OBSERVATIONS ON SOME THEOREMS OF THE MOST ILLUSTRIOUS LAGRANGE

[E587]

Opuscula Analytica 2, 1785, p. 16-41

I communicated some theorems to the illustrious Lagrange, from those which I had demonstrated quite recently [E464 : to be found in *Vol. IV*, *Foundations of the integral calculus, Supp. 5a, on this website*], in which the value of the integral formula $\int\frac{(x-1)dx}{lx}$ is found to be $=l2$, if there may be put $x=1$ on integration. He was moved by the novelty of this argument, and entered not only into the most happy success of its demonstration, but also thence deduced several other outstanding theorems, the fecundity of which may be seen to promise the greatest increases to the elucidation of the analytical science; from which kindly he communicated to me several different kinds of outstanding examples, which I studied at once with the greatest care; and because this subject matter is seen to merit attention, I am going to set out my own further considerations, which occurred to me on this occasion. But since this is, as if a new kind of analysis, which shall be concerned chiefly with integral formulas of this kind, so that a certain determined value may be attributed to the variable after the integration, then in order to avoid a tedious circumlocution of words, which the mention of such conditions always demands, I will be using a special way of indicating these, which it will be necessary to explain more carefully initially.

HYPOTHESIS

1. On being designated by this account

$$\int Pdx\begin{bmatrix}\text{from } x=a\\ \text{to } \quad x=b\end{bmatrix},$$

it may be declared for the integral $\int Pdx$ to be assumed thus, so that it may vanish on putting $x=a$, then truly to be put in place for $x=b$; with which done it is evident the value of this to be determined completely.

SCHOLIUM

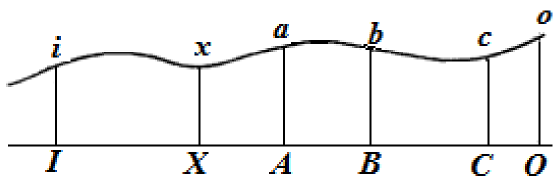

Fig. 1.

2. So that the nature of this determination may be seen more clearly, because P may denote some function of x, the nature of which we may represent by a certain curved line *ixabco* (Fig. 1) constructed above the axis IO, of which any applied line Xx of the corresponding abscissa $IX = x$ may show the function P itself, thus so that the formula of the indefinite integral $\int Pdx$ may be expressed by the area of this curve. But if now the abscissas may be taken $IA = a$, $IB = b$, to which the applied lines Aa and Bb may correspond, the proposed formula will express the area $AaBb$ intercepted between the applied lines Aa and Bb. In the same manner, if some other abscissa may be put in place $IC = c$, the area $AaCc$ will be expressed by this formula

$$\int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to} \quad x = c \end{bmatrix},$$

moreover the area $BbCc$ by this formula

$$\int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to} \quad x = b \end{bmatrix};$$

then truly by beginning from the initial I the area $IiAa$ will be indicated by this formula

$$\int Pdx \begin{bmatrix} \text{from } x = 0 \\ \text{to} \quad x = a \end{bmatrix};$$

from which the following lemmas thus will be able to be expressed succinctly.

LEMMA 1

3. $$\int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to} \quad x = b \end{bmatrix} = -\int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to} \quad x = a \end{bmatrix}.$$

For because, if b may be considered greater than a, the latter formula

$$\int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to } \quad x = a \end{bmatrix}$$

refers to the same area *AaBb* as before, but in the reverse order, this same expression will be had for the negative and thus there will be also

$$\int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = b \end{bmatrix} + \int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to } \quad x = a \end{bmatrix} = 0.$$

LEMMA 2

4.

$$\int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = b \end{bmatrix} + \int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to } \quad x = c \end{bmatrix} = \int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = c \end{bmatrix},$$

just as an inspection of the figure evidently indicates.

LEMMA. 3

5.

$$\int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = c \end{bmatrix} - \int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = b \end{bmatrix} = \int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to } \quad x = c \end{bmatrix},$$

where in the two first formulas the same term occurs *from which,* clearly $x = a$; truly of the terms *to which,* evidently $x = c$ and $x = b$, the latter $x = b$ gives for the third formula the term *from which,* truly the former term *to which.*

LEMMA. 4

6.

$$\int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = c \end{bmatrix} - \int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to } \quad x = c \end{bmatrix} = \int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = b \end{bmatrix},$$

where it may be noted the two first formulas have the same term *to which*, evidently $x = c$, but of the terms *from which* the first to give $x = a$, in the third formula to give the term *from which,* the latter term truly *to which.*

LEMMA 5

7.

$$\int Pdx \begin{bmatrix} \text{from } x = a \\ \text{to } \quad x = b \end{bmatrix} + \int Pdx \begin{bmatrix} \text{from } x = b \\ \text{to } \quad x = c \end{bmatrix} + \int Pdx \begin{bmatrix} \text{from } x = c \\ \text{to } \quad x = a \end{bmatrix} = 0.$$

SCHOLIUM

8. Therefore from these, which the Cel. Lagrange has written out for me, I may run through in order, which by themselves are especially clear, with particular arguments advanced. But initially he makes mention of a significant paradox, the nature of which does not admit itself to be seen well enough, from which therefore I will begin my considerations.

THE RESOLUTION OF A SIGNIFICANT PARADOX

9. Since the celeb. man also had discovered this general theorem

$$\int \frac{x^n - x^m}{lx} \cdot \frac{dx}{x} \begin{bmatrix} \text{from } x = 0 \\ \text{to } \quad x = 1 \end{bmatrix} = l\,\frac{n}{m},$$

the truth of which thus recently I have added to with several demonstrations, there may be put $x^n = z$ and $x^m = y$; with which done the first part $\int \frac{x^{n-1}dx}{lx}$ may be transformed into this $\int \frac{dz}{lz}$, truly in a similar manner the other part $\int \frac{x^{m-1}dx}{lx}$ into this $\int \frac{dy}{ly}$; from which with these parts themselves put in place there follows to become

$$\int \frac{dz}{lz} \begin{bmatrix} \text{from } z = 0 \\ \text{to } \quad z = 1 \end{bmatrix} - \int \frac{dy}{ly} \begin{bmatrix} \text{from } y = 0 \\ \text{to } \quad y = 1 \end{bmatrix} = l\,\frac{n}{m}.$$

Whereby since these two formulas generally shall be similar and containing the same limits of integration, why not also may it not be believed these two also to be completely equal to each other, or there shall be

$$\int \frac{dz}{lz} \begin{bmatrix} \text{from } z = 0 \\ \text{to } \quad z = 1 \end{bmatrix} = \int \frac{dy}{ly} \begin{bmatrix} \text{from } y = 0 \\ \text{to } \quad y = 1 \end{bmatrix}?$$

Yet meanwhile we see the difference between these formulas to be $l\,\frac{n}{m}$. Therefore here a question arises of the greatest interest, in what way this evident contradiction may be able to be resolved.

10. But here in the first place it is agreed to note both the quantities y and z depend on each other in a certain way. For since there shall be $y = x^m$ and $z = x^n$, there will be $y^n = z^m$, so that yet by this connection it does not prevent, whether or not $y = 0$ or $y = 1$ may be put in place, there shall also become $z = 0$ or $z = 1$. Yet meanwhile hence it may be by no means apparent, why on account of this reasoning these two formulas

$$\int \frac{dy}{ly}\begin{bmatrix}\text{from } y=0\\ \text{to } \quad y=1\end{bmatrix} \text{ and } \int \frac{dz}{lz}\begin{bmatrix}\text{from } z=0\\ \text{to } \quad z=1\end{bmatrix}$$

may be able to be said disparate; from which this observation plainly may be seen to add nothing towards resolving the doubt.

11. Indeed in short it may be seen also that this more general equation to be no less liable to doubt

$$\int \frac{dy}{ly}\begin{bmatrix}\text{from } y=a\\ \text{to } \quad y=b\end{bmatrix} = \int \frac{dz}{lz}\begin{bmatrix}\text{from } z=a\\ \text{to } \quad z=b\end{bmatrix},$$

since plainly nothing prevents, whether or not in place of z we may write y in turn ; truly most phenomena observed in analysis handle equations of this kind clearly enough meanwhile an exception to be apparent, when infinite values may arise.
But these circumstances certainly have a place in our case, since the formula of the integral $\int \frac{dy}{ly}$, if it may be extended from $y=0$ ad $y=1$, certainly may increase indefinitely, which also is held for the other $\int \frac{dz}{lz}$. For if the abscissa may become $=1$, the applied line of our curve, which is $\frac{1}{lz}$, evidently shall become infinitely great, from which the above general equality

$$\int \frac{dy}{ly}\begin{bmatrix}\text{from } y=a\\ \text{to } \quad y=b\end{bmatrix} - \int \frac{dz}{lz}\begin{bmatrix}\text{from } z=a\\ \text{to } \quad z=b\end{bmatrix} = 0$$

demands this restriction, unless either a shall be $=1$ or $b=1$, certainly in which cases each formula shall be infinite.

12. From these reflections it may seem that no doubt indeed remains for me, why not in this circumstance the true solution of the proposed paradox shall be questioned, which evidently is concerned with this, so that there shall be

$$\text{both } \int \frac{dy}{ly}\begin{bmatrix}\text{from } y=0\\ \text{to } \quad y=1\end{bmatrix} = \infty \text{ as well as } \int \frac{dz}{lz}\begin{bmatrix}\text{from } z=0\\ \text{to } \quad z=1\end{bmatrix} = \infty,$$

thus so that the difference of these infinite quantities may be put equal to any finite quantity and thus in short may not itself be seen to be determined ; but because that difference in our case shall be $l\frac{n}{m}$ and thus determined, arising thence as there shall be $y^n = z^m$.

13. Likewise it can eventuate somehow in simpler formulas, such as $\int \frac{dy}{y}$ and $\int \frac{dz}{z}$, certainly the values of which from the limit $y=0$ and $z=0$ have been taken to be

infinite, from which, even if after the integration the same limit *to which* may be put in place, as it were $y=1$ and $z=1$, yet hence in no manner does it follow the absolute difference to be equal to zero, if indeed it will have to be regarded rather to be indeterminate, since indeed according to the other limits of the integration there shall certainly be

$$\int \frac{dy}{y}\left[\begin{array}{l}\text{from } y=a\\ \text{to } \quad y=b\end{array}\right]=\int \frac{dz}{z}\left[\begin{array}{l}\text{from } z=a\\ \text{to } \quad z=b\end{array}\right],$$

provided neither a nor b were $=0$ or $=\infty$.

14. And hence also the completely similar proposed paradox can be advanced, which thus may be had

$$\int \frac{dz}{z}\left[\begin{array}{l}\text{from } z=0\\ \text{to } \quad y=\infty\end{array}\right]-\int \frac{dy}{y}\left[\begin{array}{l}\text{from } y=0\\ \text{to } \quad y=\infty\end{array}\right]=la;$$

the truth of which shall be put in a sunny place, if indeed there may be accepted $z=ay$, also the proposed paradox will be considered to be duly weakened.

LAGRANGE'S OBSERVATIONS ON THIS THEOREM

$$\int \frac{x^n-x^m}{lx}\cdot\frac{dx}{x}\left[\begin{array}{l}\text{from } x=a\\ \text{to } \quad x=b\end{array}\right]=\int (b^y-a^y)\frac{dy}{y}\left[\begin{array}{l}\text{from } y=m\\ \text{to } \quad y=n\end{array}\right]$$

15. Indeed since it is some time ago that I treated the reductions of formulas of this kind, I had not considered other limits of the integration besides from $x=0$ to $x=1$, from which this theorem of mine seemed at once to of a higher level of investigation and entirely worthwhile, which was expended with great care. Therefore in the first place I set out to inquire into its truth by a series, which operation I have carried out in the following manner.

