CONCERNING THE OUTSTANDING USE OF THE METHOD OF INTERPOLATION IN THE THEORY OF SERIES

[E555]

Opuscula Analytica 1, 1783, p. 157-210

In a method of interpolation of this kind, a relation between the two variables x and y is sought, so that, if the values a, b, c, d etc. may be attributed successively to the values of x , thence also the given values p, q, r, s etc. may be allotted to the other variable y , or, what amounts to the same, the equation for a curve of this kind is sought, which may pass through some given number of points. From which therefore, if the number of these points were greater, then the curved line would be constrained more ; yet meanwhile I have observed elsewhere now [E189], even if the number of points may be increased indefinitely, the curve passing through these is not actually to be determined [uniquely], but at this stage, infinitely many curves are able to be shown always, which equally shall be going to pass through the same points. Whereby since the method of interpolation may supply a determined curved line for any case, the most appropriate solution being required always ; truly this individual circumstance suggests a certain nature of the solution found, which deserves a more accurate consideration. But the nature of this solution depends especially on the reasoning, by which the interpolation may be put in place, or on the form, which is attributed to the general equation, in which the equation sought may be contained. Which form since may be composed in an infinite number of ways, I may restrict my investigations to this form :

$$y=\alpha x+\beta x^3+\gamma x^5+\delta x^7+\varepsilon x^9+\text{etc.},$$

which clearly may contain only odd powers of x, thus so that, any values of y which may agree with the positive values of x, the same taken with negative values may correspond with the same negative values of x ; whereby innumerable other curved lines are excluded, which might be present going to pass through the same points.

PROBLEM 1

1. *To find an equation between the two variables* x *and* y *of this form*

$$y=\alpha x+\beta x^3+\gamma x^5+\delta x^7+\text{etc.},$$

so that, if the given values a, b, c, d *etc. may be attributed to x itself, likewise it may follow the given values* p, q, r, s *etc. be attributed to the variable y.*

SOLUTION

So that the assumed general equation may be adapted more easily to this case, this form may be shown here :

$$\begin{aligned} y = Ax + Bx(xx-aa) &+ Cx(xx-aa)(xx-bb) \\ &+ Dx(xx-aa)(xx-bb)(xx-cc) \\ &+ Ex(xx-aa)(xx-bb)(xx-cc)(xx-dd) \\ &+ \text{etc.}, \end{aligned}$$

which, even if perhaps it may progress to infinity, clearly if the number of conditions shall be infinite, yet for the individual conditions proposed the following equations may be assumed finite:

I. $p = Aa,$

II. $q = Ab + Bb(bb-aa),$

III. $r = Ac + Bc(cc-aa) + Cc(cc-aa)(cc-bb),$

IV. $s = Ad + Bd(dd-aa) + Cd(dd-aa)(dd-bb) + dd-aa)(dd-bb)(dd-cc)$

etc.,

which may be represented thus:

I. $\frac{p}{a} = A,$

II. $\frac{q}{b} = A + B(bb-aa),$

III. $\frac{r}{c} = A + B(cc-aa) + C(cc-aa)(cc-bb),$

IV. $\frac{s}{d} = A + B(dd-aa) + C(dd-aa)(dd-bb) + D(dd-aa)(dd-bb)(dd-cc)$

etc.

Now the first equation may be taken from the following individual equations, and the differences divided by the coefficients of *B*, so that these equations may be produced :

$$\frac{aq-bp}{ab(bb-aa)} = q' = B,$$

$$\frac{ar-cp}{ac(cc-aa)} = r' = B + C(cc-bb),$$

$$\frac{as-dp}{ad(dd-aa)} = s' = B + C(dd-bb) + D(dd-bb)(dd-cc)$$

etc.

Again in a similar manner, subtracting the first equation from the following, and dividing the remainders themselves by the coefficients of *C,* we will arrive at these equations:

$$\frac{r'-q'}{cc-bb}=r''=C,$$
$$\frac{s'-q'}{dd-bb}=s''=C+D(dd-cc), \text{ etc.}$$

and again to this :

$$\frac{s''-r''}{dd-cc}=D.$$

Thus on account of which, the coefficients *A, B, C, D* etc. will be defined most conveniently from the given quantities *a, b, c, d* etc. and *p, q, r, s* etc. : In the first place, these :

$$P=\frac{p}{a},\ Q=\frac{q}{b},\ R=\frac{r}{c},\ S=\frac{s}{d}, \text{ etc.}$$

may be derived from these given quantities, and hence these equations will be formed:

$$Q'=\frac{Q-P}{bb-aa},\ \ R'=\frac{R-P}{cc-aa},\ \ S'=\frac{S-P}{dd-aa},\ \ T'=\frac{T-P}{ee-aa}, \text{ etc.},$$
$$R''=\frac{R'-Q'}{cc-bb},\ \ S''=\frac{S'-Q'}{dd-bb},\ T''=\frac{T'-Q}{ee-bb}, \text{ etc.},$$
$$S'''=\frac{S''-R''}{dd-cc},\ T'''=\frac{T''-R''}{ee-cc}, \text{ etc.},$$
$$T''''=\frac{T'''-S'''}{ee-dd}, \text{ etc.}$$

From which values found, we will have :

$$A=P,\ B=Q',\ C=R'',\ D=S''',\ E=T'''' \text{ etc.}$$

COROLLARY 1

2. Since there shall be $P=\frac{p}{a}$, the first coefficient will be

$$A=\frac{p}{a},$$

truly for the following, on account of

$$Q'=\frac{aq-bp}{ab(bb-aa)},\ \ R'=\frac{ar-cp}{ac(cc-aa)},\ \ S'=\frac{as-dp}{ad(dd-aa)},\ \ T'=\frac{at-ep}{ae(ee-aa)} \ \text{ etc.}$$

the second coefficient will be

$$B=\frac{aq-bp}{ab(bb-aa)},$$

or

$$B=\frac{p}{a(aa-bb)}+\frac{q}{b(bb-aa)}.$$

COROLLARY 2

3. Again since there shall be

$$R" = \frac{ar-cp}{ac(cc-aa)(cc-bb)} - \frac{aq-bp}{ab(bb-aa)(cc-bb)},$$

there becomes

$$C = \frac{p}{a(aa-bb)(aa-cc)} + \frac{q}{b(bb-aa)(bb-cc)} + \frac{r}{c(cc-aa)(cc-bb)}.$$

COROLLARY 3

4. In a similar manner, with the calculation proceeding further, there will be found

$$D = \frac{p}{a(aa-bb)(aa-cc)(aa-dd)} + \frac{q}{b(bb-aa)(bb-cc)(bb-dd)}$$
$$+ \frac{r}{c(cc-aa)(cc-bb)(cc-dd)} + \frac{s}{d(dd-aa)(dd-bb)(dd-cc)},$$

from which the form of the following quantities *E, F, G* etc. will be allowed to follow on now without risk, by conjecture .

SCHOLIUM 1

5. Moreover in general the values of the individual coefficients *A, B, C, D, E* etc. are defined more readily from the preceding. Indeed the following formulas are deduced from the fundamental equations:

$$A = \frac{p}{a},$$

$$B = \frac{q-bA}{b(bb-aa)},$$

$$C = \frac{r-cA}{c(cc-aa)(cc-bb)} - \frac{B}{cc-bb}.$$

$$D = \frac{s-dA}{d(dd-aa)(dd-bb)(dd-cc)} - \frac{B}{(dd-bb)(dd-cc)} - \frac{C}{dd-cc}$$

$$E = \frac{t-eA}{e(ee-aa)(ee-bb)(ee-cc)(ee-dd)} - \frac{B}{(ee-bb)(ee-cc)(ee-dd)}$$
$$- \frac{C}{(ee-cc)(ee-dd)} - \frac{D}{ee-dd}$$

etc.,

where generally the arrangement of this kind may be observed soon, from which the following can be derived easily, just as will be apparent from the following problems, in which I am going to apply this method to certain particular cases.

