SOME PROPOSED PROGRESSION IS SOUGHT BEGINNING FROM ONE, SO THAT THE SAME NUMBER OF ITS TERMS MAY BE REQUIRED TO BE ADDED TO THE MINIMUM IN ORDER THAT ALL [THE INTERVENING] NUMBERS MAY BE PRODUCED.

E558

Opuscula analytica 1, 1783, p. 296-309

Shown to the St. Petersburg Academy on the 22nd of March, 1773

[*Fermat had asserted : *Indeed , we ourselves are the first to have completed the prettiest and most general proposition: truly every number, either to be triangular or to be composed from two or three triangular numbers ; to be square, or to be composed from two, three, or four squares; to be pentagonal, or to be composed from two, three, four or five pentagonal numbers; and thus henceforth indefinitely for hexagons, heptagons, and polygons no matter which, clearly to be enunciated for any number of the angle by this general and wonderful proposition….* *Opera Omnia* V. I, *pp*. 303-305]

That most noteworthy theorem of Fermat * or Bachet is of this kind, because the sum either of four or fewer squares shall be a whole number, the demonstration of which the celebrated Lagrange published, which he established after FERMAT, truly just as I also have communicated not long ago to the Academy, deduced from other principles [E242]. Moreover I have confirmed the things mentioned by Fermat, to have shown these also, all triangular numbers to be the sum of three or fewer triangular numbers, then truly also five or fewer pentagonal numbers, likewise six or fewer hexagonal numbers, and thus henceforth. Truly at this point no rigorous demonstration of these theorems has been composed.

Truly recently also the illustrious Beguelin, an associate of the Berlin Academy, extended the theorems of this kind yet wider to all the pyramidal numbers or figures, which indeed arise from the sum of polygonal numbers repeated continually. For this being requiring to be explained by us, Θ will denote the number of terms required to be added to the smallest, so that clearly all the numbers may be produced, and the most ingenious man has given the values of the letter Θ for the following series added on below :

	Θ
$1,\ 1+a,\ 1+2a,\ 1+3a,\ 1+4a$ etc.	a
$1,\ 2+a,\ 3+3a,\ 4+6a,\ 5+10a$ etc.	$a+2$
$1,\ 3+a,\ 6+4a,\ 10+10a,\ 15+20a$ etc.	$a+4$
$1,\ 4+a,\ 10+5a,\ 20+15a,\ 35+35a$ etc.	$a+6$
$1,\ 5+a,\ 15+6a,\ 35+2la,\ 70+56a$ etc.	$a+8$
$1,\ 6+a,\ 21+7a,\ 56+28a,\ 126+84a$ etc.	$a+10$

etc.

from which it is concluded in general, the number to become $\Theta = a + 2n - 2$, for this series generally:

$$1, \frac{n+a}{1}, \frac{(n+1)(n+2a)}{1 \cdot 2}, \frac{(n+1)(n+2)(n+3a)}{1 \cdot 2 \cdot 3}, \frac{(n+1)(n+2)(n+3)(n+4a)}{1 \cdot 2 \cdot 3 \cdot 4} \text{ etc.},$$

, which formula include not only all the polygonal numbers, but also summable series arising from these ; from which this theorem certainly merits the greatest attention. But so great is the pain, which the discoverer himself may admit, a rigorous demonstration of this cannot be grasped by any means, but yet, as if by moving from point to point, its truth may become known with the aid of the principle of sufficient reasoning. But even if I believe clearly nothing to be attributed by this principle in investigations of this kind, yet I am forced to acknowledge the truth of this proposition on account of other arguments; but also that this same theorem is found precisely to be indicated after thirty or more years with my troubles.

[Thus, these results may have been found by the method of infinite descent, and as well, they can be represented by binomial coefficients, etc., See Dickson, p. 7: thus, the difference of two neighbouring squares is a term in the above triangle series, the difference of two neighbouring pentagonal numbers is a square number, etc.; by substituting a value for a in any series, the truth of the theorem is demonstrated for small values: yet no general proof had been found free from fallacy at this stage; Fermat promised but did not deliver such a proof; Euler's task was to provide a proof in general, if possible. Such a series of natural numbers were to be established 1, *A*, *B*, *C*, ...The table supplied by Beguelin showed the truth of the proposition for the start of the series of natural numbers in an arithmetical progression, triangular numbers, etc. on setting $a = 1$, and by extension to any number a. The interested reader would be well-advised to consult Bk. II, Ch. I, of L. E. Dickson's *History of the Theory of Numbers*, p. 12 onwards, to find the fallacious nature of these results. Euler was unable, after much trouble, to find a general proof.]

Therefore it will be observed these conspicuous properties of the numbers are to be considered carefully, without doubt with the greatest pain, even now we need a demonstration of these, and more from that, because the first principles shall be scarcely established, from which the desired demonstrations may be aimed at ; on account of which here these principles to be put in place are investigated more carefully, from which perhaps, whatever may be desired in this kind even now, at last will be allowed to be exhausted. To this end we will consider some progression starting from unity, which shall be:

$$1, A, B, C, D, E, F \text{ etc.},$$

from which there is sought, for how many of its terms there is a need in turn for the minimum number itself to be added, so that clearly all the numbers may be produced; and this number sought itself we will indicate by the letter Θ [*i.e.* the number of terms to be added to generate the next number of the same kind] ; where indeed not only different terms, but also the same are required to be understood to be repeating continually a number of times. Therefore I will try to deduce the investigation from first principles in the following manner.

1. Therefore initially we will consider the first term of the series only, which is $=1$, and hence it is evident as far as is concerned for the following term *A* all the numbers cannot be produced, unless there may be put $\Theta = A-1$, if indeed the agreeing number $A-1$ demands the first number 1 be required to be repeated from just as many units ; from which whatever our series were, now we may see clearly the number sought Θ certainly cannot be smaller than $A-1$.

