the continued fraction may be extended indefinitely, truly it will be put together contrary to a number with a finite number of terms.

9. Therefore since in this manner we will have transformed the continued fraction into an ordinary series, it will not be difficult to convert some proposed series into a continued fraction. Therefore this infinite series may be proposed

$$s=\tfrac{1}{\alpha}-\tfrac{1}{\beta}+\tfrac{1}{\gamma}-\tfrac{1}{\delta}+\text{etc.},$$

indeed of which all the numerators shall be ones with alternating $+$ and $-$ signs attached, truly the denominators may constitute some progression, because it still allows all series clearly may be contained in this form, if indeed the terms of the series $\alpha,\beta,\ \gamma,\delta$ etc. are able not only to be fractional but also to be negative numbers.

10. Therefore so that we may elicit the continued fraction equal to this series, initially we may put

$$\mathfrak{AB}=\alpha,\ \mathfrak{BC}=\beta,\ \mathfrak{CD}=\gamma \text{ and thus henceforth,}$$

from which on account of $\mathfrak{A}=1$ we will obtain the following values :

$$\mathfrak{B} = \alpha \qquad \mathfrak{C} = \frac{\beta}{\alpha},$$
$$\mathfrak{D} = \frac{\alpha\gamma}{\beta}, \qquad \mathfrak{E} = \frac{\beta\delta}{\alpha\gamma},$$
$$\mathfrak{F} = \frac{\alpha\gamma\varepsilon}{\beta\delta} \qquad \mathfrak{G} = \frac{\beta\delta\zeta}{\alpha\gamma\varepsilon},$$
$$\mathfrak{H} = \frac{\alpha\gamma\varepsilon\eta}{\beta\delta\zeta} \qquad \mathfrak{I} = \frac{\beta\delta\zeta\theta}{\alpha\gamma\varepsilon\eta},$$
etc.

Now therefore it remains only, that from these values of the Germanic letters we may elicit the indices themselves of the continued fraction *b, c, d, e* etc.

11. Moreover from the formulas, from which the Germanic letters have been determined by the indices of the continued fractions above, in turn from these letters themselves we may define the indices *b, c, d, e, f* etc. and we will find

$$b = \mathfrak{B}, \;\; c = \frac{\mathfrak{C}-\mathfrak{A}}{\mathfrak{B}}, \;\; d = \frac{\mathfrak{D}-\mathfrak{B}}{\mathfrak{C}}, \;\; e = \frac{\mathfrak{E}-\mathfrak{C}}{\mathfrak{D}}, \;\; f = \frac{\mathfrak{F}-\mathfrak{D}}{\mathfrak{E}} \;\; \text{etc.}$$

Therefore we may set out these values in order, while in place of the letters $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. we may substitute the formulas found before.

12. But initially there was $\mathfrak{B} = \alpha$, from which there becomes $b = \alpha$; then there is

$$\mathfrak{C} - \mathfrak{A} = \frac{\beta-\alpha}{\alpha},$$

from which there becomes

$$c = \frac{\beta-\alpha}{\alpha\alpha}.$$

Again there will be

$$\mathfrak{D} - \mathfrak{B} = \frac{\alpha(\gamma-\beta)}{\beta},$$

from which there becomes

$$d = \frac{\alpha\alpha(\gamma-\beta)}{\beta\beta}.$$

Then we will have

$$\mathfrak{E} - \mathfrak{C} = \frac{\beta(\delta-\gamma)}{\alpha\gamma}$$

and hence

$$e = \frac{\beta\beta(\delta-\gamma)}{\alpha\alpha\gamma\gamma}.$$

In a similar manner on account of

$$\mathfrak{F} - \mathfrak{D} = \frac{\alpha\gamma(\varepsilon-\delta)}{\beta\delta}$$

there will be

$$f=\frac{\alpha\alpha\gamma\gamma(\varepsilon-\delta)}{\beta\beta\delta\delta}.$$

In the same manner on account of

$$\mathfrak{G}-\mathfrak{E}=\frac{\beta\delta(\zeta-\varepsilon)}{\alpha\gamma\varepsilon}$$

there will be

$$g=\frac{\beta\beta\delta\delta(\zeta-\varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon}$$

etc.

Therefore by this account the indices of the continued fraction, which we have sought, will be able to be expressed in the following manner :

$$b=\alpha, \qquad c=\frac{\beta-\alpha}{\alpha\alpha},$$
$$d=\frac{\alpha\alpha(\gamma-\beta)}{\beta\beta}, \qquad e=\frac{\beta\beta(\delta-\gamma)}{\alpha\alpha\gamma\gamma},$$
$$f=\frac{\alpha\alpha\gamma\gamma(\varepsilon-\delta)}{\beta\beta\delta\delta}, \qquad g=\frac{\beta\beta\delta\delta(\zeta-\varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon},$$
$$h=\frac{\alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta-\zeta)}{\beta\beta\delta\delta\zeta\zeta}, \qquad i=\frac{\beta\beta\delta\delta\zeta\zeta(\theta-\eta)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta}$$

etc.

13. There is a need still therefore, so that we may substitute these values into the continued fraction in place of the indices *b, c, d, e, f* etc.

$$s=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}};$$

truly since these values are fractions, so that we may free the form more easily from partial fractions, first we may remove the denominators from the values found and there will be

$$b=\alpha, \qquad \alpha\alpha c=\beta-\alpha,$$
$$\beta\beta d=\alpha\alpha(\gamma-\beta), \qquad \alpha\alpha\gamma\gamma e=\beta\beta(\delta-\gamma),$$
$$\beta\beta\delta\delta f=\alpha\alpha\gamma\gamma(\varepsilon-\delta), \qquad \alpha\alpha\gamma\gamma\varepsilon\varepsilon g=\beta\beta\delta\delta(\zeta-\varepsilon),$$
$$\beta\beta\delta\delta\zeta\zeta h=\alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta-\zeta), \qquad \alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta i=\beta\beta\delta\delta\zeta\zeta(\theta-\eta)$$

etc.

14. Now we will transform the continued fraction itself thus, so that the same formulas occur in place of the index, the values of which we have assigned here. Clearly we may

multiply the second fraction above and below by $\alpha\alpha$, the third by $\beta\beta$, the fourth by $\alpha\alpha\gamma\gamma$, the fifth by $\beta\beta\delta\delta$, the sixth by $\alpha\alpha\gamma\gamma\varepsilon\varepsilon$ etc., so that this form may be produced :

$$s=\cfrac{1}{b+\cfrac{\alpha\alpha}{\alpha\alpha c+\cfrac{\alpha\alpha\beta\beta}{\beta\beta d+\cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\alpha\alpha\gamma\gamma e+\cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\beta\beta\delta\delta f+\text{etc.}}}}}}$$

15. But if now in place of these new indices $\alpha\alpha c,\ \beta\beta d,\ \alpha\alpha\gamma\gamma e$ etc. we may substitute the above values found, the following continued fraction will arise

$$s=\cfrac{1}{\alpha+\cfrac{\alpha\alpha}{\beta-\alpha+\cfrac{\alpha\alpha\beta\beta}{\alpha\alpha(\gamma-\beta)+\cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\beta\beta(\delta-\gamma)+\cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha\gamma\gamma(\varepsilon-\delta)+\text{etc.}}}}}}$$

But if we may consider this form more carefully, but if we take the third fraction able to be divided above and below by $\alpha\alpha$, then truly the fourth by $\beta\beta$, the fifth by $\gamma\gamma$, the sixth by $\delta\delta$ etc.; with which done this continued fraction will arise :

$$s=\cfrac{1}{\alpha+\cfrac{\alpha\alpha}{\beta-\alpha+\cfrac{\beta\beta}{\gamma-\beta+\cfrac{\gamma\gamma}{\delta-\gamma+\cfrac{\delta\delta}{\varepsilon-\delta+\text{etc.}}}}}}$$

Hence we may establish the following

THEOREM I

16. *If such an infinite series were proposed*

$$s=\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\frac{1}{\varepsilon}-etc.,$$

from that such a continued fraction always will be able to be formed

$$\frac{1}{s}=\alpha+\cfrac{\alpha\alpha}{\beta-\alpha+\cfrac{\beta\beta}{\gamma-\beta+\cfrac{\gamma\gamma}{\delta-\gamma+\cfrac{\delta\delta}{\varepsilon-\delta+\text{etc.}}}}}$$

17. Therefore we have elicited this reduction by several roundabout ways from the consideration of continued fractions, by which indeed we have satisfied our proposition, since we have transformed any series into a continued fraction. Truly here the direct method rightly is deserved, by which at once from the proposed series without these winding ways it may be possible to derive the continued fraction equal to that. Therefore I have presented here such a noteworthy method, certainly by which the theory of continued fractions will be illustrated.

PROBLEM I

18. *To transform the proposed infinite series*

$$s=\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\frac{1}{\varepsilon}-etc.$$

into a continued fraction.

SOLUTION

Since there shall be

$$s=\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\frac{1}{\varepsilon}-\text{etc.}$$

we may put in place

$$t=\frac{1}{\beta}-\frac{1}{\gamma}+\frac{1}{\delta}-\frac{1}{\varepsilon}+\text{etc.}$$

and

$$u=\frac{1}{\gamma}-\frac{1}{\delta}+\frac{1}{\varepsilon}-\frac{1}{\zeta}+\text{etc.}$$

etc.

Hence there will be therefore

$$s=\frac{1}{\alpha}-t=\frac{1-\alpha t}{\alpha},$$

from which there becomes:

$$\frac{1}{s}=\alpha+\frac{\alpha\alpha t}{1-\alpha t}.$$

But there is

$$\frac{\alpha\alpha t}{1-\alpha t}=\frac{\alpha\alpha}{-\alpha+\frac{1}{t}}$$

from which there becomes :

$$\frac{1}{s}=\alpha+\frac{\alpha\alpha}{-\alpha+\frac{1}{t}}.$$

Therefore in a similar manner also there will be

$$\frac{1}{t}=\beta+\frac{\beta\beta}{-\beta+\frac{1}{u}}$$

and

$$\frac{1}{u}=\gamma+\frac{\gamma\gamma}{-\gamma+\frac{1}{v}}$$

etc.,

with which values substituted the following continued fraction will be obtained

$$\frac{1}{s}=\alpha+\frac{\alpha\alpha}{\beta-\alpha+\frac{\beta\beta}{\gamma-\beta+\frac{\gamma\gamma}{\delta-\gamma+\text{etc.}}}}$$

which is the form shown in the theorem itself.

19. So that if therefore the proposed series shall be

$$s=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\text{etc.}=l2,$$

on account of

$$\alpha=1,\ \beta=2,\ \gamma=3,\ \delta=4,\ \text{etc.}$$

there will be :

$$\frac{1}{l2}=1+\frac{1\cdot1}{1+\frac{2\cdot2}{1+\frac{3\cdot3}{1+\text{etc.}}}}$$

But if we may assume this series

$$s=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.}=\tfrac{\pi}{4},$$

on account of

$$\alpha=1,\ \beta=3,\ \gamma=5,\ \delta=7,\ \text{etc.}$$

there will be

$$\frac{4}{\pi}=1+\frac{1\cdot1}{2+\frac{3\cdot3}{2+\frac{5\cdot5}{2+\text{etc.}}}}$$

which is the continued fraction advanced at one time by Brouncker.

20. We may take

$$s=\int\frac{x^{m-1}dx}{1+x^{n}},$$

and after the integration we may put $x=1$; with which done the value of s itself will be expressed by the following series

$$s=\frac{1}{m}-\frac{1}{m+n}+\frac{1}{m+2n}-\frac{1}{m+3n}+\text{etc.},$$

thus so that there shall be

$$\alpha=m,\ \beta=m+n,\ \gamma=m+2n,\ \delta=m+3n\ \text{ etc.};$$

hence therefore the following continued fraction will emerge :

$$\frac{1}{s}=m+\cfrac{mm}{n+\cfrac{(m+n)^2}{n+\cfrac{(m+2n)^2}{n+\text{etc.}}}}$$

which value I have given now in Vol. XI., Comment. V of the Proceedings of our Academy [see E123].

21. But if this series shall be proposed :

$$s=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\varepsilon}+\text{etc.},$$

of which all the terms shall be positive, there is a need still, so that in the above continued fraction in place of the letters $\beta,\ \delta,\ \zeta,\ \theta$ etc. there may be written $-\beta,\ -\delta,\ -\zeta,\ -\theta$ etc.; then truly there will become

$$\frac{1}{s}=\alpha+\cfrac{\alpha\alpha}{-\beta-\alpha+\cfrac{\beta\beta}{\gamma+\beta+\cfrac{\gamma\gamma}{-\delta-\gamma+\cfrac{\delta\delta}{\varepsilon+\delta+\text{etc.}}}}}$$

which fraction may be transformed easily into this form :

$$\frac{1}{s}=\alpha-\cfrac{\alpha\alpha}{\alpha+\beta-\cfrac{\beta\beta}{\beta+\gamma-\cfrac{\gamma\gamma}{\gamma+\delta-\text{etc.}}}}$$

22. But the proposed series itself can be transformed in several ways, from which more and more other continued fractions are elicited. Therefore we may consider here some forms of this kind. Therefore there shall be

$$\alpha=ab,\ \beta=bc,\ \gamma=cd,\ \delta=de\ \text{etc.},$$

so that this series may be had

$$s=\frac{1}{ab}-\frac{1}{bc}+\frac{1}{cd}-\frac{1}{de}+\text{ etc.},$$

and hence this continued fraction will be formed :

$$\frac{1}{s} = ab + \cfrac{aabb}{b(c-a) + \cfrac{bbcc}{c(d-b) + \cfrac{ccdd}{d(e-c) + \text{etc.}}}}$$

which is reduced easily to the following form :

$$\frac{1}{s} = ab + \cfrac{aab}{c-a + \cfrac{bc}{d-b + \cfrac{cd}{e-c + \text{etc.}}}}$$

or

$$\frac{1}{as} = b + \cfrac{ab}{c-a + \cfrac{bc}{d-b + \cfrac{cd}{e-c + \text{etc.}}}}$$

which form supplies us with the following Theorem.

