HOW THE SINES AND COSINES OF MULTIPLE ANGLES

MAY BE ABLE TO BE EXPRESSED BY PRODUCTS

Opuscula analytica 1, 1783, p. 353-363 [E562]

1. For some angle φ proposed there may be put for the sake of brevity

$$p = \cos.\varphi + \sqrt{-1}\cdot\sin.\varphi$$

and

$$q = \cos.\varphi - \sqrt{-1}\cdot\sin.\varphi,$$

there will be

$$pq = 1;$$

then truly

$$p^n = \cos.n\varphi + \sqrt{-1}\cdot\sin.n\varphi$$

and

$$q^n = \cos.n\varphi - \sqrt{-1}\cdot\sin.n\varphi,$$

from which there becomes

$$p^n + q^n = 2\cos.n\varphi$$

and

$$p^n - q^n = 2\sqrt{-1}\cdot\sin.n\varphi.$$

Therefore the matter is reduced to that, so that the formulas $p^n + q^n$ and $p^n - q^n$ may be resolved into factors.

2. Initially we will consider the formula

$$p^n + q^n = 2\cos.n\varphi,$$

which, as often as n is an odd number, it has the simple factor $p + q = 2\cos.\varphi$, thus so that in these cases $\cos.\varphi$ shall be a factor of $\cos.n\varphi$. But for the remaining cases we may put the twofold factor in general to be $pp - 2pq\cos.\omega + qq$, thus so that the formula $p^n + q^n$ may vanish on putting

$$pp - 2pq\cos.\omega + qq = 0;$$

but then there will be either

$$p = q(\cos.\omega + \sqrt{-1}\cdot\sin.\omega)$$

or

$$p = q(\cos.\omega - \sqrt{-1}\cdot\sin.\omega)$$

and hence $$p^n = q^n(\cos.n\omega \pm \sqrt{-1}\cdot \sin.n\omega)\,;$$

and thus there will have to be
$$q^n(\cos.n\omega \pm \sqrt{-1}\cdot \sin.n\omega) + q^n = 0$$
or
$$\cos.n\omega \pm \sqrt{-1}\cdot \sin.n\omega + 1 = 0\,,$$

from which there becomes fit $\sin.n\omega = 0$ and $\cos.n\omega = -1$, but then there becomes immediately $\sin.n\omega = 0$.

3. Therefore since $\cos.n\omega = -1$, the angle $n\omega$ will be either $\pi,\ 3\pi,\ 5\pi,\ 7\pi$ etc . And thus, if i may denote some odd number, there will be $n\omega = i\pi$ and hence $\omega = \frac{i\pi}{n}$, on account of which the twofold factor in general will be

$$pp - 2pq\cos.\tfrac{i\pi}{n} + qq$$

4. Now since there shall be
$$pp + qq = 2\cos.2\varphi\,,$$

on account of $pq = 1$ there will be this factor $2\cos.2\varphi - 2\cos.\frac{i\pi}{n}$, which is resolved at once into two factors. Indeed since there shall be

$$\cos.A - \cos.B = 2\sin.\tfrac{B+A}{2}\sin.\tfrac{B-A}{2}\,,$$
there will be
$$\cos.2\varphi - \cos.\tfrac{i\pi}{n} = 2\sin.(\tfrac{i\pi}{2n} + \varphi)\sin.(\tfrac{i\pi}{2n} - \varphi)$$

and thus one single factor in general will be

$$4\sin.(\tfrac{i\pi}{2n} + \varphi)\sin.(\tfrac{i\pi}{2n} - \varphi).$$

Hence by writing for i successively the numbers 1, 2, 3, 4 etc. there will be

$$2\cos.n\varphi = 4\sin.(\tfrac{\pi}{2n} + \varphi)\sin.(\tfrac{\pi}{2n} - \varphi)$$
$$\cdot 4\sin.(\tfrac{3\pi}{2n} + \varphi)\sin.(\tfrac{3\pi}{2n} - \varphi)\cdot 4\sin.(\tfrac{5\pi}{2n} + \varphi)\sin.(\tfrac{5\pi}{2n} - \varphi)\cdot \text{etc.},$$

then altogether there may be had n factors.

5. Therefore we may run through this expression according to the single values of the number n and there will be

if $n=1,\ 2\cos.\ \varphi = 2\ \sin.(\frac{\pi}{2}-\varphi)$,

if $n=2,\ 2\ \cos.\ 2\varphi = 2^2\sin.(\frac{\pi}{4}-\varphi)\sin.(\frac{\pi}{4}+\varphi)$,

if $n=3,\ 2\ \cos.3\varphi = 2^3\sin.(\frac{\pi}{6}-\varphi)\sin.(\frac{\pi}{6}+\varphi)\sin.(\frac{3\pi}{6}-\varphi)$,

if $n=4,\ 2\ \cos.\ 4\varphi = 2^4\sin.(\frac{\pi}{8}-\varphi)\sin.(\frac{\pi}{8}+\varphi)\sin.(\frac{3\pi}{8}-\varphi)\sin.(\frac{3\pi}{8}+\varphi)$,

if $n=5,\ 2\ \cos.\ 5\varphi = 2^5\sin.(\frac{\pi}{10}-\varphi)\sin.(\frac{\pi}{10}+\varphi)\sin.(\frac{3\pi}{10}-\varphi)\sin.(\frac{3\pi}{10}+\varphi)\sin.(\frac{5\pi}{10}+\varphi)$,

if $n=6,\ 2\ \cos.\ 6\varphi = 2^6\sin.(\frac{\pi}{12}-\varphi)\sin.(\frac{\pi}{12}+\varphi)\sin.(\frac{3\pi}{12}-\varphi)\sin.(\frac{3\pi}{12}+\varphi)\sin.(\frac{5\pi}{12}-\varphi)\sin.(\frac{5\pi}{12}+\varphi)$.

Moreover generally there will be

$$\cos.\ n\varphi = 2^{\mathrm{n}-1}\sin.(\frac{\pi}{2n}-\varphi)\sin.(\frac{\pi}{2n}+\varphi)\sin.(\frac{3\pi}{2n}-\varphi)\sin.(\frac{3\pi}{2n}+\varphi)\cdot\text{etc.},$$

then there may be had n factors.

