CONCERNING A RELATION BEING ESTABLISHED BETWEEN THREE OR MORE QUANTITIES

Opuscula analytica 2, 1785, p. 91-101; [E591]

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1. From two proposed quantities *A* and *B,* a relation or ratio of these is defined, provided that two whole numbers α and β may be sought, and these to be the smallest, so that there may become $\alpha A=\beta B$; from which if the quantities *A* and *B* were commensurabile between themselves, these numbers α and β always will be able to be assigned more accurately; but if they shall be incommensurabile, the numbers α and β will be given thus, so that the difference between the formulas αA and βB shall be a minimum, or thus as small as possible, so that a closer approach to equality between these formulas αA and βB cannot happen, unless greater numbers may be used for α and β. And a problem proposed at one time by Wallis in this way is accustomed to be solved, where, with the two numbers proposed for some magnitudes *A* and *B,* the ratios are required with smaller numbers, which will both express the ratio of these exactly, which can happen without using larger numbers.

2. In a similar manner if three quantities *A, B* and *C* may be proposed, three whole numbers will be required to be found α, β and γ, so that there may become $\alpha A=\pm\beta B\mp\gamma C$; and indeed all possible values will be allowed to be found for these numbers α, β, γ, from which found the minimal numbers will be able to be shown without difficulty for the numbers α, β and γ, and in this manner the relation between the three proposed quantities *A, B* and *C* will be seen to be indicated most clearly. But the method of investigating these three numbers α, β, γ will be similar to that, by which the relation between only two quantities are accustomed to be defined, and which is resolved by operations of this kind, from which the maximum common divisor of two numbers is accustomed to be sought, that which we will illustrate by the following example.

3. Therefore the three following quantities shall be proposed: $A=49, B=59$ and $C=75$, and the numbers *a, b, c* are sought so that there may become $49a+59b+75c=0$, where indeed *a, b, c* will indicate whole numbers, either positive or negative. Now that equation may be divided by the smallest of these proposed quantities, evidently by 49, and the quotients from the latter terms arising may be resolved into whole numbers and fractions and these may be shown separately; which since taken together they must be equal to zero, we may put the whole number parts equal to the number *d,* but the fractions to the same number negative number $-d$; and in this way two equations will arise:

$$a+b+c=d \text{ and } \tfrac{10b+26c}{49}=-d.$$

Now from the latter equation there shall become $10b+26c+49d=0$, which at once may be treated as the first, evidently divided by 10 it will be give

$$b+2c+4d=+3e \text{ and } \tfrac{6c+9d}{10}=-3e.$$

Evidently here, because the numbers 6 and 9 have the common divisor 3, in place of the simple letter e at once we may write $3e$, and thus the new equation will be $2c+3d+10e=0$, which divided by 2 , and set out in a similar manner, supplies these equations:

$$c+d+5e=+f \text{ and } \tfrac{d}{2}=-f,$$

which the final gives at once $d=-2f$, and here the operations are terminated, because no further fractions are present.

4. Therefore since there must be $d=-2f$, the letter e shall not be determined, but by these two preceding letters e and f will be defined by regressing in the following manner:

$$c=3f-5e,\ b=13e+2f \text{ and } a=-8e-7f.$$

Therefore the general solution of our question, or the relation between the three numbers proposed 49, 59, 75 will be contained in the following equation :

$$-(8e+7f)49+(13e+2f)59+(3f-5e)75=0,$$

where any numbers are allowed to be taken for e and f.

5. Therefore we may consider, what kind of numbers it may be convenient to take for e and f, so that this equation may become the simplest. Initially there may be assumed $f=1$ and $e=-1$, and the relation found will be

$$1\cdot 49-11\cdot 59+8\cdot 75=0;$$

but if we may assume $e=0$ et $f=-1$, the relation will be

$$7\cdot 49-2\cdot 59-3\cdot 75=0,$$

which without doubt is the simplest form of the relation. And from this example it has been seen well enough, however larger the quantities *A, B* and *C* may have been, since we come upon smaller divisors continually for divisors, finally clearly all the fractions to be removed and always whole numbers to be found for the numbers *a, b, c* .

