SMALL MISCELLANEOUS ANALYTICAL WORKS

Opuscula analytica I, 1783, p. 329-344 [E560]

Shown to the St. Petersburg Academy assembly on the 15th November 1773.

I. A THEOREM PROPOSED WITHOUT DEMONSTRATION BY THE MOST ILLUSTRIOUS WARING [now WILSON]

If n were a prime number, this continued product : $1\cdot 2\cdot 3\cdot\cdot\cdot\cdot\cdot(n-1)$, *increased by one, will always be able to be divided by that prime number n.*

DEMONSTRATION

Although the twin of this theorem now has been given by the illustrious geometer Lagrange in the *New Proc. Pruss. Acad. Sc. Berlin*, I believe Geometers may not be the least ungrateful, if I too may communicate my demonstration in a manner more familiar to me. Moreover I have shown elsewhere [E449] for some prime number *n*, which indeed may be easily enlarged as wished, always to give numbers of this kind, the individual powers of which, having an exponent less than $n-1$, divided by *n*, will provide different remainders. Therefore a number of this kind, of which the individual powers,

$$a^0, a^1, a^2, a^3, a^4, ...a^{n-2}$$

may be divided by *n* may produce just as many different remainders, since the number of which shall be $n-1$, in these remainders all the numbers $1, 2, 3, 4, ...n-1$ will occur, with which exhausted the following power a^{n-1} again divided by *n* will leave the remainder $+1$, and likewise with the first a^0. Therefore since after that, this formula $a^{n-1}-1$ finally shall be divisible *n*, on account of the even number $n-1$, which shall be $=2p$, thus so that $n=2p+1$, or the formula a^p-1, or this, a^p+1 by necessity shall be divisible by the prime number *n*. But in the first case a^p will give the remainder $+1$, which since it will be against our hypothesis, the latter formula a^p+1 will be divisible by *n*, or the power of the remainder a^p will give -1, or $n-1$. From these premises, since the individual remainders $1, 2, 3, 4, ...n-1$ arise from the powers, evidently $a^0, a^1, a^2, a^3, a^4, ...a^{n-2}$ is the product of all these remainders, as it were the product of all these proposed $1, 2, 3, 4, ...n-1$, if it may be divided by *n*, to be the remainder of the numbers remaining, since the product of all these powers $a^{0+1+2+3+\cdots+(n-2)}$, or if this power $a^{\frac{(n-1)(n-2)}{2}}$ may be given. But since there shall be $n=2p+1$, there will be $n-1=2p$ and $n-2=2p-1$, and thus this power becomes a^{2pp-p}, which is reduced to this : $a^{2p(p-1)+p}$, which is the product from the two powers $a^{2p(p-1)}$ and a^{p}. Truly now we have seen the power a^{2p}, or a^{n-1}, divided by *n* to give the remainder $+1$, which likewise will result from all the powers of this, of which the kind is $a^{2p(p-1)}$; but the other power a^{p} leaves behind -1, from which the powers of this a^{2pp-p} the remainder will be -1. And thus this formula $a^{2pp-p}+1$ will be divisible by the number *n* ; therefore from the substitution with $1\cdot 2\cdot 3\cdots\ (n-1)$ produced in place of the powers a^{2pp-p}, this formula $1\cdot 2\cdot 3\cdots\ (n-1)+1$ will be divisible by the number *n*.

COROLLARY 1

Hence several other formulas can be deduced easily, which individually equally will be divisible by the prime number *n*, which we may place here, with that formula, from which they have arisen, :

$$1\cdot 2\cdot 3\cdots\cdots(n-1)+1$$
$$1\times 1\cdot 2\cdot 3\cdots\cdots(n-2)-1$$
$$1\cdot 2\times 1\cdot 2\cdot 3\cdots\cdots(n-3)+1$$
$$1\cdot 2\cdot 3\times 1\cdot 2\cdot 3\cdots\cdots(n-4)-1 \text{ etc.}$$

But in these formulas the number *n* most not therefore be diminished all the way, as the pertaining factor may become $=0$.

COROLLARY 2

If again an arithmetical progression of *n* terms may be had, the differences of which neither shall be *n*, nor a multiple of this, in that there will be present one term divisible by *n*, with which excluded the product of the remaining terms increased by one always will be divisible by the number *n*, if indeed it were a prime number. Thus if there may be $n=7$ and this arithmetical progression with the 7 terms : $2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20$, may be formed, with the term rejected 14, this form $2\cdot 5\cdot 8\cdot 11\cdot 17\cdot 20+1$ will be able to be divided by 7 .

II. PROBLEM

To find four whole numbers such that, the products from two increased by one shall become squares.

SOLUTION

I have treated this problem further in my *Elements of Algebra* [art. 233, p.456], truly the method, which I have used, was less convenient for finding whole numbers. Moreover this question is therefore more difficult, because it is required to satisfy six conditions. Therefore now I will show the following simplest enough solution. With the two numbers *m* and *n* taken as it pleases, so that there may become $mn+1=ll$, four numbers will be sought:

$$\text{I. } m,\ \text{II. } n,\ \text{III. } m+n+2l,\ \text{IV. } 4l(l+m)(l+n),$$

from which 6 conditions will be implemented in the following way:

1°. $mn+1=ll$,

2°. $m(m+n+2l)+1=(l+m)^2$,

3°. $n(m+n+2l)+1=(l+n)^2$,

4°. $4ml(l+m)(l+n)+1=(2ll+2lm-1)^2$,

5°. $4nl(l+m)(l+n)+1=(2ll+2ln-1)^2$, and finally

6°. $4l(m+n+2l)(l+m)(l+n)+1=(4ll+2lm+2ln-1)^2$.

Here truly it will help most to have observed the number *l* can be taken positive as well as negative. Thus if there may be taken $m=3$ and $n=8$, so that there may become $mn+1=25$, and thus $l=\pm 5$, the case $l=-5$ will give these four numbers :

I. 3, II. 8, III. 1 and IV. 120.

But if there may be taken $l=+5$, the numbers will become:

I. 3, II. 8, III. 21 et IV. 2080.

ANALYSIS FOR PRODUCING THIS SOLUTION

Since the first three numbers may be found most easily *m, n* and $m+n+2l$, the fourth may be put $=z$, and with these three conditions satisfied there will become :

$$1°.\ mz+1=\square,$$
$$2°.\ nz+1=\square,$$
$$3°.\ (m+n+2l)z+1=\square.$$

Hence therefore also the product of these three formulas must be a square; but with the calculation put in place this product will be found :

$$1+2(m+n+l)z+\left((m+n+l)^2-1\right)zz+mn(m+n+2l)z^3=\square\ ,$$

the root of which, so that the three first members may be taken, may be put in place

$$1+(m+n+l)z-\tfrac{1}{2}zz\ ,$$

the square of which is :

$$1+2(m+n+l)z+((m+n+l)^2-1)zz-(m+n+l)z^3+\tfrac{1}{4}z^4,$$

from which this equation arises :

$$mn(m+n+2l)=-m-n-l+\tfrac{1}{4}z$$

$$\text{or }\ \tfrac{1}{4}z=m+n+l+mn(m+n+2l)$$

$$\text{or }\ \tfrac{1}{4}z=(mn+1)(m+n+l)+lmn,$$

or because $mn+1=ll$, we will have

$$ll(m+n+l)+lmn=l(ll+lm+ln+mn)=l(l+m)(l+n),$$

concerning which we find

$$z=4l(l+m)(l+n)\ .$$

But although here we have returned so great a square from the three formulas $mz+1,\ nz+1,\ (m+n+2l)z+1$, yet, because as if besides the expectation it has produced a whole number for *z*, from which these three formulas among themselves first will emerge, with care we are able to conclude also the three individual become squares, the roots of which thus we have shown above now.

