CERTAIN ANALYTICAL THEOREMS OF WHICH THE DEMONSTRATION IS NOW DESIRED

[E590]

Opuscula analytica 2**,** 1785, p. 76-90

1. In Diophantine analysis, which is concerned with the properties of numbers, it is well known numerous theorems occur, of which the truth cannot be doubted, even if we may not be able to confirm that by a rigorous demonstration. However, in geometry, no one hitherto has presented theorems of this kind in public, of which either the truth or falsehood of which may not be allowed to be demonstrated. But now truly in higher analysis theorems of this kind also have occurred to me recently, the demonstration of which even now in no manner have I been able to find , even if the truth of these by no means may be seen to be called into any doubt. Therefore such theorems certainly deserve our maximum attention, since clearly there should be no doubt, even if hitherto we had finished the demonstration of these with a frustrated result, why thence they may not be able to return increases of the greatest benefit to analysis.

2. But initially deservedly I attribute a place among the analytical truths to that significant property of imaginary quantities, where, whenever such quantities occur with their impossible nature, these may always be able to be taken in this form $a+b\sqrt{-1}$. For indeed the resolution of all algebraic equations depends on this truth ; certainly the roots of which unless they were real, all are presented to be contained in such a form $a+b\sqrt{-1}$, that which also has been confirmed by the illustrious D'Alembert by a most ingenious demonstration; but since which has been demanded from a consideration of the infinitely small, it is not at all without merit a clearer demonstration may be desired besides, depending on the nature of the imaginary quantities themselves. Moreover truly that demonstration itself is apparent only for algebraic expressions, since again it may be equally sure also to have a place for transcending quantities of every kind, where the reasoning, which the most celebrated man has used, cannot always be used, which will be worth the effort to have shown more clearly.

3. An algebraic curve may be considered, that were composed from however many branches, the branch *FNLMH* (Fig. 1) shall be of this kind, which relative to the axis *AK*, after it will proceed to the right from *F* as far as to *L*, hence again it extends to the left through *LMH*, thus so that, if the applied line *KL* may touch this curve at the extremity *L*, the twofold applied lines *PM* and *PN* may correspond to any abscissa *AP* smaller than *AK*.. From which if there may be put the abscissa $AP=x$, the applied line *y* will have a double value requiring to be determined from such a quadratic equation $yy=2py-q$, thus so that hence the applied line shall be either

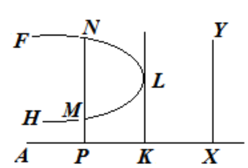

Fig. 1.

$PM = p - \sqrt{(pp-q)}$, or truly $PN = p + \sqrt{(pp-q)}$, where according to the nature of the curve the letters p and q can denote some functions of the abscissa x. Therefore as long as there were $pp > q$, the actual pair of applied lines PM and PN will arise. But provided that the abscissa x is increased as far as to K, where there may become $pp = q$, there both the applied lines will merge into one KL, thus so that here the applied line KL becomes the tangent of the curve. Because therefore the abscissa x on being increased further, there may become $q > pp$, both the applied lines will emerge imaginary. From which it is understood, if the abscissa may be taken $AX > AK$, in this place evidently no applied line to be given, or each perpendicular right line XY produced indefinitely at no time is going to cross the curve FLH, that which is to be discussed in analysis in the customary manner may indicate the same, and the applied line at this place X to become imaginary; from which it is understood more clearly by a similar imaginary notation, how they are to be addressed in analysis. Indeed since this applied line XY of the curve may never meet, even if it may be continued from the point X, where it is $=0$, both upwards as far as positive infinity as well as downwards as far as minus infinity, it is evident its value found neither to be 0, nor greater nor less than 0, from which condition the definition of the imaginary quantities itself be contained. So that if for this place we may suppose to become $q = pp - rr$, the double expression of the applied line will emerge

$$y = p \pm r\sqrt{-1}.$$

4. Therefore here it is enquired, whether hence in general certainly it may be concluded, however often imaginary numbers may occur, those can be expressed always by a formula of this kind $p + r\sqrt{-1}$. Indeed in the first place this demonstration has been demanded only from the branch FLH, while the whole curve perhaps contained by an equation between x and y perhaps will involve several other branches in addition, which in this undertaking perhaps may not be allowed to be completely ignored. But the most excellent man certainly foresaw this same objection, while the account extended only to an infinitely small portion of the curve NLM, where the extension of other branches may be ignored with care, but which not situated in the full light it may seem, so that we may not be able to desire a plainer demonstration from such a concept lacking in merit. Then truly also hence further it may not follow, how the applied lines XY infinitely close to the extreme KL are able to be expressed by such a formula $p \pm r\sqrt{-1}$, and not without merit it may be allowed to doubt, whither for the greater intervals KX also the applied lines of such formula may be able to be understood and can it be that the parts of the rest of the curve hitherto neglected may prevail in these places to be completely unchanged.

5. Besides truly that same consideration has been adapted only to equations and algebraic curves, certainly in which all the branches are not given, except which may return on themselves or may depart to infinity on each side, thus so that around the terminus L here a part of the curve may show always two parts LM and LN, from which that quadratic equation $yy = 2py - q$ has arisen, on which the whole demonstration depends. But truly branches of this kind occur between the transcending curves, which

neither are extended out to infinity nor may return into themselves, but suddenly are terminated in a certain point. A transcending curve presents such a case held by this equation

$$y = a + \tfrac{bx}{l(c-x)},$$

from which it follows a single applied line only corresponds to the individual abscissas. For on putting $x = 0$ there becomes $y = a$; and if the abscissa x may be increased continually as far as to the value $x = c$, a single applied line will be given always ; truly with the abscissa taken $x = c$ on account of $l(c-x) = -\infty$ the applied line hare will become $y = a$. And moreover immediately the abscissa x is increased beyond c , the applied line suddenly will become imaginary, therefore because the logarithms of negative quantities certainly are imaginary; whereby with the abscissa taken $x > c$ the applied line y, even if it may be produced indefinitely on both sides, yet it will never run to meet our curve. But in this case the account cited above depending on the nature of the quadratic equation ceases completely, thus so that here we may be able deservedly to doubt, or also we may be able to understand that same applied line in the formula $p + q\sqrt{-1}$. At least here we must acknowledge that same theorem to need by another demonstration and requiring to be chosen especially, so that the equation of such may be derived at once from the nature of the imaginary numbers themselves.

