NEW AIDS FOR THE RESOLUTION OF THE FORMULA

$$axx+1=yy.$$

Opuscula analytica 1, 1783, p. 310-328 [E559]

Shown to the St. Petersburg Academy assembly on the 23rd September 1773.

I. This problem, set out by the author Pell [it is well known that Euler was mistaken in identifying Pell as the solver of this equation, who was apparently Lord Brouncker; see e.g. Wikipedia, for details], whereby for some given number *a*, neither square nor negative, the numbers x are sought, so that the formula $axx+1=yy$ may become a square, and the method I have discussed and treated often now [see E323], with the aid of which the problem can be resolved much more easily than by the method thought out by Pell himself. Yet meanwhile the setting out of these cases, which demand large numbers *x*, of which kind is the case $a=61$, for which there becomes $x=226153980$ and $y=1766319049$, emerges even by my shorter method with several quite tedious operations ; from which indeed I myself have been seen to have excelled somewhat, then I have found another straight forwards way for finding these very large numbers with wonderful ease. But before finding that, it will help to have noted about the nature of the numbers *a*, which will be able to be resolved by the smaller numbers *x* and *y*, as often as *a* were a number of this form : $a=bbcc\pm 2b$, the solution to be brought out, if there may be taken $x=c$,for then there becomes $y=bcc\pm 1$; then truly also, if there were $a=bbcc\pm b$ and there may be taken $x=2c$, there will be $y=2bcc\pm 1$; from which more cases will be allowed to be set forth without calculation, which indeed also are able to be resolved readily enough by the Pell method. But the following problems will be deduced for more abstruse cases.

PROBLEM 1

2. *If there were $app-1=qq$, to find the numbers x and y, so that there may become $axx+1=yy$.*

SOLUTION

From the proposed formula there will be $app=qq+1$, from which on being multiplied by $4qq$ and on adding one there will arise

$$4appqq+1=4q^4+4qq+1=(2qq+1)^2.$$

Hence therefore for the solution of the problem we may follow
with $x=2pq$ and $y=2qq+1$.

COROLLARY 1

3. Therefore whenever it may happen, so that $app-1$ may become a square, x can be found easily, so that this formula $axx+1$ may become a square. Thus if there were $a=5$, on account of $5\cdot 1^2-1=2^2$ here there will be $p=1$ and $q=2$, from which there becomes $x=4$ and $y=9$.

COROLLARY 2

4. But this can work only for these numbers *a*, which are the sum of two squares ; otherwise it is seen this Pellian property would have to be known. Hence indeed I establish the following inverse problem, where these numbers themselves are sought.

PROBLEM 2

5. *To find the numbers a, for which there can become* $app-1=qq$ *, and hence the numbers* x *and* y *to be assigned, so that there may become* $axx+1=yy$.

SOLUTION

Since there must become $app=qq+1$, there will be $a=\frac{qq+1}{pp}$, and thus numbers of this kind must be sought for *p* and *q*, so that this fraction may become a whole number. Therefore since both *pp* as well as p must be the sum of two squares, there may be put in place

$$pp=bb+cc \quad \text{and} \quad qq+1=(bb+cc)(ff+gg),$$

so that there may become $a=ff+gg$. Now truly there will be $q=bf+cg$ and $\pm 1=bg-cf$. Therefore from the given numbers *b* and *c* the others f and *g* must be taken thus so that $bg-cf=\pm 1$, which indeed can happen easily in an infinite number of ways. Then truly there will be $q=bf+cg$, and because the number p may be considered as given, hence we may conclude to become $x=2pq$ and $y=2qq+1$, so that the use of these formulas may become more apparent, we will add the following examples, while some numbers may be taken for *p*, which indeed we will assume shall be the sum of the two squares.

EXAMPLE 1

6. Let $p=5$, there will $pp=25=bb+cc$, from which there shall be $b=3$ and $c=4$, while therefore f and g must be taken such, so that there may become $3g-4f=\pm1$, and thence we will have $a=ff+gg$ and for this number $q=3f+4g$, and finally $x=10q$ and $y=2qq+1$. Moreover we will show the values of this kind for the letters f and g in the following table:

$$4f-3g=\pm1.$$

f	1	2	4	5	7	8
g	1	3	5	7	9	11
a	2	13	41	74	130	185
q	7	18	32	43	57	68
x	70	180	680	570	430	320
y	99	649	2049	3699	6499	9249

Therefore in this table several quite difficult cases occur.

EXAMPLE 2

7. There shall be $p=13$, and thus $pp=169=52+122$, from which there becomes $b=5$ and $c=12$. Now in the first place this equation is had : $12f-5g=\pm1$; then truly there will be $a=ff+gg$ and $q=5f+12g$, from which there becomes $x=26q$ and $y=2qq+1$. Therefore hence we will show the cases arising in the following table :

$$12f-5g=\pm1$$

f	2	3	7	8
g	5	7	17	19
a	29	58	338	425
q	70	99	239	268
x	1820	2574	6214	6968
y	9801	19603	114243	143649

EXAMPLE 3

8. There shall be $p=17$, and thus $pp=289=8^2+152$, therefore $b=8$ and $c=15$, from which the first equation requiring to be implemented will be $15f-8g=\pm1$, with which done there becomes $a=ff+gg$ and $q=8f+15g$, and hence again $x=34q$ and $y=2qq+1$.
Hence the following table shows these cases arising:

$$15f - 8g = \pm 1.$$

f	1	7
g	2	13
a	5	218
q	38	251
x	1292	8534
y	2889	126003

EXAMPLE 4

9. There shall be $p = 25$, hence $pp = 625 = 72 + 242$, from which $b = 7$ and $c = 24$. Now this equation is had : $24f - 7g = \pm 1$, from which there is produced $a = ff + gg$, $q = 7f + 24g$ and $x = 2pq = 50q$ and $y = 2qq + 1$. Hence we present a single case, in which $f = 2$ and $g = 7$, hence there arises $a = 53$, then truly there becomes $q = 182$, therefore $x = 9100$ and $y = 66249$.

PROBLEM 3

10. *If there were* $app - 2 = qq$, *to find the numbers x and y, so that there may become* $axx + 1 = yy$.

SOLUTION

Therefore since there shall be $app = qq + 2$, we may multiply each side by qq, and by adding one there will be $appqq + 1 = q^4 + 2qq + 1$, from which evidently there is deduced $x = pq$ and $y = qq + 1$.

PROBLEM 4

11. *To investigate the numbers a, for which it can occur that* $app - 2 = qq$, *and hence numbers themselves x and y to be assigned, so that there may become* $axx + 1 = yy$.

SOLUTION

Since there must be $app = qq + 2$, there will be $a = \frac{qq+2}{pp}$, and thus for p and q numbers of this kind must be sought, so that fraction may become a whole number. But because the formula $qq + 2$ does not admit other divisors, unless which may have the form $bb + 2cc$, also for p other numbers may not be allowed to be accepted, unless which shall be of the same form, on account of which at once we may put $pp = bb + 2cc$, and there may become

$$qq + 2 = (bb + 2cc)(ff + 2gg),$$

so that there may be obtained $a = ff + 2gg$, then truly there will be required to be $cf - bg = 1$, and hence there may arise $q = bf + 2cg$, and finally $x = pq$ and $y = qq + 1$. But the form $bb + 2cc$ does not admit other divisors, unless which shall be either of this form: $8n + 1$, or of this form: $8n + 3$, which therefore progress thus : 3, 11, 17, 19, 41, 43 etc. and which are composed from these, on account of which we may run through the following examples.