III. PROBLEM

To find the numbers x and y, so that this formula $\left(\frac{xx+1}{x}\right)^2+\left(\frac{yy+1}{y}\right)^2$ *may become a square*.

SOLUTION

In the first place this condition will be fulfilled at once, if there may be taken $y=\frac{x+1}{x-1}$, then indeed there will become $\frac{yy+1}{y}=\frac{2(xx+1)}{xx-1}$, from which the formula proposed will be changed into this form:

$$\frac{(xx+1)^2}{xx}+\frac{4(xx+1)^2}{(xx-1)^2}=\frac{(xx+1)^4}{xx(xx-1)^2},$$

which formula now by itself is a square. Truly since this is only a special solution, from which we may obtain a more general, we may put

$$y=\frac{px-1}{x+p}, \text{ from which there will become } \frac{yy+1}{y}=\frac{(pp+1)(xx+1)}{(x+p)(px-1)},$$

whereby our formula will become :

$$\frac{(xx+1)^2}{xx}+\frac{(pp+1)^2(xx+1)^2}{(x+p)^2(px-1)^2},$$

which divided by the square $(xx+1)^2$ will change into this:

$$\frac{1}{xx}+\frac{(pp+1)^2}{(x+p)^2(px-1)^2},$$

which multiplied by the square $xx(x+p)^2(px-1)^2$ gives

$$(x+p)^2(px-1)^2+xx(pp+1)^2.$$

Now this formula must produce a square, which expanded out is reduced to this:

$$ppx^4+2p(pp-1)x^3+2\left(p^4-pp+1\right)xx-2p(pp-1)x+pp,$$

the root of which, following the known precepts, if it may be put to become

$$pxx+(pp-1)x+p,$$

this value is elicited : $x=\frac{4p}{pp-1}$, where therefore the number p can be taken as wished; then truly we will have for the other number y, as we have assumed,

$$y=\frac{px-1}{x+p}=\frac{3pp+1}{p(pp+3)}.$$

Hence therefore, if there may be taken $p=2$, there will become $x=\frac{8}{3}$ and $y=\frac{13}{14}$ and

$$\frac{xx+1}{x}=\frac{73}{24} \text{ et } \frac{yy+1}{y}=\frac{365}{182},$$

of which the sum of the squares shall be the square of the root $\frac{73\cdot 109}{2\cdot 12\cdot 91}$.

IV. PROBLEM

To find two numbers p and q, the sum of which shall be a square, truly the sum of the squares a biquadratic.

SOLUTION

This problem, proposed at one time by Leibniz, therefore is the more noteworthy, because the smallest numbers shall be exceedingly large, if indeed they may be desired to be positive. But although the solution to this problem now occurs everywhere, yet this solution may seem to be worth attention. Moreover we may put $p+q=B^2$ and $pp+qq=A^4$. Now from twice the latter equation $2pp+2qq=2A^4$ the square of the former may be subtracted $pp+2pq+qq=B^4$, and the remainder will be $pp-2pq+qq=2A^4-B^4$,
and thus

$$p-q=\sqrt{(2A^4-B^4)},$$

and thus the whole matter is reduced to this, so that the formula $2A^4-B^4$ may be rendered a square. But so that both the numbers p and q may be produced positive, it is necessary, that there shall be $B>A$. Therefore we may put in place

$$\sqrt{(2A^4-B^4)}=yy+2xy-xx,$$

which will come about, if there may be taken

$$A^2=xx+yy \text{ and } B^2=xx+2xy-yy,$$

therefore which two formulas anew will be required to be reduced to squares. But since the latter shall become $(x+y)^2-2yy$, and for each condition we may make $y=2abcd$; then truly for the former $x=aabb-ccdd$, but for the latter $x+y=aacc+2bbdd$, thus indeed there will become

$$A=aabb+ccdd \text{ and } B=aacc-2bbdd.$$

Moreover since we have

$$x=aabb-ccdd \text{ and } y=2abcd,$$

hence there will be

$$x+y=aabb-ccdd+2abcd=aacc+2bbdd,$$

from which we deduce

$$aa=\frac{2abcd-dd(2bb+cc)}{cc-bb},$$

from which the root on being extracted, we find

$$a=\frac{bcd\pm\sqrt{bbccdd-dd(2bb+cc)(cc-bb)}}{cc-bb},$$

from which by rearranging there becomes $\frac{a}{d}=\frac{bc\pm\sqrt{(2b^4-c^4)}}{cc-bb}$, or by inverting

$$\frac{d}{a}=\frac{bc\mp\sqrt{(2b^4-c^4)}}{2bb+cc}.$$

Therefore in this manner we have reduced the resolution of the formula $\sqrt{(2A^4-B^4)}$ to the resolution in short of another similar formula $\sqrt{(2b^4-c^4)}$, from which if a single case may be agreed, where such a rational formula may appear, thence immediately other cases will be able to be inferred. Therefore since this may happen initially by assuming $b=1$ and $c=1$, there will be $\frac{a}{d}=\frac{1\pm1}{0}$, or from the latter form $\frac{d}{a}=\frac{1\pm1}{3}$. Therefore there may be taken $\frac{d}{a}=\frac{2}{3}$ or $a=3$ and $d=2$, and there will become $x=5$ and $y=12$, then truly $A=13$ and $B=1$, and hence :

$$yy+2xy-xx=(x+y)^2-2xx=239,$$

which is the square root of the formula $2\cdot13^4-1$. But since here $B<A$, hence no suitable solution follows. But with this case found now we may make $b=13$ and $c=1$, and there will become $\sqrt{(2b^4-c^4)}=239$, and thus again we obtain $\frac{a}{d}=\frac{13\pm239}{-168}$, from which on account of the ambiguous sign two solutions arise: either $\frac{a}{d}=-\frac{3}{2}$ or $\frac{a}{d}=\frac{113}{84}$. From the first case therefore we have $a=3,\ b=13,\ c=1$ and $d=-2$, from which we

gather $x = 1521 - 4 = 1517$ and $y = -156$, then truly $A = 1525$ and $B = -1343$. But since only the square and the biquadratic of this letter *B* occur, it will be possible to take $B = 1343$, which since the value shall be smaller than $A = 1525$, presents no desired solution.