6. Therefore since the matter shall be evident, when the proposed quantities *A, B, C, D* are rational, or if they may be commensurable among themselves, also it is evident, if

these quantities were irrational or indeed transcendental, then the operations used here never to end, nor therefore such an exact relation able to be shown in any possible way; nevertheless, because it is noteworthy from these special cases, if the mentioned operations may be interrupted at some place, then relations of this kind are going to be produced, which indeed not exactly equal to the desired solution, yet truly may be shown truly to be approximate, since that will be able to be used most often, when a relation of this kind between quantities of this kind close to the truth and indeed with the smallest difference to be desired. But how it shall be convenient to treat the calculation in cases of this kind, we will show by some examples.

7. Therefore there shall be the three quantities $A=1,\ B=\sqrt{2}$ and $C=\sqrt{3}$, and in the firsts place, so that the operations used before may be able to find a place, we will convert these irrational quantities into decimal fractions, which indeed we will not continue beyond the sixth place. Truly there is

$$\sqrt{2}=1,414214 \text{ and } \sqrt{3}=1,732051.$$

Now by multiplying by 1000000 the whole investigation will be returned to whole numbers, since the whole relation consists of rational numbers, which maintain these relations between each other, and in this manner the principal equation $a+b\sqrt{2}+c\sqrt{3}=0$ will be transformed into this :

$$1000000a+1414214b+1732051c=0,$$

which divided by 1000000 and so that the above distributed into two parts will give

$$a+b+c=+d \text{ and } \tfrac{414214b+732051c}{1000000}=-d;$$

truly the latter part reduced to integers provides

$$414214b+732051c+1000000d=0,$$

and this equation treated in the same manner divided by 414214 leads to these equations :

$$b+c+2d=+e \text{ and } \tfrac{317837c+171572d}{414214}=-e,$$

of which the latter is reduced to this :

$$317837c+171572d+414214e=0.$$

This equation may be treated in the same manner, so that these equations may be provided :

$$d+c+2e=+f \text{ and } \tfrac{146265c+71070e}{171572}=-f,$$

of which the last part reduced to whole numbers thus is itself had:

$$146265c + 71070e + 171572f = 0,$$

which divided by 71070 provides

$$e + 2c + 2f = +g \text{ and } \tfrac{4125c+29432f}{71070} = -g;$$

the latter reduced becomes

$$4125c + 29432f + 71070g = 0\,,$$

from which on being divided by 4125 these two equations may be provided:

$$c + 7f + 17g = +h \text{ and } \tfrac{557f+945g}{4125} = -h,$$

8. In this manner these operations will be allowed to continue, as far as allowed; truly because the decimal fractions are not produced beyond the sixth figure, by these operations the final figures of our numbers become more uncertain, from which in the end an equation with two numbers hence arises.
By computation thus the true value is had :

$$557f + 945g + 4125h = 0$$
$$f + g + 7h = i,$$

there follows
$$388g + 226h + 557i = 0$$
$$g + h + 2i = k,$$
there follows
$$162g + 105i + 226k = 0$$
$$g + i + 2k = l,$$

there follows
$$57g + 16k + 105l = 0$$
$$3g + k + 6l = m,$$
there follows
$$9g + 9l + 16m = 0;$$

and hence $g = 1,\ l = -1,\ m = 0$, which is the simplest solution. From these the smallest may be obtained :
$k = 3,\ i = -8,\ h = 18,\ f = -135,\ c = 946,\ e = -1621,\ d = 2161,\ b = -\ 6889, a = 8104$.
From which we will have the approximately true equation :

$$8104 - 6889\sqrt{2} + 946\sqrt{3} = 0,$$
which agrees best with the approximate values of the roots taken.

[Unfortunately, Euler had made a small typographical error early in this calculation, rendering his conclusion incorrect; the values subsequently found in Vol. IV, Series I, *Opera Omnia,* by the editor Rudolf Fueter, have been adopted here.]