6. Therefore there will be with the logarithms taken :

$$l\cos.\ n\varphi = l2^{\mathrm{n}-1} + l\sin.(\frac{\pi}{2n}-\varphi) + l\sin.(\frac{\pi}{2n}+\varphi) + l\sin.(\frac{3\pi}{2n}-\varphi) + l\sin.(\frac{3\pi}{2n}+\varphi) + \text{etc.},$$

which equation differentiated will produce

$$\frac{nd\varphi\sin.n\varphi}{\cos.\ n\varphi} = \frac{d\varphi\cos.(\frac{\pi}{2n}-\varphi)}{\sin.(\frac{\pi}{2n}-\varphi)} - \frac{d\varphi\cos.(\frac{\pi}{2n}+\varphi)}{\sin.(\frac{\pi}{2n}-\varphi)} + \frac{d\varphi\cos.(\frac{3\pi}{2n}-\varphi)}{\sin.(\frac{3\pi}{2n}-\varphi)} - \frac{d\varphi\cos.(\frac{3\pi}{2n}+\varphi)}{\sin.(\frac{3\pi}{2n}+\varphi)} + \text{etc.},$$

that is

$$n\tan.n\varphi = \cot.(\frac{\pi}{2n}-\varphi) - \cot.(\frac{\pi}{2n}+\varphi) + \cot.(\frac{3\pi}{2n}-\varphi) - \cot.(\frac{3\pi}{2n}+\varphi) + \text{etc.},$$

from which the following noteworthy equations are deduced :

I. $\tan.\varphi = \cot.(\frac{\pi}{2} - \varphi)$,

II. $2\tan.2\varphi = \cot.(\frac{\pi}{4} - \varphi) - \cot.(\frac{\pi}{4} + \varphi) = \tan.(\frac{\pi}{4} + \varphi) - \tan.(\frac{\pi}{4} - \varphi)$,

III. $3\tan.3\varphi = \cot.(\frac{\pi}{6} - \varphi) - \cot.(\frac{\pi}{6} + \varphi) + \cot.(\frac{3\pi}{6} - \varphi)$,

or,

IV. $4\tan.4\varphi = \cot.(\frac{\pi}{8} - \varphi) - \cot.(\frac{\pi}{8} + \varphi) + \cot.(\frac{3\pi}{8} - \varphi) - \cot.(\frac{3\pi}{8} + \varphi)$,

or,

$4\tan.4\varphi = \tan.(\frac{3\pi}{8} + \varphi) - \tan.(\frac{3\pi}{8} - \varphi) + \tan.(\frac{\pi}{8} - \varphi) - \tan.(\frac{\pi}{8} - \varphi)$.

7. In the same manner we may treat the formula

$$p^n - q^n = 2\sqrt{-1} \cdot \sin.n\varphi,$$

the twofold factor of which we may put in place :

$$pp - 2pq\cos.\omega + qq,$$

in which on putting $=0$, so that there becomes as before

$$p = q(\cos.\omega \pm \sqrt{-1} \cdot \sin.\omega)$$

and hence again:

$$p^n = q^n(\cos.n\omega \pm \sqrt{-1} \cdot \sin.n\omega);$$

and thus there will have to become:

$$q^n(\cos.n\omega \pm \sqrt{-1} \cdot \sin.n\omega) - q^n = 0$$

or

$$\cos.n\omega \pm \sqrt{-1} \cdot \sin.n\omega - 1 = 0,$$

from which there must become

$$\sin.n\omega = 0 \text{ ac } \cos.n\omega = 1,$$

on account of which the angle $n\omega$ will be either $0,\ 2\pi,\ 4\pi,\ 6\pi$ or in general $2i\pi$;
and thus

$$\omega = \frac{2i\pi}{n}$$

with *i* denoting all the numbers 1, 2, 3, 4 etc. Hence therefore the twofold factor will be in be general:

$$pp - 2pq\cos.\frac{2i\pi}{n} + qq = 2\cos.2\varphi - 2\cos.\frac{2i\pi}{n}$$

which is resolved into these factors

$$2\sin.(\tfrac{i\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{i\pi}{n}+\varphi);$$

but besides the formula p^n-q^n has the simple factor

$$p-q=2\sqrt{-1}\cdot\sin.\varphi;$$

and consequently we will have

$$\sin.n\varphi=\sin.\varphi\cdot 2\sin.(\tfrac{i\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{i\pi}{n}+\varphi)\cdot\text{etc.}$$

and thus

$$\sin.n\varphi=\sin.\varphi\cdot 2\sin.(\tfrac{\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{\pi}{n}+\varphi)$$
$$\cdot 2\sin.(\tfrac{2\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{2\pi}{n}+\varphi)\cdot\text{etc.},$$

then overall n factors may be produced. Therefore there will be

$$\sin.n\varphi=2^{n-1}\sin.\varphi\cdot\sin.(\tfrac{\pi}{n}-\varphi)\sin.(\tfrac{\pi}{n}+\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}-\varphi)\cdot\sin.(\tfrac{2\pi}{n}+\varphi)\cdot\text{etc.}$$

8. Now from this general form we may deduced the following special forms :

if $n=1,\ \sin.\varphi=2^0\sin.(\varphi)$,
if $n=2,\ \sin.2\varphi=2\sin.(\varphi)\sin.(\tfrac{\pi}{2}-\varphi)$,
if $n=3,\ \sin.3\varphi=4\sin.(\varphi)\sin.(\tfrac{\pi}{3}-\varphi)\sin.(\tfrac{\pi}{3}+\varphi)$,
if $n=4,\ \sin.4\varphi=8\sin.(\tfrac{\pi}{4}-\varphi)\sin.(\tfrac{\pi}{4}+\varphi)\sin.(\tfrac{2\pi}{4}-\varphi)$,
if $n=5,\ \sin.5\varphi=16\sin.(\tfrac{\pi}{5}-\varphi)\sin.(\tfrac{\pi}{5}+\varphi)\sin.(\tfrac{2\pi}{5}-\varphi)\sin.(\tfrac{2\pi}{5}+\varphi)$,
if $n=6,\ \sin.6\varphi=32\sin.(\tfrac{\pi}{6}-\varphi)\sin.(\tfrac{\pi}{6}+\varphi)\sin.(\tfrac{2\pi}{6}-\varphi)\sin.(\tfrac{2\pi}{6}+\varphi)\sin.(\tfrac{3\pi}{6}-\varphi)$.

9. Here also as before we may take logarithms and there will be

$$l\sin.\,n\varphi=l2^{\mathrm{n}-1}+l\sin.(\tfrac{\pi}{n}-\varphi)+l\sin.(\tfrac{\pi}{n}+\varphi)+\text{etc.},$$

which equation differentiated and divided by $d\varphi$ provides :

$$\frac{n\cos.\,n\varphi}{\sin.\,n\varphi}=\frac{\cos.\varphi}{\sin.\varphi}-\frac{\cos.(\frac{\pi}{n}-\varphi)}{\sin.(\frac{\pi}{n}-\varphi)}+\frac{\cos.(\frac{\pi}{n}+\varphi)}{\sin.(\frac{\pi}{n}-\varphi)}-\frac{\cos.(\frac{2\pi}{n}-\varphi)}{\sin.(\frac{2\pi}{n}-\varphi)}+\text{etc.},$$

or

$$n\cot.n\varphi = \cot.\varphi - \cot.(\tfrac{\pi}{n} - \varphi) + \cot.(\tfrac{\pi}{n} + \varphi) - \cot.(\tfrac{2\pi}{n} - \varphi) + \text{etc.},$$

then we may have n terms.