Therefore we will consider the other case, where $\frac{a}{d} = \frac{113}{84}$, from which our four letters will be $a = 113,\ b = 13,\ c = 1$ and $d = 84$, from which if the values *x, y* and *A, B,* may be deduced, there will be $B > A$, and hence these huge numbers for *p* and *q* are deduced satisfying the problem.

V. PROBLEM

If the formula $1 + Az + Bzz + Cz^3 + Dz^4 + \text{etc.}$ *were a product from the factors* $1+\alpha z,\ 1+\beta z,\ 1+\gamma z,\ 1+\delta z$ etc., *to find the sum of the powers of all the letters* $\alpha,\ \beta,\ \gamma,\ \delta$ etc. [see E158.]

SOLUTION

This same sum, which we require, we will designate thus:

$$P = \alpha + \beta + \gamma + \delta + \text{etc.}$$
$$Q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.}$$
$$R = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \text{etc.}$$
$$S = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \text{etc.},$$

where the whole matter is reduced to this, so that the values of the letters *P, Q, R, S* etc. will be determined by the letters *A, B, C, D* etc. But here before everything it is agreed to have observed the letter *P* to depend only on the letter *A*, certainly it is equal to that; then the letter *Q* can depend only on the two letters *A* and *B*, because products from three letters, $\alpha,\ \beta,\ \gamma,\ \delta$ etc. do not enter into the composition of the squares. In the same manner the letter *R* will depend only on the three letters *A, B* and *C*, but the letter *S* will involve only these four: *A* , *B, C* and *D* ; and thus in a similar manner for the following.

1°. With these noted the letter *P* will be found in the same manner, and if the formula were only $1 + Az$ and the remaining letters *B, C, D, E, ...* may vanish ; but in this case a single factor has a place, which shall be $1 + az$, thus so that there shall be $P = a$. Now with this factor put $1 + az = 0$ or $z = -\frac{1}{a}$, that formula itself must vanish, and therefore there becomes $1 - \frac{A}{a} = 0$ or $a - A = 0$, from which there shall be $a = P = A$, as indeed is very well-known.

2°. But the letter *Q* will be chosen with the same value, and if our formula may become $1 + Az + Bzz$, with the remaining terms vanishing. But this formula has two factors, which shall be $1+az$ and $1+bz$, and hence there will be $P=a+b$ and $Q=a^2+b^2$. Now there may become $1+az=0$ or $z=-\frac{1}{a}$, and in this case our formula itself must vanish, and there will be

$$1-\frac{A}{a}+\frac{B}{a^2}=0$$

or $a^2-Aa+B=0$. In the same manner from the other factor$1+bz$ this equation will arise: $b^2-Ab+B=0$. These two equations may be added together, and by writing Q in place of a^2+b^2 and P in place of $a+b$ this equation will arise: $Q-AP+2B=0$, from which it is deduced $Q=AP-2B$.

3°. Again truly the value of R will be deduced from the formula $1+Az+Bz^2+Cz^3$, the three factors of which shall be

$$(1+az)(1+bz)(1+cz),$$

thus so that we may have

$$P=a+b\ +\ c,\ Q=a^2+b^2+c^2 \text{ et } R=a^3+b^3+c^3.$$

Any of which factors we may reduce to zero, and from the first there will become $z=-\frac{1}{a}$, from which itself the formula will be given:

$$1-\frac{A}{a}+\frac{B}{a^2}-\frac{C}{a^3}=0$$

or $a^3-Aa^2+Ba-C=0$. The two remaining factors will give in a similar manner :

$$b^3-Ab^2+Bb-C=0 \text{ et } c^3-Ac^2+Bc-C=0,,$$

which three equations added together will give :

$$R-AQ+BP-3C=0, \text{ from which } R=AQ-BP+3C.$$

4°. In the same manner the letter S is deduced from this formula:

$$1+Az+Bz^2+Cz^3+Dz^4,$$

the four factors of which shall be

$$(1+az)(1+bz)(1+cz)(1+dz),$$

and hence

$$P = a + b + c + d, \quad Q = a^2 + b^2 + c^2 + d^2,$$
$$R = a^3 + b^3 + c^3 + d^3 \text{ and } S = a^4 + b^4 + c^4 + d^4.$$

Because if now the individual factors themselves may be equated to zero and the reduction made as before, thence the four equations will arise :

$$a^4 - Aa^3 + Ba^2 - Ca + D = 0,$$
$$b^4 - Ab^3 + Bb^2 - Cb + D = 0,$$
$$c^4 - Ac^3 + Bc^2 - Cc + D = 0,$$
$$d^4 - Ad^3 + Bd^2 - Cd + D = 0,$$

which added put this formula in place:

$$S - AR + BQ - CP + 4D = 0, \text{ and hence}$$
$$S = AR - BQ + CP - 4D.$$

Hence now it is understood easily, how the higher powers also , evidently *T, U, V* etc. may be formed from the preceding, which finally we may appoint these individual values in order :

$$P = A.$$
$$Q = AP - 2B,$$
$$R = AQ - BP + 3C,$$
$$S = AR - BQ + CP - 4D,$$
$$T = AS - BR + CQ - DP + 5E,$$
$$U = AT - BS + CR - DQ + EP - 6F$$
etc.

VI. PROBLEM

8. *Thus to find five numbers of this nature, so that the products from two increased by one may become squares.*

SOLUTION

This problem must be considered to surpass the strengths of Diophantine analysis, unless in a certain individual case a solution may be rendered possible. Moreover in the first problem we have shown four numbers of this kind, and thus these integers which entertain these conditions, evidently with the two numbers *m* and *n* taken as it pleases, thus so that there may become $mn+1=ll$, the four satisfying numbers thus may themselves be had:

$$a=m,\ b=n,\ c=m+n+2l \text{ and } d=4l((l+m)(l+n)).$$

Therefore now the present question returns to this, so that the fifth number *z* may be sought, which since it may give satisfaction to these four conditions ; it is required therefore, that the following four individual formulas may be rendered squared:

$$1+az=\square,\ 1+bz=\square,\ 1+cz=\square,\ 1+dz=\square;$$

from which if the individual equations must be satisfied, insurmountable obstacles will occur. But here happily it comes from the above use, that if only the product of these four individual square formulas may be used, also the individual squares themselves shall become known. Therefore these four formulas will themselves be multiplied in turn, and for the sake of brevity the product may be put :

$$1+pz+qzz+rz^3+sz^4,$$

thus so that there shall become

$$p=a+b+c+d,\ q=ab+ac+ad+bc+bd+cd,$$
$$r=abc+abd+acd+bcd \text{ and } s=abcd.$$

Now the square root of this formula may be put in place :

$$1+\tfrac{1}{2}pz+(\tfrac{1}{2}q-\tfrac{1}{8}pp)zz,$$

so that its square may become:

$$1+pz+qz^2+p\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)z^3+\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)^2z^4,$$

where, since the three first terms at once cancel each other, the remaining terms divided by z^3 will provide this equation:

$$r+sz=p\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)+\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)^2z,$$

from which we deduce the fifth number sought :

$$z=\frac{r-p\left(\frac{1}{2}q-\frac{1}{8}pp\right)}{\left(\frac{1}{2}q-\frac{1}{8}pp\right)^2-s}.$$