SCHOLIUM 2

6. But before I may establish cases of this kind, it will help to have observed in general , how, if for some case presented, the satisfying equation were found between the two variables x and y , which I will designate in this manner :

$$y = X,$$

thus so that there shall become :

$$x = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \text{etc.}$$

then thence an equation can be formed easily, extended much more widely, and equally satisfying . For there may be put :

$$Q = x \cdot \tfrac{xx-aa}{aa} \cdot \tfrac{xx-bb}{bb} \cdot \tfrac{xx-cc}{cc} \cdot \tfrac{xx-dd}{dd} \cdot \text{etc.},$$

which quantity vanishes for all the proposed values of x :

$$x = 0, \;\; x = \pm a, \;\; x = \pm b, \;\; x = \pm c \;\; \text{etc.},$$

and likewise with all the outstanding functions of Q vanishing, together with Q itself ; from which it is evident, if there may be taken:

$$y = X + Q$$

or

$$y = X + f : Q,$$

for all the conditions to be satisfied equally. Therefore since this function $f : Q$ is arbitrary generally, provided that it may vanish on putting $Q = 0$, this equation is considered to show the most general solution:

$$y = X + f : Q,$$

PROBLEM 2

7. *Let a, b, c, d etc. be some circular arcs with the radius being* $=1$, *moreover the values p, q, r, s etc. shall be the sines of the same arcs, on account of which in this case that property is established, in order that the same sines assumed negative may correspond to negative arcs, hence the ratio between the diameter and the circumference may be defined approximately.*

SOLUTION

Since here there shall be

$$p = \sin.a, \;\; q = \sin.b, \;\; r = \sin.c \;\; \text{etc.},$$

an equation between x and y will be prepared thus, so that with the circular arc taken for x, the quantity y shall be given approximately its sine, and there may be expressed and become

$$y = \sin.x.$$

Therefore with the coefficients A, B, C, D etc. defined by the preceding problem, this equation will be obtained :

$$\sin.x = Ax + Bx(xx-aa) + Cx(xx-aa)(xx-bb) + \text{etc.},$$

which is agreed to be true, therefore, as long as there were either

$$x = 0,\ x = \pm a,\ x = \pm b, \text{ or } x = \pm c \text{ etc.}$$

Now we may establish an infinitely small arc x, and then because its sine, sin. x, may be equal to the arc x itself, this equation will arise :

$$1 = A - Baa + Caabb - Daabbcc + Eaabbccdd - \text{etc.}$$

We may substitute here the above values found for the letters A, B, C, D etc., and we will come upon this equation:

$$\begin{aligned}
1 = {} & \tfrac{p}{a}\left(1 - \tfrac{aa}{aa-bb} + \tfrac{aabb}{(aa-bb)(aa-cc)} - \tfrac{aabbcc}{(aa-bb)(aa-cc)(aa-dd)} + \text{etc.}\right) \\
& - \tfrac{q}{b}\left(\tfrac{aa}{bb-aa} - \tfrac{aabb}{(bb-aa)(bb-cc)} + \tfrac{aabbcc}{(bb-aa)(bb-cc)(bb-dd)} - \text{etc.}\right) \\
& + \tfrac{r}{c}\left(\tfrac{aabb}{(cc-aa)(cc-bb)} - \tfrac{aabbcc}{(cc-aa)(cc-bb)(cc-dd)} + \text{etc.}\right) \\
& - \tfrac{s}{d}\left(\tfrac{aabbcc}{(dd-aa)(dd-bb)(dd-cc)} - \text{etc.}\right) \\
& + \text{etc.},
\end{aligned}$$

which is reduced to this, in which all the series are similar between themselves,

$$\begin{aligned}
1 = {} & \tfrac{p}{a}\left(1 - \tfrac{aa}{bb-aa} + \tfrac{aabb}{(bb-aa)(cc-aa)} + \tfrac{aabbcc}{(bb-aa)(cc-aa)(dd-aa)} + \text{etc.}\right) \\
& - \tfrac{aaq}{b(bb-aa)}\left(1 + \tfrac{bb}{cc-bb} + \tfrac{bbcc}{(cc-bb)(dd-bb)} + \tfrac{bbccdd}{(cc-bb)(dd-bb)(ee-bb)} + \text{etc.}\right) \\
& + \tfrac{aabbr}{c(cc-aa)(cc-bb)}\left(1 + \tfrac{cc}{dd-cc} + \tfrac{ccdd}{(dd-cc)(ee-cc)} + \text{etc.}\right) \\
& - \tfrac{aabbccs}{d(dd-aa)(dd-bb)(dd-cc)}\left(1 + \tfrac{dd}{ee-dd} + \text{etc.}\right) \\
& + \text{ etc.}
\end{aligned}$$

But any of these series is summable at once; for the first two terms of the first taken together give :

$$\frac{bb}{bb-aa};$$

to which, if the third may be added, it produces :

$$\frac{bbcc}{(bb-aa)(cc-aa)}$$

and hence again the fourth added, gives :

$$\frac{bbccdd}{(bb-aa)(cc-aa)(dd-aa)}$$

and so on thus, so that the first series of our equation may emerge :

$$\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}$$

Truly for the second, there is found in a similar manner :

$$-\frac{q}{b}\cdot\frac{aa}{bb-aa}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}$$

and thus finally our equation is reduced to this form :

$$\begin{aligned}1=&\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}\\&+\frac{q}{b}\cdot\frac{aa}{aa-bb}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}\\&+\frac{r}{c}\cdot\frac{aa}{aa-cc}\cdot\frac{bb}{bb-cc}\cdot\frac{dd}{dd-cc}\cdot\frac{ee}{ee-cc}\cdot\text{etc.}\\&+\frac{s}{d}\cdot\frac{aa}{aa-dd}\cdot\frac{bb}{bb-dd}\cdot\frac{cc}{cc-dd}\cdot\frac{ee}{ee-dd}\cdot\text{etc.}\\&+\frac{t}{e}\cdot\frac{aa}{aa-ee}\cdot\frac{bb}{bb-ee}\cdot\frac{cc}{cc-ee}\cdot\frac{dd}{dd-ee}\cdot\text{etc.}\\&+\ \text{etc.}\end{aligned}$$

so that, if the given arcs *a, b, c, d* etc. may maintain a known ratio π to the semi-circumference, the value π if this quantity will be defined.

COROLLARY 1

8. If the number of these arcs *a, b, c, d* etc. were finite, then the circumference of the circle will be defined more accurately there, where that number shall be greater and likewise in which smaller arcs may occur between these. Moreover with the number of the proposed arcs increased to infinity, the true ratio of the circumference to the diameter thence will be derived.

COROLLARY 2

9. In general in a similar manner the sine of an indefinite arc x is able to be defined. For with the values found substituted in place of the coefficients A, B, C, D etc. the equation is reduced to this form

$$\begin{aligned}\frac{\sin.x}{x} &= \frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\text{etc.}\\ &+\ \text{etc.}\end{aligned}$$

which equation with the arc x assumed vanishing will go into that form.

COROLLARY 3

10. But this reduction may be extended much more wider, with nothing had in the ratio of the arcs. For if an equation of this kind may be sought between the two variables x and y, so that on accepting

$$x = 0,\ a,\ b,\ c,\ d,\ e\ \text{ etc.}$$

there may become

$$y = 0,\ p,\ q,\ r,\ s,\ t\ \text{etc.},$$

the equation thus will be able to be represented in this form :

$$\begin{aligned}\frac{y}{x} &= \frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\frac{ee-xx}{ee-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\frac{ee-xx}{ee-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\frac{ee-xx}{ee-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\frac{ee-xx}{ee-dd}\cdot\text{etc.}\\ &+\ \text{etc.},\end{aligned}$$

from which form it is evident likewise, how it may be satisfied by the particular conditions.

SCHOLIUM

11. I will not dwell on these cases here, for which the number of prescribed conditions a, b, c, d etc. is assumed finite, since thence only approximations are being supplied for the measure of the circle. Yet meanwhile it will not be any bother to have observed, if only four arcs may be taken, which shall be :

$$a=\varphi,\ b=2\varphi,\ c=3\varphi,\ d=4\varphi,$$

there shall be from the solution of the problem,

$$\begin{aligned}\varphi &= \frac{\sin.\varphi}{1}\cdot\frac{2\cdot2}{1\cdot3}\cdot\frac{3\cdot3}{2\cdot4}\cdot\frac{4\cdot4}{3\cdot5}\\ &-\frac{\sin.2\varphi}{2}\cdot\frac{1\cdot1}{1\cdot3}\cdot\frac{3\cdot3}{1\cdot5}\cdot\frac{4\cdot4}{2\cdot6}\\ &+\frac{\sin.3\varphi}{3}\cdot\frac{1\cdot1}{2\cdot4}\cdot\frac{2\cdot2}{1\cdot5}\cdot\frac{4\cdot4}{1\cdot7}\\ &-\frac{\sin.4\varphi}{4}\cdot\frac{1\cdot1}{3\cdot5}\cdot\frac{2\cdot2}{2\cdot6}\cdot\frac{3\cdot3}{1\cdot7}\\ &=\tfrac{8}{5}\sin.\varphi-\tfrac{2}{5}\sin.2\varphi+\tfrac{8}{105}\sin.3\varphi-\tfrac{1}{140}\sin.4\varphi,\end{aligned}$$