2. Now besides the first term 1 we may allow also the second term *A*, for which as far as we have seen in the established manner, there must become $\Theta = A-1$. But hence by progressing further the neighbouring number follows composed from so many terms will become $2A-2$, certainly which agrees with the second term *A* taken once and with the first term taking 1 in turn so many times, just as the formula $A-2$ indicates, since here the number can be represented thus : $A+1(A-2)$, thus so that the number of terms joined together shall be $1+A-2=A-1$. Truly, it is apparent the number immediately following $2A-1$ itself to be produced not further by the total terms, because there is $2A-1=A(1)+1(A-1)$, from which for that being produced the terms require $1+A-1=A$; on account of which thus far we have the progression $\Theta = A$. Indeed we may not yet look back on the third term *B*; for if there were $B=2A-1$, or thus smaller, then this ratio may stop and the preceding value $\Theta = A-1$ will apply at this stage ; but if the third term *B* may exceed that term $2A-1$, then certainly there will be put in place $\Theta = A$, clearly greater than the preceding case by one.

3. But we may move the third term *B* further back, and since the number $2A-1$ may require the term *A*, it is evident the number $3A-1$ is going to require the term $A+1$, since there shall be $3A-1=A+(2A-1)$, the first part of which contains the term one, truly the latter the term *A*; from which if *B* may surpass this limit $3A-1$, for that we will have as far as $\Theta = A+1$. But by progressing further the number $4A-1$ will require the term $A+2$; truly the number $5A-1$ will require the term $A+3$; and in general the number $nA-1$ will require the term $A+n-2$, if indeed the third term *B* were greater than that.

4. From these deductions, if the third term *B* may be contained between the two terms in the manner designated, we may conclude the value of the number Θ in the following way :

If B may be contained between	Θ will be:
$A-1$ and $2A-1$	$A-1$
$2A-1$ and $3A-1$	A
$3A-1$ and $4A-1$	$A+1$
$4A-1$ and $5A-1$	$A+2$
.	.
.	.
.	.
$nA-1$ et $(n+1)A-1$	$A+n-2$.

Consequently, if we may put $B > nA-1$, nevertheless $B < (n+1)A-1$, because hence it follows $n < \frac{B+1}{A}$ and finally $n > \frac{B+1}{A}-1$, with these values substituted there becomes

$$\Theta < A-2+\frac{B+1}{A}$$

and yet,

$$\Theta > A-3+\frac{B+1}{A}.$$

But it suffices for Θ to take the earlier smaller nearest whole number, or the nearest greater of the other formula. Therefore since as far s to the term A there were $\Theta = A-1$, by progressing as far as to B this increased value may be accepted, thus so that there will be had itself from the former formula :

$$\Theta < A-1+\frac{B-A+1}{A}$$

but from the latter :

$$\Theta > A-1+\frac{B-2A+1}{A}$$

5. Thus far our investigation has advanced happily; so that now we may progress further, we may put the number just found $=\vartheta$, which indicates how many terms there is needed for all the numbers being produced from unity as far as to the term B, thus so that there shall be either

$$\vartheta < A-1+\frac{B-A+1}{A},$$

or

$$\vartheta > A-1+\frac{B-2A+1}{A},$$

certainly which numbers are composed only from the two initial numbers 1 and A from unity as far as third term B being produced. But now with the third term admitted, with just as many terms or fewer joined together, we may proceed beyond B, then we may arrive at the number $B+b$, which from ϑ no greater term may be allowed to be put in place, but which it may require $\vartheta+1$ terms; then truly it is evident, by progressing further the number $2B+b$ requires $\vartheta+2$ terms; and again in a similar manner the number $3B+b$ requires $\vartheta+3$ terms; the number $4B+b$ truly $\vartheta+4$ terms; and in general the number $nB+b$ requires $\vartheta+n$ terms.

6. But if therefore the following term C does not exceed the limit $B+b$, the number ϑ may accept no increase; but if it may be greater, nor yet not greater than the second limit $2B+b$, to the number ϑ one will be required to be added ; but if it may increase further, nor still may it exceed the limit $3B+b$, it will agree with the increase $=2$; from which it is clear, if C may exceed the limit $nB+b$, nor still may it increase beyond the following $(n+1)B+b$, the increase to become $=n$. Therefore we may take $C>nB-b$, because there will become $n<\frac{C-B}{b}$, and this as far as the term C from the multitude of terms will be

$$=\vartheta+n<\vartheta+\tfrac{C-b}{B},$$

where evidently it is required to take the nearest smaller whole number. But if we may take the formula $\vartheta+\frac{C-B-b}{B}$, then the nearest greater whole number must be taken ; and thus for this as far as the term C our number sought will be

$$\vartheta=A-1+\tfrac{C-2A+1}{A}+\tfrac{C-B-b}{B},$$

where for each fraction the nearest greater whole number must be taken.

7. Again we will indicate this number by the letter ϑ, and we may progress beyond C, with just as many or fewer terms added together, then we may arrive at the number $C+c$, which may not be allowed to be produced further, but which requires $(\vartheta+1)$ terms, and by putting in place the reasoning as before it is evident as far as to the following term D that number ϑ takes the increase $\frac{D-C-c}{C}$, clearly if in place of the fraction the nearest larger whole number may be accepted; on account of which as far as to this term we obtain our number sought :

$$\Theta=A-1+\tfrac{B-2A+1}{A}+\tfrac{C-B-b}{B}+\tfrac{D-C-c}{C}.$$

8. So that if we may advance in a similar manner past the number D to the number $D+d$, which no further may be composed from such a number of terms, but may require one more, then preceding as far as to the following term E the value of the letter Θ above will accept an increase $=\frac{E-D-d}{D}$ and thus, as far as it will please, it is allowed to progress further. But this is not unlike a labour of this kind that can be undertaken only with the greatest difficulty ; indeed also, in whatever way we may have progressed now, at no time yet will we be able to be sure about the true value of Θ, if thus it may be continued indefinitely. Meanwhile yet, as far as we will have produced these operations, we can always conclude with care for the series continued indefinitely, the true value Θ

certainly not to be smaller than that, which we have found; and if we were to progress long enough, this value found will not differ far from the truth.

9. Moreover concerning these fractions, from which we have found the number Θ to be constituted, it is agreed to note, if which of these either may vanish or thus may emerge negative, then its value to be nothing, also however great its negative value were made, since on account of the following terms, the value of the original at no time can be diminished ; but when these fractions are positive, provided the values of these do not exceed unity, these equal to unity are being considered carefully again ; but when they are greater than one, yet neither of the two may they be greater, for these it is required to write 2, and thus so on. From which it can be understood, how from some initial number of terms of the series the value of Θ shall be able to be elicited, even if the series may be continued indefinitely ; clearly this may arise, if all the following fractions may emerge negative.