16. Since there shall be

$$x^\alpha=e^{\alpha lx}=1+\alpha lx+\frac{(\alpha lx)^2}{1\cdot 2}+\frac{(\alpha lx)^3}{1\cdot 2\cdot 3}+\text{etc.},$$

there will become

$$x^n-x^m=(n-m)\frac{lx}{1}+(n^2-m^2)\frac{(lx)^2}{1\cdot 2}+\frac{(n^3-m^3)(lx)^3}{1\cdot 2\cdot 3}+\text{etc.}$$

Therefore we may multiply this series by $\frac{dx}{xlx}$, and because in general :

$$\int (lx)^\lambda\frac{dx}{xlx}\left[\begin{array}{l}\text{from } x=a\\ \text{to } \quad x=b\end{array}\right]=\frac{(lb)^\lambda-(la)^\lambda}{\lambda},$$

the value of the formula written for the left-hand side will be expressed by this infinite series

$$\frac{n-m}{1}\cdot\frac{lb-la}{1}+\frac{n^2-m^2}{1\cdot 2}\cdot\frac{(lb)^2-(la)^2}{2}+\frac{n^3-m^3}{1\cdot 2\cdot 3}\cdot\frac{(lb)^3-(la)^3}{3}+\text{etc.}$$

17. In a similar manner according to the formula put in place by the infinite series for the right, there will be

$$b^y-a^y=y\frac{lb-la}{1}+y^2\frac{(lb)^2-(la)^2}{1\cdot 2}+y^3\frac{(lb)^3-(la)^3}{1\cdot 2\cdot 3}+\text{etc.},$$

which therefore may be multiplied by $\frac{dy}{y}$, and since in general there is

$$\int y^{\lambda}\frac{dy}{y}\left[\begin{array}{l}\text{from } y=m\\ \text{to } \quad y=n\end{array}\right]=\frac{n^{\lambda}-m^{\lambda}}{\lambda},$$

the value of this formula will be expressed by this infinite series

$$\frac{n-m}{1}\cdot\frac{lb-la}{1}+\frac{n^2-m^2}{1\cdot 2}\cdot\frac{(lb)^2-(la)^2}{2}+\frac{n^3-m^3}{1\cdot 2\cdot 3}\cdot\frac{(lb)^3-(la)^3}{3}+\text{etc.}$$

Therefore since this series agrees perfectly with the preceding, the truth of the theorem has been established firmly.

18. Truly hence by no means can it be seen, how the most acute author may have been led to this theorem, on account of which with the matter properly considered the way to be found from the same principles, as I have used previously, to arrive at the same formulas. Moreover the beginning is from this simplest form :

$$\int x^{\lambda}\frac{dx}{x}\left[\begin{array}{l}\text{from } x=a\\ \text{to } \quad x=b\end{array}\right]=\frac{b^{\lambda}-a^{\lambda}}{\lambda},$$

where multiplying each side $d\lambda$ with an integration put in place anew, and since, as now occasionally the demonstration is found, there shall be

$$\int d\lambda\int x^{\lambda}\frac{dx}{x}=\int\frac{dx}{x}\int x^{\lambda}d\lambda,$$

only this integral must be sought $\int x^{\lambda}d\lambda$ with the quantity x regarded as constant, thus so that λ alone shall be variable. Therefore there is :

$$\int x^{\lambda}d\lambda=\frac{x^{\lambda}}{lx}+C,$$

just as may be clear from the elements of the calculus of the exponential. Truly here the point of the matter turns on this, so that this same integral may be defined by a certain rule, as will be required to be observed henceforth in the other part. Therefore we may establish such integrals to be taken thus, so that they may vanish on putting $\lambda = 0$, and there will become :

$$\int x^{\lambda} d\lambda = \frac{x^{\lambda}-1}{lx},$$

with which agreed, we will have for the left-hand side

$$\int d\lambda \int x^{\lambda} \frac{dx}{x} = \int \frac{dx}{x} \cdot \frac{x^{\lambda}-1}{lx}.$$

19. But for the right-hand side we will have

$$\int \frac{d\lambda}{\lambda}(b^{\lambda} - a^{\lambda}),$$

which formula by the same integration rule, so that on making $\lambda = 0$, zero may be produced, this value will be represented here in the received manner:

$$\int \frac{dy}{y}(b^{y} - a^{y})\begin{bmatrix}\text{from } y = 0\\ \text{to } \quad y = \lambda\end{bmatrix},$$

Indeed here we have made no other change, except that for λ we have written y and with the integration made we have assumed the value λ to be put in place of its value y, and thus we have followed on with the formula

$$\int (x^{\lambda} - 1)\frac{dx}{xlx}\begin{bmatrix}\text{from } x = a\\ \text{to } \quad x = b\end{bmatrix} = \int \frac{dy}{y}(b^{y} - a^{y})\begin{bmatrix}\text{from } y = 0\\ \text{to } \quad y = \lambda\end{bmatrix},$$

so as considered to be a most useful theorem.

20. Therefore from the strength of this theorem we obtain the following reductions

$$\int (x^{n} - 1)\frac{dx}{xlx}\begin{bmatrix}\text{from } x = a\\ \text{to } \quad x = b\end{bmatrix} = \int \frac{dy}{y}(b^{y} - a^{y})\begin{bmatrix}\text{from } y = 0\\ \text{to } \quad y = n\end{bmatrix}$$

and

$$\int (x^{m} - 1)\frac{dx}{xlx}\begin{bmatrix}\text{from } x = a\\ \text{to } \quad x = b\end{bmatrix} = \int \frac{dy}{y}(b^{y} - a^{y})\begin{bmatrix}\text{from } y = 0\\ \text{to } \quad y = m\end{bmatrix};$$

whereby if the latter formula may be subtracted from the former, there will be

$$\int (x^n - x^m)\frac{dx}{xlx}\begin{bmatrix}\text{from } x=a\\ \text{to } \quad x=b\end{bmatrix} = \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{from } y=0\\ \text{to } \quad y=n\end{bmatrix} - \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{from } y=0\\ \text{to } \quad y=m\end{bmatrix};$$

truly this formula put to the right I have shown by the reduction in Lemma 3 to be recalled to this simper form

$$\int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{from } y=m\\ \text{to } \quad y=n\end{bmatrix};$$

from which it is apparent in this manner this conspicuous theorem also shall be able to be investigated from our principles.

21. But here for the most general the most ingenious man has used my demonstration for the theorem, in which to be shown

$$\int (x^n - x^m)\frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=1\end{bmatrix} = l\frac{n}{m};$$

indeed only there was a need, so that there may be taken $a=0$ and $b=1$, with which agreed on the formula for the right-hand of the integral put in place will be changed into

$$\int \frac{dy}{y}\begin{bmatrix}\text{from } y=m\\ \text{to } \quad y=n\end{bmatrix},$$

the value of which evidently becomes $ln - lm = l\frac{m}{n}$, which is a new demonstration of my theorem, indeed of which kind I had given several others previously.

OBSERVATIONS ON THE LAGRANGE THEOREM

$$\int \frac{(x^n - x^m)dx}{(1+x^r)lx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix} = l\frac{\text{tang.}\frac{(n+1)\pi}{2r}}{\text{tang.}\frac{(m+1)\pi}{2r}}$$

22. Because here both the exponents *m* and *n* neither depend on each other nor on the exponent *r*, it is evident according to each power x^m and x^n separately ought to have such an integral form :

$$\int \frac{x^n dx}{(1+x^r)lx} = l\text{tang.}\frac{(n+1)\pi}{2r} + C \text{ and } \int \frac{x^m dx}{(1+x^r)lx} = l\text{tang.}\frac{(m+1)\pi}{2r} + C.$$

Indeed if the latter form may be taken from the former, the constant *C* departs from the calculation and the proposed integral results. Therefore here the most value of this constant *C* present is to be determined.

23. Among these integral formulas, of which the values for the case, in which after the integration an infinite value of the variable may be put in place, I have assigned from the first principles of integral calculus, that same may be found

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} \begin{bmatrix} \text{from } x=0 \\ \text{to } \quad x=\infty \end{bmatrix} = \frac{\pi}{2k\cos.\frac{n\pi}{2k}} = \frac{\pi}{2k\sin.\frac{(k+n)\pi}{2k}},$$

but where it is assumed the exponent n not to be taken greater than k. But if now here the exponent n may be treated as a variable with x itself considered as constant and both may be multiplied by dn and integrated anew, the left-hand formula will be

$$\int dn \int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} = \int \frac{dx}{x(1+x^{2k})} \int x^{k+n} dn,$$

where the latter integral becomes

$$\int dn = \frac{x^{k+n}}{lx} + C.$$

But so that this integral may be determined, we may define the constant thus, so that it may vanish on putting $n=0$, from which there is obtained :

$$\int x^{k+n} dn = \frac{x^{k+n} - x^k}{lx},$$

thus so that the formula of the integral put for the left-hand shall become

$$\int \frac{x^{k+n} - x^k}{1+x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{from } x=0 \\ \text{to } \quad x=\infty \end{bmatrix}.$$

24. For the right part we will have this integral

$$\int \frac{\pi dn}{2k\sin.\frac{(k+n)\pi}{2k}}$$

also being taken thus, so that it may vanish on putting $n=0$. Finally in this we may put the angle $\frac{(k+n)\pi}{2k} = \varphi$, and since hence there will be $d\varphi = \frac{\pi dn}{2k}$, our formula requiring to be integrated will be $\int \frac{d\varphi}{\sin.\varphi}$, the integral of which by the known rules generally is :

$$l\text{tang.}\tfrac{1}{2}\varphi + C = l\text{tang.}\frac{(k+n)\pi}{4k} + C,$$

which on making $n=0$ will be changed into $l\text{tang.}\frac{\pi}{4}+C$. Whereby since $\text{tang.}\frac{\pi}{4}=1$ and $l1=0$, it is evident the constant C to become $=0$, thus so that this integral sought shall be $l\text{tang.}\frac{(k+n)\pi}{4k}$. Hence therefore we have followed that same general reduction

$$\int\frac{x^{k+n}-x^{k}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=l\text{tang.}\frac{(k+n)\pi}{4k},$$

but where it is to be observed properly the exponents m and n not to be allowed greater than k.

25. Therefore since by assuming another number m in place of n in a similar manner there shall be

$$\int\frac{x^{k+m}-x^{k}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=l\text{tang.}\frac{(k+m)\pi}{4k},$$

that formula may be subtracted from the preceding and that same one will be obtained :

$$\int\frac{x^{k+n}-x^{k+m}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{l\text{tang.}\frac{(k+n)\pi}{4k}}{l\text{tang.}\frac{(k+m)\pi}{4k}},$$

which evidently agrees with the proposed form, only if in place of $k+n-1$ there may be written n and m in place of $k+m-1$, but in place of the exponent $2k$ there may be written r; then indeed clearly there will become

$$\int\frac{x^{n}-x^{m}}{1+x^{r}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{l\text{tang.}\frac{(n+1)\pi}{2r}}{l\text{tang.}\frac{(m+1)\pi}{2r}}.$$

26. Since this same analysis has led us to this formula

$$\int\frac{x^{k+n}-x^{k}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=l\text{tang.}\frac{(k+n)\pi}{4k},$$

here it will be of the greatest interest to have observed to become always

$$\int\frac{x^{k}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=0,$$

that which we can show thus. There may be put $x^{k}=z$; there will become

$$x^{k-1}dx=\frac{dz}{k} \text{ and } lx=\frac{lz}{k}$$

and thus that formula will adopt this form $\int \frac{dz}{(1+zz)lz}$, where the limits of the integration even now are $z=0$ and $z=\infty$. Again there may become $z=\text{tang.}\varphi$, from which the limits of integration will be $\varphi=0$ and $\varphi=\frac{\pi}{2}$; hence moreover on account of $d\varphi=\frac{dz}{1+zz}$ this formula will arise $\int \frac{d\varphi}{l\,tang\,\varphi}\left[\begin{smallmatrix}\text{from } \varphi=0\\ \text{to } \varphi=\frac{\pi}{2}\end{smallmatrix}\right]$, the value of which must be shown to go to zero.