9. But although this relation may approach the truth, yet thence nevertheless is it allowed to be concluded that to be agreeing completely with the truth. If indeed, with *a, b, c* denoting rational numbers, there shall be exactly $a=b\sqrt{2}+c\sqrt{3}$, then with the squares taken there will become $aa=2bb+3cc+2bc\sqrt{6}$, and hence $\sqrt{6}=\frac{aa-2bb-3cc}{2bc}$, and thus $\sqrt{6}$ would become a rational number, which certainly shall be completely absurd ; and this likewise also is required to be understood for all other numbers with regard to the roots of any order, thus so that any irrational quantity may differ so much by its nature from all the other irrational numbers both of the same kind as well as of different orders, thus so that plainly no rational relation shall be able to have a place between several different surd quantities of this kind.

10. But whichever the transcending quantities, just as those which involve the periphery of a circle or logarithms, also since they may not be compared with the roots from any quantities, at this point it will appear greatly uncertain, if indeed such has not been shown by anyone to be impossible. Yet indeed it may be seen to prevail well enough the periphery of the circle π, of which the diameter $=1$, to allow no comparison with the simple square root formulas, since otherwise the continued fraction equal to π ought to have periodic indices, which still by no means is seen to happen. But we are compelled to remain in doubt, in short whether or not the quantity π in any way may be able to be compared with such formulas placed together ; on account of which we may undertake such an investigation for the relation of the quantities π, $\sqrt{2}$ and $\sqrt{3}$ in the manner set out by the method .

11. Therefore we may set out this equation in the manner explained:

$$a\sqrt{2}+b\sqrt{3}+c\pi=0,$$

which thus with approximately true whole numbers is itself had:

$$1414214a+1732051b+3141593c=0,$$

which divided by the smallest number provided these equations :

$$a+b+2c=d \text{ and } \frac{317837b+313165c}{1414214}=-d.$$

The latter equation therefore becomes :

$$317837b+313165c+1414214d=0,$$

which again may be divided by the smallest number, with which done these two equations are produced :

$$c+b+4d=e \text{ and } \tfrac{4672b+161554d}{313165}=-e,$$

which latter equation reduced becomes :

$$4672b+161554d+313165e=0.$$

On dividing by 4672 these equations are deduced :

$$b+34d+67e=f \text{ and } \tfrac{2706d+141e}{4672}=-f.$$

12. I shall not pursue these operations further, because, if a relation were given exactly, that thus without doubt would not be going to be complicated. But it would be of little help to have shown such almost true relations. It may seem certain enough for such an opinion, because the periphery of the circle may present such an unusual kind of quantity, so that since in no way may it allow itself to be compared, either with surds or with other kinds of transcending quantities.

13. But indefinitely many other kinds of transcending quantities may be given, which cannot be reduced either to the circle or logarithms, even if some affinity with these quantities may appear to hold; and if perhaps such quantities may maintain some exact relation to such an extent, whereby it may not be allowed to be defined directly from analytical principles, this method is seen to supply a single way, the benefit of which will allow relations of this kind to be investigated as if by divination.

14. Therefore here I may set out more accurately an individual case of this kind, which is seen not to reject such a relation, evidently the sum of the series of the reciprocals of the cubes

$$1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\tfrac{1}{6^3}+\text{etc.},$$

whereby at this stage not by any way have I been able to reduce either to the circle or to logarithms, since only the sums of all even powers may be able to be shown by even powers of π, but the sum of the first powers

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\text{etc.}$$

may be expressed by the logarithm of two.