10. Truly it is no wonder an investigation in this way to become involved generally with the greatest difficulties, since here we have become involved with an exceedingly complex matter, nor for any rule, by which we may consider the terms of the series may be progressing. Yet there is no doubt, why a law for that progression requiring may not be brought forwards for the number Θ especially; just as we may observe in the initial general series advanced, which includes all the number figures within itself, from which we can confirm, the number Θ always to be $=a+2n-2$; nor yet at this stage is it apparent how this determination now may be rendered more general. Indeed the illustrious Beguelin, introduced by this case, had observed for a similar law plainly for all the algebraic series starting from unity, of which clearly the general formula may be shown algebraically, the value of Θ itself to be shown in the following manner : The proposed series shall be 1, *A, B, C, D, E* etc. being referred to the order *n*, and thence successive series may be formed of the differences of all the orders, of which the initial terms shall be respectively *a, b, c, d* etc., truly the final differences of the order *n*, which shall be constants, shall be $=i$, thus so that the amount of the numbers *a, b, c, ... i* shall be $=n$;then the laudable man considered to become $\Theta=a+n-1$, which formula certainly agrees with the case of the number of figures. Indeed since here there shall be $a=A-1$, for that case our *a* will become $n+a-1$, to which if the number $n-1$ may be added on account of the order number of the series, that formula results for the given Θ.

11. But it will be a trivial matter to attend to, to consider how this rule may be broken, just as happens in this series :

$$1,\ 2,\ 4,\ 7,\ 11,\ 16,\ 22,\ 29 \text{ etc.}$$

of which the first differences are $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$ etc.,

and the second $1,\ 1,\ 1,\ 1,\ 1,\ 1$ etc.,

which therefore is of the second order, or $n=2$ and $a=1$. Therefore following the rule there must become $\Theta=2$; truly it is certain towards producing all the numbers of this series there is a need for a minimum of three terms required to be added. Then truly the illustrious man has indicated himself, when that rule may be described, with a fraction required to be added omitted, which shall be $\frac{i}{a}$, as indeed it may be omitted always, if it were less than unity, just as usually happens with all polygonal figures ; but when it may rise to unity or beyond, then its value in whole numbers, with the adjoined fraction ignored, must be added in addition. Now since in our case the final differences shall be $i=1$, there will be $\frac{i}{a}=1$, and hence there becomes $\Theta=3$, which agrees exactly with the truth, and now indeed it is to be admitted this amended rule not only agrees with polygonal progressions, but also to be satisfied more generally by innumerable other progressions.

12. Yet meanwhile it is possible to show also a general infinitude of series, for which this rule fails, which will concern most to be shown more carefully. Therefore we may begin with a second order algebraic progression, of which the second differences now are constants, thus so that only these two numbers *a* and *b* occur, from which the general form of these will be :

$$1,\ 1+a,\ 1+2a+b,\ 1+3a+3b,\ 1+4a+6b,\ 1+5a+10b \text{ etc.};$$

and since there shall be $n=2$ and $i=b$,that second rule must become $\Theta=a+1+\frac{b}{a}$, which, as we have noted now, for polygonal numbers, where $b=a-1$, agrees outstandingly well. Why not also consider the situation, whenever the number *b* may not exceed the number *a* much; and indeed at once the fraction $\frac{b}{a}$ shall be big enough, it will be easy to have shown this rule to be differing from the truth; for indeed if we take $a=1$ and $b=100$, so that we may have this progression :

$$1,\ 2,\ 103,\ 304,\ 605,\ 1006,\ 1507,\ 2108,\ 2809 \text{ etc.},$$

for which that rule gives $\Theta=102$, if we may examine this series by the method set out initially, as far as the third 103 we deduce $\Theta=51$; as far as to the fourth term 304, $\Theta=52$; hence to the following 605 it will produce as much as $\Theta=53$; which value may not increase further, even if we may progress beyond the following 1006; from which it is understood clearly by progressing as far as to higher terms this number scarcely will be going to increase beyond 54, since finally the second rule mentioned must become $\Theta=102$; but a much greater error will emerge, if greater numbers may be accepted in place of *b*.

13. Since in this case the error shall be so great, it will be worth the effort for the value $a=1$, from which this series arises

$$1,\ 2,\ 3+b,\ 4+3b,\ 5+6b,\ 6+10b,\ 7+15b,\ 8+21b \text{ etc.}$$

in place of *b* successive greater numbers are to be assumed continually, and for any of these series desired the number Θ, elicited from the observations, to be ascribed, so that it may be able to be compared with the value of the rule $\Theta=2+b$.

b	Series	Θ	error
1	1, 2, 4, 7, 11, 16, 22, ...	3	0
2	1, 2, 5, 10, 17, 26, 37, ...	4	0
3	1, 2, 6, 13, 23, 36, 52, ...	4	1
4	1, 2, 7, 16, 29, 46, 67, ...	5	1
5	1, 2, 8, 19, 35, 56, 82, ...	5	2
6	1, 2, 9, 22, 41, 66, 97, ...	6	2
7	1, 2, 10, 25, 47, 76, 112, ...	6	3
8	1, 2, 11, 28, 53, 86, 127, ...	7	3
9	1, 2, 12, 31, 59, 96, 142, ...	7	4
10	1, 2, 13, 34, 65, 106, 157, ...	8	4

Hence therefore it is apparent from the initial cases $b = 1$ and $b = 2$ the error of the rule to be nothing, hence truly by continually increasing more ; and if we may consider these values attentively, we will conclude readily for this general series the rule to become $\Theta = 3 + \frac{1}{2}b$, from which for the case $b = 100$ there becomes $\Theta = 53$.