6. But before I may abandon this argument, it will help to have shown, clearly how all the individual imaginary numbers shall be able to be represented directly on account of the circle. From the point A (Fig. 2) according to the principle axis AB assumed the perpendicular $AC = a$ may be erected ; a circle may be described with centre C and with the radius $CM = c$ and with some abscissa put in place $AP = x$ and with the applied line $PM = y$ corresponding to that, there will be

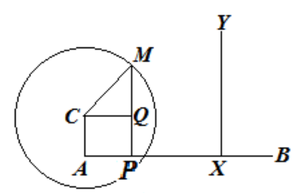

Fig. 2.

$$y = AC + QM = AC + \sqrt{(CM^2 - CQ^2)} = a + \sqrt{(cc - xx)},$$

thus so that is value shall be real always, as long as the x is taken smaller than the radius c; truly as soon as the abscissa x exceeds the radius c, or as if there may be taken $x = AX$, then certainly the applied line XY will be imaginary. But truly, since on this account the applied line cannot be shown, yet it will have the value of an imaginary number determined (for now not to be indisposed to the idea of determining imaginary quantities). For because there may be put $x > c$, there may be put $xx = cc + bb$, so that there may become $\sqrt{(cc - xx)} = b\sqrt{-1}$ and thus this same imaginary applied line $XY = a + b\sqrt{-1}$. Whereby since the formula $a + b\sqrt{-1}$ clearly may contain all imaginary quantities, these will be allowed to be represented by a applied line XY of this kind determined pertaining to some circle. Clearly with the perpendicular $AC = a$ with centre C for some arbitrary assumed radius c , a circle is described and the abscissa may be taken $AX = \sqrt{(bb+cc)}$ *;* then indeed the imaginary applied line XY will show that same formula $a+b\sqrt{-1}$ and thus accordingly in a certain miraculous way all the imaginary formulas will be able to be constructed, as if geometrically.

7. It will be worth the effort to have indicated this by a certain example. Evidently we seek the arc of a circle, of which with the sine shall be twice as great as the whole sine, which certainly will be imaginary. Therefore with the whole sine $=1$ the formula $\frac{dx}{\sqrt{(1-xx)}}$ must be integrated , thus so that the integral may vanish on putting $x=0$; then truly there will have to be taken $x=2$ and the value of this integral will give this sine itself. In the end we will attribute this form $\frac{dx\sqrt{-1}}{\sqrt{(xx-1)}}$ to the formula of the differential $\frac{dx}{\sqrt{(1-xx)}}$; but there is agreed to be

$$\int \frac{dx}{\sqrt{(xx-1)}} = l\,\frac{x+\sqrt{(xx-1)}}{\sqrt{-1}},$$

from which on putting $x=2$ the arc sought will be

$$= = \sqrt{-1}l\,\frac{2+\sqrt{3}}{\sqrt{-1}} = \sqrt{-1}l(2+\sqrt{3}) - \sqrt{-1}l\sqrt{-1}.$$

But we know the value of this latter member to be $\frac{\pi}{2}$, from which the arc of the circle, of which the sine $=2$, will be $=\frac{\pi}{2}+\sqrt{-1}l(2+\sqrt{3})$. On account of which so that we may show the applied line XY to be equal to this applied line, in our figure there may be taken the interval $AC=\frac{\pi}{2}$ and with the circle described of radius $CM=c=1$, since c is left to our choice, for the sake of brevity on putting $l(2+\sqrt{3})=b$ the abscissa may be taken $AX=\sqrt{(1+bb)}$ and thus the imaginary applied line XY will be equal to the equally imaginary arc sought, because that therefore may seem to be more noteworthy, since this same arc is of an imaginary transcending quantity.

8. Therefore the first analytical theorem, of which a plainer or at least a more direct demonstration is desired, if indeed its truth certainly now may be seen to prevail sufficiently, may be proposed in this manner.

THEOREM 1

Evidently all the imaginary quantities, whichever can occur in analytical calculations, can be recalled to that most simple form $a+b\sqrt{-1}$*, so that the letters a and b may denote real quantities.*

Therefore the demonstration of this I do not doubt is recommended especially with the most acute analysis.

9. The two following theorems consider the rectification of curved lines and thus are to be referred to the higher geometry. Indeed since now some time ago [*Acta Erud*. 1719, 1723] a geometrical method had been discovered by the celebrated Hermann of finding innumerable algebraic curves, which either shall be rectifiable or of which the rectification may depend on some given quadrature (which method henceforth I have transferred to pure analysis and greatly enriched, thus so that it may be seen to constitute a particular kind of analysis), thence certainly they have been able to have shown an infinitude of algebraic curves, on which the rectification of may depend on the quadrature of the circle. Moreover they may all be taken together to be prepared thus, with the circle excepted, so that the arc of these may be equal to a certain sum from an algebraic quantity and a circular arc, but that algebraic quantity in no way may be allowed to be reduced to zero; from which the following theorem without doubt may be proposed as true, even if I have not yet been able to show its demonstration. [Euler later found this theorem to be false, see E783]

THEOREM 2

Besides the circle no algebraic curve is given, the individual arcs of which may be able to be expressed more simply by circular arcs.

10. This theorem may be reduced to that, so that it may be shown no algebraic equation between the two orthogonal x and y be able to be shown, so that the integral formula $\int\sqrt{(dx^2+dy^2)}$ may be equal to the arc of a certain circle, of which the sine or cosine shall be some algebraic function of x and y, with one case excepted, where the equation between x and y indicates a circle. So that it may be understood more clearly, φ may denote some indefinite angle or indefinite are in the circle, of which the radius $=1$, and there may be put

$$\int\sqrt{(dx^2+dy^2)}=a\varphi$$

and thus

$$dx^2+dy^2=aad\varphi^2$$

and there may become

$$dx=apd\varphi \text{ and } dy=aqd\varphi$$

and it is necessary, that there shall be $pp+qq=1$. Truly besides it will be required both the formulas $apd\varphi$ and $aqd\varphi$ thus to be integrable, so that the integrals of these may be able to be expressed by the sine or cosine of the angle φ only, which I say cannot be done in any way, unless the curve itself were a circle.

11. Moreover clearly it will be satisfied by these conditions, if there may be taken

$$p = \sin.(n\varphi + \alpha) \text{ and } q = \cos.(n\varphi + \alpha)$$

with α denoting some constant angle, n truly some rational number; then indeed certainly there will be $pp + qq = 1$, and since there shall be

$$dx = ad\varphi \sin.(n\varphi + \alpha) \text{ and } dy = ad\varphi \cos.(n\varphi + \alpha),$$

hence on integrating there is elicited

$$x = b - \tfrac{a}{n}\cos.(n\varphi + \alpha) \text{ and } y = c + \tfrac{a}{n}\sin.(n\varphi + \alpha),$$

which formulas on account of the arbitrary letters α and n may be seen to involve innumerable curves. Truly thence since there may become

$$b - x = \tfrac{a}{n}\cos.(n\varphi + \alpha) \text{ and } y - c = +\tfrac{a}{n}\sin.(n\varphi + \alpha),$$

there will be always

$$(b - x)^2 + (y - c)^2 = \tfrac{aa}{nn},$$

which equation evidently always is for a circle. Therefore it has been required to show, for the conditions prescribed before in place of the letters p and q, other values are not able to be taken, which satisfy these.

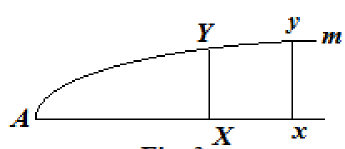

Fig. 3.

12. But since no evidence may be apparent for such a demonstration arising, or we may see it may be possible to be gained by some demonstration to the absurd. Therefore we may assume another algebraic curve to be given besides the circle, of which all the arcs may be allowed to be measured by circular arcs. Therefore let *AYym* (Fig. 3) be such an algebraic curve, of which some arc *AY* taken from the beginning *A* will be equal to some circular arc, of which the sine shall be some algebraic function of the abscissa *AX*, and in a similar manner some other arc *Ay* also will be equal to a circular arc, the sine of which will be a similar function of the abscissa *Ax;* and hence it is evident also the difference of these arcs also *Yy* can be assigned equal to a circular arc, thus so that plainly all the parts of this curve *Yy* may be allowed to be expressed by simple circular arcs, and thus it will require to be shown no such an algebraic curve evidently be shown in this manner.