THEOREM II

23. *If the proposed series were of this form*

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \frac{1}{ef} - etc.,$$

from that the following continued fraction arises

$$\frac{1}{as} = b + \cfrac{ab}{c-a + \cfrac{bc}{d-b + \cfrac{cd}{e-c + \cfrac{de}{f-d + \text{etc.}}}}}$$

24. This form, even if it may be derived easily from the preceding, thus is noteworthy, because the continued fraction forms it may produce to be greatly diverse, from which it will be worth the effort to apply the above examples introduced to this form also. Therefore since there were

$$l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.},$$

there will become:

$$l2 - 1 = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.}$$

and from these series added there arises :

$$2l2 - 1 = \frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} - \text{etc.}$$

Therefore here there is :

$$s = 2l2 - 1$$

and

$$a = 1,\ b = 2,\ c = 3,\ d = 4 \text{ etc.};$$

hence therefore this continued fraction will be formed :

$$\frac{1}{2l2-1} = 2 + \cfrac{1\cdot 2}{2 + \cfrac{2\cdot 3}{2 + \cfrac{3\cdot 4}{2 + \text{etc.}}}}$$

25. In a similar manner, because there is

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.},$$

there will be

$$\tfrac{\pi}{4} - 1 = -\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.},$$

of which the sum of the series gives :

$$\tfrac{\pi}{2} - 1 = \tfrac{2}{1\cdot 3} - \tfrac{2}{3\cdot 5} + \tfrac{2}{5\cdot 7} - \tfrac{2}{7\cdot 9} + \text{etc.}$$

or

$$\tfrac{\pi}{4} - \tfrac{1}{2} = \tfrac{1}{1\cdot 3} - \tfrac{1}{3\cdot 5} + \tfrac{1}{5\cdot 7} - \tfrac{1}{7\cdot 9} + \text{etc.}$$

Therefore here there will be :

$$s = \tfrac{\pi}{4} - \tfrac{1}{2},$$

then truly

$$a = 1,\quad b = 3,\quad c = 5,\quad d = 7 \quad \text{etc.};$$

whereby the continued fraction hence will have arisen:

$$\frac{4}{\pi-2} = 3 + \cfrac{1\cdot 3}{4 + \cfrac{3\cdot 5}{4 + \cfrac{5\cdot 7}{4 + \cfrac{7\cdot 9}{4 + \text{etc.}}}}}$$

26. Now more generally we may consider also this transformation. Therefore Δ may denote the value of the integral formula

$$s = \int \frac{x^{m-1}dx}{1+x^n},$$

by putting $x = 1$ after the integration, and since there shall be, as we saw above in § 20,

$$\Delta=\frac{1}{m}-\frac{1}{m+n}+\frac{1}{m+2n}-\text{etc.},$$

there will become :

$$\Delta-\frac{1}{m}=-\frac{1}{m+n}+\frac{1}{m+2n}-\frac{1}{m+3n}+\text{etc.},$$

with which two series added there becomes :

$$2\Delta-\frac{1}{m}=\frac{n}{m(m+n)}-\frac{n}{(m+n)(m+2n)}+\frac{n}{(m+2n)(m+3n)}-\text{etc.};$$

hence by dividing by n there will become:

$$\frac{2m\Delta-1}{mn}=\frac{1}{m(m+n)}-\frac{1}{(m+n)(m+2n)}+\frac{1}{(m+2n)(m+3n)}-\text{etc.}$$

Here therefore we have

$$s=\frac{2m\Delta-1}{mn},$$

then truly

$$a=m,\ b=m+n,\ c=m+2n,\ d=m+3n\ \text{etc.},$$

concerning which continued fraction hence the format will be

$$\frac{n}{2m\Delta-1}=m+n+\cfrac{m(m+n)}{2n+\cfrac{(m+n)(m+2n)}{2n+\cfrac{(m+2n)(m+3n)}{2n+\cfrac{(m+3n)(m+4n)}{2n+\text{etc.}}}}}$$

which form gives nothing simplified from the preceding.

27. Now we may attribute also certain numbers to the initial series assumed

$$\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\text{etc.}$$

and there shall be

$$s=\frac{a}{\alpha}-\frac{b}{\beta}+\frac{c}{\gamma}-\frac{d}{\delta}+\text{etc.}$$

and in the first Theorem in place of the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. it is required to write $\frac{\alpha}{a},\ \frac{\beta}{b},\ \frac{\gamma}{c},\ \frac{\delta}{d}$ etc., with which done the continued fraction itself will be had :

$$\frac{1}{s}=\frac{\alpha}{a}+\cfrac{\frac{\alpha\alpha}{aa}}{\frac{\beta}{b}-\frac{\alpha}{a}+\cfrac{\frac{\beta\beta}{bb}}{\frac{\gamma}{c}-\frac{\beta}{b}+\cfrac{\frac{\gamma\gamma}{cc}}{\frac{\delta}{d}-\frac{\gamma}{c}+\text{etc.}}}}$$

now for the removing the fractions the first fraction may be multiplied above and below by *ab*, the second by *bc*, the third by *cd* and so on thus; then truly on multiplying each side by *a* there will be obtained

$$\frac{a}{s}=\alpha+\cfrac{\alpha\alpha b}{a\beta-b\alpha+\cfrac{ac\beta\beta}{b\gamma-c\beta+\cfrac{bd\gamma\gamma}{c\delta-d\gamma+\text{etc.}}}}$$

Hence therefore the following may be formed :

THEOREM III

28. *If an infinite series of this form may be proposed*

$$s=\frac{a}{\alpha}-\frac{b}{\beta}+\frac{c}{\gamma}-\frac{d}{\delta}+\text{etc.}$$

from that the following continued fraction may be formed

$$\frac{a}{s}=\alpha+\cfrac{\alpha\alpha b}{a\beta-b\alpha+\cfrac{ac\beta\beta}{b\gamma-c\beta+\cfrac{bd\gamma\gamma}{c\delta-d\gamma+\text{etc.}}}}$$

29. This series shall be proposed towards illustrating this :

$$\frac{1}{1}-\frac{2}{2}+\frac{3}{3}-\frac{4}{4}+\frac{5}{5}-\text{etc.}=\frac{1}{2},$$

thus so that there shall be $s=\frac{1}{2}$; therefore the continued fraction hence arising will be :

$$2=1+\cfrac{2}{0+\cfrac{3\cdot 4}{0+\cfrac{8\cdot 9}{0+\cfrac{15\cdot 16}{0+\text{etc.}}}}}$$

which form is reduced to that same infinite product

$$2 = 1 + \frac{2\cdot 1^2\cdot 2\cdot 4\cdot 3^2\cdot 4\cdot 6\cdot 5^2\cdot 6\cdot 8\cdot 7^2\cdot \text{etc.}}{1\cdot 3\cdot 2^2\cdot 3\cdot 5\cdot 4^2\cdot 5\cdot 7\cdot 6^2\cdot 9\cdot 8^2\cdot \text{etc.}}.$$

the truth of which cannot be seen easily, because the numbers of the factors in the numerator and denominator are not put equal to each other, even if both shall be infinite. Truly there can be no doubt, why the value of this product may not be $=1$.

30. Now we may consider this series

$$s = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \text{etc.}$$

the sum of which is $s = l2 - \frac{1}{2}$. Therefore since there is

$$a = 1,\quad b = 2,\quad c = 3,\quad d = 4 \text{ etc.},$$
$$\alpha = 2,\quad \beta = 3,\quad \gamma = 4,\quad \delta = 5 \text{ etc.},$$

the continued fraction hence produced will be

$$\frac{1}{l2-\frac{1}{2}} = 2 + \cfrac{1\cdot 2\cdot 2^2}{-1 + \cfrac{1\cdot 3\cdot 3^2}{-1 + \cfrac{2\cdot 4\cdot 4^2}{-1 + \cfrac{3\cdot 5\cdot 5^2}{-1 + \text{etc.}}}}}$$

31. So that moreover we may take this series

$$s = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \text{etc.},$$

the value of which is $\frac{1}{2} + l2$, we will have

$$a = 2,\quad b = 3,\quad c = 4,\quad d = 5 \text{ etc.},$$
$$\alpha = 1,\quad \beta = 2,\quad \gamma = 3,\quad \delta = 4 \text{ etc.};$$

hence this continued fraction therefore will arise :

$$\frac{2}{\frac{1}{2}+l2} = 1 + \cfrac{1\cdot 3\cdot 1^2}{1 + \cfrac{2\cdot 4\cdot 2^2}{1 + \cfrac{3\cdot 5\cdot 3^2}{1 + \cfrac{4\cdot 6\cdot 4^2}{1 + \text{etc.}}}}}$$

or

$$\frac{4}{2l2+1} = 1 + \cfrac{1^3 \cdot 3}{1 + \cfrac{2^3 \cdot 4}{1 + \cfrac{3^3 \cdot 5}{1 + \cfrac{4^3 \cdot 6}{1 + \text{etc.}}}}}$$

PROBLEM Il

To transform this proposed infinite series

$$s = \frac{x}{\alpha} - \frac{xx}{\beta} + \frac{x^3}{\gamma} - \frac{x^4}{\delta} + etc.$$

into a continued fraction.

SOLUTION

32. The following series may be considered formed from the proposed series

$$t = \frac{x}{\beta} - \frac{xx}{\gamma} + \frac{x^3}{\delta} - \frac{x^4}{\varepsilon} + \text{etc.},$$

again

$$u = \frac{x}{\gamma} - \frac{xx}{\delta} + \frac{x^3}{\varepsilon} - \frac{x^4}{\zeta} + \text{etc.},$$

$$v = \frac{x}{\delta} - \frac{xx}{\varepsilon} + \frac{x^3}{\zeta} - \frac{x^4}{\eta} + \text{etc.}$$

and there will be

$$s = \frac{x}{\alpha} - tx = \frac{x(1-\alpha t)}{\alpha};$$

from which there becomes :

$$\frac{x}{s} = \frac{\alpha}{1-\alpha t} = \alpha + \frac{\alpha\alpha t}{1-\alpha t} = \alpha + \frac{\alpha\alpha}{-\alpha + \frac{1}{t}}.$$

Hence there will be therefore :

$$\frac{x}{s} = \alpha + \frac{\alpha\alpha x}{-\alpha x + \frac{x}{t}};$$

but in a similar manner there will be

$$\frac{x}{t}=\beta+\frac{\beta\beta x}{-\beta x+\frac{x}{u}}.$$

Therefore these values if all may be substituted in order, that same continued fraction will arise :

$$\frac{x}{s}=\alpha+\cfrac{\alpha\alpha x}{\beta-\alpha x+\cfrac{\beta\beta x}{\gamma-\beta x+\cfrac{\gamma\gamma x}{\delta-\gamma x+\text{etc.}}}}$$

33. So that if everywhere here in place x we may write $\frac{x}{y}$, so that we may have this series y

$$s=\frac{x}{\alpha y}-\frac{xx}{\beta yy}+\frac{x^3}{\gamma y^3}-\frac{x^4}{\delta y^4}+\text{etc.},$$

then the continued fraction hence will be produced :

$$\frac{x}{sy}=\alpha+\cfrac{\alpha\alpha x:y}{\beta-\frac{\alpha x}{y}+\cfrac{\beta\beta x:y}{\gamma-\frac{\beta x}{y}+\text{etc.}}}$$

which freed from the partial fractions gives

$$\frac{x}{sy}=\alpha+\cfrac{\alpha\alpha x}{\beta y-\alpha x+\cfrac{\beta\beta xy}{\gamma y-\beta x+\cfrac{\gamma\gamma xy}{\delta y-\gamma x+\text{etc.}}}}$$

From which the following arises :

THEOREM IV

34. *If an infinite series of this kind were proposed*

$$s=\frac{x}{\alpha y}-\frac{xx}{\beta yy}+\frac{x^3}{\gamma y^3}-\frac{x^4}{\delta y^4}+etc.,$$

from that it will be able to form this continued fraction

$$\frac{x}{s}=\alpha y+\cfrac{\alpha\alpha xy}{\beta y-\alpha x+\cfrac{\beta\beta xy}{\gamma y-\beta x+\cfrac{\gamma\gamma xy}{\delta y-\gamma x+\cfrac{\delta\delta xy}{\varepsilon y-\delta x+\text{etc.}}}}}$$

35. Since there shall be

$$l\left(1+\frac{x}{y}\right)=\frac{x}{y}-\frac{xx}{2yy}+\frac{x^3}{3y^3}-\frac{x^4}{4y^4}+\text{etc.}$$

on putting

$$s=l\left(1+\frac{x}{y}\right)$$

there will be

$$\alpha=1,\ \ \beta=2,\ \ \gamma=3,\ \ \delta=4\ \text{etc.};$$

and hence this continued fraction will arise

$$\frac{x}{l\left(1+\frac{x}{y}\right)}=y+\cfrac{xy}{2y-x+\cfrac{4xy}{3y-2x+\cfrac{9xy}{4y-3x+\text{etc.}}}}$$

36. Since the arc, of which the tangent is *t*, may be expressed by this series

$$\text{Atang.}t=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\frac{t^9}{9}-\text{etc.},$$

there will be

$$t\text{Atang.}t=\frac{tt}{1}-\frac{t^4}{3}+\frac{t^6}{5}-\frac{t^8}{7}+\frac{t^{10}}{9}-\text{etc.}$$

Now there may be put $tt=\frac{x}{y}$, thus so that there shall be $t=\sqrt{\frac{x}{y}}$, and there will become

$$\sqrt{\frac{x}{y}}\cdot\text{Atang.}\sqrt{\frac{x}{y}}=\frac{x}{y}-\frac{xx}{3yy}+\frac{x^3}{5y^3}-\frac{x^4}{7y^4}+\text{etc.},$$

Hence therefore there is :

$$s=\sqrt{\frac{x}{y}}\cdot\text{Atang.}\sqrt{\frac{x}{y}},$$

then truly

$$\alpha=1,\ \ \beta=3,\ \ \gamma=5,\ \ \delta=7\ \ \text{etc.};$$

whereby the continued fraction hence will arise :

$$\frac{\sqrt{xy}}{\text{Atang.}\sqrt{\frac{x}{y}}}=y+\cfrac{xy}{3y-x+\cfrac{9xy}{5y-3x+\cfrac{25xy}{7y-5x+\text{etc.}}}}$$

Just as if there were $x=1$ and $y=3$, on account of

$$\text{Atang.}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$$

this continued fraction will be had :

$$\frac{6\sqrt{3}}{\pi}=3+\cfrac{1\cdot 3}{8+\cfrac{3\cdot 9}{12+\cfrac{3\cdot 25}{16+\text{etc.}}}}$$

37. So that if in the case of the Theorem in place of the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. we may write the fractions :

$$\frac{\alpha}{a},\ \frac{\beta}{b},\ \frac{\gamma}{c},\ \frac{\delta}{d}\ \text{etc.},$$

so that we may have this series

$$s=\frac{ax}{\alpha y}-\frac{bxx}{\beta yy}+\frac{cx^3}{\gamma y^3}-\frac{dx^4}{\delta y^4}+etc.,$$

hence the continued fraction thus itself will have this form

$$\frac{x}{s}=\frac{\alpha}{a}y+\cfrac{\alpha\alpha xy:aa}{\frac{\beta}{b}y-\frac{\alpha}{a}x+\cfrac{\beta\beta xy:bb}{\frac{\gamma}{c}y-\frac{\beta}{b}x+\cfrac{\gamma\gamma xy:cc}{\frac{\delta}{d}y-\frac{\gamma}{c}x+\text{etc.}}}}$$

Here now initially each side may be multiplied by *a*, then the numerator and denominator of the first fraction may be multiplied by *ab*, of the second by *bc*, of the third by *cd* etc. and the continued fraction may adopt this form

$$\frac{ax}{s}=\alpha y+\cfrac{\alpha\alpha bxy}{a\beta y-b\alpha x+\cfrac{\beta\beta acxy}{b\gamma y-c\beta x+\cfrac{\gamma\gamma bdxy}{c\delta y-d\gamma x+\text{etc.}}}}$$

from which it is worthwhile to attach the following

THEOREM V

38. *If an infinite series of this form were proposed*

$$s = \frac{ax}{\alpha y} - \frac{bxx}{\beta yy} + \frac{cx^3}{\gamma y^3} - \frac{dx^4}{\delta y^4} + etc.,$$

then the following continued fraction will be formed

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha\alpha bxy}{a\beta y - b\alpha x + \cfrac{\beta\beta acxy}{b\gamma y - c\beta x + \cfrac{\gamma\gamma bdxy}{c\delta y - d\gamma x + \text{etc.}}}}$$

PROBLEM III

To convert this infinite series

$$s = \frac{1}{\alpha} - \frac{1}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha\beta\gamma\delta} + etc.,$$

into a continued fraction.