Truly if the nature of the 4 numbers given may be considered more accurately, we will find always to become $\frac{1}{2}q-\frac{1}{8}pp=\frac{-1-s}{2}$, from which the denominator of the fraction found emerges :

$$(\tfrac{1}{2}q-\tfrac{1}{8}pp)^2-s=\tfrac{(s-1)^2}{4},$$

and thus conveniently it comes about, that here the denominator may become a square; unless here it may have happened, the squares of the individual formulas :

$$1+az,\ 1+bz,\ 1+cz,\ 1+dz$$

may not be able to be done. Because if also we may substitute that same value $(\frac{1}{2}q-\frac{1}{8}pp)$ in the numerator, there will become $z=\frac{4r+p(s+1)}{(s-1)^2}$. But this number z found generally will satisfy the ten following conditions :

$$\begin{array}{ll} \text{I}°.\ ab+1=\square, & \text{II}°.\ ac+1=\square, \\ \text{III}°.\ ad+1=\square, & \text{IV}°.\ bc+1=\square, \\ \text{V}°.\ bd+1=\square, & \text{VI}°.\ cd+1=\square, \\ \text{VII}°.\ az+1=\square, & \text{VIII}°.\ bz+1=\square, \\ \text{IX}°.\ cz+1=\square, & \text{X}°.\ dz+1=\square, \end{array}$$

COROLLARY

But because it can be shown in the following manner, there shall be always $\frac{1}{2}q-\frac{1}{8}pp=\frac{-s-1}{2}$. There may be put for the sake of brevity $m+n+l=f$ and $l(l+m)(l+n)=k$, thus so that there shall be $k=fll+lmn$, and since there shall be $a=m,\ b=n,\ c=f+l$ and $d=4k$, we will have $a+b+c=2f$, therefore $p=2f+4k$; then, because

$$q=(a+b+c)d+(a+b)c+ab,$$

now there will become

$$q=8fk+(m+n)^2+2l(m+n)+mn,$$

which expression on account of $mn=ll-1$ will change into this : $q=8fk+ff-1$; then truly there will be $s=4mnk(f+l)$, hence there will be

$$1+q+s=8fk+ff+4mnk(f+l),$$

or which we may see shall be equal to $\frac{1}{4}pp$ itself. But there is

$$\tfrac{1}{4}pp = ff + 4fk + 4kk,$$

and with these values equated amongst themselves we will have :

$$8fk + ff + 4mnk(f+l) = ff + 4fk + 4kk, \text{ or}$$
$$4fk + 4mnk(f+l) = 4kk,$$

which equation divided by $4k$ gives :

$$f + mn(f+l) = k = fll + lmn, \text{ or}$$
$$f + fmm = fll, \text{ on account of } mn+1 = ll$$

by hypothesis, which equation shall be identical, truly there is by necessity from that : $1+q+s = \tfrac{1}{4}pp$, from which it follows, as we assumed, $\tfrac{1}{2}q - \tfrac{1}{8}pp = \frac{-s-1}{2}$.

EXAMPLE 1

We may assume $m=1$ and $n=3$, and there will be $l=2$, from which the four first numbers will be $a=1,\ b=3,\ c=8,\ d=120$; hence we deduce therefore:

$$p=132,\ q=1475,\ r=4224 \text{ and } s=2880;$$

from which values we deduce:

$$z = \frac{4\cdot 4224 + 264\cdot 2881}{(2879)^2},$$

which fraction is reduced to this $\frac{1777480}{8288641}$, and hence the ten prescribed conditions are fulfilled in the following manner:

$$1^\circ.\ ab+1 = 2^2, \qquad 2^\circ.\ ac+1 = 3^2,$$
$$3^\circ.\ ad+1 = 11^2, \qquad 4^\circ.\ bc+1 = 5^2,$$
$$5^\circ.\ bd+1 = 19^2, \qquad 6^\circ.\ cd+1 = 31^2,$$
$$7^\circ.\ az+1 = \frac{(3011)^2}{(2879)^2}, \qquad 8^\circ.\ bz+1 = \frac{(3259)^2}{(2879)^2},$$
$$9^\circ.\ cz+1 = \frac{(3809)^2}{(2879)^2}, \qquad 10^\circ.\ dz+1 = \frac{(10079)^2}{(2879)^2}.$$

EXAMPLE 2

Hence since the number z will have been produced so exceedingly large, we may set out the following case with fractions, since now we have forced to admit fractions. Therefore we may take $m=\frac{1}{2}$, $n=\frac{5}{2}$ so that there shall be $l=\frac{3}{2}$, from which the first four numbers will be

$$a=\tfrac{1}{2}\ ,\ b=\tfrac{5}{2},\ c=6 \text{ and } d=48\ ;$$

from which again we deduce :

$$p=57,\ q=451\tfrac{1}{4},\ r=931\tfrac{1}{2} \text{ and } s=360\ ;$$

therefore from these there is deduced :

$$z=\frac{4\cdot 931\frac{1}{2}+114\cdot 361}{359^2}=\frac{44880}{359^2}=\frac{44880}{128881},$$

which numbers are much smaller than the preceding ones.