10. Hence therefore we will have the following special forms:

if $n = 1$, $\cot.\varphi = \cot.\varphi$,
if $n = 2$, $2\cot.2\varphi = \cot.\varphi - \cot.(\tfrac{\pi}{2} - \varphi)$,
if $n = 3$, $3\cot.3\varphi = \cot.\varphi - \cot.(\tfrac{\pi}{3} - \varphi) + \cot.(\tfrac{\pi}{3} + \varphi)$,
if $n = 4$, $4\cot.4\varphi = \cot.\varphi - \cot.(\tfrac{\pi}{4} - \varphi) + \cot.(\tfrac{\pi}{4} + \varphi) - \cot.(\tfrac{2\pi}{4} - \varphi)$.

11. If we may differentiate anew the formula found for $\cot.n\varphi$, on account of

$$d\cot.\theta = \frac{-d\theta}{\sin^2.\theta}$$

on dividing by $-d\varphi$ we will have

$$\frac{nn}{\sin^2.n\varphi} = \frac{1}{\sin^2.\varphi} + \frac{1}{\sin^2.(\frac{\pi}{n} - \varphi)} + \frac{1}{\sin^2.(\frac{\pi}{n} + \varphi)} + \frac{1}{\sin^2.(\frac{2\pi}{n} - \varphi)} + \text{etc.},$$

then we may have n terms, from which the following cases may be observed :

if $n = 1$, $\frac{1}{\sin^2.\varphi} = \frac{1}{\sin^2.\varphi}$,
if $n = 2$, $\frac{4}{\sin^2.2\varphi} = \frac{1}{\sin^2.\varphi} + \frac{1}{\sin^2.(\frac{\pi}{2} - \varphi)}$,
if $n = 3$, $\frac{9}{\sin^2.3\varphi} = \frac{1}{\sin^2.\varphi} + \frac{1}{\sin^2.(\frac{\pi}{3} - \varphi)} + \frac{1}{\sin^2.(\frac{\pi}{3} + \varphi)}$,
if $n = 4$, $\frac{16}{\sin^2.4\varphi} = \frac{1}{\sin^2.\varphi} + \frac{1}{\sin^2.(\frac{\pi}{4} - \varphi)} + \frac{1}{\sin^2.(\frac{\pi}{4} + \varphi)} + \frac{1}{\sin^2.(\frac{2\pi}{4} - \varphi)}$.

THE EVOLUTION OF THE FORMULA $p^{2n} - 2p^n q^n \cos.\theta + q^{2n}$

12. We may assume here as before

$$p = \cos.\varphi + \sqrt{-1} \cdot \sin.\varphi$$

et

$$q = \cos.\varphi - \sqrt{-1} \cdot \sin.\varphi,$$

thus so that this formula may involve this value

$$2\cos.2n\varphi - 2\cos.\theta = 4\sin.(n\varphi + \tfrac{1}{2}\theta)\sin.(\tfrac{1}{2}\theta - n\varphi).$$

Now $pp - 2pq\cos.\omega + qq$ shall be a twofold factor of this formula, which therefore must vanish on putting

$$p = q(\cos.\omega \pm \sqrt{-1}\cdot\sin.\omega),$$

from which with the factor substitutes there will be produced

$$q^{2n}(\cos.2n\omega \pm \sqrt{-1}\cdot\sin.2n\omega) - 2q^{2n}\cos.\theta(\cos.n\omega \pm \sqrt{-1}\cdot\sin.n\omega) + q^{2n} = 0,$$

that is

$$\cos.2n\omega - 2\cos.\theta\cos.n\omega + 1 \pm \sqrt{-1}\cdot\sin.2n\omega \mp 2\cos.\theta\sqrt{-1}\cdot\sin.n\omega = 0,$$

from which these two equations arise

$$\cos.2n\omega - 2\cos.\theta\cos.n\omega + 1 = 0$$

and

$$\sin.2n\omega - 2\cos.\theta\sin.n\omega = 0.$$

Now since there shall be

$$\cos.2n\omega = 2\cos^2.n\omega - 1 \text{ and } \sin.2n\omega = 2\sin.n\omega\cos.n\omega,$$

these two equations will be

$$2\cos^2.n\omega - 2\cos.\theta\cos.n\omega = 0 \text{ and } 2\sin.n\omega\cos.n\omega - 2\cos.\theta\sin.n\omega = 0$$

or

$$\cos.n\omega - \cos.\theta = 0 \text{ and } \cos.n\omega - \cos.\theta = 0,$$

from which it follows $\cos.n\omega = \cos.\theta$. Therefore there will be either $n\omega = \theta,\ n\omega = 2\pi + \theta,\ 4\pi + \theta,\ 6\pi + \theta$, or in general $n\omega = 2i\pi + \theta$, from which there becomes in general

$$\omega = \tfrac{2i\pi + \theta}{n},$$

thus so that i may denote the numbers 0, 1, 2, 3, 4 etc.

13. Therefore in general the formulas of our twofold factor will be

$$pp + qq - 2pq\cos.\left(\tfrac{2i\pi + \theta}{n}\right).$$

Truly there is

$$pp + qq = 2\cos.2\varphi \text{ and } pq = 1,$$

from which this factor will be

$$2\left(\cos.2\varphi-\left(\tfrac{2i\pi+\theta}{n}\right)\right),$$

which is reduced to these simple factors

$$4\sin.\tfrac{2i\pi+2n\varphi+\theta}{2n}\sin.\tfrac{2i\pi+\theta-2n\varphi}{2n};$$

from which in place of *i* by writing the numbers 1, 2, 3, 4 etc. the factors of our formulas will be

$$\begin{aligned}&4\sin.\tfrac{2n\varphi+\theta}{2n}\sin.\tfrac{\theta-2n\varphi}{2n}\cdot4\sin.\tfrac{2\pi+2n\varphi+\theta}{2n}\sin.\tfrac{2\pi+\theta-2n\varphi}{2n}\\ &\cdot4\sin.\tfrac{4\pi+2n\varphi+\theta}{2n}\sin.\tfrac{4\pi+\theta-2n\varphi}{2n}\cdot4\sin.\tfrac{6\pi+2n\varphi+\theta}{2n}\sin.\tfrac{6\pi+\theta-2n\varphi}{2n}\cdot\text{etc.},\end{aligned}$$

which factors must be continued to that point, at which the number of these may become $=n$.