SOLUTION

39. We may form the following series from the proposed series

$$t = \frac{1}{\beta} - \frac{1}{\beta\gamma} + \frac{1}{\beta\gamma\delta} - \frac{1}{\beta\gamma\delta\varepsilon} + \text{etc.},$$
$$u = \frac{1}{\gamma} - \frac{1}{\gamma\delta} + \frac{1}{\gamma\delta\varepsilon} - \frac{1}{\gamma\delta\varepsilon\zeta} + \text{etc.}$$
$$\text{etc.}$$

and we will have :

$$s = \frac{1-t}{\alpha},\quad t = \frac{1-u}{\beta},\quad u = \frac{1-v}{\gamma},\quad \text{etc.};$$

hence therefore we deduce :

$$\frac{1}{s} = \frac{\alpha}{1-t} = \alpha + \frac{\alpha t}{1-t} = \alpha + \frac{\alpha}{-1+\frac{1}{t}}.$$

Moreover in a like manner there will be

$$\frac{1}{t}=\beta+\frac{\beta}{-1+\frac{1}{u}},\quad \frac{1}{u}=\gamma+\frac{\gamma}{-1+v},\ \text{etc.};$$

whereby with the latter values substituted into the former we will obtain this continued fraction

$$\frac{1}{s}=\alpha+\cfrac{\alpha}{\beta-1+\cfrac{\beta}{\gamma-1+\cfrac{\gamma}{\delta-1+\text{etc.}}}}$$

from which we deduce the following Theorem:

THEOREM VI

40. *If an infinite series of this kind were proposed*

$$s=\frac{1}{\alpha}-\frac{1}{\alpha\beta}+\frac{1}{\alpha\beta\gamma}-\frac{1}{\alpha\beta\gamma\delta}+etc.,$$

thence this continued fraction will be able to be formed:

$$\frac{1}{s}=\alpha+\cfrac{\alpha}{\beta-1+\cfrac{\beta}{\gamma-1+\cfrac{\gamma}{\delta-1+\text{etc.}}}}$$

41. If e may denote the number, of which the hyperbolic logarithm is unity, it is known

$$\frac{1}{e}=1-\frac{1}{1}+\frac{1}{1\cdot 2}-\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}-\text{etc.}$$

or

$$\frac{e-1}{e}=\frac{1}{1}-\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}-\frac{1}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

This therefore becomes $s=\frac{e-1}{e}$, then truly

$$\alpha=1,\ \beta=2,\ \gamma=3,\ \delta=4\ \text{etc.};$$

whereby the continued fraction hence arising is

$$\frac{e}{e-1}=1+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{2+\text{etc.}}}}$$

42. Therefore since there shall be

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{etc.}}}}}=\frac{1}{e\text{-}1}$$

it may be shown moreover without difficulty, if there were

$$\cfrac{a}{a+\cfrac{b}{b+\cfrac{c}{c+\text{etc.}}}}=s,$$

then to become :

$$\cfrac{a}{b+\cfrac{b}{c+\cfrac{c}{d+\text{etc.}}}}=\frac{s}{1-s},$$

for our case there will be

$$s=\tfrac{1}{e-1},\ \ a=1,\ \ b=2,\ \ c=3\ \text{ etc.},$$

with which values substituted there will become :

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}=\frac{1}{e-2}.$$

43. So that if in the series of Theorem VI in place of the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc., the fractions may be written :

$$\tfrac{\alpha}{a},\ \tfrac{\beta}{b},\ \tfrac{\gamma}{c},\ \tfrac{\delta}{d}\ \text{etc.},$$

so that there shall become :

$$s=\tfrac{a}{\alpha}-\tfrac{ab}{\alpha\beta}+\tfrac{abc}{\alpha\beta\gamma}-\tfrac{abcd}{\alpha\beta\gamma\delta}+\text{etc.},$$

the continued fraction hence produced will be

$$\frac{1}{s}=\frac{\alpha}{a}+\cfrac{\alpha:a}{\frac{\beta}{b}-1+\cfrac{\beta:b}{\frac{\gamma}{c}-1+\cfrac{\gamma:c}{\frac{\delta}{d}-1+\text{etc.}}}}$$

So that if now at first each side may be multiplied by a, then truly the first fraction above and below by b, the second by c, the third by d etc., this form may be produced

$$\frac{a}{s}=\alpha+\cfrac{\alpha b}{\beta-b+\cfrac{\beta c}{\gamma-c+\cfrac{\gamma d}{\delta-d+\text{etc.}}}}$$

so that the following Theorem may be included.

THEOREM VII

44. *If an infinite series of this kind were proposed*

$$s=\frac{a}{\alpha}-\frac{ab}{\alpha\beta}+\frac{abc}{\alpha\beta\gamma}-\frac{abcd}{\alpha\beta\gamma\delta}+etc.,$$

thence this continued fraction is deduced

$$\frac{a}{s}=\alpha+\cfrac{\alpha b}{\beta-b+\cfrac{\beta c}{\gamma-c+\cfrac{\gamma d}{\delta-d+\text{etc.}}}}$$

45. We may apply this to the following infinite series

$$s=\frac{1}{2}-\frac{1\cdot3}{2\cdot4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}-\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}+\text{etc.},$$

the sum of which is agreed to be $s=\frac{\sqrt{2}-1}{\sqrt{2}}$; then truly there will become :

$$a=1,\quad b=3,\quad c=5,\quad d=7 \text{ etc.,}$$
$$\alpha=2,\quad \beta=4,\quad \gamma=6,\quad \delta=8 \text{ etc.;}$$

the continued fraction hence arising will be :

$$\frac{\sqrt{2}}{\sqrt{2}-1}=2+\cfrac{2\cdot3}{1+\cfrac{4\cdot5}{1+\cfrac{6\cdot7}{1+\text{etc.}}}}$$

So if one may be taken away from each side, there will become

$$\frac{1}{\sqrt{2}-1}=1+\cfrac{2\cdot 3}{1+\cfrac{4\cdot 5}{1+\cfrac{6\cdot 7}{1+\text{etc.}}}}$$

from which there is deduced :

$$\sqrt{2}=1+\cfrac{1\cdot 1}{1+\cfrac{2\cdot 3}{1+\cfrac{4\cdot 5}{1+\cfrac{6\cdot 7}{1+\text{etc.}}}}}$$

PROBLEM IV

To convert an infinite series of this form

$$s=\frac{x}{\alpha}-\frac{xx}{\alpha\beta}+\frac{x^3}{\alpha\beta\gamma}-\frac{x^4}{\alpha\beta\gamma\delta}+etc.$$

into a continued fraction.

SOLUTION

46. We may put in place as thus far :

$$t=\frac{x}{\beta}-\frac{xx}{\beta\gamma}+\frac{x^3}{\beta\gamma\delta}-\frac{x^4}{\beta\gamma\delta\varepsilon}+\text{etc.}$$

and

$$u=\frac{x}{\gamma}-\frac{xx}{\gamma\delta}+\frac{x^3}{\gamma\delta\varepsilon}-\frac{x^4}{\gamma\delta\varepsilon\zeta}+\text{etc.}$$

thus so that there shall become

$$s=\frac{x-tx}{\alpha};$$

from which there becomes

$$\frac{x}{s}=\frac{\alpha}{1-t}=\alpha+\frac{\alpha t}{1-t}.$$

Truly there is

$$\frac{\alpha t}{1-t}=\frac{\alpha}{-1+\frac{1}{t}}=\frac{\alpha x}{-x+\frac{x}{t}},$$

and thus there will become:

$$\frac{x}{s}=\alpha+\frac{\alpha x}{-x+\frac{x}{t}}.$$

There may be found in a similar manner:

$$\frac{x}{t}=\beta+\frac{\beta x}{-x+\frac{x}{u}},\quad \frac{x}{u}=\gamma+\frac{\gamma x}{-x+\frac{x}{v}}\ \text{etc.}$$

So that if therefore these values may be substituted into the preceding, the following continued fraction will be found :

$$\frac{x}{s}=\alpha+\cfrac{\alpha x}{\beta-x+\cfrac{\beta x}{\gamma-x+\cfrac{\gamma x}{\delta-x+\text{etc.}}}}$$

and hence there arises

THEOREM VIII

47. *If an infinite series of this kind were proposed*

$$s=\frac{x}{\alpha}-\frac{xx}{\alpha\beta}+\frac{x^3}{\alpha\beta\gamma}-\frac{x^4}{\alpha\beta\gamma\delta}+etc.,$$

thence the following continued fraction will be formed:

$$\frac{x}{s}=\alpha+\cfrac{\alpha x}{\beta-x+\cfrac{\beta x}{\gamma-x+\cfrac{\gamma x}{\delta-x+\text{etc.}}}}$$

48. So that if here in place of x we may write $\frac{x}{y}$, so that we may have this series

$$s=\frac{x}{\alpha y}-\frac{xx}{\alpha\beta yy}+\frac{x^3}{\alpha\beta\gamma y^3}-\frac{x^4}{\alpha\beta\gamma\delta y^4}+\text{etc.},$$

hence the following continued fraction will be produced :

$$\frac{x}{s}=\alpha y+\cfrac{\alpha xy}{\beta y-x+\cfrac{\beta xy}{\gamma y-x+\cfrac{\gamma xy}{\delta y-x+\text{etc.}}}}$$

49. We may take $\alpha=1,\ \beta=2,\ \delta=3,\ \delta=4$ etc., so that

$$s=\frac{x}{y}-\frac{xx}{1\cdot 2yy}+\frac{x^3}{1\cdot 2\cdot 3y^3}-\frac{x^4}{1\cdot 2\cdot 3y^4}+\text{etc.},$$

in which there is therefore

$$s=1-e^{\frac{-x}{y}},$$

and hence this continued fraction will be formed

$$\frac{x}{1-e^{\frac{-x}{y}}}=y+\cfrac{xy}{2y-x+\cfrac{2xy}{3y-x+\cfrac{3xy}{4y-x+\text{etc.}}}}=\frac{xe^{\frac{x}{y}}}{e^{\frac{x}{y}}-1},$$

from which the following special formulas will be obtained by assuming $x=1$ and in place of y successively the numbers 1, 2, 3, 4, 5 etc.

$$\frac{e}{e-1}=1+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}$$

$$\frac{\sqrt{e}}{\sqrt{e}-1}=2+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\text{etc.}}}}$$

$$\frac{\sqrt[3]{e}}{\sqrt[3]{e}-1}=3+\cfrac{3}{5+\cfrac{6}{8+\cfrac{9}{11+\text{etc.}}}}$$

$$\frac{\sqrt[4]{e}}{\sqrt[4]{e}-1}=4+\cfrac{4}{7+\cfrac{8}{11+\cfrac{12}{15+\text{etc.}}}}$$

etc.

PROBLEM V

If in general a series of this form were proposed

$$s=\frac{x}{\alpha y}-\frac{xx}{\alpha\beta yy}+\frac{x^3}{\alpha\beta\gamma y^3}-\frac{x^4}{\alpha\beta\gamma\delta y^4}+\text{etc.},$$

to convert that into a continued fraction.

SOLUTION

50. From the series proposed we may form the following :

$$t=\frac{bx}{\beta y}-\frac{bcxx}{\beta\gamma yy}+\frac{bcdx^3}{\beta\gamma\delta y^3}-\frac{bcdex^4}{\beta\gamma\delta\varepsilon y^4}+\text{etc.}$$

and

$$u=\frac{cx}{\gamma y}-\frac{cdxx}{\gamma\delta yy}+\frac{cdex^3}{\gamma\delta\varepsilon y^3}-\frac{cdefx^4}{\gamma\delta\varepsilon\zeta y^4}+\text{etc.},$$

so that there shall become

$$s = \frac{ax}{\alpha y}(1-t)$$

and hence

$$\frac{ax}{s} = \frac{\alpha y}{1-t} = \alpha y + \frac{\alpha yt}{1-t}.$$

Truly there is

$$\frac{\alpha yt}{1-t} = \frac{\alpha yt}{-1+\frac{1}{t}} = \frac{\alpha bxy}{-bx+\frac{bx}{t}},$$

from which there becomes

$$\frac{ax}{s} = \alpha y + \frac{\alpha bxy}{-bx+\frac{bx}{t}};$$

therefore in a similar manner from the relation :

$$t = \frac{bx}{\beta y}(1-u)$$

there will become

$$\frac{bx}{t} = \beta y + \frac{\beta cxy}{-cx+\frac{ex}{u}}$$

and thus henceforth. Whereby with these values substituted continually this continued fraction will be produced :

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y-bx+\cfrac{\beta cxy}{\gamma y-cx+\cfrac{\gamma dxy}{\delta y-dx+\text{etc.}}}}$$

Hence the following

THEOREM IX

51. *If this general series were proposed*

$$s = \frac{ax}{\alpha y} - \frac{abxx}{\alpha\beta yy} + \frac{abcx^3}{\alpha\beta\gamma y^3} - \frac{abcdx^4}{\alpha\beta\gamma\delta y^4} + etc.,$$

thence the continued fraction will be formed

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y-bx+\cfrac{\beta cxy}{\gamma y-cx+\cfrac{\gamma dxy}{\delta y-dx+\text{etc.}}}}$$

52. So that we may illustrate this Theorem by a noteworthy example, we may consider this integral formula :

$$Z = \int z^{m-1}dz(1+z^n)^{\frac{k}{n}-1},$$

which integral may be taken thus, so that it may vanish on putting $z = 0$, and we may put in place

$$Z = v(1+z^n)^{\frac{k}{n}}$$

on differentiating there will be

$$dZ = z^{m-1}dz(1+z^n)^{\frac{k}{n}-1}$$
$$= dv(1+z^n)^{\frac{k}{n}} + kvz^{n-1}dz(1+z^n)^{\frac{k}{n}-1},$$

which equation divided by $(1+z^n)^{\frac{k}{n}-1}$ provides :

$$z^{m-1}dz = dv(1+z^n) + kvz^{n-1}dz,$$

and thus

$$\frac{dv}{dz}(1+z^n) + kvz^{n-1} - z^{m-1} = 0.$$

53. So that on taking *s* infinitely small there becomes

$$Z = \frac{z^m}{m} = v,$$

thence we become acquainted with the quantity *v* to be expressed by an infinite series of this kind, of which the first term shall be the power z^m, but in the following terms the exponents of z to be increased continually by the number *n* ; on account of which we may form the following infinite series for *v* :

$$v = Az^m - Bz^{m+2} + Cz^{m+2n} - Dz^{m+3n} + \text{etc.},$$

which value we may substitute into the differential equation and the like powers of z themselves we may write in the following manner :

$$\begin{array}{llllll}
\frac{dv}{dz} = mAz^{m-1} & -(m+n)Bz^{m+n-1} & +(m+2n)Cz^{m+2n-1} & -(m+3n)Dz^{m+3n-1} & +\text{etc.}, \\
\frac{z^n dv}{dz} = & +mA & -(m+n)B & +(m+2n)C & -\text{etc.}, \\
+kvz^{n-1} = & +kA & -kB & +kC & -\text{etc.} \\
-z^{m-1} = -1. & & & &
\end{array}$$

54. So that if now the individual powers of z separately may be reduced to zero, the following values will be obtained :

$$mA-1=0,\ \text{therefore } A=\tfrac{1}{m},$$
$$-(m+n)B+(m+k)A=0,\ \text{therefore } B=\tfrac{(m+k)A}{m+n},$$
$$(m+2n)C-(m+n+k)B=0,\ \text{therefore } C=\tfrac{(m+n+k)B}{m+2n},$$
$$-(m+3n)D+(m+2n+k)C=0,\ \text{therefore } D=\tfrac{(m+2n+k)C}{m+3n},$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