27. Towards demonstrating this, the axis $IH=\frac{\pi}{2}$ may be put in place (Fig. 2), upon which from the beginning I with the indefinite abscissa taken $Ip=\varphi$, the applied line shall be $=\frac{1}{l\text{tang.}\varphi}$. Therefore if this axis IH may be bisected at O, so that there shall be $IO=\frac{\pi}{4}$, at this point the applied line will be

$$=\frac{1}{l\text{tang.}\frac{\pi}{4}}=\infty.$$

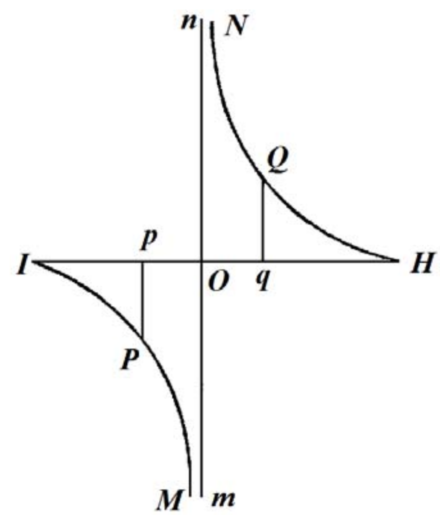

Fig. 2

Now from this point O on each side the equal intervals $Op=Oq=\omega$ may be taken, and for the point p there will be $\varphi=\frac{\pi}{4}-\omega$ and thus at this point p the applied line will be $=\frac{1}{l\text{tang.}(\frac{\pi}{4}-\omega)}$; truly there is $\text{tang.}(\frac{\pi}{4}-\omega)=\text{cot.}(\frac{\pi}{4}+\omega)$, whereby, since there shall be $l\text{cot.}=-l\text{tang.}$, the applied line at this point p will be $=\frac{-1}{l\text{tang.}(\frac{\pi}{4}+\omega)}$; but because there is $Iq=\frac{\pi}{4}+\omega$, the applied line at the point q will be $=\frac{+1}{l\text{tang.}(\frac{\pi}{4}+\omega)}$; and thus it is equal to the applied line at p, but in the opposite inclination. Thus if the applied line qQ were directed upwards, at the point p the same applied line $pP=qQ$ will be directed downwards.

28. So that if such a curve were erected on the axis $IH=\frac{\pi}{2}$, thus so that to the abscissa φ there may correspond the applied line $\frac{1}{l\text{tang.}\varphi}$, this curve thus will be considered with the two parts perfectly equal between themselves set out about the mean point O, so that the left-hand curve shall be IPM falling to infinity to the asymptote Om, but the right-hand part in a similar manner from H of the left will rise upwards to the asymptote On. Whereby since the formula of the integral $\int \frac{d\varphi}{l\,tang\,\varphi}$ extended from $\varphi=0$ to $\varphi=\frac{\pi}{2}$ may set out the whole area of this curve extended from I as far as to H, it is evident this whole to be rendered to zero, because its negative part taken is perfectly similar to the positive part taken.

29. Therefore thus by this single demonstration everything has prevailed to become always

$$\int \frac{x^k}{1+x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{from } x = 0 \\ \text{to } \quad x = \infty \end{bmatrix} = 0,$$

which certainly is a most noteworthy theorem of this kind. So that if therefore with the illustrious Lagrange we may put $2k = r$, there will become

$$\int \frac{x^{\frac{1}{2}r-1}dx}{(1+x^r)lx} = 0;$$

besides truly according to our formula § 24 has shown on account of

$$\int \frac{x^k dx}{(1+x^{2k})xlx} \begin{bmatrix} \text{from } x = 0 \\ \text{to } \quad x = \infty \end{bmatrix} = 0,$$

this same notable theorem is deduced generally

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{from } x = 0 \\ \text{to } \quad x = \infty \end{bmatrix} = l\text{tang.}\frac{(k+n)\pi}{4k},$$

which in the manner of Lagrange can be proposed thus :

$$\int \frac{x^n dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{from } x = 0 \\ \text{to } \quad x = \infty \end{bmatrix} = l\text{tang.}\frac{(n+1)\pi}{2r};$$

and thus it is apparent that constant introduced by us above (§ 22) actually to be equal to zero.

30. Since the demonstration of this theorem depends sufficiently on an unusual method, it will help for its truth to be shown by a series. But for this value of the formula

$$\int \frac{x^{\lambda-1}dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{from } x = 0 \\ \text{to } \quad x = \infty \end{bmatrix}$$

it is necessary to be split into two parts (evidently by writing $\lambda - 1$ in place of n), which shall be

$$P = \int \frac{x^{\lambda-1}dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{from } x = 0 \\ \text{to } \quad x = 1 \end{bmatrix} \quad \text{and} \quad Q = \int \frac{x^{\lambda-1}dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{from } x = 1 \\ \text{to } \quad x = \infty \end{bmatrix},$$

thus so that $P+Q$ may express the value which we seek. Now in the latter part in place of x we may write $\frac{1}{z}$ and there will become

$$Q=\int\frac{z^{-\lambda}}{1+z^{-r}}\cdot\frac{dz}{zlz}\left[\begin{array}{l}\text{from } z=1\\ \text{to} \quad z=0\end{array}\right]=\int\frac{z^{r-\lambda}}{1+z^{r}}\cdot\frac{dz}{zlz}\left[\begin{array}{l}\text{from } z=1\\ \text{to} \quad z=0\end{array}\right]$$

and with the limits of the integration interchanged

$$Q=-\int\frac{z^{r-\lambda}}{1+z^{r}}\cdot\frac{dz}{zlz}\left[\begin{array}{l}\text{from } z=0\\ \text{to} \quad z=1\end{array}\right].$$

But now in place of z we may write x ; because the limits of the integration are the same on both sides, there will be

$$P+Q=\int\frac{x^{\lambda}-x^{r-\lambda}}{1+x^{r}}\cdot\frac{dx}{xlx}\left[\begin{array}{l}\text{from } x=0\\ \text{to} \quad x=1\end{array}\right],$$

of which the value of the proposed formula therefore is equal.

31. Now we may convert the fraction $\frac{1}{1+x^{r}}$ into the infinite series

$$1-x^{r}+x^{2r}-x^{3r}+x^{4r}-\text{etc.},$$

of which the individual terms multiplied by $\frac{dx}{xlx}(x^{\lambda}-x^{r-\lambda})$ produce

$$\frac{dx}{xlx}(x^{\lambda}-x^{r-\lambda})-\frac{dx}{xlx}(x^{r+\lambda}-x^{2r-\lambda})+\frac{dx}{xlx}(x^{2r+\lambda}-x^{3r-\lambda})-\frac{dx}{xlx}(x^{3r+\lambda}-x^{4r-\lambda})+\text{etc.}$$

But since by the principal theorem in this generally there shall be

$$\int\frac{dx}{xlx}(x^{\alpha}-x^{\gamma})\left[\begin{array}{l}\text{from } x=0\\ \text{to} \quad x=1\end{array}\right]=l\,\frac{\alpha}{\beta},$$

with the individual members integrated in this manner there will be produced

$$P+Q=l\,\frac{\lambda}{r-\lambda}-l\,\frac{r+\lambda}{2r-\lambda}+l\,\frac{2r+\lambda}{3r-\lambda}-l\,\frac{3r+\lambda}{4r-\lambda}+\text{etc.}$$

32. All these logarithms will be allowed to be joined together into a single ratio with the sign of each had and in this manner there will be found

$$P+Q=l\,\frac{\lambda}{r-\lambda}\cdot\frac{2r-\lambda}{r+\lambda}\cdot\frac{2r+\lambda}{3r-\lambda}\cdot\frac{4r-\lambda}{3r+\lambda}\cdot\frac{4r+\lambda}{5r-\lambda}\cdot\frac{6r-\lambda}{5r+\lambda}\cdot\text{etc.}$$

But truly in *Introduction to Analysin Infinitorum* p. 147 to be shown

$$\text{tang.}\frac{m\pi}{2n} = \frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}\cdot\text{etc.},$$

which series clearly may be transformed into that found by putting $m=\lambda$ and $n=r$, thus so that now there shall be $P+Q=l\text{tang.}\frac{\lambda\pi}{2r}$, just as has been found above.

AN ADDITION

33. Inserted into the discourse of the Acts Book V, part I, from which I have chosen this theorem :

$$\int\frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

likewise the following occur:

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=1\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{\pi}{k}\text{tang}\frac{n\pi}{2k},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=1\end{bmatrix}=\frac{\pi}{2k}\text{tang.}\frac{n\pi}{2k},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{2\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=1\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int\frac{x^{k\pm n}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

which formulas therefore it will be worth the effort to be treated in a similar manner.

34. Therefore we may begin from the formula

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=1\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

because the preceding in short will agree with the formula now treated ; which if it may be multiplied by *dn* and thus integrated, so that the integral may vanish on putting $n=0$, because there is

$$\int x^{k-n}dn = -\frac{x^{k-n}-x^{k}}{lx} \quad \text{and} \quad \int x^{k+n}dn = \frac{x^{k+n}-x^{k}}{lx},$$

then truly, as we have seen before,

$$\int \frac{\pi dn}{2k\cos.\frac{n\pi}{2k}} = l\text{tang}.\frac{(k+n)\pi}{4k},$$

will produce this equation:

$$\int \frac{x^{k+n}-x^{k-n}}{1+x^{2k}} \cdot \frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to} \quad x=1\end{bmatrix} = l\text{tang}.\frac{(k+n)\pi}{4k},$$

which value in short agrees with that, which we have found for the formula

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to} \quad x=\infty\end{bmatrix}.$$

35. We shall treat the following formula in a similar manner

$$\int \frac{x^{k-n}-x^{k+n}}{1-x^{2k}} \cdot \frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to} \quad x=\infty\end{bmatrix} = \frac{\pi}{k}\text{tang}\frac{n\pi}{2k},$$

which multiplied by dn and integrated as above provides from the left-hand part

$$\int \frac{2x^{k}-x^{k-n}-x^{k+n}}{1-x^{2k}} \cdot \frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to} \quad x=\infty\end{bmatrix},$$

but from the right-hand part

$$\int \frac{\pi dn}{k}\text{tang}\frac{n\pi}{2k} = \int \frac{\pi dn \sin.\frac{n\pi}{2k}}{k\cos.\frac{n\pi}{2k}}.$$

For integrating this there may be put $\frac{n\pi}{2k} = \varphi$ and there will become $\frac{\pi dn}{k} = 2d\varphi$ and thus the formula being integrated will be

$$\int \frac{2d\varphi\sin.\varphi}{\cos.\varphi} = -2l\cos.\varphi + C = -2l\cos.\frac{n\pi}{2k} + C.$$

Therefore there may become $n=0$ and there must be $-2l1+C=0$ and thus the constant $C=0$, on account of which this integration provides us with the following formula:

$$\int \frac{2x^{k}-x^{k-n}-x^{k+n}}{1-x^{2k}} \cdot \frac{dx}{xlx}\begin{bmatrix}\text{from } x=0\\ \text{to} \quad x=\infty\end{bmatrix} = -2l\cos.\frac{n\pi}{2k};$$

but the following formula $\left[\begin{array}{l}\text{from } x=0\\ \text{to} \quad x=1\end{array}\right]$ for the individual expansion is not needed, since its value shall be half of this.