13. But here initially I have observed, if such a curve may be given, that certainly it cannot be extended to infinity, as I will show thus. Let *Abcde* etc. (Fig. 4) be such a curve departing to infinity with the axis *ABCDE*, and on that equal parts may be taken *Ab, bc, ce, de* etc., the measure of which shall be the quadrant of a circle, and from the applied lines

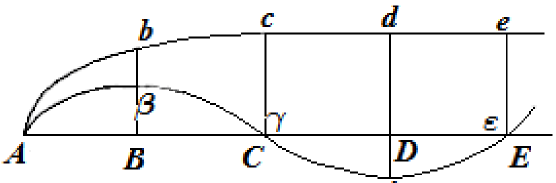

Fig. 4

Bb, Cc, Dd etc. the parts $B\beta,\ C\gamma,\ D\delta$ etc. may be removed, which shall be equal to the sines of the arcs *Ab, Ac, Ad* etc., because it may be understood these to be present intermediate on the individual applied lines ; and it is evident these individual points $\beta,\ \gamma,\ \delta$ etc. can be assigned geometrically or algebraically, thus so that the curve drawn through all these points $A\beta\gamma\delta\varepsilon$ etc. shall become algebraic. Truly because this will have an infinite number of alternative parts present above and below the axis, that may be cut from the axis itself at an infinite number of points, which cannot happen for an algebraic curve. From which it follows very well such a curve *Abcd* etc. certainly cannot be given an infinite extension; and now hence it prevails, if besides the circle algebraic curves of this kind may be given, of which the individual parts may be able to be given by circular arcs, by necessity these curves must be returning onto themselves; then indeed the absurdity in the manner shown would have to cease, thus so that in a similar manner nothing of the absurd thence would be able to be inferred.

14. Therefore *ABPQRS* (Fig. 5) shall be such an algebraic curve returning into itself, of which clearly all the parts may be allowed to be measured by circular arcs, which yet may not be a circle ; then with some part taken *AB* from which with another point *P* cut the part *PQ* will be equal to that, which still will be greatly different to that, since the curvature or the radius of osculation can differ greatly in these parts, such as are for *AB, PQ, RS* etc. But nevertheless in this indeed I am able to show no contradiction,

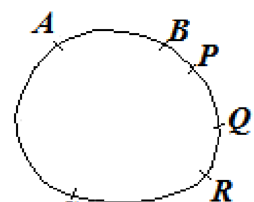

Fig. 5.

yet it can be demonstrated, if one such curve may be given from that an infinitude of others diverse among themselves can be constructed geometrically. Then truly from any of these again in a similar manner an infinitude of others truly from any of these anew an infinitude of others and thus indefinitely, thus so that there may be a multitude of such satisfying curves not only infinite in number, but thus of boundless infinitesimal powers. Whereby, since hitherto in no way will such a curve have been able to be found, hence will it not be able justly to conclude clearly no algebraic curves of this kind to be given?

15. But towards demonstrating these conspicuous properties, which the most celebrated Johan Bernoullli brought to light concerning incremental motion and equally with large curves will be able to be called into use with great success. But the foundation of this excellent method rests on this. If two curves *aym* and $a'y'm'$ may be present different in some way (Fig. 6 and 7) and on these there may be taken the arcs *ay* and $a'y'$ equally great, thus so that with the normals *yr* and $y'r'$ drawn to the points *y* and y',

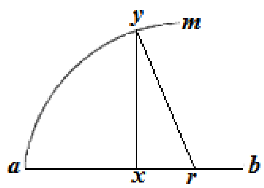

Fig.6

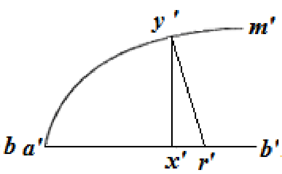

Fig. 7.

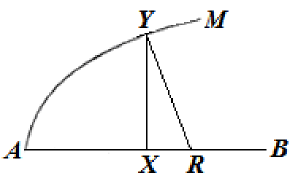

Fig. 8.

which meet the axis *ab* and $a'b'$ at *r* and r', which themselves are put in place normal to the curves at *a* and a', the angles *ary* and $a'r'y'$ may become equal to each other, from which these arcs *ay* et $a'y'$ are to be called equally great, with which in place if hence the new curve *AYM* (Fig. 8) may be constructed thus, so that with the abscissa taken $AX = m\cdot ax + n\cdot a'x'$ the applied line may be put in place $XY = m\cdot xy + n\cdot x'y'$, then also the arc *AY* of this new curve *AM* will be $= m\cdot ay + n\cdot a'y'$. But if indeed for the given curves we may put the abscissas $ax = x$ and $a'x' = x'$, the applied lines truly $xy = y$ et $x'y' = y'$, for the subnormals there will be

$$xr = \tfrac{ydy}{dx} \text{ and } x'r' = \tfrac{y'dy'}{dx'} \text{ and hence tang.} ary = \tfrac{dx}{dy} \text{ and tang.} a'r'y' = \tfrac{dx'}{dy'}.$$

Whereby since these angles shall be equal, on putting $\frac{dy}{dx} = p$ or $dy = pdx$ there will be also $\frac{dy'}{dx'} = p$ or $dy' = pdx'$. Hence therefore it is deduced the arc $ay = \int dx\sqrt{(1+pp)}$ and the arc $a'y' = \int dx'\sqrt{(1+pp)}$. Now on the curve *AY* thence constructed the abscissa $AX = X = mx + nx'$, the applied line truly $XY = Y = my + ny'$ and hence

$$dX = mdx + ndx' \text{ and } dY = mdy + ndy' = p(mdx + ndx')$$

and thus there will be $dp = dX$ and the arc *AY* will be equally great with the two preceding ones *ay* and $a'y'$; hence therefore the arc of this new curve will be

$$AY = \int dX\sqrt{(1+pp)} = m\int dx\sqrt{(1+pp)} + n\int dx'\sqrt{(1+pp)},$$

from which it is evident the arc $AY = m\cdot ay + n\cdot a'y'$.

16. Now with this foundation established if both the curves *ay* and *a' y'* were prepared thus, so that the arcs *ay* and *a' y'* may be able to be measured by circular arcs, then also thence the arc *AY* of the curve described also will be measured by a circular arc, but only if the letters *m* and *n* may denote some rational numbers. From which now it is understood from these given curves *ay* and *a' y'* innumerable curves *AY* can be constructed with the same property. But this is required to be observed, if both the given curves *ay* and *a'y'* were circles, that curve described *AY* shall become a circle also, of which the radius $RA = RY$ will be $= m \cdot ra + n \cdot r'a'$, thus so that in this case only no new curve will result, that which is itself evident. But immediately either one or the other or both of these curves *ay* and *a' y'* were not circles, then also the curve described *AY* certainly will not be a circle and thus it can be made to vary indefinitely, just as all values may be attributed to the numbers *m* and *n* .

