INVESTIGATION OF THE INTEGRAL FORMULA

$$\int \frac{x^{m-1}dx}{(1+x^k)^n}$$

IN THE CASE WHERE THERE IS PUT $x=\infty$ AFTER THE INTEGRATION .

[E588]

Opuscula analytica 2, 1785, p. 42-54

1. Now it is known well enough the integral of this formula, in the case where $n=1$, to include in part logarithms and in part circular arcs, and the logarithmic parts to constitute this progression :

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}\, l\sqrt{(1-2x\cos.\tfrac{\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}\, l\sqrt{(1-2x\cos.\tfrac{3\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}\, l\sqrt{(1-2x\cos.\tfrac{5\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\cos.\tfrac{7m\pi}{k}\, l\sqrt{(1-2x\cos.\tfrac{7\pi}{k}+xx)}$$
$$\vdots$$
$$-\tfrac{2}{k}\cos.\tfrac{im\pi}{k}\, l\sqrt{(1-2x\cos.\tfrac{i\pi}{k}+xx)},$$

where *i* denotes and odd number not greater than *k.* Hence if *k* were an even number, there will be $i=k-1$;and if k were an odd number, this progression will be required to be continued as far as to $i=k$ *,* of which truly the coefficient must be taken less by twice as much, or in place of $-\frac{2}{k}$ there must be written only $-\frac{1}{k}$, an account of this irregularity has been set out in the *Calculo Integrali* [Vol. I, § 77; see also E462].

2. Now since these parts vanish at once on putting $x=0$, at once we may put in place $x=\infty$, and since in general there shall be

$$\sqrt{(1-2xcos.\omega+xx)}=x-\cos.\omega ,$$

there will become

$$l\sqrt{(1-2xcos.\omega+xx)}=l(x-\cos.\omega)=lx-\tfrac{\cos.\omega}{x}=lx ,$$

on account of $\frac{\cos.\omega}{x}=0$; therefore all these logarithms are reduced to the same form *lx,* which on being multiplied by this series

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}-\cdots-\tfrac{2}{k}\cos.\tfrac{im\pi}{k},$$

where, as we have said, *i* may denote the maximum odd number not greater than to *k* itself, yet with this restriction, so that if *k* were odd and thus $i=k$, the final term must be reduced by half. On account of which if we may wish to investigate the sum of this progression, two cases are required to be put in place, the one, where *k* is an even number and $i=k-1$, the other indeed, where *k* is odd and $i=k$.

THE ESTABLISHMENT OF THE FIRST CASE, WHERE k IS AN ODD NUMBER AND $i=k-1$.

3. Therefore in this case, on putting $x=\infty$ the formula $-\frac{2}{k}lx$ is multiplied by this series of cosines

$$\cos.\tfrac{m\pi}{k}+\cos.\tfrac{3m\pi}{k}+\cos.\tfrac{5m\pi}{k}+\cdots+\cos.\tfrac{(k-1)m\pi}{k},$$

the sum of which we may put $=S$. We may multiply this series by $\sin.\frac{m\pi}{k}$*,* and since in general there shall be

$$\sin.\tfrac{m\pi}{k}\cos.\tfrac{i\pi}{k}=\tfrac{1}{2}\sin.\tfrac{(i+1)m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{(i-1)m\pi}{k},$$

with this done, we will have with this reduced

$$\begin{aligned}&S\sin.\tfrac{m\pi}{k}\\ &=\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}+\cdots+\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{km\pi}{k}\\ &-\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}-\cdots-\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k},\end{aligned}$$

where all the terms besides the final evidently cancel out, thus so that there shall be

$$S\sin.\tfrac{m\pi}{k}=\tfrac{1}{2}\sin.m\pi.$$

Now truly, since our coefficients *m* and *k* are supposed whole, certainly there shall be $\sin.m\pi=0$ and thus also $S=0$, unless perhaps also there were $\sin.\frac{m\pi}{k}=0$, but which place cannot occur, since in the integration of the proposed formula $\int\frac{x^{m-1}dx}{(1+x^k)^n}$, it is customary to assumed to be $m<k$. Therefore in this manner in this case, it has prevailed, where after integration there may be put $x=\infty$, all the logarithmic parts of the integral cancel each other out.

SETTING OUT THE OTHER CASE, WHERE k IS AN ODD NUMBER AND $i=k$

4. Therefore in this case by supposing $x=\infty$ the formula lx is multiplied by this series

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}-\cdots-\tfrac{1}{k}\cos.\tfrac{km\pi}{k},$$

where the penultimate term is $-\tfrac{2}{k}\cos.\tfrac{(k-2)m\pi}{k}$, for the ultimate the term truly will be $\cos.m\pi=\pm 1$ with the upper sign prevailing, if *n* shall be an even number, the lower, if odd; whereby with the final term removed we may put for the rest :

$$\cos.\tfrac{m\pi}{k}+\cos.\tfrac{3m\pi}{k}+\cos.\tfrac{5m\pi}{k}+\cdots+\cos.\tfrac{(k-2)m\pi}{k}=S,$$

thus so that the multiplier of the logarithm *x* shall be

$$-\tfrac{2S}{k}-\tfrac{1}{k}\cos.m\pi.$$

Hence by proceeding as before there will become :

$$\begin{aligned}&S\sin.\tfrac{m\pi}{k}\\ &=\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}+\cdots+\tfrac{1}{2}\sin.\tfrac{(k-3)m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{(k-1)m\pi}{k}\\ &-\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}-\cdots-\tfrac{1}{2}\sin.\tfrac{(k-3)m\pi}{k},\end{aligned}$$

where again all the terms besides the final one mutually cancel each other, thus so that hence there may be produced

$$\sin.\tfrac{m\pi}{k}=\tfrac{1}{2}\sin.\tfrac{(k-1)m\pi}{k}=\tfrac{1}{2}\sin.(m\pi-\tfrac{m\pi}{k});$$

but truly there is

$$\tfrac{1}{2}\sin.(m\pi-\tfrac{m\pi}{k})=\sin.m\pi\cos.\tfrac{m\pi}{k}-\cos.m\pi\sin.\tfrac{m\pi}{k},$$

where there may be observed to be $\sin.m\pi=0$ on account of the whole number *m* ; therefore we will have

$$S\sin.\tfrac{m\pi}{k}=-\tfrac{1}{2}\cos.m\pi\sin.\tfrac{m\pi}{k} \text{ or } ,S=-\tfrac{1}{2}\cos.m\pi,$$

consequently the multiplier of lx will be

$$=\tfrac{1}{k}\cos.m\pi-\tfrac{1}{k}\cos.m\pi=0$$

and thus it is evident, whether k shall be an odd or even number, all the logarithmic terms in our integral mutually cancel each other out, if indeed after integration we may put $x=\infty$, just as here we suppose always .

5. Now we will consider also the parts depending on the circle, from which integral our formula is composed. But these parts are prepared to constitute the following progression:

$$\begin{aligned}&\tfrac{2}{k}\sin.\tfrac{m\pi}{k}\text{Atang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}+\tfrac{2}{k}\sin.\tfrac{3m\pi}{k}\text{Atang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}\\&+\tfrac{2}{k}\sin.\tfrac{5m\pi}{k}\text{Atang.}\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}}+\tfrac{2}{k}\sin.\tfrac{7m\pi}{k}\text{Atang.}\frac{x\sin.\frac{7\pi}{k}}{1-x\cos.\frac{7\pi}{k}}\\&+\cdot\ \cdot\ \cdot\ \cdot+\tfrac{2}{k}\sin.\tfrac{im\pi}{k}\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}},\end{aligned}$$

where in the final term there is either $i=k-1$ or $i=k$ *;* evidently the first prevails, if i is an even number, the latter if odd.

6. Therefore since all these terms vanish on putting $x=0$, we may make $x=\infty$ for instituting our case. In general therefore there will become

$$\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}=\text{Atang.}\left(-\text{tang.}\tfrac{i\pi}{k}\right).$$

Truly there is

$$-\text{tang.}\tfrac{i\pi}{k}=+\text{tang.}\left(\tfrac{(k-i)\pi}{k}\right),$$

from which here the arc becomes $=\frac{(k-i)\pi}{k}$. Hence therefore by writing the successive numbers 1, 3, 5, 7 etc. in place of i these parts of the integral sought will be

$$\begin{aligned}&\tfrac{2(k-1)\pi}{kk}\sin.\tfrac{m\pi}{k}+\tfrac{2(k-3)\pi}{kk}\sin.\tfrac{3m\pi}{k}+\tfrac{2(k-5)\pi}{kk}\sin.\tfrac{5m\pi}{k}+\tfrac{2(k-7)\pi}{kk}\sin.\tfrac{7m\pi}{k}\\&+\tfrac{2(k-9)\pi}{kk}\sin.\tfrac{9m\pi}{k}+\cdots+\tfrac{2(k-i)\pi}{kk}\sin.\tfrac{im\pi}{k},\end{aligned}$$

where in the case, in which k is an even number, it will be required to progress as far as to $i=k-1$, and if k shall be an odd number, as far as to $i=k$.

7. For the sake of brevity we may put

$$(k-1)\sin.\tfrac{m\pi}{k}+(k-3)\sin.\tfrac{3m\pi}{k}+(k-5)\sin.\tfrac{5m\pi}{k}+\cdots+(k-i)\sin.\tfrac{im\pi}{k}=S,$$

thus so that the integral sought shall be $\frac{2\pi S}{kk}$, since the logarithmic parts will have cancelled mutually. Now we may multiply each side by $2\sin.\frac{m\pi}{k}$, and since in general there shall be

$$2\sin.\tfrac{m\pi}{k}\sin.\tfrac{im\pi}{k}=\cos.\tfrac{(i-1)\pi}{k}-\cos.\tfrac{(i+1)\pi}{k},$$

with the substitution made there will be

$$\begin{aligned}&2S\sin.\tfrac{m\pi}{k}=(k-1)\cos.\tfrac{0m\pi}{k}\\&+(k-3)\cos.\tfrac{2m\pi}{k}+(k-5)\cos.\tfrac{4m\pi}{k}+\cdots+(k-i)\cos.\tfrac{(i-1)m\pi}{k}\\&-(k-1)\cos.\tfrac{2m\pi}{k}-(k-3)\cos.\tfrac{4m\pi}{k}-\cdots-(k-i+2)\cos.\tfrac{(i-1)m\pi}{k}-(k-i)\cos.\tfrac{(i+1)m\pi}{k},\end{aligned}$$

which series evidently is contracted into the following

$$\begin{aligned}&2S\sin.\tfrac{m\pi}{k}=k-1-2\cos.\tfrac{2m\pi}{k}-2\cos.\tfrac{4m\pi}{k}-2\cos.\tfrac{6m\pi}{k}-\cdots-2\cos.\tfrac{(i-1)m\pi}{k}\\&-(k-i)\cos.\tfrac{(i+1)m\pi}{k},\end{aligned}$$

where with the first and final terms removed the intermediate terms constitute a regular series, for the value of which requiring to be investigated we may put

$$T=2\cos.\tfrac{2m\pi}{k}+2\cos.\tfrac{4m\pi}{k}+2\cos.\tfrac{6m\pi}{k}+\cdots+2\cos.\tfrac{(i-1)m\pi}{k},$$

thus so that there shall be

$$2S\sin.\tfrac{m\pi}{k}=k-1-2T-(k-i)\cos.\tfrac{(i+1)m\pi}{k}.$$

But here again it is convenient to consider two cases, whether k were even or odd.

ESTABLISHING THE FIRST CASE
WHERE k IS AN EVEN NUMBER AND $i=k-1$.

8. Therefore in this case we will have

$$T=\cos.\tfrac{2m\pi}{k}+\cos.\tfrac{4m\pi}{k}+\cos.\tfrac{6m\pi}{k}+\cdots+\cos.\tfrac{(k-2)m\pi}{k}.$$

We will multiply anew by $2\sin.\frac{m\pi}{k}$ and by the reductions indicated above we will have

$$2T\sin.\tfrac{m\pi}{k}=\sin.\tfrac{3m\pi}{k}+\sin.\tfrac{5m\pi}{k}+\cdots+\sin.\tfrac{(k-3)m\pi}{k}+\sin.\tfrac{(k-1)m\pi}{k}$$
$$-\sin.\tfrac{m\pi}{k}-\sin.\tfrac{3m\pi}{k}-\sin.\tfrac{5m\pi}{k}-\cdots-\sin.\tfrac{(k-3)m\pi}{k};$$

therefore with the terms mutually cancelling each other deleted there will be

$$2T\sin.\tfrac{m\pi}{k}=-\sin.\tfrac{m\pi}{k}+\sin.\tfrac{(k-1)m\pi}{k}.$$

Truly there is

$$\sin.\tfrac{(k-1)m\pi}{k}=\sin.\left(m\pi-\tfrac{m\pi}{k}\right)=\sin.m\pi\cos.\tfrac{m\pi}{k}-\cos.m\pi\sin.\tfrac{m\pi}{k},$$

where $\sin.m\pi=0$, on account of which there will become

$$2T=-1-\cos.m\pi.$$

9. With the value found for T there is gathered to become

$$2S\sin.\tfrac{m\pi}{k}=k \text{ and thus } S=\frac{k}{2\sin.\frac{m\pi}{k}}.$$

Finally indeed the value itself of our integral formula, which we seek, will be $\frac{2\pi S}{kk}$ and now it is clear in the case of our integral formula, where S is an even number, to become $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, if indeed after the integration there may be put $x=\infty$.

THE ESTABLISHMENT OF THE OTHER CASE WHERE k IS AN ODD NUMBER AND $i=k$.

10. Therefore in this case there is

$$T=\cos.\tfrac{2m\pi}{k}+\cos.\tfrac{4m\pi}{k}+\cos.\tfrac{6m\pi}{k}+\cdots+\cos.\tfrac{(k-1)m\pi}{k},$$

which series multiplied by $2\sin.\frac{m\pi}{k}$ will produce as before:

$$2T\sin.\tfrac{m\pi}{k}=\sin.\tfrac{3m\pi}{k}+\sin.\tfrac{5m\pi}{k}+\cdots+\sin.\tfrac{(k-2)m\pi}{k}+\sin.\tfrac{km\pi}{k}$$
$$-\sin.\tfrac{m\pi}{k}-\sin.\tfrac{3m\pi}{k}-\sin.\tfrac{5m\pi}{k}-\cdots-\sin.\tfrac{(k-2)m\pi}{k},$$

from which with the mutually cancelling terms deleted there will be found

$$2T\sin.\tfrac{m\pi}{k} = -\sin.\tfrac{m\pi}{k} + \sin.m\pi$$

and thus

$$2T = -1 + \frac{\sin.m\pi}{\sin.\frac{m\pi}{k}} = -1$$

on account of $\sin.m\pi = 0$, and hence again there will become

$$2S\sin.\tfrac{m\pi}{k} = k;$$

whereby since the value of the integral sought shall be $\frac{2\pi S}{kk}$, also there will be in this case our integral $= \frac{\pi}{k\sin.\frac{m\pi}{k}}$, just as in the preceding case. Hence we deduce the following

THEOREM

11. *If this differential formula*

$$\frac{x^{m-1}dx}{1+x^k}$$

may be integrated thus, so that on putting $x = 0$ *the integral may vanish, then truly there may be put* $x = \infty$, *thence the value always will be*

$$\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

whether k *shall be an even or odd number.*

The demonstration of this theorem is evident from the preceding.