which expression therefore approaches closer to the truth, when the arc φ may be taken smaller; yet meanwhile, even if it may be increased as far as to the quadrant, so that there shall be

$$\varphi=\tfrac{\pi}{2},$$

the error shall not become immense ; for there will be produced

$$\tfrac{\pi}{2}=\tfrac{8}{5}-\tfrac{8}{105}=\tfrac{32}{21}$$

and thus

$$\pi=3\tfrac{1}{21}.$$

But if we may take

$$\varphi=30^0=\tfrac{\pi}{6},$$

there becomes

$$\tfrac{\pi}{6}=\tfrac{8}{5}\cdot\tfrac{1}{2}-\tfrac{2}{5}\cdot\tfrac{\sqrt{3}}{2}+\tfrac{8}{105}-\tfrac{1}{140}\cdot\tfrac{\sqrt{3}}{2}$$

or

$$\pi=\tfrac{184}{35}-\tfrac{171\sqrt{3}}{140},$$

which value differs from the true value by only some hundred thousands of one [*i.e.* 3.1415665…] . Truly I shall run through some cases derived from this speculation, where the number of the proposed arcs *a, b, c, d* etc. is infinite, for a certain law of progression.

EXAMPLE I

12. *The arcs a, b, c, d etc. may be progressing following the series of natural numbers, and there shall be*

$$a=\varphi,\quad b=2\varphi,\quad c=3\varphi,\quad d=4\varphi \text{ etc. indefinitely;}$$

and truly from the sines of which p, q, r etc., the length of the arc φ must be determined.

Therefore the solution of the problem for this case supplies this equation :

$$\begin{aligned}\varphi = &\frac{\sin.\varphi}{1}\cdot\frac{2\cdot2}{1\cdot3}\cdot\frac{3\cdot3}{2\cdot4}\cdot\frac{4\cdot4}{3\cdot5}\cdot\frac{5\cdot5}{4\cdot6}\cdot\text{etc.}\\ &-\frac{\sin.2\varphi}{2}\cdot\frac{1\cdot1}{1\cdot3}\cdot\frac{3\cdot3}{1\cdot5}\cdot\frac{4\cdot4}{2\cdot6}\cdot\frac{5\cdot5}{3\cdot7}\cdot\text{etc.}\\ &+\frac{\sin.3\varphi}{3}\cdot\frac{1\cdot1}{2\cdot4}\cdot\frac{2\cdot2}{1\cdot5}\cdot\frac{4\cdot4}{1\cdot7}\cdot\frac{5\cdot5}{2\cdot8}\cdot\text{etc.}\\ &-\frac{\sin.4\varphi}{4}\cdot\frac{1\cdot1}{3\cdot5}\cdot\frac{2\cdot2}{2\cdot6}\cdot\frac{3\cdot3}{1\cdot7}\cdot\frac{5\cdot5}{1\cdot9}\cdot\text{etc.}\\ &+\frac{\sin.5\varphi}{5}\cdot\frac{1\cdot1}{4\cdot6}\cdot\frac{2\cdot2}{3\cdot7}\cdot\frac{3\cdot3}{2\cdot8}\cdot\frac{4\cdot4}{1\cdot9}\cdot\text{etc.}\\ &+\ \text{etc.};\end{aligned}$$

but all these products are found to have the same value $=2$, thus so that there shall be

$$\tfrac{1}{2}\varphi = \sin.\varphi - \tfrac{1}{2}\sin.2\varphi + \tfrac{1}{3}\sin.3\varphi - \tfrac{1}{4}\sin.4\varphi + \tfrac{1}{5}\sin.5\varphi - \text{etc.},$$

the truth of which series in the case, where the angle φ is infinitely small, by itself is evident.
Therefore we may set out the following cases :

1. Let

$$\varphi = 90^0 = \tfrac{\pi}{2},$$

and the Leibniz series will be produced [*vis* : *Acta Erud*. 1682, *De vera proportione…* , and also on this website] :

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.}$$

2. Let

$$\varphi = 45^0 = \tfrac{\pi}{4}$$

and this series will arise :

$$\tfrac{\pi}{8} = \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3\sqrt{2}} * - \frac{1}{5\sqrt{2}} + \frac{1}{6} - \frac{1}{7\sqrt{2}} * + \frac{1}{\sqrt{2}} - \frac{1}{10} + \frac{1}{11\sqrt{2}} - \text{etc.},$$

which is resolved into these two

$$\begin{aligned}\tfrac{\pi}{8} = &\tfrac{1}{\sqrt{2}}(1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15} + \text{etc.})\\ &-\tfrac{1}{2}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}),\end{aligned}$$

thus so that there shall be

$$1+\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\text{etc.}=\tfrac{\pi}{2\sqrt{2}}.$$

3. Let

$$\varphi=60^0=\tfrac{\pi}{3}$$

and there will become :

$$\tfrac{\pi}{6}=\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}\cdot\tfrac{\sqrt{3}}{2}*+\tfrac{1}{4}\cdot\tfrac{\sqrt{3}}{2}-\tfrac{1}{5}\cdot\tfrac{\sqrt{3}}{2}+\text{etc.}$$

or

$$\tfrac{\pi}{3\sqrt{3}}=1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{10}-\tfrac{1}{11}+\text{etc.}$$

4. Let

$$\varphi=30^0=\tfrac{\pi}{6}$$

and there becomes:

$$\tfrac{\pi}{12}=\tfrac{1}{2}-\tfrac{1}{2}\cdot\tfrac{\sqrt{3}}{2}+\tfrac{1}{3}-\tfrac{1}{4}\cdot\tfrac{\sqrt{3}}{2}+\tfrac{1}{5}\cdot\tfrac{1}{2}*-\tfrac{1}{7}\cdot\tfrac{1}{2}+\tfrac{1}{8}\cdot\tfrac{\sqrt{3}}{2}-\text{etc.}$$

or

$$\begin{aligned}\tfrac{\pi}{12}=&\tfrac{1}{2}(1+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+\text{etc.})\\&-\tfrac{\sqrt{3}}{4}(1+\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{8}-\tfrac{1}{10}-\tfrac{1}{11}+\text{etc.})\\&+\tfrac{1}{3}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\text{etc.}),\end{aligned}$$

of which the series becomes finally $=\tfrac{\pi}{12}$; hence it is concluded :

$$1+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\text{etc.}=\tfrac{\sqrt{3}}{2}(1+\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{8}-\text{etc.})$$

Moreover each series is equal to the arc $\tfrac{\pi}{3}$, which indeed initially was established by Leibniz.

COROLLARY 1

13. From the equation found here :

$$\tfrac{1}{2}\varphi=\sin.\varphi-\tfrac{1}{2}\sin.2\varphi+\tfrac{1}{3}\sin.3\varphi-\tfrac{1}{4}\sin.4\varphi+\tfrac{1}{5}\sin.5\varphi-\text{etc.},$$

several others not less noteworthy can be derived. Just as with the differential put in place there is produced :

$$\tfrac{1}{2}=\cos.\varphi-\cos.2\varphi+\cos.3\varphi-\cos.4\varphi+\text{etc.},$$

the ratio of which is evident, because by multiplying each side by $2\cos.\tfrac{1}{2}\varphi$ the identity equation is produced $\cos.\tfrac{1}{2}\varphi=\cos.\tfrac{1}{2}\varphi$.

COROLLARY 2

14. But if we may integrate that equation multiplied by $-d\varphi$, there comes about

$$C-\tfrac{1}{4}\varphi\varphi=\cos.\varphi-\tfrac{1}{4}\cos.2\varphi+\tfrac{1}{9}\cos.3\varphi-\tfrac{1}{16}\cos.4\varphi+\text{etc.},$$

where in the case $\varphi=0$ the constant introduced by the integration is determined, evidently

$$C=1-\tfrac{1}{4}+\tfrac{1}{9}-\tfrac{1}{16}+\text{etc.}=\tfrac{\pi\pi}{12},$$

thus so that there shall be

$$\tfrac{\pi\pi}{12}-\tfrac{1}{4}\varphi\varphi=\cos.\varphi-\tfrac{1}{4}\cos.2\varphi+\tfrac{1}{9}\cos.3\varphi-\tfrac{1}{16}\cos.4\varphi+\text{etc.},$$

which series on taking $\varphi=\frac{\pi}{\sqrt{3}}$ becomes $=0$. Moreover there is approximately,

$$\tfrac{\pi}{\sqrt{3}}=103^\circ\ 55'\ 23''\ \text{and}\ \cos.\tfrac{\pi}{\sqrt{3}}=-\ 0,2406185.$$

COROLLARY 3

15. If we may again integrate this equation multiplied by $d\varphi$, this new summation will arise :

$$\tfrac{1}{12}\pi\pi\varphi-\tfrac{1}{12}\varphi^3=\sin.\varphi-\tfrac{1}{8}\sin.2\varphi+\tfrac{1}{27}\sin.3\varphi-\tfrac{1}{64}\sin.4\varphi+\text{etc.},$$

from which on taking the arc

$$\varphi=90^0=\tfrac{\pi}{2}$$

there will be obtained :

$$\tfrac{1}{32}\pi^3=1-\tfrac{1}{27}+\tfrac{1}{125}-\tfrac{1}{343}+\text{etc.},$$

as now has been noted elsewhere.

