CONSIDERATIONS ON A THEOREM OF FERMAT CONCERNED WITH THE RESOLUTION OF NUMBERS INTO POLYGONAL NUMBERS

[E586]

Opuscula analytica 2, 1785, p. 3-15
[Presented to the Academy on the 12th of December, 1774]

1. This theorem of FERMAT includes the following assertions for an infinite number of cases:

I. *Every number to be the sum of three or fewer triangular numbers.*
II. *Every number to be the sum of four or fewer tetragonal or square numbers.*
III. *Every number to be the sum of five or fewer pentagonal numbers*.
IV. *Every number to be the sum of six or fewer hexagonal numbers.*
V. *Every number to be the sum of seven or fewer heptagonal numbers.*
etc.

Since Fermat himself may have completed the demonstration of these theorems requiring to be found, surely it cannot be doubted its demonstration would have been based on the most certain principles; from which therefore it is the more to be lamented that evidently the proof to have perished after his death, as plainly no vestige had been found, and since without doubt most Geometers have labored in vain in finding his demonstrations. Hence indeed with the second assertion excepted concerning the resolution of numbers into square numbers, the complete demonstration of which has been brought into the light by the most ingenious Lagrange, but which has been deduced from principles of this kind to be shown, so that thence clearly no aid may be expected for the remaining demonstrations.

2. Therefore it is agreed a huge distinction intervenes between the resolution into squares and the remaining polygonal numbers, which consists mainly in this, because the resolution into four squares plainly may itself be extend to all numbers, both fractions as well as whole numbers, since the resolution into other polygonal numbers may be restricted to whole numbers only, and thus for the truth to be agreed upon only under a certain limitation. Indeed the resolution into three trigonal numbers only is restricted to whole numbers only, since indefinitely many fractions may be given, which in no way will be allowed to be resolved into three parts, contained in the formula $\frac{xx+x}{2}$; or as if for the fraction $\frac{1}{2}$ we may wish to put in place :

$$\frac{1}{2}=\frac{xx+x}{2}+\frac{yy+y}{2}+\frac{zz+z}{2},$$

on multiplying by 8 must become

$$4=4xx+4x+4yy+4y+4zz+4zzz,$$

and hence with three ones added there will become

$$7=(2x+1)^2+(2y+1)^2+(2z+1)^2.$$

But in no manner can it be shown that the number 7 to be the sum of three squares. Whereby if by which, the demonstration of these parts thus were desired to put in place, so that, for some proposed number *N,* this equation may be proposed :

$$N=\tfrac{xx+x}{2}+\tfrac{yy+y}{2}+\tfrac{zz+z}{2}$$

itself being required to be resolved, the time and the effort would have been wasted.

3. Truly also again the resolution into pentagonal numbers is restricted to whole numbers, but it arises from triangular numbers used by another longer way. For if we wish to admit fractions in the formula $\tfrac{3xx-x}{2}$ also to be contained among pentagonal numbers, then plainly thus all numbers will be allowed to be split up into four pentagonal numbers. Indeed for some proposed number *N* if we may put

$$N=\tfrac{3xx-x}{2}+\tfrac{3yy-y}{2}+\tfrac{3zz-z}{2}+\tfrac{3vv-v}{2},$$

on multiplying by 24 it will become

$$24N=36xx-12x+36yy-12y+36zz-12z+36vv-12v,$$

from which with four units added there will become

$$24N+4=(6x-1)^2+(6y-1)^2+(6z-1)^2+(6v-1)^2.$$

Therefore since the number $24N+4$ certainly shall be the sum of four squares, which shall be $aa+bb+cc+dd$, hence we will find :

$$x=\tfrac{a+1}{6},\quad y=\tfrac{b+1}{6},\quad z=\tfrac{c+1}{6},\quad v=\tfrac{d+1}{6},$$

and the pentagonal numbers formed from these roots summed together will be equal to the proposed number *N*. Truly if we may admit only whole numbers with nothing less, as FERMAT evidently postulates, certainly numbers of this kind may be given, which can by no means be resolved into fewer than five pentagonal numbers.

4. Truly besides, even if hence we may exclude fractions and we may wish to admit only whole numbers, yet a new limitation to be introduced, and from the order of the pentagons, we must exclude all these of which the roots are negative numbers, from all these orders of pentagonal numbers. Indeed since the general formula of pentagonal numbers $\tfrac{3xx-x}{2}$ with negative roots *x* taken shall produce these number: 2, 7, 15, 26, 40

etc., if also we may wish to admit these, no further five, but plainly only three pentagonal numbers will suffice for the production of all pentagonal numbers, and with such a twin limitation, much more is necessary for the remaining polygonal numbers, so that for the truth shall be agreed on for the Fermat theorems, which limitation without doubt is in the cause, because at this stage none of the Geometers after Fermat has been allowed to enter into the demonstration of these cases.

5. Therefore since in the demonstrations which are desired, by necessity an account of these restrictions shall be required to be had, we will represent that theorem of Fermat in another form, which now may contain these same limitations, which may be seen to happen in the following most convenient manner.

FOR THE RESOLUTION IN TRIGONAL NUMBERS

6. The series of powers may be considered

$$1+x+x^3+x^6+x^{10}+x^{15}+x^{21}+x^{28}+x^{36}+\ x^{45}+\text{etc.},$$

in which the powers of x themselves are progressing according to these trigonal numbers

$$0, 1, 3, 6, 10, 15, 21, 28 \text{ etc.},$$

and initially it is evident, if the square of this series may be taken, other powers of x cannot occur, unless the exponents of these shall be the sum of two triangular numbers. In the same manner it is understood, if the cube of the same series is taken, in that other powers of that do not occur, unless the exponents of these shall be the sum of three triangular numbers. On account of which Fermat's first theorem is reduced to this, so that, if the series may be assumed for the cube, this series may be established by all the ascending powers of x :

$$1+Ax+Bxx+ax^3+Dx^4+Ex^5+Fx^6+Gx^7+Hx^8+\text{etc.},$$

it may be shown in this series plainly no coefficient to be equal to nothing. But from that form it is evident none of these coefficients can become negative.