12. In the establishment of this formula we assume to be $m < k$, because otherwise the logarithmic parts will not cancel each other ; but truly now lest indeed there is no need for this limitation. Indeed in the case, where there becomes $m = k$, the integral of the formula $\frac{x^{m-1}dx}{1+x^k}$ may become $\frac{1}{k}l(1+x^k)$, whereby on making $x = \infty$ also may become ∞; truly from this our integral indicates likewise to be $\frac{\pi}{k\sin.\frac{m\pi}{k}} = \infty$. Therefore provided m were not greater than k, our formula is agreed to be true always.

13. Also indeed it may not be necessary, that the exponents m and k shall be whole numbers, provided there were not $m > k$; if indeed there were $m = \frac{\mu}{\lambda}$ and $k = \frac{\chi}{\lambda}$, the

value will be by our formula $\frac{\lambda\pi}{\chi\sin.\frac{\mu\pi}{\chi}}$, the truth of which is shown thus. Because in this case the formula being integrated is

$$\int\frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\chi}{\lambda}}}\cdot\frac{dx}{x},$$

there may be put $x=y^{\lambda}$; there will become $\frac{dx}{x}=\lambda\frac{dy}{y}$; and the formula will become

$$\int\frac{y^{\mu}}{1+y^{\chi}}\cdot\frac{\lambda dy}{y}=\lambda\int\frac{y^{\mu-1}dy}{1+y^{\chi}},$$

the value of which certainly will be $\frac{\lambda\pi}{\chi\sin.\frac{\mu\pi}{\chi}}$.

ANOTHER DEMONSTRATION OF THE THEOREM

14. Let *P* denote the value of the integral $\int\frac{x^{m}}{1+x^{k}}\cdot\frac{dx}{x}$ from the limit $x=0$ as far as to $x=1$, but Q the value of the same integral from the limit $x=1$ as far as to $x=\infty$, thus so that $P+Q$ may present that value itself, which is contained in the theorem. Now for the value *Q* requiring to be found there may be put $x=\frac{1}{y}$, from which there becomes $\frac{dx}{x}=-\frac{dy}{y}$, and there will become

$$Q=\int\frac{y^{-m}}{1+y^{-k}}\cdot\frac{-dy}{y}=-\int\frac{y^{k-m}}{1+y^{k}}\cdot\frac{dy}{y},$$

from the limit $y=1$ as far as to $y=0$. Hence therefore with the limits interchanged there will be

$$Q=+\int\frac{y^{k-m}}{1+y^{k}}\cdot\frac{dy}{y},$$

from the limit $y=0$ as far as to $y=1$. Now because for this integral to be performed, the letter *y* is removed from this calculation, and in place of *y* it will be permitted to write *x,* thus so that

$$Q=\int\frac{x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x},$$

with which done we will have

$$P+Q=\int\frac{x^{m}+x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x}$$

from the limit $x=0$ as far as to the limit $x=1$. Truly thus not long ago I have shown the value of this integral formula contained between the limits $x=0$ to $x=1$ to be $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$. Hence therefore the following none the less noteworthy theorem arises.

THEOREM

15. *The value of this integral formula*

$$\int\frac{x^{m}+x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x}$$

contained within the limits $x=0$ *and* $x=1$ *is equal to the value of this integral*

$$\int\frac{x^{m}}{1+x^{k}}\cdot\frac{dx}{x}$$

contained between the limits $x=0$ *and* $x=\infty$.

16. We may approach the integral formula proposed from these set out carefully in the title, and where we may reduce that to the form treated formerly, we may call into help the following reduction

$$\int\frac{x^{m-1}dx}{(1+x^{k})^{\lambda+1}}=\frac{Ax^{m}}{(1+x^{k})^{\lambda}}+B\int\frac{x^{m-1}dx}{(1+x^{k})^{\lambda}},$$

from which with the differentiation done bring forth the following equation

$$\frac{x^{m-1}dx}{(1+x^{k})^{\lambda+1}}=\frac{mAx^{m-1}dx}{(1+x^{k})^{\lambda}}-\frac{\lambda kAx^{m+k-1}dx}{(1+x^{k})^{\lambda+1}}+B\frac{x^{m-1}dx}{(1+x^{k})^{\lambda}},$$

which equation divided by $x^{m-1}dx$ and multiplied by $\left(1+x^{k}\right)^{\lambda}$, and by transposing the negative term from the right to the left, there will be

$$\frac{1+\lambda kAx^{k}}{1+x^{k}}=mA+B,$$

which equation cannot exist , unless there shall be $\lambda kA=1$ or $A=\frac{1}{\lambda k}$,
from which there shall be $1=mA+B=\frac{m}{\lambda k}+B$, and thus there will become $B=1-\frac{m}{\lambda k}$.

17. With these values found for the letters A and B we may assume the integrals thus taken, so that they may vanish by putting $x=0$; then truly by putting $x=\infty$, because the exponent *m* is supposed to be smaller than *k,* the absolute term associated with the letter A vanishes at once, thus so that in the case $x=\infty$ there may become

$$\int\frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}}=\left(1-\frac{m}{\lambda k}\right)\int\frac{x^{m-1}dx}{(1+x^k)^{\lambda}}.$$

So that if now initially we may take $\lambda=1$, because before we found for the same case $x=\infty$ to be

$$\int\frac{x^{m-1}dx}{1+x^k}=\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

we will have the value of this integral

$$\int\frac{x^{m-1}dx}{\left(1+x^k\right)^2}=(1-\tfrac{m}{k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

if indeed the integral may be extended from the limit $x=0$ as far as to the limit $x=\infty$.

18. But if now in a similar manner we may put $\lambda=2$, there will be found from the same limits of the integration

$$\int\frac{x^{m-1}dx}{\left(1+x^k\right)^3}=(1-\tfrac{m}{k})(1-\tfrac{m}{2k})\frac{\pi}{k\sin.\frac{m\pi}{k}};$$

in the same manner if the letters λ may be attributed continually greater values, the following formulas of the of the integral will be found with all worthy of attention :

$$\int\frac{x^{m-1}dx}{\left(1+x^k\right)^4}=(1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int\frac{x^{m-1}dx}{\left(1+x^k\right)^5}=(1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})(1-\tfrac{m}{4k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int\frac{x^{m-1}dx}{\left(1+x^k\right)^6}=(1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})(1-\tfrac{m}{4k})(1-\tfrac{m}{5k})\frac{\pi}{k\sin.\frac{m\pi}{k}}$$

etc.

19. Whereby if the letter *n* may denote some whole number for the formula expressed in the title, if its integral may be extended from the limit $x=0$ as far as to $x=\infty$, its value will be had in the following manner:

$$(1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})(1-\tfrac{m}{4k})\cdots(1-\tfrac{m}{(n-1)k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

which therefore will agree with the integral of this formula

$$\int \frac{x^{m-1}dx}{\left(1+x^k\right)^n}.$$

20. Here indeed by necessity other numbers besides integers cannot be accepted for n ; but truly by the method of interpolation, which has been explained further now some time ago [see E254], this integration can be extended also for the case, in which the exponent n is a fractional number. So that indeed the following integrals may be extended from the limit $y=0$ as far as to $y=1$, in general the value of our proposed formula will be able to be represented thus:

$$\int \frac{x^{m-1}dx}{\left(1+x^k\right)^n} = \frac{\pi}{k\sin.\frac{m\pi}{k}} \cdot \frac{\int y^{nk-m-1}dy(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}dy(1-y^k)^{\frac{m}{k}-1}}.$$

From which, if there were $m=1$ and $k=2$, there follows to become :

$$\int \frac{dx}{\left(1+xx\right)^n} = \frac{\pi}{2}\int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}} : \int \frac{dy}{\sqrt{(1-yy)}} = \int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}.$$

Thus, if $n=\frac{3}{2}$, there will be

$$\int \frac{dx}{(1+xx)^{\frac{3}{2}}} = \int \frac{ydy}{\sqrt{(1-yy)}},$$

the truth of which is apparent at once, because the first integral produced generally is $\frac{x}{\sqrt{(1+xx)}}$, the latter truly $=1-\sqrt{(1-yy)}$, which on making $x=\infty$ and $y=1$ certainly become equal. Otherwise for this general equation it will help to have observed the exponent cannot be accepted less than one, because otherwise the values of both integrals will increase to infinity.

INVESTIGATIO FORMULAE INTEGRALIS
$$\int\frac{x^{m-1}dx}{(1+x^k)^n}$$
CASU QUO POST INTEGRATIONEM STATUITUR $x=\infty$

Commentatio 588 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 42-54

1. Iam satis notum est huius formulae integrale [casu, quo $n=1$] partim logarithmos, partim arcus circulares complecti et partes logarithmicas hanc progressionem constituere

$$\begin{aligned}
&-\tfrac{2}{k}cos.\tfrac{m\pi}{k}l\sqrt{(1-2xcos.\tfrac{\pi}{k}+xx)}\\
&-\tfrac{2}{k}cos.\tfrac{3m\pi}{k}l\sqrt{(1-2xcos.\tfrac{3\pi}{k}+xx)}\\
&-\tfrac{2}{k}cos.\tfrac{5m\pi}{k}l\sqrt{(1-2xcos.\tfrac{5\pi}{k}+xx)}\\
&-\tfrac{2}{k}cos.\tfrac{7m\pi}{k}l\sqrt{(1-2xcos.\tfrac{7\pi}{k}+xx)}\\
&\quad\vdots\\
&-\tfrac{2}{k}cos.\tfrac{im\pi}{k}l\sqrt{(1-2xcos.\tfrac{i\pi}{k}+xx)},
\end{aligned}$$

ubi *i* denotat numerum imparem non maiorem quam *k.* Hinc si *k* fuerit numerus par, erit $i=k-1$; ac si *k* fuerit numerus impar, hanc progressionem continuari oportet usque ad $i=k$, eius vero coefficiens duplo minor capi debet seu loco $-\frac{2}{k}$ tantum scribi debet $-\frac{1}{k}$, cuius irregularitatis ratio in *Calculo Integrali* est exposita.

2. Cum hae partes sponte iam evanescant posito $x=0$, statuamus statim $x=\infty$, et cum in genere sit
$$\sqrt{(1-2xcos.\omega+xx)}=x-\cos.\omega,$$
erit
$$l\sqrt{(1-2xcos.\omega+xx)}=l(x-\cos.\omega)=lx-\tfrac{\cos.\omega}{x}=lx,$$
ob $\frac{\cos.\omega}{x}=0$; omnes ergo illi logarithmi reducuntur ad eandem formam *lx,* quae multiplicanda est per hanc seriem
$$-\tfrac{2}{k}cos.\tfrac{m\pi}{k}-\tfrac{2}{k}cos.\tfrac{3m\pi}{k}-\tfrac{2}{k}cos.\tfrac{5m\pi}{k}-\cdots-\tfrac{2}{k}cos.\tfrac{im\pi}{k},$$

ubi, ut diximus, i denotat maximum numerum imparem ipso k non maiorem, hac tamen restrictione, ut, si k fuerit impar ideoque $i=k$, ultimum membrum ad dimidium reduci debeat. Quamobrem si huius progressionis summam investigare velimus, duo casus erunt constituendi, alter, quo k est numerus par et $i=k-1$, alter vero, quo k est impar et $i=k$.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i=k-1$

3. Hoc ergo casu posito $x=\infty$ formula $-\frac{2}{k}lx$ multiplicatur per hanc cosinuum seriem

$$\cos.\tfrac{m\pi}{k}+\cos.\tfrac{3m\pi}{k}+\cos.\tfrac{5m\pi}{k}+\cdots+\cos.\tfrac{(k-1)m\pi}{k},$$

cuius summam statuamus $=S$. Ducamus hanc seriem in $\sin.\frac{m\pi}{k}$, et cum in genere sit

$$\sin.\tfrac{m\pi}{k}\cos.\tfrac{i\pi}{k}=\tfrac{1}{2}\sin.\tfrac{(i+1)m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$\begin{aligned}&S\sin.\tfrac{m\pi}{k}\\ &=\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}+\cdots+\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{km\pi}{k}\\ &-\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}-\cdots-\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k},\end{aligned}$$

ubi omnes termini praeter ultimum manifesto se destruunt, ita ut sit

$$S\sin.\tfrac{m\pi}{k}=\tfrac{1}{2}\sin.m\pi.$$

Iam vero, quia nostri coefficientes m et k supponuntur integri, utique erit $\sin.m\pi=0$ ideoque etiam $S=0$, nisi forte etiam fuerit $\sin.\frac{m\pi}{k}=0$, qui autem casus locum habere nequit, quoniam in integratione formulae propositae $\int\frac{x^{m-1}dx}{(1+x^k)^n}$ semper assumi solet esse $m<k$. Hoc igitur modo evictum est casu, quo post integrationem statuitur $x=\infty$, omnes partes logarithmicas integralis se destruere.

EVOLUTIO CASUS ALTERIUS QUO EST k NUMERUS IMPAR ET $i=k$

4. Hoc ergo casu sumto $x=\infty$ formula lx multiplicatur per hanc seriem

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}-\cdots-\tfrac{1}{k}\cos.\tfrac{km\pi}{k},$$

ubi terminus penultimus est $-\frac{2}{k}cos.\frac{(k-2)m\pi}{k}$, pro ultimo vero termino erit $\cos.m\pi = \pm 1$ signo superiore valente, si n sit numerus par, inferiore, si impar; quare remoto termino ultimo pro reliquis ponamus

$$\cos.\tfrac{m\pi}{k} + \cos.\tfrac{3m\pi}{k} + \cos.\tfrac{5m\pi}{k} + \cdots + \cos.\tfrac{(k-2)m\pi}{k} = S,$$

ita ut multiplicator ipsius logarithmi x sit

$$-\tfrac{2S}{k} - \tfrac{1}{k}\cos.m\pi.$$

Hinc procedendo ut ante fiet

$$\begin{aligned} &S\sin.\tfrac{m\pi}{k} \\ &= \tfrac{1}{2}\sin.\tfrac{2m\pi}{k} + \tfrac{1}{2}\sin.\tfrac{4m\pi}{k} + \tfrac{1}{2}\sin.\tfrac{6m\pi}{k} + \cdots + \tfrac{1}{2}\sin.\tfrac{(k-3)m\pi}{k} + \tfrac{1}{2}\sin.\tfrac{(k-1)m\pi}{k} \\ &- \tfrac{1}{2}\sin.\tfrac{2m\pi}{k} - \tfrac{1}{2}\sin.\tfrac{4m\pi}{k} - \tfrac{1}{2}\sin.\tfrac{6m\pi}{k} - \cdots - \tfrac{1}{2}\sin.\tfrac{(k-3)m\pi}{k}, \end{aligned}$$

ubi iterum omnes termini praeter ultimum se mutuo tollunt, ita ut hinc prodeat

$$\sin.\tfrac{m\pi}{k} = \tfrac{1}{2}\sin.\tfrac{(k-1)m\pi}{k} = \tfrac{1}{2}\sin.(m\pi - \tfrac{m\pi}{k});$$

at vero est

$$\tfrac{1}{2}\sin.(m\pi - \tfrac{m\pi}{k}) = \sin.m\pi\cos.\tfrac{m\pi}{k} - \cos.m\pi\sin.\tfrac{m\pi}{k},$$

ubi notetur esse $\sin.m\pi = 0$ ob m numerum integrum; habebimus ergo

$$S\sin.\tfrac{m\pi}{k} = -\tfrac{1}{2}\cos.m\pi\sin.\tfrac{m\pi}{k} \text{ sive } ,S = -\tfrac{1}{2}\cos.m\pi,$$

consequenter multiplicator ipsius lx erit

$$= \tfrac{1}{k}\cos.m\pi - \tfrac{1}{k}\cos.m\pi = 0$$

sicque manifestum est, sive k sit numerus par sive impar, omnia membra logarithmica in nostro integrali se mutuo destruere, siquidem post integrationem statuamus $x = \infty$, quemadmodum hic semper supponimus.

5. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componitur. Hae autem partes sequentem progressionem constituere sunt compertae:

$$\tfrac{2}{k}\sin.\tfrac{m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}+\tfrac{2}{k}\sin.\tfrac{3m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}$$
$$+\tfrac{2}{k}\sin.\tfrac{5m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}}+\tfrac{2}{k}\sin.\tfrac{7m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{7\pi}{k}}{1-x\cos.\frac{7\pi}{k}}$$
$$+\cdot\;\cdot\;\cdot\;\cdot+\tfrac{2}{k}\sin.\tfrac{im\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}},$$

ubi in ultimo membro est vel $i=k-1$ vel $i=k$*;* prius scilicet valet, si *i* est numerus par, posterius, si impar.

6. Cum etiam omnia haec membra evanescant posito $x=0$, faciamus pro instituto nostro $x=\infty$. In genere igitur fiet

$$\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}=\text{Atang.}\left(-\text{tang.}\tfrac{i\pi}{k}\right).$$

Est vero

$$-\text{tang.}\tfrac{i\pi}{k}=+\text{tang.}\left(\tfrac{(k-i)\pi}{k}\right),$$

ex quo hic arcus fit $=\frac{(k-i)\pi}{k}$. Hinc ergo loco *i* scribendo successive numeros 1, 3, 5, 7 etc. istae partes nostri integralis quaesiti erunt

$$\tfrac{2(k-1)\pi}{kk}\sin.\tfrac{m\pi}{k}+\tfrac{2(k-3)\pi}{kk}\sin.\tfrac{3m\pi}{k}+\tfrac{2(k-5)\pi}{kk}\sin.\tfrac{5m\pi}{k}+\tfrac{2(k-7)\pi}{kk}\sin.\tfrac{7m\pi}{k}$$
$$+\tfrac{2(k-9)\pi}{kk}\sin.\tfrac{9m\pi}{k}+\cdots+\tfrac{2(k-i)\pi}{kk}\sin.\tfrac{im\pi}{k},$$

ubi casu, quo *k* est numerus par, progredi oportet usque ad $i=k-1$, ac si *k* sit numerus impar, usque ad $i=k$.

7. Statuamus brevitatis gratia

$$(k-1)\sin.\tfrac{m\pi}{k}+(k-3)\sin.\tfrac{3m\pi}{k}+(k-5)\sin.\tfrac{5m\pi}{k}+\cdots+(k-i)\sin.\tfrac{im\pi}{k}=S,$$

ita ut integrale quaesitum sit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae se mutuo destruxerunt. Multiplicemus nunc utrinque per $2\sin.\frac{m\pi}{k}$, et cum in genere sit

$$2\sin.\tfrac{m\pi}{k}\sin.\tfrac{im\pi}{k}=\cos.\tfrac{(i-1)\pi}{k}-\cos.\tfrac{(i+1)\pi}{k},$$

facta substitutione erit

$$2S\sin.\tfrac{m\pi}{k}=(k-1)\cos.\tfrac{0m\pi}{k}$$
$$+(k-3)\cos.\tfrac{2m\pi}{k}+(k-5)\cos.\tfrac{4m\pi}{k}+\cdots+(k-i)\cos.\tfrac{(i-1)m\pi}{k}$$
$$-(k-1)\cos.\tfrac{2m\pi}{k}-(k-3)\cos.\tfrac{4m\pi}{k}-\cdots-(k-i+2)\cos.\tfrac{(i-1)m\pi}{k}-(k-i)\cos.\tfrac{(i+1)m\pi}{k},$$

quae series manifesto contrahitur in sequentem

$$2S\sin.\tfrac{m\pi}{k}=k-1-2\cos.\tfrac{2m\pi}{k}-2\cos.\tfrac{4m\pi}{k}-2\cos.\tfrac{6m\pi}{k}-\cdots-2\cos.\tfrac{(i-1)m\pi}{k}$$
$$-(k-i)\cos.\tfrac{(i+1)m\pi}{k},$$

ubi primo et ultimo membro sublatis regularem termini intermedii constituunt seriem, pro cuius valore investigando ponamus

$$T=2\cos.\tfrac{2m\pi}{k}+2\cos.\tfrac{4m\pi}{k}+2\cos.\tfrac{6m\pi}{k}+\cdots+2\cos.\tfrac{(i-1)m\pi}{k},$$

ita ut sit

$$2S\sin.\tfrac{m\pi}{k}=k-1-2T-(k-i)\cos.\tfrac{(i+1)m\pi}{k}.$$

Hic autem iterum convenit duos casus perpendere, prout k fuerit par vel impar.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i=k-1$

8. Hoc ergo casu habebimus

$$T=\cos.\tfrac{2m\pi}{k}+\cos.\tfrac{4m\pi}{k}+\cos.\tfrac{6m\pi}{k}+\cdots+\cos.\tfrac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2\sin.\frac{m\pi}{k}$ et per reductiones supra indicatas habebimus

$$2T\sin.\tfrac{m\pi}{k}=\sin.\tfrac{3m\pi}{k}+\sin.\tfrac{5m\pi}{k}+\cdots+\sin.\tfrac{(k-3)m\pi}{k}+\sin.\tfrac{(k-1)m\pi}{k}$$
$$-\sin.\tfrac{m\pi}{k}-\sin.\tfrac{3m\pi}{k}-\sin.\tfrac{5m\pi}{k}-\cdots-\sin.\tfrac{(k-3)m\pi}{k};$$

deletis igitur terminis se mutuo tollentibus erit

$$2T\sin.\tfrac{m\pi}{k}=-\sin.\tfrac{m\pi}{k}+\sin.\tfrac{(k-1)m\pi}{k}.$$

Est vero

$$\sin.\tfrac{(k-1)m\pi}{k}=\sin.\left(m\pi-\tfrac{m\pi}{k}\right)=\sin.m\pi\cos.\tfrac{m\pi}{k}-\cos.m\pi\sin.\tfrac{m\pi}{k},$$

ubi $\sin.m\pi=0$, quamobrem fiet

$$2T = -1 - \cos.m\pi.$$

9. Invento valore pro T colligitur fore

$$2S\sin.\tfrac{m\pi}{k} = k \quad \text{ideoque} \quad S = \frac{k}{2\sin.\frac{m\pi}{k}}.$$

Denique vero ipse valor formulae nostrae integralis, quem quaerimus, erit $\frac{2\pi S}{kk}$ et nunc manifestum est integrale nostrae formulae casu, quo S est numerus par, fore $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, siquidem post integrationem statuatur $x = \infty$.

EVOLUTIO ALTERIUS CASUS QUO k EST NUMERUS IMPAR ET $i = k$

10. Hoc ergo casu est

$$T = \cos.\tfrac{2m\pi}{k} + \cos.\tfrac{4m\pi}{k} + \cos.\tfrac{6m\pi}{k} + \cdots + \cos.\tfrac{(k-1)m\pi}{k},$$

quae series multiplicata per $2\sin.\frac{m\pi}{k}$ producet ut ante

$$\begin{aligned}2T\sin.\tfrac{m\pi}{k} &= \sin.\tfrac{3m\pi}{k} + \sin.\tfrac{5m\pi}{k} + \cdots + \sin.\tfrac{(k-2)m\pi}{k} + \sin.\tfrac{km\pi}{k}\\ &- \sin.\tfrac{m\pi}{k} - \sin.\tfrac{3m\pi}{k} - \sin.\tfrac{5m\pi}{k} - \cdots - \sin.\tfrac{(k-2)m\pi}{k},\end{aligned}$$

unde deletis terminis se mutuo tollentibus reperietur

$$2T\sin.\tfrac{m\pi}{k} = -\sin.\tfrac{m\pi}{k} + \sin.m\pi$$

ideoque

$$2T = -1 + \frac{\sin.m\pi}{\sin.\frac{m\pi}{k}} = -1$$

ob $\sin.m\pi = 0$, hincque porro fiet

$$2S\sin.\tfrac{m\pi}{k} = k;$$

quare cum valor integralis quaesitus sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $= \frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti praecedente casu. Hinc ergo deducimus sequens

THEOREMA

11. *Si haec formula differentialis*

$$\frac{x^{m-1}dx}{1+x^k}$$

ita integretur, ut posito $x=0$ *integrale evanescat, tum vero statuatur* $x=\infty$, *valor inde resultans semper erit*

$$\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

sive k sit numerus par sive impar.

Huius theorematis demonstratio ex praecedentibus est manifesta.

12. In evolutione huius formulae assumsimus esse $m<k$, quia alioquin membra logarithmica se non destruissent; at vero ne hac quidem limitatione nunc amplius est opus. Casu enim, quo foret $m=k$, integrale formulae $\frac{x^{m-1}dx}{1+x^k}$ esset $\frac{1}{k}l(1+x^k)$, quo facto $x=\infty$ eret etiam ∞; verum hoc idem indicat nostrum integrale esse $\frac{\pi}{k\sin.\frac{m\pi}{k}}=\infty$. Dummodo ergo *m* non fuerit maius quam *k,* nostra formula veritati semper est consentanea.

13. Quin etiam ne quidem necesse est, ut exponentes *m* et *k* sint numeri integri, dummodo non fuerit $m>k$; si enim fuerit $m=\frac{\mu}{\lambda}$ et $k=\frac{\chi}{\lambda}$, erit valor per nostram formulam $\frac{\lambda\pi}{\chi\sin.\frac{\mu\pi}{\chi}}$, cuius veritas ita ostenditur. Quia hoc casu formula integranda est

$$\int\frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\chi}{\lambda}}}\cdot\frac{dx}{x},$$

statuatur $x=y^{\lambda}$; erit $\frac{dx}{x}=\lambda\frac{dy}{y}$; et formula fiet

$$\int\frac{y^{\mu}}{1+y^{\chi}}\cdot\frac{\lambda dy}{y}=\lambda\int\frac{y^{\mu-1}dy}{1+y^{\chi}},$$

cuius valor utique erit $\frac{\lambda\pi}{\chi\sin.\frac{\mu\pi}{\chi}}$.

ALIA DEMONSTRATIO THEOREMATIS

14. Denotet *P* valorem integralis $\int\frac{x^m}{1+x^k}\cdot\frac{dx}{x}$ a termino $x=0$ usque ad $x=1$, at Q valorem eiusdem integralis a termino $x=1$ usque ad $x=\infty$, ita ut $P+Q$ praebeat eum

ipsum valorem, qui in theoremate continetur. Nunc pro valore Q inveniendo statuatur $x=\frac{1}{y}$, unde fit $\frac{dx}{x}=-\frac{dy}{y}$, fietque

$$Q=\int\frac{y^{-m}}{1+y^{-k}}\cdot\frac{-dy}{y}=-\int\frac{y^{k-m}}{1+y^{k}}\cdot\frac{dy}{y},$$

a termino $y=1$ usque ad $y=0$. Hinc igitur commutatis terminis erit

$$Q=+\int\frac{y^{k-m}}{1+y^{k}}\cdot\frac{dy}{y},$$

a termino $y=0$ usque ad $y=1$. Iam quia hoc integrali expedito littera *y* ex calculo egreditur, loco *y* scribere licebit *x,* ita ut sit

$$Q=\int\frac{x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x},$$

quo facto habebimus

$$P+Q=\int\frac{x^{m}+x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x},$$

a termino $x=0$ usque ad terminum $x=1$. Verum non ita pridem demonstravi valorem huius formulae integralis intra terminos $x=0$ et $x=1$ contentum esse $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$. Hinc igitur nascitur sequens theorema non minus notatu dignum.

THEOREMA

15. *Valor huius formulae integralis*

$$\int\frac{x^{m}+x^{k-m}}{1+x^{k}}\cdot\frac{dx}{x}$$

intra terminos $x=0$ *et* $x=1$ *contentus aequalis est valori istius integralis*

$$\int\frac{x^{m}}{1+x^{k}}\cdot\frac{dx}{x}$$

intra terminos $x=0$ *et* $x=\infty$ *contento.*

16. His expensis formulam integralem in titulo propositam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in subsidium vocemus sequentem reductionem

$$\int\frac{x^{m-1}dx}{(1+x^{k})^{\lambda+1}}=\frac{Ax^{m}}{(1+x^{k})^{\lambda}}+B\int\frac{x^{m-1}dx}{(1+x^{k})^{\lambda}},$$

unde facta differentiatione prodit sequens aequatio

$$\frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \frac{mAx^{m-1}dx}{(1+x^k)^{\lambda}} - \frac{\lambda kAx^{m+k-1}dx}{(1+x^k)^{\lambda+1}} + B\frac{x^{m-1}dx}{(1+x^k)^{\lambda}},$$

quae aequatio per $x^{m-1}dx$ divisa ac per $(1+x^k)^{\lambda}$ multiplicata terminum negativum a dextra ad sinistram transponendo erit

$$\frac{1+\lambda kAx^k}{1+x^k} = mA + B,$$

quae aequatio manifesto subsistere nequit, nisi sit $\lambda kA = 1$ sive $A = \frac{1}{\lambda k}$,
unde erit $1 = mA + B = \frac{m}{\lambda k} + B$, sicque erit $B = 1 - \frac{m}{\lambda k}$.

17. Inventis his valoribus pro litteris *A* et *B* primum assumimus integralia ita capi, ut evanescant posito $x = 0$; tum vero posito $x = \infty$, quia exponens *n* minor supponitur quam *k,* membrum absolutum littera *A* affectum sponte evanescit, ita ut hoc casu $x = \infty$ fiat

$$\int\frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \left(1 - \frac{m}{\lambda k}\right)\int\frac{x^{m-1}dx}{(1+x^k)^{\lambda}}.$$

Quodsi iam primo capiamus $\lambda = 1$, quia ante invenimus pro eodem casu $x = \infty$ esse

$$\int\frac{x^{m-1}dx}{1+x^k} = \frac{\pi}{k\sin.\frac{m\pi}{k}},$$

habebimus valorem istius integralis

$$\int\frac{x^{m-1}dx}{\left(1+x^k\right)^2} = (1 - \tfrac{m}{k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

siquidem integrale etiam a termino $x = 0$ usque ad terminum $x = \infty$ extendatur.