EXAMPLE 1

12. Let there be $p = 3$, hence $pp = 9 = 1^2 + 2 \cdot 2^2$, from which there becomes $b = 1$ and $c = 2$; therefore the equation requiring to be resolved is $2f - g = \pm 1$, with which done there will be $a = ff + 2gg$, $q = f + 4g$ and hence $x = 3q$ and $y = qq + 1$ etc.. Hence the cases arising include the following table:

$$2f - g = \pm 1$$

f	1	1	2	2	3	3	4
g	1	3	3	5	5	7	7
a	3	19	22	54	59	107	114
q	5	13	14	22	23	31	32
x	15	39	42	66	69	93	96
y	26	170	197	485	530	962	1025

EXAMPLE 2

13. Let there be $p = 9$, there will be $pp = 81 = 7^2 + 2 \cdot 4^2$, and thus $b = 7$ and $c = 4$; hence our equation becomes $4f - 7g = \pm 1$; then truly there will be $a = ff + 2gg$ and $q = 7f + 8g$, and hence again $x = 9q$ and $y = qq + 1$. Behold therefore the cases, which hence arise :

$$4f - 7g = \pm 1$$

f	2	5	9	12
g	1	3	5	7
a	6	43	131	242
q	22	59	103	140
x	198	531	927	1260
y	485	3482	10610	19601

EXAMPLE 3

14. Let there be $p = 11$, there will be $pp = 121 = 72 + 2 \cdot 6^2$, and thus $b = 7$ and $c = 6$, therefore our equation becomes $6f - 7g = \pm 1$, then truly $a = ff + 2gg$ and $q = 71 + 12g$, from which $x = 11q$ and $y = qq + 1$. Therefore the cases will be:

$$6f-7g=\pm1.$$

f	1	6	8
g	1	5	7
a	3	86	162
q	19	102	140
x	209	1122	1540
y	362	10405	19601

EXAMPLE 4

15. Let there be $p=17$, there will be $pp=289=12+2\cdot12^2$, and thus $b=1$ and $c=12$, from which the equation $121-g=\pm1$, then truly we will have $a=ff+2gg$, $q=f+24g$, $x=17q$ and $y=qq+1$, from which it suffices to have observed two cases :

$$f=1,\ g=11,\ a=243,\ q=265,\ x=4505\quad\text{and}\quad y=265^2+1,$$
$$f=1,\ g=13,\ a=339,\ q=313,\ x=17\cdot313\ \text{and}\ y=313^2+1.$$

EXAMPLE 5

16. There shall be $p=19$, there will be $pp=361=17^2+2\cdot6^2$, and thus $b=17$ and $c=6$, and the equation will be $6f-17g=\pm1$, hence there becomes

$$a=ff+2gg,\ q=17f+12g,\ x=19q\ \text{and}\ y=qq+1,$$

from which the following cases arise:

If $f=3$ and $g=1$, there will be $a=11$, $q=63$, $x=1197$ and $y=3970$.
If $f=14$ and $g=5$, there will be $a=246$, $q=298$, $x=5662$, $y=298^2+1$.

PROBLEM 5

17. *If there were* $app+2=qq$, *to find the numbers x and y, so that there may become* $axx+1=yy$.

SOLUTION

Since there shall be $app=qq-2$, it is evident to become $x=pq$ and $y=qq-1$, from which we may progress to the following problems.

PROBLEM 6

18. *To find the numbers a, for which it shall be able to arise,* $app+2=qq$ *, and hence to assign the numbers x and y, so that there may become* $axx+1 = yy$.

SOLUTION

From the equation $app+2=qq$ there is deduced $a=\frac{qq-2}{pp}$, thus so that pp must be a divisor of the formula $qq-2$, that which cannot eventuate, unless *pp* and thus *p* shall be a number of the form $bb-2cc$. On account of this matter, we may put $pp=bb-2cc$ and $qq-2=(bb-2cc)(ff-2gg)$, so that which may be able to happen, there must be $cf-bg=\pm 1$; then truly there will be $q=bf-2cg$ and $a=ff-2gg$, and we will have $x=pq$ and $y=qq-1$. Now truly it will be agreed to be observed other prime numbers cannot be taken for *p*, unless contained either of these formulas : $8n+1$ and $8n-1$. Because if now there may be taken $p=1$, there will be $a=qq-2$, which is the most noteworthy case itself, and there must become $x=q$ and $y=qq-1$, whereby we progress to the following examples.

EXAMPLE 1

19. Let $p=7$, there will be $pp=49=9^2-2\cdot 4^2$, hence $b=9$ and $c=4$, therefore our equation $4f-9g=\pm 1$; then truly there will be $a=ff-2gg$ and $q=9f-8g$, $x=7q$ et $y=qq-1$, from which we may present the following cases:

1^o. $f=2,\ g=1,\ a=2,\ q=10,\ x=70$ and $y=99$,
2^o. $f=7,\ g=3,\ a=31,\ q=39,\ x=273$ and $y=1520$,
3^o. $f=11,\ g=5,\ a=71,\ q=59,\ x=413$ and $y=3480$,
4^o. $f=16,\ g=7,\ a=158,\ q=88,\ x=616$ and $y=7743$.

But here it must be noted all numbers of the form $bb-2cc$ can be continued in the same form in an infinite number of ways. Thus with $p=7$ remaining there will be also $pp=49=11^2-2\cdot 6^2$, thus so that now there shall be $b=11$ and $c=6$, and thus our equation will be $6f-11g=\pm 1$, then truly as before $a=ff-2gg$, but truly $q=11f-12g$ and $x=7q$ and $y=qq-1$. Hence we may establish the following cases:

$1°$. $f=2,\ g=1,\ a=2,\ q=10,\ x=70$ and $y=99$,
$2°$. $f=9,\ g=5,\ a=31,\ q=39,\ x=273$ and $y=1520$;

hence moreover it is evident the same cases are going to be produced, which we have now found before.

EXAMPLE 2

20. Let there be $p=17$, there will be $pp=289=19^2-2\cdot 6^2$, from which our equation becomes $6f-19g=\pm 1$, then there will be

$$a=f-2gg,\ q=191-12g \text{ and } x=17q \text{ and } y=qq-1.$$

$$1°.\ f=3,\ g=1,\ a=7,\ q=45,\ x=765 \text{ and } y=2024,$$

$$2°.\ f=16,\ g=5,\ a=206,\ q=244,\ x=4148 \text{ and } y=244^2-1.$$

EXAMPLE 3

21. Let there be $p=23$, there will be $pp=529=27^2-2\cdot 10^2$, and thus $b=27$ and $c=10$. Therefore our equation will be $101-27g=\pm 1$, and hence $a=ff-2gg$, $q=27f-20g$, $x=23q$ and $y=qq-1$. Hence if $f=8$, there will be $g=3$, $a=46$, $q=156$, $x=3588$ and $y=156^2-1$.

PROBLEM 7

22. *If there were* $app+4=qq$ *, to find the numbers x and y, so that there may become* $axx+1=yy$.