FOR THE RESOLUTION IN SQUARE OR TETRAGONAL NUMBERS

7. Here this series of powers may be considered:

$$P=1+x+x^4+x^9+x^{16}+x^{25}+x^{36}+\text{etc.},$$

of which the biquadratic P^4, if it may be set out, will include all the powers of x, of which the exponents are the sum of four squares; and thus the coefficient of each shows, in how many different ways the exponent of x may be able to be distributed into four squares; whereby for this case it will be required to shown, if there may be put

$$P^4 = 1 + Ax + Bxx + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.},$$

which series clearly rises through all the powers of x, none of the coefficients *A, B, C, D, E* etc. to be going to vanish. Indeed if this were shown, plainly likewise it will prevail all numbers can be resolved into four squares.

FOR THE RESOLUTION INTO PENTAGONAL NUMBERS

8. The series of powers of x may be considered, the exponents of which increase following the pentagonal numbers, which shall be

$$P = 1 + x + x^5 + x^{12} + x^{22} + x^{35} + x^{51} + \text{etc.},$$

and the fifth power of this may be set out, which shall be

$$P^5 = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \text{etc.}$$

plainly rising through all the powers of x, and it is required to be shown in this series evidently no coefficient is to be found, which shall be equal to zero.

FOR THE RESOLUTION INTO ANY POLYGONAL NUMBERS

9. Let π be the number of sides of a polygonal number, and since the general formula consisting of all these polygonal numbers shall be $= \frac{1}{2}(\pi - 2)zz - \frac{1}{2}(\pi - 4)z$, evidently with the root put $= z$, all the polygonal numbers resulting from this order shall be $0,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta$ etc., where indeed there may be agreed to be $\alpha = 1,\ \beta = \pi,\ \gamma = 3\pi - 3,\ \delta = 6\pi - 8,\ \varepsilon = 10\pi - 15,\ \zeta = 15\pi - 24,\ \eta$ etc.; then truly we may consider the infinite series

$$P = 1 + x^{\alpha} + x^{\beta} + x^{\gamma} + x^{\delta} + x^{\varepsilon} + x^{\zeta} + \text{etc.}$$

and the exponent of this series may be taken to be $= \pi$, which shall become :

$$P^{\pi} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \text{etc.}$$

increasing through all the powers of x, and it will be required to be shown in this series no coefficient occurs equal to zero; from which it is apparent, if only it may be possible to be demonstrated in general, likewise all the theorems of Fermat to be demonstrated.

10. Therefore in this matter perhaps it will be quite useful, if I may demonstrate how to establish the powers P^{π} in general, and likewise I will show, how all the coefficients of the series set out may depend on the preceding series, and from which these will be determined. In the end these coefficients of the individual powers of x we may designate with suitable characters, and the coefficient of the power x^{n} shall be $[n]$; since in this manner it may be understood at once, to which power and to which coefficient it may refer. Now therefore we will investigate, how any coefficient n may be determined from the preceding, which shall be $[n-1]$, $[n-2]$, $[n-3]$, $[n-4]$ etc. Indeed in this way a judgement may be put in place most easily, whether which coefficient may be able to be put equal to zero, that which perhaps will be able to be shown for that, since it is certain no coefficient may be able to become negative.

11. Therefore since on putting

$$P=1+x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+\text{etc.},$$

the power of the exponent π of this series must be sought, we may put $S=P^{\pi}$, and there will be with logarithms taken $lS=\pi lP$, and hence by differentiation $\frac{dS}{S}=\frac{\pi dP}{P}$, from which this equation will be formed :

$$P\frac{xdS}{dx}=\frac{\pi SxdP}{dx}$$

for which therefore there will be

$$\frac{xdP}{dx}=\alpha x^{\alpha}+\beta x^{\beta}+\gamma x^{\gamma}+\delta x^{\delta}+\varepsilon x^{\varepsilon}+\zeta x^{\zeta}+\text{etc.}$$

therefore which series multiplied by λS must produce the same generally, which arises, if that series

$$P=1+x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+\text{etc.}$$

may be multiplied by $\frac{xdS}{dx}$.

12. Therefore the following characters described before may be put in place :

$$S=1+[1]x+[2]x^{2}+[3]x^{3}+[4]x^{4}+\text{etc.},$$

where it will be noted the first term 1 to be equivalent to the term [0], from which there becomes :

$$\frac{xdS}{dx}=[1]x^{1}+2[2]x^{2}+3[3]x^{3}+4[4]x^{4}+\text{etc.}$$

and from these series put in place we may consider, in how many ways the proposed power x^n may occur in each product $\frac{PxdS}{dx}$ and $\frac{\pi SxdP}{dx}$.