14. Therefore since this product shall be equal to the formula

$$4\sin.(n\varphi+\tfrac{1}{2}\theta)4\sin.(n\varphi-\tfrac{1}{2}\theta)$$

and in our product the numerical factor shall be, on dividing by 4 we will have this equation

$$\begin{aligned}&\sin.(n\varphi+\tfrac{1}{2}\theta)\sin.(n\varphi-\tfrac{1}{2}\theta)=2^{2n-2}\sin.\tfrac{2n\varphi+\theta}{2n}\sin.\tfrac{\theta-2n\varphi}{2n}\\ &\cdot\sin.\tfrac{2\pi+2n\varphi+\theta}{2n}\sin.\tfrac{2\pi+\theta-2n\varphi}{2n}\cdot\sin.\tfrac{4\pi+2n\varphi+\theta}{2n}\sin.\tfrac{4\pi+\theta-2n\varphi}{2n}\cdot\text{etc.},\end{aligned}$$

which equation so that it may be returned more concisely, we may put $\theta=2n\alpha$ and there will be

$$\begin{aligned}&\sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi)=2^{2n-2}\sin.(\alpha+\varphi)\sin.(\alpha-\varphi)\\ &\cdot\sin.(\tfrac{\pi}{n}+\alpha+\varphi)\sin.(\tfrac{\pi}{n}+\alpha+\varphi)\cdot\sin.(\tfrac{2\pi}{n}+\alpha+\varphi)\sin.(\tfrac{2\pi}{n}+\alpha-\varphi)\cdot\text{etc.},\end{aligned}$$

15. But this equation is not new, but now may be contained in the preceding one, which was

$$\sin.n\varphi=2^{n-1}\sin.\varphi\cdot\sin.(\tfrac{\pi}{n}-\varphi)\sin.(\tfrac{\pi}{n}+\varphi)\sin.(\tfrac{2\pi}{n}-\varphi)\sin.(\tfrac{2\pi}{n}+\varphi)\cdot\text{etc.},$$

and since there shall be

$$\sin.(o-\varphi)=\sin.(\pi+o+\varphi),$$

there will be

$$\sin.(\tfrac{\pi}{n}-\varphi)=\sin.(\tfrac{(n-1)\pi}{n}+\varphi),$$
$$\sin.(\tfrac{2\pi}{n}-\varphi)=\sin.(\tfrac{(n-2)\pi}{n}+\varphi),$$
$$\sin.(\tfrac{3\pi}{n}-\varphi)=\sin.(\tfrac{(n-3)\pi}{n}+\varphi);$$

from which that expression may be reduced to this form

$$\sin.n\varphi=2^{n-1}\sin.\varphi\cdot\sin.(\tfrac{\pi}{n}+\varphi)\sin.(\tfrac{2\pi}{n}+\varphi)$$
$$\cdot\sin.(\tfrac{3\pi}{n}+\varphi)\cdots\sin.(\tfrac{(n-1)\pi}{n}+\varphi),$$

where the arcs are continued in an arithmetical progression. So that if now here in place of φ we may write initially $\alpha+\varphi$, then $\alpha-\varphi$, hence the two following formulas may arise:

$$\sin.n(\alpha+\varphi)=2^{n-1}\sin.(\alpha+\varphi)\sin.(\tfrac{\pi}{n}+\alpha+\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}+\alpha+\varphi)\sin.(\tfrac{3\pi}{n}+\alpha+\varphi)\cdot\text{etc.},$$

$$\sin.n(\alpha-\varphi)=2^{n-1}\sin.(\alpha-\varphi)\sin.(\tfrac{\pi}{n}+\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}+\alpha-\varphi)\sin.(\tfrac{3\pi}{n}+\alpha-\varphi)\cdot\text{etc.},$$

which two equations multiplied together present

$$\sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi)=2^{2n-2}\sin.(\alpha+\varphi)\sin.(\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{\pi}{n}+\alpha+\varphi)\sin.(\tfrac{\pi}{n}+\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}+\alpha+\varphi)\sin.(\tfrac{2\pi}{n}+\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{3\pi}{n}+\alpha+\varphi)\text{ etc.}$$

16. If now we may attend to the origin of our formulas, since from our formula

$$p^{2n}-2p^nq^n\cos.\theta+q^{2n}$$

this has arisen

$$4\sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi)$$

with

$$p=\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$$

and

$$q=\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi,$$

being present, if we may put

$$f = \cos.(\alpha+\varphi) + \sqrt{-1}\cdot\sin.(\alpha+\varphi)$$

and

$$g = \cos.\ (\alpha+\varphi) - \sqrt{-1}\cdot\sin.(\alpha+\varphi).$$

then there will be

$$f^n - g^n = 2\sqrt{-1}\cdot\sin.n(\alpha+\varphi).$$

Thence if we put

$$h = \cos.(\alpha-\varphi) + \sqrt{-1}\cdot\sin.(\alpha-\varphi)$$

and

$$k = \cos.\ (\alpha-\varphi) - \sqrt{-1}\cdot\sin.(\alpha-\varphi),$$

in a like manner there will be

$$h^n - k^n = 2\sqrt{-1}\cdot\sin.n(\alpha-\varphi),$$

from which there will become

$$\begin{aligned}(f^n - g^n)(h^n - k^n) &= -4\sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi)\\ &= -p^{2n} + 2p^n q^n\cos.n\alpha - q^{2n}.\end{aligned}$$

For this requiring to be demonstrated it is to be observed

$$\begin{aligned} f &= p(\cos.\alpha + \sqrt{-1}\cdot\sin.\alpha),\\ g &= q(\cos.\alpha - \sqrt{-1}\cdot\sin.\alpha),\\ h &= q(\cos.\alpha + \sqrt{-1}\cdot\sin.\alpha),\\ k &= p(\cos.\alpha - \sqrt{-1}\cdot\sin.\alpha),\end{aligned}$$

from which there becomes

$$\begin{aligned} f^n &= p^n(\cos.n\alpha + \sqrt{-1}\cdot\sin.n\alpha),\\ g^n &= q^n(\cos.n\alpha - \sqrt{-1}\cdot\sin.n\alpha),\\ h^n &= q^n(\cos.n\alpha + \sqrt{-1}\cdot\sin.n\alpha),\\ k^n &= p^n(\cos.n\alpha - \sqrt{-1}\cdot\sin.n\alpha).\end{aligned}$$

We may put for brevity, therefore

$$\cos.n\alpha + \sqrt{-1}\cdot\sin.n\alpha = A, \quad \cos.n\alpha - \sqrt{-1}\cdot\sin.n\alpha = B,$$

so that there shall be

$$f^n = Ap^n,\ g^n = Bq^n,\ h^n = Aq^n \text{ et } k^n = Bp^n,$$

and hence again:

$$f^n - g^n = Ap^n - Bq^n,$$

and

$$h^n - k^n = Aq^n - Bp^n,$$

which two formulas multiplied present

$$(f^n - g^n)(h^n - k^n) = (A^2 + B^2)p^n q^n - AB(p^{2n} + q^{2n});$$

where since there shall be

$$AB = 1 \text{ and } AA + BB = 2\cos.2n\alpha,$$

this product will be

$$-p^{2n} + 2p^n q^n \cos.n\alpha - q^{2n},$$

which is that itself, which we have found.

COROLLARY

Hence therefore we understand the formula

$$p^{2n} - 2p^n q^n \cos.n\alpha + q^{2n}$$

to be resolved into these two factors

$$(Ap^n - Bq^n) \text{ and } (Bp^n - Aq^n)$$

with

$$A = \cos.na + \sqrt{-1}\cdot\sin.n\alpha,$$
$$B = \cos.na - \sqrt{-1}\cdot\sin.n\alpha.$$

being present.