SOLUTION

Because the number 4 is a square, in the desired form there will be $\frac{app}{4}+1=\frac{qq}{4}$, thus so that now there shall be $x=\frac{p}{2}$ and $y=\frac{q}{2}$, if indeed *p* and *q* must be even numbers, which therefore must be the most obvious case. But if *p* and *q* shall be odd numbers, it is required to observe from this case the following known rule this following case is able to be concluded: $x=\frac{pq}{2}$ and $y=\frac{qq}{2}-1$, but which numbers even now are fractions. Hence moreover a third case may be deduced, while the values of the second case are multiplied by *q* and the first values are subtracted, hence there will be produced

$$x=p(\tfrac{qq-1}{2}) \text{ and } y=q(\tfrac{qq-3}{2}),$$

both which values are whole on account of the odd number *q*, on account of which this has presented a solution of the problem, so that there shall be
$x=p(\frac{qq-1}{2})$ and $y=q(\frac{qq-3}{2})$,

COROLLARY

23. If there were $a = bb - 4$, there can be taken $p = 1$, and there will be $q = b$, and hence $x = \frac{bb-1}{2}$ and $y = b(\frac{bb-3}{2})$, where indeed it will be required to accept odd numbers for *b*, from which it will help to have established the following cases :

1°. Let $b = 3$, there will be $a = 5$ and $x = 4$ and $y = 9$,
2°. let $b = 5$, there will be $a = 21$, hence $x = 12$ and $y = 55$,
3°, let $b = 7$, there will be $a = 45$, hence $x = 24$ and $y = 161$,
4°. let $b = 9$, there will be $a = 77$ and $x = 40$ and $y = 351$,
5°. let $b = 11$, there will be $a = 117$ and $x = 60$ and $y = 649$.

PROBLEM 8

24. *To investigate the numbers a, for which it may be possible fore* $app + 4 = qq$ *to occur, and hence the numbers x and y to be assigned, so that there may become* $axx + 1 = yy$.

SOLUTION

Hence since it shall happen that $app = qq - 4$, there becomes $a = \frac{qq-4}{pp}$, where it is allowed to accept all the odd numbers for *p*, since the numerator $qq - 4$ is the difference of two squares. Therefore always it will be able to put $pp = bb - cc$ in place, by putting $b = \frac{pp+1}{2}$ and $c = \frac{pp-1}{2}$; then truly there may become $qq - 4 = (bb - cc)(ff - gg)$, so that hence there may arise $a = ff - gg$; truly for this it is required, so that there shall be $cf - bg = \pm 2$ and $q = bf - cg$, with which done there will be found $x = p\left(\frac{qq-1}{2}\right)$ and $y = q\left(\frac{qq-3}{2}\right)$. Hence therefore we may present the following examples.

EXAMPLE 1

25. There may be taken $p = 3$, and there becomes $b = 5$ and $c = 4$, hence $4f - 5g = \pm 2$, again $a = ff - gg$, $q = 5f - 4g$, $x = 3(qq - 1)$ and $y = q(qq - 3)$; from which we will consider the following cases :

1°. $f=2,\ g=2,$ hence $a=0,$ from which therefore nothing follows.

2°. $f=3,\ g=2,$ hence $a=5,\ q=7,\quad x=72$ and $y=161.$

3°. $f=7,\ g=6,$ hence $a=13,\ q=11,\ x=180$ and $y=649.$

4°. $f=8,\ g=6,$ hence $a=28,\ q=16,\ x=3\cdot\frac{255}{2},$ which case cannot be used.

5°. $f=12,\ g=10,$ from which equally nothing can be deduced, for it is necessary that the number f shall be odd.

6°. $f=13$ and $g=10,$ thence $a=69,\ q=25,$ therefore $x=936$ et $y=7775.$

7°. $f=17$ and $g=14,$ thence $a=93,\ q=29,$ therefore $x=1260$ et $y=12151.$

8°. $f=23$ and $g=18,$ thence $a=205,\ q=43,$ therefore $x=2772$ et $y=39689.$

Here too the value of q in general can be assigned from a single known value $q=2$. For there may be put $q=9n\pm2$, and there will be $qq-4=81nn\pm36n$, from which there becomes $a=9nn\pm4n$, then truly there will be

$$x=3\left(\frac{81nn\pm36n+3}{2}\right)\text{ and }y=(9n\pm2)\left(\frac{81nn\pm36n+3}{2}\right),$$

where any odd number is allowed to be taken for n.

EXAMPLE 2

26. Let $p=5$, and there will be $b=13$ and $c=12$, and thus $12f-13g=\pm2$, from which there becomes $a=ff-gg$ and $q=13f-12g,\ x=5\left(\frac{qq-1}{2}\right)$, and $y=q\left(\frac{qq-3}{2}\right),$ where again f must be some odd number. There shall be

1°. $f=11$ and $g=10,$ hence $a=21,\ q=23,\ x=1320$ and $y=6049.$

2°. $f=15$ and $g=14,$ hence $a=29,\ q=27,\ x=1820$ and $y=9801.$

But at once without the aid of the letters f and g it will be able to establish $q=25n\pm2$, and there will be $a=25nn\pm4n$; then truly as before there will be $x=5\left(\frac{qq-1}{2}\right)$ and $y=q\left(\frac{qq-3}{2}\right)$; and here now for n it is allowed to take any odd number, of which the individuals on account of the sign ambiguity will give two solutions.

1°. If $n=1$ there will be $a=25\pm4$ and $q=25\pm2.$

2°. If $n=3$ there will be $a=225\pm12$ and $q=75\pm2.$

3°. If $n=5$ there will be $a=625\pm20$ and $q=125\pm2.$

4°. If $n=7$ there will be $a=1225\pm28$ and $q=175\pm2.$

GENERAL DEVELOPMENT

27. If p were some odd number, there may be taken $q = npp \pm 2$, and there will be $a = \frac{qq-4}{pp} = nnpp \pm 4n$, then truly there will be had $x = p\left(\frac{qq-1}{2}\right)$, $y = q\left(\frac{qq-3}{2}\right)$, which has a single solution in place, when p will be a prime number ; truly if p may involve prime factors, in addition other solutions will have a place. For let $p = rs$, and there will become $pp = rrss = bb - cc$, by taking $b = \frac{rr+ss}{2}$ and $c = \frac{rr-ss}{2}$; then the numbers f and g may be sought, in order that there may become $cf - bg = \pm 2$ and there may be put in place $bf - cg = k$, and there will be $kk - 4 = (bb - cc)(ff - gg) = rrss(ff - gg)$, and hence $kk - 4\; pp = ff - gg$. Now there may be taken $q = npp \pm k$, and there will be found $a = \frac{qq-4}{pp} = nnpp \pm 2nk + ff - gg$, and here again there will be $x = p\left(\frac{qq-1}{2}\right)$ and $y = q\left(\frac{qq-3}{2}\right)$.

Just as if there were $p = 15 = 5 \cdot 3$, or $r = 5$ and $s = 3$, by the first method there is found at once $q = 225n \pm 2$, and hence $a = 225nn \pm 4n$, truly from the second solution we have $b = 17$ and $c = 8$, and now there becomes $8f - 17g = \pm 2$, on taking $f = 4$ and $g = 2$, and thus $ff - gg = 12$ and $k = 52$. Therefore if there may be taken $q = 225n \pm 52$, there will be produced $a = 225nn \pm 104n + 12$, then truly for each case there will be $x = 15\left(\frac{qq-1}{2}\right)$ and $y = q\left(\frac{qq-3}{2}\right)$.

PROBLEM 9

28. *If there were* $arr - 4 = ss$, *to find the numbers x and y, so that there may become* $axx + 1 = yy$.

SOLUTION

Since there shall be $arr = ss + 4$, on multiplying by ss and adding 4 there will be produced $arrss + 4 = s^4 + 4ss + 4 = (ss + 2)^2$, and thus we are returned to the previous case, where there was $app + 4 = qq$; evidently now there will be $p = rs$ and $q = ss + 2$, from which values we deduce as before $x = p\left(\frac{qq-1}{2}\right)$ and $y = q\left(\frac{qq-3}{2}\right)$.