55. Therefore we may substitute these values found and we will find the following infinite series for v :

$$v=\tfrac{1}{m}z^{m}-\tfrac{m+k}{m(m+n)}z^{m+n}+\tfrac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{m+2n}$$
$$-\tfrac{(m+k)(m+n+k)(m+2n+k)}{m(m+n)(m+2n)(m+3n)}z^{m+3n}+\text{etc.},$$

which series, so that we may reduce to the form of our Theorem, we may represent in this manner :

$$v=\tfrac{z^{m-n}}{m}\left\{\begin{array}{l} z^{n}-\tfrac{m+k}{m+n}z^{2n}+\tfrac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{3n} \\ -\tfrac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n}+\text{etc.}\end{array}\right\}$$

56. Now since there shall be

$$Z=\int z^{m-1}dz(1+z^{n})^{\frac{k}{n}-1},$$

we may put in place

$$V=\frac{mZ}{z^{m-n}(1+z^{n})^{\frac{k}{n}}},$$

so that there may become

$$V=z^{n}-\tfrac{m+k}{m+n}z^{2n}+\tfrac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{3n}-\tfrac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n}+\text{etc.};$$

therefore the function V of z will be required to be elicited by the integration of the differential formula, which value therefore may be arrived at for any determined value of z , if indeed we may thus put the integral taken, so that it may vanish on making $z=0$. Therefore so that we may assign these values of v more easily, when fractional values may be attributed to the variable z , in general we may establish :

$$z^n = \frac{x}{y},$$

thus so that we may obtain this form

$$V = \frac{x}{y} - \frac{(m+k)xx}{(m+n)yy} + \frac{(m+k)(m+n+k)x^3}{m(m+n)(m+2n)y^3} - \frac{(m+k)(m+n+k)(m+2n+k)x^4}{(m+n)(m+2n)(m+3n)y4} + \text{etc.},$$

which series brought together with our Theorem provides

$$s = V,$$

then truly

$$a = 1,\ \ b = m+k,\ \ c = m+n+k \text{ etc.},$$
$$\alpha = 1,\ \beta = m+n,\ \gamma = m+2n \quad \text{etc.}$$

57. With these noted the integral formula assumed provides us with the following continued fraction :

$$\frac{x}{V} = y + \cfrac{(m+k)xy}{(m+n)y-(m+k)x+\cfrac{(m+n)(m+n+k)xy}{(m+2n)y-(m+n+k)x+\cfrac{(m+2n)(m+2n+k)xy}{(m+3n)y-(m+n+k)x+\text{etc.}}}}$$

the value of which will be

$$\frac{xz^{m-n}(1+z^n)^{\frac{k}{n}}}{mZ}.$$

58. For the sake of an example we may assume this form

$$\int \frac{dz}{\sqrt{(1+zz)}} = l(z + (\sqrt{(1+zz)}));$$

therefore there will be $m = 1,\ n = 2,\ k = 1$: and there will become

$$V = \frac{zl(z+(\sqrt{(1+zz)}))}{\sqrt{(1+zz)}},$$

which value is equal to this series :

$$zz - \frac{2}{3}z^4 + \frac{2\cdot4}{3\cdot5}z^6 - \frac{2\cdot4\cdot6}{3\cdot5\cdot7}z^8 + \text{etc.},$$

which, since now in place of zz we may write $\frac{x}{y}$, will become

$$V=\frac{\sqrt{x}}{\sqrt{(x+y)}}l\frac{\sqrt{x}+\sqrt{(x+y)}}{\sqrt{y}}=\frac{x}{y}-\frac{2}{3}\frac{xx}{yy}+\frac{2\cdot 4}{3\cdot 5}\frac{x^3}{y^3}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\frac{x^4}{y^4}+\text{etc.};$$

whereby the continued fraction hence will be formed

$$\frac{\sqrt{x(x+y)}}{l\frac{\sqrt{x}+\sqrt{(x+y)}}{\sqrt{y}}}=y+\cfrac{1\cdot 2xy}{3y-2x+\cfrac{3\cdot 4xy}{5y-4x+\cfrac{5\cdot 6xy}{7y-6x+\cfrac{7\cdot 8xy}{9y-8x+\text{etc.}}}}}$$

59. But if now therefore we may suppose $x=1$ et $y=1$, we will have that continued fraction itself :

$$\frac{\sqrt{2}}{l(1+\sqrt{2})}=1+\cfrac{1\cdot 2}{1+\cfrac{3\cdot 4}{1+\cfrac{5\cdot 6}{1+\text{etc.}}}}$$

with this infinite series present

$$\frac{l(1+\sqrt{2})}{\sqrt{2}}=1-\frac{2}{3}+\frac{2\cdot 4}{3\cdot 5}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}-\text{etc.}$$

But if with $x=1$ remaining we may take $y=2$, the infinite series will become :

$$\frac{1}{2}-\frac{2}{3}\cdot\frac{1}{4}+\frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{8}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{16}+\text{etc.},$$

truly with this continued fraction :

$$\frac{\sqrt{3}}{l\frac{1+\sqrt{3}}{\sqrt{2}}}=2+\cfrac{1\cdot 2\cdot 2}{4+\cfrac{3\cdot 4\cdot 2}{6+\cfrac{5\cdot 6\cdot 2}{8+\text{etc.}}}}$$

from which the following form is deduced

$$\frac{\sqrt{3}}{2l\frac{1+\sqrt{3}}{\sqrt{2}}}=1+\cfrac{1}{2+\cfrac{6}{3+\cfrac{15}{4+\cfrac{28}{5+\text{etc.}}}}}$$

where the numerators are the alternate triangular numbers taken.

60. At this point we may set out the case, where $x=1$ et $y=3$, since the irrational numbers may be removed in this manner; but in this case there will be

$$\frac{1}{4}l3=\frac{1}{3}-\frac{2}{3}\cdot\frac{1}{9}+\frac{2\cdot4}{3\cdot5}\cdot\frac{1}{27}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cdot\frac{1}{81}+\text{etc.};$$

moreover the continued fraction produced hence will be

$$\frac{4}{l3}=3+\cfrac{1\cdot2\cdot3}{7+\cfrac{3\cdot4\cdot3}{11+\cfrac{5\cdot6\cdot3}{15+\cfrac{7\cdot8\cdot3}{19+\text{etc.}}}}}$$

61. Therefore since here clearly I have found a new method of transforming any infinite series into continued fractions, deservedly I may be seen to have enhanced the teaching of continued fractions to some extent. Therefore to these finally I may attach a most noteworthy theorem, where above in § 42 we have transformed the fraction

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{etc.}}}}}=\frac{1}{e-1}$$

into this :

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}=\frac{1}{e-2},$$

which thus itself appears to be had more widely.

THEOREM X

62. *If there were*

$$s=\cfrac{aA}{\alpha A+\cfrac{bB}{\beta B+\cfrac{cC}{\gamma C+\text{etc.}}}}$$

there will be

$$\frac{bs}{a-\alpha s}=\beta A+\cfrac{cA}{\gamma B+\cfrac{dB}{\delta C+\cfrac{eC}{\varepsilon D+\text{etc.}}}}$$

DEMONSTRATION

Indeed since there shall be

$$s=\cfrac{aA}{\alpha A+\cfrac{bB}{\beta B+\cfrac{cC}{\gamma C+\text{etc.}}}}$$

if we my divider the first fraction by *A,* the second by *B,* the third by *C*, etc , there will be produced :

$$s=\cfrac{a}{\alpha+\cfrac{b:A}{\beta+\cfrac{c:B}{\gamma+\cfrac{d:C}{\delta+\text{etc.}}}}}$$

Now we may multiply the second fraction of this form above and below by *A*, the third by *B*, the fourth by *C* , and thus in turn, and we will obtain this formula

$$s=\cfrac{a}{\alpha+\cfrac{b:A}{\beta+\cfrac{c:B}{\gamma+\cfrac{d:C}{\delta+\text{etc.}}}}}$$

whereby if we may put

$$t=\beta A+\cfrac{cA}{\gamma B+\cfrac{dB}{\delta C+\text{etc.}}}$$

there will be

$$s=\frac{a}{\alpha+\frac{b}{t}}=\frac{at}{\alpha t+b},$$

from which there will be found

$$t=\frac{bs}{a-\alpha a},$$

Q.E.D.

DE TRANSFORMATIONE SERIERUM IN FRACTIONES CONTINUAS UBI SIMUL HAEC THEORIA NON MEDIOCRITER AMPLIFICATUR

[E593]
Opuscula analytica 2, 1785, p. 138-177

1. Consideremus fractionem continuam quamcunque, quae sit

$$s=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}$$

ac primo quaeramus fractiones simplices, quae continuo propius ad valorem ipsius s accedant, quas ita formemus, ut sit

$$\frac{A}{\mathfrak{A}}=a,\ \frac{B}{\mathfrak{B}}=a+\frac{1}{b},\ \frac{C}{\mathfrak{C}}=a+\cfrac{1}{b+\cfrac{1}{c}},\ \frac{D}{\mathfrak{D}}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}\ \text{etc.}$$

Harum igitur fractionum ultima verum valorem fractionis continuae propositae exprimet. Hinc igitur statim patet fore

$$\frac{A}{\mathfrak{A}}=\frac{a}{1},\ \frac{B}{\mathfrak{B}}=\frac{ab+1}{b},\ \frac{C}{\mathfrak{C}}=\frac{abc+a+c}{bc+1}.$$

Quemadmodum autem hae fractiones ulterius progrediantur, sequenti modo inquiramus.

2. Evidens hic est ex fractione prima secundam oriri, si loco a scribatur

$$a+\frac{1}{b},$$

similique modo ex secunda oriri tertiam, si loco b scribatur

$$b+\frac{1}{c},$$

ex tertia vero quartam, si loco c scribatur

$$c+\frac{1}{d},$$

et ita porro. Hinc ergo, si indefinite fractio $\frac{P}{\mathfrak{P}}$ formata sit ex indicibus *a, b, c, d, ... p* binaeque sequentes ponantur $\frac{Q}{\mathfrak{Q}}$ et $\frac{R}{\mathfrak{R}}$, quae respondeant indicibus *a, b, c, d, ... q* et *a, b, c, d, e, . . . r,* manifestum est ex fractione $\frac{P}{\mathfrak{P}}$ reperiri sequentem $\frac{Q}{\mathfrak{Q}}$, si loco *p* scribatur

$$p+\frac{1}{q},$$

ex hac vero $\frac{Q}{\mathfrak{Q}}$ oriri sequentem $\frac{R}{\mathfrak{R}}$, si loco *q* scribatur

$$q+\frac{1}{r}.$$

Nunc vero facile patet in fractione $\frac{P}{\mathfrak{P}}$ tam numeratorem *P quam* denominatorem $\mathfrak{P}$ omnes litteras *a, b, c, d, ... p* ita involvere, ut nulla earum ultra primam dimensionem exsurgat. Si enim omnes indices *a, b, c, d, e* etc. ut inaequales spectentur, nullius quadratum vel altior potestas usquam occurrere poterit.

3. Quamobrem tam in *P* quam in $\mathfrak{P}$ duplicis generis occurrent termini, dum alii indicem *p* plane non continent, alii vero eum tanquam factorem involvunt; unde numerator *P* huiusmodi habebit formam $M+Np$ similique modo denominator $\mathfrak{P}$ hanc $\mathfrak{M}+\mathfrak{N}p$, ita ut sit

$$\frac{P}{\mathfrak{P}}=\frac{M+Np}{\mathfrak{M}+\mathfrak{N}p}.$$

In hac igitur forma loco *p* scribamus

$$p+\frac{1}{q},$$

ut obtineamus fractionem $\frac{Q}{\mathfrak{Q}}$, quae ergo, postquam supra et infra per *q* multiplicaverimus, erit

$$\frac{Q}{\mathfrak{Q}}=\frac{Mq+Npq+N}{\mathfrak{M}q+\mathfrak{N}pq+\mathfrak{N}}=\frac{(M+Np)q+N}{\mathfrak{N}+(\mathfrak{M}+\mathfrak{N}p)q}.$$

Nunc ut hinc sequentem fractionem $\frac{R}{\mathfrak{R}}$ obtineamus, loco q scribamus

$$q+\frac{1}{r},$$

et postquam supra et infra per r multiplicaverimus, orietur

$$\frac{R}{\mathfrak{R}}=\frac{Nr+(M+Np)qr+M+Np}{\mathfrak{N}r+(\mathfrak{M}+\mathfrak{N}p)qr+\mathfrak{M}+\mathfrak{N}p},$$

sive

$$\frac{R}{\mathfrak{R}}=\frac{M+Np+(N+Mq+Npq)r}{\mathfrak{M}+\mathfrak{N}p+(\mathfrak{N}+\mathfrak{M}q+\mathfrak{N}pq)r}.$$

Cum igitur sit $P=M+Np$ et $Q=N+(M+Np)q$, erit

$$R=P+Qr.$$

Simili modo cum sit $\mathfrak{P}=\mathfrak{M}+\mathfrak{N}p$ et $\mathfrak{Q}=\mathfrak{N}+(\mathfrak{M}+\mathfrak{N}p)q,$ erit

$$\mathfrak{R}=\mathfrak{P}+\mathfrak{Q}r.$$

Sicque patet, quomodo quaelibet nostrarum simplicium fractionum ex binis praecedentibus facile formari possit.

4. Ecce igitur demonstrationem satis planam et dilucidam regulae notissimae pro conversione fractionis continuae in fractiones simplices, ubi tam numeratores quam denominatores secundum eandem legem ex binis praecedentibus formantur. Cum igitur pro ambabus primis fractionibus sit $A=a,\ \mathfrak{A}=1$, tum vero $B=ab+1$ et $\mathfrak{B}=b$, ex his duabus fractionibus sequentes omnes facili negotio formari poterunt. Quod quo clarius appareat, singulis indicibus *a, b, c, d, e* etc. fractiones respondentes ordine subscribamus

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & \text{etc.} \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} & \frac{F}{\mathfrak{F}} & \frac{G}{\mathfrak{G}} & \text{etc.} \end{array}$$

ac tam numeratores quam denominatores secundum eandem legem ex binis praecedentibus sequenti modo determinabuntur:

Pro numeratoribus	Pro denominatoribus
$A=a$	$\mathfrak{A}=1$
$B=Ab+1$	$\mathfrak{B}=b$
$C=Bc+A$	$\mathfrak{C}=\mathfrak{B}c+\mathfrak{A}$
$D=Cd+B$	$\mathfrak{D}=\mathfrak{C}d+\mathfrak{B}$

$E = De + C$	$\mathfrak{E} = \mathfrak{D}e + \mathfrak{C}$
$F = Ef + D$	$\mathfrak{F} = \mathfrak{E}f + \mathfrak{D}$
etc.	etc.

Unde perspicuum est in serie numeratorum terminum primo anteriorem ex lege progressionis esse debere $=1$, in serie autem denominatorum terminum primo anteriorem esse debere $=0$, ita ut fractio primam praecedens sit $\frac{1}{0}$.

