VARIOUS OBSERVATIONS CONCERNING ANGLES ADVANCING IN GEOMETRIC PROGRESSION

[E561]

Opuscula analytica 1**,** 1783, p. 345-352

1. Since most significant properties, which hitherto have been investigated concerning the angles or the sines, cosines, tangents, cotangents, secants and cosecants of the arcs of these, may be derived from a consideration of the arcs increasing in an arithmetic progression, these properties may be considered none the less noteworthy which may be deduced from a consideration of the arcs of these proceeding in a geometric progression, especially since the truth of these and several more much more hidden may be seen; on account of which I have decided to set out more properties of this kind.

2. The most noteworthy formula

$$\sin.2\varphi = 2\sin.\varphi \cdot \cos.\varphi ,$$

reveals the first source for us for investigations of this kind : from which, if s may denote the arc or some angle, there will become

$$\sin.s = 2\sin\tfrac{1}{2}s \cdot \cos.\tfrac{1}{2}s ,$$

then truly in a like manner there will be

$$\sin.\tfrac{1}{2}s = 2\sin.\tfrac{1}{4}s \cdot \cos.\tfrac{1}{4}s,$$

which value substituted there provides

$$\sin.s = 4\sin.\tfrac{1}{4}s \cdot \cos.\tfrac{1}{2}s \cdot \cos.\tfrac{1}{4}.$$

Thence, since again there

$$\sin.\tfrac{1}{4}s = 2\sin.\tfrac{1}{8}s \cdot \cos.\tfrac{1}{8}s,$$

with this value substituted there will be

$$\sin.s = 8\sin.\tfrac{1}{8}s\cdot\cos.\tfrac{1}{2}s\cdot\cos.\tfrac{1}{4}s\cdot\cos\tfrac{1}{8}s.$$

By progressing in a like manner there is :

$$\sin.s = 16\sin.\tfrac{1}{16}s\cdot\cos.\tfrac{1}{2}s\cdot\cos.\tfrac{1}{4}s\cdot\cos.\tfrac{1}{8}s\cdot\cos.\tfrac{1}{16}s,$$

and we may progress indefinitely in this manner, with i an infinite number or rather [$\frac{1}{i}$] an infinitesimal power of two 2 we will have

$$\sin.s = i\sin.\tfrac{1}{i}s\cdot\cos.\tfrac{1}{2}s\cdot\cos.\tfrac{1}{4}s\cdot\cos.\tfrac{1}{8}s\cdot\cos.\tfrac{1}{16}s\cdot\text{etc.},$$

where, since the arc $\frac{s}{i}$ is infinitely small, there will be $\sin.\frac{s}{i}=\frac{s}{i}$ and therefore $i\sin.\frac{s}{i}=s$, from which we arrive at this conspicuous property, so that there shall become:

$$\sin.s = s\cos.\tfrac{1}{2}s\cdot\cos.\tfrac{1}{4}s\cdot\cos.\tfrac{1}{8}s\cdot\cos.\tfrac{1}{16}s\cdot\cos.\tfrac{1}{32}s\cdot\text{etc. ad infinitum.}$$

3. Hence therefore this arc s itself is defined thus most beautifully by its sine and by the cosines of the arcs thus continually descending in a two fold ratio, thus so that there shall become

$$s = \frac{\sin.s}{\cos.\frac{1}{2}s\cdot\cos.\frac{1}{4}s\cdot\cos.\frac{1}{8}s\cdot\cos.\frac{1}{16}s\cdot\cos.\frac{1}{32}s\cdot\text{etc.}};$$

but since $\frac{1}{\cos.\varphi}=\sec.\varphi$, there will become by this whole expression

$$s = \sin.s\cdot\sec.\tfrac{1}{2}s\cdot\sec.\tfrac{1}{4}s\cdot\sec.\tfrac{1}{8}s\cdot\sec.\tfrac{1}{16}s\cdot\sec.\tfrac{1}{32}s\cdot\text{etc.},$$

which expression can be represented geometrically well enough, as I have shown elsewhere [see E74].

4. Because here the arc s is expressed by a product, with logarithms taken we will have :

$$ls = l\sin.s + l\sec.\tfrac{1}{2}s + l\sec.\tfrac{1}{4}s + l\sec.\tfrac{1}{8}s + l\sec.\tfrac{1}{16}s + l\sec.\tfrac{1}{32}s + \text{etc.},$$

from which, if we may take $s=\frac{\pi}{2}=90^\circ$, there will become :

$$l\tfrac{\pi}{2} = 0 + l\sec.45^\circ + l\sec.\,22^\circ30' + l\sec.\,11^\circ15' + l\sec.5^\circ37\tfrac{1}{2}' + \text{etc.},$$

from which calculation put in place there will be :

$$l\sec.45° = 0,1505150$$
$$l\sec.22°30' = 0,0343847$$
$$l\sec.11°15' = 0,0084261$$
$$l\sec.5°37' = 0,0020963$$
$$l\sec.2°48' = 0,0005234$$
$$l\sec.1°24' = 0,0001308$$
$$l\sec.0°42' = 0,0000327$$
$$l\sec.0°21' = 0,0000082$$
$$\text{reliqui omnes} = 0,0000027$$
$$l\tfrac{\pi}{2} = 0,1961199$$
$$l2 = 0,3010300$$
$$l\pi = 0,4971499$$

and hence $\pi = 3,1415928$ exactly enough, as it may be agreed.