18. Quodsi iam simili modo ponamus $\lambda = 2$, reperietur pro iisdem terminis integrationis

$$\int\frac{x^{m-1}dx}{\left(1+x^k\right)^3} = (1 - \tfrac{m}{k})(1 - \tfrac{m}{2k})\frac{\pi}{k\sin.\frac{m\pi}{k}};$$

eodem modo si litterae λ continuo maiores valores tribuantur, reperientur sequentes integralium formae omni attentione dignae

$$\int \frac{x^{m-1}dx}{\left(1+x^k\right)^4} = (1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}dx}{\left(1+x^k\right)^5} = (1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})(1-\tfrac{m}{4k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}dx}{\left(1+x^k\right)^6} = (1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})(1-\tfrac{m}{4k})(1-\tfrac{m}{5k})\frac{\pi}{k\sin.\frac{m\pi}{k}}$$

etc.

19. Quare si littera n denotet numerum quemcunque integrum pro formula in titulo expressa, si eius integrale a termino $x=0$ usque ad $x=\infty$ extendatur, eius valor sequenti modo se habebit:

$$(1-\tfrac{m}{k})(1-\tfrac{m}{2k})(1-\tfrac{m}{3k})(1-\tfrac{m}{4k})\cdots(1-\tfrac{m}{(n-1)k})\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

qui ergo conveniet huic formulae integrali

$$\int \frac{x^{m-1}dx}{\left(1+x^k\right)^n}.$$

20. Hic quidem necessario pro n alii numeri praeter integros accipi non licet; at vero per methodum interpolationum, quae fusius iam passim est explicata, hanc integrationem etiam ad casus, quibus exponens n est numerus fractus, extendere licet. Quodsi enim sequentes formulae integrales a termino $y=0$ usque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit

$$\int \frac{x^{m-1}dx}{\left(1+x^k\right)^n} = \frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\int y^{nk-m-1}dy(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}dy(1-y^k)^{\frac{m}{k}-1}}.$$

Unde, si fuerit $m=1$ et $k=2$, sequitur fore

$$\int \frac{dx}{(1+xx)^n} = \frac{\pi}{2}\int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}} : \int \frac{dy}{\sqrt{(1-yy)}} = \int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}.$$

Ita, si $n=\frac{3}{2}$, erit

$$\int \frac{dx}{(1+xx)^{\frac{3}{2}}} = \int \frac{ydy}{\sqrt{(1-yy)}},$$

cuius veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{(1+xx)}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae facto $x=\infty$ et $y=1$ utique fiunt aequalia. Caeterum pro hac integratione generali notasse iuvabit exponentem unitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

INVESTIGATION OF THE VALUE OF THE INTEGRAL

$$\int \frac{x^{m-1}dx}{1-2x^k\cos.\theta+x^{2k}}$$

EXTENDED FROM THE LIMIT $x=0$ AS FAR AS TO $x=\infty$

[E589]

Opuscula analytica 2, 1785, p. 55-75

1. At first we seek the indefinite integral of the proposed formula and accordingly we shall derive all the analytical operations from first principles. And indeed in the first place, since the denominator cannot be resolved into simple real factors, and in general its duplicate factor shall be for some $1-2x\cos.\omega+xx$; for it is evident the denominator to be the product from *k* duplicate factors of this kind. Therefore since on putting this factor $=0$ there may become $x=\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega$, also the twofold manner of the denominator itself will have to vanish, if there may be put

$$x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega \text{ or } x=\cos.\omega-\sqrt{-1}\cdot\sin.\omega.$$

Moreover it is agreed all the powers of these formulas thus may be expressed conveniently, so that there shall be

$$(\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega)^{\lambda}=\cos.\lambda\omega\pm\sqrt{-1}\cdot\sin.\lambda\omega;$$

hence therefore there will become

$$x^k=\cos.k\omega\pm\sqrt{-1}\cdot\sin.k\omega \text{ and } x^{2k}=\cos.2k\omega\pm\sqrt{-1}\cdot\sin.2k\omega$$

Therefore we may substitute these values and our denominator will appear :

$$1-2\cos.\theta\cos.k\omega+\cos.2k\omega\mp2\sqrt{-1}\cdot\cos.\theta\sin.k\omega\pm\sqrt{-1}\cdot\sin.2k\omega.$$

2. Therefore it is evident of this equation both the real terms as well as the imaginary ones must be removed separately mutually among themselves, from which these two equations arise

$$\text{I. } 1-2\cos.\theta\cos.k\omega+\cos.2k\omega=0,$$
$$\text{II. } -2\cos.\theta\sin.k\omega+\sin.2k\omega=0.$$

Therefore since there shall be

$$\sin.2k\omega = 2\sin.k\omega\cos.k\omega,$$

the latter equation will adopt this form

$$-2\cos.\theta\sin.k\omega + 2\sin.k\omega\cos.k\omega = 0,$$

which divided by $2\sin.k\omega$ gives

$$\cos.k\omega = \cos.\theta$$

and thus

$$\cos.2k\omega = \cos.2\theta = \cos.^2\theta - \sin.^2\theta = 2\cos.^2\theta - 1,$$

which values substituted into the former equation present an identical equation, thus so that for each equation it may be satisfied by assuming $\cos.k\omega = \cos.\theta$.

3. Therefore it will be required to assume an angle of this kind for ω, so that there may become $\cos.k\omega = \cos.\theta$, from which indeed at once there is deduced $k\omega = \theta$ and thus $\omega = \frac{\theta}{k}$. In truth since an infinitude of angles are given having the same cosine, which besides the angle θ itself are $2\pi \pm \theta,\ 4\pi \pm \theta,\ 6\pi \pm \theta$ etc., and thus in general $2i\pi \pm \theta$ with *i* denoting all the whole numbers, for which our angle sought will be satisfied by making $k\omega = 2i\pi \pm \theta$, from which the angle deduced $\omega = \frac{2i\pi \pm \theta}{k}$ and thus for ω we may obtain innumerable satisfying angles, but of which it will suffice to have assumed just as many as the exponent *k* contains unity; therefore we may attribute the following values to the angle ω

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\frac{4\pi+\theta}{k},\frac{6\pi+\theta}{k},\frac{8\pi+\theta}{k},\cdots\frac{2(k-1)\pi+\theta}{k}.$$

So that if therefore these individual values may be attributed to the angle ω successively, of which the number is $= k$, the formula $1 - 2x\cos.\omega + xx$ will supply all the double factors of our denominator $1 - 2x^k\cos.\theta + x^{2k}$, of which the number will be $= k$.

4. Now with all the double factors of our denominator $\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}$ found the fraction must be resolved into just as many partial fractions, the denominators of which shall be these duplicate factors, of which the number is *k,* thus so that in general such a partial fraction shall be going to have such a form

$$\frac{A+Bx}{1-2x\cos.\omega+xx}$$

as we resolve above into the two simple fractions even if imaginary, and since there shall be

$$xx - 2x\cos.\omega + 1 = (x - \cos.\omega + \sqrt{-1}\cdot\sin.\omega)(x - \cos.\omega - \sqrt{-1}\cdot\sin.\omega),$$

both these partial fractions may be put in place :

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega}+\frac{g}{x-\cos.\omega+\sqrt{-1}\cdot\sin.\omega},$$

thus so that the whole business of the resolution may correspond to this, so that both the numerators *f* and *g* will be determined; indeed with these found the sum of both fractions will be had

$$\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}\cdot(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

where therefore there will be :

$$B=f+g \text{ and } A=(f-g)\sqrt{-1}\cdot\sin.\omega-(f+g)\cos.\omega.$$

5. Therefore we may establish any fractions requiring to be resolved into simple fractions by the method

$$\frac{x^{m-1}}{1-2x^{k}\cos.\omega+x^{2k}}=\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega}+R,$$

where *R* contains all the remaining partial fractions. Hence on multiplying by

$$x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$$

there will be had

$$\frac{x^{m}-x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{1-2x^{k}\cos.\theta+x^{2k}}=f+R(x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega);$$

since which equation must be true, whatever value of *x* may be attributed, we may set $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$, so that the latter part may be taken at once from the calculation ; then truly in the left-hand part, since the formula $x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$ likewise is a factor of the denominator, with this substitution made both the numerator as well as the denominator will become zero, thus so that nothing may be seen to be concluded.

6. Therefore here we may use a most noteworthy rule and in place of the numerator as well as the denominator we may write the differentials of these, from which our equation may take the following form

$$\frac{mx^{m-1}-(m-1)x^{m-2}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^{k-1}\cos.\theta+2kx^{2k-1}}$$
$$=\frac{mx^{m}-(m-1)x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^{k}\cos.\theta+2kx^{2k}}=f,$$

evidently on putting $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$. But then there will be

$$x^{m}=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

and

$$x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)=x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

and for the denominator

$$x^k=\cos.k\omega+\sqrt{-1}\cdot\sin.k\omega \text{ and } x^{2k}=\cos.2k\omega+\sqrt{-1}\cdot\sin.2k\omega;$$

from which the numerator becomes

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

and the denominator

$$-2k\cos.\theta\cos.k\omega+2k\cos.2k\omega-2k\sqrt{-1}\cdot\cos.\theta\sin.k\omega+2k\sqrt{-1}\cdot\sin.2k\omega.$$

7. With the denominator being reduced we may consider $\cos.k\omega=\cos.\theta$ now found above to be, from which there becomes $\sin.k\omega=-\sin.\theta$, then truly

$$\cos.2k\omega=\cos.2\theta=2\cos.^2\theta-1 \text{ and } \sin.2k\omega=2\sin.\theta\cos.\theta,$$

with which values used our denominator will be

$$2k\cos.^2\theta-2k+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta=-2k\sin.^2\theta+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta$$
$$=-2k\sin.\theta(\sin.\theta-\sqrt{-1}\cdot\cos.\theta),$$

on this account we will have this value to be used:

$$f=\frac{\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega}{2k\sin.\theta(\sqrt{-1}\cdot\cos.\theta-\sin.\theta)}.$$

Truly likewise hence without a new calculation we will deduce the value *g*, certainly which may differ from *f* only on account of the sign $\sqrt{-1}$, and thus there will be

$$g=\frac{\cos.m\omega-\sqrt{-1}\cdot\sin.m\omega}{-2k\sin.\theta(\sin.\theta+\sqrt{-1}\cdot\cos.\theta)}.$$

8. Moreover from these letters found *f* and *g* for the letters *A* and *B* we will gather in the first place

$$f+g=\frac{\cos.\theta\sin.m\omega-\sin.\theta\cos.m\omega}{k\sin\theta}=\frac{\sin.(m\omega-\theta)}{k\sin\theta},$$

then truly there will be

$$f-g=-\frac{\sqrt{-1}\cdot\cos.(m\omega-\theta)}{k\sin\theta}.$$

Therefore from these we will find :

$$B=\frac{\sin.(m\omega-\theta)}{k\sin\theta}$$

and

$$A=\frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta}=-\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta},$$

there where the imaginary parts at once mutually have cancelled each other out.

9. With these values A and B found it will be required to investigate from the partial integral

$$\int\frac{(A+Bx)dx}{1-2x\cos.\omega+xx},$$

where, since the differential of the denominator shall be

$$2xdx-2dx\cos.\omega=2dx(x-\cos.\omega),$$

we may establish

$$A+Bx=B(x-cos.\omega)+C$$

and there will be $C=A+B\cos.\omega$; hence therefore there will be

$$C=\frac{\cos.\omega\sin.(m\omega-\theta)-\sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Truly since $-\sin.((m\omega-\theta)-\omega)=-\sin.(m\omega-\theta)cos.\omega+cos.(m\omega-\theta)\sin.\omega,$
there will be

$$C=\frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Therefore with this form used the formula requiring to be integrated $\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ may be split into these two parts

$$\frac{B(x-\cos.\omega)dx}{1-2x\cos.\omega+xx}+\frac{Cdx}{1-2x\cos.\omega+xx}.$$

Therefore here evidently the integral of the first part is

$$Bl\sqrt{(1-2x\cos.\omega+xx)},$$

truly of the other part it is readily apparent the integral is going to be expressed by the arc of a circle of which the tangent shall be $\frac{x\sin.\omega}{1-x\cos.\omega}$. Towards finding this integral we may put

$$\int\frac{Cdx}{1-2x\cos.\omega+xx}=D\ \text{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

and with the differentials taken, since $d.\text{Atang.}t$ is equal to $\frac{dt}{1+tt}$ we will have,

$$\frac{Cdx}{1-2x\cos.\omega+xx}=D\frac{dx\sin.\omega}{1-2x\cos.\omega+xx},$$

from which evidently there becomes

$$D=\frac{C}{\sin.\omega}=\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

10. Therefore in place of B and D we may substitute the values found, and from the individual factors of the denominator $1-2x^k\cos.\theta+x^{2k}$, of which the formula is $1-2x\cos.\omega+x^2$, the part of the integral agreeing with the logarithmic member and with the circular arc, which will be

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}l\sqrt{(1-2x\cos.\omega+xx)}+\frac{\cos.(m\omega-\theta)}{k\sin.\theta}\text{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

which vanishes on setting $x=0$. Therefore in this formula there is still a need, so that in place of ω we may write the successive values indicated above, evidently

$$\omega=\frac{\theta}{k},\quad \frac{2\pi+\theta}{k},\quad \frac{4\pi+\theta}{k},\quad \frac{6\pi+\theta}{k}\ \text{etc.},$$

while we may come to $\frac{2(k-1)\pi+\theta}{k}$; then truly the sum of all these formulas will provide the whole indefinite integral of the proposed formula.

11. Therefore after we have elicited the indefinite integral, nothing else remains, except that in that we may put $x=\infty$, with which done the logarithmic part on account of

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega$$

will be $Bl(x-\cos.\omega)$. Truly there is

$$l(x-\cos.\omega)=lx-\frac{\cos.\omega}{x}=lx$$

on account of $\frac{\cos.\omega}{x}=0$; wherefore by making $x=\infty$ any logarithmic part will have this form $\frac{\sin.(m\omega-\theta)}{k\sin.\theta}lx$. Then for the part depending on the circle on making $x=\infty$ there becomes

$$\frac{x\sin.\omega}{1-x\cos.\omega}=-\text{tang.}\omega=\text{tang.}(\pi-\omega)$$

and thus the arc, of which this is the tangent, will be $=\pi-\omega$ and hence any part of the circle will become $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$.

12. Since any value of the angle ω in general may have this form $\frac{2i\pi+\theta}{k}$, there will be the angle

$$m\omega-\theta=\frac{2im\pi-\theta(k-m)}{k} \text{ and } \pi-\omega=\frac{\pi(k-2i)-\theta}{k}.$$

For the sake of brevity we may put

$$\frac{\theta(k-m)}{k}=\zeta \text{ and } \frac{m\pi}{k}=\alpha,$$

so that there shall be

$$m\omega-\theta=2i\alpha-\zeta ,$$

where in place of i the successive numbers 0, 1, 2, 3 etc. must be written as far as to $k-1$. Hence therefore, if we may gather all the logarithmic parts into one, thus this can be represented thus :

$$\frac{lx}{k\sin.\theta}\left\{\begin{matrix}-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\sin.(6\alpha-\zeta)\\ +\sin.(8\alpha-\zeta)+\cdots+\sin.(2(k-1)\alpha-\zeta)\end{matrix}\right\};$$

where indeed from these, which have been treated up to this point, it will be easily allowed to suspect this whole progression to be returned to zero. Truly it is necessary to demonstrate this supposition itself rigorously.