5. Quoniam per se satis est perspicuum has fractiones

$$\tfrac{A}{\mathfrak{A}},\ \tfrac{B}{\mathfrak{B}},\ \tfrac{C}{\mathfrak{C}},\ \tfrac{D}{\mathfrak{D}}\ \text{etc.}$$

continuo propius ad veritatem accedere ac tandem verum valorem fractionis continuae exhaurire, necesse est, ut differentiae inter harum fractionum binas proximas continuo fiant minores, quamobrem has differentias ordine evolvamus. Primo igitur habebimus

$$\text{II} - \text{I} = \tfrac{B\mathfrak{A} - A\mathfrak{B}}{\mathfrak{A}\mathfrak{B}}.$$

iam hic loco B et $\mathfrak{B}$ valores ex tabula substituantur ac prodibit numerator $A\mathfrak{A}b + \mathfrak{A} - Ab$, quae forma ob $\mathfrak{A} = 1$ abit in 1, ita ut sit

$$\tfrac{B}{\mathfrak{B}} - \tfrac{A}{\mathfrak{A}} = \tfrac{1}{\mathfrak{A}\mathfrak{B}}.$$

Porro erit

$$\text{III} - \text{II} = \tfrac{C\mathfrak{B} - B\mathfrak{C}}{\mathfrak{B}\mathfrak{C}},$$

cuius numerator, si loco C et $\mathfrak{C}$ valores assignati scribantur, praebet

$$\mathfrak{B}(Bc + A) - B(\mathfrak{B}c + \mathfrak{A}) = A\mathfrak{B} - B\mathfrak{A}.$$

Modo autem vidimus esse $A\mathfrak{B} - B\mathfrak{A} = 1$, unde iste numerator erit -1 ideoque

$$\tfrac{C}{\mathfrak{C}} - \tfrac{B}{\mathfrak{B}} = -\tfrac{1}{\mathfrak{B}\mathfrak{C}}.$$

Porro est

$$\text{IV} - \text{III} = \tfrac{D\mathfrak{C} - C\mathfrak{D}}{\mathfrak{C}\mathfrak{D}},$$

ubi, si loco D et $\mathfrak{D}$ valores assignati scribantur, erit

$$\mathfrak{C}D - C\mathfrak{D} = \mathfrak{C}(CD + B) - C(\mathfrak{C}D + \mathfrak{B}) = B\mathfrak{C} - C\mathfrak{B}.$$

Modo autem vidimus esse $C\mathfrak{B} - B\mathfrak{C} = -1$, unde concluditur

$$\tfrac{D}{\mathfrak{D}} - \tfrac{C}{\mathfrak{C}} = +\tfrac{1}{\mathfrak{C}\mathfrak{D}}.$$

Simili modo reperietur pro sequentibus

$$\frac{E}{\mathfrak{E}}-\frac{D}{\mathfrak{D}}=-\frac{1}{\mathfrak{DE}},\ \frac{F}{\mathfrak{F}}-\frac{E}{\mathfrak{E}}=+\frac{1}{\mathfrak{EF}}\text{ etc.}$$

6. Hinc igitur singulas nostras fractiones ex sola prima $\frac{A}{\mathfrak{A}}=a$ et fractionibus solas litteras germanicas involventibus definire poterimus, quandoquidem habebimus

$$\begin{aligned}
\frac{B}{\mathfrak{B}}&=a+\frac{1}{\mathfrak{AB}},\\
\frac{C}{\mathfrak{C}}&=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}},\\
\frac{D}{\mathfrak{D}}&=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}},\\
\frac{E}{\mathfrak{E}}&=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}}-\frac{1}{\mathfrak{DE}},\\
\frac{F}{\mathfrak{F}}&=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}}-\frac{1}{\mathfrak{DE}}+\frac{1}{\mathfrak{EF}}\\
&\text{etc.}
\end{aligned}$$

7. Cum igitur harum fractionum ultima seu infinitesima verum valorem fractionis continuae propositae, quem designamus littera *s,* exhibeat, erit

$$s=a+\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}}-\frac{1}{\mathfrak{DE}}+\frac{1}{\mathfrak{EF}}-\frac{1}{\mathfrak{FG}}+\text{etc.};$$

sicque fractionem continuam reduximus ad seriem infinitam fractionum, quarum omnes numeratores sunt alternatim $+1$ et -1, denominatores vero per solas litteras germanicas determinantur, ita ut non opus sit valores litterarum *A, B, C* etc. evolvere, sed sufficiat sequentes formulas expedivisse

$$\mathfrak{A}=1,\ \mathfrak{B}=b,\ \mathfrak{C}=\mathfrak{B}c+\mathfrak{A},\ \mathfrak{D}=\mathfrak{C}d+\mathfrak{B},\ \mathfrak{E}=\mathfrak{D}e+\mathfrak{C}\text{ etc.}$$

8. Cum igitur utraque expressio incipiat a quantitate *a,* ea prorsus ex calculo egredietur, quoniam litterae germanicae ab ea prorsus non pendent; unde, quae hactenus invenimus, huc redeunt, ut proposita fractione continua

$$s=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}$$

si ex eius indicibus *b, c, d, e* etc. definiantur litterae germanicae, ubi quidem continuo est $\mathfrak{A}=1$, semper futurum sit

$$s=\frac{1}{\mathfrak{AB}}-\frac{1}{\mathfrak{BC}}+\frac{1}{\mathfrak{CD}}-\frac{1}{\mathfrak{DE}}+\text{etc.},$$

quae progressio in infinitum progreditur, si fractio continua in infinitum extendatur, contra vero finito terminorum numero constabit.

9. Cum igitur hoc modo fractionem continuam in seriem ordinariam transformaverimus, haud difficile erit seriem quamcunque propositam in fractionem continuam convertere. Proposita igitur sit ista series infinita

$$s=\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\text{etc.},$$

cuius quidem numeratores omnes sint unitates signo $+$ et $-$ alternatim affectae, denominatores vero progressionem quamcunque constituant, quod tamen non obstat, quominus omnes plane series in hac forma contineantur, siquidem termini seriei $\alpha, \beta, \gamma, \delta$ etc. non solum numeri fracti, sed etiam negativi evadere possunt.

10. Quo igitur fractionem continuam isti seriei aequalem eruamus, primo faciamus

$$\mathfrak{AB}=\alpha,\ \mathfrak{BC}=\beta,\ \mathfrak{CD}=\gamma \text{ et ita porra,}$$

unde ob $\mathfrak{A}=1$ sequentes nanciscemur valores:

$$\mathfrak{B}=\alpha \qquad \mathfrak{C}=\frac{\beta}{\alpha},$$
$$\mathfrak{D}=\frac{\alpha\gamma}{\beta}, \qquad \mathfrak{E}=\frac{\beta\delta}{\alpha\gamma},$$
$$\mathfrak{F}=\frac{\alpha\gamma\varepsilon}{\beta\delta} \qquad \mathfrak{G}=\frac{\beta\delta\zeta}{\alpha\gamma\varepsilon},$$
$$\mathfrak{H}=\frac{\alpha\gamma\varepsilon\eta}{\beta\delta\zeta} \qquad \mathfrak{I}=\frac{\beta\delta\zeta\theta}{\alpha\gamma\varepsilon\eta},$$

etc.

Nunc igitur tantum superest, ut ex his valoribus litterarum germanicarum ipsos indices *b, c, d, e* etc. fractionis continuae eliciamus.

11. Ex formulis autem, quibus supra litterae germanicae per indices fractionis continuae sunt determinatae, vicissim ex his litteris ipsos indices *b, c, d, e, f* etc. definiamus ac reperiemus

$$b=\mathfrak{B},\ \ c=\frac{\mathfrak{C}-\mathfrak{A}}{\mathfrak{B}},\ \ d=\frac{\mathfrak{D}-\mathfrak{B}}{\mathfrak{C}},\ \ e=\frac{\mathfrak{E}-\mathfrak{C}}{\mathfrak{D}},\ \ f=\frac{\mathfrak{F}-\mathfrak{D}}{\mathfrak{E}} \ \text{ etc.}$$

Hos igitur valores ordine evolvamus, dum loco litterarum $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. formulas ante inventas substituemus.

12. Primo autem erat $\mathfrak{B}=\alpha$, unde fit $b=\alpha$; deinde est

$$\mathfrak{C}-\mathfrak{A}=\frac{\beta-\alpha}{\alpha},$$

unde fit

$$c=\frac{\beta-\alpha}{\alpha\alpha}.$$

Porro erit

$$\mathfrak{D}-\mathfrak{B}=\frac{\alpha(\gamma-\beta)}{\beta},$$

unde fit

$$d=\frac{\alpha\alpha(\gamma-\beta)}{\beta\beta}.$$

Deinde habebimus

$$\mathfrak{E}-\mathfrak{C}=\frac{\beta(\delta-\gamma)}{\alpha\gamma}$$

hincque

$$e=\frac{\beta\beta(\delta-\gamma)}{\alpha\alpha\gamma\gamma}.$$

Simili modo ob

$$\mathfrak{F}-\mathfrak{D}=\frac{\alpha\gamma(\varepsilon-\delta)}{\beta\delta}$$

erit

$$f=\frac{\alpha\alpha\gamma\gamma(\varepsilon-\delta)}{\beta\beta\delta\delta}.$$

Eodem modo ob

$$\mathfrak{G}-\mathfrak{E}=\frac{\beta\delta(\zeta-\varepsilon)}{\alpha\gamma\varepsilon}$$

erit

$$g=\frac{\beta\beta\delta\delta(\zeta-\varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon}$$

etc.

Hac igitur ratione indices fractionis continuae, quam quaerimus, sequenti modo erunt expressi

$$b=\alpha, \qquad c=\frac{\beta-\alpha}{\alpha\alpha},$$

$$d=\frac{\alpha\alpha(\gamma-\beta)}{\beta\beta}, \qquad e=\frac{\beta\beta(\delta-\gamma)}{\alpha\alpha\gamma\gamma},$$

$$f=\frac{\alpha\alpha\gamma\gamma(\varepsilon-\delta)}{\beta\beta\delta\delta}, \qquad g=\frac{\beta\beta\delta\delta(\zeta-\varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon},$$

$$h=\frac{\alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta-\zeta)}{\beta\beta\delta\delta\zeta\zeta}, \qquad i=\frac{\beta\beta\delta\delta\zeta\zeta(\theta-\eta)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta}$$

etc.

13. Tantum igitur opus est, ut isti valores loco indicum *b, c, d, e, f* etc. in fractione continua

$$s=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}$$

substituamus; quoniam vero isti valores sunt fracti, quo facilius formam a fractionibus partialibus liberemus, primum ex valoribus inventis denominatores tollamus eritque

$$b=\alpha, \qquad \alpha\alpha c=\beta-\alpha,$$
$$\beta\beta d=\alpha\alpha(\gamma-\beta), \qquad \alpha\alpha\gamma\gamma e=\beta\beta(\delta-\gamma),$$
$$\beta\beta\delta\delta f=\alpha\alpha\gamma\gamma(\varepsilon-\delta), \qquad \alpha\alpha\gamma\gamma\varepsilon\varepsilon g=\beta\beta\delta\delta(\zeta-\varepsilon),$$
$$\beta\beta\delta\delta\zeta\zeta h=\alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta-\zeta), \qquad \alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta i=\beta\beta\delta\delta\zeta\zeta(\theta-\eta)$$
$$\text{etc.}$$

14. Nunc ipsam fractionem continuam ita transformerons, ut loco indicum eaedem formulae occurrant, quarum valores hic assignavimus. Secundam scilicet fractionem multiplicemus supra et infra per $\alpha\alpha$, tertiam per $\beta\beta$, quartam per $\alpha\alpha\gamma\gamma$, quintam per $\beta\beta\delta\delta$, sextam per $\alpha\alpha\gamma\gamma\varepsilon\varepsilon$ etc., ut prodeat ista forma

$$s=\cfrac{1}{b+\cfrac{\alpha\alpha}{\alpha\alpha c+\cfrac{\alpha\alpha\beta\beta}{\beta\beta d+\cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\alpha\alpha\gamma\gamma e+\cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\beta\beta\delta\delta f+\text{etc.}}}}}}$$

15. Quodsi iam loco horum novorum indicum $\alpha\alpha c$, $\beta\beta d$, $\alpha\alpha\gamma\gamma e$ etc. valores supra inventos substituamus, sequens orietur fractio continua

$$s=\cfrac{1}{\alpha+\cfrac{\alpha\alpha}{\beta-\alpha+\cfrac{\alpha\alpha\beta\beta}{\alpha\alpha(\gamma-\beta)+\cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\beta\beta(\delta-\gamma)+\cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha\gamma\gamma(\varepsilon-\delta)+\text{etc.}}}}}}$$

Quodsi hanc formam attentius consideremus, deprehendimus tertiam fractionem supra et infra deprimi posse per $\alpha\alpha$, tum vero quartam per $\beta\beta$, quintam per $\gamma\gamma$ sextam per $\delta\delta$ etc.; quo facto orietur haec fractio continua

$$s=\cfrac{1}{\alpha+\cfrac{\alpha\alpha}{\beta-\alpha+\cfrac{\beta\beta}{\gamma-\beta+\cfrac{\gamma\gamma}{\delta-\gamma+\cfrac{\delta\delta}{\varepsilon-\delta+\text{etc.}}}}}}$$

Hinc igitur stabiliamus sequens

THEOREMA I

16. *Si proposita fuerit talis series infinita*

$$s=\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\frac{1}{\varepsilon}-etc.,$$

ex ea semper formari poterit talis fractio continua

$$\frac{1}{s}=\alpha+\cfrac{\alpha\alpha}{\beta-\alpha+\cfrac{\beta\beta}{\gamma-\beta+\cfrac{\gamma\gamma}{\delta-\gamma+\cfrac{\delta\delta}{\varepsilon-\delta+\text{etc.}}}}}$$

17. Hanc igitur reductionem per plures ambages ex consideratione fractionis continuae elicuimus, quo quidem proposito nostro satisfecimus, quandoquidem seriem quamcunque in fractionem continuam transformavimus. Verum hic merito desideratur methodus directa, qua immediate ex serie proposita sine illis ambagibus fractio continua illi aequalis derivari possit. Talem igitur methodum, quippe qua theoria fractionum continuarum non mediocriter illustrabitur, hic sum expositurus.

PROBLEMA I

18. *Propositam seriem infinitam*

$$s=\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\frac{1}{\varepsilon}-etc.$$

in fractionem continuam transformare.

SOLUTIO

Cum sit

$$s=\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\frac{1}{\varepsilon}-\text{etc.}$$

statuamus

$$t=\frac{1}{\beta}-\frac{1}{\gamma}+\frac{1}{\delta}-\frac{1}{\varepsilon}+\text{etc.}$$

et

$$u = \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \frac{1}{\zeta} + \text{etc.}$$

etc.

Hinc ergo erit

$$s = \frac{1}{\alpha} - t = \frac{1-\alpha t}{\alpha},$$

unde fit

$$\frac{1}{s} = \alpha + \frac{\alpha\alpha t}{1-\alpha t}.$$

Est autem

$$\frac{\alpha\alpha t}{1-\alpha t} = \frac{\alpha\alpha}{-\alpha + \frac{1}{t}}$$

unde fit

$$\frac{1}{s} = \alpha + \frac{\alpha\alpha}{-\alpha + \frac{1}{t}}.$$

Simili ergo modo erit etiam

$$\frac{1}{t} = \beta + \frac{\beta\beta}{-\beta + \frac{1}{u}}$$

et

$$\frac{1}{u} = \gamma + \frac{\gamma\gamma}{-\gamma + \frac{1}{v}}$$

etc.,

quibus valoribus substitutis obtinebitur sequens fractio continua

$$\frac{1}{s} = \alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\beta\beta}{\gamma - \beta + \cfrac{\gamma\gamma}{\delta - \gamma + \text{etc.}}}}$$

quae est ipsa forma in theoremate exhibita.