13. Therefore since in both the multipliers P and $\frac{xdP}{dx}$, other powers of x^n may not be present, unless the exponents of these are $\alpha,\ \beta,\ \gamma,\ \delta$ etc., it is evident the term $[n]x^n$ of the series S cannot result from these preceding terms, besides these : $[n-\alpha]x^\alpha,\ [n-\beta]x^\beta,\ [n-\gamma]x^\gamma$ etc.; on account of which with the remaining terms omitted we will consider only those, thus, so that we may have only those :

$$S=[n]x^n\cdots\cdot+[n-\alpha]x^{n-\alpha}\cdots\cdot+[n-\beta]x^{n-\beta}\cdots\cdot+[n-\gamma]x^{n-\gamma}\cdots\cdot+\text{etc.},$$

were it is clear these terms themselves therefore must be continued only as far, so that the exponents $[n-\alpha],\ [n-\beta],\ [n-\gamma]$ etc. etc. may not become negative. Therefore this form may be multiplied by

$$\frac{\pi xdP}{dx}=\pi\alpha x^\alpha+\pi\beta x^\beta+\pi\gamma x^\gamma+\text{etc.}$$

and the product will supply the following terms containing the power x^n :

$$\pi\alpha[n-\alpha]x^n+\pi\beta[n-\beta]x^n+\pi\gamma[n-\gamma]x^n+\text{etc.}$$

14. Then with the same terms retained there will be

$$\frac{xdS}{dx}=n[n]x^n\cdots\cdot+(n-\alpha)[n-\alpha]x^{n-\alpha}\cdots\cdot+(n-\beta)[n-\beta]x^{n-\beta}\cdots\cdot+[n-\gamma]x^{n-\gamma}\cdots\cdot$$
$$+(n-\gamma)[n-\gamma]x^{n-\gamma}\cdots\cdot+(n-\delta)[n-\delta]x^{n-\gamma}\cdots\cdot+\text{etc.}$$

which form multiplied into the series

$$P=1+x^\alpha+x^\beta+x^\gamma+x^\delta+\text{etc.}$$

will present the following terms containing the powers of x^n :

$$n[n]x^n+(n-\alpha)[n-\alpha]x^n+(n-\beta)[n-\beta]x^n+(n-\gamma)[n-\gamma]x^n+\text{etc.}$$

On account of which, since this must be equal to the first product, we will obtain the following equation:

$$n[n]+(n-\alpha)[n-\alpha]+(n-\beta)[n-\beta]+(n-\gamma)[n-\gamma]+\text{etc.}$$
$$=\pi\alpha[n-\alpha]+\pi\beta[n-\beta]+\pi\gamma[n-\gamma]x^{n-\gamma}+\pi\delta[n-\delta]+\text{etc}$$

15. Hence therefore it is apparent the coefficient [*n*] depends only on these preceding, of which the characters are

$$[n-\alpha],\ [n-\beta],\ [n-\gamma]\ \text{etc.},$$

thus so that there shall be

$$n[n]=(\pi\alpha-(n-\alpha))[n-\alpha]+(\pi\beta-(n-\beta))[n-\beta]+(\pi\gamma-(n-\gamma))[n-\gamma]+\text{etc.}$$
$$=\pi\alpha[n-\alpha]+\pi\beta[n-\beta]+\pi\gamma[n-\gamma]x^{n-\gamma}+\pi\delta[n-\delta]+\text{etc}$$

Where indeed in the first place it is not to be feared, lest on account of negative terms at some time the value of [*n*] itself going to be produced shall be negative, since this would disagree with the nature of the matter; and because the value of [*n*] itself certainly is a whole number, it is evident all the terms in the right hand member taken together always must present the same value, either *n, 2n, 3n,* or *4n,* etc., unless perhaps thence there may be produced 0, therefore which is required to be shown at no time can eventuate, if indeed the Fermat theorem were agreed to be true, and the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. will denote in order all the polygonal numbers for a given side *n.*

16. But it is evident this determination of the coefficient [*n*] to be quite general, nor only to be extended to polygonal numbers. Indeed any series of numbers may be accepted for the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc., the equation found will always have a place, and the coefficient [*n*] will indicate, in how many ways the number may be able to be the sum of *n* terms of this series :

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta\ \text{etc.},$$

and hence the number π thus will be able to be defined, so that no coefficient [*n*] of this kind may be produced equal to zero. Indeed it is evident the number *n* always can be taken only, so that the right hand member of our equation at no time may be produced negative. But the minimum value is accustomed to be sought always, which for *n* assumed this shall become outstanding; from which it is apparent this method is able to be applied to infinitely many other questions of this kind perhaps with success.

17. Moreover in general it will help to be known, unless there were $\alpha=1$, always to become $[n]=0$, provided there were $n<\alpha$ *;*from which in all the questions of this kind it is necessary, that there shall be $\alpha=1$, which term indeed is accustomed always to proceed from the term $=0$, if indeed the series assumed for *P* may start from one.

APPLICATION TO TRIGONAL NUMBERS

18. So that the nature of the equation found may be seen more clearly, we will apply that to trigonal numbers, for which there will be $\alpha=1,\ \beta=3,\ \gamma=6,\ \delta=10,\ \varepsilon=15\ \text{etc.}$, while truly, with the exponent taken $\pi=3$, our formula found will adopt this form :

$$n[n]=(4-n)[n-1]+(12-n)[n-3]+(24-n)[n-6]$$
$$+(40-n)[n-10]+\text{etc.}$$

Hence therefore, by beginning from the smallest numbers, we will obtain the following reductions :