15. Therefore since this series of the reciprocals of the cubes

$$1-\tfrac{1}{2^3}+\tfrac{1}{3^3}-\tfrac{1}{4^3}+\tfrac{1}{5^3}-\tfrac{1}{6^3}+\text{etc.}$$

may include within itself the cube of this series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\text{etc.},$$

it may well be seen, that $(\log.2)^3$ must occur in its sum, nor yet it must be equal to a certain multiple of this quantity. Truly then, since the same series includes within itself the product from the two preceding, evidently:

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\text{etc.}=l2 \text{ and } 1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}=\tfrac{\pi\pi}{6},$$

it is allowed to suspect, the product of the same $\frac{\pi\pi}{6}l2$ to occur also ; on account of which it will be worth the effort to enquire, whether perhaps the sum of the reciprocals of the cubes may be equal to such a formula composed of :

$$\alpha(l2)^3+\beta\tfrac{\pi\pi}{6}\cdot l2,$$

thus, so that α and β shall be rational numbers.

16. But by approximations I have thus assigned at one time the sum of the reciprocals of the cubes thus :

$$1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\text{etc.}=1,202056903,$$

from which if its fourth part is subtracted, the sum of this series will be produced:

$$1-\tfrac{1}{2^3}+\tfrac{1}{3^3}-\tfrac{1}{4^3}+\tfrac{1}{5^3}-\text{etc.}=0,901542677,$$

which for the sake of brevity we may put $=A$, and we may seek the numbers *a, b, c,* so that there may become

$$aA+b(l2)^3+cl2\cdot\tfrac{\pi\pi}{6}=0,$$

where since there shall be

$$\tfrac{\pi\pi}{6}=1,644934066 \text{ and } l2=0,693147180,$$

we deduce to become approximately

$$(l2)^3=0,333025 \text{ et } l2\cdot\tfrac{\pi\pi}{6}=1,140182,$$

from which the relation requiring to be set out will be

$$901543a + 333025b + 1140182c = 0.$$

17. Therefore for this equation the operations may be put in place as above, and there will be on dividing by 333025 :

$$b + 2a + 3c = d \text{ et } \frac{235493a + 141107c}{333025} = -d;$$

but the latter reduced to indices provides :

$$235493a + 141107c + 333025d = 0,$$

from which on dividing by 141107 we deduce these equations :

$$a + c + 2d = e \text{ and } \frac{94386a + 50811d}{141107} = -e$$

or

$$94386a + 50811d + 141107e = 0,$$

with which equation divided by 50811 there is deduced :

$$a + d + 2e = f \text{ and } \frac{43575a + 39485e}{50811} = -f;$$

but this latter equation reduced to integers becomes

$$43575a + 39485e + 50811f = 0,$$

from which again by dividing by the smallest number these equations arise:

$$a + e + f = g \text{ et } \frac{4090a + 11326f}{39485} = -g$$

or

$$4090a + 11326f + 39485g = 0,$$

from which these equations may be formed:

$$a + 2f + 9g = +h \text{ and } \frac{3146f + 2675g}{4090} = -h$$

etc.

18. It would be superfluous to continue these operations further, because now it can hence be understood well enough no such neat relation to be given between the three quantities assumed, so that it might be able to be considered to agree with the truth. Therefore since now I shall have attempted the investigation of this sum of the reciprocals of the cubes by so many ways in vain, and this method also shall be useless in being called upon to help, and it may seem to be a despairing reward from such a discovery.

DE RELATIONE INTER TERNAS PLURESVE QUANTITATIES INSTITUENDA

Commentatio 591 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 91-101
[Conventui exhibita die 14. augusti 1775]

1. Propositis duabus quantitatibus *A* et *B,* earum relatio seu ratio definitur, dum quaeruntur duo numeri integri α et β, iique minimi, ut fiat $\alpha A=\beta B$; unde si quantitates *A* et *B* fuerint inter se commensurabiles, istos numeros α et β semper accurate assignare licebit; sin autem sint incommensurabiles, numeros α et β ita dare licebit, ut discrimen inter formulas αA et βB sit minimum, vel ita parvum, ut propius ad aequalitatem inter has formulas αA et βB accedi nequeat, nisi pro α et β maiores numeri adhibeantur. Hocque modi solvi solet problema olim a WALLISIO propositum, quo, propositis duobus numeris quantumvis magnis *A* et *B,* rationes in minoribus numeris requiruntur, qui tam exacte eorum rationem exprimant, quam fieri potest numeris non maioribus adhibendis.