19. Quodsi ergo series proposita sit

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = l2,$$

ob

$$\alpha = 1,\ \beta = 2,\ \gamma = 3,\ \delta = 4,\ \text{etc.}$$

erit

$$\frac{1}{l2} = 1 + \cfrac{1\cdot 1}{1 + \cfrac{2\cdot 2}{1 + \cfrac{3\cdot 3}{1 + \text{etc.}}}}$$

Sin autem assumamus hanc seriem

$$s = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.} = \tfrac{\pi}{4},$$

ob

$$\alpha = 1,\ \beta = 3,\ \gamma = 5,\ \delta = 7,\ \text{etc.}$$

erit

$$\frac{4}{\pi} = 1 + \cfrac{1\cdot 1}{2 + \cfrac{3\cdot 3}{2 + \cfrac{5\cdot 5}{2 + \text{etc.}}}}$$

quae est ipsa fractio continua olim a BROUNCKERO prolata.

20. Sumamus

$$s = \int \frac{x^{m-1}dx}{1+x^n},$$

et post integrationem statuamus $x = 1$; quo facto valor ipsius s per sequentem seriem exprimetur

$$s = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \text{etc.},$$

ita ut sit

$$\alpha = m,\ \beta = m+n,\ \gamma = m+2n,\ \delta = m+3n\ \text{ etc.};$$

hinc ergo sequens fractio continua emerget

$$\frac{1}{s} = m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \text{etc.}}}}$$

quem valorem iam XI. Tom. Commentar. V et nostrae Academiae dedi.

21. Sin autem proposita sit ista series

$$s = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} + \frac{1}{\varepsilon} + \text{etc.}$$

cuius omnes termini sint positivi, tantum opus est, ut in superiore fractione continua loco litterarum $\beta,\ \delta,\ \zeta,\ \theta$ etc. scribatur $-\beta,\ -\delta,\ -\zeta,\ -\theta$ etc.; tum igitur fiet

$$\frac{1}{s} = \alpha + \cfrac{\alpha\alpha}{-\beta - \alpha + \cfrac{\beta\beta}{\gamma + \beta + \cfrac{\gamma\gamma}{-\delta - \gamma + \cfrac{\delta\delta}{\varepsilon + \delta + \text{etc.}}}}}$$

quae fractio facile transmutatur in hanc formam

$$\frac{1}{s}=\alpha-\cfrac{\alpha\alpha}{\alpha+\beta-\cfrac{\beta\beta}{\beta+\gamma-\cfrac{\gamma\gamma}{\gamma+\delta-\text{etc.}}}}$$

22. Pluribus autem modis ipsa series proposita transformari potest, unde continuo aliae atque aliae fractiones continuae eliciuntur. Nonnullas igitur huiusmodi formas hic perpendamus. Sit ergo

$$\alpha=ab,\ \beta=bc,\ \gamma=cd,\ \delta=de \text{ etc.},$$

ut habeatur ista series

$$s=\frac{1}{ab}-\frac{1}{bc}+\frac{1}{cd}-\frac{1}{de}+\text{ etc.},$$

hincque formabitur ista fractio continua

$$\frac{1}{s}=ab+\cfrac{aabb}{b(c-a)+\cfrac{bbcc}{c(d-b)+\cfrac{ccdd}{d(e-c)+\text{etc.}}}}$$

quae facile reducitur ad formam sequentem

$$\frac{1}{s}=ab+\cfrac{aab}{c-a+\cfrac{bc}{d-b+\cfrac{cd}{e-c+\text{etc.}}}}$$

sive

$$\frac{1}{as}=b+\cfrac{ab}{c-a+\cfrac{bc}{d-b+\cfrac{cd}{e-c+\text{etc.}}}}$$

quae forma nobis suppeditat sequens Theorema.

THEOREMA II

23. *Si proposita fuerit series huius formae*

$$s=\frac{1}{ab}-\frac{1}{bc}+\frac{1}{cd}-\frac{1}{de}+\frac{1}{ef}-etc.,$$

ex ea sequens oritur fractio continua

$$\frac{1}{as}=b+\cfrac{ab}{c-a+\cfrac{bc}{d-b+\cfrac{cd}{e-c+\cfrac{de}{f-d+\text{etc.}}}}}$$

24. Haec forma, etsi facile ex praecedente derivatur, ideo est notatu digna, quod fractionem continuam formae maxime diversae praebeat, unde operae pretium erit exempla supra allata etiam ad hanc formam accommodare.
Cum igitur fuerit

$$l2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \text{etc.},$$

erit

$$l2 - 1 = -\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}$$

et his seriebus addendis oritur

$$2l2 - 1 = \tfrac{1}{1\cdot 2} - \tfrac{1}{2\cdot 3} + \tfrac{1}{3\cdot 4} - \tfrac{1}{4\cdot 5} + \tfrac{1}{5\cdot 6} - \text{etc.}$$

Hic ergo est

$$s = 2l2 - 1$$

et

$$a = 1,\ b = 2,\ c = 3,\ d = 4 \ \text{ etc.};$$

hinc igitur formabitur ista fractio continua

$$\frac{1}{2l2-1} = 2 + \cfrac{1\cdot 2}{2 + \cfrac{2\cdot 3}{2 + \cfrac{3\cdot 4}{2 + \text{etc.}}}}$$

25. Simili modo, quia est

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.},$$

erit

$$\tfrac{\pi}{4} - 1 = -\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.},$$

quarum serierum summa dat

$$\tfrac{\pi}{2} - 1 = \tfrac{2}{1\cdot 3} - \tfrac{2}{3\cdot 5} + \tfrac{2}{5\cdot 7} - \tfrac{2}{7\cdot 9} + \text{etc.}$$

sive

$$\tfrac{\pi}{4} - \tfrac{1}{2} = \tfrac{1}{1\cdot 3} - \tfrac{1}{3\cdot 5} + \tfrac{1}{5\cdot 7} - \tfrac{1}{7\cdot 9} + \text{etc.}$$

Hic igitur erit

$$s = \tfrac{\pi}{4} - \tfrac{1}{2},$$

tum vero

$$a = 1,\ \ b = 3,\ \ c = 5,\ \ d = 7 \ \ \text{ etc.};$$

quare fractio continua hinc nata erit

$$\frac{4}{\pi-2}=3+\cfrac{1\cdot 3}{4+\cfrac{3\cdot 5}{4+\cfrac{5\cdot 7}{4+\cfrac{7\cdot 9}{4+\text{etc.}}}}}$$

26. Generalius nunc etiam hanc transformationem contemplemur. Denotet igitur Δ valorem formulae integralis

$$s=\int\frac{x^{m-1}dx}{1+x^n},$$

posito post integrationem $x=1$, et cum sit, ut supra § 20 vidimus,

$$\Delta=\frac{1}{m}-\frac{1}{m+n}+\frac{1}{m+2n}-\text{etc.},$$

erit

$$\Delta-\frac{1}{m}=-\frac{1}{m+n}+\frac{1}{m+2n}-\frac{1}{m+3n}+\text{etc.},$$

quibus seriebus additis fit

$$2\Delta-\frac{1}{m}=\frac{n}{m(m+n)}-\frac{n}{(m+n)(m+2n)}+\frac{n}{(m+2n)(m+3n)}-\text{etc.};$$

hinc dividendo per n erit

$$\frac{2m\Delta-1}{mn}=\frac{1}{m(m+n)}-\frac{1}{(m+n)(m+2n)}+\frac{1}{(m+2n)(m+3n)}-\text{etc.}$$

Hic igitur habemus

$$s=\frac{2m\Delta-1}{mn},$$

tum vero

$$a=m,\ b=m+n,\ c=m+2n,\ d=m+3n\ \text{etc.},$$

quocirca fractio continua hinc formata erit

$$\frac{n}{2m\Delta-1}=m+n+\cfrac{m(m+n)}{2n+\cfrac{(m+n)(m+2n)}{2n+\cfrac{(m+2n)(m+3n)}{2n+\cfrac{(m+3n)(m+4n)}{2n+\text{etc.}}}}}$$

quae forma praecedenti simplicitate nihil cedit.

27. Tribuamus nunc etiam seriei initio assumtae

$$\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\delta}+\text{etc.}$$

numeratores quoscunque sitque

$$s=\frac{a}{\alpha}-\frac{b}{\beta}+\frac{c}{\gamma}-\frac{d}{\delta}+\text{etc.}$$

atque in Theoremate primo loco litterarum $\alpha,\ \beta,\ \gamma,\ \delta$ etc. scribi oportet $\frac{\alpha}{a},\ \frac{\beta}{b},\ \frac{\gamma}{c},\ \frac{\delta}{d}$ etc., quo facto fractio continua ita se habebit

$$\frac{1}{s}=\frac{\alpha}{a}+\cfrac{\frac{\alpha\alpha}{aa}}{\frac{\beta}{b}-\frac{\alpha}{a}+\cfrac{\frac{\beta\beta}{bb}}{\frac{\gamma}{c}-\frac{\beta}{b}+\cfrac{\frac{\gamma\gamma}{cc}}{\frac{\delta}{d}-\frac{\gamma}{c}+\text{etc.}}}}$$

iam ad fractiones tollendas prima fractio supra et infra multiplicetur per *ab* secunda per *bc,* tertia per *cd* et ita porro; tum vero utrinque per *a* multiplicando obtinebitur

$$\frac{a}{s}=\alpha+\cfrac{\alpha\alpha b}{a\beta-b\alpha+\cfrac{ac\beta\beta}{b\gamma-c\beta+\cfrac{bd\gamma\gamma}{c\delta-d\gamma+\text{etc.}}}}$$

Hinc igitur formetur sequens

THEOREMA III

28. *Si proposita fuerit series infinita huius formae*

$$s=\frac{a}{\alpha}-\frac{b}{\beta}+\frac{c}{\gamma}-\frac{d}{\delta}+\text{etc.}$$

ex ea formabitur sequens fractio continua

$$\frac{a}{s}=\alpha+\cfrac{\alpha\alpha b}{a\beta-b\alpha+\cfrac{ac\beta\beta}{b\gamma-c\beta+\cfrac{bd\gamma\gamma}{c\delta-d\gamma+\text{etc.}}}}$$

29. Ad hoc illustrandum proposita sit haec series

$$\frac{1}{1}-\frac{2}{2}+\frac{3}{3}-\frac{4}{4}+\frac{5}{5}-\text{etc.}=\frac{1}{2},$$

ita ut sit $s=\frac{1}{2}$; fractio ergo continua hinc orta erit

$$2 = 1 + \cfrac{2}{0 + \cfrac{3\cdot 4}{0 + \cfrac{8\cdot 9}{0 + \cfrac{15\cdot 16}{0 + \text{etc.}}}}}$$

quae forma reducitur ad istud productum infinitum

$$2 = 1 + \frac{2\cdot 1^2\cdot 2\cdot 4\cdot 3^2\cdot 4\cdot 6\cdot 5^2\cdot 6\cdot 8\cdot 7^2\cdot \text{etc.}}{1\cdot 3\cdot 2^2\cdot 3\cdot 5\cdot 4^2\cdot 5\cdot 7\cdot 6^2\cdot 9\cdot 8^2\cdot \text{etc.}}.$$

cuius veritas non facile perspicitur, quoniam numeri factorum in numeratore et denominatore non aequales statui possunt, etiamsi ambo sint infiniti. Nullum vero dubium esse potest, quin valor istius producti sit $=1$.

30. Consideremus nunc istam seriem

$$s = \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{3}{4} - \tfrac{4}{5} + \tfrac{5}{6} - \text{etc.}$$

cuius summa est $s = l2 - \tfrac{1}{2}$. Quia igitur est

$$a = 1,\quad b = 2,\quad c = 3,\quad d = 4 \text{ etc.},$$
$$\alpha = 2,\quad \beta = 3,\quad \gamma = 4,\quad \delta = 5 \text{ etc.},$$

fractio continua hinc nata erit

$$\frac{1}{l2 - \frac{1}{2}} = 2 + \cfrac{1\cdot 2\cdot 2^2}{-1 + \cfrac{1\cdot 3\cdot 3^2}{-1 + \cfrac{2\cdot 4\cdot 4^2}{-1 + \cfrac{3\cdot 5\cdot 5^2}{-1 + \text{etc.}}}}}$$

31. Quodsi autem hanc accipiamus seriem

$$s = \tfrac{2}{1} - \tfrac{3}{2} + \tfrac{4}{3} - \tfrac{5}{4} + \tfrac{6}{5} - \text{etc.},$$

cuius valor est $\tfrac{1}{2} + l2$, habebimus

$$a = 2,\quad b = 3,\quad c = 4,\quad d = 5 \text{ etc.},$$
$$\alpha = 1,\quad \beta = 2,\quad \gamma = 3,\quad \delta = 4 \text{ etc.};$$

hinc ergo orietur haec fractio continua

$$\frac{2}{\frac{1}{2}+l2}=1+\cfrac{1\cdot3\cdot1^2}{1+\cfrac{2\cdot4\cdot2^2}{1+\cfrac{3\cdot5\cdot3^2}{1+\cfrac{4\cdot6\cdot4^2}{1+\text{etc.}}}}}$$

sive

$$\frac{4}{2l2+1}=1+\cfrac{1^3\cdot3}{1+\cfrac{2^3\cdot4}{1+\cfrac{3^3\cdot5}{1+\cfrac{4^3\cdot6}{1+\text{etc.}}}}}$$

PROBLEMA Il

Propositam seriem infinitam

$$s=\frac{x}{\alpha}-\frac{xx}{\beta}+\frac{x^3}{\gamma}-\frac{x^4}{\delta}+etc.$$

in fractionem continuam transformare.