13. Towards showing this we may put

$$S=-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\cdots+\sin.(2(k-1)\alpha-\zeta);$$

we may multiply each side by $2\sin.\alpha$, and since there shall be

$$2\sin.\alpha\sin.\varphi=\cos.(\alpha-\varphi)-\cos.(\alpha+\varphi),$$

with the aid of this reduction we will obtain the following expression

$$\begin{aligned}2S\sin.\alpha&=\cos.(\alpha+\zeta)\\ &-\cos.(\alpha-\zeta)-\cos.(3\alpha-\zeta)-\cos.(5\alpha-\zeta)-\cdots\\ &+\cos.(\alpha-\zeta)+\cos.(3\alpha-\zeta)+(5\alpha-\zeta)+\cdots\\ &-\cos.((2k-1)\alpha-\zeta),\end{aligned}$$

from which, with the terms mutually cancelling each other, we will have

$$2S\sin.\alpha = \cos.(\alpha+\zeta) - \cos.((2k-1)\alpha - \zeta).$$

14. We may put these two angles which remain,

$$\alpha+\zeta = p \text{ and } (2k-1)\alpha - \zeta = q$$

and the sum of these will be $p+q=2\alpha k$. Again since there is $\alpha = \frac{m\pi}{k}$, there will be $p+q=2m\pi$, that is a multiple of the whole periphery of the circle on account of the whole number m. Whereby since there shall be $q=2m\pi - p$, there will be $\cos.q = \cos.p$; from which it is apparent the sum found to be equal to zero and thus it is clear all the logarithmic parts, which have been introduced into our integral formula, in the case $x=\infty$ mutually cancel out.

15. Therefore we may progress to the circular parts, the general form of which, as we have seen, is $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$, which on putting $\alpha = \frac{m\pi}{k}$ and $\zeta = \frac{\theta(k-m)}{k}$, becomes

$$\tfrac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\pi - \tfrac{2i\pi+\theta}{k}) = \tfrac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\pi - \tfrac{2i\pi}{k} + \tfrac{\theta}{k}).$$

Here again there may be put $\frac{\pi}{k} = \beta$ and $\pi - \frac{\theta}{k} = \gamma$, so that the general formula shall be

$$\tfrac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma - 2i\beta).$$

Whereby if in place of i we may write the values 0, 1, 2, 3, 4,… in order as far as to $k-1$, all the circular parts will constitute this progression

$$\begin{aligned}&\tfrac{1}{k\sin.\theta}(\gamma\cos.\zeta + (\gamma-2\beta)\cos.(2\alpha-\zeta) + (\gamma-4\beta)\cos.(4\alpha-\zeta) + \cdots\\ &\quad + (\gamma - 2(k-1)\beta)\cos.(2(k-1)\alpha - \zeta)).\end{aligned}$$

Therefore we may put

$$\begin{aligned}S &= \gamma\cos.\zeta + (\gamma-2\beta)\cos.(2\alpha-\zeta) + (\gamma-4\beta)\cos.(4\alpha-\zeta) + \cdots\\ &\quad + (\gamma - 2(k-1)\beta)\cos.(2(k-1)\alpha - \zeta),\end{aligned}$$

so that the sum of all the circular parts shall be $\frac{S}{k\sin.\theta}$, which therefore will present the value of the integral formula proposed sought in the case, where after the integration there may be put $x=\infty$, thus so that the whole matter in investigating the value of S may be returned.

16. In the end we will multiply these on both sides by $2\sin.\alpha$, and since in general there shall be

$$2\sin.\alpha\cos.\varphi = \sin.(\alpha+\varphi)-\sin.(\varphi-\alpha),$$

with this reduction made into individual terms we will arrive at this equation :

$$\begin{aligned}&2\mathrm{S}\sin.\alpha = \gamma\sin.(\alpha+\zeta)\\ &+\gamma\sin.(\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)+(\gamma-4\beta)\sin.(5\alpha-\zeta)+\cdots\\ &-(\gamma-2\beta)\sin.(\alpha-\zeta)-(\gamma-4\beta)\sin.(3\alpha-\zeta)-(\gamma-6\beta)\sin.(5\alpha-\zeta)-\cdots\\ &+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta),\end{aligned}$$

where besides the first and the last terms all the remaining can be contracted, thus so that there may be produced

$$\begin{aligned}&2\mathrm{S}\sin.\alpha = \gamma\sin.(\alpha+\zeta)+2\beta\sin.(\alpha-\zeta)+2\beta\sin.(3\alpha-\zeta)+2\beta\sin.(5\alpha-\zeta)\\ &+\cdots+2\beta\sin.((2k-3)\alpha-\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta).\end{aligned}$$

17. Now for this series requiring to be summed we may put again

$$T = 2\sin.(\alpha-\zeta)+2\sin.(3\alpha-\zeta)+2\sin.(5\alpha-\zeta)+\cdots+2\sin.((2k-3)\alpha-\zeta),$$

so that we may have

$$2S\sin.\alpha = \gamma\sin.(\alpha+\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta)+\beta T.$$

Now we will multiply as thus far by $\sin.\alpha$, and since there shall be

$$2\sin.\alpha\sin.\varphi = \cos.(\varphi-\alpha)-\cos.(\varphi+\alpha),$$

with this reduction made we obtain

$$\begin{aligned}&\mathrm{T}\sin.\alpha = +\cos.\zeta\\ &+\cos.(2\alpha-\zeta)+\cos.(4\alpha-\zeta)+\cdots+\cos.(2(k-2)\alpha-\zeta)\\ &-\cos.(2\alpha-\zeta)-\cos.(4\alpha-\zeta)-\cdots-\cos.(2(k-2)\alpha-\zeta)\\ &-\cos.(2(k-1)\alpha-\zeta),\end{aligned}$$

from which with the terms deleted, which cancel each other mutually, only this same expression will remain

$$T\sin.\alpha = \cos.\zeta-\cos.(2(k-1)\alpha-\zeta).$$

Therefore since there shall be $\alpha=\frac{m\pi}{k}$, there will be $2(k-1)\alpha=2m\pi-\frac{2m\pi}{k}$, in place of which there may be written $-\frac{2m\pi}{k}$, from which on account of $\zeta=\frac{\theta(k-m)}{k}$, there will be

$$T\sin.\alpha=\cos.\frac{\theta(k-m)}{k}-\cos.\frac{2m\pi+\theta(k-m)}{k}.$$

18. Now truly there may be observed in general to be

$$\cos.p-\cos.q=2\sin.\tfrac{q+p}{2}\sin.\tfrac{q-p}{2};$$

whereby since there shall be

$$p=\frac{\theta(k-m)}{k} \text{ and } q=\frac{2m\pi+\theta(k-m)}{k},$$

there will be

$$\frac{q+p}{2}=\frac{m\pi+\theta(k-m)}{k} \text{ et } \frac{q-p}{2}=\frac{m\pi}{k},$$

from which there follows to become

$$T\sin.\alpha=2\sin.\frac{m\pi+\theta(k-m)}{k}\sin.\frac{m\pi}{k}$$

and thus

$$T=2\sin.\frac{m\pi+\theta(k-m)}{k}$$

on account of $\alpha=\frac{m\pi}{k}$.

19. Therefore with this value T found again we will find

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta)$$
$$+2\beta\sin.\frac{m\pi+\theta(k-m)}{k},$$

which on account of $\frac{m\pi+\theta(k-m)}{k}=\alpha+\zeta$, is reduced to this form

$$2S\sin.\alpha=(\gamma+2\beta)\sin.(\alpha+\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta),$$

which can be represented thus :

$$2S\sin.\alpha=(\gamma+2\beta)(\sin.(\alpha+\zeta)+\sin.((2k-1)\alpha-\zeta))-2k\beta\sin.((2k-1)\alpha-\zeta),$$

where for the first part on account of

$$\sin.p+\sin.q=2\sin.\tfrac{p+q}{2}\cos.\tfrac{p-q}{2}$$

there will be

$$\tfrac{p+q}{2}=\alpha k \text{ and } \tfrac{p-q}{2}=(k-1)\alpha-\zeta,$$

from which that same first part becomes

$$2(\gamma+2\beta)\sin.\alpha k\cos.((k-1)\alpha-\zeta);$$

where since there shall be $\alpha k=m\pi$, there will become $\sin.\alpha k=0$, thus so that there shall remain only

$$2S\sin.\alpha=-2\beta k\sin.((2k-1)\alpha-\zeta),$$

and hence

$$S=-\tfrac{\beta k\sin.((2k-1)\alpha-\zeta)}{\sin.\alpha}$$

Truly there is

$$(2k-1)\alpha-\zeta=2m\pi-\tfrac{m\pi}{k}-\tfrac{\theta(k-m)}{k};$$

therefore with the term $2m\pi$ omitted there will be :

$$S=+\frac{\pi\sin.\frac{m\pi+\theta(k-m)}{k}}{\sin.\frac{m\pi}{k}}$$

and thus the value sought will be

$$\frac{S}{k\sin.\theta}=+\frac{\pi\sin.\frac{m\pi+\theta(k-m)}{k}}{k\sin.\theta\sin.\frac{m\pi}{k}},$$

which form is reduced to this

$$\frac{\pi\sin.\frac{m(\pi-\theta)+k\theta}{k}}{k\sin.\theta\sin.\frac{m\pi}{k}}.$$

20. We will consider here before everything the case, where $\theta=\frac{\pi}{2}$, and the integral formula proposed will go into this :

$$\int\tfrac{x^{m-1}dx}{1+x^{2k}},$$

the value of which therefore, if there may be put $x=\infty$ after the integration, will emerge

$$=\frac{\pi\sin.(\frac{\pi}{2}+\frac{m\pi}{2k})}{k\sin.\frac{m\pi}{k}}=\frac{\pi\cos.\frac{m\pi}{2k}}{k\sin.\frac{m\pi}{k}}.$$

Therefore because there is $\sin.\frac{m\pi}{k}=2\sin.\frac{m\pi}{k}\cos.\frac{m\pi}{k}$, this value itself will be produced

$$=\frac{\pi}{2k\sin.\frac{m\pi}{2k}},$$

which value agrees especially with that, which thus not long ago we have assigned for the formula $\int\frac{x^{m-1}dx}{1+x^{2k}}$, if indeed there may be written $2k$ in place of k.

21. Also we may set out the case, where $\theta=\pi$, and our integral formula will change into this

$$\int\frac{x^{m-1}dx}{(1+x^{2k})^2},$$

therefore the value of which on making $x=\infty$ will be

$$\frac{\pi\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{k\sin.\theta\sin.\frac{m\pi}{k}}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\sin.\theta}.$$

But in the case $\theta=\pi$ of this latter fraction both the numerator as well as the denominator vanish ; whereby so that its true value may be elicited, in place of each we may write its differential, with which done this same fraction will be changed into this

$$\frac{d\theta\left(1-\frac{m}{k}\right)\cos.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{d\theta\cos.\theta},$$

the value of which on making $\theta=\pi$ now clearly is $1-\frac{m}{k}$; and thus the value of the integral sought will be $(1-\frac{m}{k})\frac{\pi}{k\sin.\frac{m\pi}{k}}$, precisely as we have found in the above dissertation.

22. But so that we may render the general value found more conveniently, we may put $\pi-\theta=\eta$ and there will become $\sin.\theta=\sin.\eta$ et $\cos.\theta=-\cos.\eta$; then truly the angle will be

$$\frac{m(\pi-\theta)}{k}+\theta=\frac{m\eta}{k}+\pi-\eta,$$

of which the sine is $\sin.(1-\frac{m}{k})\eta$, from which the value sought of our formula will be

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\eta\sin.\frac{m\pi}{k}},$$

and hence finally we have arrived at the following

THEOREM

23. *If this integral formula*

$$\int \frac{x^{m-1}dx}{1+2x^k \cos.\eta + x^{2k}}$$

may be extended from the term $x=0$ *as far as to* $x=\infty$, *its value will be*

$$=\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}},$$

or if since there shall bet

$$\sin.(1-\tfrac{m}{k})\eta = \sin.\eta \cos.\tfrac{m\eta}{k} - \cos.\eta \sin.\tfrac{m\eta}{k},$$

this same value can also be expressed in this manner

$$\frac{\pi \cos.\frac{m\eta}{k}}{k \sin.\frac{m\pi}{k}} - \frac{\pi \sin.\frac{m\eta}{k}}{k tang.\eta \sin.\frac{m\pi}{k}}.$$

24. We will now consider another way this integral formula

$$\int \frac{x^{m-1}dx}{1+2x^k \cos.\eta + x^{2k}},$$

of which the value from the limit $x=0$ as far as to $x=1$ may be put $=P$, truly the value of the same from $x=1$ as far as to $x=\infty$ may be put $=Q$, thus so that $P+Q$ must show th value itself found before. Now truly for finding the value Q we may put $x=\frac{1}{y}$ and our formula represented thus

$$\frac{x^m}{1+2x^k \cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

on account $\frac{dx}{x} = -\frac{dy}{y}$ will adopt this form

$$-\int \frac{y^{-m}dx}{1+2y^{-k}\cos.\eta + y^{-2k}} \cdot \frac{dy}{y} = -\int \frac{y^{2k-m-1}dy}{y^{2k}+2y^k \cos.\eta + 1},$$

the value of which must be extended from the limit $y=1$ as far as to $y=0$. Therefore with these limits interchanged we will have

$$Q = +\int \frac{y^{2k-m-1}dy}{y^{2k}+2y^k \cos.\eta + 1}$$

from the limit $y=0$ as far as to $y=1$.

25. Since the same condition of integration is prescribed in each form for P and Q, from the term 0 as far as to 1, nothing stands in the way, why in the latter we may not write y in place of x, from which we will have this integral form for $P+Q$:

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}\,dx,$$

the value of which extended from the limit $x=0$ as far as to $x=1$ will be equal to this expression

$$\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\frac{m\pi}{k}}.$$

Therefore with these two integral formulas compared we will obtain the following most noteworthy theorem.