36. We may set out the case, where $k=2$ and $n=1$, and from the left-hand part we have

$$-\int\frac{(1-x)^2}{(1-x^4)}\cdot\frac{dx}{lx}=-\int\frac{1-x}{(1+x)(1+xx)}\cdot\frac{dx}{lx}\left[\begin{array}{l}\text{from } x=0\\ \text{to} \quad x=\infty\end{array}\right];$$

but truly from the right-hand part $-2l\cos.\frac{\pi}{4}=2l\sqrt{2}=l2.$ Truly the fraction $\frac{1-x}{(1+x)(1+xx)}$ is resolved into these two parts $\frac{1}{1+x}-\frac{x}{1+xx}$, from which our formula is resolved into these two

$$-\int\frac{dx}{(1+x)lx}+\int\frac{xdx}{(1+xx)lx}=l2.$$

But from the general formula:

$$\int\frac{x^{\lambda-1}dx}{(1+x^r)lx}=l\text{tang.}\frac{\lambda\pi}{2r}$$

and the value of each formula increases to infinity and thus nothing hinders, that the difference be $=l2$.

37. But if here in the latter formula we may put $xx=z$, that will be changed into this $\int\frac{dz}{(1+z)lz}$ which is entirely similar to the former and is contained within the same limits of integration. But here again in short the paradox occurs again similar to that, which was mentioned by the illust. Lagrange; clearly here the two forms are found completely equal $\int\frac{dx}{(1+x)lx}$ and $\int\frac{dz}{(1+z)lz}$, each of which will be required to be integrated from the limit 0 to ∞; yet the difference of these is not zero , but nothing less, as we have seen to be $=l2$. And hence the solution of this paradox evidently is situated in that, because the value of each integral increases to infinity.

38. But if the two latter formulas are treated in the same manner and multiplied by *dn* we wish to integrate, this integral formula results from the left-hand side

$$\int\frac{x^{k+n}-x^{k-n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{xlx}\left[\begin{array}{l}\text{from } x=0\\ \text{to} \quad x=\infty\end{array}\right];$$

but from the right-hand side we obtain this integral formula

$$\int \frac{2\pi dn \sin.\frac{n\eta}{k}}{k\sin.\eta \sin.\frac{n\pi}{k}}$$

being extended from the limit $n=0$. Truly this integration cannot succeed in any way ; if indeed we may put $\frac{n\pi}{k}=\varphi$, there becomes $\frac{n\varphi}{k}=\frac{\eta\varphi}{\pi}=\alpha\varphi$ on putting $\frac{\eta}{\pi}=\alpha$, from which the formula being integrated will be $\frac{2}{\sin.\eta}\int\frac{d\varphi\sin.\alpha\varphi}{\sin.\varphi}$, the value of which cannot be expressed other than by the sign of the summation, and thus with no neat theorem hence allowed to be derived.

39. But just as here by regarding the exponent *n* as a variable we have put in place transformations by integration, thus also outstanding transformations will be supplied by differentiation, which argument it will suffice to illustrate by a single formula of the principal. Evidently we will consider this formula

$$\int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

which with the exponent *n* as the single variable to be differentiated continually, where it is to be noted $d.x^{k+n}=x^{k+n}dnlx.$ But truly for the formula $\frac{\pi}{2k\cos.\frac{n\pi}{2k}}$ we may write the letter *v,* which therefore by being regarded as if a function of *n,* of which therefore the differentials of any order are in our power. Hence therefore we will follow with the following reductions

$$\int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}lx=\frac{dv}{dn}$$

or

$$\int \frac{x^{k+n-1}dxlx}{1+x^{2k}}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{dv}{dn},$$

$$\int \frac{x^{k+n-1}dx(lx)^2}{1+x^{2k}}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{ddv}{dn^2},$$

$$\int \frac{x^{k+n-1}dx(lx)^3}{1+x^{2k}}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{d^3v}{dn^3},$$

$$\int \frac{x^{k+n-1}dx(lx)^4}{1+x^{2k}}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{d^4v}{dn^4},$$

$$\int \frac{x^{k+n-1}dx(lx)^5}{1+x^{2k}}\begin{bmatrix}\text{from } x=0\\ \text{to } \quad x=\infty\end{bmatrix}=\frac{d^5v}{dn^5},$$

etc.

40. Therefore hence since the whole matter for the continued differentials of v may be reduced, that will be allowed to be found in the most convenient manner. For since there shall be

$$v = \frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

there will be $v\cos.\frac{n\pi}{2k} = \frac{\pi}{2k}$ and hence by continual differentiation we will obtain the following formulas

$$\frac{dv}{dn}\cos.\frac{n\pi}{2k} - \frac{\pi}{2k}v\sin.\frac{n\pi}{2k} = 0,$$
$$\frac{ddv}{dn^2}\cos.\frac{n\pi}{2k} - \frac{2\pi}{2k}\frac{dv}{dn}\sin.\frac{n\pi}{2k} - \frac{\pi\pi}{4kk}v\cos.\frac{n\pi}{2k} = 0,$$
$$\frac{d^3v}{dn^3}\cos.\frac{n\pi}{2k} - \frac{3\pi}{2k}\frac{ddv}{dn^2}\sin.\frac{n\pi}{2k} - \frac{3\pi\pi}{4kk}\frac{dv}{dn}\cos.\frac{n\pi}{2k} + \frac{\pi^3}{8k^3}v\sin.\frac{n\pi}{2k} = 0,$$
$$\frac{d^4v}{dn^4}\cos.\frac{n\pi}{2k} - \frac{4\pi}{2k}\frac{d^3v}{dn^3}\sin.\frac{n\pi}{2k} - \frac{6\pi\pi}{4kk}\frac{ddv}{dn^2}\cos.\frac{n\pi}{2k} + \frac{4\pi^3 dv}{8k^3}\sin.\frac{n\pi}{2k} + \frac{\pi^4}{16k^4}v\cos.\frac{n\pi}{2k} = 0,$$
etc.

from which the individual higher differentials are able to be formed from the lower ones.

41. But so that these operations may be raised more, we may put for the sake of brevity $\frac{\pi}{2k} = \alpha$, so that there shall be $v = \frac{\alpha}{\cos.\alpha n}$, and the individual differentials will be determined from the above equations in the following manner

$$\frac{dv}{dn} = \alpha v\,\text{tang}.\alpha n,$$
$$\frac{ddv}{dn^2} = 2\alpha\frac{dv}{dn}\text{tang}.\alpha n + \alpha\alpha v,$$
$$\frac{d^3v}{dn^3} = 3\alpha\frac{ddv}{dn^2}\text{tang}.\alpha n + 3\alpha\alpha v\frac{dv}{dn} - \alpha^3 v\,\text{tang}.\alpha n,$$
$$\frac{d^4v}{dn^4} = 4\alpha\frac{d^3v}{dn^3}\text{tang}.\alpha n + 6\alpha\alpha\frac{ddv}{dn^2} - 4\alpha^3\frac{dv}{dn}\text{tang}.\alpha n - \alpha^4 v,$$
$$\frac{d^5v}{dn^5} = 5\alpha\frac{d^4v}{dn^4}\text{tang}.\alpha n + 10\alpha\alpha\frac{d^3v}{dn^3} - 10\alpha^3\frac{ddv}{dn^2}\text{tang}.\alpha n - 5\alpha^4\frac{dv}{dn} + \alpha^5 v,$$
etc.

But if for the sake of brevity we may put above $\text{tang}.\alpha n = t$ and we may substitute the preceding values into the following, we will find

$$\frac{dv}{dn} = \alpha vt,$$
$$\frac{ddv}{dn^2} = \alpha\alpha v(2tt+1),$$
$$\frac{d^3v}{dn^3} = \alpha^3 v(6t^3+5t),$$
$$\frac{d^4v}{dn^4} = \alpha^4 v(24t^4+28tt+5),$$
$$\frac{d^5v}{dn^5} = \alpha^5 v(120t^5+180t^3+61t),$$
$$\frac{d^6v}{dn^6} = \alpha^6 v(720t^6+1320t^4+662tt+61),$$
etc.

42. From the consideration of these expressions it will be easy to elicit the operation, with the aid of which any of these expressions can be deduced in the following. Indeed there shall be for a differential of the indefinite order

$$\frac{d^\lambda v}{dn^\lambda} = \alpha^\lambda vP,$$

but for the following order

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}} = \alpha^{\lambda+1}vQ,$$

and because we have seen the value of P to have such a form

$$P = At^\lambda + Bt^{\lambda-2} + Ct^{\lambda-4} + Dt^{\lambda-6} + \text{etc.},$$

then truly the value of Q will be composed from the two following series

$$\begin{aligned} Q = (\lambda+1)At^{\lambda+1} &+ (\lambda-1)Bt^{\lambda-1} + (\lambda-3)Ct^{\lambda-3} + (\lambda-5)Dt^{\lambda-5} + \text{etc.} \\ &\lambda At^{\lambda-1} + (\lambda-2)Bt^{\lambda-3} + (\lambda-4)Ct^{\lambda-5} + \text{etc.}, \end{aligned}$$

from which it is apparent this determination can be represented thus, so that there shall be

$$Q = \frac{td.Pt}{dt} + \frac{dP}{dt}.$$

43. Truly this formula, where from the known value P the following Q is derived, also from that same the nature of the following equation only can be shown. Since by the hypothesis there shall be

$$\frac{d^\lambda v}{dn^\lambda} = \alpha^\lambda vP,$$

by differentiating there will be

$$\frac{d^{\lambda+1}v}{dn^{\lambda}} = \alpha^{\lambda} Pdv + \alpha^{\lambda} vdP;$$

but we have seen initially to be $\frac{dv}{dn} = \alpha vt$ or $dv = \alpha vtdn$, so that with the value substituted there becomes

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}} = \alpha^{\lambda+1} vPt + \alpha^{\lambda} v\frac{dP}{dn};$$

then truly we have taken $t = \text{tang.}\alpha n$, from which by differentiating there becomes $adn = \frac{dt}{1+tt}$, with which value substituted into the latter term there will be obtained

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}} = \alpha^{\lambda+1} vPt + \alpha^{\lambda+1} v\frac{dP(1+tt)}{dt} = \alpha^{\lambda+1} v\left(1 + \frac{dP(1+tt)}{dt}\right),$$

which expression evidently is reduced to this

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}} = \alpha^{\lambda+1} v\frac{td.Pt+dP}{dt},$$

thus so that there shall be

$$Q = \frac{td.Pt+dP}{dt} = Pt + \frac{dP(1+tt)}{dt};$$

from which it is understood, if there may be taken $tt + 1 = 0$, with which made in our formulas with the signs of the terms will be alternating, and with the letter *t* omitted to become $Q = P$; from which it is apparent in this case all the above formulas are going to be given the same value, that which also is evident from the formulas shown above, from which there will be $2 - 1 = 1$, $6 - 5 = 1$, $24 - 28 + 5 = 1$, $120 - 180 + 61 = 1$, $720 - 1320 + 662 - 61 = 1$ etc., from which the significant criterion is obtained, each of these formulas shall be defined correctly by the calculation.