MISCELLANEA ANALYTICA

Commentatio 560 indicis ENESTROEMIANI

Opuscula analytica 1, 1783, p. 329-344
[Conventui exhibita die 15. novembris 1773]

I. THEOREMA A CLARISSIMO WARING SINE DEMONSTRATIONE PROPOSITUM

Si n fuerit numerus primus, hoc productum continuum: $1\cdot 2\cdot 3\cdot\cdot\cdot\cdot\cdot(n-1)$, *unitate auctum, semper dividi potest per illum numerum primum n.*

DEMONSTRATIO

Quanquam illustris Geometra DE LAGRANGE iam geminam huius theorematis demonstrationem in novis Actis Academiae regiae scientiarum borussicae dedit, Geometris haud ingratum fore arbitror, si etiam meam demonstrationem more mihi familiari communicavero. Pro posito autem quocunque numero primo n alio loco ostendi, quod quidem quilibet facile largietur, semper dari huiusmodi numeros, quorum singulae potestates, exponentem minorem quam $n-1$habentes, per *n* divisae, diversa praebeant residua. Sit igitur a huiusmodi numerus, cuius singulae potestates

$$a^0, a^1, a^2, a^3, a^4, ...a^{n-2}$$

per *n* divisae totidem diversa residua producant, quorum numerus cum sit $n-1$, in his residuis omnes occurrent numeri $1,\ 2,\ 3,\ 4,\ .....n-1$, quibus exhaustis sequens potestas a^{n-1} iterum per *n* divisa relinquet residuum $+1$, perinde ac prima a^0. Cum ergo post hanc haec formula $a^{n-1}-1$demum per *n* sit divisibilis, ob $n-1$ numerum parem, qui sit $=2p$, ita ut $n=2p+1$, vel formula a^p-1vel haec a^p+1 per numerum primum *n* divisibilis sit necesse est. Priore autem casu a^p residuum daret + 1, quod cum nostrae hypothesi adversetur, posterior formula a^p+1 per *n* erit divisibilis, sive potestas a^presiduum dabit -1, seu $n-1$. His praemissis, quoniam singula residua $1,\ 2,\ 3,\ 4,\ .....n-1$ oriuntur ex potestatibus $a^0, a^1, a^2, a^3, a^4, ...a^{n-2}$, manifestum est productum omnium eorum residuorum, scilicet productum propositum $1,\ 2,\ 3,\ 4,\ .....n-1$, si per *n* dividatur, idem residuum esse relicturum, quod productum omnium earum potestatum $a^{0+1+2+3+\cdots+(n-2)}$, sive haec potestas $a^{\frac{(n-1)(n-2)}{2}}$ esset daturum. Cum autem sit $n=2p+1$, erit $n-1=2p$ et $n-2=2p-1$, ideoque haec potestas fiet

a^{2pp-p}, quae reducitur ad hanc: $a^{2p(p-1)+p}$, quae est productum ex his duabus potestatibus $a^{2p(p-1)}$ et a^p. Verum iam vidimus potestatem a^{2p}, sive a^{n-1}, per n divisam residuum dare $+1$, quod idem resultabit ex omnibus eius potestatibus, cuiusmodi est $a^{2p(p-1)}$; at altera potestas a^p residuum relinquit -1, unde ipsius potestatis a^{2pp-p} residuum erit -1. Sicque haec formula $a^{2pp-p}+1$ per numerum n erit divisibilis; substituto igitur producto $1\cdot 2\cdot 3\cdots\,(n-1)$ in locum potestatis a^{2pp-p}, haec formula $1\cdot 2\cdot 3\cdots\,(n-1)+1$ per numerum primum n erit divisibilis.

COROLLARIUM 1

Hinc facile plures aliae similes formulae deduci possunt, quae singulae pariter per numerum primum n erunt divisibiles, quas cum ipsa formula, unde sunt natae, hic apponamus:

$$1\cdot 2\cdot 3\cdots\cdots(n-1)+1$$
$$1\times 1\cdot 2\cdot 3\cdots\cdots(n-2)-1$$
$$1\cdot 2\times 1\cdot 2\cdot 3\cdots\cdots(n-3)+1$$
$$1\cdot 2\cdot 3\times 1\cdot 2\cdot 3\cdots\cdots(n-4)-1 \text{ etc.}$$

In his autem formulis numerus n non eo usque diminui debet, donec fiat factor pertinens $=0$.

COROLLARIUM 2

Si porro habeatur progressio arithmetica n terminorum, cuius differentia neque sit n, neque multiplum ipsius, in ea inerit unus terminus per n divisibilis, quo excluso productum reliquorum terminorum unitate auctum semper erit divisibile per numerum n, siquidem fuerit numerus primus. Ita si sit $n=7$ et formetur haec progressio arithmetica 7 terminorum: $2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20$, reiecto termino 14 haec forma $2\cdot 5\cdot 8\cdot 11\cdot 17\cdot 20+1$ dividi poterit per 7 .

II. PROBLEMA

Invenire quatuor numeros integros tales, ut producta ex binis unitate aucta fiant quadrata.

SOLUTIO

Problema hoc in Elementis meis Algebrae fusius tractavi, methodus vero, qua sum usus, minus erat accommodata ad numeros integros inveniendos. Haec autem quaestio eo est difficilior, quod sex conditionibus satisfieri oportet. Nunc igitur sequentem solutionem satis simplicem exhibeo. Sumtis pro lubitu duobus numeris m et n, ut fiat $mn+1=ll$, quatuor numeri quaesiti erunt:

I. m, II. n, III. $m+n+2l$, IV. $4l(l+m)(l+n)$,
quibus 6 conditiones praescriptae sequenti modo implentur:

1°. $mn+1=ll$,

2°. $m(m+n+2l)+1=(l+m)^2$,

3°. $n(m+n+2l)+1=(l+n)^2$,

4°. $4ml(l+m)(l+n)+1=(2ll+2lm-1)^2$,

5°. $4nl(l+m)(l+n)+1=(2ll+2ln-1)^2$, denique

6°. $4l(m+n+2l)(l+m)(l+n)+1=(4ll+2lm+2ln-1)^2$.

Hic vero plurimum observasse iuvabit numerum l tam positive quam negative accipi posse. Ita si sumatur $m=3$ et $n=8$, ut fiat $mn+1=25$, ideoque $l=\pm 5$, casus $l=-5$ dabit hos quatuor numeros:

I. 3, II. 8, III. 1 et IV. 120.

Sin autem capiatur $l=+5$, numeri erunt

I. 3, II. 8, III. 21 et IV. 2080.

ANALYSIS AD HANC SOLUTIONEM PERDUCENS

Cum tres priores numeri m, n et $m+n+2l$ facillime inveniantur, ponatur quartus $=z$, atque his tribus conditionibus satisfieri debebit:

$$1°.\ mz+1=\square,$$
$$2°.\ nz+1=\square,$$
$$3°.\ (m+n+2l)z+1=\square.$$

Hinc ergo etiam harum trium formularum productum debebit esse quadratum; at calculo instituto hoc productum reperietur:

$$1+2(m+n+l)z+\left((m+n+l)^2-1\right)zz+mn(m+n+2l)z^3=\square,$$

cuius radix, ut tria priora membra tollantur, statuatur

$$1+(m+n+l)z-\tfrac{1}{2}zz,$$

cuius quadratum est:

$$1+2(m+n+l)z+((m+n+l)^2-1)zz-(m+n+l)z^3+\tfrac{1}{4}z^4,$$

unde nascitur haec aequatio:

$$mn(m+n+2l)=-m-n-l+\tfrac{1}{4}z$$
$$\text{sive } \tfrac{1}{4}z=m+n+l+mn(m+n+2l)$$
$$\text{sive } \tfrac{1}{4}z=(mn+1)(m+n+l)+lmn,$$

vel quia $mn+1=ll$, habebimus

$$ll(m+n+l)+lmn=l(ll+lm+ln+mn)=l(l+m)(l+n),$$

quo circa invenimus

$$z=4l(l+m)(l+n).$$

Quanquam autem hic tantum productum ex tribus formulis $mz+1,\ nz+1,\ (m+n+2l)z+1$ reddidimus quadratum, tamen, quia pro z quasi praeter exspectationem prodiit numerus integer, unde tres istae formulae inter se evadent primae, tuto concludere possumus etiam singulas ternas formulas fieri quadrata, quorum radices adeo iam supra exhibuimus.