SOLUTIO

32. Considerentur sequentes series ex proposita serie formatae

$$t=\frac{x}{\beta}-\frac{xx}{\gamma}+\frac{x^3}{\delta}-\frac{x^4}{\varepsilon}+\text{etc.},$$

porro

$$u=\frac{x}{\gamma}-\frac{xx}{\delta}+\frac{x^3}{\varepsilon}-\frac{x^4}{\zeta}+\text{etc.},$$

$$v=\frac{x}{\delta}-\frac{xx}{\varepsilon}+\frac{x^3}{\zeta}-\frac{x^4}{\eta}+\text{etc.}$$

eritque

$$s=\frac{x}{\alpha}-tx=\frac{x(1-\alpha t)}{\alpha};$$

unde fit

$$\frac{x}{s}=\frac{\alpha}{1-\alpha t}=\alpha+\frac{\alpha\alpha t}{1-\alpha t}=\alpha+\frac{\alpha\alpha}{-\alpha+\frac{1}{t}}.$$

Hinc ergo erit

$$\frac{x}{s}=\alpha+\frac{\alpha\alpha x}{-\alpha x+\frac{x}{t}};$$

simili autem modo erit

$$\frac{x}{t}=\beta+\frac{\beta\beta x}{-\beta x+\frac{x}{u}}.$$

Hi ergo valores si omnes ordine substituantur, orietur ista fractio continua

$$\frac{x}{s}=\alpha+\cfrac{\alpha\alpha x}{\beta-\alpha x+\cfrac{\beta\beta x}{\gamma-\beta x+\cfrac{\gamma\gamma x}{\delta-\gamma x+\text{etc.}}}}$$

33. Quodsi hic ubique loco x scribamus $\frac{x}{y}$, ut habeamus hanc seriem y

$$s=\frac{x}{\alpha y}-\frac{xx}{\beta yy}+\frac{x^3}{\gamma y^3}-\frac{x^4}{\delta y^4}+\text{etc.},$$

tum fractio continua hinc nata erit

$$\frac{x}{sy}=\alpha+\cfrac{\alpha\alpha x:y}{\beta-\frac{\alpha x}{y}+\cfrac{\beta\beta x:y}{\gamma-\frac{\beta x}{y}+\text{etc.}}}$$

quae a fractionibus partialibus liberata dat

$$\frac{x}{sy}=\alpha+\cfrac{\alpha\alpha x}{\beta y-\alpha x+\cfrac{\beta\beta xy}{\gamma y-\beta x+\cfrac{\gamma\gamma xy}{\delta y-\gamma x+\text{etc.}}}}$$

Unde nascitur sequens

THEOREMA IV

34. *Si proposita fuerit huiusmodi series infinita*

$$s=\frac{x}{\alpha y}-\frac{xx}{\beta yy}+\frac{x^3}{\gamma y^3}-\frac{x^4}{\delta y^4}+etc.,$$

ex ea formari poterit ista fractio continua

$$\frac{x}{s}=\alpha y+\cfrac{\alpha\alpha xy}{\beta y-\alpha x+\cfrac{\beta\beta xy}{\gamma y-\beta x+\cfrac{\gamma\gamma xy}{\delta y-\gamma x+\cfrac{\delta\delta xy}{\varepsilon y-\delta x+\text{etc.}}}}}$$

35. Cum sit

$$l\left(1+\frac{x}{y}\right)=\frac{x}{y}-\frac{xx}{2yy}+\frac{x^3}{3y^3}-\frac{x^4}{4y^4}+\text{etc.}$$

posito

$$s=l\left(1+\frac{x}{y}\right)$$

erit

$$\alpha=1,\ \beta=2,\ \gamma=3,\ \delta=4 \text{ etc.};$$

hincque nascetur ista fractio continua

$$\frac{x}{l\left(1+\frac{x}{y}\right)}=y+\cfrac{xy}{2y-x+\cfrac{4xy}{3y-2x+\cfrac{9xy}{4y-3x+\text{etc.}}}}$$

36. Cum arcus, cuius tangens *t*, hac serie exprimatur

$$\text{Atang.}t=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\frac{t^9}{9}-\text{etc.},$$

erit

$$t\text{Atang.}t=\frac{tt}{1}-\frac{t^4}{3}+\frac{t^6}{5}-\frac{t^8}{7}+\frac{t^{10}}{9}-\text{etc.}$$

Nunc ponatur $tt=\frac{x}{y}$, ita ut sit $t=\sqrt{\frac{x}{y}}$, fietque

$$\sqrt{\frac{x}{y}}\text{Atang.}\sqrt{\frac{x}{y}}=\frac{x}{y}-\frac{xx}{3yy}+\frac{x^3}{5y^3}-\frac{x^4}{7y^4}+\text{etc.},$$

Hinc ergo est

$$s=\sqrt{\frac{x}{y}}\text{Atang.}\sqrt{\frac{x}{y}},$$

tum vero

$$\alpha=1,\ \beta=3,\ \gamma=5,\ \delta=7 \text{ etc.};$$

quare fractio continua hinc nata erit

$$\frac{\sqrt{xy}}{\text{Atang.}\sqrt{\frac{x}{y}}}=y+\cfrac{xy}{3y-x+\cfrac{9xy}{5y-3x+\cfrac{25xy}{7y-5x+\text{etc.}}}}$$

Veluti si fuerit $x=1$ et $y=3$, ob

$$\text{Atang.}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$$

habebitur ista fractio continua

$$\frac{6\sqrt{3}}{\pi}=3+\cfrac{1\cdot 3}{8+\cfrac{3\cdot 9}{12+\cfrac{3\cdot 25}{16+\text{etc.}}}}$$

37. Quodsi in casu Theorematis loco litterarum $\alpha,\ \beta,\ \gamma,\ \delta$ etc. scribamus fractiones

$$\frac{\alpha}{a},\ \frac{\beta}{b},\ \frac{\gamma}{c},\ \frac{\delta}{d}\ \text{etc.},$$

ut habeamus hanc seriem

$$s=\frac{ax}{\alpha y}-\frac{bxx}{\beta yy}+\frac{cx^3}{\gamma y^3}-\frac{dx^4}{\delta y^4}+etc.,$$

fractio continua hinc formata ita se habebit

$$\frac{x}{s}=\frac{\alpha}{a}y+\cfrac{\alpha\alpha xy:aa}{\frac{\beta}{b}y-\frac{\alpha}{a}x+\cfrac{\beta\beta xy:bb}{\frac{\gamma}{c}y-\frac{\beta}{b}x+\cfrac{\gamma\gamma xy:cc}{\frac{\delta}{d}y-\frac{\gamma}{c}x+\text{etc.}}}}$$

Hic iam primo utrinque multiplicetur per *a,* deinde primae fractionis numerator et denominator multiplicentur per *ab,* secundae per *bc,* tertiae per *cd* etc. et fractio continua hanc induet formam

$$\frac{ax}{s}=\alpha y+\cfrac{\alpha\alpha bxy}{a\beta y-b\alpha x+\cfrac{\beta\beta acxy}{b\gamma y-c\beta x+\cfrac{\gamma\gamma bdxy}{c\delta y-d\gamma x+\text{etc.}}}}$$

unde operae pretium erit sequens apponere

THEOREMA V

38. *Si proposita fuerit series infinita huius formae*

$$s=\frac{ax}{\alpha y}-\frac{bxx}{\beta yy}+\frac{cx^3}{\gamma y^3}-\frac{dx^4}{\delta y^4}+etc.,$$

inde formabitur sequens fractio continua

$$\frac{ax}{s}=\alpha y+\cfrac{\alpha\alpha bxy}{a\beta y-b\alpha x+\cfrac{\beta\beta acxy}{b\gamma y-c\beta x+\cfrac{\gamma\gamma bdxy}{c\delta y-d\gamma x+\text{etc.}}}}$$

PROBLEMA III

Propositam hanc seriem infinitam

$$s=\frac{1}{\alpha}-\frac{1}{\alpha\beta}+\frac{1}{\alpha\beta\gamma}-\frac{1}{\alpha\beta\gamma\delta}+etc.,$$

in fractionem continuam convertere.

SOLUTIO

39. Ex serie proposita formemus sequentes series

$$t=\frac{1}{\beta}-\frac{1}{\beta\gamma}+\frac{1}{\beta\gamma\delta}-\frac{1}{\beta\gamma\delta\varepsilon}+\text{etc.},$$
$$u=\frac{1}{\gamma}-\frac{1}{\gamma\delta}+\frac{1}{\gamma\delta\varepsilon}-\frac{1}{\gamma\delta\varepsilon\zeta}+\text{etc.}$$
etc.

atque habebimus

$$s=\frac{1-t}{\alpha},\quad t=\frac{1-u}{\beta},\quad u=\frac{1-v}{\gamma},\quad \text{etc.};$$

hinc igitur deducimus

$$\frac{1}{s}=\frac{\alpha}{1-t}=\alpha+\frac{\alpha t}{1-t}=\alpha+\frac{\alpha}{-1+\frac{1}{t}}.$$

Simili autem modo erit

$$\frac{1}{t}=\beta+\frac{\beta}{-1+\frac{1}{u}},\quad \frac{1}{u}=\gamma+\frac{\gamma}{-1+v},\quad \text{etc.};$$

quare posterioribus valoribus in prioribus substitutis obtinebitur ista fractio continua

$$\frac{1}{s}=\alpha+\cfrac{\alpha}{\beta-1+\cfrac{\beta}{\gamma-1+\cfrac{\gamma}{\delta-1+\text{etc.}}}}$$

unde deducimus sequens Theorema.

THEOREMA VI

40. *Si proposita fuerit huiusmodi series infinita*

$$s=\frac{1}{\alpha}-\frac{1}{\alpha\beta}+\frac{1}{\alpha\beta\gamma}-\frac{1}{\alpha\beta\gamma\delta}+etc.,$$

exinde formari poterit haec fractio continua

$$\frac{1}{s}=\alpha+\cfrac{\alpha}{\beta-1+\cfrac{\beta}{\gamma-1+\cfrac{\gamma}{\delta-1+\text{etc.}}}}$$

41. Si e denotet numerum, cuius logarithmus hyperbolicus est unitas, notum est esse

$$\frac{1}{e}=1-\frac{1}{1}+\frac{1}{1\cdot 2}-\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}-\text{etc.}$$

sive

$$\frac{e-1}{e}=\frac{1}{1}-\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}-\frac{1}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

Hic igitur fit $s=\frac{e-1}{e}$, tum vero

$$\alpha=1,\ \beta=2,\ \gamma=3,\ \delta=4\ \text{etc.};$$

quare fractio continua hinc oriunda est

$$\frac{e}{e-1}=1+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{2+\text{etc.}}}}$$

42. Cum igitur sit

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{etc.}}}}}=\frac{1}{e-1}$$

haud difficulter autem demonstrari queat, si fuerit

$$\cfrac{a}{a+\cfrac{b}{b+\cfrac{c}{c+\text{etc.}}}}=s,$$

tum fore

$$\cfrac{a}{b+\cfrac{b}{c+\cfrac{c}{d+\text{etc.}}}}=\frac{s}{1-s},$$

pro nostro casu erit

$$s=\frac{1}{e-1},\ a=1,\ b=2,\ c=3\ \text{etc.},$$

quibus valoribus substitutis fiet

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}=\frac{1}{e-2}.$$

43. Quodsi in serie Theorematis VI loco litterarum $\alpha,\ \beta,\ \gamma,\ \delta$ etc. scribantur fractiones

$$\frac{\alpha}{a},\ \frac{\beta}{b},\ \frac{\gamma}{c},\ \frac{\delta}{d}\ \text{etc.},$$

ut sit

$$s=\frac{a}{\alpha}-\frac{ab}{\alpha\beta}+\frac{abc}{\alpha\beta\gamma}-\frac{abcd}{\alpha\beta\gamma\delta}+\text{etc.},$$

fractio continua hinc nata erit

$$\frac{1}{s}=\frac{\alpha}{a}+\cfrac{\alpha:a}{\frac{\beta}{b}-1+\cfrac{\beta:b}{\frac{\gamma}{c}-1+\cfrac{\gamma:c}{\frac{\delta}{d}-1+\text{etc.}}}}$$

Quodsi iam primo multiplicetur utrinque per *a*, tum vero prima fractio supra et infra per *b*, secunda per *c*, tertia per *d* etc., orietur ista forma

$$\frac{a}{s}=\alpha+\cfrac{\alpha b}{\beta-b+\cfrac{\beta c}{\gamma-c+\cfrac{\gamma d}{\delta-d+\text{etc.}}}}$$

quod sequenti Theoremati includatur.

THEOREMA VII

44. *Si proposita fuerit huiusmodi series infinita*

$$s=\frac{a}{\alpha}-\frac{ab}{\alpha\beta}+\frac{abc}{\alpha\beta\gamma}-\frac{abcd}{\alpha\beta\gamma\delta}+etc.,$$

inde deducitur haec fractio continua

$$\frac{a}{s}=\alpha+\cfrac{\alpha b}{\beta-b+\cfrac{\beta c}{\gamma-c+\cfrac{\gamma d}{\delta-d+\text{etc.}}}}$$

45. Applicemus hoc ad sequentem seriem infinitam

$$s=\frac{1}{2}-\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}-\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}+\text{etc.},$$

cuius summam constat esse $s=\frac{\sqrt{2}-1}{\sqrt{2}}$; tum igitur erit

$$a=1,\quad b=3,\quad c=5,\quad d=7 \text{ etc.},$$
$$\alpha=2,\ \beta=4,\ \gamma=6,\ \delta=8 \text{ etc.};$$

fractio ergo continua hinc nata erit

$$\frac{\sqrt{2}}{\sqrt{2}-1}=2+\cfrac{2\cdot 3}{1+\cfrac{4\cdot 5}{1+\cfrac{6\cdot 7}{1+\text{etc.}}}}$$

Si utrinque unitas auferatur, erit

$$\frac{1}{\sqrt{2}-1}=1+\cfrac{2\cdot 3}{1+\cfrac{4\cdot 5}{1+\cfrac{6\cdot 7}{1+\text{etc.}}}}$$

unde deducitur

$$\sqrt{2}=1+\cfrac{1\cdot 1}{1+\cfrac{2\cdot 3}{1+\cfrac{4\cdot 5}{1+\cfrac{6\cdot 7}{1+\text{etc.}}}}}$$

PROBLEMA IV

Propositam seriem infinitam huius formae

$$s=\frac{x}{\alpha}-\frac{xx}{\alpha\beta}+\frac{x^3}{\alpha\beta\gamma}-\frac{x^4}{\alpha\beta\gamma\delta}+etc.$$

in fractionem continuam convertere.

SOLUTIO

46. Statuamus ut hactenus

$$t=\frac{x}{\beta}-\frac{xx}{\beta\gamma}+\frac{x^3}{\beta\gamma\delta}-\frac{x^4}{\beta\gamma\delta\varepsilon}+\text{etc.}$$

et

$$u=\frac{x}{\gamma}-\frac{xx}{\gamma\delta}+\frac{x^3}{\gamma\delta\varepsilon}-\frac{x^4}{\gamma\delta\varepsilon\zeta}+\text{etc.}$$

ita ut sit

$$s=\frac{x-tx}{\alpha};$$

unde fit

$$\frac{x}{s}=\frac{\alpha}{1-t}=\alpha+\frac{\alpha t}{1-t}.$$

Est vero

$$\frac{\alpha t}{1-t}=\frac{\alpha}{-1+\frac{1}{t}}=\frac{\alpha x}{-x+\frac{x}{t}},$$

sicque erit

$$\frac{x}{s}=\alpha+\frac{\alpha x}{-x+\frac{x}{t}}.$$

Simili igitur modo reperietur

$$\frac{x}{t}=\beta+\frac{\beta x}{-x+\frac{x}{u}},\quad \frac{x}{u}=\gamma+\frac{\gamma x}{-x+\frac{x}{v}}\text{ etc.}$$

Quodsi ergo hi valores continuo in praecedentibus substituantur, obtinebitur sequens fractio continua

$$\frac{x}{s}=\alpha+\cfrac{\alpha x}{\beta-x+\cfrac{\beta x}{\gamma-x+\cfrac{\gamma x}{\delta-x+\text{etc.}}}}$$

hincque nascitur

THEOREMA VIII

47. *Si proposita fuerit huiusmodi series infinita*

$$s=\frac{x}{\alpha}-\frac{xx}{\alpha\beta}+\frac{x^3}{\alpha\beta\gamma}-\frac{x^4}{\alpha\beta\gamma\delta}+etc.,$$

inde formabitur sequens fractio continua

$$\frac{x}{s}=\alpha+\cfrac{\alpha x}{\beta-x+\cfrac{\beta x}{\gamma-x+\cfrac{\gamma x}{\delta-x+\text{etc.}}}}$$

48. Quodsi hic loco x scribamus $\frac{x}{y}$, ut habeamus hanc seriem

$$s=\frac{x}{\alpha y}-\frac{xx}{\alpha\beta yy}+\frac{x^3}{\alpha\beta\gamma y^3}-\frac{x^4}{\alpha\beta\gamma\delta y^4}+\text{etc.},$$

hinc nascetur sequens fractio continua

$$\frac{x}{s}=\alpha y+\cfrac{\alpha xy}{\beta y-x+\cfrac{\beta xy}{\gamma y-x+\cfrac{\gamma xy}{\delta y-x+\text{etc.}}}}$$

49. Sumamus $\alpha=1,\ \beta=2,\ \delta=3,\ \delta=4$ etc., ut sit

$$s = \frac{x}{y} - \frac{xx}{1\cdot 2yy} + \frac{x^3}{1\cdot 2\cdot 3y^3} - \frac{x^4}{1\cdot 2\cdot 3y^4} + \text{etc.},$$

ubi igitur est

$$s = 1 - e^{\frac{-x}{y}},$$

hincque formabitur ista fractio continua

$$\frac{x}{1-e^{\frac{-x}{y}}} = y + \cfrac{xy}{2y-x+\cfrac{2xy}{3y-x+\cfrac{3xy}{4y-x+\text{etc.}}}} = \frac{xe^{\frac{x}{y}}}{e^{\frac{x}{y}}-1},$$

unde obtinebuntur sequentes formulae speciales sumendo $x = 1$ et loco y successive numeros 1, 2, 3, 4, 5 etc.

$$\frac{e}{e-1} = 1 + \cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}$$

$$\frac{\sqrt{e}}{\sqrt{e}-1} = 2 + \cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\text{etc.}}}}$$

$$\frac{\sqrt[3]{e}}{\sqrt[3]{e}-1} = 3 + \cfrac{3}{5+\cfrac{6}{8+\cfrac{9}{11+\text{etc.}}}}$$

$$\frac{\sqrt[4]{e}}{\sqrt[4]{e}-1} = 4 + \cfrac{4}{7+\cfrac{8}{11+\cfrac{12}{15+\text{etc.}}}}$$

etc.