5. But so that hence we may deduce new relations, we may differentiate the last logarithmic equation, and since the following equation shall arise on dividing by *ds* :

$$\tfrac{1}{s} = \cot.s + \tfrac{1}{2}\tan.\tfrac{1}{2}s + \tfrac{1}{4}\tan.\tfrac{1}{4}s + \tfrac{1}{8}\tan.\tfrac{1}{8}s + \tfrac{1}{16}\tan.\tfrac{1}{16}s + \text{etc.},$$

since which series so that it converges the most quickly, which will the following example will make clearer, in which we may take $s = 90° = \frac{\pi}{2}$, from which there will become:

$$\tfrac{2}{\pi} = \tfrac{1}{2}\tan.45° + \tfrac{1}{4}\tan.22°30' + \tfrac{1}{8}\tan.11°15' + \tfrac{1}{16}\tan.5°37\tfrac{1}{2}' + \text{etc.},$$

which values taken from tables will give :

$$\tfrac{1}{2}\tan.45° = 0,5000000$$
$$\tfrac{1}{4}\tan.22°30' = 0,1035534$$
$$\tfrac{1}{8}\tan.11°15' = 0,0248640$$
$$\tfrac{1}{16}\tan.5°37\tfrac{1}{2}' = 0,0061557$$
$$\tfrac{1}{32}\tan.2°48\tfrac{3}{4}' = 0,0015352$$
$$\tfrac{1}{64}\tan.1°24\tfrac{3}{8}' = 0,0003836$$
$$\text{with the rest} = 0,0001279$$
$$\tfrac{2}{\pi} = 0,6366198$$

hence $\pi = \frac{2}{0,6366198} = \frac{1}{0,3183099}$.

6. If we may differentiate the last equation again, we will arrive at a series much more convergent; since indeed there shall be

$$d.\cot.\varphi = \frac{-d\varphi}{\sin^2.\varphi} \text{ and } d.\tan.\varphi = \frac{d\varphi}{\cos^2.\varphi} = d\varphi\sec^2.\varphi,$$

we will find

$$-\frac{1}{ss} = \frac{-1}{\sin^2.s} + \frac{1}{4}sec^2.\tfrac{1}{2}s + \frac{1}{16}sec^2.\tfrac{1}{4}s + \frac{1}{64}sec^2.\tfrac{1}{8}\text{etc.}$$

or

$$\frac{1}{4}sec^2.\tfrac{1}{2}s + \frac{1}{16}sec^2.\tfrac{1}{4}s + \frac{1}{64}sec^2.\tfrac{1}{8} + \frac{1}{256}sec^2.\tfrac{1}{16} + \text{etc.} = \frac{1}{\sin^2.s} - \frac{1}{ss}.$$

7. We may apply the same reasoning to the triple ratio, following which the arcs may decrease; finally we will consider this formula

$$\sin.3\varphi = 4\sin.\varphi\cos^2.\varphi - \sin.\varphi = \sin.\varphi(3 - 4\sin^2.\varphi),$$

which gives

$$\sin.3\varphi = 3\sin.\varphi(1 - \tfrac{4}{3}\sin^2.\varphi);$$

from which, if s may denote some arc, there will become

$$\sin.s = 3\sin.\tfrac{1}{3}s(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{3});$$

and in a similar manner there will become :

$$\sin.\tfrac{1}{3}s = 3\sin.\tfrac{s}{9}(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{9}),$$

thus so that now there shall be

$$\sin.s = 9\sin.\tfrac{1}{9}s(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{3})(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{9}).$$

If such substitutions may be continued indefinitely, finally so that this expression will be come upon as before

$$\sin.s = s(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{3})(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{9})(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{27})\cdot\text{etc.}$$

8. These exceedingly complicated factors will be allowed to be resolved into simpler ones in the following manner ; for indeed the general form general

$$1 - \tfrac{4}{3}\sin^2.\varphi \text{ ob } \sin^2.\varphi = \tfrac{1}{2} - \tfrac{1}{2}\cos.2\varphi$$

is reduced to this

$$\tfrac{1}{3}+\tfrac{2}{3}\cos.2\varphi,$$

which may be referred to in this way:

$$\frac{2\cos.60^\circ+2\cos.2\varphi}{3}.$$

Now since there shall be

$$\cos.a+\cos.b=2\cos.\tfrac{a+b}{2}\cos.\tfrac{a-b}{2},$$

there will be

$$\cos.60^\circ+\cos.2\varphi=2\cos.(30^\circ+\varphi)\cos.(30^\circ-\varphi),$$

which form multiplied by $\frac{2}{3}$ will present

$$1-\tfrac{4}{3}\sin^2.\varphi=\tfrac{4}{3}\cos.(30^\circ+\varphi)\cos.(30^\circ-\varphi).$$

Whereby if this reduction may be applied to the individual factors found above, we will have the following infinite product

$$\begin{aligned}\frac{\sin.s}{s}=&\tfrac{4}{3}\cos.(30^\circ+\tfrac{s}{3})\cos.(30^\circ-\tfrac{s}{3})\cdot\\&\tfrac{4}{3}\cos.(30^\circ+\tfrac{s}{9})\cos.(30^\circ-\tfrac{s}{9})\cdot\\&\tfrac{4}{3}\cos.(30^\circ+\tfrac{s}{27})\cos.(30^\circ-\tfrac{s}{27})\cdot\\&\text{etc.},\end{aligned}$$

which will be shown by the secants thus :

$$\begin{aligned}\frac{s}{\sin.s}=&\tfrac{3}{4}\sec.(30^\circ+\tfrac{s}{3})\sec.(30^\circ-\tfrac{s}{3})\cdot\\&\tfrac{3}{4}\sec.(30^\circ+\tfrac{s}{9})\sec.(30^\circ-\tfrac{s}{9})\cdot\\&\tfrac{3}{4}\sec.(30^\circ+\tfrac{s}{27})\sec.(30^\circ-\tfrac{s}{27})\cdot\\&\text{etc.},\end{aligned}$$

which factors, by which the arc s is diminished more, therefore will approach closer to one.