COROLLARY

29. Here before everything the case occurs where $a = ee + 4$, with e denoting some odd number, for which there will be $r = 1$ and $s = e$. Hence there becomes therefore $p = e$ and $q = ee + 2$, and hence again $x = p\left(\frac{qq-1}{2}\right)$ and $y = q\left(\frac{qq-3}{2}\right)$, from which we may set out the following cases :

Let 1°. $e=1,$ there will be $a=5,$ $p=1,$ $q=3,$ and thus $x=4$ and $y=9,$
2°. $e=3,$ there will be $a=13,$ $p=3,$ $q=11,$ from which $x=180$ and $y=649,$
3°. $e=5,$ there will be $a=29,$ $p=5,$ $q=27,$ therefore $x=1820$ and $y=9801,$
4°. $e=7,$ there will be $a=53,$ $p=7,$ $q=51,$ therefore $x=9100$ and $y=66249,$
5°. $e=9,$ there will be $a=85,$ $p=9,$ $q=83,$ therefore $x=30996$ and $y=285769.$

PROBLEM 10

30. T*o find the remaining numbers, so that it may be possible for there to become* $arr-4=ss$ *, and thence to assign the numbers x and y, so that there may become* $axx+1=yy$.

SOLUTION

Since there shall be $arr=ss+4$, there will be $a=\frac{ss+4}{rr}$; from which it is apparent not to be possible to assume other odd numbers for *r*, unless which shall be the sum of two squares. Therefore there shall be $rr=bb+cc$ and there may be put $ss+4=(bb+cc)(ff+gg)$, which it is required to put in the end $cf-bg=\pm2$, then truly there will be $s=bf+cg$; but from the known numbers *r* and *s* there will be $p=rs$ and $q=ss+2$, from which again it is concluded $x=p\left(\frac{qq-1}{2}\right)$ and $y=q\left(\frac{qq-3}{2}\right)$; therefore other numbers cannot be accepted for *r*, unless they shall be either 5, 13, 17, 25 or 29 etc., which cases we will establish in the following examples.

EXAMPLE 1

31. Let $r=5$, and thus $rr=25=4^2+3^2$, from which $b=4$ and $c=3$. Hence there must be $3f-4g=\pm2$, then truly there becomes $a=ff+gg$ and $s=4f+3g$, from which the solution is performed as before.

1°. Let $f=2$ and $g=1,$ there will be $a=5,$ $s=11$, and hence again $p=55$ and $q=123$, from which large numbers for *x* and *y* satisfying the problem arise, but not small ones.
2°. If $f=2$ and $g=2$, there will be $a=8$; but the case, for which *a* is an even number, we exclude here.
3°. If $f=6$ and $g=5$, there will be $a=61$, then truly $s=39$, and hence again $p=195$ and $q=1523$; and hence the values $x=226153980$ and $y=1766319049$ are concluded.

But from the first case, where there was $s=11$ and $\frac{11^2+4}{25}=5$, there can be put at once $s=25n\pm11$, from which there is deduced $a=25nn\pm22n+5$; then truly there will be $p=5(25n\pm11)$ and $q=(25n\pm11)^2+2$, from which finally there is deduced

$x = p\left(\frac{qq-1}{2}\right)$ and $y = q\left(\frac{qq-3}{2}\right)$. Bur hence it will suffice to have derived the values of the number *a*, where indeed it will be required to take even numbers for *n*, lest an even number may be produced for *a*.

1°. If $n = 0,$ there becomes $a = 5,$
2°. if $n = 2,$ there becomes either $a = 61$ or $a = 149,$
3°. if $n = 4,$ there becomes either $a = 317$ or $a = 493,$

therefore with these cases the numbers *x* and *y* will increase into immense numbers.

EXAMPLE 2

32. Now let there be
$r = 13,$ there will be $rr = 169 = 12^2 + 5^2,$ from which $b = 12$ and $c = 5,$ thus so that there may become $5f - 12g = \pm 2,$ then truly there will be $a = ff + gg$ and $s = 12f + 5g$. But the most simple case is $f = 2$ and $g = 1$, which provides $a = 5$ and $s = 29$, from which generally at once there can be put in place $s = 169n \pm 29$, from which there is deduced $a = 169nn \pm 58n + 5$, for which number there will become
$p = 13(169n \pm 29)$ et $q = (169n \pm 29)^2 + 2$, from which finally there is deduced
$x = p\left(\frac{qq-1}{2}\right)$ and $y = q\left(\frac{qq-3}{2}\right)$. But for the numbers *n* even numbers will require to be taken, so that *a* may become an odd number.

1°. If $n = 0$ there becomes $a = 5$ and $s = 29,$ and hence $p = 13 \cdot 29,$ but which case by itself is known.
2°. Let $n = 2,$ there will be either $a = 565$ or $a = 797.$

EXAMPLE 3

33. There shall be $r = 17$ and $rr = 289 = 15^2 + 8^2$, from which $b = 15$ and $c = 8$, and there must become $8f - 15g = \pm 2$, and hence $a = ff + gg$ and $s = 15f + 8g$. For the simplest case we may take $f = 4$ and $g = 2$, from which there becomes $a = 20$ et $s = 76$; therefore for the general solution we may put $s = 289n \pm 76$, and there will become

$$a = 289nn \pm 152n + 20\,,$$

where it is agreed to have taken odd numbers for *n*.

1°. If $n = 1$, there will become $a = 309 \pm 152$ and $s = 289 \pm 76$, which values now are exceedingly large, as it will be worth the effort to have set these out.

EXAMPLE 4

34. Let $r = 25$ and $rr = 625 = 24^2 + 7^2$, and thus $b = 24$ and $c = 7$. Now there must be $7f - 24g = \pm 2$, and hence there will be $a = ff + gg$ and $s = 24f + 7g$. The simplest case gives $f = 10$ and $g = 3$, and hence $a = 109$ and $s = 261$; from which if there may be taken $s = 625n \pm 261$, generally there will be found $a = 625nn \pm 522n + 109$. Moreover we may set out the case $a = 109$ numerically, which by the common method requires a most tiresome calculation; therefore since there shall be
s = 261, on account of $r = 25$ there will be $p = 6525$ and $q = 261^2 + 2 = 68123$, from which the most desired numbers are deduced :

$$x = 6525\left(\tfrac{68123^2 - 1}{2}\right) = 15140424455100,$$

$$y = 68123\left(\tfrac{68123^2 - 3}{2}\right) = 158070671986249.$$

EXAMPLE 5

35. Let $r = 29$ and $rr = 841 = 21^2 + 20^2$, and thus $b = 21$ and $c = 20$, and there must become $20f - 21g = \pm 2$, hence there will be $a = ff + gg$ and $s = 21f + 20g$. But in the simplest case there will be $f = 2$ and $g = 2$, hence $a = 8$ and $s = 82$; therefore there may be put $s = 841n \pm 82$, and there becomes $a = 841nn \pm 164n + s$, from which moreover the numbers arising are to increase exceedingly greatly.

Therefore from these it is abundantly clear how, with the aid of these subsidiary cases, some of the difficulty of resolving Pell problems may be able to be resolved readily enough.

NOVA SUBSIDIA PRO RESOLUTIONE FORMULAE

$axx+1=yy$.