PROBLEMA V

Si in genere proposita fuerit series huius formae

$$s = \frac{x}{\alpha y} - \frac{xx}{\alpha\beta yy} + \frac{x^3}{\alpha\beta\gamma y^3} - \frac{x^4}{\alpha\beta\gamma\delta y^4} + \text{etc.},$$

eam in fractionem continuam convertere.

SOLUTIO

50. Ex serie proposita formemus sequentes

$$t = \frac{bx}{\beta y} - \frac{bcxx}{\beta\gamma yy} + \frac{bcdx^3}{\beta\gamma\delta y^3} - \frac{bcdex^4}{\beta\gamma\delta\varepsilon y^4} + \text{etc.}$$

et

$$u = \frac{cx}{\gamma y} - \frac{cdxx}{\gamma\delta yy} + \frac{cdex^3}{\gamma\delta\varepsilon y^3} - \frac{cdefx^4}{\gamma\delta\varepsilon\zeta y^4} + \text{etc.},$$

ita ut sit

$$s = \frac{ax}{\alpha y}(1-t)$$

hincque

$$\frac{ax}{s} = \frac{\alpha y}{1-t} = \alpha y + \frac{\alpha yt}{1-t}.$$

Est vero

$$\frac{\alpha yt}{1-t} = \frac{\alpha yt}{-1+\frac{1}{t}} = \frac{\alpha bxy}{-bx+\frac{bx}{t}},$$

unde fit

$$\frac{ax}{s} = \alpha y + \frac{\alpha bxy}{-bx+\frac{bx}{t}};$$

simili igitur modo ex relatione

$$t = \frac{bx}{\beta y}(1-u)$$

fiet

$$\frac{bx}{t} = \beta y + \frac{\beta cxy}{-cx+\frac{cx}{u}}$$

sicque porro. Quare his valoribus continuo substitutis orietur ista fractio continua

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y - bx + \cfrac{\beta cxy}{\gamma y - cx + \cfrac{\gamma dxy}{\delta y - dx + \text{etc.}}}}$$

Hinc sequens

THEOREMA IX

51. *Si proposita fuerit ista series generalis*

$$s = \frac{ax}{\alpha y} - \frac{abxx}{\alpha\beta yy} + \frac{abcx^3}{\alpha\beta\gamma y^3} - \frac{abcdx^4}{\alpha\beta\gamma\delta y^4} + etc.,$$

inde formabitur fractio continua

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y - bx + \cfrac{\beta cxy}{\gamma y - cx + \cfrac{\gamma dxy}{\delta y - dx + \text{etc.}}}}$$

52. Ut hoc Theorema per exemplum notatu dignum illustremus, consideremus hanc formulam integralem

$$Z=\int z^{m-1}dz(1+z^n)^{\frac{k}{n}-1},$$

quod integrale ita sumatur, ut evanescat posito $z=0$, ac statuamus

$$Z=v(1+z^n)^{\frac{k}{n}}$$

eritque differentiando

$$dZ=z^{m-1}dz(1+z^n)^{\frac{k}{n}-1}$$
$$=dv(1+z^n)^{\frac{k}{n}}+kvz^{n-1}dz(1+z^n)^{\frac{k}{n}-1},$$

quae aequatio per $(1+z^n)^{\frac{k}{n}-1}$ divisa praebet

$$z^{m-1}dz=dv(1+z^n)+kvz^{n-1}dz,$$

ideoque

$$\tfrac{dv}{dz}(1+z^n)+kvz^{n-1}-z^{m-1}=0.$$

53. Quoniam sumto *s* infinite parvo fit

$$Z=\tfrac{z^m}{m}=v,$$

inde discimus quantitatem *v* per eiusmodi seriem infinitam exprimi, cuius primus terminus sit potestas z^m, in sequentibus autem terminis exponentes ipsius *z* continuo numero *n* augeri; quamobrem pro *v* fingamus sequentem seriem infinitam

$$v=Az^m-Bz^{m+2}+Cz^{m+2n}-Dz^{m+3n}+\text{etc.},$$

quem valorem in aequatione differentiali substituamus similesque potestates ipsius *z* sibi subscribamus sequenti modo

$$\begin{array}{llllll}
\tfrac{dv}{dz}=mAz^{m-1}-(m+n)Bz^{m+n-1}+(m+2n)Cz^{m+2n-1}-(m+3n)Dz^{m+3n-1}+\text{etc.}, \\
\tfrac{z^n dv}{dz}= & + & mA & -(m+n)B & +(m+2n)C & -\text{etc.}, \\
+kvz^{n-1}= & + & kA & -kB & +kC & -\text{etc.} \\
-z^{m-1}=-1. & & & & &
\end{array}$$

54. Quodsi nunc singulae ipsius *z* potestates seorsim ad nihilum redigantur,

obtinebuntur sequentes valores:

$$\begin{aligned} mA-1=0,&\ \text{ergo } A=\tfrac{1}{m},\\ -(m+n)B+(m+k)A=0,&\ \text{ergo } B=\tfrac{(m+k)A}{m+n},\\ (m+2n)C-(m+n+k)B=0,&\ \text{ergo } C=\tfrac{(m+n+k)B}{m+2n},\\ -(m+3n)D+(m+2n+k)C=0,&\ \text{ergo } D=\tfrac{(m+2n+k)C}{m+3n},\\ \text{etc.}&\qquad\qquad \text{etc.}\end{aligned}$$

55. Substituamus igitur hos valores inventos ac pro v reperiemus sequentem seriem infinitam:

$$v=\tfrac{1}{m}z^{m}-\tfrac{m+k}{m(m+n)}z^{m+n}+\tfrac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{m+2n}$$
$$-\tfrac{(m+k)(m+n+k)(m+2n+k)}{m(m+n)(m+2n)(m+3n)}z^{m+3n}+\text{etc.},$$

quam seriem, ut ad formam nostri Theorematis reducamus, hoc modo repraesentemus

$$v=\tfrac{z^{m-n}}{m}\left\{\begin{aligned}&z^{n}-\tfrac{m+k}{m+n}z^{2n}+\tfrac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{3n}\\&-\tfrac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n}+\text{etc.}\end{aligned}\right\}$$

56. Cum iam sit

$$Z=\int z^{m-1}dz(1+z^{n})^{\frac{k}{n}-1},$$

statuamus

$$V=\frac{mZ}{z^{m-n}(1+z^{n})^{\frac{k}{n}}},$$

ut fiat

$$V=z^{n}-\tfrac{m+k}{m+n}z^{2n}+\tfrac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{3n}-\tfrac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n}+\text{etc.};$$

erit igitur V functio ipsius z per integrationem formulae differentialis eruenda, quae ergo pro quovis valore ipsius z determinatum adipiscetur valorem, siquidem integrale ita capi ponimus, ut evanescat facto $z=0$. Quo igitur facilius istos valores ipsius v assignare queamus, quando variabili z valores fracti tribuantur, statuamus in genere

$$z^{n}=\tfrac{x}{y},$$

ita ut hanc formam nanciscamur

$$V=\frac{x}{y}-\frac{(m+k)xx}{(m+n)yy}+\frac{(m+k)(m+n+k)x^3}{m(m+n)(m+2n)y^3}-\frac{(m+k)(m+n+k)(m+2n+k)x^4}{(m+n)(m+2n)(m+3n)y4}+\text{etc.},$$

quae series cum nostro Theoremate collata praebet

$$s=V,$$

tum vero

$$a=1,\ \ b=m+k,\ \ c=m+n+k \text{ etc.},$$
$$\alpha=1,\ \beta=m+n,\ \gamma=m+2n \quad \text{etc.}$$

57. His notatis formula integralis assumta sequentem nobis suppeditat fractionem continuam:

$$\frac{x}{V}=y+\cfrac{(m+k)xy}{(m+n)y-(m+k)x+\cfrac{(m+n)(m+n+k)xy}{(m+2n)y-(m+n+k)x+\cfrac{(m+2n)(m+2n+k)xy}{(m+3n)y-(m+n+k)x+\text{etc.}}}}$$

cuius valor erit

$$\frac{xz^{m-n}(1+z^n)^{\frac{k}{n}}}{mZ}.$$

58. Exempli gratia sumamus hanc formam

$$\int\frac{dz}{\sqrt{(1+zz)}}=l(z+(\sqrt{(1+zz)}));$$

erit igitur $m=1,\ n=2,\ k=1$: fietque

$$V=\frac{zl(z+(\sqrt{(1+zz)}))}{\sqrt{(1+zz)}},$$

qui valor aequatur huic seriei

$$zz-\frac{2}{3}z^4+\frac{2\cdot4}{3\cdot5}z^6-\frac{2\cdot4\cdot6}{3\cdot5\cdot7}z^8+\text{etc.},$$

quae, si iam loco *zz* scribamus $\frac{x}{y}$, fiet

$$V=\frac{\sqrt{x}}{\sqrt{(x+y)}}l\frac{\sqrt{x}+\sqrt{(x+y)}}{\sqrt{y}}=\frac{x}{y}-\frac{2}{3}\frac{xx}{yy}+\frac{2\cdot4}{3\cdot5}\frac{x^3}{y^3}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\frac{x^4}{y^4}+\text{etc.};$$

quare fractio continua hinc formata erit

$$\frac{\sqrt{x(x+y)}}{l\frac{\sqrt{x}+\sqrt{(x+y)}}{\sqrt{y}}}=y+\cfrac{1\cdot 2xy}{3y-2x+\cfrac{3\cdot 4xy}{5y-4x+\cfrac{5\cdot 6xy}{7y-6x+\cfrac{7\cdot 8xy}{9y-8x+\text{etc.}}}}}$$

59. Quodsi ergo sumamus $x=1$ et $y=1$, habebimus istam fractionem continuam

$$\frac{\sqrt{2}}{l(1+\sqrt{2})}=1+\cfrac{1\cdot 2}{1+\cfrac{3\cdot 4}{1+\cfrac{5\cdot 6}{1+\text{etc.}}}}$$

ipsa serie infinita existente

$$\frac{l(1+\sqrt{2})}{\sqrt{2}}=1-\frac{2}{3}+\frac{2\cdot 4}{3\cdot 5}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}-\text{etc.}$$

Sin autem manente $x=1$ sumamus $y=2$, series infinita erit

$$\frac{1}{2}-\frac{2}{3}\cdot\frac{1}{4}+\frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{8}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{16}+\text{etc.},$$

fractio vero continua haec

$$\frac{\sqrt{3}}{l\frac{1+\sqrt{3}}{\sqrt{2}}}=2+\cfrac{1\cdot 2\cdot 2}{4+\cfrac{3\cdot 4\cdot 2}{6+\cfrac{5\cdot 6\cdot 2}{8+\text{etc.}}}}$$

unde sequens forma deducitur

$$\frac{\sqrt{3}}{2l\frac{1+\sqrt{3}}{\sqrt{2}}}=1+\cfrac{1}{2+\cfrac{6}{3+\cfrac{15}{4+\cfrac{28}{5+\text{etc.}}}}}$$

ubi numeratores sunt numeri trigonales alternatim sumti.

60. Evolvamus adhuc casum, quo $x=1$ et $y=3$, quoniam irrationalitas hoc modo tolletur; erit autem hoc casu

$$\frac{1}{4}l3=\frac{1}{3}-\frac{2}{3}\cdot\frac{1}{9}+\frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{27}-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{81}+\text{etc.};$$

fractio autem continua hinc nata erit

$$\frac{4}{l3}=3+\cfrac{1\cdot2\cdot3}{7+\cfrac{3\cdot4\cdot3}{11+\cfrac{5\cdot6\cdot3}{15+\cfrac{7\cdot8\cdot3}{19+\text{etc.}}}}}$$

61. Quoniam igitur hic novam plane methodum aperui series quascunque infinitas in fractiones continuas transformandi, merito equidem mihi videor doctrinam fractionum continuarum haud mediocriter locupletasse. His igitur tantum subiungam theorema notatu dignissimum, quo supra § 42 fractionem transformavimus

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{etc.}}}}}=\frac{1}{e-1},$$

in hanc

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}}=\frac{1}{e-2},$$

quod multo latius patens ita se habet.

THEOREMA X

62. *Si fuerit*

$$s=\cfrac{aA}{\alpha A+\cfrac{bB}{\beta B+\cfrac{cC}{\gamma C+\text{etc.}}}}$$

erit

$$\frac{bs}{a-\alpha s}=\beta A+\cfrac{cA}{\gamma B+\cfrac{dB}{\delta C+\cfrac{eC}{\varepsilon D+\text{etc.}}}}$$

DEMONSTRATIO

Cum enim sit

$$s=\cfrac{aA}{\alpha A+\cfrac{bB}{\beta B+\cfrac{cC}{\gamma C+\text{etc.}}}}$$

si primam fractionem per A, secundam per B, tertiam per c etc. deprimamus, prodibit

$$s=\cfrac{a}{\alpha+\cfrac{b:A}{\beta+\cfrac{c:B}{\gamma+\cfrac{d:C}{\delta+\text{etc.}}}}}$$

Nunc huius formae secundam fractionem supra et infra per *A*, tertiam per *B*, quartam per *C* multiplicemus et ita porro et nanciscemur hanc formam

$$s=\cfrac{a}{\alpha+\cfrac{b:A}{\beta+\cfrac{c:B}{\gamma+\cfrac{d:C}{\delta+\text{etc.}}}}}$$

quare si statuamus

$$t=\beta A+\cfrac{cA}{\gamma B+\cfrac{dB}{\delta C+\text{etc.}}}$$

erit

$$s=\frac{a}{\alpha+\frac{b}{t}}=\frac{at}{\alpha t+b},$$

unde reperietur

$$t=\frac{bs}{a-\alpha a},$$

Q.E.D.