17. Hence if that curve mentioned above therefore may be taken for the curve *ay*, clearly we suppose its individual arcs able to be measured by circles, and that beginning from some point *A*, but for the other some circle *a' y'* , the construction in the manner treated will supply innumerable curves *AY* to us endowed with the same nature, so that a circular arc may be able to be assigned equal to the arc *AY* . Then truly also with the curve *ay* extended to the branch of that figure from the point *A*, truly for the curve *a' y'* some other branch of the same curve extended from another point *P* hence also innumerable other new curves *AY* will be able to be described, which certainly all will be algebraic too; from which it is clear, if the branches of these new curves in place of the other curve give *ay* or also of each may be substituted, then in this manner infinitely many other kinds of curves can be generated, as the multiplication according to that will be allowed to increase indefinitely. Whereby, since at this point no curve of this kind has been able to be elicited different from the circle, it is especially plausible and perhaps in short can be seen as a strong demonstration in the nature of things no such algebraic curve of this kind differing from a circle to be given.

18. What hitherto has been advanced concerning the circle, also can be extended to logarithms, certainly which is allowed to be prepared from circular imaginary arcs, from which I support the following theorem just as with the geometers equally certain and worthy of mention as to recommend the preceding derivation.

THEOREM 3

Evidently no algebraic curve is given, of which the individual arcs may be able to be expressed simpler by logarithms.

Thus so that this theorem evidently will demand no exception just as the preceding.

19. It has been observed the rectification of the parabola depends on logarithms, truly its single arc is not expressed by simple logarithms, but by a sum from a logarithm and by some algebraic quantity, thus so that hence with no exception may the theorem be

inferred. But here in the first place it is to be observed as before, if such an algebraic curve may be given *AYy* (Fig. 3), all its arcs must be able to be assigned by logarithms from the end point *A*, so that for example there may become $AY = alP$ and $Ay = alp$, thus so that P et p shall be certain algebraic functions embracing the coordinates AX, XY and *Ax*, *xy*, then also the difference of these arcs can be expressed by a logarithm, since there may become

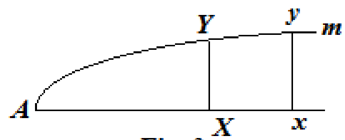

Fig. 3.

$Yy = al\frac{p}{P}$. Hence therefore on putting the abscissa $AX = x$ and the applied line $XY = y$, it is required to show no algebraic equation to be given between x and y, so that thence there may become

$$\int \sqrt{(dx^2 + dy^2)} = alv$$

with v denoting a certain function of x and y themselves; from which if we may put $dx = \frac{apdv}{v}$ and $dy = \frac{aqdv}{v}$, it is necessary that there may become $pp + qq = 1$. Truly besides it is required, that both the formulas $\int \frac{pdv}{v}$ and $\int \frac{qdv}{v}$, shall become algebraically integrable, of which it will be required to show the impossibility.

20. Just as it was required for me to show for the preceding theorem no curve to be given extending to infinity satisfying that, thus here in a similar manner it can be shown no algebraic curve to be given returning on itself, which may agree with this theorem. Indeed let *AYBDA* (Fig. 9) be a curve returning into itself, of which all the arcs *AY* may be able to be shown by logarithms, thus so that on the applied line *XY*, if there is a need, the point *Z* shall be able to be assigned to an algebraic product , so that the arc *AY* may become $= \log.XZ$; therefore then, because the curve is returning into itself and the same coordinates AX and XY may be agreed for the arc $AYBDAY$, the other will give the point *Z* also, of which the logarithm maybe equal to this arc. And if the total circumference of the curve may be put $= c$, and infinitude of such intervals *XZ*, *XZ'*, *XZ"*, *XZ'''* etc. will be able to be assigned, of which the logarithms may be equal to the arcs $AY,\ AY + c,\ AY + 2c,\ AY + 3c$ and in general $AY + nc$ with n denoting some whole number both negative as well as positive ; and because all these points will be contained by a similar algebraic formula, all will be present also on the same algebraic curve, which therefore may be cut from the same applied line *XY* produced at an infinite number of points, that which is contrary to the nature of an algebraic curve.

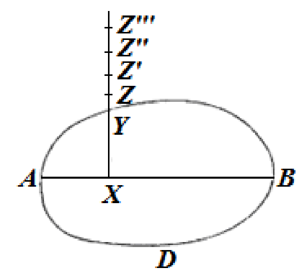

Fig.9.

21. So that if therefore such a curve may be given, of which the individual arcs may be allowed to be measured by logarithms, that certainly will depart to infinity. Now truly from a single such curve with the aid of the fundamental proposition from one such curve with the aid of the fundamental proposition concerning large curves equal to the pair by

the method advanced above, so that we may proceed from there, an infinitude of infinite new kinds of such curves will be able to be shown ; from which, since no curve of such a kind will have been able to be elicited, if not precisely sure, at least certainly the likelihood is clearly no algebraic curves of this kind to be given.

22. It follows if only the following theorem were shown more rigorously, also the demonstration of this would be had for completion. Indeed since an element of the arc of a circle, of which the radius $=a$ and the sine $=x$, shall be $\frac{adx}{\sqrt{(aa-xx)}}$, if we may consider the radius imaginary, so that there shall be $a=c\sqrt{-1}$, the element of arc may become

$$\frac{cdx\sqrt{-1}}{\sqrt{(-cc-xx)}}=-\frac{cdx}{\sqrt{(cc+xx)}},$$

which therefore will be real, even if the radius of the circle shall be imaginary, and thus its integral will be

$$cl\frac{\sqrt{(cc+xx)}-x}{c},$$

where it can be the greatest wonder to be seen, because the arcs of an imaginary circle non the less shall be real and indeed assignable by logarithms. And hence now we will be able to conclude with care, since besides the circle no other curved lines are given, of which a single circular arc may be allowed to be measured by circles, thus also besides the imaginary circle no algebraic curves to be given, of which it may be allowed to measure the individual arcs by logarithms. But since an imaginary circle clearly cannot exist, certainly no algebraic curves are considered able to be shown, of which an individual may be expressed by logarithms.

THEOREMATA QUAEDAM ANALYTICA
QUORUM DEMONSTRATIO ADHUC DESIDERATUR

Opuscula analytica 2**,** 1785, p. 76-90

1. In Analysi diophantea, quae circa proprietates numerorum versatur, notissimum est plurima occurrere theoremata, de quorum veritate dubitare non licet, etiamsi ea demonstratione rigida confirmare non valeamus. In Geometria autem nemo adhuc eiusmodi theoremata in medium produxit, quorum vel veritatem vel falsitatem demonstrare non liceat. At vero in Analysi sublimiori iam dudum etiam eiusmodi theoremata se mihi obtulerunt, quorum demonstrationem nullo modo etiam nunc invenire potui, etiamsi eorum veritas nequaquam in dubium vocari videatur. Talia igitur theoremata utique summam attentionem merentur, cum nullum plane sit dubium, quin, si eorum demonstrationem adhuc frustra acquisitam detexeremus, inde maximi momenti incrementa in Analysin sint redundatura.