OBSERVATIONES IN ALIQUOT THEOREMATA ILLUSTRISSIMI DE LA GRANGE

Opuscula. analytica 2, 1785, p. 16-41

Postquam aliquod theorema ex iis, quae non ita pridem demonstravi, quo ostendi formulae integralis $\int\frac{(x-1)dx}{lx}$, si post integrationem ponatur $x=1$, valorem esse $=l2$, cum illustri Domino DE LA GRANGE communicassem, is novitate huius argumenti permotus non solum felicissimo successu eius demonstrationem penetravit, sed etiam plurima alia praeclara inventa inde deduxit, quorum uberior enucleatio scientiae analyticae maxima incrementa polliceri videtur, ex quo genera aliquot praeclarissima specimina mecum benevole communicavit, quae statim summo studio sum perscrutatus; et quoniam haec materia attentionem mereri videtur, meas meditationes, quae se mihi hac occasione obtulerunt, fusius sum expositurus. Cum autem hoc quasi novum Analyseos genus potissimum in eiusmodi formulis integralibus versetur, in quibus variabili post integrationem certus valor determinatus tribuitur, ad taediosas verborum ambages evitandas, quas perpetua talium conditionum commemoratio postularet, peculiarem signandi modum adhibebo, quem ante omnia accuratius explicare necesse erit.

HYPOTHESIS

1. Hac signandi ratione

$$\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}$$

esse assumtum, ut evanescat posito $x=a$, tum vero statui est eius valorem penitus fore determinatum.

SCHOLION

ninationis clarius perspiciatur, quoniam P denotat eius naturam repraesentemus linea quadam curva *ixabco*

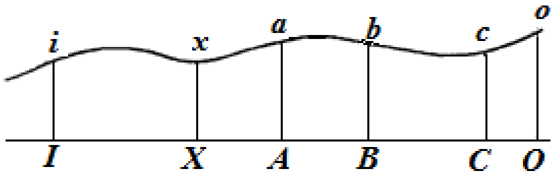

Fig. 1.

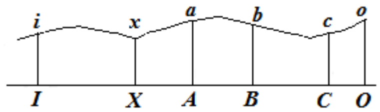

Fig. 1.

super axe *IO* exstructa, cuius quaecunque applicata *Xx* abscissae *IX*= *x* respondens exhibeat ipsam functionem. *P,* ita ut formula integralis $\int Pdx$ indefinite exprimat aream huius curvae. Quodsi iam capiantur abscissae $IA = a$, $IB = b$, quibus respondeant applicatae *Aa* et *Bb,* formula proposita exprimet aream *AaBb* inter applicatas *Aa* et *Bb* interceptam. Eodem modo, si alia quaepiam abscissa statuatur *IC* = *c,* area *AaCc* exprimetur hac formula

$$\int Pdx\begin{bmatrix}\text{ab } x = a\\ \text{ad } x = c\end{bmatrix},$$

area autem *BbCc* ista formula

$$\int Pdx\begin{bmatrix}\text{ab } x = b\\ \text{ad } x = b\end{bmatrix};$$

tum vero ab initio *I* incipiendo area *IiAa* indicabitur per hanc formulam

$$\int Pdx\begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = a\end{bmatrix};$$

unde sponte fiuunt sequentia lemmata ita succincte expressa.

LEMMA 1

3. $$\int Pdx\begin{bmatrix}\text{ab } x = a\\ \text{ad } x = b\end{bmatrix} = -\int Pdx\begin{bmatrix}\text{ab } x = b\\ \text{ad } x = a\end{bmatrix}.$$

Quoniam enim, si *b* ut maius spectetur quam *a,* formula posterior

$$\int Pdx\begin{bmatrix}\text{ab } x = b\\ \text{ad } x = a\end{bmatrix}$$

eandem aream *AaBb* refert quam prior, sed ordine retrogrado, ista expressio pro negativa erit habenda sicque erit quoque

$$\int Pdx\begin{bmatrix}\text{ab } x = a\\ \text{ad } x = b\end{bmatrix} + \int Pdx\begin{bmatrix}\text{ab } x = b\\ \text{ad } x = a\end{bmatrix} = 0.$$

LEMMA 2

4.

$$\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}+\int Pdx\begin{bmatrix}\text{ab } x=b\\ \text{ad } x=c\end{bmatrix}=\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=c\end{bmatrix},$$

quemadmodum inspectio figurae manifesto declarat.

LEMMA. 3

5.

$$\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=c\end{bmatrix}-\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int Pdx\begin{bmatrix}\text{ab } x=b\\ \text{ad } x=c\end{bmatrix},$$

ubi in binis prioribus formulis idem occurrit terminus *a quo,* scilicet $x=a$; terminorum vero *ad quem,* scilicet $x=c$ et $x=b$, posterior $x=b$ dat pro tertia formula terminum *a quo,* prior vero terminum *ad quem.*

LEMMA. 4

6.

$$\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=c\end{bmatrix}-\int Pdx\begin{bmatrix}\text{ab } x=b\\ \text{ad } x=c\end{bmatrix}=\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix},$$

ubi notetur binas formulas priores eundem habere terminum *ad quem,* scilicet $x=c$, terminorum autem *a quo* priorem $x=a$ dare in tertia formula terminum *a quo,* posteriorem vero terminum *ad quem.*

LEMMA 5

7.

$$\int Pdx\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}+\int Pdx\begin{bmatrix}\text{ab } x=b\\ \text{ad } x=c\end{bmatrix}+\int Pdx\begin{bmatrix}\text{ab } x=c\\ \text{ad } x=a\end{bmatrix}=0.$$

SCHOLION

8. His igitur, quae per se sunt maxime perspicua, praemissis argumenta praecipua, quae Celeb. DE LA GRANGE mihi perscripsit, ordine percurram. Primo autem mentionem insignis paradoxi facit, cuius indolem ipse non satis perspicere fatetur, a quo igitur meas meditationes inchoabo.

RESOLUTIO INSIGNIS PARADOXI

9. Cum Vir celeb. etiam invenisset hoc theorema generale

$$\int \frac{x^n - x^m}{lx} \cdot \frac{dx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{n}{m},$$

cuius veritatem non ita pridem pluribus demonstrationibus adstruxi, posuit $x^n = z$ et $x^m = y$; quo facto pars prior $\int \frac{x^{n-1}dx}{lx}$ transformatur in hanc $\int \frac{dz}{lz}$, simili vero modo altera $\int \frac{x^{m-1}dx}{lx}$ in hanc $\int \frac{dy}{ly}$; unde his partibus seorsim positis sequitur fore

$$\int \frac{dz}{lz} \begin{bmatrix} \text{ab } z = 0 \\ \text{ad } z = 1 \end{bmatrix} - \int \frac{dy}{ly} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y = 1 \end{bmatrix} = l\frac{n}{m}.$$

Quare cum hae duae formulae omnino sint similes atque iisdem terminis integrationis contentae, quis non crederet eos etiam inter se perfecte fore aequales sive esse

$$\int \frac{dz}{lz} \begin{bmatrix} \text{ab } z = 0 \\ \text{ad } z = 1 \end{bmatrix} = \int \frac{dy}{ly} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y = 1 \end{bmatrix}?$$

Interim tamen vidimus differentiam inter has formulas esse $l\frac{n}{m}$. Hic igitur se offert quaestio maximi momenti, quemadmodum istam manifestam contradictionem dirimere oporteat.

10. Primo autem hic observari convenit ambas quantitates y et z certo quodam modo a se invicem pendere. Cum enim sit $y = x^m$ et $z = x^n$, erit $y^n = z^m$, quo tamen nexu non impeditur, quominus posito sive $y = 0$ sive $y = 1$ etiam fiat $z = 0$ sive $z = 1$. Interim tamen hinc neutiquam patet, cur ob hanc rationem istae binae formulae

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y = 1 \end{bmatrix} \text{ et } \int \frac{dz}{lz} \begin{bmatrix} \text{a } \ z = 0 \\ \text{ad } z = 1 \end{bmatrix}$$

dispares prodire queant; unde haec observatio ad dubium solvendum nihil plane conferre videtur.

11. Quin etiam nullo prorsus dubio obnoxia videtur haec aequatio multo generalior

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab } y = a \\ \text{ad } y = b \end{bmatrix} = \int \frac{dz}{lz} \begin{bmatrix} \text{a } \ z = a \\ \text{ad } z = b \end{bmatrix},$$

quandoquidem nihil plane impedit, quominus loco z scribamus y vel vicissim; verum plurima phaenomena in analysi observata satis luculenter docent huiusmodi aequalitates interdum exceptionem pati, quando valores evadunt infiniti.
Haec autem circumstantia nostro casu utique locum habet, cum formula integralis

$\int \frac{dy}{ly}$, si ab $y = 0$ ad $y = 1$ extendatur, utique in infinitum excrescat, quod etiam de altera $\int \frac{dz}{lz}$ est tenendum. Si enim [abscissa] fiat $= 1$, applicata nostrae curvae, quae est $\frac{1}{lz}$, manifesto fit infinite magna, unde superior aequalitas generalis

$$\int \frac{dy}{ly}\left[\begin{array}{l}\text{ab } y = a\\ \text{ad } y = b\end{array}\right] - \int \frac{dz}{lz}\left[\begin{array}{l}\text{a } \ z = a\\ \text{ad } z = b\end{array}\right] = 0$$

hanc restrictionem postulat, nisi vel a sit $= 1$ vel $b = 1$, quippe quibus casibus utraque formula fit infinita.

12. His perpensis nullum plane dubium mihi quidem superesse videtur, quin in hac circumstantia vera solutio propositi paradoxi sit quaerenda, quae scilicet in eo versatur, quod sit

$$\text{tam } \int \frac{dy}{ly}\left[\begin{array}{l}\text{ab } y = 0\\ \text{ad } y = 1\end{array}\right] = \infty \ \text{ quam } \int \frac{dz}{lz}\left[\begin{array}{l}\text{a } \ z = 0\\ \text{ad } z = 1\end{array}\right] = \infty,$$

ita ut horum infinitorum differentia possit aequari quantitati finitae cuicunque ideoque in se spectata prorsus non determinetur; quod autem ista differentia nostro casu sit $l\frac{n}{m}$ ideoque determinata, inde venit, quod sit $y^n = z^m$.

13. Simile aliquid evenire potest in formulis simplicioribus, quales sunt $\int \frac{dy}{y}$ et $\int \frac{dz}{z}$, , quippe quarum valores a termino $y = 0$ et $z = 0$ sumti sunt infiniti, unde, etiamsi post integrationem idem terminus *ad quem* statuatur, scilicet $y = 1$ et $z = 1$, tamen hinc nullo modo sequitur differentiam absolute nihilo aequari, quin potius tanquam indeterminata spectari debebit, cum quidem pro aliis terminis integrationis certo sit

$$\int \frac{dy}{y}\left[\begin{array}{l}\text{ab } y = a\\ \text{ad } y = b\end{array}\right] = \int \frac{dz}{z}\left[\begin{array}{l}\text{a } \ z = a\\ \text{ad } z = b\end{array}\right],$$

dummodo neque a neque b fuerit $= 0$ vel $= \infty$.

14. Atque hinc etiam paradoxon proposito penitus simile proferri potest, quod ita se habet

$$\int \frac{dz}{z}\left[\begin{array}{l}\text{ab } z = 0\\ \text{ad } y = \infty\end{array}\right] - \int \frac{dy}{y}\left[\begin{array}{l}\text{ab } y = 0\\ \text{ad } y = \infty\end{array}\right] = la;$$

cuius veritas cum in aprico sit posita, siquidem accipiatur $z = ay$, etiam paradoxon propositum rite dilutum erit censendum.