9. If now we may take the logarithms and differentiate the individual terms, the factors for the calculation will exceed the number $\frac{3}{4}$ entirely; and because, as we have seen above, there is

$$d.l.\sec.\varphi=d\varphi\tan.\varphi,$$

the following equation will be obtained

$$\frac{1}{s}=\operatorname{cotan}.s+\tfrac{1}{3}\tan.(30^\circ+\tfrac{s}{3})+\tfrac{1}{9}\tan.(30^\circ+\tfrac{s}{9})+\tfrac{1}{27}\tan.(30^\circ+\tfrac{s}{27})+\text{etc.}$$
$$-\tfrac{1}{3}\tan.(30^\circ-\tfrac{s}{3})-\tfrac{1}{9}\tan.(30^\circ-\tfrac{s}{9})-\tfrac{1}{27}\tan.(30^\circ-\tfrac{s}{27})-\text{etc.},$$

it will help to have proven that by an example. Therefore let $s=\frac{\pi}{2}$ and there will be

$$\tfrac{2}{\pi}=\tfrac{1}{3}\tan.60^\circ+\tfrac{1}{9}\tan.40^\circ+\tfrac{1}{27}\tan.(33^\circ 20')+\tfrac{1}{81}\tan.(31^\circ 6\tfrac{2}{3}')+\tfrac{1}{243}\tan.(30^\circ 22\tfrac{2}{3}')+\text{etc.}$$
$$-\tfrac{1}{3}\tan.0^\circ-\tfrac{1}{9}\operatorname{tag}.20^\circ-\tfrac{1}{27}\tan.(26^\circ 40')-\tfrac{1}{81}\tan.(28^\circ 53\tfrac{1}{3}')-\tfrac{1}{243}\tan.(29^\circ 37\tfrac{7}{9}')-\text{etc.}$$

10. This last series therefore is seen to be the most well-known, because it will have prevailed to demonstrate scarcely any of its truth, unless it were by the same method used. But this series without doubt is of a much higher level of investigation than that, for which we have deduced by the preceding expansion, which was

$$\tfrac{1}{s}=\cot.s+\tfrac{1}{2}\tan.\tfrac{1}{2}s+\tfrac{1}{4}\tan.\tfrac{1}{4}s+\tfrac{1}{8}\tan.\tfrac{1}{8}s+\tfrac{1}{16}\tan.\tfrac{1}{16}s+\text{etc.};$$

of which the truth follows from the most well-known formula

$$2\cot.2\varphi=\cot.\varphi-\tan.\varphi,$$

from which we have

$$\tan.\varphi=\cot.\varphi-2\cot.2\varphi.$$

Hence if in place of the individual values of the tangents, the due values may be substituted, the operation may be composed in the following manner:

$$\frac{1}{s}=\begin{cases}\cot.s+\frac{1}{2}\cot.\frac{1}{2}s+\frac{1}{4}\cot.\frac{1}{4}s+\frac{1}{8}\cot.\frac{1}{8}s+\cdots+\frac{1}{i}\cot.\frac{s}{i}\\ -\cot.s-\frac{1}{2}\cot.\frac{1}{2}s-\frac{1}{4}\cot.\frac{1}{4}s-\frac{1}{8}\cot.\frac{1}{8}s-\cdots,\end{cases}$$

where all the terms evidently cancel each other, as far as to the final $\frac{1}{i}\cot.\frac{s}{i}$, which may be reduced to that form

$$\frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}},$$

with *i* denoting an infinite number. Now because the arc $\frac{s}{i}$ is infinitely small, there will be $\cos.\frac{s}{i}=1$, truly equal to the sine of this itself $\frac{s}{i}$, from which the final term itself becomes $=\frac{1}{s}$, which is equal to the value found of this series.

11. Yet meanwhile also for the case presented the demonstration is shown directly from the formula in a similar manner, in which the tangent of the triple angle is expressed by being repeated. Indeed if there may be put

$$\tan.\varphi = t,$$

because there is obtained

$$\tan.3\varphi = \frac{3t-t^3}{1-3tt},$$

there will become

$$\cot.3\varphi = \frac{1-3tt}{3t-t^3} \text{ and } 3\cot.3\varphi = \frac{3-9tt}{t(3-tt)}.$$

Hence $\cot.\varphi = \frac{1}{t}$ may be subtracted and there will become

$$3\cot.3\varphi - \cot.\varphi = \frac{-8tt}{t(3-tt)} = \frac{-8t}{3-tt};$$

hence in place of t we may substitute its value $\frac{\sin.\varphi}{\cos.\varphi}$ and we will have

$$3\cot.3\varphi - \cot.\varphi = \frac{-8\sin.\varphi\cos.\varphi}{3\cos^2.\varphi - \sin^2.\varphi},$$

of which fraction we will treat the numerator and denominator in the following manner: Since there shall be

$$\cos^2.\varphi = \tfrac{1}{2} + \tfrac{1}{2}\cos.2\varphi \text{ and } \sin^2.\varphi = \tfrac{1}{2} - \tfrac{1}{2}\cos.2\varphi,$$

the denominator will adopt this form $1 + 2\cos.2\varphi$, which therefore can be referred to thus :

$$2\cos.60° + 2\cos.2\varphi,$$

which again on account of

$$\cos.a + \cos.b = 2a\cos.\tfrac{a+b}{2}\cos.\tfrac{a-b}{2}$$

is reduced to this :

$$4\cos.(30° + \varphi)\cos.(30° - \varphi),$$

but the numerator clearly shall become $-4\sin.2\varphi$, thus so that now we may have

$$3\cot.3\varphi - \cot.\varphi = \frac{-\sin.2\varphi}{\cos.(30°+\varphi)\cos.(30°-\varphi)}.$$

Since now in general there is

$$\sin.2\varphi = \sin.(a+\varphi)\cos.(a-\varphi) - \cos.(a+\varphi)\sin.(a-\varphi),$$

we may suppose $a = 30°$ and we will have the following equation

$$3\cot.3\varphi-\cot.\varphi=\frac{-\sin.(30^\circ+\varphi)\cos.(30^\circ-\varphi)+\cos.(30^\circ+\varphi)\sin.(30^\circ-\varphi)}{\cos.(30^\circ+\varphi)\cos.(30^\circ-\varphi)}$$
$$=-\tan.(30^\circ+\varphi)+\tan.(30^\circ-\varphi),$$

whereby we reach this noteworthy equation

$$\cot.3\varphi=\tfrac{1}{3}\cot.\varphi-\tfrac{1}{3}\tan.(30^\circ+\varphi)+\tfrac{1}{3}\tan.(30^\circ-\varphi)\,.$$