14. We will consider in a similar manner the kind, in which $a = 2$, and our series will become :

$$1,\ 3,\ 5+b,\ 7+3b,\ 9+6b,\ 11+10b,\ 13+15b,\ 15+21b,\ 17+28b \text{ etc.},$$

from which the rule of the cel. Beguelin gives $\Theta = 3 + \frac{b}{2}$. Therefore we may set out the following series with the values of Θ as before :

b	Series	Θ	error
1	1, 3, 6, 10, 15, 21, 28,	3	0
2	1, 3, 7, 13, 21, 31, 43,	4	0
3	1, 3, 8, 16, 27, 41, 58,	4	0
4	1, 3, 9, 19, 33, 51, 73,	5	0
5	1, 3, 10, 22, 39, 61, 88,	5	0
6	I, 3, 11, 25, 45, 71, 103,	5	1
7	1, 3, 12, 28, 51, 81, 118,	5	1
8	1, 3, 13, 31, 57, 91, 133,	6	1
9	1, 3, 14, 34, 63, 101, 148,	6	1
10	1, 3, 15, 37, 69, 111, 163,	7	1

But if we may take $b = 100$, for this series :

$$1,\ 3,\ 105,\ 307,\ 609,\ 1011,\ 1513,\ 2115 \text{ etc.}$$

we deduce $\Theta = 37$, since from that rule there must become $\Theta = 53$.

15. Now we may progress to a progression of the third order, where $n = 3$, and the differences of the third series may begin from these numbers : *a, b* and *c,* thus so that $i = c$, and the following rule $\Theta = a + 2 + \frac{c}{a}$; therefore from that the progression will be :

$$1,\ 1+a,\ 1+2a+b,\ 1+3a+3b+c,\ 1+4a+6b+4c,$$
$$1+5a+10b+10c,\ 1+6a+15b+20c \text{ etc.}$$

and this at once may be seen to merit suspicion, because the value of this Θ plainly does not depend on the letter *b* ; for it is evident, if *b* were a very large number, also the number Θ is going to increase moderately. Why not also confirm in general the number Θ cannot be allowed to be less than a certain number, than if there may be $c = 0$; but in this case by the same rule there may become $\Theta = a + 1 + \frac{b}{a}$; whereby, as often as this formula $\Theta = a + 1 + \frac{b}{a}$ is greater than $\Theta = a + 2 + \frac{c}{a}$, that rule necessarily must fail, if there were $\frac{b}{a} > 1 + \frac{c}{a}$ or $b > a + c$.

16. Towards showing this we may take $a = 2,\ c = 1$ and $b = 6$; from which this series arises :

$$1,\ 3,\ 11,\ 26,\ 49,\ 81,\ 123,\ 176,\ 241 \text{ etc.},$$

for which there must be $\Theta = 4 + \frac{1}{2} = 4$, but since the number 21 still cannot be distributed in fewer than 5 terms ; but a greater error is going to be produced, where the number *b* will be increased more.

17. Since the number *b* has supplied one source of error, thus also from the number *c,* if it may be taken large enough, significant errors will arise. Evidently we may take $a = 2,\ b = 1,\ c = 10$, so that this series may be produced:

$$1,\ 3,\ 6,\ 20,\ 55,\ 121,\ 228 \text{ etc.}$$

and thus by the rule $\Theta = 9$; but by testing it will soon become apparent this value cannot be greater than 6 ; from which it is evident for larger numbers *c* larger errors are going to be produced; where it will help to have noted, if *b* were a very small number, the rule to err by being deficient ; but if *c* were a very large number, by being excessive.

18. Therefore without doubt the most careful way of any certainty in which this matter will be concluded, so that more cases of this kind both from the second and third orders, as we have made, as well as also from all the following study may be investigated, and anyone who will have wished to undertake this work, perhaps will find a more widely

apparent and surer rule, by which the resolutions of numbers of this kind may be able to be defined.

19. Perhaps it will help also somewhat to have examined geometrical progressions, for which our number Θ increases more, by which we may progress further. Thus in the twofold progression $1,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5$ etc. if we may stop at the term 2^5, there will be $\Theta=5$; and in general for all the numbers as far as to 2^n there becomes $\Theta=n$; then truly in the threefold progression $1,\ 3,\ 3^2,\ 3^3,\ 3^4$ etc. as far as to the term 3^n our number $\Theta=2n$ will be found ; and in general for the progression $1,\ m,\ m^2,\ m^3,\ m^4,\ m^5,\ m^6$ etc. as far as to the term m^n it may be concluded the number $\Theta=(m-1)^n$. Truly here with the continued differences of these first terms taken being noted to become $m-1,\ (m-1)^2,\ (m-1)^3$ etc., thus so that the above letters used shall become here $a=\ m-1,\ b=(m-1)^2,\ c=(m-1)^3,\ d=\left(m-1\right)^4$ etc.; from which there may be able to conclude to be $\Theta=\frac{a}{1}+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{e}{d}$ etc., but this rule may fail in many other cases. So that also in the hypergeometric progression of Wallis

$$1\ ,\ 2\ ,\ 6\ ,\ 24\ ,\ 120\ ,\ 720\ ,\ 5040 \text{ etc.}$$

as far as to the term 6 there is $\Theta=3$; as far as to the term 24 there is $\Theta=6$; and as far as to the term $1.\ 2.\ 3\ ......n$ there will be $\Theta=\frac{n(n-1)}{2}$.

20. Truly the memorable recurring series also gives us an example for the number Θ. Thus for this most noteworthy series:

Indices 0, 1, 2, 3, 4, 5, 6, 7, 8,
terms 1, 2, 3, 5, 8, 13, 21, 34, 55 etc.,

of which any term is the sum of the two preceding terms, if we may progress as far as the term with the index *n*, there will be $\Theta=\frac{n}{2}$ where if *n* were odd, where the factor with unity is being removed. Indeed I shall not be idle in undertaking further enquiry of this argument.

PROPOSITA QUACUNQUE PROGRESSIONE AB UNITATE INCIPIENTE QUAERITUR QUOT EIUS TERMINOS AD MINIMUM ADDI OPORTEAT UT OMNES NUMERI PRODUCANTUR

E558
Opuscula analytica 1, 1783, p. 296-309
Conventui exhibita die 22. martii 1773

Huius generis est notissimum illud theorema FERMATII sive BACHETTI, quod omnis numerus sit summa vel quatuor vel pauciorum quadratorum, cuius demonstrationem post FERMATIANUM, quae intercidit, Cel. LA GRANGE in medium attulit , ego vero etiam non ita pridem ex aliis principiis deductam cum Academia communicavi. Memoratus autem FERMATIUS praeterea affirmaverat, se quoque demonstrasse, omnes numeros esse summas vel trium pauciorumve numerorum trigonalium, tum vero etiam quinque pauciorumve numerorum pentagonalium, item sex pauciorumve sexagonalium et ita porro. Horum autem theorematum nullum adhuc solide demonstratum comparavit.