2. Simili modo si tres proponantur quantitates *A, B* et *C,* reperiri poterunt tres numeri integri α, β et γ, ut fiat $\alpha A=\pm\beta B\mp\gamma C$; et quidem omnes possibiles valores pro his numeris α, β, γ assignare licebit, quibus inventis haud difficile erit minimos numeros pro α, β et γ exhibere, atque hoc modo relatio inter ternas quantitates propositas *A, B* et *C* planissime indicari videtur. Methodus autem hos tres numeros α, β, γ investigandi similis erit illi, qua relatio inter duas tantum quantitates definiri solet, et quae eiusmodi operationibus absolvitur, quibus maximus communis divisor duorum numerorum indagari solet, id quod sequenti exemplo illustremus.

3. Propositae igitur sint tres sequentes quantitates: $A=49, B=59$ et $C=75$, et quaerantur numeri *a, b, c,* ut fiat $49a+59b+75c=0$, ubi quidem *a, b, c* numeros integros, sive positivos sive negativos significent. Iam dividatur aequatio illa per minimam propositarum quantitatum, scilicet per 49, et quoti ex posterioribus terminis o;riundi resolvantur in partes integras et fractas ac seorsim exhibeantur; quae quoniam iunctim sumtae nihilo debent aequari, partes integras statuamus aequales numero integro *d,* fractae autem eidem numero negativo $-d$; hocque modo duae hinc nascentur aequationes

$$a+b+c=d \text{ et } \tfrac{10b+26c}{49}=-d.$$

Iam ex postrema aequatione fit $10b+26c+49d=0$, quae prorsus ut prima tractetur, scilicet per 10 divisa dabit

$$b+2c+4d=+3e \text{ et } \tfrac{6c+9d}{10}=-3e.$$

Hic scilicet, quia numeri 6 et 9 communem divisorem habent 3, loco simplicis litterae *e* statim scripsimus 3*e,* sicque nova aequatio erit $2c+3d+10e=0$, quae, per 2 divisa, similique modo distributa, suppeditat has aequationes:

$$c+d+5e=+f \text{ et } \tfrac{d}{2}=-f,$$

quae ultima statim dat $d=-2f$, atque hic operationes terminantur, quoniam nullae amplius insunt fractiones.

4. Cum igitur esse debeat $d=-2f$, littera autem *e* non sit determinata, per has duas litteras *e* et *f* praecedentes sequenti modo regrediendo definientur:

$$c=3f-5e,\ b=13e+2f \text{ et } a=-8e-7f.$$

Solutio ergo generalis nostrae quaestionis, sive relatio inter ternos numeros propositos 49, 59, 75 sequenti aequatione continebitur:

$$-(8e+7f)49+(13e+2f)59+(3f-5e)75=0,$$

ubi pro *e* et *f* numeros quoscunque accipere licet.

5. Videamus igitur, quales numeros pro *e* et *f* accipi conveniat, ut haec aequatio fiat simplicissima. Sumatur primo $f=1$ et $e=-1$, et relatio inventa erit

$$1\cdot 49-11\cdot 59+8\cdot 75=0;$$

at si sumamus $e=0$ et $f=-1$, relatio erit

$$7\cdot 49-2\cdot 59-3\cdot 75=0,$$

quae sine dubio est simplicissima forma relationis. Atque ex hoc exemplo iam satis perspicuum est, quantumvis magnae fuerint quantitates *A, B* et *C,* quoniam continuo ad divisores minores devenimus, tandem omnes plane fractiones tolli ac pro numeris *a, b, c* semper numeros integros obtineri.