12. Now for our case by writing s in place of 3φ we obtain at once

$$\cot.s=\tfrac{1}{3}\cot.\tfrac{s}{3}-\tfrac{1}{3}\tan.(30^\circ+\tfrac{s}{3})+\tfrac{1}{3}\tan.(30^\circ-\tfrac{s}{3}).$$

Further in a similar manner there will be

$$\tfrac{1}{3}\cot.\tfrac{s}{3}=\tfrac{1}{9}\cot.\tfrac{s}{9}-\tfrac{1}{9}\tan.(30^\circ+\tfrac{s}{9})+\tfrac{1}{9}\tan.(30^\circ-\tfrac{s}{9}).$$

Again in the same manner there becomes

$$\tfrac{1}{3}\cot.\tfrac{s}{9}=\tfrac{1}{27}\cot.\tfrac{s}{27}-\tfrac{1}{27}\tan.(30^\circ+\tfrac{s}{27})+\tfrac{1}{27}\tan.(30^\circ-\tfrac{s}{27}),$$

and if we may progress indefinitely in this manner, finally we will arrive at the cotangent of this kind

$$\tfrac{1}{i}\cot.\tfrac{s}{i}=\frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}};$$

whereby our equation produced will be according to this form

$$\cot.s=-\tfrac{1}{3}\tan.(30^\circ+\tfrac{s}{3})-\tfrac{1}{9}\tan.(30^\circ+\tfrac{s}{9})-\tfrac{1}{27}\tan.(30^\circ+\tfrac{s}{27})-\cdots-\tfrac{1}{s}$$
$$+\tfrac{1}{3}\tan.(30^\circ-\tfrac{s}{3})+\tfrac{1}{9}\tan.(30^\circ-\tfrac{s}{9})+\tfrac{1}{27}\tan.(30^\circ-\tfrac{s}{27})+\cdots,$$

from which we deduce our same equation being demonstrated

$$\tfrac{1}{s}=\cot.s+\tfrac{1}{3}\tan.(30^\circ+\tfrac{s}{3})+\tfrac{1}{9}\tan.(30^\circ+\tfrac{s}{9})+\tfrac{1}{27}\tan.(30^\circ+\tfrac{s}{27})+\cdots$$
$$-\tfrac{1}{3}\tan.(30^\circ-\tfrac{s}{3})-\tfrac{1}{9}\tan.(30^\circ-\tfrac{s}{9})-\tfrac{1}{27}\tan.(30^\circ-\tfrac{s}{27})-\cdots,$$

13. So that also in a similar manner it will be allowed to show series of this kind for greater ratios, for which the arc s is diminished continually. Since indeed there shall be

$$\sin.4\varphi=8\sin.\varphi\cos.(45^\circ+\varphi)\cos.(45^\circ-\varphi)\cos.\varphi,$$

for the quadruple ratio there will be

$$\begin{aligned}\tfrac{1}{s} = \cot.s &+ \tfrac{1}{4}\tan.\tfrac{s}{4} + \tfrac{1}{16}\tan.\tfrac{s}{16} + \tfrac{1}{64}\tan.\tfrac{s}{64} + \text{etc.}\\ &+ \tfrac{1}{4}\tan.(45^\circ + \tfrac{s}{4}) + \tfrac{1}{16}\tan.(45^\circ + \tfrac{s}{16}) + \tfrac{1}{64}\tan.(45^\circ + \tfrac{s}{64}) + \text{etc.}\\ &- \tfrac{1}{4}\tan.(45^\circ - \tfrac{s}{4}) - \tfrac{1}{16}\tan.(45^\circ - \tfrac{s}{16}) - \tfrac{1}{64}\tan.(45^\circ - \tfrac{s}{64}) - \text{etc.}\end{aligned}$$

Again since there shall be

$$\sin.5\varphi = 16\sin.\varphi\cos.(18^\circ + \varphi;)\cos.(18^\circ - \varphi)\cos.\left(54^\circ + \varphi\right)\cos.(54^\circ - \varphi)$$

we will find for the quintuple ratio

$$\begin{aligned}\tfrac{1}{s} = \cot.s &+ \tfrac{1}{5}\tan.(18^\circ + \tfrac{s}{4}) + \tfrac{1}{25}\tan.(18^\circ + \tfrac{s}{5}) + \text{etc.}\\ &- \tfrac{1}{5}\tan.(18^\circ - \tfrac{s}{4}) - \tfrac{1}{25}\tan.(18^\circ - \tfrac{s}{5}) - \text{etc.}\\ &+ \tfrac{1}{5}\tan.(54^\circ + \tfrac{s}{5}) + \tfrac{1}{25}\tan.(54^\circ + \tfrac{s}{25}) + \text{etc.}\\ &- \tfrac{1}{5}\tan.(54^\circ - \tfrac{s}{5}) - \tfrac{1}{25}\tan.(54^\circ - \tfrac{s}{25}) - \text{etc.}\end{aligned}$$

It will be permitted to progress further in a similar manner, truly the series will result to become exceedingly complex, so that which may be considered worthy of attention.

VARIAE OBSERVATIONES CIRCA ANGULOS IN PROGRESSIONE GEOMETRICA PROGREDIENTES

Commentatio 561 indicis ENESTROEMIANI
Opuscula analytica 1**,** 1783, p. 345-352

1. Cum pleraeque insignes proprietates, quae adhuc circa angulos sive arcus eorumque sinus, cosinus, tangentes, cotangentes, secantes et cosecantes sunt investigatae, ex consideratione arcuum in arithmetica progressione crescentium sint derivatae, non minus notatu dignae videntur illae proprietates, quas ex consideratione arcuum in geometrica progressione procedentium deducere licet, imprimis cum earum veritas plerumque multo magis abscondita videatur; quocirca hoc loco plures eiusmodi proprietates evolvere constitui.