SCHOLIUM

16. Doubt can arise concerning the series found :

$$\tfrac{1}{2}\varphi=\sin.\varphi-\tfrac{1}{2}\sin.2\varphi+\tfrac{1}{3}\sin.3\varphi-\text{etc.},$$

because with the arc taken $\varphi = 180° = \pi$, the individual terms of the series vanish and thus the sum cannot be equal to $\frac{1}{2}\pi$. Truly towards resolving this doubt at first there may be put $\varphi = \pi - \omega$ and this equation will result :

$$\tfrac{\pi-\omega}{2} = \sin.\omega + \tfrac{1}{2}\sin.2\omega + \tfrac{1}{3}\sin.3\omega + \tfrac{1}{4}\sin.4\omega + \text{etc.}$$

now truly the arc ω may be taken infinitely small, from which we come upon this equation:

$$\tfrac{\pi-\omega}{2} = \omega + \omega + \omega + \omega + \omega + \text{ etc.},$$

which holds no further absurdity. Just as the same situation arises, if we wish to take $\varphi = 2\pi$ or $\varphi = 3\pi$ etc. [Thus, a note of perplexity creeps into Euler's derivations here, where evidently the series does not always converge to the stated value.]

EXAMPLE II

17. *If the arcs a, b, c, d etc. may constitute some arithmetical progression, so that there shall become*

$$a = n\varphi,\ b = (n+1)\varphi, c = (n+2)\varphi, d = (n+3)\varphi,\ \ etc.,$$

to define the length of the arc φ from the sines of these.

The general solution shown before for this case gives

$$\begin{aligned}\varphi = &\frac{\sin.n\varphi}{n}\cdot\frac{(n+1)^2}{1(1+2n)}\cdot\frac{(n+2)^2}{2(2+2n)}\cdot\frac{(n+3)^2}{3(3+2n)}\cdot\frac{(n+4)^2}{4(4+2n)}\cdot\frac{(n+5)^2}{5(5+2n)}\cdot\text{etc.}\\ &-\frac{\sin.(n+1)\varphi}{n+1}\cdot\frac{n^2}{1(1+2n)}\cdot\frac{(n+2)^2}{1(3+2n)}\cdot\frac{(n+3)^2}{2(4+2n)}\cdot\frac{(n+4)^2}{3(5+2n)}\cdot\frac{(n+5)^2}{4(6+2n)}\cdot\text{etc.}\\ &+\frac{\sin.(n+2)\varphi}{n+2}\cdot\frac{n^2}{2(2+2n)}\cdot\frac{(n+1)^2}{1(3+2n)}\cdot\frac{(n+3)^2}{1(5+2n)}\cdot\frac{(n+4)^2}{2(6+2n)}\cdot\frac{(n+5)^2}{3(7+2n)}\cdot\text{etc.}\\ &-\frac{\sin.(n+3)\varphi}{n+3}\cdot\frac{n^2}{3(3+2n)}\cdot\frac{(n+1)^2}{2(4+2n)}\cdot\frac{(n+2)^2}{1(5+2n)}\cdot\frac{(n+4)^2}{1(7+2n)}\cdot\frac{(n+5)^2}{2(8+2n)}\cdot\text{etc.}\\ &+\frac{\sin.(n+4)\varphi}{n+4}\cdot\frac{n^2}{4(4+2n)}\cdot\frac{(n+1)^2}{3(5+2n)}\cdot\frac{(n+2)^2}{2(6+2n)}\cdot\frac{(n+3)^2}{1(7+2n)}\cdot\frac{(n+5)^2}{1(9+2n)}\cdot\text{etc.}\\ &+\qquad\qquad\text{etc.}\end{aligned}$$

We may put, for the sake of brevity, the values of these products requiring to be investigated departing to infinity

$$\varphi = \mathfrak{A}\frac{\sin.n\varphi}{n} - \mathfrak{B}\frac{\sin.(n+1)\varphi}{n+1} + \mathfrak{C}\frac{\sin.(n+2)\varphi}{n+2} - \mathfrak{D}\frac{\sin.(n+3)\varphi}{n+3} + \text{etc.}$$

and these coefficients may be compared amongst themselves in the following manner

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{nn}{(n+1)^2}\cdot\frac{2(2+2n)}{1(3+2n)}\cdot\frac{3(3+2n)}{2(4+2n)}\cdot\frac{4(4+2n)}{3(5+2n)}\cdot\text{etc.}$$

which value is reduced to

$$\frac{nn}{(n+1)^2}\cdot\frac{(i-1)(2+2n)}{1(i+2n)},$$

with i denoting some infinite number, and thus there will be

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{2nn}{n+1}.$$

There is deduced in a similar manner

$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1(1+2n)}{2(2+2n)}\cdot\frac{(n+1)^2}{(n+2)^2}\cdot\frac{(i-3)(4+2n)}{1(i+2n)} = \frac{(n+1)(2n+1)}{2(n+2)},$$

then again truly,

$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2)(2n+2)}{3(n+3)},\quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3)(2n+3)}{4(n+4)}$$

and thus henceforth; from which there follows to become

$$\mathfrak{B} = \frac{2nn}{1(n+1)}\mathfrak{A},$$
$$\mathfrak{C} = \frac{2nn(2n+1)}{1\cdot 2(n+2)}\mathfrak{A},$$
$$\mathfrak{D} = \frac{2nn(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)}\mathfrak{A},$$
$$\mathfrak{E} = \frac{2nn(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)}\mathfrak{A}$$
etc.

and thus the whole concern reduces to the finding of the first letter

$$\mathfrak{A} = \frac{(n+1)^2}{1(2n+1)}\cdot\frac{(n+2)^2}{2(2n+2)}\cdot\frac{(n+3)^2}{3(2n+3)}\cdot\frac{(n+3)^2}{4(2n+4)}\cdot\text{etc.}$$

But now I have shown formerly, the value of this general product to be expressed thus :

$$\frac{a(b+c)}{b(a+c)}\cdot\frac{(a+d)(b+c+d)}{(b+d)(a+c+d)}\cdot\frac{(a+2d)(b+c+2d)}{(b+2d)(a+c+2d)}\cdot\text{etc.},$$

so that it shall become

$$= \frac{\int x^{b-1}dx(1-x^d)^{\frac{c-d}{d}}}{\int x^{a-1}dx(1-x^d)^{\frac{c-d}{d}}}$$

clearly with each integration extended from the limit $x = 0$ to the limit $x = 1$. Whereby since in our case it may be required to take

$$a = n+1,\ b+c = n+1,\ b = 1,\ c = n \ \text{ and } \ d = 1,$$

we will have

$$\mathfrak{A} = \frac{\int dx(1-x)^{n-1}}{\int x^n dx(1-x)^{n-1}} = \frac{1}{n\int x^n dx(1-x)^{n-1}}$$

and hence the following expression for the arc φ:

$$\begin{aligned}\varphi\int x^n dx(1-x)^{n-1} &= \tfrac{1}{nn}\sin.n\varphi - \tfrac{2n}{1(n+1)^2}\sin.(n+1)\varphi \\ &+ \tfrac{2n(2n+1)}{1\cdot 2(n+2)^2}\sin.(n+2)\varphi - \tfrac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}\sin.(n+3)\varphi \\ &+ \tfrac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)^2}\sin.(n+4)\varphi + \text{etc.}\end{aligned}$$