QUOMODO SINUS ET COSINUS ANGULORUM MULTIPLORUM PER PRODUCTA EXPRIMI QUEANT

Commentatio 562 indicis ENESTROEMlANI
Opuscula analytica 1, 1783, p. 353-363

1. Proposito angulo quocunque φ ponatur brevitatis gratia

$$p=\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$$
et
$$q=\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi,$$

erit

$$pq=1;$$

tum vero

$$p^n=\cos.n\varphi+\sqrt{-1}\cdot\sin.n\varphi$$
et
$$q^n=\cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi,$$

unde fit

$$p^n+q^n=2\cos.n\varphi$$
et
$$p^n-q^n=2\sqrt{-1}\cdot\sin.n\varphi.$$

Res igitur eo redit, ut formulae p^n+q^n et p^n-q^n in factores resolvantur.

2. Consideremus primo formulam

$$p^n+q^n=2\cos.n\varphi,$$

quae, quoties n est numerus impar, factorem habet simplicem $p+q=2\cos.\varphi$, ita ut his casibus $\cos.\varphi$ sit factor ipsius $\cos.n\varphi$. Pro reliquis factoribus autem ponamus factorem duplicem in genere esse $pp-2pq\cos.\omega+qq$, ita ut formula p^n+q^n evanescat posito

$$pp-2pq\cos.\omega+qq=0;$$

tum autem erit vel

$$p=q(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)$$

vel

$$p=q(\cos.\omega-\sqrt{-1}\cdot\sin.\omega)$$

hincque

$$p^n = q^n(\cos.n\omega \pm \sqrt{-1}\cdot\sin.n\omega)\,;$$

sicque debebit esse

$$q^n(\cos.n\omega \pm \sqrt{-1}\cdot\sin.n\omega) + q^n = 0$$

sive

$$\cos.n\omega \pm \sqrt{-1}\cdot\sin.n\omega + 1 = 0,$$

unde fit $\sin.n\omega = 0$ et $\cos.n\omega = -1$, tum autem sponte fit $\sin.n\omega = 0$.

3. Quia igitur $\cos.n\omega = -1$, angulus $n\omega$ erit vel π vel 3π vel 5π vel 7π vel etc. Sicque, si i denotet numerum imparem quemcunque, erit $n\omega = i\pi$ hincque $\omega = \frac{i\pi}{n}$, quocirca factor duplex in genere erit

$$pp - 2pq\cos.\tfrac{i\pi}{n} + qq$$

4. Cum nunc sit

$$pp + qq = 2\cos.2\varphi\,,$$

ob $pq = 1$ erit iste factor $2\cos.2\varphi - 2\cos.\frac{i\pi}{n}$, qui sponte in duos factores resolvitur. Cum enim sit

$$\cos.A - \cos.B = 2\sin.\tfrac{B+A}{2}\sin.\tfrac{B-A}{2}\,,$$

erit

$$\cos.2\varphi - \cos.\tfrac{i\pi}{n} = 2\sin.(\tfrac{i\pi}{2n}+\varphi)\sin.(\tfrac{i\pi}{2n}-\varphi)$$

sicque unus factor in genere erit

$$4\sin.(\tfrac{i\pi}{2n}+\varphi)\sin.(\tfrac{i\pi}{2n}-\varphi).$$

Hinc pro i successive numeros 1, 2, 3, 4 etc. scribendo erit

$$2\cos.n\varphi = 4\sin.(\tfrac{\pi}{2n}+\varphi)\sin.(\tfrac{\pi}{2n}-\varphi)$$
$$\cdot 4\sin.(\tfrac{3\pi}{2n}+\varphi)\sin.(\tfrac{3\pi}{2n}-\varphi)\cdot 4\sin.(\tfrac{5\pi}{2n}+\varphi)\sin.(\tfrac{5\pi}{2n}-\varphi)\cdot \text{etc.},$$

donec omnino habeantur n factores.

5. Percurramus igitur hanc expressionem secundum singulos valores numeri n eritque

si $n=1,\ 2\cos.\ \varphi = 2\ \sin.(\frac{\pi}{2}-\varphi),$

si $n=2,\ 2\ \cos.\ 2\varphi = 2^2\sin.(\frac{\pi}{4}-\varphi)\sin.(\frac{\pi}{4}+\varphi),$

si $n=3,\ 2\ \cos.3\varphi = 2^3\sin.(\frac{\pi}{6}-\varphi)\sin.(\frac{\pi}{6}+\varphi)\sin.(\frac{3\pi}{6}-\varphi),$

si $n=4,\ 2\ \cos.\ 4\varphi = 2^4\sin.(\frac{\pi}{8}-\varphi)\sin.(\frac{\pi}{8}+\varphi)\sin.(\frac{3\pi}{8}-\varphi)\sin.(\frac{3\pi}{8}+\varphi),$

si $n=5,\ 2\ \cos.\ 5\varphi = 2^5\sin.(\frac{\pi}{10}-\varphi)\sin.(\frac{\pi}{10}+\varphi)\sin.(\frac{3\pi}{10}-\varphi)\sin.(\frac{3\pi}{10}+\varphi)\sin.(\frac{5\pi}{10}+\varphi),$

si $n=6,\ 2\ \cos.\ 6\varphi = 2^6\sin.(\frac{\pi}{12}-\varphi)\sin.(\frac{\pi}{12}+\varphi)\sin.(\frac{3\pi}{12}-\varphi)\sin.(\frac{3\pi}{12}+\varphi)\sin.(\frac{5\pi}{12}-\varphi)\sin.(\frac{5\pi}{12}+\varphi).$

Generaliter autem erit

$$\cos.\ n\varphi = 2^{\mathrm{n}-1}\sin.(\tfrac{\pi}{2n}-\varphi)\sin.(\tfrac{\pi}{2n}+\varphi)\sin.(\tfrac{3\pi}{2n}-\varphi)\sin.(\tfrac{3\pi}{2n}+\varphi)\cdot \text{etc.},$$

donec habeantur *n* factores.

6. Sumendis igitur logarithmis erit

$$l\cos.\ n\varphi = l2^{\mathrm{n}-1} + l\sin.(\tfrac{\pi}{2n}-\varphi) + l\sin.(\tfrac{\pi}{2n}+\varphi) + l\sin.(\tfrac{3\pi}{2n}-\varphi) + l\sin.(\tfrac{3\pi}{2n}+\varphi) + \text{etc.},$$

quae aequatio differentiata praebet

$$\frac{nd\varphi\sin.n\varphi}{\cos.\ n\varphi} = \frac{d\varphi\cos.(\frac{\pi}{2n}-\varphi)}{\sin.(\frac{\pi}{2n}-\varphi)} - \frac{d\varphi\cos.(\frac{\pi}{2n}+\varphi)}{\sin.(\frac{\pi}{2n}-\varphi)} + \frac{d\varphi\cos.(\frac{3\pi}{2n}-\varphi)}{\sin.(\frac{3\pi}{2n}-\varphi)} - \frac{d\varphi\cos.(\frac{3\pi}{2n}+\varphi)}{\sin.(\frac{3\pi}{2n}+\varphi)} + \text{etc.},$$

hoc est

$$n\tan.n\varphi = \cot.(\tfrac{\pi}{2n}-\varphi) - \cot.(\tfrac{\pi}{2n}+\varphi) + \cot.(\tfrac{3\pi}{2n}-\varphi) - \cot.(\tfrac{3\pi}{2n}+\varphi) + \text{etc.},$$

unde deducuntur sequentes aequalitates memoratu dignae

I. $\tan.\varphi = \cot.(\frac{\pi}{2} - \varphi)$,

II. $2\tan.2\varphi = \cot.(\frac{\pi}{4} - \varphi) - \cot.(\frac{\pi}{4} + \varphi) = \tan.(\frac{\pi}{4} + \varphi) - \tan.(\frac{\pi}{4} - \varphi)$,