2. Primum fontem ad huiusmodi speculationes nobis aperit notissima formula

$$\sin.2\varphi = 2\sin.\varphi \cdot \cos.\varphi ,$$

unde, si s denotet arcum sive angulum quemcunque, erit

$$\sin.s = 2\sin\tfrac{1}{2}s \cdot \cos.\tfrac{1}{2}s ,$$

tum vero simili modo erit

$$\sin.\tfrac{1}{2}s = 2\sin.\tfrac{1}{4}s \cdot \cos.\tfrac{1}{4}s,$$

qui valor ibi substitutus praebet

$$\sin.s = 4\sin.\tfrac{1}{4}s \cdot \cos.\tfrac{1}{2}s \cdot \cos.\tfrac{1}{4}.$$

Deinde, quia porro est

$$\sin.\tfrac{1}{4}s = 2\sin.\tfrac{1}{8}s \cdot \cos.\tfrac{1}{8}s,$$

hoc valore substituto erit

$$\sin.s = 8\sin.\tfrac{1}{8}s \cdot \cos.\tfrac{1}{2}s \cdot \cos.\tfrac{1}{4}s \cdot \cos\tfrac{1}{8}s.$$

Pari modo progrediendo est

$$\sin.s = 16\sin.\tfrac{1}{16}s \cdot \cos.\tfrac{1}{2}s \cdot \cos.\tfrac{1}{4}s \cdot \cos.\tfrac{1}{8}s \cdot \cos.\tfrac{1}{16}s,$$

atque si hoc modo in infinitum progrediamur, denotante i numerum infinitum seu potius infinitesimam potestatem ipsius 2 habebimus

$$\sin.s = i\sin.\tfrac{1}{i}s \cdot \cos.\tfrac{1}{2}s \cdot \cos.\tfrac{1}{4}s \cdot \cos.\tfrac{1}{8}s \cdot \cos.\tfrac{1}{16}s \cdot \text{etc.},$$

ubi, quia arcus $\frac{s}{i}$ est infinite parvus, erit $\sin.\frac{s}{i}=\frac{s}{i}$ eoque $i\sin.\frac{s}{i}=s$,
unde adipiscimur hanc insignem proprietatem, ut sit

$$\sin.s = s\cos.\tfrac{1}{2}s\cdot\cos.\tfrac{1}{4}s\cdot\cos.\tfrac{1}{8}s\cdot\cos.\tfrac{1}{16}s\cdot\cos.\tfrac{1}{32}s\cdot\text{etc. in infinitum.}$$

3. Hinc igitur ipse arcus s per eius sinum et cosinus arcuum continuo in ratione dupla decrescentium ita pulcherrime definitur, ut sit

$$s=\frac{\sin.s}{\cos.\frac{1}{2}s\cdot\cos.\frac{1}{4}s\cdot\cos.\frac{1}{8}s\cdot\cos.\frac{1}{16}s\cdot\cos.\frac{1}{32}s\cdot\text{etc.}};$$

at quia $\frac{1}{\cos.\varphi}=\sec.\varphi$, erit per expressionem integram

$$s=\sin.s\cdot\sec.\tfrac{1}{2}s\cdot\sec.\tfrac{1}{4}s\cdot\sec.\tfrac{1}{8}s\cdot\sec.\tfrac{1}{16}s\cdot\sec.\tfrac{1}{32}s\cdot\text{etc.},$$

quae expressio satis commode geometrice repraesentari potest, quemadmodum iam alio loco ostendi.

4. Quia hic arcus s per productum exprimitur, sumendis logarithmis habebimus

$$ls = l\sin.s + l\sec.\tfrac{1}{2}s + l\sec.\tfrac{1}{4}s + l\sec.\tfrac{1}{8}s + l\sec.\tfrac{1}{16}s + l\sec.\tfrac{1}{32}s + \text{etc.},$$

unde, si accipiamus $s=\frac{\pi}{2}=90^\circ$, fiet

$$l\tfrac{\pi}{2}=0+l\sec.45^\circ+l\sec.\,22^\circ30'+l\sec.\,11^\circ15'+l\sec.5^\circ37\tfrac{1}{2}'+\text{etc.},$$

unde calculo instituto erit

$$\begin{aligned}
l\sec.\,45^\circ &= 0{,}1505150\\
l\sec.\,22^\circ\,30' &= 0{,}034384\,7\\
l\sec.\,11^\circ\,15' &= 0{,}0084261\\
l\sec.\,5^\circ\,37\tfrac{1}{2}' &= 0{,}0020963\\
l\sec.\,2^\circ\,48\tfrac{3}{4}' &= 0{,}0005234\\
l\sec.\,1^\circ\,24\tfrac{3}{8}' &= 0{,}0001308\\
l\sec.\,0^\circ\,42:\tfrac{3}{16}' &= 0{,}0000327\\
l\sec.\,0^\circ\,21\tfrac{3}{32}' &= 0{,}0000082\\
\text{reliqui omnes} &= 0{,}0000027\\
l\tfrac{\pi}{2} &= 0{,}1961199\\
l2 &= 0{,}3010300
\end{aligned}$$

$$l\pi = 0{,}4971499$$

hincque $\pi = 3{,}1415928$ satis exacte, uti constat.