THEOREM

26. *This integral formula*

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}\,dx,$$

extended from the limit $x=0$ *as far as to the limit* $x=1$ *is equal to this integral formula*

$$\int \frac{x^{m-1}dx}{1+2x^{k}\cos.\eta+x^{2k}}$$

extended from the limit $x=0$ *as far as to the limit* $x=\infty$*; indeed the value of each will be*

$$\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\eta\sin.\frac{m\pi}{k}}.$$

27. So that if we may expand out this fraction $\frac{\sin.\eta}{1+2x^{k}\cos.\eta+x^{2k}}$ into an infinite series, which shall be

$$\sin.\eta+Ax^{k}+Bx^{2k}+Cx^{3k}+Dx^{4k}+Ex^{5k}+\text{etc.},$$

on being multiplied by the denominator we will come upon this infinite expression :

$$\begin{array}{lllllll}
\sin.\eta = \sin.\eta + & Ax^{k} + & Bx^{2k} + & Cx^{3k} + & Dx^{4k} + & Ex^{5k} & + \text{etc.}, \\
+2\sin.\eta\cos.\eta & +2A\cos.\eta & +2B\cos.\eta & +2C\cos.\eta & +2D\cos.\eta & & + \text{etc.} \\
 & + \sin.\eta & + A & + B & + C & & + \text{etc.}
\end{array}$$

from which we will find with the individual terms reduced to zero

$$\begin{array}{lll}
1.\ A + 2\sin.\eta\cos.\eta = 0 & \text{hincque} & A = -\sin.2\eta, \\
2.\ B. + 2A\cos.\eta + \sin.\eta = 0, & \text{unde fit} & B = \sin.3\eta, \\
3.\ C + 2B\cos.\eta + A = 0, & \text{unde fit} & C = -\sin 4\eta, \\
4.\ D + 2C\cos.\eta + B = 0, & \text{unde fit} & D = \sin.5\eta \\
\text{etc.} & & \text{etc.},
\end{array}$$

thus so that our fraction $\frac{\sin.\eta}{1+2x^{k}\cos.\eta + x^{2k}}$ may be resolved into this series

$$\sin.\eta - x^{k}\sin.2\eta + x^{2k}\sin.3\eta - x^{3k}\sin.4\eta + x^{4k}\sin.5\eta - \text{etc.}$$

28. Now we may multiply this series by

$$x^{m-1} + x^{2k-m-1}$$

and after integration we may make $x = 1$, so that we may obtain the value of this formula

$$\sin.\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}dx$$

for the case $x = 1$, and in this manner we will arrive at the two following series

$$\frac{\sin.\eta}{m} - \frac{\sin.2\eta}{m+k} + \frac{\sin.3\eta}{m+2k} - \frac{\sin.4\eta}{m+3k} + \frac{\sin.5\eta}{m+4k} - \text{etc.},$$
$$\frac{\sin.\eta}{2k-m} - \frac{\sin.2\eta}{3k-m} + \frac{\sin.3\eta}{4k-m} - \frac{\sin.4\eta}{5k-m} + \frac{\sin.5\eta}{6k-m} - \text{etc.}$$

Therefore the sum of these two series taken together will be equal to this value

$$\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\frac{m\pi}{k}},$$

from which we may adjoin besides this theorem.

THEOREM

29. *If η may denote some angle, truly the letters m and k may be taken as it pleases, and from these the two following series may be formed :*

$$P=\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$
$$Q=\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.},$$

indeed the sum of neither can be shown, but the sum of each taken together will be

$$P+Q=\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\frac{m\pi}{k}}.$$

COROLLARY

30. But if therefore we may take the angle η infinitely small, so that there may become

$$\sin.\eta=\eta,\ \sin.2\eta=2\eta,\ \sin.3\eta=3\eta\ \text{etc.},$$

because in the formula of the sum there will become

$$\sin.(1-\tfrac{m}{k})\eta=(1-\tfrac{m}{k})\eta,$$

if we may divide both sides by η , we will obtain the following double series

$$\frac{1}{m}-\frac{2}{m+k}+\frac{3}{m+2k}-\frac{4}{m+3k}+\frac{5}{m+4k}-\text{etc.},$$
$$\frac{1}{2k-m}-\frac{2}{3k-m}+\frac{3}{4k-m}-\frac{4}{5k-m}+\frac{5}{6k-m}-\text{etc.}$$

the sum of which will be $(1-\frac{m}{k})\frac{\pi}{k\sin.\frac{m\pi}{k}}$; where it will be observed both these series not inconsistently can be contracted into this simple series

$$\frac{2k}{m(2k-m)}-\frac{8k}{(m+k)(3k-m)}+\frac{18k}{(m+2k)(4k-m)}-\frac{32k}{(m+3k)(5k-m)}+\text{etc.},$$

where the numerators are the squared numbers doubled.

31. But the formulas, the values of which we have just found, can be expressed much more concisely and elegantly, if we may write $k-n$ in place of the exponent *m ;* for then in the value of the integral found there will become $(1-\frac{m}{k})\eta=\frac{n\eta\pi}{k}$, but truly for the

denominator there will become $\frac{m\pi}{k}=\pi-\frac{n\pi}{k}$, the sine of which will be $\sin.\frac{n\pi}{k}$; and thus our formula found will adopt this form $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}$, which therefore will express the value of this integral formula

$$\int\frac{x^{k-n-1}dx}{1+2x^k\cos.\eta+x^{2k}}$$

from $x=0$ as far as to $x=\infty$, and as well of this formula

$$\int\frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k\cos.\eta+x^{2k}}dx$$

from the term $x=0$ as far as to the term $x=1$; and because the value of each is $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}$, it is evident that remains the same, even if there may be written $-n$ in place of n, from which the first formula will be able to be represented thus

$$\int\frac{x^{k\pm n-1}dx}{1+2x^k\cos.\eta+x^{2k}};$$

but the second formula on account of this ambiguity plainly is allowed no change.

32. Also by putting $m=k-n$ our double series may take a prettier form; indeed there will be had

$$\begin{aligned}&\frac{\sin.\eta}{k-n}-\frac{\sin.2\eta}{2k-n}+\frac{\sin.3\eta}{3k-n}-\frac{\sin.4\eta}{4k-n}+\text{etc.}\\+&\frac{\sin.\eta}{k+n}-\frac{\sin.2\eta}{2k+n}+\frac{\sin.3\eta}{3k+n}-\frac{\sin.4\eta}{4k+n}+\text{etc.},\end{aligned}$$

therefore the sum of which will be $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}}$. Then truly if these double series may be contracted into one and each may be divided by $2k$, the following noteworthy summation will be obtained

$$\frac{\pi\sin.\frac{n\eta}{k}}{2kk\sin.\frac{n\pi}{k}}=\frac{\sin.\eta}{kk-nn}-\frac{2\sin.2\eta}{4kk-nn}+\frac{3\sin.3\eta}{9kk-nn}-\frac{4\sin.4\eta}{16kk-nn}+\text{etc.}$$

33. But if this latter series may be differentiated by assuming only the angle η variable, on account of $d\ \sin.\frac{n\eta}{k}=\frac{nd\eta}{k}\cos.\frac{n\eta}{k}$ we will have

$$\frac{\pi n\cos.\frac{n\eta}{k}}{2k^3\sin.\frac{n\pi}{k}}=\frac{\cos.\eta}{kk-nn}-\frac{4\cos.2\eta}{4kk-nn}+\frac{9\cos.3\eta}{9kk-nn}-\frac{16\cos.4\eta}{16kk-nn}+\text{etc.}$$

From which if there may be taken $\eta = 0$, this summation thus will arise

$$\frac{\pi n}{2k^3 \sin.\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}$$

but if there may be taken $\eta = 90° = \frac{\pi}{2}$, there will be

$$\cos.\eta = 0,\ \cos.2\eta = -1,\ \cos.3\eta = 0,\ \cos.4\eta = +1 \text{ etc.},$$

from which the following series arises

$$\frac{n\pi \cos.\frac{n\pi}{2k}}{2k^3 \sin.\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

But since $\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, the sum of the same series will be $\frac{n\pi}{4k^3 \sin.\frac{n\pi}{2k}}$.

34. But if that series shown in § 32 may be multiplied by $d\eta$ and integrated, on account of $\int d\eta \sin.\frac{n\eta}{k} = -\frac{n}{k}\cos.\frac{n\eta}{k}$ there will become

$$C - \frac{\pi \cos.\frac{n\eta}{k}}{2nk \sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} - \text{etc.}$$

But so that here we may define the constant *C* being added, we may take $\eta = 0$ and there will become

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.};$$

whereby if the sum of this series may be known from elsewhere, the constant *C* will be able to be defined. But this doubled series is able to be resolved in the following twofold manner

$$2nC - \frac{\pi}{k \sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$

35. Therefore since in the *Introductione in Analysin lnfinitorum* pag. 142 I may have come upon the series

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.} = \frac{\pi}{2kn \sin.\frac{n\pi}{k}} - \frac{1}{2nn}$$

(here evidently in place of the letters *m* and *n* being used there to have written *n* and *k*), with this value used our equation will be

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk \sin.\frac{n\pi}{k}},$$

from which there becomes $C = \frac{1}{2nn}$. Hence therefore we will have that same summation

$$\frac{\pi \cos.\frac{n\eta}{k}}{2nk \sin.\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos.\eta}{kk-nn} - \frac{\cos.2\eta}{4kk-nn} + \frac{\cos.3\eta}{9kk-nn} - \frac{\cos.4\eta}{16kk-nn} + \text{etc.},$$

which series certainly may be considered worthy of every attention.

INVESTIGATIO VALORIS INTEGRALIS

$$\int \frac{x^{m-1}dx}{1-2x^{k}\cos.\theta+x^{2k}}$$

A TERMINO $x=0$ USQUE AD $x=\infty$ EXTENSI

Commentatio 589 indicis ENESTROEMIANI
Opuscula analytica. 2, 1785, p. 55-75

1. Quaeramus primo integrale formulae propositae indefinitum atque adeo omnes operationes ex primis Analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolvi nequit, sit in genere eius factor duplicatus quicunque $1-2x\cos.\omega+xx$; evidens enim est denominatorem fore productum ex k huiusmodi factoribus duplicatis. Cum igitur posito hoc factore $=0$ fiat $x=\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega$, etiam ipse denominator duplici modo evanescere debebit, sive si ponatur

$$x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega \text{ sive } x=\cos.\omega-\sqrt{-1}\cdot\sin.\omega .$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, ut sit

$$(\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega)^{\lambda}=\cos.\lambda\omega\pm\sqrt{-1}\cdot\sin.\lambda\omega;$$

hinc igitur erit

$$x^{k}=\cos.k\omega\pm\sqrt{-1}\cdot\sin.k\omega \text{ et } x^{2k}=\cos.2k\omega\pm\sqrt{-1}\cdot\sin.2k\omega$$

Substituamus ergo hos valores et denominator noster evadet

$$1-2\cos.\theta\cos.k\omega+\cos.2k\omega\mp2\sqrt{-1}\cdot\cos.\theta\sin.k\omega\pm\sqrt{-1}\cdot\sin.2k\omega.$$

2. Perspicuum igitur est huius aequationis tam terminos reales quam imaginarios seorsim se mutuo tollere debere, unde nascuntur hae duae aequationes

$$\text{I. } 1-2\cos.\theta\cos.k\omega+\cos.2k\omega=0,$$
$$\text{II. } -2\cos.\theta\sin.k\omega+\sin.2k\omega=0.$$

Cum igitur sit

$$\sin.2k\omega=2\sin.k\omega\cos.k\omega,$$

posterior aequatio induet hanc formam

$$-2\cos.\theta\sin.k\omega+2\sin.k\omega\cos.k\omega=0,$$

quae per $2\sin.k\omega$ divisa dat

$$\cos.k\omega = \cos.\theta$$

ideoque

$$\cos.2k\omega = \cos.2\theta = \cos.^2\theta - \sin.^2\theta = 2\cos.^2\theta - 1,$$

qui valores in aequatione priore substituti praebent aequationem identicam, ita ut utrique aequationi satisfiat sumendo $\cos.k\omega = \cos.\theta$.

3. Pro ω igitur eiusmodi angulum assumi oportet, ut fiat $\cos.k\omega = \cos.\theta$, unde quidem statim deducitur $k\omega = \theta$ ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem cosinum habentes, qui praeter ipsum angulum θ sunt $2\pi \pm \theta,\ 4\pi \pm \theta,\ 6\pi \pm \theta$etc. atque adeo in genere $2i\pi \pm \theta$ denotante i omnes numeros integros, quaesito nostro satisfiet faciendo $k\omega = 2i\pi \pm \theta$, unde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$ sicque pro ω nancisceremur numerabiles angulos satisfacientes, quorum autem sufficiet tot assumsisse, quot exponens k continet unitates; successive igitur angulo ω sequentes tribuamus valores

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k}, \frac{4\pi+\theta}{k}, \frac{6\pi+\theta}{k}, \frac{8\pi+\theta}{k}, \cdots \frac{2(k-1)\pi+\theta}{k}.$$

Quodsi ergo angulo ω successive singulos istos valores, quorum numerus est $= k$, tribuamus, formula $1 - 2x\cos.\omega + xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1 - 2x^k\cos.\theta + x^{2k}$, quorum numerus erit $= k$.

4. Inventis iam omnibus factoribus duplicatis nostri denominatoris $\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}$ fractio resolvi debet in tot fractiones partiales, quarum denominatores sint ipsi isti factores duplicati, quorum numerus est k, ita ut in genera talis fractio partialis habitura sit talem formam

$$\frac{A+Bx}{1-2x\cos.\omega+xx}$$

quam insuper resolvamus in binas simplices etsi imaginarias, et cum sit

$$xx - 2x\cos.\omega + 1 = (x - \cos.\omega + \sqrt{-1}\cdot\sin.\omega)(x - \cos.\omega - \sqrt{-1}\cdot\sin.\omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega} + \frac{g}{x-\cos.\omega+\sqrt{-1}\cdot\sin.\omega},$$

ita ut totum resolutionis negotium huc redeat, ut ambo numeratores f et g determinentur; iis enim inventis habebitur summa ambarum fractionum

$$\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}\cdot(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

ubi igitur erit

$$B = f + g \text{ et } A = (f - g)\sqrt{-1} \cdot \sin.\omega - (f + g)\cos.\omega.$$

5. Per methodum igitur fractiones quascunque in fractiones simplices resolvendi statuamus

$$\frac{x^{m-1}}{1-2x^k\cos.\omega+x^{2k}} = \frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega} + R,$$

ubi R complectatur omnes reliquas fractiones partiales. Hinc per

$$x - \cos.\omega - \sqrt{-1} \cdot \sin.\omega$$

multiplicando habebitur

$$\frac{x^m - x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{1-2x^k\cos.\theta+x^{2k}} = f + R(x - \cos.\omega - \sqrt{-1} \cdot \sin.\omega);$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x = \cos.\omega + \sqrt{-1} \cdot \sin.\omega$, ut membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x - \cos.\omega - \sqrt{-1} \cdot \sin.\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita ut hinc nihil concludi posse videatur.