Which series therefore deserves more attention, because it involves the integral formula

$$\int x^n dx(1-x)^{n-1}.$$

COROLLARY 1

18. Concerning this integral formula

$$\int x^n dx(1-x)^{n-1}$$

initially it will help to have observed, if in the case $n = \lambda$ its value were $= \varDelta$, then that case

$$n = \lambda + 1$$

to become

$$= \tfrac{\lambda}{2(2\lambda+1)}\varDelta.$$

Thus, since in the case $n = 1$ there shall be

$$\int xdx = \tfrac{1}{2},$$

there will become

$$\int x^2(1-x)dx = \tfrac{1}{2}\cdot\tfrac{1}{2\cdot 3},\quad \int x^3(1-x)^2 dx = \tfrac{1}{2}\cdot\tfrac{1}{2\cdot 3}\cdot\tfrac{2}{2\cdot 5} \ \text{ etc.}$$

COROLLARY 2

19. Therefore if in general there may be put

$$\int x^n dx(1-x)^{n-1} = f:n,$$

since its value can be regarded as a function of n , there will become

$$f:1=\tfrac{1}{2},\ f:2=\tfrac{1}{2}\cdot\tfrac{1}{6},\quad f:3=\tfrac{1}{2}\cdot\tfrac{1}{6}\cdot\tfrac{2}{10},\quad f:4=\tfrac{1}{2}\cdot\tfrac{1}{6}\cdot\tfrac{2}{10}\cdot\tfrac{3}{14}$$

and in general

$$f:(n+1)=\tfrac{2}{2(2n+1)}f:n.$$

from which, as often as n is a whole number, the value of this formula $f:n$ is assigned easily.

COROLLARY 3

20. Now let there be $n=\tfrac{1}{2}$, and there will be

$$f:\tfrac{1}{2}=2\int\tfrac{dx\sqrt{x}}{\sqrt{(1-x)}}=2\int\tfrac{yydy}{\sqrt{(1-yy)}}$$

on putting $x=yy$*;* but

$$\int\tfrac{yydy}{\sqrt{(1-yy)}}=\tfrac{1}{2}\int\tfrac{dy}{\sqrt{(1-yy)}}=\tfrac{\pi}{4},$$

from which there becomes

$$f:\tfrac{1}{2}=\tfrac{\pi}{2}$$

and hence again:

$$f:\tfrac{3}{2}=\tfrac{1}{8}\cdot\tfrac{\pi}{2},\ f:\tfrac{5}{2}=\tfrac{1}{8}\cdot\tfrac{3}{16}\cdot\tfrac{\pi}{2},\ f:\tfrac{7}{2}=\tfrac{1}{8}\cdot\tfrac{3}{16}\cdot\tfrac{5}{24}\cdot\tfrac{\pi}{2}\ \text{ etc.}$$

But if in general there shall be $n=\tfrac{\mu}{\nu}$, there is found :

$$f:\tfrac{\mu}{\nu}=\int x^{\frac{\mu}{\nu}}dx(1-x)^{\frac{\mu}{\nu}-1}=\nu\int y^{\mu+\nu-1}dy(1-y^{\nu})^{\frac{\mu}{\nu}-1}$$

on putting $x=y^{\nu}$, and hence with the reduction made,

$$f:\tfrac{\mu}{\nu}=\tfrac{\nu}{2}\int y^{\mu-1}dy(1-y^{\nu})^{\frac{\mu}{\nu}-1},$$

which form involves all the general transcending quantities.

COROLLARY 4

21. But the value of this integral formula

$$\int x^{n}dx(1-x)^{n-1}$$

in the case $x=1$ in turn may be determined from the series elegantly enough; for with the differentiation made by regarding the arc φ alone as the variable, it produces :

$$\int x^{n}dx(1-x)^{n-1}=\tfrac{1}{n}\cos.n\varphi-\tfrac{2n}{1(n+1)}\cos.(n+1)\varphi$$
$$+\tfrac{2n(2n+1)}{1\cdot 2(n+2)}\cos.(n+2)\varphi-\tfrac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)}\cos.(n+3)\varphi+\text{etc.},$$

which therefore is equal to this series, and from which the customary expansion arises

$$\int x^{n}dx(1-x)^{n-1}=\tfrac{1}{n+1}-\tfrac{n-1}{1(n+2)}+\tfrac{(n-1)(n-2)}{1\cdot 2(n+3)}-\tfrac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3(n+4)}+\text{etc.},$$

SCHOLIUM 1

22. Because we have set out the case $n=1$ in the preceding example, we will consider here chiefly the case

$$n=\tfrac{1}{2},$$

in which we have seen to be

$$\int x^{n}dx(1-x)^{n-1}=\tfrac{\pi}{2},$$

and therefore there will be

$$\tfrac{\varphi\pi}{2}=\tfrac{4}{1}\sin.\tfrac{1}{2}\varphi-\tfrac{4}{9}\sin.\tfrac{3}{2}\varphi+\tfrac{4}{25}\sin.\tfrac{5}{2}\varphi-\tfrac{4}{49}\sin.\tfrac{7}{2}\varphi+\text{etc.}$$

We may put $\varphi=2\omega$ and this neater series will be produced:

$$\tfrac{\pi\omega}{4}=\tfrac{1}{2}\sin.\omega-\tfrac{1}{9}\sin.3\omega+\tfrac{1}{25}\sin.5\omega-\tfrac{1}{49}\sin.7\omega+\text{etc.}$$

which in the first place, if the arc ω may be taken vanishing, gives

$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\text{etc.}$$

But if there shall be

$$\omega=\tfrac{\pi}{2}$$

this known series arises also [*see* E61] :

$$\frac{\pi\pi}{8}=\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\text{etc.}$$

But on taking the arc

$$\omega=45^0=\frac{\pi}{4}$$

there comes about:

$$\frac{\pi\pi}{8\sqrt{2}}=1-\frac{1}{9}-\frac{1}{25}+\frac{1}{49}+\frac{1}{81}-\frac{1}{121}-\frac{1}{169}+\text{etc.}$$

Let

$$\omega=30^0=\frac{\pi}{6};$$

there will become :

$$\begin{aligned}\frac{\pi\pi}{24}=&\frac{1}{2}(1+\frac{1}{7^2}+\frac{1}{13^2}+\frac{1}{19^2}+\frac{1}{25^2}+\text{etc.})\\&-1(\frac{1}{3^2}+\frac{1}{9^2}+\frac{1}{15^2}+\frac{1}{21^2}+\text{etc.})\\&+\frac{1}{2}(\frac{1}{5^2}+\frac{1}{11^2}+\frac{1}{17^2}+\frac{1}{23^2}+\text{etc.}),\end{aligned}$$

where the middle term $=\frac{\pi\pi}{72}$; and the reckoning of the rest is evident. Then by differentiation of our series, there arises this noteworthy form

$$\frac{\pi}{4}=\frac{1}{1}\cos.\omega-\frac{1}{3}\cos.3\omega+\frac{1}{5}\cos.5\omega-\frac{1}{7}\cos.7\omega+\text{etc.},$$

since clearly all the arcs taken for ω produce the same sum. Then truly by differentiating again, there is produced

$$0=\sin.\omega-\sin.3\omega+\sin.5\omega-\sin.7\omega+\text{etc.}$$

But by integration we find

$$C-\frac{\pi\omega^2}{8}=\frac{1}{1}\cos.\omega-\frac{1}{3^3}\cos.3\omega+\frac{1}{5^3}\cos.5\omega-\frac{1}{7^3}\cos.7\omega+\text{etc.},$$

where, on taking $\omega=0$ there shall be

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\text{etc.}=\frac{\pi^3}{32},$$

there will be

$$C=\frac{\pi^3}{32},$$

thus so that there shall be

$$\frac{\pi}{8}(\frac{\pi\pi}{4}-\omega\omega)=\frac{1}{1}\cos.\omega-\frac{1}{3^3}\cos.3\omega+\frac{1}{5^3}\cos.5\omega-\frac{1}{7^3}\cos.7\omega+\text{etc.},$$

SCHOLIUM 2

23. Now in general we may put

$$\varphi = \pi,$$

and since there shall be

$$\sin.(n+1)\pi = -\sin.n\pi,\ \sin.(n+2)\pi = +\sin.n\pi \text{ etc.},$$

our equation divided by $\sin.n\pi$ will adopt this form:

$$\frac{\pi}{\sin.n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} + \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2} + \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2} + \text{etc.};$$

but on taking

$$\varphi = 2\pi,$$

in a similar manner there will be :

$$\frac{2\pi}{\sin.2n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} - \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2} + \text{etc.};$$

of which therefore that series divided by this however many times becomes $= \cos.n\pi$, which is seen to be incongruous, since the quotient shall be greater than one. But we have now resolved a similar difficulty above, which has arisen on putting $\varphi = 2\pi$; for if we may put $\varphi = 3\pi$, the former series arising is going to have the sum

$$\frac{3\pi}{\sin.3n\pi}\int x^n dx(1-x)^{n-1},$$

which is not equal to that, unless n shall be a vanishing ratio. Whereby only the first series is agreed to be consistent; so that we may investigate the sum of which from its very nature, we may put

$$s = \frac{1}{n^2}t^n + \frac{2n}{(n+1)^2}t^{n+1} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2}t^{n+2} + \text{etc.},$$

and hence there will be

$$\frac{d.tds}{dt^2} = 1t^{n-1} + \frac{2n}{1}t^n + \frac{2n(2n+1)}{1\cdot 2}t^{n+2} + \text{etc.},$$

thus sum of which series evidently is

$$= t^{n-1}(1-t)^{-2n},$$

thus so that there shall be

$$\frac{tds}{dt} = \int t^{n-1}dt(1-t)^{-2n}$$

and

$$s = \int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t)^{2n}},$$

and thus on putting $t=1$ after the integration, there will be had

$$\frac{\pi}{\sin.n\pi}\int x^n dx(1-x)^{n-1} = \int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t)^{2n}}.$$

Which comparison of the two integral formulas is therefore the more memorable, because between most others, which have been uncovered at this stage, no such comparisons of this kind may be found.