$1\ [1]=\ 3\ [0],$	therefore $[1]=3$
$2\ [2]=\ 2\ [1]=6,$	therefore $[2]=3$
$3\ [3]=\ 1\ [2]+9\ [0]=12,$	therefore $[3]=4$
$4\ [4]=\ 0\ [3]+8\ [1]=24,$	therefore $[4]=6$
$5\ [5]=-1\ [4]+7\ [2]=15,$	therefore $[5]=3$
$6\ [6]=-2\ [5]+6\ [3]+18[0]=36,$	therefore $[6]=6$
$7\ [7]=-3\ [6]+5\ [4]+17[1]=63,$	therefore $[7]=9$
$8\ [8]=-4\ [7]+4\ [5]+16[2]=24,$	therefore $[8]=3$
$9\ [9]=-5[8]+3\ [6]+15[3]=63,$	therefore $[9]=7$
$10[10]=-6[9]+2\ [7]+14[4]+30[0]=90,$	therefore $[10]=9$
$11[11]=-7[10]+1\ [8]+13[5]+29[1]=66,$	therefore $[11]=6$
$12[12]=-8[11]+0\ [9]+12[6]+28[2]=108,$	therefore $[12]=9$
$13[I3]=-\ 9[12]-1[10]+11[7]+27[3]=117,$	therefore $[13]=9$
$14[14]=-10[13]-2[11]+10[8]+26[4]=84,$	therefore $[14]=6$
$15[15]=-11[14]-3[12]+\ 9[9]+25[5]+45[0]=\ 90,$	therefore $[15]=6$
$16(16]=-12[15]+4[13]+8[10]+24[6]+44[1]=240,$	therefore $[16]=15$
$17[17]=-13[16]-5[14]+7[11]+23[7]+43[2]=153,$	therefore $[17]=\ 9$
$18[18]=-14[17]-6[15]+6[12]+22[8]+42[3]=126,$	therefore $[18]=\ 7$
$19[19]=-15[18]-7[16]+5[13]+21[9]+41[4]=228,$	therefore $[19]=12$
$20[20]=-16[19]-8[17]+4[14]+20[10]+40[5]=60,$	therefore $[20]=3$

Hence therefore it is apparent the cube of the series $1+x+x^3+x^6+x^{10}+x^{15}+\text{etc.}$ to be changed into this series:

$$1+3x^1+3x^2+4x^3+6x^4+3x^5+6x^6+9x^7+3x^8+7x^9+9x^{10}$$
$$+6x^{11}+9x^{12}+9x^{13}+6x^{14}+6x^{15}+15x^{16}+9x^{17}+7x^{18}+\text{etc.}$$

19. In establishing this it was pleasing to observe, how the first negative members were always overcome by the following positive members and the excess would always be divisible by the number *n*, that which by necessity likewise must arise in the general form used. So that this may be seen more clearly, we will show the general equation according to this form:

$$n[n]=((\pi+1)\alpha-n)[n-\alpha]+((\pi+1)\beta-n)[n-\beta]$$
$$+((\pi+1)\gamma-n)[n-\gamma]+\text{etc.},$$

from which, with the negative parts moved to the left, there will become :

$$n([n]+[n-\alpha]+[n-\beta]+[n-\gamma]+\text{etc.})$$
$$=(\pi+1)(\alpha[n-\alpha]+\beta[n-\beta]+\gamma[n-\gamma]+\delta[n-\delta]+\text{etc.}).$$

Hence therefore we understand : in the first place the sum of all these values

$$\alpha[n-\alpha]+\beta[n-\beta]+\gamma[n-\gamma]+\text{etc.}$$

always to be divisible by the number *n,* unless perhaps $\pi+1$ were divisible by n ;
then the sum of these values

$$[n]+[n-\alpha]+[n-\beta]+[n-\gamma]+\text{etc.}$$

always to be divisible by the number $\pi+1$, unless perhaps the number *n* itself may be allowed to be divided by that.

20. Nor truly do these outstanding properties only have a place in polygonal numbers, which here we consider mainly, but also they ought to be observed more generally, whatever series of numbers may be assumed for the letters $\alpha,\ \beta,\ \gamma$ etc., even if that may not be restricted to a certain rule. Truly according to this circumstance it may be considered not much to be promised for our goal, since at last they must be considered agreeing with the truth of Fermat's theorem, when for the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. polygonal numbers themselves are written in order; thus so that, if the order of these may be disturbed a little, the demonstration itself being desired also must be made defective; on account of which by necessity there will be a need for a legitimate demonstration of these theorems requiring to be found, so that likewise that law of the progression of the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. also may be lead into the computation, because since whether by that method it may be able to be taken together conveniently, or not, is not so easy seen. Yet meanwhile these considerations themselves perhaps will be able to provide some light for others, by which more happily they may prevail to penetrate these truths.

[Fermat's Polygonal Number Theorem was finally proven in its entirety by Cauchy in 1813; as mentioned, the quadrangle case had been solved originally by Lagrange, and the trigonal case later was solved by Gauss.]

CONSIDERATIONES SUPER THEOREMATE FERMATIANO DE RESOLUTIONE NUMERORUM IN NUMEROS POLYGONALES

Opuscula analytica. 2, 1785, p. 3-15
[Conventui exhibits. die 12. decembris 1774]

1. Hoc theorema FERMATIANUM in se complectitur sequentes assertiones numero infinitas:

I. *Omnem numerum esse summam trium trigonalium vel pauciorum.*

II. *Omnem numerum esse summam quatuor tetragonalium seu quadratorum vel pauciorum.*

III. *Omnem numerum esse summam quinque pentagonalium, vel pau*cior~ tm.

IV. *Omnem numerum esse summam sex hexagonalium vel pauciorum.*

V. *Omnem numerum esse summam septem heptagonalium vel pauciorum.*

etc.

Quorum theorematum cum FERMATIUS asseverasset demonstrationem a se esse inventam, dubitari certe nequit eius demonstrationem certissimis principiis fuisse innixam; ex quo eo magis dolendum est eam post eius obitum prorsus periisse, ut nullum plane vestigium reperiri potuerit, cum sine dubio plerique Geometrae in his demonstrationibus investigandis frustra desudaverint. Hinc quidem excipienda est secunda assertio de resolutione numerorum in quatuor quadrata, cuius perfecta demonstratio ab ingeniosissimo LA GRANGE in lucem est protracta, quae autem ex eiusmodi prin- cipiis est deducta, ut inde nullum plane subsidium ad reliqua demonstranda exspectari possit.