OBSERVATIONES IN HOC THEOREMA D. DE LA GRANGE

$$\int \frac{x^n - x^m}{lx}\cdot\frac{dx}{x}\left[\begin{array}{l}\text{ab } x = a\\ \text{ad } x = b\end{array}\right] = \int (b^y - a^y)\frac{dy}{y}\left[\begin{array}{l}\text{ab } y = m\\ \text{ad } y = n\end{array}\right]$$

15. Cum equidem ante aliquod tempus reductiones huiusmodi formularum tractassem, alios terminos integrationis praeterquam ab $x = 0$ ad $x = 1$ non sum contemplatus, unde hoc theorema mihi statim altioris indaginis est visum atque omnino dignum, quod summa cura expendatur. Primum igitur in eius veritatem per series inquirere constitui, quod negotium sequenti modo peregi.

16. Cum sit

$$x^\alpha = e^{\alpha lx} = 1 + \alpha lx + \frac{(\alpha lx)^2}{1\cdot 2} + \frac{(\alpha lx)^3}{1\cdot 2\cdot 3} + \text{etc.},$$

erit

$$x^n - x^m = (n-m)\frac{lx}{1} + (n^2 - m^2)\frac{(lx)^2}{1\cdot 2} + \frac{(n^3-m^3)(lx)^3}{1\cdot 2\cdot 3} + \text{etc.}$$

Hanc ergo seriem ducamus in $\frac{dx}{xlx}$, et quia in genere

$$\int (lx)^\lambda \frac{dx}{xlx}\left[\begin{array}{l}\text{ab } x = a\\ \text{ad } x = b\end{array}\right] = \frac{(lb)^\lambda - (la)^\lambda}{\lambda},$$

formulae ad sinistram partem scriptae valor per hanc seriem infinitam exprimetur

$$\frac{n-m}{1}\cdot\frac{lb-la}{1} + \frac{n^2-m^2}{1\cdot 2}\cdot\frac{(lb)^2-(la)^2}{2} + \frac{n^3-m^3}{1\cdot 2\cdot 3}\cdot\frac{(lb)^3-(la)^3}{3} + \text{etc.}$$

17. Simili modo pro formula ad dextram posita per seriem infinitam erit

$$b^y - a^y = y\frac{lb-la}{1} + y^2\frac{(lb)^2-(la)^2}{1\cdot 2} + y^3\frac{(lb)^3-(la)^3}{1\cdot 2\cdot 3} + \text{etc.},$$

quae ergo ducatur in $\frac{dy}{y}$, et quia in genere est

$$\int y^\lambda \frac{dy}{y}\left[\begin{array}{l}\text{ab } y = m\\ \text{ad } y = n\end{array}\right] = \frac{n^\lambda - m^\lambda}{\lambda},$$

valor istius formulae per seriem hanc infinitam exprimetur

$$\frac{n-m}{1}\cdot\frac{lb-la}{1} + \frac{n^2-m^2}{1\cdot 2}\cdot\frac{(lb)^2-(la)^2}{2} + \frac{n^3-m^3}{1\cdot 2\cdot 3}\cdot\frac{(lb)^3-(la)^3}{3} + \text{etc.}$$

Quia igitur haec series cum praecedente perfecte congruit, veritas theorematis firmiter est evicta.

18. Verum hinc neutiquam perspicitur, quomodo sagacissimus Auctor ad hoc theorema sit perductus, quamobrem rebus probe perpensis viam inveni ex iisdem principiis, quibus antehac sum usus, ad easdem formulas perveniendi. Inchoandum autem est ab hac forma simplicissima

$$\int x^{\lambda}\frac{dx}{x}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\frac{b^{\lambda}-a^{\lambda}}{\lambda},$$

ubi utrinque per $d\lambda$ multiplicans denuo integrationem instituo, et cum, uti iam passim demonstratum reperitur, sit

$$\int d\lambda\int x^{\lambda}\frac{dx}{x}=\int\frac{dx}{x}\int x^{\lambda}d\lambda,$$

quaeri tantum debet hoc integrale $\int x^{\lambda}d\lambda$ spectata quantitate x ut constante, ita ut sola λ sit variabilis. Est vero

$$\int x^{\lambda}d\lambda=\frac{x^{\lambda}}{lx}+C,$$

quemadmodum ex elementis calculi exponentialis liquet. Hic vero cardo rei in hoc versatur, ut istud integrale certa lege definiatur, quam deinceps etiam in altera parte observari oportet. Statuamus ergo talia integralia ita capi, ut evanescant posito $\lambda=0$, eritque

$$\int x^{\lambda}d\lambda=\frac{x^{\lambda}-1}{lx},$$

quo pacto pro sinistra parte habebimus

$$\int d\lambda\int x^{\lambda}\frac{dx}{x}=\int\frac{dx}{x}\cdot\frac{x^{\lambda}-1}{lx}.$$

19. Pro parte autem dextra habebimus

$$\int\frac{d\lambda}{\lambda}(b^{\lambda}-a^{\lambda}),$$

qua formula eadem lege integrata, ut facto $\lambda=0$ prodeat nihilum, hunc valorem more hic recepto repraesentare licebit

$$\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=\lambda\end{bmatrix},$$

Hic enim nil aliud fecimus, nisi quod pro λ scripsimus y et facta integratione loco y eius valorem λ restitui assumsimus, sicque assecuti sumus sequentem formulam

$$\int (x^{\lambda}-1)\frac{dx}{xlx}\left[\begin{array}{l}\text{ab } x=a\\ \text{ad } x=b\end{array}\right]=\int \frac{dy}{y}(b^{y}-a^{y})\left[\begin{array}{l}\text{ab } y=0\\ \text{ad } y=\lambda\end{array}\right],$$

quam tanquam theorema utilissimum spectare licet.

20. Vi ergo huius theorematis nanciscimur sequentes reductiones

$$\int (x^{n}-1)\frac{dx}{xlx}\left[\begin{array}{l}\text{ab } x=a\\ \text{ad } x=b\end{array}\right]=\int \frac{dy}{y}(b^{y}-a^{y})\left[\begin{array}{l}\text{ab } y=0\\ \text{ad } y=n\end{array}\right]$$

et

$$\int (x^{m}-1)\frac{dx}{xlx}\left[\begin{array}{l}\text{ab } x=a\\ \text{ad } x=b\end{array}\right]=\int \frac{dy}{y}(b^{y}-a^{y})\left[\begin{array}{l}\text{ab } y=0\\ \text{ad } y=m\end{array}\right];$$

quare si formula posterior a priore subtrahatur, erit

$$\int (x^{n}-x^{m})\frac{dx}{xlx}\left[\begin{array}{l}\text{ab } x=a\\ \text{ad } x=b\end{array}\right]=\int \frac{dy}{y}(b^{y}-a^{y})\left[\begin{array}{l}\text{ab } y=0\\ \text{ad } y=n\end{array}\right]-\int \frac{dy}{y}(b^{y}-a^{y})\left[\begin{array}{l}\text{ab } y=0\\ \text{ad } y=m\end{array}\right];$$

verum ista formula ad dextram posita per reductionem in Lemmate 3 ostensam revocatur ad hanc formam simpliciorem

$$\int \frac{dy}{y}(b^{y}-a^{y})\left[\begin{array}{l}\text{ab } y=m\\ \text{ad } y=n\end{array}\right];$$

unde patet hoc modo ipsum hoc insigne theorema etiam ex nostris principiis investigari potuisse.

21. Hoc autem theoremate generalissimo Vir ingeniosissimus est usus ad theorema meum demonstrandum, quo ostendi esse

$$\int (x^{n}-x^{m})\frac{dx}{xlx}\left[\begin{array}{l}\text{ab } x=0\\ \text{ad } x=1\end{array}\right]=l\frac{n}{m};$$

tantum enim opus erat, ut caperetur $a=0$ et $b=1$, quo pacto formula ad dextram posita integralis abit in

$$\int \frac{dy}{y}\left[\begin{array}{l}\text{ab } y=m\\ \text{ad } y=n\end{array}\right],$$

cuius valor manifesto fit $ln-lm=l\frac{m}{n}$, quae est nova demonstratio mei theorematis, cuiusmodi quidem dudum plures alias dederam.

OBSERVATIONS IN THEOREMA D. DE LA GRANGE

$$\int \frac{(x^n - x^m)dx}{(1+x^r)lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = l\,\frac{\text{tang.}\frac{(n+1)\pi}{2r}}{\text{tang.}\frac{(m+1)\pi}{2r}}$$

22. Quia hic ambo exponentes m et n neque a se invicem neque ab exponente r pendent, manifestum est pro utraque potestate x^m et x^n seorsim integrale talem formam habere debere

$$\int \frac{x^n dx}{(1+x^r)lx} = l\text{tang.}\frac{(n+1)\pi}{2r} + C \text{ et } \int \frac{x^m dx}{(1+x^r)lx} = l\text{tang.}\frac{(m+1)\pi}{2r} + C.$$

Si enim posterior forma a priore subtrahatur, constans C ex calculo egreditur et ipsum integrale propositum resultat. Hic igitur plurimum intererit valorem istius constantis C determinasse.

23. Inter formulas integrales, quarum valores pro casu, quo post integrationem variabilis infinita statuitur, ex primis principiis calculi integralis assignavi, reperitur ista

$$\int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = \frac{\pi}{2k\cos.\frac{n\pi}{2k}} = \frac{\pi}{2k\sin.\frac{(k+n)\pi}{2k}},$$

ubi autem assumitur exponentem n non maiorem capi quam k. Quodsi iam hic exponens n ut variabilis tractetur spectata ipsa x ut constante et utrinque per dn multiplicetur denuoque integretur, formula sinistra erit

$$\int dn \int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x} = \int \frac{dx}{x(1+x^{2k})}\int x^{k+n}dn,$$

ubi postremum integral fit

$$\int dn = \frac{x^{k+n}}{lx} + C.$$

Ut autem hoc integrale determinetur, constantem ita definiamus, ut id evanescat posito $n=0$, unde obtinetur

$$\int x^{k+n}dn = \frac{x^{k+n} - x^k}{lx},$$

ita ut formula integralis ad sinistram posita futura sit

$$\int \frac{x^{k+n} - x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}.$$

24. Pro parte dextra autem habebimus hoc integrale

$$\int \frac{\pi dn}{2k\sin.\frac{(k+n)\pi}{2k}}$$

etiam ita sumendum, ut evanescat posito $n=0$. Hunc in finem statuamus angulum $\frac{(k+n)\pi}{2k}=\varphi$, et quia hinc erit $d\varphi=\frac{\pi dn}{2k}$, formula nostra integranda erit $\int\frac{d\varphi}{\sin.\varphi}$, cuius integral per regulas notas in genera est

$$l\text{tang.}\tfrac{1}{2}\varphi+C=l\text{tang.}\frac{(k+n)\pi}{4k}+C,$$

quod facto $n=0$ abit in $l\text{tang.}\frac{\pi}{4}+C$. Quare cum $\text{tang.}\frac{\pi}{4}=1$ et $l1=0$, evidens est constantem C fore $=0$, ita ut integrale hoc quaesitum sit $l\text{tang.}\frac{(k+n)\pi}{4k}$. Hinc ergo assecuti sumus istam reductionem generalem

$$\int\frac{x^{k+n}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\text{tang.}\frac{(k+n)\pi}{4k},$$

ubi autem probe notari oportet exponentes *m* et *n* maiores capi non licere quam *k.*

25. Cum igitur loco *n* alium numerum *m* sumendo simili modo sit

$$\int\frac{x^{k+m}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\text{tang.}\frac{(k+m)\pi}{4k},$$

subtrahatur ista formula a praecedente et obtinebitur ista

$$\int\frac{x^{k+n}-x^{k+m}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{l\text{tang.}\frac{(k+n)\pi}{4k}}{l\text{tang.}\frac{(k+m)\pi}{4k}},$$

quae manifesto cum forma proposita congruit, si modo loco $k+n-1$ scribatur *n* et *m* loco $k+m-1$, at loco exponentis $2k$ scribatur *r;* tum enim manifesto fiet

$$\int\frac{x^{n}-x^{m}}{1+x^{r}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{l\text{tang.}\frac{(n+1)\pi}{2r}}{l\text{tang.}\frac{(m+1)\pi}{2r}}.$$

26. Quoniam ista analysis nos perduxit ad hanc formam

$$\int\frac{x^{k+n}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\text{tang.}\frac{(k+n)\pi}{4k},$$

hic maximi momenti erit observasse semper fore

$$\int \frac{x^k}{1+x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = 0,$$

id quod ita ostendere possum. Ponatur $x^k = z$; erit

$$x^{k-1}dx = \frac{dz}{k} \text{ et } lx = \frac{lz}{k}$$

sicque ista formula induet hanc formam $\int \frac{dz}{(1+zz)lz}$, ubi termini integrationis etiamnunc sunt $z=0$ et $z=\infty$. Fiat porro $z = \text{tang.}\varphi$, unde termini integrationis erunt $\varphi = 0$ et $\varphi = \frac{\pi}{2}$.; hinc autem ob $d\varphi = \frac{dz}{1+zz}$ nascetur ista formula $\int \frac{d\varphi}{l\,tang\,\varphi} \begin{bmatrix} \text{a } \varphi=0 \\ \text{ad } \varphi=\frac{\pi}{2} \end{bmatrix}$, cuius valorem in nihilum abire ostendi debet.