SCHOLIUM 3

24. In general we may put

$$\varphi=\frac{\pi}{2}$$

and there will become :

$$\sin.n\varphi=\sin.\frac{n\pi}{2},\quad \sin.(n+1)\varphi=\cos.\frac{n\pi}{2},$$
$$\sin.(n+2)\varphi=-\sin.\frac{\pi}{2},\quad \sin.(n+3)\varphi=-\cos.\frac{n\pi}{2},\ \text{etc.}$$

from which this equation results :

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1}=\sin.\frac{n\pi}{2}\left(\frac{1}{nn}-\frac{2n(2n+1)}{1\cdot 2(n+2)^2}+\frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)^2}-\text{etc.}\right)$$
$$-\cos.\frac{n\pi}{2}\left(\frac{2n}{1(n+1)^2}-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}+\text{etc.}\right).$$

But from the above reduction it is evident to become :

$$1-\frac{2n(2n+1)}{1\cdot 2}t^2+\frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4}t^2-\text{etc.}$$
$$=\frac{(1+t\sqrt{-1})^{-2n}+(1-t\sqrt{-1})^{-2n}}{2},$$
$$\frac{2n}{1}t-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}t^3+\text{etc.}$$
$$=\frac{(1+t\sqrt{-1})^{-2n}-(1-t\sqrt{-1})^{-2n}}{2},$$

and hence it is deduced :

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1}=$$
$$\frac{1}{2}\sin.\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}}+\frac{1}{2}\sin.\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}}$$
$$-\frac{1}{2\sqrt{-1}}\cos.\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}}+\frac{1}{2\sqrt{-1}}\cos.\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}}$$

where indeed after the integration it is required to put $t = 1$. But so that we may free this expression from imaginary numbers, we may put

$$t = \text{tang}.\omega = \tfrac{\sin.\omega}{\cos.\omega};$$

there will be

$$dt = \frac{d\omega}{\cos.\omega^2}, \quad \frac{dt}{t} = \frac{d\omega}{\sin.\omega\cos.\omega}, \quad t^{n-1}dt = \frac{d\omega\sin.\omega^{n-1}}{\cos.\omega^{n+1}},$$

then truly

$$\begin{aligned}(1+t\sqrt{-1})^{-2n} &= \cos.\omega^{2n}(\cos.\omega + \sqrt{-1}\cdot\sin.\omega)^{-2n}\\ &= \cos.\omega^{2n}(\cos.2n\omega - \sqrt{-1}\cdot\sin.2n\omega),\\ (1-t\sqrt{-1})^{-2n} &= \cos.\omega^{2n}(\cos.\omega - \sqrt{-1}\cdot\sin.\omega)^{-2n}\\ &= \cos.\omega^{2n}(\cos.2n\omega + \sqrt{-1}\cdot\sin.2n\omega).\end{aligned}$$

With which values substituted, the imaginary terms mutually cancel each other, and this equation will be produced :

$$\begin{aligned}&\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} =\\ &\sin.\tfrac{n\pi}{2}\int\tfrac{d\omega}{\sin.\omega\cos.\omega}\int d\omega\sin.\omega^{n-1}\cos.\omega^{n-1}\cos.2n\omega\\ &+\cos.\tfrac{n\pi}{2}\int\tfrac{d\omega}{\sin.\omega\cos.\omega}\int d\omega\sin.\omega^{n-1}\cos.\omega^{n-1}\sin.2n\omega,\end{aligned}$$

which may be condensed into this simpler form

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \int\tfrac{d\omega}{\sin.\omega\cos.\omega}\int d\omega\sin.\omega^{n-1}\cos.\omega^{n-1}\sin.(\tfrac{n\pi}{2}+2n\omega)$$

or on account of $\sin.\omega\cos.\omega = \frac{1}{2}\sin.2\omega$ into this form:

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \tfrac{1}{2^n}\int\tfrac{2d\omega}{\sin.2\omega}\int 2d\omega\sin.2\omega^{n-1}\sin.(\tfrac{n\pi}{2}+2n\omega).$$

Now let the angle $2\omega = \theta$, so that there become more neatly

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \tfrac{1}{2^n}\int\tfrac{d\theta}{\sin.\theta}\int d\theta\sin.\theta^{n-1}\sin.n(\tfrac{\pi}{2}+\theta),$$

where after the integration it is required to put $\theta = 90° = \frac{\pi}{2}$, so that then there may become $\omega = 45°$ and $t = \text{tang}.\omega = 1$.

EXAMPLE III

25. *If the arcs a, b, c, d etc. may constitute a finite arithmetical progression, so that there shall be*

$$a = m\varphi,\ b = n\varphi,\ c = (1+m)\varphi,\ d = (1+n)\varphi,$$
$$e = (2+m)\varphi,\ f = (2+n)\varphi\ etc.,$$

to define the length of the arc p from the sines of these.

The general solution supplied above gives this equation (§ 17) :

$$\varphi = \frac{\sin.m\varphi}{m}\cdot\frac{nn}{(n-m)(n+m)}\cdot\frac{(1+m)^2}{1(1+2m)}\cdot\frac{(1+n)^2}{(1+n-m)(1+n+m)}$$
$$\cdot\frac{(2+m)^2}{2(2+2m)}\cdot\frac{(n+2)^2}{(2+n-m)(2+n+m)}\cdot\text{etc.}$$
$$-\frac{\sin.n\varphi}{n}\cdot\frac{mm}{(n-m)(n+m)}\cdot\frac{(1+m)^2}{(1+m-n)(1+m+n)}\cdot\frac{(1+n)^2}{1(1+2n)}$$
$$\cdot\frac{(2+m)^2}{(2+m-n)(2+m+n)}\cdot\frac{(2+n)^2}{2(2+2n)}\cdot\text{etc.}$$
$$+\frac{\sin.(1+m)\varphi}{1+m}\cdot\frac{mm}{1(1+2m)}\cdot\frac{nn}{(1+m-n)(1+m+n)}\cdot\frac{(1+n)^2}{(n-m)(2+m+n)}$$
$$\cdot\frac{(2+m)^2}{1(3+2n)}\cdot\frac{(2+n)^2}{(1+n-m)(3+n+m)}\cdot\text{etc.}$$
$$-\frac{\sin.(1+n)\varphi}{1+n}\cdot\frac{mm}{(1+n-m)(1+n+m)}\cdot\frac{nn}{1(1+2n)}\cdot\frac{(1+m)^2}{(n-m)(2+n+m)}$$
$$\cdot\frac{(2+m)^2}{(1+m-n)(3+m+n)}\cdot\frac{(2+n)^2}{1(3+2n)}\cdot\text{etc.}$$
$$+\frac{\sin.(2+m)\varphi}{2+m}\cdot\frac{mm}{2(2+2m)}\cdot\frac{nn}{(2+m-n)(2+m+n)}\cdot\frac{(1+m)^2}{1(3+2m)}$$
$$\cdot\frac{(1+n)^2}{(1+m-n)(3+m+n)}\cdot\frac{(2+n)^2}{(n-m)(4+m+n)}\cdot\text{etc.}$$
$$-\ \text{etc.}$$