Commentatio 559 indicia ENESTROEMIANI

Opuscula analytica 1, 1783, p. 310-328

Conventui exhibita die 23 septembris 1773

I. Problema hoc, ab auctore PELLIANUM dictum, quo pro dato quocunque numero *a*, neque quadrato neque negativo, numeri quaeruntur *x*, ut formula $axx+1=yy$ fiat quadratum, iam saepius pertractavi methodumque tradidi, cuius ope multo facilius resolvi potest quam methodo ab ipso PELLIO excogitata. Interim tamen evolutio eorum casuum, qui pro *x* numeros praegrandes postulant, cuiusmodi est casus $a=61$, pro quo fit $x=226153980$ et $y=1766319049$, etiam mea methodo succincta plurimas non parum taediosas operationes exigit; unde equidem non parum praestitisse mihi videor, dum aliam prorsus viam detexi hos ipsos praemagnos numeros mira facilitate inveniendi. Ante autem quam eam apperiam, circa indolem numerorum *a*, quos per minores numeros *x* et *y* resolvere licet, notasse iuvabit, quoties *a* fuerit numerus huius formae: $a=bbcc\pm 2b$, solutionem in promtu esse, si capiatur $x=c$, tum enim fit $y=bcc\pm 1$; tum vero etiam, si fuerit $a=bbcc\pm b$ et capiatur $x=2c$, erit $y=2bcc\pm 1$; unde plurimos casus sine ulteriori calculo expedire licebit, qui quidem etiam methodo PELLIANA satis prompte resolvi possunt. Sequentia autem problemata ad casus magis abstrusos deducent.

PROBLEMA 1

2. *Si fuerit* $app-1=qq$, *invenire numeros x et y, ut fiat* $axx+1=yy$.

SOLUTIO

Ex pro posita formula erit $app=qq+1$, unde per $4qq$ multiplicando et unitatem addendo orietur $4appqq+1=4q^4+4qq+1=(2qq+1)^2$.
Hinc igitur pro solutione problematis consequimur $x=2pq$ et $y=2qq+1$.

COROLLARIUM 1

3. Quoties ergo evenit, ut fiat $app-1$ quadratum, facile inveniri potest *x*, ut haec formula $axx+1$ fiat quadratum. Ita si fuerit $a=5$, ob $5\cdot 1^2-1=2^2$ erit hic $p=1$ et $q=2$, unde fit $x=4$ et $y=9$.

COROLLARIUM 2

4. Hoc autem tantum pro iis numeris a locum habere potest, qui sunt summae duorum quadratorum; caeterum haec pro prietas iam ipsi PELLIO cognita fuisse videtur. Hinc autem sequens problema inversum, quo ipsi isti numeri quaeruntur, evolvam.

PROBLEMA 2

5. *lnvestigare numeros a, pro quibus fieri potest* $app-1=qq$ *, hincque ipsos numeros x et y assignare, ut fiat* $axx+1=yy$.

SOLUTIO

Cum debeat esse $app=qq+1$, erit $a=\frac{qq+1}{pp}$, sicque pro p et q eiusmodi numeri quaeri debent, ut haec fractio praebeat numerum integrum. Quia igitur tam pp quam p debet esse summa duorum quadratorum, statuatur

$$pp=bb+cc \text{ et } qq+1=(bb+cc)(ff+gg),$$

ut fiat $a=ff+gg$. Iam vero erit $q=bf+cg$ et $\pm1=bg-cf$. Ex datis ergo numeris b et c alteri f et g ita accipi debent, ut fiat $bg-cf=\pm1$, quod quidem infinitis modis facile fieri potest. Tum igitur erit $q=bf+cg$, et quia numerus p ut datus spectatur, hinc concludimus fore $x=2pq$ et $y=2qq+1$, quarum formularum usus quo clarius appareat, sequentia exempla adiiciemus, dum pro p nonnullos numeros, qui quidem sint summae duorum quadratorum, assumemus.

EXEMPLUM 1

6. Sit $p=5$, erit $pp=25=bb+cc$, unde fit $b=3$ et $c=4$, tum ergo f et g tales sumi debent, ut fiat $3g-4f=\pm1$, indeque habebimus $a=ff+gg$ et pro hoc numero $q=3f+4g$, ac denique $x=10q$ et $y=2qq+1$. Huiusmodi autem valores pro litteris f et g in sequenti tabella exhibemus:

$$4f-3g=\pm1.$$

f	1	2	4	5	7	8
g	1	3	5	7	9	11
a	2	13	41	74	130	185
q	7	18	32	43	57	68
x	70	180	680	570	430	320
y	99	649	2049	3699	6499	9249

In hac tabula occurrunt ergo casus alioquin non parum difficiles.

EXEMPLUM 2

7. Sit $p=13$, ideoque $pp=169=52+122$, unde fit $b=5$ et $c=12$. Nunc primo habetur ista aequatio: $12f-5g=\pm 1$; tum vero erit $a=ff+gg$ et $q=5f+12g$, unde fit $x=26q$ et $y=2qq+1$.
Casus ergo hinc oriundos in sequenti tabula exhibemus:

$$12f-5g=\pm 1$$

f	2	3	7	8
g	5	7	17	19
a	29	58	338	425
q	70	99	239	268
x	1820	2574	6214	6968
y	9801	19603	114243	143649

EXEMPLUM 3

8. Sit $p=17$, ideoque $pp=289=8^2+152$, ergo $b=8$ et $c=15$, unde prima aequatio adimplenda erit $15f-8g=\pm 1$, quo facto fiet $a=ff+gg$ et $q=8f+15g$, hincque porro $x=34q$ et $y=2qq+1$.
Casus hinc oriundos sequens tabula ostendit:

$$15f-8g=\pm 1.$$

f	1	7
g	2	13
a	5	218
q	38	251
x	1292	8534
y	2889	126003

EXEMPLUM 4

9. Sit $p=25$, hinc $pp=625=72+242$, unde $b=7$ et $c=24$. Iam habetur ista aequatio: $24f-7g=\pm 1$, ex qua prodit $a=ff+gg$, $q=7f+24g$ et $x=2pq=50q$ atque $y=2qq+1$. Hinc unicum casum evolvamus, quo $f=2$ et $g=7$, hinc oritur $a=53$, tum vero fit $q=182$, ergo $x=9100$ et $y=66249$.

PROBLEMA 3

10. *Si fuerit* $app-2=qq$ *, invenire numeros x et y, ut fiat* $axx+1=yy$.

SOLUTIO

Cum igitur sit $app=qq+2$, multiplicemus utrinque per qq, et adiecta unitate erit $appqq+1=q^4+2qq+1$, unde manifesto colligitur $x=pq$ et $y=qq+1$.

PROBLEMA 4

11. *Investigare numeros a, pro quibus fieri potest* $app-2=qq$ *, hincque ipsos numeros x et y assignare, ut fiat* $axx+1=yy$.

SOLUTIO

Cum debeat esse $app=qq+2$, erit $a=\frac{qq+2}{pp}$, sicque pro p et q eius modi numeri quaeri debent, ut illa fractio praebeat numerum integrum. Quia autem formula $qq+2$ alios divisores non admittit, nisi qui habeant formam $bb+2cc$, etiam pro p alios numeros accipere non licet, nisi qui sint eiusdem formae, quocirca ponamus statim $pp=bb+2cc$, fiatque

$$qq+2=(bb+2cc)(ff+2gg),$$

ut obtineatur $a=ff+2gg$, tum vero esse oportet $cf-bg=1$, hincque orietur $q=bf+2cg$, ac tandem $x=pq$ et $y=qq+1$. At forma $bb+2cc$ alios divisores primos non admittit, nisi qui sint vel huius formae: $8n+1$, vel huius: $8n+3$, qui ergo ita progrediuntur: 3, 11, 17, 19, 41, 43 etc. et qui ex his componuntur, quamobrem sequentia exempla percurramus.