27. Ad hoc demonstrandum statuatur axis $IH = \frac{\pi}{2}$ (Fig. 2), super quo ab initio I sumta abscissa indefinita $Ip = \varphi$ applicata sit $= \frac{1}{l\text{tang.}\varphi}$. Quodsi ergo hic axis IH in O bisecetur, ut sit $IO = \frac{\pi}{4}$, in hoc puncto applicata erit

$$= \frac{1}{l\text{tang.}\frac{\pi}{4}} = \infty.$$

Iam ab hoc puncto O utrinque capiantur intervalla aequalia $Op = Oq = \omega$ et pro puncto p erit $\varphi = \frac{\pi}{4} - \omega$ sicque in hoc puncto p applicata erit

$$= \frac{1}{l\text{tang.}(\frac{\pi}{4} - \omega)};$$

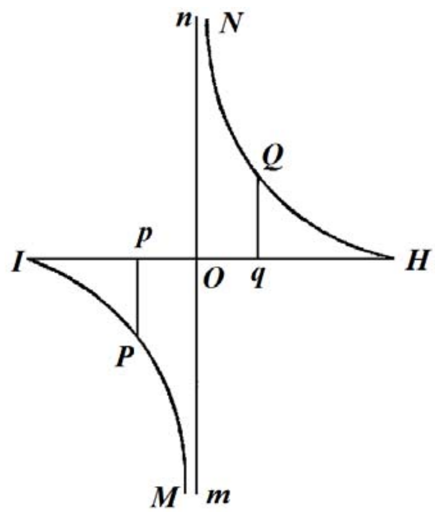

Fig. 2

est vero $\text{tang.}(\frac{\pi}{4} - \omega) = \text{cot.}(\frac{\pi}{4} + \omega)$, quare, cum sit $l\text{cot.} = -l\text{tang.}$, applicata in hoc puncto p erit

$$= \frac{-1}{l\text{tang.}(\frac{\pi}{4} + \omega)};$$

at quia est $Iq = \frac{\pi}{4} + \omega$, erit applicata in puncto q

$$= \frac{+1}{l\text{tang.}(\frac{\pi}{4} + \omega)}$$

sicque aequalis est applicatae in p, sed in contrarium vergens. Ita si applicata sursum directa fuerit qQ, in puncto p eadem applicata deorsum erit directa $pP = qQ$.

28. Quodsi ergo talis curva super axe $IH = \frac{\pi}{2}$; exstruatur, ita ut abscissae φ respondeat applicata $\frac{1}{l\text{tang.}\varphi}$, haec curva ex duabus portionibus inter se perfecte aequalibus constabit

circa punctum medium *O* ita dispositis, ut curva sinistra sit *IPM* in infinitum descendens ad asymptotam *Om,* pars autem dextra simili modo a *H* sinistrorsum sursum ascendet ad asymptotam *On.* Quare cum formula integralis $\int \frac{d\varphi}{l\,tang\,\varphi}$ a $\varphi = 0$ ad $\varphi = \frac{\pi}{2}$ extensa exprimat totius huius curvae ab *I* usque ad *H* protensae aream, evidens est totam hanc aream ad nihilum redigi, quia portio eius negative sumenda perfecte similis est portioni positive sumendae.

29. Sic igitur per demonstrationem omnino singularem evictum est semper esse

$$\int \frac{x^k}{1+x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = 0,$$

quod certe est theorema in hoc genere maxime notatu dignum. Quodsi ergo cum illustri D. DE LA GRANGE statuamus $2k = r$, erit

$$\int \frac{x^{\frac{1}{2}r-1} dx}{(1+x^r)lx} = 0;$$

praeterea vero pro nostra formula § 24 exhibita ob

$$\int \frac{x^k dx}{(1+x^{2k})xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = 0$$

deducitur istud theorema omnino notabile

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = l\text{tang.} \frac{(k+n)\pi}{4k},$$

quod more D. DE LA GRANGE ita proponi potest

$$\int \frac{x^n dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = l\text{tang.} \frac{(n+1)\pi}{2r};$$

sicque patet constantem illam supra (§ 22) a nobis inductam revera nihilo aequari.

30. Quoniam demonstratio huius theorematis methodo satis insueta innititur, eius veritatem per series ostendisse iuvabit. Ad hoc autem valorem formulae

$$\int \frac{x^{\lambda-1} dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$

In duas partes divelli necesse est (scilicet loco *n* scribendo $\lambda - 1$), quae sint

$$P = \int \frac{x^{\lambda-1} dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} \quad \text{et} \quad Q = \int \frac{x^{\lambda-1} dx}{(1+x^r)lx} \cdot \begin{bmatrix} \text{ab } x = 1 \\ \text{ad } x = \infty \end{bmatrix},$$

ita ut $P+Q$ exprimat valorem, quem quaerimus. Nunc in posteriore parte loco x scribamus $\frac{1}{z}$ fietque

$$Q=\int\frac{z^{-\lambda}}{1+z^{-r}}\cdot\frac{dz}{zlz}\left[\begin{matrix}\text{ab } z=1\\ \text{ad } z=0\end{matrix}\right]=\int\frac{z^{r-\lambda}}{1+z^{r}}\cdot\frac{dz}{zlz}\left[\begin{matrix}\text{ab } z=1\\ \text{ad } z=0\end{matrix}\right]$$

et commutatis terminis integrationis

$$Q=-\int\frac{z^{r-\lambda}}{1+z^{r}}\cdot\frac{dz}{zlz}\left[\begin{matrix}\text{ab } z=0\\ \text{ad } z=1\end{matrix}\right].$$

Nunc autem loco z scribamus x; quia termini integrationis utrinque sunt iidem, erit

$$P+Q=\int\frac{x^{\lambda}-x^{r-\lambda}}{1+x^{r}}\cdot\frac{dx}{xlx}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right],$$

cuius ergo valor formulae propositae est aequalis.

31. Iam fractionem $\frac{1}{1+x^{r}}$ in seriem infinitam convertamus

$$1-x^{r}+x^{2r}-x^{3r}+x^{4r}-\text{etc.},$$

cuius singuli termini in $\frac{dx}{xlx}(x^{\lambda}-x^{r-\lambda})$ ducti producunt

$$\frac{dx}{xlx}(x^{\lambda}-x^{r-\lambda})-\frac{dx}{xlx}(x^{r+\lambda}-x^{2r-\lambda})+\frac{dx}{xlx}(x^{2r+\lambda}-x^{3r-\lambda})-\frac{dx}{xlx}(x^{3r+\lambda}-x^{4r-\lambda})+\text{etc.}$$

Cum autem per theorema principale in hoc genere sit

$$\int\frac{dx}{xlx}(x^{\alpha}-x^{\gamma})\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right]=l\frac{\alpha}{\beta},$$

singulis membris hoc modo integratis prodibit

$$P+Q=l\frac{\lambda}{r-\lambda}-l\frac{r+\lambda}{2r-\lambda}+l\frac{2r+\lambda}{3r-\lambda}-l\frac{3r+\lambda}{4r-\lambda}+\text{etc.}$$

32. Omnes hos logarithmos in unicum compingere licebit ratione habita signi cuiusque hocque modo reperietur fore

$$P+Q=l\frac{\lambda}{r-\lambda}\cdot\frac{2r-\lambda}{r+\lambda}\cdot\frac{2r+\lambda}{3r-\lambda}\cdot\frac{4r-\lambda}{3r+\lambda}\cdot\frac{4r+\lambda}{5r-\lambda}\cdot\frac{6r-\lambda}{5r+\lambda}\cdot\text{etc.}$$

At vero in *Introductione in Analysin Infinitorum* p. 147 ostendi esse

$$\text{tang.}\tfrac{m\pi}{2n}=\tfrac{m}{n-m}\cdot\tfrac{2n-m}{n+m}\cdot\tfrac{2n+m}{3n-m}\cdot\tfrac{4n-m}{3n+m}\cdot\tfrac{4n+m}{5n-m}\cdot\text{etc.},$$

quae series manifesto in inventam transformatur statuendo $m=\lambda$ et $n=r$, ita ut nunc sit $P+Q=l\text{tang.}\tfrac{\lambda\pi}{2r}$, prorsus uti supra est inventum.

ADDITAMENTUM

33. In dissertatione Actorum Tomo V, parte I, inserta, unde desumsi hoc theorema

$$\int\frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

simul occurrunt sequentia

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{k}\text{tang}\frac{n\pi}{2k},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2k}\text{tang.}\frac{n\pi}{2k},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{2\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int\frac{x^{k\pm n}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

quas formulas ergo simili modo tractare operae pretium erit.

34. Incipiamus igitur a formula

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

quia praecedens cum formula iam tractata prorsus conveniret; quae si ducatur in dn et ita integretur, ut integrale evanescat posito $n=0$, quoniam est

$$\int x^{k-n}dn=-\frac{x^{k-n}-x^{k}}{lx}\quad\text{et}\quad\int x^{k+n}dn=\frac{x^{k+n}-x^{k}}{lx},$$

tum vero, ut ante vidimus,

$$\int \frac{\pi dn}{2k\cos.\frac{n\pi}{2k}} = l\text{tang.}\frac{(k+n)\pi}{4k},$$

prodibit haec integratio

$$\int \frac{x^{k+n}-x^{k-n}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\text{tang.}\frac{(k+n)\pi}{4k},$$

qui ergo valor prorsus convenit cum eo, quem pro formula

$$\int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}$$

invenimus.

35. Simili modo tractemus sequentem formulam

$$\int \frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = \frac{\pi}{k}\text{tang}\frac{n\pi}{2k},$$

quae ducta in dn et ut supra integrata praebet a parte sinistra

$$\int \frac{2x^{k}-x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix},$$

a parte autem dextra

$$\int \frac{\pi dn}{k}\text{tang}\frac{n\pi}{2k} = \int \frac{\pi dn\sin.\frac{n\pi}{2k}}{k\cos.\frac{n\pi}{2k}}.$$

:

Ad hoc integrandum fiat $\frac{n\pi}{2k}=\varphi$ eritque $\frac{\pi dn}{k}=2d\varphi$ sicque formula integranda erit

$$\int \frac{2d\varphi\sin.\varphi}{\cos.\varphi} = -2l\cos.\varphi + C = -2l\cos.\frac{n\pi}{2k}+C.$$

Fiat igitur $n=0$ esseque debebit $-2l1+C=0$ ideoque constans $C=0$,
quocirca haec integratio nobis suppeditat sequentem formulam

$$\int \frac{2x^{k}-x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = -2l\cos.\frac{n\pi}{2k};$$

sequens autem formula $\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}$ singulari evolutione non indiget, cum eius valor sit huius semissis.