6. Cum igitur res sit manifesta, quando quantitates propositae *A, B, C, D* sunt rationales, sive commensurabiles inter se, etiam evidens est, si istae quantitates fuerint irrationales vel adeo transcendentes, tum operationes hic usitatas nunquam terminari, neque idcirco talem relationem exactam ullo modo exhiberi posse; veruntamen, quod his casibus imprimis est notandum, si memoratae operationes alicubi abrumpantur, tum eius modi relationes esse prodituras, quae quidem rem non exacte, attamen vero proxime exhibeant, id quod saepenumero usui esse poterit, quando inter huiusmodi quantitates relatio tantum proxime vera, et quidem in minimis desideratur. Quomodo autem huiusmodi casibus calculum tractare conveniat, nonnullis exemplis ostendemus.

7. Sint igitur ternae quantitates $A=1,\ B=\sqrt{2}$ et $C=\sqrt{3}$, ac primo, ut operationes ante adhibitae locum invenire possint, has quantitates irrationales in fractiones decimales convertamus, quas quidem non ultra sextam notam continuemus. Est vero

$$\sqrt{2}=1,414214 \text{ et } \sqrt{3}=1,732051.$$

Iam per 1000000 multiplicando tota investigatio ad numeros integros revocetur, quandoquidem tota relatio in rationibus, quas hae quantitates inter se tenent, subsistit, hocque modo aequatio principalis $a+b\sqrt{2}+c\sqrt{3}=0$ transformabitur in hanc:

$$1000000a+1414214b+1732051c=0,$$

quae divisa per 1000000 et ut supra in binas partes distributa dabit

$$a+b+c=+d \text{ et } \tfrac{414214b+732051c}{1000000}=-d;$$

postrema igitur ad integras reducta praebet

$$414214b+732051c+1000000d=0,$$

haecque aequatio per 414214 divisa eodemque modo tractata deducit ad has aequationes:

$$b+c+2d=+e \text{ et } \tfrac{317837c+171572d}{414214}=-e,$$

quarum postrema reducitur ad hanc:

$$317837c+171572d+414214e=0.$$

Tractetur ista aequatio eodem modo, ut prodeant hae aequationes:

$$d+c+2e=+f \text{ et } \tfrac{146265c+71070e}{171572}=-f,$$

quarum postrema ad integras reducta ita se habet:

$$146265c+71070e+171572f=0,$$

quae per 71070 divisa praebet

$$e+2c+2f=+g \text{ et } \tfrac{4125c+29432f}{71070}=-g;$$

posterior reducta fit

$$4125c+29432f+71070g=0\,,$$

unde per 4125 dividendo se produnt hae duae aequationes:

$$c+7f+17g=+h \text{ et } \tfrac{575f+945g}{4125}=-h,$$

at haec posterior ad integros reducta praebet istam:

$$575f+945g+4125h=0.$$

Dividatur nunc per 575 prodibitque

$$f+g+7h=+i \text{ et } \tfrac{370g+100h}{575}=-i$$

sive

$$\tfrac{74g+20h}{115}=-i,$$

quae reducta fit

$$74g+20h+115i=0,$$

unde per 20 dividendo hae oriuntur aequationes:

$$h+3g+5i=k \text{ et } \tfrac{14g+15i}{20}=-k.$$

8. Hoc modo has operationes continuari liceret, quousque libuerit; verum quia fractiones decimales non ultra sextam figuram sunt productae, per has operationes ultimae numerorum nostrorum figurae continuo magis fiunt incertae, unde in ultima aequatione binos numeros 14 et 15 tanquam aequales inter se spectare licebit, unde capi poterit g = 1 et i = -1, fletque
k = 0, atque hinc regrediendo sequentes valores reperientur:
$h=2$, f=-16, c=97, e=-161, d=209, b=-676, a=788,
sicque relatio quaesita ita se habebit:

Computatio vera ita se habet:

$$557f+945g+4125h=0$$
$$f+\quad g+\quad 7h=i$$

sequitur $$388g+226h+557i=0$$
$$g+\quad h+\quad 2i=k$$
sequitur $$162g+105i+226k=0$$
$$g+\quad i+\quad 2k=l$$

sequitur $57g+16k+105l=0$

$3g+\quad k+\quad 6l=m$

sequitur $9g\ +9l\quad +16m=0$

hincque $g=1,\ l=-1,\ m=0$, quae est solutio simplicissima. Ex his paulatim obtinetur: $k=3,\ i=-8,\ h=18,\ f=-135,\ c=946,\ e=-1621,\ d=2161,\ b=-\ 6889, a=8104$. Unde habebitur haec aequatio proxime vera

$$8104-6889\sqrt{2}+946\sqrt{3}=0,$$

9. Quantumvis autem haec relatio ad veritatem accedat, tamen inde neutiquam concludere licet eam penitus veritati esse consentaneam. Si enim, denotantibus *a, b, c* numeros rationales, esset exacte $a=b\sqrt{2}+c\sqrt{3}$, tum sumtis quadratis foret $aa=2bb+3cc+2bc\sqrt{6}$, hincque $\sqrt{6}=\frac{aa-2bb-3cc}{2bc}$, ideoque $\sqrt{6}$ foret numerus rationalis, quod utique maxime esset absurdum; atque hoc idem etiam de omnibus aliis numeris radicalibus cuiuscunque ordinis est tenendum, ita ut quaelibet quantitas irrationalis natura sua tantopere discrepet ab omnibus aliis irrationalibus tam eiusdem quam diversorum graduum, ut nulla plane relatio rationalis inter plures huiusmodi quantitates surdas diversas locum habere possit.

10. Utrum autem quantitates transcendentes, veluti qui peripheriam circuli involvunt sive logarithmi, etiam cum nullis quantitatibus radicalibus comparari queant, adhuc maxime incertum videtur, siquidem a nemine adhuc talis impossibilitas est ostensa. Tantum quidem satis evictum videtur peripheriam π circuli, cuius diameter $=1$, nullam comparationem cum formulis radicalibus quadraticis simplicibus admittere, quoniam aliter fractio continua ipsi π aequalis indices periodicos habere deberet, quod tamen neutiquam evenire videtur. Num autem quantitas π cum talibus formulis compositis nullo prorsus modo comparari queat, in dubio relinquere cogimur; quamobrem talem investigationem pro relatione quantitatum $\pi,\ \sqrt{2}$ et $\sqrt{3}$ methodo modo exposita suscipiamus.

11. Evolvamus igitur modo explicato hanc aequationem:

$$a\sqrt{2}+b\sqrt{3}+c\pi=0,$$

quae in numeris integris proxime veris ita se habet:

$$1414214a+1732051b+3141593c=0,$$

quae per minimum numerum divisa praebet has aequationes:

$$a+b+2c=d \text{ et } \frac{317837b+313165c}{1414214}=-d.$$

Postrema aequatio ergo in integris fit

$$317837b+313165c+1414214d=0,$$

quae iterum per minimum numerum dividatur, quo facto produnt hae duae aequationes:

$$c+b+4d=e \text{ et } \tfrac{4672b+161554d}{313165}=-e,$$

quae posterior reducta fit

$$4672b+161554d+313165e=0.$$

Dividendo per 4672 hae colliguntur aequationes:

$$b+34d+67e=f \text{ et } \tfrac{2706d+141e}{4672}=-f.$$

12. Operationes has ulterius non prosequor, quoniam, si exacta daretur relatio, ea sine dubio non adeo complicata esset futura. Prope veras autem tales relationes exhibuisse parum iuvaret. Unde sententia satis certa videtur, quod peripheria circuli tam peculiare genus quantitatum transcendentium constituat, ut cum nullis aliis quantitatibus, sive surdis, sive alius generis transcendentibus, nullo modo se comparari patiatur.

13. Infinita autem alia dantur transcendentium genera, quae neque ad circulum neque ad logarithmos reduci possunt, etiamsi quampiam affinitatem cum his quantitatibus tenere videantur; ac si forte tales quantitates cum hactenus cognitis exactam quandum relationem tenerent, quaro directe ex principiis analyticis definira non licet, haec methodus unicam viam suppeditare videtur, cuius beneficio huiusmodi relationes quasi divinando explorare licebit.