6. Hic igitur utamur regula notissima et loco tam numeratoris quam denominatoris eorum differentialia scribamus, unde nostra aequatio accipiet sequentem formam

$$\frac{mx^{m-1}-(m-1)x^{m-2}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^{k-1}\cos.\theta+2kx^{2k-1}}$$
$$= \frac{mx^m-(m-1)x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^k\cos.\theta+2kx^{2k}} = f,$$

posito scilicet $x = \cos.\omega + \sqrt{-1} \cdot \sin.\omega$. Tum autem erit

$$x^m = \cos.m\omega + \sqrt{-1} \cdot \sin.m\omega$$

et

$$x^{m-1}(\cos.\omega + \sqrt{-1} \cdot \sin.\omega) = x^m = \cos.m\omega + \sqrt{-1} \cdot \sin.m\omega$$

et pro denominatore

$$x^k = \cos.k\omega + \sqrt{-1} \cdot \sin.k\omega \text{ et } x^{2k} = \cos.2k\omega + \sqrt{-1} \cdot \sin.2k\omega;$$

unde fit numerator

$$x^m = \cos.m\omega + \sqrt{-1} \cdot \sin.m\omega$$

et denominator

$$-2k\cos.\theta\cos.k\omega+2k\cos.2k\omega-2k\sqrt{-1}\cdot\cos.\theta\sin.k\omega+2k\sqrt{-1}\cdot\sin.2k\omega.$$

7. Pro denominatore reducendo recordemur iam supra inventum esse $\cos.k\omega=\cos.\theta$, unde fit $\sin.k\omega=-\sin.\theta$, tum vero

$$\cos.2k\omega=\cos.2\theta=2\cos.^{2}\theta-1 \text{ et } \sin.2k\omega=2\sin.\theta\cos.\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\cos.^{2}\theta-2k+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta=-2k\sin.^{2}\theta+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta$$
$$=-2k\sin.\theta(\sin.\theta-\sqrt{-1}\cdot\cos.\theta),$$

quamobrem hoc valore adhibito habebimus

$$f=\frac{\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega}{2k\sin.\theta(\sqrt{-1}\cdot\cos.\theta-\sin.\theta)}.$$

Simul vero hinc sine novo calculo deducemus valorem *g*, quippe qui ab *f* ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g=\frac{\cos.m\omega-\sqrt{-1}\cdot\sin.m\omega}{-2k\sin.\theta(\sin.\theta+\sqrt{-1}\cdot\cos.\theta)}.$$

8. Inventis autem his litteris *f* et *g* pro litteris *A* et *B* colligemus primo

$$f+g=\frac{\cos.\theta\sin.m\omega-\sin.\theta\cos.m\omega}{k\sin\theta}=\frac{\sin.(m\omega-\theta)}{k\sin\theta},$$

tum vero erit

$$f-g=-\frac{\sqrt{-1}\cdot\cos.(m\omega-\theta)}{k\sin\theta}.$$

Ex his igitur reperiemus

$$B=\frac{\sin.(m\omega-\theta)}{k\sin\theta}$$

et

$$A=\frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta}=-\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta},$$

ubi ergo imaginaria sponte se mutuo destruxerunt.

9. Inventis his valoribus *A* et *B* investigari oportet integrale partiale

$$\int\frac{(A+Bx)dx}{1-2x\cos.\omega+xx},$$

ubi, cum denominatoris differentiale sit

$$2xdx-2dx\cos.\omega=2dx(x-\cos.\omega),$$

statuamus

$$A+Bx=B(x-cos.\omega)+C$$

eritque $C=A+B\cos.\omega$; hinc igitur erit

$$C=\frac{\cos.\omega\sin.(m\omega-\theta)-\sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Quia vero $-\sin.((m\omega-\theta)-\omega)=-\sin.(m\omega-\theta)cos.\omega+cos.(m\omega-\theta)\sin.\omega,$
erit

$$C=\frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Hac ergo forma adhibita formula integranda $\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ discerpatur in has duas partes

$$\frac{B(x-\cos.\omega)dx}{1-2x\cos.\omega+xx}+\frac{Cdx}{1-2x\cos.\omega+xx}.$$

Hic igitur prioris partis integrale manifesto est

$$Bl\sqrt{(1-2x\cos.\omega+xx)},$$

alterius vero partis facile patet integrale per arcum circuli expressum iri,
cuius tangens sit $\frac{x\sin.\omega}{1-x\cos.\omega}$. Ad hoc integrale inveniendum ponamus

$$\int\frac{Cdx}{1-2x\cos.\omega+xx}=D\ \text{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

et sumtis differentialibus, quia $d.\text{Atang.}t$ aequale est $\frac{dt}{1+tt}$ habebimus,

$$\frac{Cdx}{1-2x\cos.\omega+xx}=D\frac{dx\sin.\omega}{1-2x\cos.\omega+xx},$$

unde manifesto fit

$$D=\frac{C}{\sin.\omega}=\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

10. Substituamus igitur loco B et D valores modo inventos et ex singulis factoribus denominatoris $1-2x^k\cos.\theta+x^{2k}$, quorum forma est $1-2x\cos.\omega+x^2$, oritur pars integralis constans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}l\sqrt{(1-2x\cos.\omega+xx)}+\frac{\cos.(m\omega-\theta)}{k\sin.\theta}\text{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

quae evanescit sumto $x=0$. In hac igitur forma tantum opus est, ut loco ω successive scribamus valores supra indicatos, scilicet

$$\omega=\frac{\theta}{k},\quad \frac{2\pi+\theta}{k},\quad \frac{4\pi+\theta}{k},\quad \frac{6\pi+\theta}{k}\ \text{etc.},$$

donec perveniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim summa omnium harum forumarum praebebit totum integrale indefinitum formulae propositae.

11. Postquam igitur integrale indefinitum elicuimus, nihil aliud superest, nisi ut in eo faciamus $x=\infty$, quo facto pars logarithmica ob

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega$$

erit $Bl(x-\cos.\omega)$. Est vero

$$l(x-\cos.\omega)=lx-\frac{\cos.\omega}{x}=lx$$

ob $\frac{\cos.\omega}{x}=0$; quamobrem facto $x=\infty$ quaelibet pars logarithmica habebit hanc formam $\frac{\sin.(m\omega-\theta)}{k\sin.\theta}lx$. Deinde pro partibus a circulo pendentibus facto $x=\infty$ fit

$$\frac{x\sin.\omega}{1-x\cos.\omega}=-\text{tang.}\omega=\text{tang.}(\pi-\omega)$$

sicque arcus, cuius haec est tangens, erit $=\pi-\omega$ hincque pars circularis quaecunque fiet $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$.

12. Cum quilibet valor anguli ω in genere hanc habeat formam

$\frac{2i\pi+\theta}{k}$, erit angulus

$$m\omega-\theta=\frac{2im\pi-\theta(k-m)}{k}\quad\text{et}\quad \pi-\omega=\frac{\pi(k-2i)-\theta}{k}.$$

Ponamus brevitatis gratia

$$\frac{\theta(k-m)}{k}=\zeta\quad\text{et}\quad\frac{m\pi}{k}=\alpha,$$

ut sit

$$m\omega-\theta=2i\alpha-\zeta\ ,$$

ubi loco *i* scribi debent successive numeri 0, 1, 2, 3 etc. usque ad $k-1$. Hinc igitur, si omnes partes logarithmicas in unam summam colligamus, ea ita repraesentari poterit

$$\frac{lx}{k\sin.\theta}\left\{\begin{matrix}-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\sin.(6\alpha-\zeta)\\ +\sin.(8\alpha-\zeta)+\cdots+\sin.(2(k-1)\alpha-\zeta)\end{matrix}\right\};$$

ubi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

13. Ad hoc ostendendum ponamus

$$S=-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\cdots+\sin.(2(k-1)\alpha-\zeta);$$

multiplicemus utrinque per $2\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\varphi=\cos.(\alpha-\varphi)-\cos.(\alpha+\varphi),$$

huius reductionis ope obtinebimus sequentem expressionem

$$\begin{aligned}2S\sin.\alpha=\cos.(\alpha+\zeta)\\ -\cos.(\alpha-\zeta)-\cos.(3\alpha-\zeta)-\cos.(5\alpha-\zeta)-\cdots\\ +\cos.(\alpha-\zeta)+\cos.(3\alpha-\zeta)+(5\alpha-\zeta)+\cdots\\ -\cos.((2k-1)\alpha-\zeta),\end{aligned}$$

unde deletis terminis se mutuo destruentibus habebitur

$$2S\sin.\alpha=\cos.(\alpha+\zeta)-\cos.((2k-1)\alpha-\zeta),$$

14. Ponamus hos duos angulos, qui sunt relicti,

$$\alpha+\zeta=p \text{ et } (2k-1)\alpha-\zeta=q$$

eritque eorum summa $p+q=2\alpha k$. Quia porro est $\alpha=\frac{m\pi}{k}$, erit $p+q=2m\pi$, hoc est multiplo totius circuli peripheriae ob *m* numerum integrum. Quare cum sit $q=2m\pi-p$, erit $\cos.q=\cos.p$; unde patet summam inventam nihilo esse aequalem sicque manifestum est omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x=\infty$ se mutuo destruere.

15. Progrediamur igitur ad partes circulares, quarum forma generalis, ut vidimus, est $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$, quae posito $\alpha=\frac{m\pi}{k}$ et $\zeta=\frac{\theta(k-m)}{k}$, fit

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi+\theta}{k}\right)=\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi}{k}+\frac{\theta}{k}\right).$$

Hic ponatur porro $\frac{\pi}{k}=\beta$ et $\pi-\frac{\theta}{k}=\gamma$, ut forma generalis sit

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma-2i\beta).$$

Quare si loco *i* scribamus ordine valores 0, 1, 2, 3, 4 usque ad $k-1$, omnes partes circulares hanc progressionem constituent

$$\frac{1}{k\sin.\theta}(\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)+\cdots$$
$$+(\gamma-2(k-1)\beta)\cos.(2(k-1)\alpha-\zeta)).$$

Ponamus igitur

$$S=\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)+\cdots$$
$$+(\gamma-2(k-1)\beta)\cos.(2(k-1)\alpha-\zeta),$$

ut summa omnium partium circularium sit $\frac{S}{k\sin.\theta}$, quae ergo praebebit valorem quaesitum formulae integralis propositae casu, quo post integrationem statuitur $x=\infty$, ita ut totum negotium in investigando valore ipsius *S* versetur.

16. Hunc in finem multiplicemus utrinque per $2\sin.\alpha$, et cum in genere sit

$$2\sin.\alpha\cos.\varphi=\sin.(\alpha+\varphi)-\sin.(\varphi-\alpha),$$

hac reductione in singulis terminis facta perveniemus ad hanc aequationem

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)$$
$$+\gamma\sin.(\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)+(\gamma-4\beta)\sin.(5\alpha-\zeta)+\cdots$$
$$-(\gamma-2\beta)\sin.(\alpha-\zeta)-(\gamma-4\beta)\sin.(3\alpha-\zeta)-(\gamma-6\beta)\sin.(5\alpha-\zeta)-\cdots$$
$$+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta),$$

ubi praeter primum et ultimum terminum omnes reliqui contrahi possunt, ita ut prodeat

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)+2\beta\sin.(\alpha-\zeta)+2\beta\sin.(3\alpha-\zeta)+2\beta\sin.(5\alpha-\zeta)+\cdots$$
$$+2\beta\sin.((2k-3)\alpha-\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta).$$

17. Iam pro hac serie summanda ponamus porro

$$T=2\sin.(\alpha-\zeta)+2\sin.(3\alpha-\zeta)+2\sin.(5\alpha-\zeta)+\cdots+2\sin.((2k-3)\alpha-\zeta),$$

ut habeamus

$$2S\sin.\alpha = \gamma\sin.\left(\alpha+\zeta\right)+(\gamma-2(k-1)\beta)\sin.(\left(2k-1\right)\alpha-\zeta)+\beta T.$$

Iam multiplicemus ut hactenus per $\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\varphi = \cos.(\varphi-\alpha)-\cos.(\varphi+\alpha),$$

facta hac reductione nanciscimur

$$\begin{aligned}
\mathrm{T}\sin.\alpha = &+\cos.\zeta \\
&+\cos.(2\alpha-\zeta)+\cos.(4\alpha-\zeta)+\cdots+\cos.(2\left(k-2\right)\alpha-\zeta) \\
&-\cos.(2\alpha-\zeta)-\cos.(4\alpha-\zeta)-\cdots-\cos.(2\left(k-2\right)\alpha-\zeta) \\
&\qquad -\cos.(2\left(k-1\right)\alpha-\zeta),
\end{aligned}$$

unde deletis terminis, quae se mutuo destruunt, remanebit tantum ista expressio

$$T\sin.\alpha = \cos.\zeta-\cos.(2(k-1)\alpha-\zeta).$$

Cum igitur sit $\alpha=\frac{m\pi}{k}$, erit $2(k-1)\alpha = 2m\pi-\frac{2m\pi}{k}$, cuius loco scribere licet $-\frac{2m\pi}{k}$ unde ob $\zeta=\frac{\theta(k-m)}{k}$, erit

$$T\sin.\alpha = \cos.\frac{\theta(k-m)}{k}-\cos.\frac{2m\pi+\theta(k-m)}{k}.$$

18. Nunc vero notetur in genere esse

$$\cos.p-\cos.q = 2\sin.\tfrac{q+p}{2}\sin.\tfrac{q-p}{2};$$

quare cum sit

$$p=\frac{\theta(k-m)}{k} \quad \text{et} \quad q=\frac{2m\pi+\theta(k-m)}{k},$$

erit

$$\frac{q+p}{2}=\frac{m\pi+\theta(k-m)}{k} \quad \text{et} \quad \frac{q-p}{2}=\frac{m\pi}{k},$$

unde sequitur fore

$$T\sin.\alpha = 2\sin.\frac{m\pi+\theta(k-m)}{k}\sin.\frac{m\pi}{k}$$

ideoque

$$T = 2\sin.\frac{m\pi+\theta(k-m)}{k}$$

ob $\alpha=\frac{m\pi}{k}$.