2. Ingens igitur discrimen inter resolutionem in quadrata et reliquos numeros polygonales intercedere est censendum, quod potissimum in hoc consistit, quod resolutio in quaterna quadrata ad omnes plane numeros tarn fractos quam integros se extendat, cum resolutio in alios polygonales tantum ad numeros integros restringatur, atque adeo nonnisi sub certa limitatione veritati sit consentanea. Resolutio enim in ternos trigonales manifesto tantum ad numeros integros adstringitur, cum infinitae dentur fractiones, quas nullo modo in ternas partes, in formula $\frac{xx+x}{2}$ contentas, resolvere licet; vel uti si pro fractione $\frac{1}{2}$ statuere vellemus

$$\frac{1}{2}=\frac{xx+x}{2}+\frac{yy+y}{2}+\frac{zz+z}{2}$$

multiplicando per 8 deberet esse

$$4=4xx+4x+4yy+4y+4zz+4zzz,$$

hincque tribus unitatibus additis fieret

$$7=(2x+1)^2+\left(2y+1\right)^2+\left(2z+1\right)^2.$$

Demonstratum autem est numerum 7 nullo modo esse posse summam trium quadratorum. Quocirca si quis demonstrationem huius partis ita instituere voluerit, ut, proposito numero quocunque *N,* hanc aequationem:

$$N=\tfrac{xx+x}{2}+\tfrac{yy+y}{2}+\tfrac{zz+z}{2}$$

sibi resolvendam proponeret, oleum atque operam perdiderit.

3. Porro vero etiam resolutio in quinque pentagonales ad numeros integros adstringitur, sed longe alio modo atque in trigonalibus usu venit. Nam si inter numeros pentagonal etiam fractiones in formula $\tfrac{3xx-x}{2}$ contentas admittere velimus, turn omnes plane numeros adeo in quatuor pentagonales discerpere liceret. Proposito enim numero quocunque *N* si statuamus

$$N=\tfrac{3xx-x}{2}+\tfrac{3yy-y}{2}+\tfrac{3zz-z}{2}+\tfrac{3vv-v}{2}$$

per 24 multiplicando fiet

$$24N=36xx-12x+36yy-12y+36zz-12z+36vv-12v,$$

unde quatuor unitatibus additis fiet

$$24N+4=(6x-1)^2+(6y-1)^2+(6z-1)^2+(6v-1)^2.$$

Cum igitur numerus $24N+4$ certo sit summa quatuor quadratorum, quae sit $aa+bb+cc+dd$, hinc reperiemus

$$x=\tfrac{a+1}{6},\;\; y=\tfrac{b+1}{6},\;\; z=\tfrac{c+1}{6},\;\; v=\tfrac{d+1}{6},$$

atque horum radicum numeri pentagonales iunctim sumti numero proposito *N* aequabuntur. Nihilo vero minus si tantum numeros integros admittamus, uti FERMATIUS manifesto postulat, utique dantur eiusmodi numeri, quos in pauciores quam quinque pentagonales neutiquam resolvere licet.

4. Praeterea vero, etiamsi hinc fractiones excludamus et tantum numeros integros admittere velimus, tamen novam limitationem adiicere, atque ex ordine pentagonalium omnes eos, quorum radices sunt numeri negativi, excludere debemus. Cum enim formula generalis numerorum pentagonalium $\tfrac{3xx-x}{2}$ pro radicibus *x* negative sumtis praebeat hos numeros: 2, 7, 15, 26, 40 etc., si etiam hos admittere vellemus, non amplius quinque, sed tantum tres numeri pentagonales sufficerent omnibus plane numeris producendis, atque talis gemina limitatio multo magis pro sequentibus numeris polygonalibus est necessaria, ut theoremata FERMATIANA veritati sint consentanea, quae limitatio sine dubio in

caussa est, quod nulli adhuc Geometrae post FERMATIUM ad demonstrationem horum casuum penetrare licuerit.

5. Cum igitur in demonstrationibus, quae desiderantur, harum restrictionum ratio necessario sit habenda, ipsa theoremata FERMATIANA sub alia forma repraesentemus, quae istas limitationes iam in se contineat, quod commodissime sequenti modo fieri posse videtur.

PRO RESOLUTIONE IN NUMEROS TRIGONALES

6. Consideretur series potestatum

$$1+x+x^3+x^6+x^{10}+x^{15}+x^{21}+x^{28}+x^{36}+x^{45}+\text{etc.},$$

in qua potestates ipsius x progrediuntur secundum ipsos numeros trigonales

0, 1, 3, 6, 10, 15, 21, 28 etc.,

ac primo manifestum est, si huius seriei quadratum capiatur, alias potestates ipsius x occurrere non posse, nisi quarum exponentes sint summae duorum numerorum trigonalium. Eodem modo intelligitur, si eiusdem seriei capiatur cubus, in eo alias potestates non occurrere, nisi quarum exponentes sint summae trium numerorum trigonalium. Quocirca primum theorema FERMATIT hue reducitur, ut, si pro cubo assumtae seriei statuatur haec series per omnes potestates ipsius x ascendens:

$$1+Ax+Bxx+ax^3+Dx^4+Ex^5+Fx^6+Gx^7+Hx^8+\text{etc.},$$

demonstretur in hac serie nullum plane coefficientem nihilo fore aequalem. Ex ipsa autem formatione manifestum est nullum horum coefficientium fieri posse negativum.