5. Quo autem hinc novas relationes deducamus, differentiemus postremam aequationem logarithmicam, et cum sit

orietur per *ds* dividendo sequens aequatio

$$\frac{1}{s} = \cot.s + \frac{1}{2}\tan.\frac{1}{2}s + \frac{1}{4}\tan.\frac{1}{4}s + \frac{1}{8}\tan.\frac{1}{8}s + \frac{1}{16}\tan.\frac{1}{16}s + \text{etc.},$$

quae series quam citissime convergit, id quod sequenti exemplo clarius patebit, in quo sumamus $s = 90° = \frac{\pi}{2}$, unde fiet

$$\frac{2}{\pi} = \frac{1}{2}\tan.45° + \frac{1}{4}\tan.22°30' + \frac{1}{8}\tan.11°15' + \frac{1}{16}\tan.5°37\tfrac{1}{2}' + \text{etc.},$$

qui valores ex tabulis desumti dabunt

$$\begin{aligned}
\tfrac{1}{2}\tan.45° &= 0{,}5000000\\
\tfrac{1}{4}\tan.\,22°30' &= 0{,}1035534\\
\tfrac{1}{8}\tan.11°15' &= 0{,}0248640\\
\tfrac{1}{16}\tan.5°37\tfrac{1}{2}' &= 0{,}0061557\\
\tfrac{1}{32}\tan.2°48\tfrac{3}{4}' &= 0{,}0015352\\
\tfrac{1}{64}\tan.1°\,24\tfrac{3}{8}' &= 0{,}0003836\\
\text{pro reliquis} &= 0{,}0001279\\
\tfrac{2}{\pi} &= 0{,}6366198
\end{aligned}$$

hinc $\pi = \frac{2}{0{,}6366198} = \frac{1}{0{,}3183099}$.

6. Si postremam aequationem denuo differentiemus, ad seriem perveniemus multo magis convergentem; cum enim sit

$$d.\cot.\varphi = \frac{-d\varphi}{\sin^2.\varphi} \text{ et } d.\tan.\varphi = \frac{d\varphi}{\cos^2.\varphi} = d\varphi\sec^2.\varphi,$$

reperiemus

$$-\frac{1}{ss} = \frac{-1}{\sin^2.s} + \frac{1}{4}sec^2.\frac{1}{2}s + \frac{1}{16}sec^2.\frac{1}{4}s + \frac{1}{64}sec^2.\frac{1}{8}\text{etc.}$$

sive

$$+\frac{1}{4}sec^2.\frac{1}{2}s + \frac{1}{16}sec^2.\frac{1}{4}s + \frac{1}{64}sec^2.\frac{1}{8} + \frac{1}{256}sec^2.\frac{1}{16} + \text{etc.} = \frac{1}{\sin^2.s} - \frac{1}{ss}.$$

7. Accommodemus eadem ratiocinia ad rationem triplam, secundum quam arcus decrescant; hunc in finem consideremus formulam

$$\sin.3\varphi = 4\sin.\varphi\cos^2.\varphi - \sin.\varphi = \sin.\varphi(3 - 4\sin^2.\varphi),$$

quae dat

$$\sin.3\varphi = 3\sin.\varphi(1 - \tfrac{4}{3}\sin^2.\varphi);$$

unde, si s denotet arcum quemcunque, erit

$$\sin.s = 3\sin.\tfrac{1}{3}s(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{3});$$

similique modo erit

$$\sin.\tfrac{1}{3}s = 3\sin.\tfrac{s}{9}(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{9}),$$

ita ut nunc sit

$$\sin.s = 9\sin.\tfrac{1}{9}s(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{3})(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{9}).$$

Si tales substitutiones in infinitum continuentur, pervenietur tandem ut ante ad hanc expressionem

$$\sin.s = s(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{3})(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{9})(1 - \tfrac{4}{3}\sin^2.\tfrac{s}{27}) \cdot \text{etc.}.$$

8. Factores hos nimis complicatos sequenti modo in simpliciores resolvere licet ; namque forma general $1 - \frac{4}{3}\sin^2.\varphi$ ob $\sin^2.\varphi = \frac{1}{2} - \frac{1}{2}\cos.2\varphi$
reducitur ad hanc

$$\tfrac{1}{3} + \tfrac{2}{3}\cos.2\varphi,$$

quam hoc modo referre licet

$$\frac{2\cos.60^\circ + 2\cos.2\varphi}{3}.$$

Cum iam sit

$$\cos.a + \cos.b = 2\cos.\tfrac{a+b}{2}\cos.\tfrac{a-b}{2},$$

erit

$$\cos.60^\circ + \cos.2\varphi = 2\cos.(30^\circ + \varphi)\cos.(30^\circ - \varphi),$$

quae forma in $\frac{2}{3}$ ducta praebebit

$$1 - \tfrac{4}{3}\sin^2.\varphi = \tfrac{4}{3}\cos.(30^\circ + \varphi)\cos.(30^\circ - \varphi).$$

Quare si haec reductio ad singulos factores supra inventos applicetur, habebimus sequens productum infinitum

$$\frac{\sin.s}{s} = \tfrac{4}{3}\cos.(30° + \tfrac{s}{3})\cos.(30° - \tfrac{s}{3}) \cdot$$
$$\tfrac{4}{3}\cos.(30° + \tfrac{s}{9})\cos.(30° - \tfrac{s}{9}) \cdot$$
$$\tfrac{4}{3}\cos.(30° + \tfrac{s}{27})\cos.(30° - \tfrac{s}{27}) \cdot$$
$$\text{etc.,}$$

quod per secantes ita exhibebitur

$$\frac{s}{\sin.s} = \tfrac{3}{4}\sec.(30° + \tfrac{s}{3})\sec.(30° - \tfrac{s}{3}) \cdot$$
$$\tfrac{3}{4}\sec.(30° + \tfrac{s}{9})\sec.(30° - \tfrac{s}{9}) \cdot$$
$$\tfrac{3}{4}\sec.(30° + \tfrac{s}{27})\sec.(30° - \tfrac{s}{27}) \cdot$$
$$\text{etc.,}$$

qui factores, quo magis arcus s diminuitur, eo propius ad unitatem accedunt.