III. PROBLEMA

Invenire numeros x et y, ut haec formula $\left(\frac{xx+1}{x}\right)^2+\left(\frac{yy+1}{y}\right)^2$ *fiat quadratum.*

SOLUTIO

Primo haec conditio sponte adimplebitur, si capiatur $y=\frac{x+1}{x-1}$, tum enim fiet $\frac{yy+1}{y}=\frac{2(xx+1)}{xx-1}$, unde formula proposita abibit in hanc formam:

$$\frac{(xx+1)^2}{xx}+\frac{4(xx+1)^2}{(xx-1)^2}=\frac{(xx+1)^4}{xx(xx-1)^2},$$

quae formula iam per se est quadratum. Verum quia haec solutio tantum est specialis, quo generaliorem obtineamus, statuamus

$$y=\frac{px-1}{x+p},\ \text{unde fiet } \frac{yy+1}{y}=\frac{(pp+1)(xx+1)}{(x+p)(px-1)},$$

quare formula nostra evadet

$$\frac{(xx+1)^2}{xx}+\frac{(pp+1)^2(xx+1)^2}{(x+p)^2(px-1)^2},$$

quae per quadratum $(xx+1)^2$ divisa abit in hanc:

$$\frac{1}{xx}+\frac{(pp+1)^2}{(x+p)^2(px-1)^2},$$

quae per quadratum $xx(x+p)^2(px-1)^2$ multiplicata dat

$$(x+p)^2(px-1)^2+xx(pp+1)^2.$$

Haec iam formula quadratum effici debet, quae evoluta reducitur ad istam:

$$ppx^4+2p(pp-1)x^3+2\left(p^4-pp+1\right)xx-2p(pp-1)x+pp,$$

cuius radix secundum praecepta cognita si statuatur

$$pxx+(pp-1)x+p,$$

elicietur iste valor: $x=\frac{4p}{pp-1}$, ubi ergo numerus p pro lubitu accipi potest; tum vero pro altero numero y habebimus, uti assumsimus,

$$y=\frac{px-1}{x+p}=\frac{3pp+1}{p(pp+3)}.$$

Hinc ergo, si sumatur $p=2,$ fiet $x=\frac{8}{3}$ et $y=\frac{13}{14}$ et

$$\frac{xx+1}{x}=\frac{73}{24}\text{ et }\frac{yy+1}{y}=\frac{365}{182},$$

quorum quadratorum summa sit quadratum radicis $\frac{73\cdot 109}{2\cdot 12\cdot 91}$.

IV. PROBLEMA

Invenire duos numeros p et q, quorum summa sit quadratum, summa vero quadratorum biquadratum.

SOLUTIO

Hoc Problema, a Leibnizio olim propositum, eo magis est notatu dignum, quod minimi numeri sint vehementer magni, siquidem positivi desiderentur. Quamvis autem hoc problema iam passim occurrat solutum, tamen haec solutio attentione non indigna videtur. Ponamus autem

$p+q=B^2$ et $pp+qq=A^4$. Iam a duplo posterioris aequationis $2pp+2qq=2A^4$ subtrahatur quadratum prioris $pp+2pq+qq=B^4$ et residuum erit
$pp-2pq+qq=2A^4-B^4$,
ideoque

$$p-q=\sqrt{(2A^4-B^4)}\ ,$$

sicque totum negotium huc reducitur, ut formula $2\mathrm{A}^4-\mathrm{B}^4$ quadratum reddatur.
Ut autem ambo numeri *p* et *q* prodeant positivi, necesse est, ut sit $\mathrm{B}>\mathrm{A}$. Statuamus igitur

$$\sqrt{(2A^4-B^4)}=yy+2xy-xx\ ,$$

quod eveniet, si capiatur

$$A^2=xx+yy \text{ et } B^2=xx+2xy-yy\ ,$$

quas ergo binas formulas denuo ad quadrata redigi oportet. Cum autem posterior sit $(x+y)^2-2yy$, pro utraque conditione faciamus $y=2abcd$; tum vero pro priore $x=aabb-ccdd$, pro posteriore autem $x+y=aacc+2bbdd$, sic enim fiet

$$A=aabb+ccdd \text{ et } B=aacc-2bbdd.$$

Quia autem habemus

$$x=aabb-ccdd \text{ et } y=2abcd\ ,$$

erit hinc

$$x+y=aabb-ccdd+2abcd=aacc+2bbdd\ ,$$

unde deducimus

$$aa=\frac{2abcd-dd(2bb+cc)}{cc-bb},$$

unde radicem extrahendo reperimus

$$a=\frac{bcd\pm\sqrt{bbccdd-dd(2bb+cc)(cc-bb)}}{cc-bb},$$

unde per evolutionem fit $\frac{a}{d}=\frac{bc\pm\sqrt{(2b^4-c^4)}}{cc-bb}$, vel per conversionem $\frac{d}{a}=\frac{bc\mp\sqrt{(2b^4-c^4)}}{2bb+cc}$.
Hoc igitur modo resolutionem formulae $\sqrt{(2A^4-B^4)}$ reduximus ad resolutionem alius formulae prorsus similis $\sqrt{(2b^4-c^4)}$, unde si unicus casus constet, quo talis formula rationalis evadit, inde continuo alius casus concludi poterit. Cum igitur hoc primo eveniat

sumendo $b=1$ et $c=1$, erit $\frac{a}{d}=\frac{1\pm 1}{0}$, seu ex posteriori forma $\frac{d}{a}=\frac{1\pm 1}{3}$. Sumatur ergo $\frac{d}{a}=\frac{2}{3}$ sive $a=3$ et $d=2$, fietque $x=5$ et $y=12$, tum vero $A=13$ et $B=1$, atque hinc:

$$yy+2xy-xx=(x+y)^2-2xx=239,$$

quae est radix quadrata formulae $2\cdot 13^4-1$. Quia autem hic $B<A$, hinc nulla solutio idonea sequitur. Hoc autem casu reperto faciamus nunc $b=13$ et $c=1$, fietque $\sqrt{(2b^4-c^4)}=239$, sicque porro nanciscimur $\frac{a}{d}=\frac{13\pm 239}{-168}$, unde ob signum ambiguum binae solutiones oriuntur: vel $\frac{a}{d}=-\frac{3}{2}$ vel $\frac{a}{d}=\frac{113}{84}$. Ex priore ergo casu habemus $a=3$, $b=13$, $c=1$ et $d=-2$, unde colligimus $x=1521-4=1517$ et $y=-156$, tum vero $A=1525$ et $B=-1343$. Quia autem huius litterae B tantum quadratum et biquadratum occurrit, sumi poterit $B=1343$, qui valor cum minor sit quam $A=1525$, nullam solutionem desideratam praebet.