III. $3\tan.3\varphi = \cot.(\frac{\pi}{6} - \varphi) - \cot.(\frac{\pi}{6} + \varphi) + \cot.(\frac{3\pi}{6} - \varphi)$,

sive,

IV. $4\tan.4\varphi = \cot.(\frac{\pi}{8} - \varphi) - \cot.(\frac{\pi}{8} + \varphi) + \cot.(\frac{3\pi}{8} - \varphi) - \cot.(\frac{3\pi}{8} + \varphi)$,

sive,

$4\tan.4\varphi = \tan.(\frac{3\pi}{8} + \varphi) - \tan.(\frac{3\pi}{8} - \varphi) + \tan.(\frac{\pi}{8} - \varphi) - \tan.(\frac{\pi}{8} - \varphi)$.

7. Eodem modo tractemus formulam

$$p^n - q^n = 2\sqrt{-1} \cdot \sin.n\varphi,$$

cuius factorem duplicem statuamus

$$pp - 2pq\cos.\omega + qq,$$

quo posito $= 0$ fit ut ante

$$p = q(\cos.\omega \pm \sqrt{-1} \cdot \sin.\omega)$$

hincque porro

$$p^n = q^n(\cos.n\omega \pm \sqrt{-1} \cdot \sin.n\omega);$$

sicque debebit esse

$$q^n(\cos.n\omega \pm \sqrt{-1} \cdot \sin.n\omega) - q^n = 0$$

sive

$$\cos.n\omega \pm \sqrt{-1} \cdot \sin.n\omega - 1 = 0,$$

unde fieri debet

$$\sin.n\omega = 0 \text{ ac } \cos.n\omega = 1,$$

quamobrem angulus $n\omega$ erit vel 0 vel 2π vel 4π vel 6π vel in genere $2i\pi$ ideoque

$$\omega = \frac{2i\pi}{n}$$

denotante i numeros omnes 1, 2, 3, 4 etc. Hinc igitur factor duplex in genere erit

$$pp - 2pq\cos.\frac{2i\pi}{n} + qq = 2\cos.2\varphi - 2\cos.\frac{2i\pi}{n}$$

qui resolvitur in hos factores

$$2\sin.(\tfrac{i\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{i\pi}{n}+\varphi);$$

praeterea autem formula $p^n - q^n$ habet factorem simplicem

$$p-q=2\sqrt{-1}\cdot\sin.\varphi;$$

consequenter habebimus

$$\sin.n\varphi=\sin.\varphi\cdot 2\sin.(\tfrac{i\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{i\pi}{n}+\varphi)\cdot\text{etc.}$$

ideoque

$$\sin.n\varphi=\sin.\varphi\cdot 2\sin.(\tfrac{\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{\pi}{n}+\varphi)$$
$$\cdot 2\sin.(\tfrac{2\pi}{n}-\varphi)\cdot 2\sin.(\tfrac{2\pi}{n}+\varphi)\cdot\text{etc.},$$

donec omnino prodeant n factores. Erit ergo

$$\sin.n\varphi=2^{n-1}\sin.\varphi\cdot\sin.(\tfrac{\pi}{n}-\varphi)\sin.(\tfrac{\pi}{n}+\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}-\varphi)\cdot\sin.(\tfrac{2\pi}{n}+\varphi)\cdot\text{etc.}$$

8. Iam ex hac forma generali sequentes deducamus formas speciales:

si $n=1,\ \sin.\varphi=2^0\sin.(\varphi),$

si $n=2,\ \sin.2\varphi=2\sin.(\varphi)\sin.(\tfrac{\pi}{2}-\varphi),$

si $n=3,\ \sin.3\varphi=4\sin.(\varphi)\sin.(\tfrac{\pi}{3}-\varphi)\sin.(\tfrac{\pi}{3}+\varphi),$

si $n=4,\ \sin.4\varphi=8\sin.(\tfrac{\pi}{4}-\varphi)\sin.(\tfrac{\pi}{4}+\varphi)\sin.(\tfrac{2\pi}{4}-\varphi),$

si $n=5,\ \sin.5\varphi=16\sin.(\tfrac{\pi}{5}-\varphi)\sin.(\tfrac{\pi}{5}+\varphi)\sin.(\tfrac{2\pi}{5}-\varphi)\sin.(\tfrac{2\pi}{5}+\varphi),$

si $n=6,\ \sin.6\varphi=32\sin.(\tfrac{\pi}{6}-\varphi)\sin.(\tfrac{\pi}{6}+\varphi)\sin.(\tfrac{2\pi}{6}-\varphi)\sin.(\tfrac{2\pi}{6}+\varphi)\sin.(\tfrac{3\pi}{6}-\varphi).$

9. Sumamus hic etiam ut ante logarithmos eritque

$$l\sin.\,n\varphi=l2^{\mathrm{n-1}}+l\sin.(\tfrac{\pi}{n}-\varphi)+l\sin.(\tfrac{\pi}{n}+\varphi)+\text{etc.},$$

quae aequatio differentiata et per $d\varphi$ divisa praebet

$$\frac{n\cos.\,n\varphi}{\sin.\,n\varphi}=\frac{\cos.\varphi}{\sin.\varphi}-\frac{\cos.(\frac{\pi}{n}-\varphi)}{\sin.(\frac{\pi}{n}-\varphi)}+\frac{\cos.(\frac{\pi}{n}+\varphi)}{\sin.(\frac{\pi}{n}-\varphi)}-\frac{\cos.(\frac{2\pi}{n}-\varphi)}{\sin.(\frac{2\pi}{n}-\varphi)}+\text{etc.},$$

sive

$$n\cot.n\varphi=\cot.\varphi-\cot.(\tfrac{\pi}{n}-\varphi)+\cot.(\tfrac{\pi}{n}+\varphi)-\cot.(\tfrac{2\pi}{n}-\varphi)+\text{etc.},$$

donec habeantur n termini.