2. Inter huiusmodi autem veritates analyticas merito primum locum tribuo insigni illi proprietati quantitatum imaginariarum, quod, ubicunque tales quantitates natura sua impossibiles occurrant, eae semper in formula hac $a+b\sqrt{-1}$ comprehendi queant. Huic quidem veritati innititur resolutio omnium aequationum algebraicarum; quippe quarum radices nisi fuerint reales, omnes in tali formula $a+b\sqrt{-1}$ contineri perhibentur, id quod etiam illustris D'Alembert demonstratione perquam ingeniosa confirmavit; quae autem quoniam ex consideratione infinite parvorum est petita, haud immerito adhuc demonstratio planior ex ipsa natura imaginariorum petenda desideratur. Praeterea vero ista demonstratio tantum ad expressiones algebraicas patet, cum tamen aeque certum sit eam etiam in omnis generis quantitatibus transcendentibus locum habere, ubi ratiocinium, quo Vir celeberr. est usus, non semper adhiberi potest, id quod operae pretium erit clarius ostendisse.

3. Consideretur curva algebraica, ex quotcunque ramis fuerit composita, cuiusmodi sit ramus *FNLMH* (Fig. 1), qui ad axem *AK* relatus, postquam ab *F* dextrorsum usque ad *L* processerit, hinc iterum sinistrorsum per *LMH* porrigatur, ita ut, si applicata *KL* hanc curvam in extremitate *L* tangat, abscissae cuilibet *AP* minori quam *AK* duplex respondeat applicata *PM* et *PN*. Unde si ponatur abscissa $AP = x$, applicata *y* duplicem habebit valorem ex tali aequatione quadratica $yy = 2py - q$ determinandum, ita ut hinc sit altera applicata $PM = p - \sqrt{(pp-q)}$, altera vero $PN = p + \sqrt{(pp-q)}$, ubi pro indole curvae litterae *p* et *q* functiones quascunque abscissae *x* denotare possunt. Quamdiu igitur fuerit $pp > q$, revera gemina orietur applicata *PM* et *P N*. Dum autem abscissa *x* usque in *K* augetur, ubi fiat $pp = q$, ibi ambae applicatae in unam *KL* coalescent, ita ut hic applicata *KL* evadat

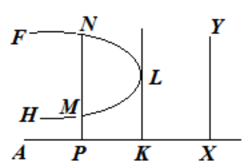

Fig. 1.

curvae tangens. Quodsi ergo abscissam x ulterius augendo fiat $q > pp$, ambae applicatae evadent imaginariae. Unde intelligitur, si capiatur abscissa $AX > AK$, in hoc loco nullam prorsus dari applicatam seu rectam in hoc loco perpendicularem *XY* utrinque etiam in infinitum productam nusquam curvae *FLH* esse occursuram, id quod more loquendi in Analysi recepto idem significat ac applicatam in hoc loco *X* esse imaginariam; unde simul notio imaginariorum, uti in Analysi adpellantur, clarius intelligitur. Cum enim haec applicata *XY* curvae nusquam occurrat, etiamsi a puncto *X*, ubi est $=0$, tam sursum usque in infinitum positivum quam deorsum usque in infinitum negativum continuetur, evidens est eius valorem inventum neque esse 0 neque maiorem quam 0 neque minorem quam 0, qua conditione definitio ipsa quantitatum imaginariarum continetur. Quodsi ergo pro hoc loco sumamus fieri $q = pp - rr$, gemina expressio applicatae evadet $y = p \pm r\sqrt{-1}$.

4. Hic igitur quaeritur, num hinc certo in genere concludi possit, quotiescunque imaginaria occurrant, ea semper huiusmodi formula $p + r\sqrt{-1}$ exprimi posse. Primo enim haec demonstratio tantum ex ramo *FLH* est petita, dum tota curva aequatione inter x et y contenta fortasse plures insuper alios ramos involvat, quos in hoc negotio penitus negligere fortasse non licet. Hanc autem obiectionem Vir excell. utique ipse praevidit, dum hoc ratiocinium tantum ad portiunculam curvae infinite parvam *NLM* extendit, ubi ulteriorem ramorum extensionem tuto negligere liceat, quod autem non adeo in aprico situm videtur, ut non planiorem demonstrationem a tali conceptu immunem merito desiderare queamus. Tum vero etiam hinc plus non sequeretur, quam applicatas *XY* extremae *KL* infinite propinquas tali formula $p \pm r\sqrt{-1}$ exprimi posse, ac non immerito dubitare liceret, an pro intervallis maioribus *KX* etiam applicatae tali formula comprehendi queant et annon reliquae curvae partes hactenus neglectae indolem imaginarii in his locis penitus immutare valeant.

5. Praeterea vero ista consideratio tantum ad aequationes et curvas algebraicas est accommodata, in quibus utique alii rami non dantur, nisi qui vel in se redeant vel utrinque in infinitum excurrant, ita ut circa terminum *L* portio curvae hic semper binas portiones *LM* et *LN* exhibeat, unde aequatio illa quadratica $yy = 2py - q$ est nata, cui tota demonstratio innititur. At vero inter curvas transcendentes eiusmodi rami occurrunt, qui neque utrinque in infinitum protenduntur neque in se redeunt, sed subito in quopiam puncto terminantur. Talem casum praebet curva transcendens hac aequatione contenta

$$y = a + \tfrac{bx}{l(c-x)},$$

ex qua sequitur singulis abscissis unicam tantum applicatam respondere. Posito enim $x = 0$ fit $y = a$; ac si abscissa x continuo augeatur usque ad valorem $x = c$, perpetuo unica dabitur applicata; sumta vero abscissa $x = c$ ob $l(c - x) = -\infty$ fiet applicata in hoc loco $y = a$. Statim autem atque abscissa x ultra c augetur, applicata subito fiet imaginaria, propterea quod logarithmi quantitatum negativarum certo sunt imaginarii;

quare sumta abscissa $x > c$ applicata y, etiamsi utrinque in infinitum producatur, curvae tamen nostrae nusquam occurret. Hoc autem casu ratio supra allegata et naturae aequationis quadraticae innixa penitus cessat, ita ut hic merito dubitare possimus, an ista applicata imaginaria etiam in formula $p+q\sqrt{-1}$ comprehendi queat. Saltem hic agnoscere debemus istud theorema alia demonstratione indigere ideoque maxime optandum esse, ut talis aequatio immediate ex ipsa natura imaginariorum derivetur.

6. Ante autem quam hoc argumentum deseram, ostendisse iuvabit, quomodo omnia plane imaginaria singulari prorsus ratione per circulum repraesentari possint. Ex puncto A (Fig. 2) pro principio axis AB assumto erigatur perpendiculum $AC = a$; centro C radio $CM = c$ describatur circulus ac posita abscissa quacunque $AP = x$ eique respondente applicata $PM = y$ erit

$$y = AC + QM = AC + \sqrt{(CM^2 - CQ^2)} = a + \sqrt{(cc - xx)},$$

ita ut eius valor semper sit realis, quamdiu abscissa x minor capitur quam radius c; simulac vero abscissa x radium c superat, veluti si sumatur $x = AX$, tum applicata XY certe erit imaginaria. At vero, quanquam ob hanc ipsam causam applicata exhiberi nequit, tamen determinatum habet valorem imaginarium (iam enim evictum est notionem determinati notioni imaginarii non adversari). Quoniam enim ponitur $x > c$, statuatur $xx = cc + bb$, ut fiat $\sqrt{(cc - xx)} = b\sqrt{-1}$ ideoque applicata ista imaginaria $XY = a + b\sqrt{-1}$. Quare cum formula $a + b\sqrt{-1}$ omnes plane quantitates imaginarias contineat, eas per huiusmodi applicatam determinatam XY ad circulum quendam pertinentem repraesentare licebit. Posito scilicet perpendiculo $AC = a$ centro C radio pro arbitrio assumto c describatur circulus ac sumatur abscissa $AX = \sqrt{(bb + cc)}$; tum enim applicata imaginaria XY istam formulam $a + b\sqrt{-1}$ exhibebit sicque mirabili quodam modo omnes adeo formulas imaginarias quasi geometrice construere licebit.