Nuper vero etiam ill. BEGUELIN, Berolinensis Academiae socius, huiusmodi theoremata adhuc multo latius ad omnes numeros pyramidales seu figuratas extendit, qui quidem ex summatione numerorum polygonalium continue repetita nascuntur. Ad haec explicanda denotet nobis Θ numerum terminorum ad minimum addendorum, ut omnes plane numeri producantur, atque Vir ingeniosissimus pro seriebus sequentibus valores litterae Θ subnexos dedit:

$$\begin{array}{l|l}
1,\ 1+a,\ 1+2a,\ 1+3a,\ 1+4a \text{ etc.} & a \\
1,\ 2+a,\ 3+3a,\ 4+6a,\ 5+10a \text{ etc.} & a+2 \\
1,\ 3+a,\ 6+4a,\ 10+10a,\ 15+20a \text{ etc.} & a+4 \\
1,\ 4+a,\ 10+5a,\ 20+15a,\ 35+35a \text{ etc.} & a+6 \\
1,\ 5+a,\ 15+6a,\ 35+21a,\ 70+56a \text{ etc.} & a+8 \\
1,\ 6+a,\ 21+7a,\ 56+28a,\ 126+84a \text{ etc.} & a+10
\end{array}$$

etc.

unde in genere concluditur, fore pro hac serie generali:

$$1, \frac{n+a}{1}, \frac{(n+1)(n+2a)}{1\cdot 2}, \frac{(n+1)(n+2)(n+3a)}{1\cdot 2\cdot 3}, \frac{(n+1)(n+2)(n+3)(n+4a)}{1\cdot 2\cdot 3\cdot 4} \text{ etc.}$$

numerum $\Theta = a+2n-2$, quae forma non solum complectitur omnes numeros polygonales, sed etiam omnes series summatrices ex illis natas; ex quo hoc theorema utique maximam attentionem meretur. Tantum autem est dolendum, quod ipse inventor fateatur, se solidam eius demonstrationem neutiquam possidere, sed tantum, quasi per transennam, eius veritatem ope principii rationis sufficientis agnoscere. Etsi autem huic principio in huiusmodi investigationibus nihil plane tribuendum esse arbitror, veritatem tamen istius pro positionis ob alias rationes agnoscere cogor; quin etiam hoc ipsum theorema iam ante triginta et plures annos in adversariis meis consignatum reperitur.

Has igitur insignes numerorum pro prietates perpendenti sine dubio maxime dolendum videbitur nos etiamnunc earum demonstratione indigere, idque eo magis, quod vix adhuc prima principia sint stabilita, ex quibus demonstrationes desideratae peti queant; quamobrem constitui hic ista principia adcuratius investigare, unde fortasse, quicquid in hoc genere etiam-num desideratur, tandem haurire licebit. Hunc in finem consideremus progressionem quamcunque ab unitate incipientem, quae sit:

1, *A*, *B*, *C*, *D*, *E*, *F* etc.,

de qua, quot eius terminis ad minimum sibi invicem addendis opus sit, ut omnes plane numeri producantur, quaeritur; atque hunc ipsum numerum quaesitum littera Θ indicemus; quo quidem non solum termini diversi, sed et iidem aliquoties repetiti contineri sunt intelligendi. Hanc igitur investigationem sequenti modo ex primis principiis deducere conabor.

1. Consideremus igitur initio solum primum seriei terminum, qui est $=1$, ac manifestum est hinc usque ad secundum terminum *A* omnes numeros produci non posse, nisi statuatur $\Theta = A-1$, siquidem numerus $A-1$ ex totidem unitatibus constans toties primum terminum 1 repetendum postulat; unde quaecunque fuerit nostra series, iam clare perspicimus numerum quaesitum Θ certe minorem esse non posse quam $A-1$.

2. Nunc praeter primum terminum 1 admittamus quoque secundum *A*, ad quem usque modo vidimus statui debere $\Theta = A-1$. Hinc autem ulterius progrediendo numerus proxime sequens ex totidem terminis compositus erit $2A-2$, quippe qui constat ex secundo termino *A* semel sumto et primo termino 1 tot vicibus sumto, quot formula $A-2$ indicat, quandoquidem hic numerus ita repraesentari potest: $A+1(A-2)$, ita ut multitudo terminorum iunctorum sit $1+A-2=A-1$. Numerus vero immediate sequens $2A-1$ non amplius per tot terminos se produci patitur, quia est $2A-1=A(1)+1(A-1)$, unde ad eum producendum requiruntur termini $1+A-1=A$; quocirca hucusque progressi habemus $\Theta = A$. Nondum enim ad tertium terminum *B* respicimus; si enim foret $B=2A-1$, vel adeo minor, tum haec ratio cessaret et praecedens valor $\Theta = A-1$ adhuc subsisteret; sin autem tertius terminus *B* superet istum limitem $2A-1$, tum certe statui debebit $\Theta = A$, unitate scilicet maior quam casu praecedente.

3. Removeamus autem tertium terminum *B* ulterius, et cum numerus $2A-1$ requirat *A* terminos, evidens est numerum $3A-1$ requisiturum esse $A+1$ terminos, cum sit $3A-1=A+(2A-1)$, quarum pars prior unum continet terminum, posterior vero *A* terminos; unde si *B* superet hunc limitem $3A-1$, ad eum usque habebimus $\Theta = A+1$. Ulterius autem progrediendo numerus $4A-1$ requiret terminos $A+2$; numerus vero $5A-1$ requiret terminos $A+3$; atque in genere numerus $nA-1$ requiret terminos $A+n-2$, siquidem tertius terminus *B* illo fuerit maior.