PRO RESOLUTIONE IN NUMEROS TETRAGONALES SED QUADRATOS

7. Hic consideretur ista series potestatum:

$$P=1+x+x^4+x^9+x^{16}+x^{25}+x^{36}+\text{etc.},$$

cuius biquadratum P^4 , si evolvatur, omnes complectetur potestates ipsius x, quarum exponentes sunt summae quatuor quadratorum; atque adeo cuiusque coefficiens ostendet, quot variis modis exponens ipsius x in quatuor quadrata distribui queat; quare pro hoc casu demonstrari oportet, si ponatur

$$P^4=1+Ax+Bxx+Cx^3+Dx^4+Ex^5+Fx^6+\text{etc.},$$

quae scilicet series per omnes potestates ipsius x ascendat, nullum coefficientium

A, B, C, D, E etc. esse evaniturum. Hoc enim si fuerit demonstratum, simul erit evictum omnes plane numeros in quatuor quadrata resolvi posse.

8. Consideretur series potestatum ipsius *x,* quarum exponentes secundum numeros pentagonales ascendant, quae sit

$$P=1+x+x^{5}+x^{12}+x^{22}+x^{35}+x^{51}+\text{etc.},$$

eiusque evolvatur potestas quinta, quae sit

$$P^{5}=1+Ax+Bx^{2}+Cx^{3}+Dx^{4}+Ex^{5}+\text{etc.}$$

per omnes plane potestates ipsius *x* ascendens, ac demonstrandum est in hac serie nullum prorsus coefficientem reperiri, qui sit nihilo aequalis.

PRO RESOLUTIONE IN NUMEROS POLYGONALES QUOSCUNQUE

9. Sit *n* numerus laterum polygonorum, et cum formula generalis omnes istos numeros polygonales complectens sit $=\frac{1}{2}(\pi-2)zz-\frac{1}{2}(\pi-4)z$, posita scilicet radice $=z$, sint omnes numeri polygonales hinc ordine resultantes $0,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta$ etc., ubi quidem constat esse

$$\alpha=1,\ \beta=\pi,\ \gamma=3\pi-3,\ \delta=6\pi-8,\ \varepsilon=10\pi-15,\ \zeta=15\pi-24,\ \eta\ \text{etc.};$$

tum vero consideretur series infinita

$$P=1+x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+x^{\zeta}+\text{etc.}$$

huiusque seriei sumatur potestas exponentis $=\pi$, quae sit

$$P^{\pi}=1+Ax+Bx^{2}+Cx^{3}+Dx^{4}+Ex^{5}+\text{etc.}$$

per omnes plane potestates ipsius *x* ascendens, ac demonstrandum erit in hac serie nullum occurrere coefficientem nihilo aequalem; unde patet, si modo hoc demonstrari in genere posset, simul omnia theoremata FERMATIANA fore demonstrata.

10. In hoc igitur negotio non parum fortasse proderit, si evolutionem istius potestatis P^{π} in genere docuero, simulque ostendero, quomodo omnes coefficientes seriei evolutae a praecedentibus pendeant, ex iisque determinentur. Hunc in finem coefficientes singularum potestatum ipsius *x* idoneis characteribus designemus, sitque potestatis x^{n} coefficiens [*n*]; quandoquidem hoc modo statim perspicitur, ad quamnam potestatem quisque coefficiens referatur. Nunc igitur investigemus, quomodo quilibet coefficiens [n]

ex praecedentibus, qui sunt $[n-1]$, $[n-2]$, $[n-3]$, $[n-4]$ etc., determinetur. Hoc enim modo iudicium facillime instituetur, num quis coefficiens nihilo aequalis fieri possit, id quod forsan eo facilius ostendi poterit, cum certum sit nullum coefficientem fieri posse negativum.

11. Cum igitur posito

$$P=1+x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+\text{etc.}$$

quaeri debeat istius seriei potestas exponentis π, statuamus $S=P^{\pi}$, eritque sumtis logarithmis $lS=\pi lP$, hincque differentiando $\frac{dS}{S}=\frac{\pi dP}{P}$, unde formetur ista aequatio:

$$P\frac{xdS}{dx}=\frac{\pi SxdP}{dx},$$

pro qua ergo erit

$$\frac{xdP}{dx}=\alpha x^{\alpha}+\beta x^{\beta}+\gamma x^{\gamma}+\delta x^{\delta}+\varepsilon x^{\varepsilon}+\zeta x^{\zeta}+\text{etc.}$$

quae ergo series per λS multiplicata idem productum generare debet, quod oritur, si ipsa series

$$P=1+x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+\text{etc.}$$

per formulam $\frac{xdS}{dx}$ multiplicetur.

12. Ponatur igitur secundum characteres ante descriptos:

$$S=1+[1]x+[2]x^{2}+[3]x^{3}+[4]x^{4}+\text{etc.},$$

ubi notetur primum terminum 1 aequivalere termino [0], unde fiet

$$\frac{xdS}{dx}=[1]x^{1}+2[2]x^{2}+3[3]x^{3}+4[4]x^{4}+\text{etc.},$$

hisque seriebus constitutis perpendamus, quot modis proposita potestas x^{n} in utroque producto $\frac{PxdS}{dx}$ et $\frac{\pi SxdP}{dx}$ occurrat.