But hence in general nothing worthy of attention can be concluded; so that I may set out the especially memorable case, in which there is

$$n = 1-m;$$

for which I may put, for the sake of brevity :

$$\varphi = \frac{\mathfrak{A}\sin.m\varphi}{m} - \frac{\mathfrak{B}\sin.(1-m)\varphi}{1-m} + \frac{\mathfrak{C}\sin.(1+m)\varphi}{1+m} - \frac{\mathfrak{D}\sin.(2-m)\varphi}{2-m} + \text{etc.},$$

so that there shall become

$$\mathfrak{A}=\frac{(1-m)^2}{1(1-2m)}\cdot\frac{(1+m)^2}{1(1+2m)}\cdot\frac{(2-m)^2}{2(2-2m)}\cdot\frac{(2+m)^2}{2(2+2m)}\cdot\frac{(3-m)^2}{3(3-2m)}\cdot\text{etc.},$$

$$\frac{\mathfrak{B}}{\mathfrak{A}}=\frac{mm}{(1-m)^2}\cdot\frac{1(1+2m)}{2\cdot 2m}\cdot\frac{2(2-2m)}{1(3-2m)}\cdot\frac{2(2+2m)}{3(1+2m)}\cdot\frac{3(3-2m)}{2(4-2m)}\cdot\text{etc.},$$

$$\frac{\mathfrak{C}}{\mathfrak{B}}=\frac{1(1-2m)}{1(1+2m)}\cdot\frac{(1-m)^2}{(1+m)^2}\cdot\frac{1(3-2m)}{3(1-2m)}\cdot\frac{3(1+2m)}{1(3+2m)}\cdot\frac{2(1-2m)}{4(2-2m)}\cdot\text{etc.},$$

$$\frac{\mathfrak{D}}{\mathfrak{C}}=\frac{1(1+2m)}{2(2-2m)}\cdot\frac{2\cdot 2m}{1(3-2m)}\cdot\frac{(1+m)^2}{(2-m)^2}\cdot\frac{1(3+2m)}{4\cdot 2m}\cdot\frac{4(2-2m)}{1(5-2m)}\cdot\text{etc.},$$

$$\frac{\mathfrak{E}}{\mathfrak{D}}=\frac{2(2-2m)}{2(2+2m)}\cdot\frac{1(3-2m)}{3(1+2m)}\cdot\frac{3(1-2m)}{1(3+2m)}\cdot\frac{(2-m)^2}{(2+m)^2}\cdot\frac{1(5-2m)}{5(1-2m)}\cdot\text{etc.},$$

etc.

And from the above reduction there is found

$$\mathfrak{A}=\frac{\int x^{m-1}dx(1-x)^{-2m}}{m\int x^m dx(1-x)^{m-1}\cdot\int x^{m-1}dx(1-x)^{-m}};$$

then truly for the remainder there is deduced from the form of the products themselves

$$\frac{\mathfrak{B}}{\mathfrak{A}}=\frac{m}{1-m},\quad \frac{\mathfrak{C}}{\mathfrak{B}}=\frac{1-m}{1+m},\quad \frac{\mathfrak{D}}{\mathfrak{C}}=\frac{1+m}{2-m},\quad \frac{\mathfrak{E}}{\mathfrak{D}}=\frac{2-m}{2+m}\ \text{etc.},$$

thus so that there shall be

$$\mathfrak{B}=\frac{m}{1-m}\mathfrak{A},\ \ \mathfrak{C}=\frac{m}{1+m}\mathfrak{A},\ \ \mathfrak{D}=\frac{m}{2-m}\mathfrak{A},\ \ \mathfrak{E}=\frac{m}{2+m}\mathfrak{A}\ \text{etc.}$$

Therefore we may put for the sake of brevity

$$\int x^m dx(1-x)^{m-1}\cdot\frac{\int x^{m-1}dx(1-x)^{-m}}{\int x^{m-1}dx(1-x)^{-2m}}=M$$

and there will be, so that it follows,

$$M\varphi=\frac{\sin.m\varphi}{m^2}-\frac{\sin.(1-m)\varphi}{(1-m)^2}+\frac{\sin.(1+m)\varphi}{(1+m)^2}-\frac{\sin.(2-m)\varphi}{(2-m)^2}+\frac{\sin.(2+m)\varphi}{(2+m)^2}-\text{etc.},$$

from which by differentiation we conclude to become :

$$M=\frac{\cos.m\varphi}{m}-\frac{\cos.(1-m)\varphi}{1-m}+\frac{\cos.(1+m)\varphi}{1+m}-\frac{\cos.(2-m)\varphi}{2-m}+\frac{\cos.(2+m)\varphi}{2+m}-\text{etc.},$$

which series is noteworthy on account of the conspicuous simplicity, since indeed on putting $\varphi=0$, we deduce

$$M=\tfrac{1}{m}-\tfrac{1}{1-m}+\tfrac{1}{1+m}-\tfrac{1}{2-m}+\tfrac{1}{2+m}-\tfrac{1}{3-m}+\tfrac{1}{3+m}-\text{etc.},$$

the sum of which series at some time I have shown to be [see E130, also *Introd. Analysis.Infin.* Book I. §178.]

$$M=\tfrac{\pi\cos.m\pi}{\sin.m\pi},$$

from which we deduce this elegant comparison

$$\int x^m dx(1-x)^{m-1}=\tfrac{\pi\cos.m\pi}{\sin.m\pi}\cdot\tfrac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}},$$

which again is reduced to this :

$$\int x^m dx(1-x)^{m-1}=\tfrac{(1-m)\pi\cos.m\pi}{\sin.m\pi}\cdot\tfrac{\int x^{m}dx(1-x)^{-2m}}{\int x^{m}dx(1-x)^{-m}},$$

or more neatly to this :

$$\int x^m dx(1-x)^{m-1}=\tfrac{2\pi\cos.m\pi}{\sin.m\pi}\cdot\tfrac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}}.$$

COROLLARY 1

26. Behold therefore some significant theorems, which the establishment of this example presents to us, of which the first is :

If φ may denote some angle, there will be

$$\tfrac{\pi\cos.m\pi}{\sin.m\pi}=\tfrac{\cos.m\varphi}{m}-\tfrac{\cos.(1-m)\varphi}{1-m}+\tfrac{\cos.(1+m)\varphi}{1+m}-\tfrac{\cos.(2-m)\varphi}{2-m}+\text{etc.},$$

which equality also can be shown thus, so that there shall be

$$\tfrac{\pi\cos.m\pi}{\sin.m\pi}=\cos.m\varphi(\tfrac{1}{m}-\tfrac{2m\cos.\varphi}{1-mm}-\tfrac{2m\cos.2\varphi}{4-mm}-\tfrac{2m\cos.3\varphi}{9-mm}-\text{etc.})$$
$$-2\sin.m\varphi(\tfrac{\sin.\varphi}{1-mm}+\tfrac{2\sin.2\varphi}{4-mm}+\tfrac{3m\sin.3\varphi}{9-mm}+\tfrac{4m\sin.4\varphi}{16-mm}+\text{etc.}),$$

from which, if

$$m\varphi=90^\circ=\tfrac{\pi}{2}\text{ and thus }\varphi=\tfrac{\pi}{2m},$$

there will be

$$-\tfrac{\pi\cos.m\pi}{\sin.m\pi}=\tfrac{\sin.\frac{\pi}{2m}}{1-mm}+\tfrac{2\sin.\frac{2\pi}{2m}}{4-mm}+\tfrac{3\sin.\frac{3\pi}{2m}}{9-mm}+\tfrac{4\sin.\frac{4\pi}{2m}}{16-mm}+\text{etc.}$$

COROLLARY 2

27. The second theorem may be enunciated thus :

If φ may denote some angle, there will be

$$\frac{\pi\varphi\cos.m\pi}{\sin.m\pi}=\frac{\sin.m\varphi}{mm}-\frac{\sin.(1-m)\varphi}{(1-m)^2}+\frac{\sin.(1+m)\varphi}{(1+m)^2}-\frac{\sin.(2-m)\varphi}{(2-m)^2}+\text{etc.}$$

Whereby on taking $\varphi=\pi$, there will be

$$\frac{\pi\pi\cos.m\pi}{\sin.m\pi}=\frac{\sin.m\pi}{mm}-\frac{\sin.m\pi}{(1-m)^2}-\frac{\sin.m\pi}{(1+m)^2}+\frac{\sin.m\pi}{(2-m)^2}+\frac{\sin.m\pi}{(2+m)^2}-\text{etc.},$$

or

$$\frac{\pi\pi}{\sin.m\pi\tang.m\pi}=\frac{1}{m^2}-\frac{1}{(1-m)^2}-\frac{1}{(1+m)^2}+\frac{1}{(2-m)^2}+\frac{1}{(2+m)^2}-\text{etc.}$$