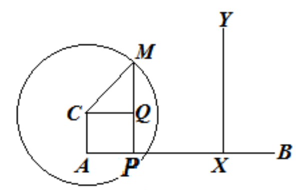

Fig. 2.

7. Operae pretium erit hoc exemplo quodam declarasse. Quaeramus scilicet arcum circuli, cuius sinus duplo maior sit sinu toto, qui ergo certe erit imaginarius. Posito ergo sinu toto $=1$ integrari debet formula $\frac{dx}{\sqrt{(1-xx)}}$, ita ut integrale evanescat posito $x = 0$; tum vero sumi debebit $x = 2$ et valor integralis dabit ipsum arcum. Hunc in finem formulae differentiali $\frac{dx}{\sqrt{(1-xx)}}$ tribuamus hanc formam $\frac{dx\sqrt{-1}}{\sqrt{(xx-1)}}$; constat autem esse

$$\int \frac{dx}{\sqrt{(xx-1)}} = l\,\frac{x+\sqrt{(xx-1)}}{\sqrt{-1}},$$

unde posito $x = 2$ arcus quaesitus erit

$$= = \sqrt{-1}l\,\tfrac{2+\sqrt{3}}{\sqrt{-1}} = \sqrt{-1}l(2+\sqrt{3}) - \sqrt{-1}l\sqrt{-1}.$$

Novimus autem huius postremi membri valorem esse $\frac{\pi}{2}$, unde arcus circuli, cuius sinus $=2$, erit $=\frac{\pi}{2}+\sqrt{-1}l(2+\sqrt{3}).$ Quamobrem ut huic arcui imaginario aequalem applicatam XY exhibeamus, in nostra figura capiatur intervallum $AC=\frac{\pi}{2}$ ac descripto circulo radii $CM = c = 1$, quia c arbitrio nostro relinquitur, posito brevitatis gratia $l(2+\sqrt{3}) = b$ capiatur abscissa $AX = \sqrt{(1+bb)}$ atque applicata imaginaria XY aequalis erit ipsi arcui quaesito pariter imaginario, id quod eo magis notatu dignum videtur, quod iste arcus est imaginarium transcendens.

8. Primum igitur theorema analyticum, cuius demonstratio planior vel saltem magis directa desideratur, siquidem eius veritas quibusdam iam satis evicta videatur, hoc modo proponatur:

THEOREMA 1

Omnes plane quantitates imaginariae, quaecunque in calculo analytico occurrere possunt, ad hanc formam simplicissimam $a+b\sqrt{-1}$ *ita revocari possunt, ut litterae a et b quantitates reales denotent.*

Eius igitur demonstrationem sagacissimis analystis imprimis commendare non dubito.

9. Sequentia duo theoremata rectificationem linearum curvarum respiciunt ideoque ad geometriam sublimiorem sunt referenda. Cum enim iam pridem a celeb. HERMANNO methodus geometrica sit reperta innumerabiles curvas algebraicas inveniendi, quae vel sint rectificabiles vel quarum rectificatio a data quacunque quadratura pendeat (quam methodum deinceps ad analysin puram transtuli et plurimum locupletavi, ita ut peculiarem speciem analyseos infinitorum constituere videatur), inde utique infinitae curvae algebraice exhiberi possunt, quarum rectificatio a quadratura circuli pendeat. Omnes autem excepto circulo ita comparatae deprehenduntur, ut earum arcus aggregato cuipiam ex quantitate algebraica et arcu circulari aequentur, quantitatem autem illam algebraicam nullo modo ad nihilum redigere liceat; unde sequens theorema tanquam verum proponere non dubito, etiamsi eius demonstrationem exhibere nondum potuerim.

THEOREMA 2

Praeter circulum nulla datur curva algebraica, cuius singuli arcus per arcus circulares simpliciter exprimi queant.

10. Hoc theorema igitur eo redit, ut demonstretur nullam aequationem algebraicam inter binas coordinatas orthogonales x et y exhiberi posse, ut formula integralis $\int\sqrt{(dx^2+dy^2)}$ aequetur arcui cuipiam circulari, cuius sinus vel cosinus sit functio quaepiam algebraica ipsarum x et y, solo casu excepto, quo aequatio inter x et y circulum indicat. Quod quo clarius intelligatur, denotet φ angulum seu arcum quemcunque indefinitum in circulo, cuius radius $=1$, ac ponatur

$$\int\sqrt{(dx^2+dy^2)}=a\varphi$$

ideoque

$$dx^2+dy^2=aad\varphi^2$$

fiatque

$$dx=apd\varphi \text{ et } dy=aqd\varphi$$

atque necesse est, ut sit $pp+qq=1$. Praeterea vero ambas formulas $apd\varphi$ et $aqd\varphi$ ita integrabiles esse oportet, ut earum integralia per solos sinus vel cosinus anguli φ exprimi queant, quod dico nullo modo fieri posse, nisi curva fuerit ipse circulus.

11. His autem conditionibus manifesto satisfiet, si capiatur

$$p=\sin.(n\varphi+\alpha) \text{ et } q=\cos.(n\varphi+\alpha)$$

denotante α angulum quemcunque constantem, n vero numerum rationalem quemcunque; tum enim utique erit $pp+qq=1$, et cum sit

$$dx=ad\varphi\sin.(n\varphi+\alpha) \text{ et } dy=ad\varphi\cos.(n\varphi+\alpha),$$

hinc integrando elicitur

$$x=b-\tfrac{a}{n}\cos.(n\varphi+\alpha) \text{ et } y=c+\tfrac{a}{n}\sin.(n\varphi+\alpha),$$

quae formulae ob litteras α et n arbitrarias innumeras curvas involvere videntur. Verum cum inde fiat

$$b-x=\tfrac{a}{n}\cos.(n\varphi+\alpha) \text{ et } y-c=+\tfrac{a}{n}\sin.(n\varphi+\alpha),$$

semper erit

$$(b-x)^2+(y-c)^2=\tfrac{aa}{nn},$$

quae aequatio manifesto semper est pro circulo. Demonstrandum igitur est pro conditionibus ante praescriptis loco litterarum p et q alios valores accipi non posse, qui iis satisfaciant.

12. Cum autem nullum vestigium appareat ad talem demonstrationem perveniendi, videamus, an per demonstrationem ad absurdum quicquam lucrari possit. Assumamus igitur praeter circulum aliam dari curvam algebraicam, cuius omnes arcus per arcus circulares metiri liceat. Sit igitur *AYym* (Fig. 3) talis curva algebraica, cuius quilibet arcus *AY* ab initio *A* captus aequetur arcui cuipiam circulari, cuius sinus sit functio quaecunque algebraica abscissae *AX*, ac simili modo alius arcus quicunque *Ay* etiam aequabitur arcui circulari, cuius sinus erit similis functio abscissae *Ax;* hincque manifestum est etiam differentiae horum arcuum *Yy* aequalem arcum circularem assignari posse, ita ut huius curvae omnes plane portiones *Yy* per simplices arcus circulares exprimi queant, sicque demonstrari oportebit talem curvam algebraicam nullo prorsus modo exhiberi posse.