9. Si nunc logarithmos sumamus et singulos terminos differentiemus, factores illi numerici $\frac{3}{4}$ utpote constantes penitus ex calculo excedent; et quia, ut supra vidimus, est

$$d.l.\sec.\varphi = d\varphi \tan.\varphi,$$

obtinebitur sequens aequatio

$$\frac{1}{s} = \cot an.s + \tfrac{1}{3}\tan.(30° + \tfrac{s}{3}) + \tfrac{1}{9}\tan.(30° + \tfrac{s}{9}) + \tfrac{1}{27}\tan.(30° + \tfrac{s}{27}) + \text{etc.}$$
$$-\tfrac{1}{3}\tan.(30° - \tfrac{s}{3}) - \tfrac{1}{9}\tan.(30° - \tfrac{s}{9}) - \tfrac{1}{27}\tan.(30° - \tfrac{s}{27}) - \text{etc.,}$$

id quod exemplo comprobasse iuvabit. Sit igitur $s = \frac{\pi}{2}$ eritque

$$\frac{2}{\pi} = \tfrac{1}{3}\tan.60° + \tfrac{1}{9}\tan.40° + \tfrac{1}{27}\tan.(33°20') + \tfrac{1}{81}\tan.(31°6\tfrac{2}{3}') + \tfrac{1}{243}\tan.(30°22\tfrac{2}{3}') + \text{etc.}$$
$$-\tfrac{1}{3}\tan.0° - \tfrac{1}{9}\text{tag}.20° - \tfrac{1}{27}\tan.(26°40') - \tfrac{1}{81}\tan.(28°53\tfrac{1}{3}') - \tfrac{1}{243}\tan.(29°37\tfrac{7}{9}') - \text{etc.}$$

10. Haec postrema series eo magis videtur notatu digna, quod vix quisquam eius veritatem demonstrare valuerit, nisi eadem methodo fuerit usus. Haec autem series sine dubio multo altioris est indaginis quam ea, ad quam per evolutionem praecedentis casus sumus deducti, quae erat

$$\frac{1}{s} = \cot.s + \tfrac{1}{2}\tan.\tfrac{1}{2}s + \tfrac{1}{4}\tan.\tfrac{1}{4}s + \tfrac{1}{8}\tan.\tfrac{1}{8}s + \tfrac{1}{16}\tan.\tfrac{1}{16}s + \text{etc.};$$

cuius veritas ex notissima formula

$$2\cot.2\varphi = \cot.\varphi - \tan.\varphi$$

sequitur, unde habemus

$$\tan.\varphi = \cot.\varphi - 2\cot.2\varphi.$$

Hinc si loco singularum tangentium valores debiti substituantur, operatio sequenti modo instruetur

$$\frac{1}{s} = \begin{cases} \cot.s + \frac{1}{2}\cot.\frac{1}{2}s + \frac{1}{4}\cot.\frac{1}{4}s + \frac{1}{8}\cot.\frac{1}{8}s + \cdots + \frac{1}{i}\cot.\frac{s}{i} \\ -\cot.s - \frac{1}{2}\cot.\frac{1}{2}s - \frac{1}{4}\cot.\frac{1}{4}s - \frac{1}{8}\cot.\frac{1}{8}s - \cdots, \end{cases}$$

ubi omnes termini manifesto se mutuo destruunt, usque ad ultimum $\frac{1}{i}\cot.\frac{s}{i}$, qui ad hanc formam redigatur

$$\frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}},$$

denotante i numerum infinitum. Iam quia arcus $\frac{s}{i}$ est infinite parvus, erit $\cos.\frac{s}{i} = 1$, sinus vero ipsi arcui $\frac{s}{i}$ aequalis, ex quo ultimus iste terminus fit $= \frac{1}{s}$, qui est ipse valor huic seriei aequalis inventus.

11. Interim tamen etiam pro casu praesenti demonstratio directa simili modo exhibetur ex formula, qua tangens anguli tripli exprimitur, repetenda. Si enim ponatur

$$\tan.\varphi = t,$$

quia habetur

$$\tan.3\varphi = \frac{3t-t^3}{1-3tt},$$

erit

$$\cot.3\varphi = \frac{1-3tt}{3t-t^3} \quad \text{et} \quad 3\cot.3\varphi = \frac{3-9tt}{t(3-tt)}.$$

Hinc subtrahatur $\cot.\varphi = \frac{1}{t}$ fietque

$$3\cot.3\varphi - \cot.\varphi = \frac{-8tt}{t(3-tt)} = \frac{-8t}{3-tt};$$

hinc loco t substituamus eius valorem $\frac{\sin.\varphi}{\cos.\varphi}$ et habebimus

$$3\cot.3\varphi - \cot.\varphi = \frac{-8\sin.\varphi\cos.\varphi}{3\cos^2.\varphi - \sin^2.\varphi},$$

cuius fractionis numeratorem et denominatorem sequenti modo tractemus:
Cum sit

$$\cos^2.\varphi = \frac{1}{2} + \frac{1}{2}\cos.2\varphi \quad \text{et} \quad \sin^2.\varphi = \frac{1}{2} - \frac{1}{2}\cos.2\varphi,$$

induet denominator hanc formam $1+2\cos.2\varphi,$ quae propterea ita referri poterit

$$2\cos.60°+2\cos.2\varphi,$$

quae porro ob

$$\cos.a+\cos.b=2a\cos.\tfrac{a+b}{2}\cos.\tfrac{a-b}{2}$$

reducitur ad hanc

$$4\cos.(30°+\varphi)\cos.(30°-\varphi),$$

numerator autem manifesto fit $-4\sin.2\varphi$, ita ut iam habeamus

$$3\cot.3\varphi-\cot.\varphi=\frac{-\sin.2\varphi}{\cos.(30°+\varphi)\cos.(30°-\varphi)}.$$