10. Hinc igitur sequentes obtinebimus formas speciales:

$$\text{si } n=1,\ \cot.\varphi=\cot.\varphi,$$
$$\text{si } n=2,\ 2\cot.2\varphi=\cot.\varphi-\cot.(\tfrac{\pi}{2}-\varphi),$$
$$\text{si } n=3,\ 3\cot.3\varphi=\cot.\varphi-\cot.(\tfrac{\pi}{3}-\varphi)+\cot.(\tfrac{\pi}{3}+\varphi),$$
$$\text{si } n=4,\ 4\cot.4\varphi=\cot.\varphi-\cot.(\tfrac{\pi}{4}-\varphi)+\cot.(\tfrac{\pi}{4}+\varphi)-\cot.(\tfrac{2\pi}{4}-\varphi).$$

11. Si formulam pro $\cot.n\varphi$*;* inventam denuo differentiemus, ob

$$d\cot.\theta=\frac{-d\theta}{\sin^2.\theta}$$

per $-d\varphi$ dividendo habebimus

$$\frac{nn}{\sin^2.n\varphi}=\frac{1}{\sin^2.\varphi}+\frac{1}{\sin^2.(\frac{\pi}{n}-\varphi)}+\frac{1}{\sin^2.(\frac{\pi}{n}+\varphi)}+\frac{1}{\sin^2.(\frac{2\pi}{n}-\varphi)}+\text{etc.},$$

donec habeantur n termini, unde sequentes casus notentur:

$$\text{si } n=1,\quad \frac{1}{\sin^2.\varphi}=\frac{1}{\sin^2.\varphi},$$
$$\text{si } n=2,\quad \frac{4}{\sin^2.2\varphi}=\frac{1}{\sin^2.\varphi}+\frac{1}{\sin^2.(\frac{\pi}{2}-\varphi)},$$
$$\text{si } n=3,\quad \frac{9}{\sin^2.3\varphi}=\frac{1}{\sin^2.\varphi}+\frac{1}{\sin^2.(\frac{\pi}{3}-\varphi)}+\frac{1}{\sin^2.(\frac{\pi}{3}+\varphi)},$$
$$\text{si } n=4,\quad \frac{16}{\sin^2.4\varphi}=\frac{1}{\sin^2.\varphi}+\frac{1}{\sin^2.(\frac{\pi}{4}-\varphi)}+\frac{1}{\sin^2.(\frac{\pi}{4}+\varphi)}+\frac{1}{\sin^2.(\frac{2\pi}{4}-\varphi)}.$$

EVOLUTIO FORMULAE $p^{2n}-2p^nq^n\cos.\theta+q^{2n}$

12. Sumamus hic ut ante

$$p=\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$$
et
$$q=\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi,$$

ita ut ista formula involvat hunc valorem

$$2\cos.2n\varphi-2\cos.\theta=4\sin.(n\varphi+\tfrac{1}{2}\theta)\sin.(\tfrac{1}{2}\theta-n\varphi).$$

Iam sit $pp-2pq\cos.\omega+qq$ factor duplex huius formulae, quae ergo evanescere debet posito

$$p=q(\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega),$$

unde facta substitutione prodibit

$$q^{2n}(\cos.2n\omega\pm\sqrt{-1}\cdot\sin.2n\omega)-2q^{2n}\cos.\theta(\cos.n\omega\pm\sqrt{-1}\cdot\sin.n\omega)+q^{2n}=0,$$

hoc est

$$\cos.2n\omega-2\cos.\theta\cos.n\omega+1\pm\sqrt{-1}\cdot\sin.2n\omega\mp2\cos.\theta\sqrt{-1}\cdot\sin.n\omega=0,$$

unde nascuntur hae duae aequationes

$$\cos.2n\omega-2\cos.\theta\cos.n\omega+1=0$$

et

$$\sin.2n\omega-2\cos.\theta\sin.n\omega=0.$$

Cum nunc sit

$$\cos.2n\omega=2\cos^2.n\omega-1 \text{ et } \sin.2n\omega=2\sin.n\omega\cos.n\omega,$$

hae duae aequationes erunt

$$2\cos^2.n\omega-2\cos.\theta\cos.n\omega=0 \text{ et } 2\sin.n\omega\cos.n\omega-2\cos.\theta\sin.n\omega=0$$

sive

$$\cos.n\omega-\cos.\theta=0 \text{ et } \cos.n\omega-\cos.\theta=0,$$

unde sequitur $\cos.n\omega=\cos.\theta$. Erit ergo vel $n\omega=\theta$ vel $n\omega=2\pi+\theta$ vel $4\pi+\theta$ vel $6\pi+\theta$ vel in genere $n\omega=2i\pi+\theta$, unde fit in genere

$$\omega=\tfrac{2i\pi+\theta}{n},$$

ita ut *i* denotet numeros 0, 1, 2, 3, 4 etc.

13. Formulae igitur nostrae factor duplex in genere erit

$$pp+qq-2pq\cos.\left(\tfrac{2i\pi+\theta}{n}\right).$$

Est vero

$$pp+qq=2\cos.2\varphi \text{ et } pq=1,$$

unde hic factor erit

$$2\left(\cos.2\varphi-\left(\tfrac{2i\pi+\theta}{n}\right)\right),$$

qui reducitur ad hos factores simplices

$$4\sin.\tfrac{2i\pi+2n\varphi+\theta}{2n}\sin.\tfrac{2i\pi+\theta-2n\varphi}{2n};$$

unde loco *i* scribendo numeros 1, 2, 3, 4 etc. factores nostrae formulae erunt

$$4\sin.\tfrac{2n\varphi+\theta}{2n}\sin.\tfrac{\theta-2n\varphi}{2n}\cdot 4\sin.\tfrac{2\pi+2n\varphi+\theta}{2n}\sin.\tfrac{2\pi+\theta-2n\varphi}{2n}$$
$$\cdot 4\sin.\tfrac{4\pi+2n\varphi+\theta}{2n}\sin.\tfrac{4\pi+\theta-2n\varphi}{2n}\cdot 4\sin.\tfrac{6\pi+2n\varphi+\theta}{2n}\sin.\tfrac{6\pi+\theta-2n\varphi}{2n}\cdot \text{etc.},$$

qui factores eousque continuari debent, donec eorum numerus fiat $=n$.

14. Cum igitur hoc productum aequale sit formulae

$$4\sin.(n\varphi+\tfrac{1}{2}\theta)4\sin.(n\varphi-\tfrac{1}{2}\theta)$$

et in nostro producto factor numericus sit $4^n=2^{2n}$, per 4 dividendo habebimus hanc aequationem

$$\sin.(n\varphi+\tfrac{1}{2}\theta)\sin.(n\varphi-\tfrac{1}{2}\theta)=2^{2n-2}\sin.\tfrac{2n\varphi+\theta}{2n}\sin.\tfrac{\theta-2n\varphi}{2n}$$
$$\cdot\sin.\tfrac{2\pi+2n\varphi+\theta}{2n}\sin.\tfrac{2\pi+\theta-2n\varphi}{2n}\cdot\sin.\tfrac{4\pi+2n\varphi+\theta}{2n}\sin.\tfrac{4\pi+\theta-2n\varphi}{2n}\cdot\text{etc.},$$

quae aequatio quo concinnior reddatur, ponamus $\theta=2n\alpha$ et erit

$$\sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi)=2^{2n-2}\sin.(\alpha+\varphi)\sin.(\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{\pi}{n}+\alpha+\varphi)\sin.(\tfrac{\pi}{n}+\alpha+\varphi)\cdot\sin.(\tfrac{2\pi}{n}+\alpha+\varphi)\sin.(\tfrac{2\pi}{n}+\alpha-\varphi)\cdot\text{etc.},$$