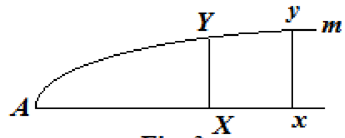

Fig. 3.

13. Primo hic autem observo, si daretur talis curva, eam certe non in infinitum extendi posse, id quod ita ostendo. Sit *Abcde* etc. (Fig. 4) talis curva cum axe *ABCDE* in infinitum excurrens in eaque accipiantur portiones aequales *Ab, bc, ce, de* etc., quarum mensura sit quadrans circuli, atque in applicatis *Bb, Cc, Dd* etc. abscindantur portiones $B\beta$, $C\gamma$, $D\delta$ etc., quae sint sinibus arcuum *Ab, Ac, Ad* etc. aequales, id quod etiam in singulis applicatis intermediis fieri intelligatur; ac manifestum est singula haec puncta β, γ, δ etc. geometrice seu algebraice assignari posse, ita ut curva per omnia haec puncta ducta $A\beta\gamma\delta\varepsilon$ etc. futura esset algebraica. Quoniam vero ea habebit infinitas portiones alternatim supra et infra axem existentes, ea ab axe ipso in infinitis punctis intersecaretur, id quod in nulla curva algebraica locum habere potest. Unde luculenter sequitur talem curvam *Abed* etc. in infinitum extensam certe non dari posse; atque hinc iam est evictum, si darentur praeter circulum eiusmodi curvae algebraicae, quarum singulae portiones per arcus circulares mensurari queant, necessario eas in se redeuntes esse debere; tum enim absurditas modo ostensa cessare posset, ita ut simili modo nihil absurdi inde inferri possit.

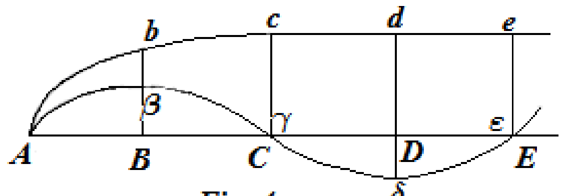

Fig. 4

14. Sit igitur *AB PQRS* (Fig. 5) talis curva algebraica in se rediens, cuius omnes plane portiones per arcus circulares metiri liceat, quae tamen non sit circulus; tum sumta quacunque portione *AB* a quovis alio puncto *P* abscindi poterit portio *PQ* illi aequalis, quae tamen illi maxime erit dissimilis, quandoquidem curvamen seu radius osculi maxime differre potest in his portionibus, quales sunt *AB, PQ, RS* etc. Quanquam autem in hoc equidem nullam contradictionem ostendere possum, tamen demonstrari potest, si unica talis curva daretur, ex ea infinitas alias inter se diversas geometrice construi posse. Tum vero ex qualibet earum porro simili modo infinitas alias ex earumque denuo qualibet infinitas alias sicque in infinitum, ita ut multitudo talium curvarum satisfacientium foret

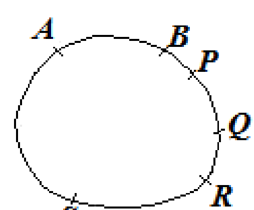

Fig. 5.

non solum numerus infinitus, sed adeo potestas infinitesima infiniti. Quare, cum adhuc nullo modo talis curva reperiri potuerit, nonne hinc iure concludere licebit nullas plane dari huiusmodi curvas algebraicas?

15. Ad hoc autem demonstrandum insignes illae proprietates, quas Vir celeberr. JOANNES BERNOULLI de motu reptorio et curvis aeque amplis in lucem produxit, summo cum successu in usum vocari poterunt. Fundamentum autem huius eximiae methodi in hoc consistit. Si habeantur duae curvae utcunque diversae *aym* et *a' y' m'* (Fig. 6 et 7) in iisque capiantur arcus *ay* et *a' y'* aeque ampli, ita ut ductis ad puncta *y* et *y'*

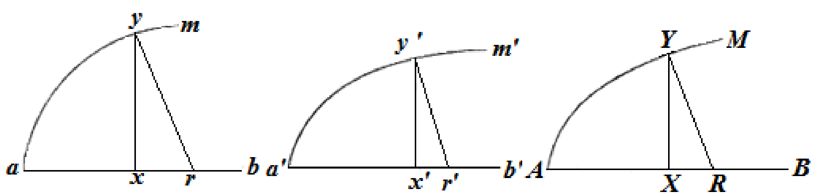

Fig.6 *Fig. 7.* *Fig. 8.*

normalibus *yr* et *y' r'*, quae axibus *ab* et *a'b'* in *r* et *r'* occurrant, qui ipsi ad curvas normales supponuntur in *a* et *a'*, anguli *ary* et *a' r' y'* fiant inter se aequales, ex quo hi arcus *ay* et *a' y'* aeque ampli sunt appellati, quibus positis si hinc nova curva *AYM* (Fig. 8) ita construatur, ut sumta abscissa $AX = m\cdot ax + n\cdot a'x'$ constituatur applicata $XY = m\cdot xy + n\cdot x'y'$, tum etiam huius novae curvae *AM* arcus *AY* erit $= m\cdot ay + n\cdot a'y'$. Quodsi enim pro curvis datis ponamus abscissas $ax = x$ et $a'x' = x'$, applicatas vero $xy = y$ et $x'y' = y'$, erit subnormalis
$xr = \frac{ydy}{dx}$ et $x'r' = \frac{y'dy'}{dx'}$ hincque tang.$ary = \frac{dx}{dy}$ et tang.$a'r'y' = \frac{dx'}{dy'}$. Quare cum hi anguli sint aequales, posito $\frac{dy}{dx} = p$ seu $dy = pdx$ erit etiam $\frac{dy'}{dx'} = p$ sive $dy' = pdx'$. Hinc igitur colligitur arcus $ay = \int dx\sqrt{(1+pp)}$ et arcus $a'y' = \int dx'\sqrt{(1+pp)}$. Iam in curva inde constructa *AY* erit abscissa $AX = X = mx + nx'$, applicata vero $XY = Y = my + ny'$ hincque

$$dX = mdx + ndx' \text{ et } dY = mdy + ndy' = p(mdx + ndx')$$

ideoque erit $dp = dX$ et arcus *AY* aeque amplus erit ac duo praecedentes *ay* et *a' y';* hinc ergo huius novae curvae arcus erit

$$AY = \int dX\sqrt{(1+pp)} = m\int dx\sqrt{(1+pp)} + n\int dx'\sqrt{(1+pp)},$$

unde manifestum est fore arcum $AY = m\cdot ay + n\cdot a'y'$.