Quia nunc in genere est

$$\sin.2\varphi=\sin.(a+\varphi)\cos.\left(a-\varphi\right)-\cos.(a+\varphi)\sin.(a-\varphi)\,,$$

sumamus $a=30°$ et habemus sequentem aequationem

$$3\cot.3\varphi-\cot.\varphi=\frac{-\sin.(30°+\varphi)\cos.(30°-\varphi)+\cos.(30°+\varphi)\sin.(30°-\varphi)}{\cos.(30°+\varphi)\cos.(30°-\varphi)}$$
$$=-\tan.(30°+\varphi)+\tan.(30°-\varphi),$$

quocirca pertingimus ad hanc aequationem notatu dignam

$$\cot.3\varphi=\tfrac{1}{3}\cot.\varphi-\tfrac{1}{3}\tan.(30°+\varphi)+\tfrac{1}{3}\tan.(30°-\varphi)\,.$$

12. Iam pro nostro casu loco 3φ scribendo s statim nanciscimur

$$\cot.s=\tfrac{1}{3}\cot.\tfrac{s}{3}-\tfrac{1}{3}\tan.(30°+\tfrac{s}{3})+\tfrac{1}{3}\tan.(30°-\tfrac{s}{3}).$$

Simili vero modo ulterius erit

$$\tfrac{1}{3}\cot.\tfrac{s}{3}=\tfrac{1}{9}\cot.\tfrac{s}{9}-\tfrac{1}{9}\tan.(30°+\tfrac{s}{9})+\tfrac{1}{9}\tan.(30°-\tfrac{s}{9}).$$

Eodem porro modo fit

$$\tfrac{1}{3}\cot.\tfrac{s}{9}=\tfrac{1}{27}\cot.\tfrac{s}{27}-\tfrac{1}{27}\tan.(30°+\tfrac{s}{27})+\tfrac{1}{27}\tan.(30°-\tfrac{s}{27}),$$

et si hoc modo in infinitum progrediamur, perveniemus tandem ad huiusmodi cotangentem

$$\tfrac{1}{i}\cot.\tfrac{s}{i}=\frac{\cos.\frac{s}{i}}{i\sin.\frac{s}{i}};$$

quamobrem nostra aequatio perducta erit ad hanc formam

$$\begin{aligned}\cot.s=&-\tfrac{1}{3}\tan.(30^\circ+\tfrac{s}{3})-\tfrac{1}{9}\tan.(30^\circ+\tfrac{s}{9})-\tfrac{1}{27}\tan.(30^\circ+\tfrac{s}{27})-\cdots-\tfrac{1}{s}\\&+\tfrac{1}{3}\tan.(30^\circ-\tfrac{s}{3})+\tfrac{1}{9}\tan.(30^\circ-\tfrac{s}{9})+\tfrac{1}{27}\tan.(30^\circ-\tfrac{s}{27})+\cdots,\end{aligned}$$

ex qua deducimus ipsam aequationem nostram demonstrandam

$$\begin{aligned}\tfrac{1}{s}=\cot.s&+\tfrac{1}{3}\tan.(30^\circ+\tfrac{s}{3})+\tfrac{1}{9}\tan.(30^\circ+\tfrac{s}{9})+\tfrac{1}{27}\tan.(30^\circ+\tfrac{s}{27})+\cdots\\&-\tfrac{1}{3}\tan.(30^\circ-\tfrac{s}{3})-\tfrac{1}{9}\tan.(30^\circ-\tfrac{s}{9})-\tfrac{1}{27}\tan.(30^\circ-\tfrac{s}{27})-\cdots,\end{aligned}$$

13. Quin etiam simili modo huiusmodi series pro maioribus rationibus, quibus arcus s continuo diminuitur, exhibere licet. Cum enim sit

$$\sin.4\varphi=8\sin.\varphi\cos.(45^\circ+\varphi)\cos.(45^\circ-\varphi)\cos.\varphi,$$

pro ratione quadrupla erit

$$\begin{aligned}\tfrac{1}{s}=\cot.s&+\tfrac{1}{4}\tan.\tfrac{s}{4}\qquad+\tfrac{1}{16}\tan.\tfrac{s}{16}\qquad+\tfrac{1}{64}\tan.\tfrac{s}{64}\qquad+\text{etc.}\\&+\tfrac{1}{4}\tan.(45^\circ+\tfrac{s}{4})+\tfrac{1}{16}\tan.(45^\circ+\tfrac{s}{16})+\tfrac{1}{64}\tan.(45^\circ+\tfrac{s}{64})+\text{etc.}\\&-\tfrac{1}{4}\tan.(45^\circ-\tfrac{s}{4})-\tfrac{1}{16}\tan.(45^\circ-\tfrac{s}{16})-\tfrac{1}{64}\tan.(45^\circ-\tfrac{s}{64})-\text{etc.}\end{aligned}$$

Porro cum sit

$$\sin.5\varphi=16\ \sin.\varphi\cos.(18^\circ+\varphi;)\cos.(18^\circ-\varphi)\cos.\left(54^\circ+\varphi\right)\cos.(54^\circ-\varphi)$$

reperiemus pro ratione quintupla

$$\begin{aligned}\tfrac{1}{s}=\cot.s&+\tfrac{1}{5}\tan.(18^\circ+\tfrac{s}{4})+\tfrac{1}{25}\tan.(18^\circ+\tfrac{s}{5})+\text{etc.}\\&-\tfrac{1}{5}\tan.(18^\circ-\tfrac{s}{4})-\tfrac{1}{25}\tan.(18^\circ-\tfrac{s}{5})-\text{etc.}\\&+\tfrac{1}{5}\tan.(54^\circ+\tfrac{s}{5})+\tfrac{1}{25}\tan.(54^\circ+\tfrac{s}{25})+\text{etc.}\\&-\tfrac{1}{5}\tan.(54^\circ-\tfrac{s}{5})-\tfrac{1}{25}\tan.(54^\circ-\tfrac{s}{25})-\text{etc.}\end{aligned}$$

Pari modo ulterius progredi liceret, verum series resultarent nimis perplexae, quam ut attentione dignae viderentur.