Consideremus ergo alterum casum, quo $\frac{a}{d}=\frac{113}{84}$, unde nostrae quaternae litterae erunt $a=113$, $b=13$, $c=1$ et $d=84$, ex quibus si colligantur valores x, y et A, B, erit $B>A$, hincque enormes illi numeri pro p et q colliguntur problemati satisfacientes.

V. PROBLEMA

Si formula $1+Az+Bzz+Cz^3+Dz^4+\text{etc.}$ *fuerit productum ex factoribus* $1+\alpha z$, $1+\beta z$, $1+\gamma z$, $1+\delta z$ etc., *invenire summam potestatum omnium litterum* α, β, γ, δ etc.

SOLUTIO

Summas istas, quas quaerimus, ita designemus:

$$\begin{aligned}P&=\alpha+\beta+\gamma+\delta+\text{etc.}\\Q&=\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}\\R&=\alpha^3+\beta^3+\gamma^3+\delta^3+\text{etc.}\\S&=\alpha^4+\beta^4+\gamma^4+\delta^4+\text{etc.},\end{aligned}$$

ubi totum negotium huc redit, ut valores litterarum P, Q, R, S etc. per litteras A, B, C, D etc. determinentur. Hic autem ante omnia observari convenit litteram P a sola littera A pendere, quippe cui est aequalis; deinde littera Q tantum a duabus litteris A et B pendere potest, quoniam producta ex ternis litteris α, β, γ, δ etc. non ingrediuntur in compositionem quadratorum. Eodem modo littera R tantum pendebit a tribus litteris A, B et C, at littera S involvet has tantum quatuor: A, B, C et D; et ita simili modo de sequentibus.

1°. His praenotatis littera P eodem modo reperietur, ac si formula esset tantum $1+Az$ et litterae reliquae $B, C, D, E, \ldots$ evanescerent; hoc autem casu unicus factor locum habet, qui sit $1+az$, ita ut sit $P=a$. Iam posito hoc factore $1+az=0$ sive $z=-\frac{1}{a}$, ipsa formula evanescere debet, eritque idcirco $1-\frac{A}{a}=0$ sive $a-A=0$, unde fit $a=P=A$, uti quidem notissimum est.

2°. Littera autem Q eundem valorem sortietur, ac si formula nostra foret $1+Az+Bzz$, reliquis terminis evanescentibus. Haec autem formula duos habet factores, qui sint $1+az$ et $1+bz$, hincque erit $P=a+b$ et $Q=a^2+b^2$. Fiat nunc $1+az=0$ sive $z=-\frac{1}{a}$, et hoc casu ipsa nostra formula debet evanescere, eritque

$$1-\frac{A}{a}+\frac{B}{a^2}=0$$

sive $a^2-Aa+B=0$. Eodem modo ex altero factore $1+bz$ orietur haec aequatio: $b^2-Ab+B=0$. Addantur hae duae aequationes, et loco a^2+b^2 scribendo Q et P loco $a+b$ orietur haec aequatio: $Q-AP+2B=0$, unde colligitur $Q=AP-2B$.

3°. Verus porro valor ipsius R deducetur ex formula $1+Az+Bz^2+Cz^3$, cuius tres factores sint

$$(1+az)(1+bz)(1+cz),$$

ita ut habeamus

$$P=a+b+c,\ Q=a^2+b^2+c^2 \text{ et } R=a^3+b^3+c^3.$$

Quemlibet horum factorum redigamus ad nihilum, et ex primo fiet $z=-\frac{1}{a}$, unde ipsa formula praebebit

$$1-\frac{A}{a}+\frac{B}{a^2}-\frac{C}{a^3}=0$$

sive $a^3-Aa^2+Ba-C=0$. Simili modo bini reliqui factores dabunt

$$b^3-Ab^2+Bb-C=0 \text{ et } c^3-Ac^2+Bc-C=0,,$$

quae tres aequationes iunctae dabunt

$$R-AQ+BP-3C=0, \text{ unde } R=AQ-BP+3C.$$

4°, Pari modo littera S ex hac formula:

$$1+Az+Bz^2+Cz^3+Dz^4$$

colligitur, cuius quatuor factores sint

$$(1+az)(1+bz)(1+cz)(1+dz),$$

hincque

$$P=a+b+c+d,\qquad Q=a^2+b^2+c^2+d^2,$$
$$R=a^3+b^3+c^3+d^3 \text{ et } S=a^4+b^4+c^4+d^4.$$

Quod si nunc singuli factores seorsim nihilo aequentur et reductio fiat ut ante, orientur inde 4 sequentes aequationes:

$$a^4-Aa^3+Ba^2-Ca+D=0,$$
$$b^4-Ab^3+Bb^2-Cb+D=0,$$
$$c^4-Ac^3+Bc^2-Cc+D=0,$$
$$d^4-Ad^3+Bd^2-Cd+D=0,$$

quae additae hanc formulam suppeditant:

$$S-AR+BQ-CP+4D=0, \text{ hincque}$$
$$S=AR-BQ+CP-4D.$$

Hinc iam facile intelligitur, quomodo etiam superiores potestates, scilicet *T*, *U*, *V* etc. ex praecedentibus formantur, quem in finem singulos hos valores ordine apponamus:

$$P=A.$$
$$Q=AP-2B,$$
$$R=AQ-BP+3C,$$
$$S=AR-BQ+CP-4D,$$
$$T=AS-BR+CQ-DP+5E,$$
$$U=AT-BS+CR-DQ+EP-6F$$

etc.