15. Haec autem expressio non est nova, sed iam in praecedente continetur, quae erat

$$\sin.n\varphi=2^{n-1}\sin.\varphi\cdot\sin.(\tfrac{\pi}{n}-\varphi)\sin.(\tfrac{\pi}{n}+\varphi)\sin.(\tfrac{2\pi}{n}-\varphi)\sin.(\tfrac{2\pi}{n}+\varphi)\cdot\text{etc.},$$

et cum sit

$$\sin.(o-\varphi)=\sin.(\pi+o+\varphi),$$

erit

$$\sin.(\tfrac{\pi}{n}-\varphi)=\sin.(\tfrac{(n-1)\pi}{n}+\varphi),$$
$$\sin.(\tfrac{2\pi}{n}-\varphi)=\sin.(\tfrac{(n-2)\pi}{n}+\varphi),$$
$$\sin.(\tfrac{3\pi}{n}-\varphi)=\sin.(\tfrac{(n-3)\pi}{n}+\varphi);$$

unde illa expressio reducetur ad hanc formam

$$\sin.n\varphi = 2^{n-1}\sin.\varphi \cdot \sin.(\tfrac{\pi}{n}+\varphi)\sin.(\tfrac{2\pi}{n}+\varphi)$$
$$\cdot \sin.(\tfrac{3\pi}{n}+\varphi)\cdots\sin.(\tfrac{(n-1)\pi}{n}+\varphi),$$

ubi arcus in progressione arithmetica continua progrediuntur. Quodsi iam hic loco φ scribamus primo $\alpha+\varphi$, deinde $\alpha-\varphi$, hinc duae formulae sequentes nascentur:

$$\sin.n(\alpha+\varphi) = 2^{n-1}\sin.(\alpha+\varphi)\sin.(\tfrac{\pi}{n}+\alpha+\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}+\alpha+\varphi)\sin.(\tfrac{3\pi}{n}+\alpha+\varphi)\cdot \text{etc.},$$

$$\sin.n(\alpha-\varphi) = 2^{n-1}\sin.(\alpha-\varphi)\sin.(\tfrac{\pi}{n}+\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}+\alpha-\varphi)\sin.(\tfrac{3\pi}{n}+\alpha-\varphi)\cdot \text{etc.},$$

quae duae aequationes in se invicem ductae praebent

$$\sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi) = 2^{2n-2}\sin.(\alpha+\varphi)\sin.(\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{\pi}{n}+\alpha+\varphi)\sin.(\tfrac{\pi}{n}+\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{2\pi}{n}+\alpha+\varphi)\sin.(\tfrac{2\pi}{n}+\alpha-\varphi)$$
$$\cdot\sin.(\tfrac{3\pi}{n}+\alpha+\varphi)\ \text{etc.}$$

16. Si nunc attendamus ad originem harum formularum, quandoquidem ex nostra formula

$$p^{2n} - 2p^nq^n\cos.\theta + q^{2n}$$

nata est haec

$$4\ \sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi)$$

existente

$$p = \cos.\varphi + \sqrt{-1}\cdot\sin.\varphi$$
et
$$q = \cos.\varphi - \sqrt{-1}\cdot\sin.\varphi,$$

si ponamus

$$f = \cos.(\alpha+\varphi) + \sqrt{-1}\cdot\sin.(\alpha+\varphi)$$
et
$$g = \cos.\ (\alpha+\varphi) - \sqrt{-1}\cdot\sin.(\alpha+\varphi).$$

tum erit

$$f^n - g^n = 2\sqrt{-1}\cdot \sin.n(\alpha+\varphi).$$

Deinde si ponamus

$$h = \cos.(\alpha-\varphi) + \sqrt{-1}\cdot \sin.(\alpha-\varphi)$$

et

$$k = \cos.\ (\alpha-\varphi) - \sqrt{-1}\cdot \sin.(\alpha-\varphi),$$

erit simili modo

$$h^n - k^n = 2\sqrt{-1}\cdot \sin.n(\alpha-\varphi),$$

unde erit

$$(f^n - g^n)(h^n - k^n) = -4\sin.n(\alpha+\varphi)\sin.n(\alpha-\varphi)$$
$$= -p^{2n} + 2p^n q^n \cos.n\alpha - q^{2n}.$$

.
Ad hoc demonstrandum notetur esse

$$\begin{aligned} f &= p(\cos.\alpha + \sqrt{-1}\cdot \sin.\alpha),\\ g &= q(\cos.\alpha - \sqrt{-1}\cdot \sin.\alpha),\\ h &= q(\cos.\alpha + \sqrt{-1}\cdot \sin.\alpha),\\ k &= p(\cos.\alpha - \sqrt{-1}\cdot \sin.\alpha), \end{aligned}$$

unde fit

$$\begin{aligned} f^n &= p^n(\cos.n\alpha + \sqrt{-1}\cdot \sin.n\alpha),\\ g^n &= q^n(\cos.n\alpha - \sqrt{-1}\cdot \sin.n\alpha),\\ h^n &= q^n(\cos.n\alpha + \sqrt{-1}\cdot \sin.n\alpha),\\ k^n &= p^n(\cos.n\alpha - \sqrt{-1}\cdot \sin.n\alpha). \end{aligned}$$

Ponamus brevitatis ergo

$$\cos.n\alpha + \sqrt{-1}\cdot \sin.n\alpha = A, \quad \cos.n\alpha - \sqrt{-1}\cdot \sin.n\alpha = B,$$

ut sit

$$f^n = Ap^n,\ g^n = Bq^n,\ h^n = Aq^n \text{ et } k^n = Bp^n,$$

hincque porro

$$f^n - g^n = Ap^n - Bq^n,$$

et

$$h^n - k^n = Aq^n - Bp^n,$$

quae duae formulae multiplicatae praebent

$$(f^n - g^n)(h^n - k^n) = (A^2 + B^2)p^n q^n - AB(p^{2n} + q^{2n});$$

ubi cum sit

$$AB = 1 \text{ et } AA + BB = 2\cos.2n\alpha,$$

hoc productum erit

$$-p^{2n} + 2p^n q^n \cos.n\alpha - q^{2n},$$

quod est id ipsum, quod invenimus.

COROLLARIUM

Hinc igitur intelligimus formulam

$$p^{2n} - 2p^n q^n \cos.n\alpha + q^{2n}$$

resolvi in hos duos factores

$$(Ap^n - Bq^n) \text{ et } (Bp^n - Aq^n)$$

existente

$$A = \cos.na + \sqrt{-1}\cdot\sin.n\alpha,$$

$$B = \cos.na - \sqrt{-1}\cdot\sin.n\alpha.$$