36. Evolvamus casum, quo $k=2$ et $n=1$, et ex parte sinistra habemus

$$-\int \frac{(1-x)^2}{(1-x^4)}\cdot\frac{dx}{lx} = -\int \frac{1-x}{(1+x)(1+xx)}\cdot\frac{dx}{lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix};$$

at vero ex dextra parte $-2l\cos.\frac{\pi}{4}=2l\sqrt{2}=l2.$ Verum fractio $\frac{1-x}{(1+x)(1+xx)}$ resolvitur in has duas $\frac{1}{1+x}-\frac{x}{1+xx}$, unde formula nostra resolvitur in has duas

$$-\int\frac{dx}{(1+x)lx}+\int\frac{xdx}{(1+xx)lx}=l2.$$

Sed ex forma generali

$$\int\frac{x^{\lambda-1}dx}{(1+x^r)lx}=l\text{tang.}\frac{\lambda\pi}{2r}$$

utriusque formulae valor in infinitum excrescit sicque nihil impedit, quominus differentia $=l2$.

37. Quodsi hic in posteriore formula statuamus $xx=z$, ea abibit in hanc $\int\frac{dz}{(1+z)lz}$ quae priori omnino est similis atque sub iisdem terminis integrationis continetur. Hic igitur iterum occurrit paradoxon prorsus simile illi, quod ab Illustr. DE LA GRANGE fuit memoratum; duae scilicet hic habentur formae prorsus pares $\int\frac{dx}{(1+x)lx}$ et $\int\frac{dz}{(1+z)lz}$ quarum utramque a termino 0 ad ∞ integrari oportet; nihilo tamen minus earum differentia non est nulla, sed, uti vidimus, $=l2$. Atque hinc solutio huius paradoxi in eo manifesto est sita, quod utriusque integralis valor in infinitum excrescit.

38. Quodsi binas postremas formulas eodem modo tractare et per *dn* multiplicatas integrare velimus, a parte sinistra resultat ista formula integralis

$$\int\frac{x^{k+n}-x^{k-n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix};$$

pro dextra autem parte nanciscimur hanc formulam integralem

$$\int\frac{2\pi dn\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}$$

a termino $n=0$ extendendam. Verum haec integratio nullo modo succedit; si enim ponamus $\frac{n\pi}{k}=\varphi$, fiet $\frac{n\varphi}{k}=\frac{\eta\varphi}{\pi}=\alpha\varphi$ ponendo $\frac{\eta}{\pi}=\alpha$, unde formula integranda erit

$\frac{2}{\sin.\eta}\int\frac{d\varphi\sin.\alpha\varphi}{\sin.\varphi}$, cuius valor aliter nisi per signum summatorium exprimi non potest, sicque nulla concinna theoremata hinc derivare licet.

39. Quemadmodum autem hic exponentem n ut variabilem spectando transformationes per integrationem instituimus, ita etiam differentiatio egregias transformationes suppeditabit, quod argumentum unica formula principali illustrasse sufficiet. Consideremus scilicet hanc formulam

$$\int\frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

quae sumto exponente n ut solo variabili continuo differentietur, ubi notandum est esse $d.x^{k+n}=x^{k+n}dnlx.$ At vero pro formula $\frac{\pi}{2k\cos.\frac{n\pi}{2k}}$ scribamus litteram v, quae ergo spectanda erit tanquam functio ipsius n, cuius ergo differentialia cuiusque ordinis sunt in nostra potestate. Hinc igitur sequentes reductiones consequemur

$$\int\frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}lx=\frac{dv}{dn}$$

sive

$$\int\frac{x^{k+n-1}dxlx}{1+x^{2k}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{dv}{dn},$$

$$\int\frac{x^{k+n-1}dx(lx)^2}{1+x^{2k}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{ddv}{dn^2},$$

$$\int\frac{x^{k+n-1}dx(lx)^3}{1+x^{2k}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{d^3v}{dn^3},$$

$$\int\frac{x^{k+n-1}dx(lx)^4}{1+x^{2k}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{d^4v}{dn^4},$$

$$\int\frac{x^{k+n-1}dx(lx)^5}{1+x^{2k}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{d^5v}{dn^5},$$

etc.

40. Cum igitur hinc totum negotium ad differentialia continua ipsius v reducatur, ea sequenti modo commodissime reperire licebit. Cum enim sit

$$v=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

erit $v\cos.\frac{n\pi}{2k}=\frac{\pi}{2k}$ hincque continuo differentiando obtinebimus sequentes formulas

$$\frac{dv}{dn}\cos.\frac{n\pi}{2k}-\frac{\pi}{2k}v\sin.\frac{n\pi}{2k}=0,$$
$$\frac{ddv}{dn^2}\cos.\frac{n\pi}{2k}-\frac{2\pi}{2k}\frac{dv}{dn}\sin.\frac{n\pi}{2k}-\frac{\pi\pi}{4kk}v\cos.\frac{n\pi}{2k}=0,$$
$$\frac{d^3v}{dn^3}\cos.\frac{n\pi}{2k}-\frac{3\pi}{2k}\frac{ddv}{dn^2}\sin.\frac{n\pi}{2k}-\frac{3\pi\pi}{4kk}\frac{dv}{dn}\cos.\frac{n\pi}{2k}+\frac{\pi^3}{8k^3}v\sin.\frac{n\pi}{2k}=0,$$
$$\frac{d^4v}{dn^4}\cos.\frac{n\pi}{2k}-\frac{4\pi}{2k}\frac{d^3v}{dn^3}\sin.\frac{n\pi}{2k}-\frac{6\pi\pi}{4kk}\frac{ddv}{dn^2}\cos.\frac{n\pi}{2k}+\frac{4\pi^3dv}{8k^3}\sin.\frac{n\pi}{2k}+\frac{\pi^4}{16k^4}v\cos.\frac{n\pi}{2k}=0,$$
etc.

unde singula differentialia altiora ex inferioribus formari possunt.

41. Quo autem hae operationes magis subleventur, statuamus brevitatis gratia $\frac{\pi}{2k}=\alpha$, ut sit $v=\frac{\alpha}{\cos.\alpha n}$, atque singula differentialia ex superioribus aequationibus sequenti modo determinabuntur

$$\frac{dv}{dn}=\alpha v\,\text{tang}.\alpha n,$$
$$\frac{ddv}{dn^2}=2\alpha\frac{dv}{dn}\text{tang}.\alpha n+\alpha\alpha v,$$
$$\frac{d^3v}{dn^3}=3\alpha\frac{ddv}{dn^2}\text{tang}.\alpha n+3\alpha\alpha v\frac{dv}{dn}-\alpha^3v\,\text{tang}.\alpha n,$$
$$\frac{d^4v}{dn^4}=4\alpha\frac{d^3v}{dn^3}\text{tang}.\alpha n+6\alpha\alpha\frac{ddv}{dn^2}-4\alpha^3\frac{dv}{dn}\text{tang}.\alpha n-\alpha^4v,$$
$$\frac{d^5v}{dn^5}=5\alpha\frac{d^4v}{dn^4}\text{tang}.\alpha n+10\alpha\alpha\frac{d^3v}{dn^3}-10\alpha^3\frac{ddv}{dn^2}\text{tang}.\alpha n-5\alpha^4\frac{dv}{dn}+\alpha^5v,$$
etc.

Quodsi brevitatis gratia insuper statuamus $\text{tang}.\alpha n=t$ et praecedentes valores in sequentibus substituamus, reperiemus

$$\frac{dv}{dn}=\alpha vt,$$
$$\frac{ddv}{dn^2}=\alpha\alpha v(2tt+1),$$
$$\frac{d^3v}{dn^3}=\alpha^3v(6t^3+5t),$$
$$\frac{d^4v}{dn^4}=\alpha^4v(24t^4+28tt+5),$$
$$\frac{d^5v}{dn^5}=\alpha^5v(120t^5+180t^3+61t),$$
$$\frac{d^6v}{dn^6}=\alpha^6v(720t^6+1320t^4+662tt+61),$$
etc.

42. Ex consideratione harum expressionum facilis erui potest operatio, cuius ope ex qualibet earum expressionum sequens colligi potest. Sit enim pro differentiali ordinis indefiniti

$$\frac{d^{\lambda}v}{dn^{\lambda}}=\alpha^{\lambda}vP,$$

at pro ordine sequente

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}}=\alpha^{\lambda+1}vQ,$$

et quoniam vidimus valorem ipsius P talem habere formam

$$P=At^{\lambda}+Bt^{\lambda-2}+Ct^{\lambda-4}+Dt^{\lambda-6}+\text{etc.},$$

tum valor ipsius Q ex sequentibus binis seriebus erit compositus

$$\begin{aligned}Q=(\lambda+1)At^{\lambda+1}+(\lambda-1)Bt^{\lambda-1}+(\lambda-3)Ct^{\lambda-3}+(\lambda-5)Dt^{\lambda-5}+\text{etc.}\\ \lambda At^{\lambda-1}+(\lambda-2)Bt^{\lambda-3}+(\lambda-4)Ct^{\lambda-5}+\text{etc.},\end{aligned}$$

unde patet hanc determinationem ita repraesentari posse, ut sit

$$Q=\frac{td.Pt}{dt}+\frac{dP}{dt}.$$

43. Haec vero formula, qua ex cognito valore P sequens Q derivatur, etiam ex ipsa natura rei sequenti modo ostendi potest. Cum per hypothesin sit

$$\frac{d^{\lambda}v}{dn^{\lambda}}=\alpha^{\lambda}vP,$$

erit differentiando

$$\frac{d^{\lambda+1}v}{dn^{\lambda}}=\alpha^{\lambda}Pdv+\alpha^{\lambda}vdP;$$

initio autem vidimus esse $\frac{dv}{dn}=\alpha vt$ sive $dv=\alpha vtdn$, quo valore substituto fit

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}}=\alpha^{\lambda+1}vPt+\alpha^{\lambda}v\frac{dP}{dn};$$

tum vero assumsimus $t=\text{tang.}\alpha n$, unde differentiando fit $adn=\frac{dt}{1+tt}$, quo valore in postremo termino substituto obtinebitur

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}}=\alpha^{\lambda+1}vPt+\alpha^{\lambda+1}v\frac{dP(1+tt)}{dt}=\alpha^{\lambda+1}v\left(1+\frac{dP(1+tt)}{dt}\right),$$

quae forma manifesto reducitur ad hanc

$$\frac{d^{\lambda+1}v}{dn^{\lambda+1}} = \alpha^{\lambda+1}v\frac{td.Pt+dP}{dt},$$

ita ut sit

$$Q = \frac{td.Pt+dP}{dt} = Pt + \frac{dP(1+tt)}{dt};$$

unde intelligitur, si sumatur $tt+1=0$, quo facto in nostris formulis signa terminorum alternabuntur, et omissa littera *t* fieri $Q = P$; unde patet hoc casu omnes formulas superiores eundem valorem esse adepturas, id quod etiam ex formulis supra exhibitis manifestum est, ex quibus erit $2-1=1$, $6-5=1$, $24-28+5=1$, $120-180+61=1$, $720-1320+662-61=1$ etc., unde insigne criterium obtinetur, utrum formulae istae recte sint per calculum definitae.