VI. PROBLEMA

8. *Invenire adeo quinque numeros huius indolis, ut producta ex binis unitate aucta fiant quadrata.*

SOLUTIO

Problema hoc vires analyseos DIOPHANTEAE superare censeri deberet, nisi casu quodam singulari solutio possibilis redderetur. In primo autem problemate iam exhibuimus quatuor huiusmodi numeros, eosque adeo integros, qui his conditionibus gaudent, scilicet sumtis pro lubitu duobus numeris *m* et *n*, ita ut fiat $mn+1=ll$, quatuor numeri satisfacientes ita se habebunt:

$$a=m,\ b=n,\ c=m+n+2l \text{ et } d=4l((l+m)(l+n)).$$

Nunc igitur praesens quaestio huc redit, ut quaeratur quintus numerus z, qui cum istis quatuor conditionibus praescriptis satisfaciat; requiritur ergo, ut sequentes quatuor formulae singulae reddantur quadrata:

$$1+az=\square,\ 1+bz=\square,\ 1+cz=\square,\ 1+dz=\square;$$

quibus si singulis satisfieri deberet, insuperabilia obstacula occurrerent. Hic autem uti supra feliciter usu venit, ut, si modo productum harum quatuor formularum quadratum efficiatur, etiam singulae seorsim quadrata sint futura. Multiplicentur igitur hae quatuor formulae in se invicem, ac ponatur brevitatis gratia productum:

$$1+pz+qzz+rz^3+sz^4,$$

ita ut sit

$$p=a+b+c+d,\ q=ab+ac+ad+bc+bd+cd,$$
$$r=abc+abd+acd+bcd \text{ et } s=abcd.$$

Nunc statuatur radix quadrata istius formulae

$$1+\tfrac{1}{2}pz+(\tfrac{1}{2}q-\tfrac{1}{8}pp)zz,$$

ut eius quadratum fiat

$$1+pz+qz^2+p\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)z^3+\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)^2z^4,$$

ubi, cum tres priores termini sponte se tollant, reliqui per z^3 divisi suppeditabunt hanc aequalitatem:

$$r+sz=p\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)+\left(\tfrac{1}{2}q-\tfrac{1}{8}pp\right)^2z,$$

unde colligimus quintum numerum quaesitum:

$$z=\frac{r-p\left(\frac{1}{2}q-\frac{1}{8}pp\right)}{\left(\frac{1}{2}q-\frac{1}{8}pp\right)^2-s}.$$

Verum si indolem 4 numerorum datorum accuratius perpendamus, reperiemus semper fore $\tfrac{1}{2}q-\tfrac{1}{8}pp=\frac{-1-s}{2}$, unde denominator inventae fractionis evadet:

$$(\tfrac{1}{2}q-\tfrac{1}{8}pp)^2-s=\frac{(s-1)^2}{4},$$

sicque commode evenit, ut hic denominator fiat quadratum; nisi enim hoc contigisset, singulae formulae:

$$1+az,\ 1+bz,\ 1+cz,\ 1+dz$$

quadrata fieri non potuissent. Quod si etiam in numeratore istum valorem loco $(\frac{1}{2}q-\frac{1}{8}pp)$ substituamus, fiet $z=\frac{4r+p(s+1)}{(s-1)^2}$. Hoc autem numero z invento omnino decem sequentibus conditionibus satisfiet:

$$\begin{array}{ll} \text{I}°.\,ab+1=\square, & \text{II}°.\,ac+1=\square, \\ \text{III}°.\,ad+1=\square, & \text{IV}°.\,bc+1=\square, \\ \text{V}°.\,bd+1=\square, & \text{VI}°.\,cd+1=\square, \\ \text{VII}°.\,az+1=\square, & \text{VIII}°.\,bz+1=\square, \\ \text{IX}°.\,cz+1=\square, & \text{X}°.\,dz+1=\square, \end{array}$$

COROLLARIUM

Quod autem semper sit $\frac{1}{2}q-\frac{1}{8}pp=\frac{-s-1}{2}$, sequenti modo ostendi potest. Ponatur brevitatis gratia $m+n+l=f$ et $l(l+m)(l+n)=k$, ita ut sit $k=fll+lmn$, et cum sit $a=m,\ b=n,\ c=f+l$ et $d=4k$, habebimus $a+b+c=2f$, ergo $p=2f+4k$; deinde, quia

$$q=(a+b+c)d+(a+b)c+ab,$$

fiet nunc

$$q=8fk+(m+n)^2+2l(m+n)+mn,$$

quae expressio ob $mn=ll-1$ abit in hanc: $q=8fk+ff-1$; tum vero erit $s=4mnk(f+l)$, hinc erit

$$1+q+s=8fk+ff+4mnk(f+l),$$

quod an aequale sit ipsi $\frac{1}{4}pp$, videamus. At est

$$\tfrac{1}{4}pp=ff+4fk+4kk,$$

hisque valoribus inter se aequatis habebimus

$$8fk+ff+4mnk(f+l)=ff+4fk+4kk,\ \text{sive}$$
$$4fk+4mnk(f+l)=4kk,$$

quae aequatio per $4k$ divisa dat

$$f+mn(f+l)=k=fll+lmn, \text{ sive}$$
$$f+fmm=fll, \text{ ob } mn+1=ll$$

per hypothesin, quae aequatio cum sit identica, illa: $1+q+s=\frac{1}{4}pp$ necessario est vera, unde sequitur, quo assumsimus, $\frac{1}{2}q-\frac{1}{8}pp=\frac{-s-1}{2}$.

EXEMPLUM 1

Sumamus $m=1$ et $n=3$, eritque $l=2$, unde quatuor numeri priores erunt $a=1,\ b=3,\ c=8,\ d=120$; hinc ergo colligimus:

$$p=132,\ q=1475,\ r=4224 \text{ et } s=2880;$$

ex quibus valoribus deducimus:

$$z=\frac{4\cdot 4224+264\cdot 2881}{(2879)^2}$$

quae fractio reducitur ad hanc $\frac{1777480}{8288641}$, atque hinc decem conditiones praescriptae sequenti modo adimplentur:

$$1°.\ ab+1=2^2, \qquad 2°. ac+1=3^2,$$
$$3°.\ ad+1=11^2, \qquad 4°. bc+1=5^2,$$
$$5°.\ bd+1=19^2, \qquad 6°. cd+1=31^2,$$
$$7°.\ az+1=\frac{(3011)^2}{(2879)^2}, \qquad 8°. bz+1=\frac{(3259)^2}{(2879)^2},$$
$$9°.\ cz+1=\frac{(3809)^2}{(2879)^2}, \qquad 10°. dz+1=\frac{(10079)^2}{(2879)^2}.$$

EXEMPLUM 2

Cum numerus z hinc prodierit tam vehementer magnus, evolvamus sequentem casum in fractionibus, quandoquidem fractiones admittere nunc sumus coacti. Sumatur igitur $m=\frac{1}{2},\ n=\frac{5}{2}$ ut sit $l=\frac{3}{2}$, unde quatuor numeri priores erunt

$$a=\frac{1}{2},\ b=\frac{5}{2},\ c=6 \text{ et } d=48;$$

unde porro deducimus:

$$p=57,\ q=451\tfrac{1}{4},\ r=931\tfrac{1}{2} \text{ et } s=360;$$

ex his ergo deducitur:

$$z=\frac{4\cdot 931\frac{1}{2}+114\cdot 361}{359^2}=\frac{44880}{359^2}=\frac{44880}{128881},$$

qui numeri multo sunt minores quam praecedentes.