4. His colligendis, si tertius terminus *B* contineatur inter binos limites modo assignatos, valorem numeri *e* concludimus sequenti modo:

Si B contineatur intra	erit Θ
$A+n-2A-1$ et $2A-1$	$A-1$
$2A-1$ et $3A-1$	A
$3A-1$ et $4A-1$	$A+1$
$4A-1$ et $5A-1$	$A+2$
.	.
.	.
.	.
$nA-1$ et $(n+1)A-1$	$A+n-2$.

Consequenter, si ponamus esse $B>nA-1$, attamen $B<(n+1)A-1$, quia hinc sequitur $n<\frac{B+1}{A}$ et tamen $n>\frac{B+1}{A}-1$, his valoribus substitutis fiet

$$\Theta<A-2+\frac{B+1}{A}$$

et tamen

$$\Theta>A-3+\frac{B+1}{A}.$$

Sufficit autem pro Θ sumere numerum integrum proxime minorem priori, vel proxime maiorem altera formula. Cum igitur usque ad terminum A fuerit $\Theta=A-1$, usque ad B progrediendo hic valor augmentum accipiet, quod ita se habebit ex formula priore:

$$\Theta<A-1+\frac{B-A+1}{A}$$

ex posteriore autem:

$$\Theta>A-1+\frac{B-2A+1}{A}$$

5. Hucusque nostra investigatio feliciter successit; ut nunc ulterius progrediamur, ponamus numerum hactenus inventum $=\theta$, qui denotat, quot terminis opus est ad omnes numeros ab unitate usque ad terminum B producendos, ita ut sit vel

$$\theta<A-1+\frac{B-A+1}{A},$$

vel

$$\theta>A-1+\frac{B-2A+1}{A},$$

quippe qui numeri tantum ex binis numeris initialibus 1 et A componuntur. Nunc autem admisso etiam tertio termino B, totidem terminis, vel paucioribus, iungendis, ultra B procedamus, donec perveniamus ad numerum $B+b$, quem non amplius ex *{}* terminis componere liceat, sed qui requirat $\theta+1$ terminos; tum vero manifestum est, ulterius progrediendo numerum $2B+b$ requirere $\theta+2$ terminos; similique modo porro numerus $3B+b$ requiret $\theta+3$ terminos; numerus $4B+b$ vero $\theta+4$ terminos; et in genere numerus $nB+b$ requiret $\theta+n$ terminos.

6. Quodsi ergo sequens terminus C non superet limitem $B+b$, numerus θ nullum accipiet augmentum; sin autem maior sit, neque tamen maior secundo limite $2B+b$, ad numerum θ unitas erit addenda; sin autem ulterius crescat, neque tamen limitem $3B+b$ superet, augmentum accedet $=2$; unde palam est, si C excedat limitem $nB+b$, neque tamen ultra sequentem $(n+1)B+b$ crescat, augmentum fore $=n$. Sumamus ergo $C>nB-b$, quia erit $n<\frac{C-B}{b}$, ad hunc usque terminum C multitudo terminorum erit

$$=\theta+n<\theta+\frac{C-b}{B},$$

ubi scilicet numerum integrum proxime minorem capi oportet. Sin autem sumeremus formula $\theta+\frac{C-B-b}{B}$, tum numerus integer proxime maior capi deberet; sicque ad hunc usque terminum C numerus noster quaesitus erit

$$\theta=A-1+\frac{C-2A+1}{A}+\frac{C-B-b}{B},$$

ubi pro utraque fractione numerus integer proxime maior sumi debet.

7. Istum numerum denuo indicemus littera θ, atque ultra C progrediamur, tot vel paucioribus terminis iungendis, donec perveniamus ad numerum $C+c$, quem non amplius producere liceat, sed qui requirat $(\theta+1)$ terminos, ac ratiocinium ut ante instituendo evidens est usque ad terminum sequentem D numerum illum θ augmentum capere $\frac{D-C-c}{C}$, si scilicet loco fractionis numerus integer proxime maior accipiatur; quocirca usque ad hunc terminum nanciscimur numerum nostrum quaesitum:

$$\Theta=A-1+\frac{B-2A+1}{A}+\frac{C-B-b}{B}+\frac{D-C-c}{C}.$$

8. Quodsi simili modo post terminum D pergamus usque ad numerum $D+d$, qui non amplius ex tot terminis componatur, sed uno plus requirat, tum usque ad terminum sequentem E praecedens valor litterae Θ insuper augmentum accipiet $=\frac{E-D-d}{D}$ sicque, quousque libuerit, ulterius procedere licet. Hic autem non est dissimulandum huiusmodi laborem non nisi summa cum molestia suscipi posse; quin etiam, quocunque iam fuerimus progressi, nunquam tamen certi esse poterimus de vero valore ipsius Θ, si progressio adeo in infinitum continuetur. Interim tamen, quousque has operationes produxerimus, semper tuto concludere poterimus pro serie in infinitum continuata verum valorem Θ certe non fore minorem eo, quem invenimus; ac si satis longe fuerimus progressi, plerumque iste valor inventus haud a veritate aberrabit.

9. Ceterum circa illas fractiones, quibus numerum Θ constitui invenimus, notari convenit, si quae earum vel evanescat vel adeo negativa evadat, tum eius valorem esse nullum, quantumvis etiam eius valor negativus fuerit magnus, quoniam ob terminos sequentes valor praecedens nunquam potest diminui; quando autem istae fractiones sunt

positivae, quamdiu earum valores unitatem non superant, eae unitati aequales sunt reputandae; quando autem unitate sunt maiores, neque tamen binarium superent, pro iis 2 scribi oportet, et ita porro. Ex quo intelligere licet, quomodo ex aliquot terminis seriei initialibus verus valor ipsius Θ elici possit, etiamsi series in infinitum continuetur; hoc scilicet eveniet, si omnes fractiones sequentes evadant negativae.