13. Cum igitur ambo multiplicatores P et $\frac{xdP}{dx}$ alias potestates ipsius x non contineant, nisi quarum exponentes sunt $\alpha,\ \beta,\ \gamma,\ \delta$ etc., manifestum est seriei S terminum $[n]x^{n}$ ex aliis terminis praecedentibus resultare non posse, praeter hos:

$$[n-\alpha]x^{\alpha},\ [n-\beta]x^{\beta},\ [n-\gamma]x^{\gamma};$$

quamobrem omissis reliquis terminis consideremus tantum istos, ita ut habeamus:

$$S=[n]x^n\cdots+[n-\alpha]x^{n-\alpha}\cdots+[n-\beta]x^{n-\beta}\cdots+[n-\gamma]x^{n-\gamma}\cdots+\text{etc.},$$

ubi per se manifestum est hos terminos tantum eo usque continuari debere, quoad exponentes $[n-\alpha],\ [n-\beta],\ [n-\gamma]$ etc. non fiunt negativi. Multiplicetur igitur ista forma per

$$\tfrac{\pi xdP}{dx}=\pi\alpha x^{\alpha}+\pi\beta x^{\beta}+\pi\gamma x^{\gamma}+\text{etc.}$$

14. Deinde iisdem terminis retentis erit

$$\tfrac{xdS}{dx}=n[n]x^n\cdots+(n-\alpha)[n-\alpha]x^{n-\alpha}\cdots+(n-\beta)[n-\beta]x^{n-\beta}\cdots+[n-\gamma]x^{n-\gamma}\cdots$$
$$+(n-\gamma)[n-\gamma]x^{n-\gamma}\cdots+(n-\delta)[n-\delta]x^{n-\gamma}\cdots+\text{etc.}$$

quae forma ducta in seriem

$$P=1+x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+\text{etc.}$$

sequentes praebebit terminos potestatem x^n continentes:

$$n[n]x^n+(n-\alpha)[n-\alpha]x^n+(n-\beta)[n-\beta]x^n+(n-\gamma)[n-\gamma]x^n+\text{etc.}$$

Quamobrem cum hoc productum priori debeat esse aequale, obtinebimus sequentem aequationem:

$$n[n]+(n-\alpha)[n-\alpha]x+(n-\beta)[n-\beta]x+(n-\gamma)[n-\gamma]x+\text{etc.}$$
$$=\pi\alpha[n-\alpha]+\pi\beta[n-\beta]+\pi\gamma[n-\gamma]x^{n-\gamma}+\pi\delta[n-\delta]+\text{etc}$$

15. Hinc igitur patet coefficientem $[n]$ ab iis tantum praecedentium, quorum characteres sunt

$$[n-\alpha],\ [n-\beta],\ [n-\gamma]\ \text{ etc.},$$

pendere, ita ut sit

$$n[n]=(\pi\alpha-(n-\alpha))[n-\alpha]+(\pi\beta-(n-\beta))[n-\beta]+(\pi\gamma-(n-\gamma))[n-\gamma]+\text{etc.}$$
$$=\pi\alpha[n-\alpha]+\pi\beta[n-\beta]+\pi\gamma[n-\gamma]x^{n-\gamma}+\pi\delta[n-\delta]+\text{etc}$$

Ubi quidem primo non est metuendum, ne ob terminos negativos unquam valor ipsius $[n]$ proditurus sit negativus, quandoquidem hoc naturae rei repugnaret; et quia valor ipsius $[n]$ certe est numerus integer, evidens est omnes terminos in dextro membro iunctim sumtos semper valorem praebere debere vel *n,* vel *2n,* vel *3n,* vel *4n,* vel etc., nisi forte inde prodeat 0, quod igitur demonstrandum est nunquam eveniri posse, si quidem theorema FERMATIANUM fuerit veritati consentaneum, atque literae $\alpha,\ \beta,\ \gamma,\ \delta$ etc. denotent ordine omnes numeros polygonales pro numero laterum *n*.

16. Manifestum autem est hanc determinationem coefficientis [*n*] maxime esse generalem, neque tantum ad numeros polygonales extendi. Quaecunque enim series numerorum pro litteris $\alpha,\ \beta,\ \gamma,\ \delta$ etc. accipiatur, aequatio inventa semper habebit locum, et coefficiens [*n*] indicabit, quot variis modis numerus *n* possit esse summa *n* terminorum istius seriei: $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta$ etc., hincque numerus π ita definiri poterit, ut nullus huiusmodi coefficiens [*n*] prodeat nihilo aequalis. Evidens enim est numerum *n* semper tantum accipi posse, ut dextrum membrum nostrae aequationis nunquam prodeat negativum. Quaeri autem semper solet minimus valor, qui pro *n* assumtus hoc sit praestaturus; unde patet hanc methodum ad infinitas alias quaestiones huius generis pari fortasse successu applicari posse.

17. Caeterum in genere notasse iuvabit, nisi fuerit $\alpha=1$, semper fore $[n]=0$, quamdiu fuerit $n<\alpha$; unde in omnibus huiusmodi quaestionibus necesse est, ut sit $\alpha=1$, quem quidem terminum semper praecedere solet terminus $=0$, siquidem series pro P assumta ab unitate incipiat.

APPLICATIO AD NUMEROS TRIGONALES

18. Quo natura aequationis inventae clarius percipiatur, eam ad numeros trigonales applicemus, pro quibus erit $\alpha=1,\ \beta=3,\ \gamma=6,\ \delta=10,\ \varepsilon=15$ etc., tum vero, sumto exponente $\pi=3$, nostra formula inventa hanc induet formam:

$$\begin{aligned} n[n] &= (4-n)[n-1]+(12-n)[n-3]+(24-n)[n-6] \\ &+(40-n)[n-10]+\text{etc.} \end{aligned}$$