16. Hoc iam fundamento stabilito si ambae curvae *ay* et *a' y'* ita fuerint comparatae, ut arcus *ay* et *a' y'* per arcus circulares mensurari queant, tum etiam curvae inde descriptae arcus *AY* etiam per arcum circularem mensurabitur, si modo litterae *m* et *n* denotent numeros rationales quoscunque. Ex quo iam intelligitur ex illis curvis datis *ay* et *a' y'* innumerabiles curvas *AY* eiusdem proprietatis construi posse. Hic autem observandum est, si ambae curvae datae *ay* et *a'y'* fuerint circuli, curvam illam descriptam *AY* fore quoque circulum, cuius radius $RA = RY$ erit $= m\cdot ra + n\cdot r'a'$, ita ut hoc solo casu nulla nova curva resultet, id quod per se est perspicuum. Statim autem ac vel altera earum curvarum *ay* et *a' y'* vel etiam ambae non fuerint circuli, tum quoque curva descripta *AY* certe non erit circulus atque adeo in infinitum variari poterit, prouti numeris *m* et *n* alii atque alii valores tribuantur.

17. Hinc ergo si pro curva *ay* accipiatur curva illa supra memorata, cuius scilicet singulos arcus per circulares mensurare posse assumimus, eamque a puncto quocunque *A* incipientem, pro altera autem *a' y'* circulum quemcunque, constructio modo tradita nobis suppeditabit innumerabiles curvas *AY* eadem indole praeditas, ut arcui *AY* aequalis arcus circularis assignari queat. Tum vero etiam sumta curva *ay* aequali ramo figurae illius a puncto *A* extenso, pro curva vero *a' y'* alius quicunque eiusdem curvae ramus ab alio puncto *P* protensus hinc etiam innumerabiles aliae novae curvae *AY* describi poterunt, quae utique omnes quoque erunt algebraicae; unde manifestum est, si harum novarum curvarum rami in locum alterius curvae datae *ay* vel etiam utriusque substituantur, tum hoc modo infinita alia curvarum genera construi posse, quam multiplicationem ad eo in infinitum augere licebit. Quare, cum nulla adhuc eiusmodi curva a circulo diversa erui potuerit, maxime verisimile est ac fortasse tanquam rigide demonstratum spectari potest nullam prorsus in rerum natura dari huiusmodi curvam algebraicam a circulo diversam.

18. Quod hactenus de circulo est allatum, etiam ad logarithmos extendi potest, quippe quos cum arcubus circularibus imaginariis comparare licet, unde sequens theorema geometris tanquam aeque certum et memoratu dignum ac praecedens commendare sustineo
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THEOREMA 3

Nulla prorsus datur curva algebraica, cuius singuli arcus simpliciter per logarithmos exprimi queant.

Ita ut hoc theorema nullam prorsus exceptionem quemadmodum praecedens postulet.

19. Notum est rectificationem parabolae a logarithmis pendere, verum singuli eius arcus non per simplices logarithmos, sed per aggregatum ex logarithmo et quapiam quantitate algebraica exprimuntur, ita ut hinc nulla exceptio theoremati inferatur. Hic autem primo observandum est ut ante, si talis daretur curva algebraica *AYy* (Fig. 3), cuius omnes arcus in puncto *A* terminati per logarithmos assignari possent, ut verbi gratia esset $AY = alP$ et $Ay = alp$, ita ut *P* et *p* essent certae functiones algebraicae ambarum coordinatarum *AX, XY* et *Ax, xy*, tum etiam differentiam horum arcuum logarithmo exprimi posse, quandoquidem foret $Yy = al\frac{p}{P}$ Hinc ergo posita abscissa $AX = x$ et applicata $XY = y$ demonstrandum est nullam dari aequationem algebraicam inter *x* et *y*, ut inde fiat

$$\int\sqrt{(dx^2+dy^2)} = alv$$

denotante *v* functionem quampiam algebraicam ipsarum *x* et *y;* unde si ponamus $dx = \frac{apdv}{v}$ et $dy = \frac{aqdv}{v}$, necesse est, ut fiat $pp + qq = 1$. Praeterea vero requiritur, ut ambae formulae $\int\frac{pdv}{v}$ et $\int\frac{qdv}{v}$, fiant algebraice integrabiles, cuius ergo impossibilitatem demonstrari oportet.

20. Quemadmodum mihi pro praecedente theoremate licuit ostendere nullam dari curvam in infinitum extensam illi satisfacientem, ita hic simili modo ostendi potest nullam dari curvam in se redeuntem algebraicam, quae huic theoremati conveniat. Sit enim curva *AYBDA* (Fig. 9) curva in se rediens, cuius omnes arcus *AY* per logarithmos exhiberi queant, ita ut in applicata *XY*, si opus est, producta algebraice assignari possit punctum *Z*, ut arcus *AY* fiat $= \log.XZ$; tum ergo, quia curva in se est rediens et arcui *AYBDAY* eaedem coordinatae *AX* et *XY* conveniunt, aliud quoque dabitur punctum *Z*, cuius logarithmus huic arcui aequetur. Ac si circumferentia totius curvae ponatur $= c$, infinita talia spatia *XZ, XZ', XZ'', XZ'''* etc. assignari poterunt, quorum logarithmi aequentur arcubus AY, $AY + c$, $AY + 2c$, $AY + 3c$ et in genere $AY + nc$ denotante *n* numerum integrum quemcunque tam negativum quam positivum; atque quia omnia haec puncta simili formula algebraica continebuntur, omnia quoque in eadem curva algebraica existerent, quae ergo a singulis applicatis *XY* productis in infinitis punctis secaretur, id quod naturae curvarum algebraicarum adversatur.

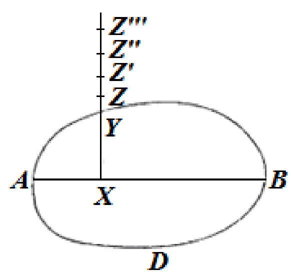

Fig.9.

21. Quodsi ergo daretur talis curva, cuius singulos arcus logarithmis metiri liceret, ea certe in infinitum excurreret. Iam vero ex unica tali curva ope propositionis fundamentalis circa curvas aeque amplas supra allatae pari modo, quo ibi processimus,

infinities-infinita nova genera talium curvarum exhiberi possent; unde, cum nulla adhuc talis curva erui potuerit, si non prorsus certum, saltem maxime verisimile est nullas plane dari eiusmodi curvas algebraicas.

22. Ceterum si modo theorema secundum firmiter fuerit demonstratum, etiam huius demonstratio pro confecta esset habenda. Cum enim elementum arcus circuli, cuius radius $= a$ et sinus $= x$, sit $\frac{adx}{\sqrt{(aa-xx)}}$, si radium ita imaginarium concipiamus, ut sit $a = c\sqrt{-1}$, elementum arcus fiet

$$\frac{cdx\sqrt{-1}}{\sqrt{(-cc-xx)}} = -\frac{cdx}{\sqrt{(cc+xx)}},$$

quod ergo erit reale, etiamsi radius circuli sit imaginarius, eiusque adeo integrale erit

$$cl\frac{\sqrt{(cc+xx)}-x}{c},$$

ubi maxime mirum videri potest, quod arcus circuli imaginarii nihilo minus sint reales et quidem per logarithmos assignabiles. Atque hinc iam tuto concludere poterimus, quemadmodum praeter circulum nullae aliae dantur lineae curvae, cuius singulos arcus per circulares metiri liceat, ita etiam praeter circulum imaginarium nullas dari curvas algebraicas, quarum singulos arcus per logarithmos metiri liceat. Quoniam autem circulus imaginarius plane existere nequit, prorsus nullae curvae algebraicae exhiberi posse sunt censendae, quarum singulos arcus per logarithmos exprimere liceat.