10. Mirum vero non est hoc modo investigationem istam summis plerumque difficultatibus fore implicatam, quandoquidem hic rem nimis generaliter sumus complexi, neque ad ullam legem, qua termini seriei progrediantur, respeximus. Nullum enim est dubium, quin ipsa progressionis lex plurimum ad numerum Θ reperiendum conferat; quemadmodum vidimus in serie generali initio allata, quae omnes numeros figuratos in se complectitur, de qua adfirmare possumus, numerum Θ semper esse $=a+2n-2$; neque tamen adhuc patet quomodo haec determinatio adhuc generalior reddi possit. Ill. quidem BEGUELIN, hoc casu inductus, erat arbitratus simili lege pro omnibus plane seriebus algebraicis ab unitate incipientibus, quarum scilicet terminum generalem formula algebraica exhibere liceat, valorem ipsius Θ sequenti modo exhiberi posse: Sit series pro posita 1, *A*, *B*, *C*, *D*, *E* etc. ad ordinem *n* referenda, et formentur inde successive series differentiarum omnium ordinum, quarum termini initiales sint respective
a, *b*, *c*, *d* etc., ultimae vero differentiae ordinis *n*, quae sunt constantes, sint $=i$, ita ut multitudo numerorum *a*, *b*, *c*, ... *i* sit $=n$; tum vir laudatus putavit fore $\Theta=a+n-1$, quae formula utique cum casu numerorum figuratorum congruit. Cum enim hic sit $a=A-1$, pro illo casu nostrum *a* erit $n+a-1$, cui si addatur ob ordinem seriei numerus $n-1$, ipsa illa formula pro Θ data resultat.

11. Leviter attendenti autem facile erit eiusmodi casus excogitare, quibus ista regula refragetur, veluti evenit in hac serie:

$$1,\ 2,\ 4,\ 7,\ 11,\ 16,\ 22,\ 29 \text{ etc.}$$

cuius differentiae primae $\quad 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$ etc.

et secundae $\quad 1,\ 1,\ 1,\ 1,\ 1,\ 1$ etc.,

quae igitur est ordinis secundi, sive $n=2$ et $a=1$. Secundum regulam igitur deberet esse $\Theta=2$; certum vero est ad producendos omnes numeros ad minimum ternis terminis istius seriei addendis opus esse. Deinceps vero vir illustris significavit se, cum istam regulam perscriberet, omisisse fractionem insuper addendam, quae sit $\frac{i}{a}$, quam quidem semper negligere liceat, si fuerit unitate minor, quemadmodum id in omnibus numeris figuratis usu venit; quando autem ad unitatem vel ultra ascendat, tum eius valorem in integris, negligendo fractionem annexam, superaddi debere. Cum nunc nostro casu differentiae ultimae sint $i=1$, erit $\frac{i}{a}=1$, hincque fiet $\Theta=3$, quod cum veriatate egregie consentit, et nunc quidem fatendum est hanc regulam emendatam non solum numeris figuratis, sed et innumeris aliis progressionum generibus satisfacere.

12. Interim tamen etiam infinita serierum genera exhibere licet, quibus haec regula fallit, id quod plurimum intererit adcuratius ostendisse. Incipiamus ergo a secundo ordine

progressionum algebraicarum, cuius differentiae secundae iam sunt constantes, ita ut hi duo tantum numeri *a* et *b* occurrant, unde earum forma generalis erit:
$1,\ 1+a,\ 1+2a+b,\ 1+3a+3b,\ 1+4a+6b,\ 1+5a+10b$ etc.; et cum sit $n=2$ et $i=b$, secundum regulam illam deberet esse $\Theta = a+1+\frac{b}{a}$, quod, uti iam notavimus, pro numeris polygonalibus, ubi $b=a-1$, egregie convenit. Quin etiam locum habere deprehenditur, quoties numerus *b* non multum superat numerum *a;* statim enim ac fractio $\frac{b}{a}$ fit satis magna, facile ostendi potest hanc regulam a veritate esse aberraturam; namque si sumamus $a=1$ et $b=100$, ut habeatur haec progressio:

$$1,\ 2,\ 103,\ 304,\ 605,\ 1006,\ 1507,\ 2108,\ 2809 \text{ etc.},$$

pro qua regula illa dat $\Theta=102$, si hanc seriem methodo initio exposita examinemus, usque ad tertium 103 colligimus $\Theta=51$; ad quartum terminum 304 usque $\Theta=52$; hinc porro ad sequentem 605 prodit tantum $\Theta=53$; qui valor non amplius augetur, etiamsi ultra sequentem 1006 progrediamur; ex quo facile intelligitur ad terminos ulteriores usque progrediendo hunc numerum vix ultra 54 auctum iri, cum tamen secundum regulam memoratam esse deberet $\Theta=102$; error autem multo magis enormis evadet, si loco *b* numeri maiores accipiantur.

13. Cum hoc casu error sit tantopere enormis, operae pretium erit pro valore $a=1$, unde oritur series

$$1,\ 2,\ 3+b,\ 4+3b,\ 5+6b,\ 6+10b,\ 7+15b,\ 8+21b \text{ etc.}$$

loco *b* successive numeros continuo maiores assumere, et cuilibet harum serierum numerum Θ, ex observationibus erutum, adscribere, ut cum valore regulae $\Theta=2+b$ comparari possit.

b	Series	Θ	error
1	1, 2, 4, 7, 11, 16, 22, ...	3	0
2	1, 2, 5, 10, 17, 26, 37, ...	4	0
3	1, 2, 6, 13, 23, 36, 52, ...	4	1
4	1, 2, 7, 16, 29, 46, 67, ...	5	1
5	1, 2, 8, 19, 35, 56, 82, ...	5	2
6	1, 2, 9, 22, 41, 66, 97, ...	6	2
7	1, 2, 10, 25, 47, 76, 112, ...	6	3
8	1, 2, 11, 28, 53, 86, 127, ...	7	3
9	1, 2, 12, 31, 59, 96, 142, ...	7	4
10	1, 2, 13, 34, 65, 106, 157, ...	8	4

Hinc igitur patet ab initio casibus $b=1$ et $b=2$ regulae errorem esse nullum, hinc vero continuo magis increscere; ac si hos valores attente consideremus, facile concludemus pro hoc serierum genere fore $\Theta=3+\frac{1}{2}b$, unde pro casu $b=100$ fit $\Theta=53$.