14. Huiusmodi igitur casum singularem, qui talem relationem non respuere videtur, hic accuratius evolvam, scilicet summam seriei reciprocae cuborum

$$1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\tfrac{1}{6^3}+\text{etc.},$$

quaro nullo adhuc modo sive ad circulum sive ad logarithmos reducere potui, cum tamen summae potestatum parium omnes per potestates pares ipsius π exhiberi queant, summa autem primarum potestatum

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\text{etc.}$$

logarithmum binarii exprimat.

15. Cum igitur haec series reciproca cuborum

$$1-\tfrac{1}{2^3}+\tfrac{1}{3^3}-\tfrac{1}{4^3}+\tfrac{1}{5^3}-\tfrac{1}{6^3}+\text{etc.}$$

cubum huius seriei

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\text{etc.}$$

in se complectatur, probabile videtur, in eius summa $(\log.2)^3$ occurrere debere, neque tamen cuipiam multiplo huius quantitatis aequari certum est. Deinde vero, cum eadem series in se complectatur productum ex binis praecedentibus, scilicet:

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\text{etc.}=l2 \;\text{ et }\; 1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}=\tfrac{\pi\pi}{6},$$

suspicari licet, ibidem quoque productum $\frac{\pi\pi}{6}l2$ occurrere; quamobrem operae pretium erit inquirere, num forte summa seriei reciprocae cuborum tali formulae compositae:

$$\alpha(l2)^3+\beta\tfrac{\pi\pi}{6}\cdot l2$$

aequetur, ita ut α et β sint numeri rationales.

16. Per approximationes autem olim summam seriei reciprocae cuborum ita assignavi :

$$1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\text{etc.}=1,202056903,$$

unde si eius pars quarta subtrahatur, prodit summa huius seriei:

$$1-\tfrac{1}{2^3}+\tfrac{1}{3^3}-\tfrac{1}{4^3}+\tfrac{1}{5^3}-\text{etc.}=0,901542677,$$

quam brevitatis gratia ponamus $=A$, et quaeramus numeros *a, b, c,* ut fiat

$$aA+b(l2)^3+cl2\cdot\tfrac{\pi\pi}{6}=0,$$

ubi cum sit

$$\tfrac{\pi\pi}{6}=1,644934066 \text{ et } l2=0,693147180,$$

colligimus fore proxime

$$(l2)^3=0,333025 \text{ et } l2\cdot\tfrac{\pi\pi}{6}=1,140182,$$

unde relatio evolvenda erit

$$901543a+333025b+1140182c=0.$$

17. Pro hac igitur aequatione operationes instituantur ut supra, eritque dividendo per 333025

$$b+2a+3c=d \text{ et } \tfrac{235493a+141107c}{333025}=-d;$$

at posterior ad integros reducta praebet

$$235493a+141107c+333025d=0,$$

unde dividendo per 141107 deducimus has aequationes:

$$a+c+2d=e \text{ et } \frac{94386a+50811d}{141107}=-e$$

sive

$$94386a+50811d+141107e=0,$$

qua aequatione divisa per 50811 colligitur

$$a+d+2e=f \text{ et } \frac{43575a+39485e}{50811}=-f;$$

at haec posterior reducta ad integros fit

$$43575a+39485e+50811f=0,$$

unde porro dividendo per minimum numerum oriuntur hae aequationes:

$$a+e+f=g \text{ et } \frac{4090a+11326f}{39485}=-g$$

sive

$$4090a+11326f+39485g=0,$$

unde formentur hae aequationes:

$$a+2f+9g=+h \text{ et } \frac{3146f+2675g}{4090}=-h$$

etc.

18. Superfluum foret has operationes ulterius continuare, quoniam hinc iam satis intelligere licet nullam dari relationem tam concinnam inter ternas quantitates assumtas, ut veritati consentanea censeri posset. Cum igitur investigationem huius summae reciprocae cuborum tot variis modis frustra explorare tentassem atque haec methodus etiam inutiliter sit in usum vocata, merito de tali inventione desperandum videtur.