19. Hoc igitur valore T invento reperiemus porro

$$2S\sin.\alpha = \gamma\sin.(\alpha+\zeta) + (\gamma - 2(k-1)\beta)\sin.\ ((2k-1)\alpha - \zeta)$$
$$+\ 2\beta\sin.\tfrac{m\pi+\theta(k-m)}{k},$$

quae ob $\frac{m\pi+\theta(k-m)}{k} = \alpha+\zeta$ reducitur as hanc formam

$$2S\sin.\alpha = (\gamma+2\beta)\sin.(\alpha+\zeta) + (\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta),$$

quae ita repraesentari potest

$$2S\sin.\alpha = (\gamma+2\beta)(\sin.(\alpha+\zeta)+\sin.((2k-1)\alpha-\zeta)) - 2k\beta\sin.((2k-1)\alpha-\zeta),$$

ubi pro parte priore ob

$$\sin.p+\sin.q = 2\sin.\tfrac{p+q}{2}\cos.\tfrac{p-q}{2}$$

erit

$$\tfrac{p+q}{2} = \alpha k \quad \text{et} \quad \tfrac{p-q}{2} = (k-1)\alpha - \zeta,$$

unde pars ipsa prior fit

$$2(\gamma+2\beta)\sin.\alpha k\cos.((k-1)\alpha-\zeta);$$

ubi cum sit $\alpha k = m\pi$, erit $\sin.\alpha k = 0$, ita ut tantum supersit

$$2S\sin.\alpha = -2\beta k\sin.((2k-1)\alpha-\zeta),$$

hincque

$$S = -\frac{\beta k\sin.((2k-1)\alpha-\zeta)}{\sin.\alpha}$$

Est vero

$$(2k-1)\alpha-\zeta = 2m\pi - \tfrac{m\pi}{k} - \tfrac{\theta(k-m)}{k};$$

omisso termino *2mπ* erit igitur

$$S = +\frac{\pi\sin.\frac{m\pi+\theta(k-m)}{k}}{\sin.\frac{m\pi}{k}}$$

ideoque valor quaesitus erit

$$\frac{S}{k\sin.\theta} = +\frac{\pi\sin.\frac{m\pi+\theta(k-m)}{k}}{k\sin.\theta\sin.\frac{m\pi}{k}},$$

quae forma reducitur ad hanc

$$\frac{\pi\sin.\frac{m(\pi-\theta)+k\theta}{k}}{k\sin.\theta\sin.\frac{m\pi}{k}}.$$

20. Contemplemur hic ante omnia casum, quo $\theta=\frac{\pi}{2}$, et formula integralis proposita abit in hanc

$$\int\frac{x^{m-1}dx}{1+x^{2k}},$$

cuius ergo valor, si post integrationem ponatur $x=\infty$, evadet

$$=\frac{\pi\sin.(\frac{\pi}{2}+\frac{m\pi}{2k})}{k\sin.\frac{m\pi}{k}}=\frac{\pi\cos.\frac{m\pi}{2k}}{k\sin.\frac{m\pi}{k}}.$$

Quia igitur est $\sin.\frac{m\pi}{k}=2\sin.\frac{m\pi}{k}\cos.\frac{m\pi}{k}$, prodibit iste valor

$$=\frac{\pi}{2k\sin.\frac{m\pi}{2k}},$$

qui valor egregie convenit cum eo, quem non ita pridem pro formula $\int\frac{x^{m-1}dx}{1+x^{2k}}$ assignavimus, siquidem loco *k* scribatur 2*k*.

21. Evolvamus etiam casum, quo $\theta=\pi$, et formula nostra integralis abit in hanc

$$\int\frac{x^{m-1}dx}{(1+x^{2k})^2},$$

cuius ergo facto $x=\infty$ valor erit

$$\frac{\pi\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{k\sin.\theta\sin.\frac{m\pi}{k}}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\sin.\theta}.$$

Huius autem posterioris fractionis casu $\theta=\pi$ tam numerator quam denominator evanescit; quare ut eius verus valor eruatur, loco utriusque eius differentiale scribamus, quo facto ista fractio abibit in hanc

$$\frac{d\theta\left(1-\frac{m}{k}\right)\cos.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{d\theta\cos.\theta},$$

cuius valor facto $\theta=\pi$ nunc manifesto est $1-\frac{m}{k}$; sicque valor integralis quaesitus erit $(1-\frac{m}{k})\frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti in superiore dissertatione invenimus.

22. Quo autem valorem generalem inventum commodiorem reddamus, ponamus $\pi-\theta=\eta$ fietque $\sin.\theta=\sin.\eta$ et $\cos.\theta=-\cos.\eta$; tum vero erit angulus

$$\frac{m(\pi-\theta)}{k}+\theta=\frac{m\eta}{k}+\pi-\eta\,,$$

cuius sinus est $\sin.(1-\frac{m}{k})\eta$, unde valor quaesitus nostrae formulae erit

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\eta\sin.\frac{m\pi}{k}},$$

atque hinc tandem sequens adepti sumus

THEOREMA

23. *Si , haec formula integralis*

$$\int\frac{x^{m-1}dx}{1+2x^{k}\cos.\eta+x^{2k}}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extendatur, eius valor erit*

$$=\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\eta\sin.\frac{m\pi}{k}},$$

sive cum sit

$$\sin.(1-\tfrac{m}{k})\eta=\sin.\eta\cos.\frac{m\eta}{k}-\cos.\eta\sin.\frac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest

$$=\frac{\pi\cos.\frac{m\eta}{k}}{k\sin.\frac{m\pi}{k}}-\frac{\pi\sin.\frac{m\eta}{k}}{k\,tang.\eta\sin.\frac{m\pi}{k}}.$$

24. Consideremus nunc alio modo hanc formulam integralem

$$\int\frac{x^{m-1}dx}{1+2x^{k}\cos.\eta+x^{2k}},$$

cuius valor a termino $x=0$ usque ad $x=1$ ponatur $=P$, eiusdem vero valor ab $x=1$ usque ad $x=\infty$ ponatur $=Q$, ita ut $P+Q$ exhibere debeat ipsum valorem ante inventum. Nunc vero pro valore Q inveniendo ponamus $x=\frac{1}{y}$ et formula nostra ita repraesentata

$$\frac{x^{m}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{dx}{x}$$

ob $\frac{dx}{x}=-\frac{dy}{y}$ induet hanc formam

$$-\int \frac{y^{-m}dx}{1+2y^{-k}\cos.\eta+y^{-2k}}\cdot\frac{dy}{y}=-\int\frac{y^{2k-m-1}dy}{y^{2k}+2y^{k}\cos.\eta+1},$$

cuius valor a termino $y=1$ usque ad $y=0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q=+\int\frac{y^{2k-m-1}dy}{y^{2k}+2y^{k}\cos.\eta+1}$$

a termino $y=0$ usque ad $y=1$.

25. Quia in utraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 usque ad 1, nihil impedit, quominus in posteriore loco y scribamus x, unde pro $P+Q$ habebimus hanc formam integralem

$$\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}dx,$$

cuius valor a termino $x=0$ usque ad $x=1$ extensus aequabitur huic expressioni

$$\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\frac{m\pi}{k}}.$$

Comparatis igitur his binis formulis integralibus nanciscemur sequens theorema notatu maxime dignum.

THEOREMA

26. *Haec formula integralis*

$$\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}dx,$$

a termino $x=0$ *usque ad terminum* $x=1$ *extensa aequalis est huic formuae integrali*

$$\int\frac{x^{m-1}dx}{1+2x^{k}\cos.\eta+x^{2k}}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extensae; utriusque enim valor erit*

$$\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\eta\sin.\frac{m\pi}{k}}.$$

27. Quodsi hanc fractionem $\frac{\sin.\eta}{1+2x^{k}\cos.\eta+x^{2k}}$ in seriem infinitam evolvamus, quae sit

$$\sin.\eta + Ax^k + Bx^{2k} + Cx^{3k} + Dx^{4k} + Ex^{5k} + \text{etc.},$$

per denominatorem multiplicando perveniemus ad hanc expressionem infinitam

$$\begin{array}{lllllll}
\sin.\eta = \sin.\eta + & Ax^k + & Bx^{2k} + & Cx^{3k} + & Dx^{4k} + & Ex^{5k} & + \text{etc.}, \\
+ 2\sin.\eta\cos.\eta & + 2A\cos.\eta & + 2B\cos.\eta & + 2C\cos.\eta & + 2D\cos.\eta & & + \text{etc.} \\
+ & \sin.\eta & + A & + B & + C & & + \text{etc.}
\end{array}$$

unde singulis terminis ad nihilum reductis reperiemus

$$\begin{array}{llll}
1. & A + 2\sin.\eta\cos.\eta = 0 & \text{hincque} & A = -\sin.2\eta, \\
2. & B. + 2A\cos.\eta + \sin.\eta = 0, & \text{unde fit} & B = \sin.3\eta, \\
3. & C + 2B\cos.\eta + A = 0, & \text{unde fit} & C = -\sin4\eta, \\
4. & D + 2C\cos.\eta + B = 0, & \text{unde fit} & D = \sin.5\eta \\
& \text{etc.} & & \text{etc.},
\end{array}$$

ita ut nostra fractio $\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}$ resolvatur in hanc seriem

$$\sin.\eta - x^k\sin.2\eta + x^{2k}\sin.3\eta - x^{3k}\sin.4\eta + x^{4k}\sin.5\eta - \text{etc.}$$

28. Multiplicemus nunc hanc seriem per

$$x^{m-1} + x^{2k-m-1}$$

et post integration faciamus $x = 1$, ut obtineamus valorem huius formulae

$$\sin.\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos.\eta+x^{2k}}dx$$

pro casu $x = 1$, hocque modo perveniemus ad geminas sequentes series

$$\frac{\sin.\eta}{m} - \frac{\sin.2\eta}{m+k} + \frac{\sin.3\eta}{m+2k} - \frac{\sin.4\eta}{m+3k} + \frac{\sin.5\eta}{m+4k} - \text{etc.},$$
$$\frac{\sin.\eta}{2k-m} - \frac{\sin.2\eta}{3k-m} + \frac{\sin.3\eta}{4k-m} - \frac{\sin.4\eta}{5k-m} + \frac{\sin.5\eta}{6k-m} - \text{etc.}$$

Aggregatum igitur harum duarum serierum iunctim sumtarum aequabitur huic valori

$$\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\frac{m\pi}{k}},$$

unde subiungamus adhuc istud theorema.

THEOREMA

29. *Si η denotet angulum quemcunque, litterae vero m et k pro lubitu accipiantur ex iisque binae sequentes series formentur*

$$P=\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$
$$Q=\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.},$$

neutrius quidem summa exhiberi potest, utriusque autem iunctim sumtae summa erit

$$P+Q=\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\frac{m\pi}{k}}.$$

COROLLARIUM

30. Quodsi ergo angulum η infinite parvum capiamus, ut fiat

$$\sin.\eta=\eta,\ \sin.2\eta=2\eta,\ \sin.3\eta=3\eta\ \text{etc.},$$

quia in formula summae fiet

$$\sin.(1-\tfrac{m}{k})\eta=(1-\tfrac{m}{k})\eta,$$

si utrinque per η dividamus, obtinebimus sequentem seriem geminatam

$$\frac{1}{m}-\frac{2}{m+k}+\frac{3}{m+2k}-\frac{4}{m+3k}+\frac{5}{m+4k}-\text{etc.},$$
$$\frac{1}{2k-m}-\frac{2}{3k-m}+\frac{3}{4k-m}-\frac{4}{5k-m}+\frac{5}{6k-m}-\text{etc.}$$

cuius ergo summa erit $(1-\frac{m}{k})\frac{\pi}{k\sin.\frac{m\pi}{k}}$; ubi notetur ambas istas series non incongrue in hanc simplicem contrahi posse

$$\frac{2k}{m(2k-m)}-\frac{8k}{(m+k)(3k-m)}+\frac{18k}{(m+2k)(4k-m)}-\frac{32k}{(m+3k)(5k-m)}+\text{etc.},$$

ubi numeratores sunt numeri quadrati duplicati.

31. Formulae autem, quarum valores hactenus invenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis *m* scribamus $k-n$; tum enim in valore

integrali invento fiet $(1-\frac{m}{k})\eta=\frac{n\eta\pi}{k}$, at vero pro denominatore fiet $\frac{m\pi}{k}=\pi-\frac{n\pi}{k}$, cuius sinus erit $\sin.\frac{n\pi}{k}$; sicque nos formula inventa hanc induet formam $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}$, quae ergo exprimet valorem huius formulae integralis

$$\int\frac{x^{k-n-1}dx}{1+2x^k\cos.\eta+x^{2k}}$$

ab $x=0$ usque ad $x=\infty$, ut et huius formulae

$$\int\frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k\cos.\eta+x^{2k}}dx$$

a termino $x=0$ usque ad terminum $x=1$; et quia utriusque valor est $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}$, perspicuum est eum manere eundem, etsi loco *n* scribatur $-n$, ex quo prior formula ita repraesentari poterit

$$\int\frac{x^{k\pm n-1}dx}{1+2x^k\cos.\eta+x^{2k}};$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

32. Ponendo $m=k-n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\sin.\eta}{k-n}-\frac{\sin.2\eta}{2k-n}+\frac{\sin.3\eta}{3k-n}-\frac{\sin.4\eta}{4k-n}+\text{etc.}$$
$$\frac{\sin.\eta}{k+n}-\frac{\sin.2\eta}{2k+n}+\frac{\sin.3\eta}{3k+n}-\frac{\sin.4\eta}{4k+n}+\text{etc.},$$

cuius ergo summa erit $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}}$. Tum vero si hae geminae series in unam contrahantur et utrinque per $2k$ dividatur, obtinebitur sequens summatio memoratu digna

$$\frac{\pi\sin.\frac{n\eta}{k}}{2kk\sin.\frac{n\pi}{k}}=\frac{\sin.\eta}{kk-nn}-\frac{2\sin.2\eta}{4kk-nn}+\frac{3\sin.3\eta}{9kk-nn}-\frac{4\sin.4\eta}{16kk-nn}+\text{etc.}$$

33. Quodsi haec postrema series differentietur sumendo solum angulum η variabilem, ob $d\,\sin.\frac{n\eta}{k}=\frac{nd\eta}{k}\cos.\frac{n\eta}{k}$ habebimus

$$\frac{\pi n\cos.\frac{n\eta}{k}}{2k^3\sin.\frac{n\pi}{k}}=\frac{\cos.\eta}{kk-nn}-\frac{4\cos.2\eta}{4kk-nn}+\frac{9\cos.3\eta}{9kk-nn}-\frac{16\cos.4\eta}{16kk-nn}+\text{etc.}$$

Unde si sumatur $\eta=0$, orietur ista summatio

$$\frac{\pi n}{2k^3 \sin.\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}$$

sin autem sumatur $\eta = 90° = \frac{\pi}{2}$, erit

$$\cos.\eta = 0,\ \cos.2\eta = -1,\ \cos.3\eta = 0,\ \cos.4\eta = +1 \text{ etc.},$$

unde nascitur sequens series

$$\frac{n\pi \cos.\frac{n\pi}{2k}}{2k^3 \sin.\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

Quia autem $\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, erit eiusdem seriei summa $\frac{n\pi}{4k^3 \sin.\frac{n\pi}{2k}}$.

34. At si series illa § 32 exhibita in $d\eta$ ducatur et integretur, ob $\int d\eta \sin.\frac{n\eta}{k} = -\frac{n}{k}\cos.\frac{n\eta}{k}$ erit

$$C - \frac{\pi \cos.\frac{n\eta}{k}}{2nk \sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} - \text{etc.}$$

Ut autem hic constantem addendam *C* definiamus, sumamus $\eta = 0$ fietque

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.};$$

quare si huius seriei summa aliunde pateat, constans *C* definiri poterit.
Series autem haec in sequentem geminatam resolvi potest

$$2nC - \frac{\pi}{k \sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$

35. Cum igitur in *Introductione in Analysin lnfinitorum* pag. 142 ad hanc pervenissem seriem

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.} = \frac{\pi}{2kn \sin.\frac{n\pi}{k}} - \frac{1}{2nn}$$

(hic scilicet loco litterarum ibi adhibitarum *m* et *n* scripsi *n* et *k*), hoc valore adhibito nostra aequatio erit

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk \sin.\frac{n\pi}{k}},$$

unde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem

$$\frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}}-\frac{1}{2nn}=\frac{\cos.\eta}{kk-nn}-\frac{\cos.2\eta}{4kk-nn}+\frac{\cos.3\eta}{9kk-nn}-\frac{\cos.4\eta}{16kk-nn}+\text{etc.},$$

quae series utique omni attentione digna videtur.