EXEMPLUM 1

12. Sit $p=3$, hinc $pp=9=1^2+2\cdot 2^2$, unde fit $b=1$ et $c=2$; aequatio ergo resolvenda est $2f-g=\pm 1$, quo facto erit $a=ff+2gg$, $q=f+4g$ atque hinc $x=3q$ et $y=qq+1$. Casus hinc oriundos sequens tabula complectitur:

$$2f-g=\pm 1$$

f	1	1	2	2	3	3	4
g	1	3	3	5	5	7	7
a	3	19	22	54	59	107	114
q	5	13	14	22	23	31	32
x	15	39	42	66	69	93	96
y	26	170	197	485	530	962	1025

EXEMPLUM 2

13. Sit $p=9$, erit $pp=81=7^2+2\cdot 4^2$, ideoque $b=7$ et $c=4$; hinc aequatio nostra $4f-7g=\pm 1$; tum vero erit $a = ff+2gg$ et $q=7f+8g$, hincque porro $x=9q$ et $y=qq+1$. Ecce ergo casus, qui hinc oriuntur:

$$4f-7g=\pm 1$$

f	2	5	9	12
g	1	3	5	7
a	6	43	131	242
q	22	59	103	140
x	198	531	927	1260
y	485	3482	10610	19601

EXEMPLUM 3

14. Sit $p=11$, erit $pp=121=72+2\cdot 6^2$, ideoque $b=7$ et $c=6$, ergo aequatio nostra $6f-7g=\pm 1$, tum vero $a= ff+2gg$ et $q=71+12g$, unde $x=11q$ et $y=qq+1$. Casus ergo erunt:

$$6f-7g=\pm 1.$$

f	1	6	8
g	1	5	7
a	3	86	162
q	19	102	140
x	209	1122	1540
y	362	10405	19601

EXEMPLUM 4

15. Sit $p=17$, erit $pp=289=12+2\cdot 12^2$, sicque $b=1$ et $c=12$, unde aequatio $121-g=\pm 1$, tum vero habebimus $a= ff+2gg$, $q= f+24g$, $x=17q$ et $y=qq+1$, unde duos casus notasse sufficiet:

$$f=1,\ g=11,\ a=243,\ q=265,\ x=4505 \quad \text{et} \ y=265^2+1,$$
$$f=1,\ g=13,\ a=339,\ q=313,\ x=17\cdot 313 \ \text{et} \ y=313^2+1.$$

EXEMPLUM 5

16. Sit $p=19$, erit $pp=361=17^2+2\cdot 6^2$, ideoque $b=17$ et $c=6$, et aequatio erit $6f-17g=\pm 1$, hinc fiet

$$a=ff+2gg,\ q=17f+12g,\ x=19q \text{ et } y=qq+1,$$

unde sequentes casus nascuntur:

Si $f=3$ et $g=1$, erit $a=11$, $q=63$, $x=1197$ et $y=3970$.
Si $f=14$ et $g=5$, erit $a=246$, $q=298$, $x=5662$, $y=298^2+1$.

PROBLEMA 5

17. *Si fuerit* $app+2=qq$, *invenire numeros x et y, ut* fiat $axx+1=yy$.

SOLUTIO

Cum sit $app=qq-2$, manifestum est fore $x=pq$ et $y=qq-1$, unde ad sequens problema progredimur.

PROBLEMA 6

18. *Investigare numeros a, pro quibus fieri possit* $app+2=qq$, *hincque numeros x et y assignare, ut fiat* $axx+1=yy$.

SOLUTIO

Ex aequatione $app+2=qq$ deducitur $a=\frac{qq-2}{pp}$, ita ut pp debeat esse divisor formulae $qq-2$, id quod evenire nequit, nisi sit pp ideoque et p numerus formae $bb-2cc$. Hanc ob rem statuamus $pp=bb-2cc$ et $qq-2=(bb-2cc)(ff-2gg)$, quod ut fieri possit, debet esse $cf-bg=\pm 1$; tum vero erit $q=bf-2cg$ et $a=ff-2gg$, atque habebimus $x=pq$ et $y=qq-1$. Nunc vero observari convenit pro p alios numeros primos accipi non posse, nisi in alterutra harum formularum: $8n+1$ et $8n-1$ contentos. Quod si iam sumeretur $p=1$, foret $a=qq-2$, qui est casus per se notissimus, fieretque $x=q$ et $y=qq-1$, quare ad sequentia exempla progredimur.

EXEMPLUM 1

19. Sit $p=7$, erit $pp=49=9^2-2\cdot 4^2$, hinc $b=9$ et $c=4$, ergo aequatio nostra $4f-9g=\pm 1$; tum vero erit $a=ff-2gg$ et $q=9f-8g$, $x=7q$ et $y=qq-1$, unde casus sequentes evolvamus:

1^{o}. $f=2,\ g=1,\ a=2,\ q=10,\ x=70$ et $y=99,$

2^{o}. $f=7,\ g=3,\ a=31,\ q=39,\ x=273$ et $y=1520,$

3^{o}. $f=11,\ g=5,\ a=71,\ q=59,\ x=413$ et $y=3480,$

4^{o}. $f=16,\ g=7,\ a=158,\ q=88,\ x=616$ et $y=7743.$

Hic autem notari debet omnes numeros formae $bb-2cc$ infinitis modis in eadem forma contineri posse. Ita manente $p=7$ erit quoque $pp=49=11^2-2\cdot 6^2$, ita ut nunc sit $b=11$ et $c=6$, sicque nostra aequatio erit $6f-11g=\pm 1$, tum vero ut ante $a=ff-2gg$, at vero $q=11f-12g$ et $x=7q$ atque $y=qq-1$. Hinc sequentes casus evolvamus:

1°. $f=2,\ g=1,\ a=2,\ q=10,\ x=70$ et $y=99,$

2°. $f=9,\ g=5,\ a=31,\ q=39,\ x=273$ et $y=1520;$

hinc autem manifestum est eosdem casus esse prodituros, quos iam ante invenimus.

EXEMPLUM 2

20. Sit $p=17,$ erit $pp=289=19^2-2\cdot 6^2$, unde nostra aequatio fit $6f-19g=\pm 1$, tum vero erit $a=f-2gg,\ q=191-12g$ et $x=17q$ et $y=qq-1$.

1°. $f=3,\ g=1,\ a=7,\ q=45,\ x=765$ et $y=2024,$

2°. $f=16,\ g=5,\ a=206,\ q=244,\ x=4148$ et $y=244^2-1.$

EXEMPLUM 3

21. Sit $p=23$, erit $pp=529=27^2-2\cdot 10^2$, ideoque $b=27$ et $c=10$. Aequatio ergo nostra erit $101-27g=\pm 1$, hincque $a=ff-2gg,\ q=27f-20g,\ x=23q$ et $y=qq-1$. Hinc si $f=8,$ erit $g=3,\ a=46,\ q=156,\ x=3588$ et $y=156^2-1$.

PROBLEMA 7

22. *Si fuerit* $app+4=qq$, *invenire numeros x et y, ut fiat* $axx+1=yy$.