14. Consideremus simili modo genus, quo $a=2$, nostraque series erit:

$$1,\ 3,\ 5+b,\ 7+3b,\ 9+6b,\ 11+10b,\ 13+15b,\ 15+21b,\ 17+28b \text{ etc.},$$

pro qua regula Cel. BEGUELIN dat $\Theta = 3+\frac{b}{2}$. Exponamus igitur sequentes series cum valoribus ipsius Θ ut ante:

b	Series	Θ	error
1	1, 3, 6, 10, 15, 21, 28,	3	0
2	1, 3, 7, 13, 21, 31, 43,	4	0
3	1, 3, 8, 16, 27, 41, 58,	4	0
4	1, 3, 9, 19, 33, 51, 73,	5	0
5	1, 3, 10, 22, 39, 61, 88,	5	0
6	I, 3, 11, 25, 45, 71, 103,	5	1
7	1, 3, 12, 28, 51, 81, 118,	5	1
8	1, 3, 13, 31, 57, 91, 133,	6	1
9	1, 3, 14, 34, 63, 101, 148,	6	1
10	1, 3, 15, 37, 69, 111, 163,	7	1

Sin autem sumamus $b = 100$, pro hac serie:

$$1,\ 3,\ 105,\ 307,\ 609,\ 1011,\ 1513,\ 2115 \text{ etc.}$$

colligimus $\Theta = 37$, cum ex illa regula esse deberet $\Theta = 53$.

15. Progrediamur nunc ad progressionum ordinem tertium, ubi $n = 3$, et ternae series differentiarum incipiant ab his numeris: *a, b* et *c,* ita ut $i = c$, et secundum regulam $\Theta = a+2+\frac{c}{a}$; ipsa igitur progressio erit:

$$1,\ 1+a,\ 1+2a+b,\ 1+3a+3b+c,\ 1+4a+6b+4c,$$
$$1+5a+10b+10c,\ 1+6a+15b+20c \text{ etc.}$$

atque hic statim merito suspectum videtur, quod valor ipsius Θ plane non pendeat a littera b ; evidens enim est, si b fuerit numerus praemagnus, etiam numerum Θ non mediocriter auctum iri. Quin etiam in genere affirmare licet numerum Θ minorem certe esse non posse, quam si esset $c = 0$; hoc autem casu per eandem regulam foret $\Theta = a+1+\frac{b}{a}$; quare, quoties haec formula $\Theta = a+1+\frac{b}{a}$ maior est quam $\Theta = a+2+\frac{c}{a}$, regula ista necessario fallere debet, si fuerit $\frac{b}{a} > 1+\frac{c}{a}$ sive $b > a+c$.

16. Ad hoc ostendendum sumamus $a = 2,\ c = 1$ et $b = 6$; unde haec series nascitur:

$$1,\ 3,\ 11,\ 26,\ 49,\ 81,\ 123,\ 176,\ 241 \text{ etc.},$$

pro qua esse deberet $\Theta = 4 + \frac{1}{2} = 4$, cum tamen numerus 21 non in pauciores quam 5 terminos distribui possit; error autem maior erit proditurus, quo magis numerus *b* augebitur.

17. Quemadmodum numerus *b* suppeditavit unum errorum fontem, ita etiam ex numero *c*, si capiatur satis magnus, errores insignes nascentur. Sumamus scilicet $a = 2,\ b = 1,\ c = 10$, ut prodeat ista series:

$$1,\ 3,\ 6,\ 20,\ 55,\ 121,\ 228 \text{ etc.}$$

ideoque per regulam $\Theta = 9$; tentanti autem mox patebit hunc valorem non superare 6 ; ex quo manifestum est pro maioribus numeris *c* errores adhuc maiores esse prodituros; ubi notasse iuvabit, si *b* fuerit numerus praeparvus, regulam errare in defectu; sin autem *c* fuerit numerus praemagnus, in excessu.

18. Tutissima igitur via quicquam certi in hac re concludendi erit sine dubio, ut plures huiusmodi casus tam ex secundo et tertio progressionum ordine, uti fecimus, quam etiam ex sequentibus omni studio explorentur, et qui hunc laborem suscipere voluerit, detegit fortasse regulam quandam latius patentem et certiorem, qua huiusmodi numerorum resolutiones definiri queant.

19. Forsitan etiam non parum iuvabit progressiones geometricas considerasse, pro quibus numerus noster Θ, quo ulterius progrediamur, magis increscit. Ita in progressione dupla $1,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5$ etc. si in termino 2^5 subsistamus, erit $\Theta = 5$; et in genere pro omnibus numeris usque ad 2^n fit $\Theta = n$; tum vero in progressione tripla $1,\ 3,\ 3^2,\ 3^3,\ 3^4$ etc. usque ad terminum 3^n numerus noster reperitur $\Theta = 2n$; atque in genere pro progressione $1,\ m,\ m^2,\ m^3,\ m^4,\ m^5,\ m^6$ etc. usque ad terminum m^n concluditur numerus $\Theta = (m-1)^n$. Hic vero notandum differentiis continuis sumendis earum terminos primos fore $m-1,\ (m-1)^2,\ (m-1)^3$ etc., ita ut litterae supra usurpatae sint hic $a = m-1,\ b = (m-1)^2,\ c = (m-1)^3,\ d = (m-1)^4$ etc.; unde concludi posset fore $\Theta = \frac{a}{1} + \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d}$ etc., haec autem regula in aliis casibus multum falleret. Quin etiam in progressione hypergeometrica WALLISII

$$1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 5040 \text{ etc.}$$

usque ad terminum 6 est $\Theta = 3$; usque ad 24 est $\Theta = 6$; usque ad terminum $1.\ 2.\ 3\ \ldots\ldots n$ erit $\Theta = \frac{n(n-1)}{2}$.

20. Verum etiam series recurrentes memorabilia nobis suppeditant exempla pro numero Θ. Ita pro hac serie notissima:

Indices 0, 1, 2, 3, 4, 5, 6, 7, 8,
termini 1, 2, 3, 5, 8, 13, 21, 34, 55 etc.,

cuius quilibet terminus est summa duorum praecedentium, si usque ad terminum indice n signatum progrediamur, erit $\Theta = \frac{n}{2}$ ubi si n fuerit impar, fractio adiuncta pro unitate est reputanda. Ulteriorem huius argumenti disquisitionem mihi quidem suscipere non vacat.