Hinc ergo, a numeris minimis incipiendo, sequentes nanciscemur reductiones:

$$1\ [1] = 3\ [0], \qquad \text{ergo}\quad [1] = 3$$
$$2\ [2] = 2\ [1] = 6, \qquad \text{ergo}\quad [2] = 3$$
$$3\ [3] = 1\ [2] + 9\ [0] = 12, \qquad \text{ergo}\quad [3] = 4$$
$$4\ [4] = 0\ [3] + 8\ [1] = 24, \qquad \text{ergo}\quad [4] = 6$$
$$5\ [5] = -1\ [4] + 7\ [2] = 15, \qquad \text{ergo}\quad [5] = 3$$
$$6\ [6] = -2\ [5] + 6\ [3] + 18[0] = 36, \qquad \text{ergo}\quad [6] = 6$$
$$7\ [7] = -3\ [6] + 5\ [4] + 17[1] = 63, \qquad \text{ergo}\quad [7] = 9$$
$$8\ [8] = -4\ [7] + 4\ [5] + 16[2] = 24, \qquad \text{ergo}\quad [8] = 3$$
$$9\ [9] = -5[8] + 3\ [6] + 15[3] = 63, \qquad \text{ergo}\quad [9] = 7$$
$$10[10] = -6[9] + 2\ [7] + 14[4] + 30[0] = 90, \qquad \text{ergo}\ [10] = 9$$
$$11[11] = -7[10] + 1\ [8] + 13[5] + 29[1] = 66, \qquad \text{ergo}\ [11] = 6$$
$$12[12] = -8[11] + 0\ [9] + 12[6] + 28[2] = 108, \qquad \text{ergo}\ [12] = 9$$
$$13[I3] = -\ 9[12] - 1[10] + 11[7] + 27[3] = 117, \qquad \text{ergo}\ [13] = 9$$
$$14[14] = -10[13] - 2[11] + 10[8] + 26[4] = 84, \qquad \text{ergo}\ [14] = 6$$
$$15[15] = -11[14] - 3[12] + 9[9] + 25[5] + 45[0] = 90, \quad \text{ergo}\ [15] = 6$$
$$16(16] = -12[15] + 4[13] + 8[10] + 24[6] + 44[1] = 240, \quad \text{ergo}\ [16] = 15$$
$$17[17] = -13[16] - 5[14] + 7[11] + 23[7] + 43[2] = 153, \quad \text{ergo}\ [17] = 9$$
$$18[18] = -14[17] - 6[15] + 6[12] + 22[8] + 42[3] = 126, \quad \text{ergo}\ [18] = 7$$
$$19[19] = -15[18] - 7[16] + 5[13] + 21[9] + 41[4] = 228, \quad \text{ergo}\ [19] = 12$$
$$20[20] = -16[19] - 8[17] + 4[14] + 20[10] + 40[5] = 60, \quad \text{ergo}\ [20] = 3$$

Hinc igitur patet seriei $1 + x + x^3 + x^6 + x^{10} + x^{15} + \text{etc.}$ cubum evolvi in hanc seriem:

$$\begin{aligned}&1 + 3x^1 + 3x^2 + 4x^3 + 6x^4 + 3x^5 + 6x^6 + 9x^7 + 3x^8 + 7x^9 + 9x^{10}\\ &+ 6x^{11} + 9x^{12} + 9x^{13} + 6x^{14} + 6x^{15} + 15x^{16} + 9x^{17} + 7x^{18} + \text{etc.}\end{aligned}$$

19. In hac evolutione non iniucundum erat videre, quomodo priora membra negativa a sequentibus positivis semper superentur atque excessus semper per numerum *n* divisibilis prodierit, id quod etiam perinde in forma generali necessaria usu venire debet. Quod quo clarius in oculos incurrat, aequationem generalem sub hac forma exhibeamus:

$$\begin{aligned}n[n] = ((\pi+1)\alpha - n)[n-\alpha] &+ ((\pi+1)\beta - n)[n-\beta]\\ &+ ((\pi+1)\gamma - n)[n-\gamma] + \text{etc.},\end{aligned}$$

unde partibus negativis ad sinistram translatis erit

$$n([n]+[n-\alpha]+[n-\beta]+[n-\gamma]+\text{etc.})$$
$$=(\pi+1)(\alpha[n-\alpha]+\beta[n-\beta]+\gamma[n-\gamma]+\delta[n-\delta]+\text{etc.}).$$

Hinc igitur intelligimus: primo summam omnium horum valorum

$$\alpha[n-\alpha]+\beta[n-\beta]+\gamma[n-\gamma]+\text{etc.}$$

semper fore divisibilem per numerum *n,* nisi forte $\pi+1$ fuerit per *n* divisibile; deinde summam horum valorum

$$[n]+[n-\alpha]+[n-\beta]+[n-\gamma]+\text{etc.}$$

semper esse divisibilem per numerum $\pi+1$, nisi forte ipse numerus *n* per eum divisionem admittat.

20. Neque vero hae eximae proprietates tantum in numeris polygonalibus, quos hic potissimum contemplamur, locum habent, sed etiam generalissime observari debent, quaecunque numerorum series pro litteris $\alpha,\ \beta,\ \gamma$ etc. assumatur, etiamsi ea nulli certae legi fuerit adstricta. Ad vero haec ipsa circumstantia non multum pro scopo nostro polliceri videtur, quandoquidem theoremata FERMATIANA tum demum veritati consentanea censeri debent, quando pro litteris $\alpha,\ \beta,\ \gamma,\ \delta$ etc. ipsi numeri polygonales ordine scribuntur; ita ut, si earum ordo tantillum perturbaretur, demonstratio etiam ipsa hinc petenda claudicare deberet; quocirca ad legitimam demonstrationem horum theorematum inveniendam necessario opus erit, ut simul etiam ipsa lex progressionis litterarum $\alpha,\ \beta,\ \gamma,\ \delta$ etc. in computum ducatur, id quod utrum cum ista methodo commode coniungi queat, necne, non tam facile perspicitur. Interim tamen istae considerationes fortasse aliis aliquam lucem accendere poterunt, quo felicius ad has veritates penetrare valeant.