SOLUTIO

Quia numerus 4 est quadratum, erit in forma desiderata $\frac{app}{4}+1=\frac{qq}{4}$, ita ut iam esset $x=\frac{p}{2}$ et $y=\frac{q}{2}$, si quidem essent *p* et *q* numeri pares, qui ergo casus foret maxime obvius. At si *p* et *q* sint numeri impares, notandum est ex hoc casu concludi posse secundum praecepta cognita hunc secundum casum: $x=\frac{pq}{2}$ et $y=\frac{qq}{2}-1$, qui numeri autem etiamnum sunt fracti. Hinc autem tertius casus deducatur, dum valores secundi casus per *q* multiplicantur et primi subtrahuntur, hinc autem prodibit

$$x=p(\frac{qq-1}{2}) \text{ et } y=q(\frac{qq-3}{2}),$$

qui ambo valores ob *q* numerum imparem erunt integri, quocirca hanc sumus adepti solutionem problematis, ut sit $x=p(\frac{qq-1}{2})$ et $y=q(\frac{qq-3}{2})$,

COROLLARIUM

23. Si ergo fuerit $a=bb-4$, capi poterit $p=1$, eritque $q=b$, hincque $x=\frac{bb-1}{2}$ et $y=b(\frac{bb-3}{2})$, ubi quidem pro b numeros impares accipi oportet, unde sequentes casus evolvisse iuvabit:

1°. Sit $b=3$, erit $a=5$ et $x=4$ atque $y=9$,
2°. sit $b=5$, erit $a=21$, hinc $x=12$ atque $y=55$,
3°, sit $b=7$, erit $a=45$, hinc $x=24$ atque $y=161$,
4°. sit $b=9$, erit $a=77$ et $x=40$ atque $y=351$,
5°. sit $b=11$, erit $a=117$ et $x=60$ atque $y=649$.

PROBLEMA 8

24. *Investigare numeros a, pro quibus fieri possit* $app+4=qq$, *hincque numeros x et y assignare, ut fiat* $axx+1=yy$.

SOLUTIO

Cum hinc sit $app=qq-4$, fiet $a=\frac{qq-4}{pp}$, ubi pro p omnes numeros impares accipere licet, quandoquidem numerator $qq-4$ est differentia duorum quadratorum. Semper igitur statui poterit $pp = bb - cc$, ponendo $b = \frac{pp+1}{2}$ et $c = \frac{pp-1}{2}$; tum vero fiat $qq - 4 = (bb - cc)(ff - gg)$, ut hinc oriatur $a = ff - gg$; ad hoc vero requiritur, ut sit $cf - bg = \pm 2$ et $q = bf - cg$, quo facto reperietur $x = p\left(\frac{qq-1}{2}\right)$ et $y = q\left(\frac{qq-3}{2}\right)$. Hinc ergo sequentia exempla evolvamus.

EXEMPLUM 1

25. Sumatur $p = 3$, fietque $b = 5$ et $c = 4$, hinc $4f - 5g = \pm 2$, porro $a = ff - gg$, $q = 5f - 4g$, $x = 3(qq - 1)$ et $y = q(qq - 3)$; unde casus sequentes consideremus:

1°. $f = 2$, $g = 2$, hinc $a = 0$, unde ergo nihil sequitur.

2°. $f = 3$, $g = 2$, hinc $a = 5$, $q = 7$, $x = 72$ et $y = 161$.

3°. $f = 7$, $g = 6$, hinc $a = 13$, $q = 11$, $x = 180$ et $y = 649$.

4°. $f = 8$, $g = 6$, hinc $a = 28$, $q = 16$, $x = 3 \cdot \frac{255}{2}$, qui casus inutilis.

5°. $f = 12$, $g = 10$, unde pariter nihil colligitur, necesse enim est, ut numerus f sit impar.

6°. $f = 13$ et $g = 10$, unde $a = 69$, $q = 25$, ergo $x = 936$ et $y = 7775$.

7°. $f = 17$ et $g = 14$, unde $a = 93$, $q = 29$, ergo $x = 1260$ et $y = 12151$.

8°. $f = 23$ et $g = 18$, unde $a = 205$, $q = 43$, ergo $x = 2772$ et $y = 39689$.

Hic quoque valor ipsius q in genere assignari potest ex unico valore cognito $q = 2$. Ponatur enim $q = 9n \pm 2$, eritque $qq - 4 = 81nn \pm 36n$, unde fit $a = 9nn \pm 4n$, tum vero erit

$$x = 3\left(\frac{81nn \pm 36n + 3}{2}\right) \text{ et } y = (9n \pm 2)\left(\frac{81nn \pm 36n + 3}{2}\right),$$

ubi pro n quemlibet numerum imparem assumere licet.

EXEMPLUM 2

26. Sit $p = 5$, eritque $b = 13$ et $c = 12$, ideoque $12f - 13g = \pm 2$, unde fit $a = ff - gg$ et $q = 13f - 12g$, $x = 5\left(\frac{qq-1}{2}\right)$, atque $y = q\left(\frac{qq-3}{2}\right)$, ubi iterum f debet esse numerus impar. Sit

1°. $f = 11$ et $g = 10$, hinc $a = 21$, $q = 23$, $x = 1320$ et $y = 6049$.

2°. $f = 15$ et $g = 14$, hinc $a = 29$, $q = 27$, $x = 1820$ et $y = 9801$.

Statim autem sine litterarum *f* et *g* ope statui poterit $q=25n\pm 2$, eritque $a=25nn\pm 4n$; tum vero ut ante erit $x=5\left(\frac{qq-1}{2}\right)$ et $y=q\left(\frac{qq-3}{2}\right)$; atque hic iam pro *n* quosvis numeros impares assumere licet, quorum singuli ob signum ambiguum binas dabunt solutiones.

1°. Si $n=1$ erit $a=25\pm 4$ et $q=25\pm 2$.
2°. Si $n=3$ erit $a=225\pm 12$ et $q=75\pm 2$.
3°. Si $n=5$ erit $a=625\pm 20$ et $q=125\pm 2$.
4°. Si $n=7$ erit $a=1225\pm 28$ et $q=175\pm 2$.

EVOLUTIO GENERALIS

27. Si *p* fuerit numerus impar quicunque, sumatur $q=npp\pm 2$, eritque $a=\frac{qq-4}{pp}=nnpp\pm 4n$, tum vero habebitur $x=p\left(\frac{qq-1}{2}\right)$, $y=q\left(\frac{qq-3}{2}\right)$, quae unica solutio locum habet, quando *p* erit numerus primus; verum si *p* involvat factores inter se primos, aliae insuper solutiones locum habere possunt. Sit enim $p=rs$, fierique poterit $pp=rrss=bb-cc$, sumendo $b=\frac{rr+ss}{2}$ et $c=\frac{rr-ss}{2}$; tum quaerantur numeri *f* et *g*, ut fiat $cf-bg=\pm 2$ et statuatur $bf-cg=k$, eritque $kk-4=(bb-cc)(ff-gg)=rrss(ff-gg)$, hincque $kk-4pp=ff-gg$. Iam sumatur $q=npp\pm k$, ac reperietur $a=\frac{qq-4}{pp}=nnpp\pm 2nk+ff-gg$, atque hic iterum erit $x=p\left(\frac{qq-1}{2}\right)$ et $y=q\left(\frac{qq-3}{2}\right)$.
Vel uti si fuerit $p=15=5\cdot 3$, sive $r=5$ et $s=3$, priori modo statim habetur $q=225n\pm 2$, hincque $a=225nn\pm 4n$, ex posteriore vero solutione habemus $b=17$ et $c=8$, nuncque fiet $8f-17g=\pm 2$, sumendo $f=4$ et $g=2$, ideoque $ff-gg=12$ et $k=52$. Si ergo capiatur $q=225n\pm 52$, prodibit $a=225nn\pm 104n+12$, tum vero pro utroque casu erit $x=15\left(\frac{qq-1}{2}\right)$ et $y=q\left(\frac{qq-3}{2}\right)$.

PROBLEMA 9

28. *Si fuerit $arr-4=ss$, invenire numeros x et y, ut fiat $axx+1=yy$.*

SOLUTIO

Cum sit $arr=ss+4$, multiplicando per *ss* et 4 addendo prodit $arrss+4=s^4+4ss+4=(ss+2)^2$, sicque ad casum praecedentem revolvimur, quo erat $app+4=qq$; erit scilicet nunc $p=rs$ et $q=ss+2$, ex quibus valoribus colligimus ut ante $x=p\left(\frac{qq-1}{2}\right)$ et $y=q\left(\frac{qq-3}{2}\right)$.

COROLLARIUM

29. Hic ante omnia occurrit casus quo $a = ee + 4$, denotante e numerum imparem quemcunque, pro quo erit $r = 1$ et $s = e$. Hinc igitur fiet $p = e$ et $q = ee + 2$, hincque porro $x = p\left(\frac{qq-1}{2}\right)$ et $y = q\left(\frac{qq-3}{2}\right)$ unde sequentes casus evolvamus:

Sit 1°. $e = 1$, erit $a = 5$, $p = 1$, $q = 3$, ideoque $x = 4$ et $y = 9$,
2°. $e = 3$, erit $a = 13$, $p = 3$, $q = 11$, unde $x = 180$ et $y = 649$,
3°. $e = 5$, erit $a = 29$, $p = 5$, $q = 27$, ergo $x = 1820$ et $y = 9801$,
4°. $e = 7$, erit $a = 53$, $p = 7$, $q = 51$, ergo $x = 9100$ et $y = 66249$,
5°. $e = 9$, erit $a = 85$, $p = 9$, $q = 83$, ergo $x = 30996$ et $y = 285769$.

PROBLEMA 10

30. *Investigare reliquos numeros, ut fieri possit* $arr - 4 = ss$, *indeque assignare numeros x et y, ut fiat* $axx + 1 = yy$.

SOLUTIO

Cum sit $arr = ss + 4$, erit $a = \frac{ss+4}{rr}$; unde patet pro r alios numeros impares assumi non posse, nisi qui sint summae duorum quadratorum. Sit igitur $rr = bb + cc$ et ponatur $ss + 4 = (bb + cc)(ff + gg)$, quem in finem esse oportet $cf - bg = \pm 2$, tum vero erit $s = bf + cg$; ex cognitis autem numeris r et s erit $p = rs$ et $q = ss + 2$, unde porro concluditur $x = p\left(\frac{qq-1}{2}\right)$ et $y = q\left(\frac{qq-3}{2}\right)$; pro r ergo alios numeros accipere non licet, nisi vel 5 vel 13 vel 17 vel 25 vel 29 etc., quos casus in sequentibus exemplis evolvemus.

EXEMPLUM 1

31. Sit $r = 5$, ideoque $rr = 25 = 4^2 + 3^2$, unde $b = 4$ et $c = 3$. Hinc esse debet $3f - 4g = \pm 2$, tum vero fit , unde $a = ff + gg$ et $s = 4f + 3g$ solutio conficitur ut ante.
1°. Sit $f = 2$ et $g = 1$, erit $a = 5$, $s = 11$, hincque porro $p = 55$ et $q = 123$, unde pro x et y ingentes prodeunt numeri problemati satisfacientes, sed non minimi.
2°. Si $f = 2$ et $g = 2$, erit $a = 8$; casus autem, quibus a numerus par, hic excludimus.
3°. Sit $f = 6$ et $g = 5$, erit $a = 61$, tum vero $s = 39$, hincque porro $p = 195$ et $q = 1523$; atque hinc concluduntur valores $x = 226153980$ et $y = 1766319049$.

Ex casu autem primo, quo erat $s = 11$ et $\frac{11^2+4}{25} = 5$, statim poni potest $s = 25n \pm 11$, unde deducitur $a = 25nn \pm 22n + 5$; tum vero erit $p = 5(25n \pm 11)$ et $q = (25n \pm 11)^2 + 2$, unde denique deducitur $x = p\left(\frac{qq-1}{2}\right)$ et $y = q\left(\frac{qq-3}{2}\right)$. Hinc autem suffecerit valores numeri a derivasse, ubi quidem pro n numeros pares sumi oportet, ne a prodeat numerus par.

1°. Si $n = 0$, fit $a = 5$,

2°. si $n = 2$, fit vel $a = 61$ vel $a = 149$,

3°. si $n = 4$, fit vel $a = 317$ vel $a = 493$,

pro his ergo casibus numeri x et y in immensum excrescent.

EXEMPLUM 2

32. Sit nunc $r = 13$, erit $rr = 169 = 12^2 + 5^2$, unde $b = 12$ et $c = 5$, ita ut fieri debeat $5f - 12g = \pm 2$, tum vero erit $a = ff + gg$ et $s = 12f + 5g$. Casus autem simplicissimus est $f = 2$ et $g = 1$, qui praebet $a = 5$ et $s = 29$, ex quo statim generaliter statui potest $s = 169n \pm 29$, unde deducitur $a = 169nn \pm 58n + 5$, pro quo numero fiet $p = 13(169n \pm 29)$ et $q = (169n \pm 29)^2 + 2$, unde denique colligitur $x = p\left(\frac{qq-1}{2}\right)$ et $y = q\left(\frac{qq-3}{2}\right)$. At pro n numeros pares accipi oportet, ut quidem fiat a numerus impar.

1°. Si $n = 0$ fit $a = 5$ et $s = 29$, hincque $p = 13 \cdot 29$, qui autem casus per se est notus.

2°. Sit $n = 2$, erit vel $a = 565$ vel $a = 797$.

EXEMPLUM 3

33. Sit $r = 17$ et $rr = 289 = 15^2 + 8^2$, unde $b = 15$ et $c = 8$, fierique debet $8f - 15g = \pm 2$, hincque $a = ff + gg$ et $s = 15f + 8g$. Pro casu simplicissimo sumamus $f = 4$ et $g = 2$, unde fit $a = 20$ et $s = 76$; pro solutione ergo generali ponamus $s = 289n \pm 76$, fietque

$$a = 289nn \pm 152n + 20,$$

ubi pro n numeros impares capi convenit.

1°. Si $n = 1$, fiet $a = 309 \pm 152$ et $s = 289 \pm 76$, qui valores iam nimis sunt magni, quam quos operi pretium sit evolvisse.

EXEMPLUM 4

34. Sit $r = 25$ et $rr = 625 = 24^2 + 7^2$, ideoque $b = 24$ et $c = 7$. Iam esse debet $7f - 24g = \pm 2$, hincque erit $a = ff + gg$ et $s = 24f + 7g$. Casus simplicissimus dat $f = 10$ et $g = 3$, hincque $a = 109$ et $s = 261$; unde si statuatur $s = 625n \pm 261$, reperietur generatim $a = 625nn \pm 522n + 109$. Evolvamus autem numerice casum $a = 109$, qui methodo vulgari molestissimos calculos requirit; cum ergo sit s = 261, ob $r = 25$ erit $p = 6525$ et $q = 2612 + 2 = 68123$, ex quibus desiderati numeri maximi deducuntur

$$x = 6525\left(\frac{68123^2 - 1}{2}\right) = 15140424455100,$$

$$y = 68123\left(\frac{68123^2 - 3}{2}\right) = 158070671986249.$$

EXEMPLUM 5

35. Sit $r = 29$ et $rr = 841 = 21^2 + 20^2$, ideoque $b = 21$ et $c = 20$, fierique debet $20f - 21g = \pm 2$, hinc erit $a = ff + gg$ et $s = 21f + 20g$. Casu autem simplicissimo erit $f = 2$ et $g = 2$, hinc $a = 8$ et $s = 82$; statuatur ergo $s = 841n \pm 82$, fietque $a = 841nn \pm 164n + s$, unde autem numeri vehementer magni nascuntur.

Ex his igitur abunde perspicitur, quemadmodum ope horum subsidiorum casus problematis PELLIANI alioquin difficillimi satis expedite resolvi queant.