But on putting

$$m\varphi=\pi$$

there will be had:

$$\frac{\pi\pi\cos.m\pi}{m\sin.m\pi}=\frac{\sin.\frac{\pi}{m}}{(1-m)^2}-\frac{\sin.\frac{\pi}{m}}{(1+m)^2}+\frac{\sin.\frac{2\pi}{m}}{(2-m)^2}-\frac{\sin.\frac{2\pi}{m}}{(2+m)^2}+\text{etc.}$$

or in this manner :

$$\frac{\pi\pi\cos.m\pi}{4mm\sin.m\pi}=\frac{1\sin.\frac{\pi}{m}}{(1-mm)^2}+\frac{2\sin.\frac{2\pi}{m}}{(4-mm)^2}+\frac{3\sin.\frac{3\pi}{m}}{(9-m)^2}+\text{etc.}$$

COROLLARY 3

28. The third theorem considers the comparison of integral formulas, and may be enunciated thus :
If the integration of the following formulas may be from the limit $x=0$ and may be extended as far as to the limit $x=1$, there will be always :

$$\int x^{m-1}dx(1-x)^{m-1}\cdot\int x^{m-1}dx(1-x)^{-m}=\frac{2\pi\cos.m\pi}{\sin.m\pi}\cdot\int x^{m-1}dx(1-x)^{-2m},$$

or if there may be put $m=\frac{\lambda}{n}$ and $x=y^n$, there will become

$$\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{n-\lambda}}}\cdot\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{\lambda}}}=\frac{2\pi\cos.\frac{\lambda\pi}{n}}{n\sin.\frac{\lambda\pi}{n}}\cdot\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}$$

SCHOLIUM

29. The demonstration of this latter theorem is seen with some difficulty; yet meanwhile by that, which formerly I have discussed concerning integral formulas of this kind [*see, e.g. Instut. Calc. Integ., Vol. I, Ch. 8*], the truth of this can be shown in the following way. For we may indicate, as I have done there, this integral formula

$$\int \frac{y^{p-1}dy}{\sqrt[n]{(1-y^n)^{n-q}}}$$

by this symbol $\left(\frac{p}{q}\right)$, to be required to show

$$\left(\frac{\lambda}{\lambda}\right)\left(\frac{\lambda}{n-\lambda}\right)=\frac{2\pi\cos.\frac{\lambda\pi}{n}}{n\sin.\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right).$$

Now in the first place I have shown, if there were

$$q+r=n,$$

to become

$$\left(\frac{q}{r}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}},$$

from which it follows at once :

$$\left(\frac{\lambda}{n-\lambda}\right)=\int \frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{\lambda}}}=\frac{\pi}{n\sin.\frac{\lambda\pi}{n}},$$

thus so that it shall remain to be shown :

$$\left(\frac{\lambda}{\lambda}\right)=2\cos.\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

Truly I have shown in the same place, if there were

$$p+q+r=n,$$

to become :

$$\frac{1}{\sin.\frac{r\pi}{n}}\left(\frac{p}{q}\right)=\frac{1}{\sin.\frac{q\pi}{n}}\left(\frac{p}{r}\right)=\frac{1}{\sin.\frac{p\pi}{n}}\left(\frac{q}{r}\right).$$

Therefore we may take :

$$p=\lambda,\quad q=\lambda$$

and there will be

$$r=n-2\lambda,$$

where, on account of which

$$\sin.\frac{(n-2\lambda)\pi}{n}=\sin.\frac{2\lambda\pi}{n}$$

we deduce

$$\frac{1}{\sin.\frac{2\lambda\pi}{n}}\left(\frac{\lambda}{\lambda}\right)=\frac{1}{\sin.\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right),$$

thus so that

$$\sin.\tfrac{2\lambda\pi}{n}=2\sin.\tfrac{\lambda\pi}{n}\cos.\tfrac{\lambda\pi}{n}$$

actually, there shall become

$$\left(\frac{\lambda}{\lambda}\right)=2\cos.\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

But much more obstruse is the resolution of the above theorem (§ 23) , because there shall become, under the same limits of the integration:

$$\frac{\pi}{\sin.n\pi}\int x^{n}dx(1-x)^{n-1}=\int\frac{dx}{x}\int\frac{x^{n-1}dx}{(1-x)^{2n}}.$$

or

$$\frac{\pi}{2\sin.n\pi}\int x^{n-1}dx(1-x)^{n-1}=\int\frac{dx}{x}\int\frac{x^{n-1}dx}{(1-x)^{2n}};$$

which equation, so that it may be reduced to that form, we may write $\frac{\lambda}{n}$ in place of n and there shall be $x=y^{n}$, from which there becomes :

$$\frac{\pi}{2n\sin.\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^{n})^{n-\lambda}}}=\int\frac{dy}{y}\int\frac{y^{\lambda-1}}{\sqrt[n]{(1-y^{n})^{2\lambda}}}.$$

But just as we have seen to be :

$$\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^{n})^{n-\lambda}}}=2\cos.\frac{\lambda\pi}{n}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^{n})^{2\lambda}}},$$

and thus, on the strength of this theorem, we deduce to be :

$$\frac{\pi}{n\,\text{tang.}\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^{n})^{2\lambda}}}=\int\frac{dy}{y}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^{n})^{2\lambda}}},$$

and hence again this same no less noteworthy theorem :

$$\frac{\pi}{n\,\text{tang.}\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^{n})^{2\lambda}}}=-\int\frac{y^{\lambda-1}dy\cdot ly}{\sqrt[n]{(1-y^{n})^{2\lambda}}},$$

from which on taking $\lambda=1$ we elicit the following proportion

$$\frac{\pi}{n} : \text{tang.}\frac{\lambda\pi}{n} = \int \frac{dyl\frac{1}{y}}{\sqrt[n]{(1-y^n)^2}} : \int \frac{dy}{\sqrt[n]{(1-y^n)^2}}.$$

PROBLEM 3

30. *To find the equation of this kind for be curved line between the two variables, the abscissa x and the applied line y, so that with the abscissas in an arithmetical progression they may agree with the given applied lines, evidently:*
If there shall be

$$x = n\theta,\ (n+1)\theta,\ (n+2)\theta,\ (n+3)\theta,\ (n+4)\theta,\ etc.$$

so that there may become:

$$y = p, \quad q, \quad r, \quad s, \quad t \quad etc.$$

SOLUTION

In general we may put

$$x = \theta\omega$$

and from the general solution given in § 10 we follow this equation :

$$\begin{aligned}\frac{y}{\omega} = &\frac{p}{n}\cdot\frac{(n+1-\omega)(n+1+\omega)}{1(2n+1)}\cdot\frac{(n+2-\omega)(n+2+\omega)}{2(2n+2)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{3(2n+3)}\cdot\text{etc.}\\ &-\frac{q}{n+1}\cdot\frac{(n-\omega)(n+\omega)}{1(2n+1)}\cdot\frac{(n+2-\omega)(n+2+\omega)}{1(2n+3)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{2(2n+4)}\cdot\text{etc.}\\ &+\frac{r}{n+2}\cdot\frac{(n-\omega)(n+\omega)}{2(2n+2)}\cdot\frac{(n+1-\omega)(n+1+\omega)}{1(2n+3)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{1(2n+5)}\cdot\text{etc.}\\ &+\ \text{etc.},\end{aligned}$$

which equation for the sake of brevity we may represent thus :

$$\frac{y}{\omega} = \mathfrak{A}\cdot\frac{p}{n} - \mathfrak{B}\cdot\frac{q}{n+1} + \mathfrak{C}\cdot\frac{r}{n+2} - \mathfrak{D}\cdot\frac{s}{n+3} + \text{etc.};$$

and initially indeed by eliciting for the value of $\mathfrak{A}$ itself from the general form § 17 advanced for this case, we will have

$$a = n+1-\omega,\ b = 1,\ c = n-\omega \text{ and } d = 1,$$

from which by the integral formulas being extended from the limit $z = 0$ to $z = 1$ we gather:

$$\mathfrak{A} = \frac{\int dz(1-z)^{n-\omega-1}}{\int z^{n-\omega}dz(1-z)^{n-\omega-1}} = \frac{1}{(n-\omega)\int z^{n-\omega}dz(1-z)^{n-\omega-1}}$$

or

$$\mathfrak{A} = \frac{2}{(n-\omega)\int z^{n-\omega-1}dz(1-z)^{n-\omega-1}},$$

with which integration conceded the rest may be extricated easily. Indeed there shall be as in § 17 above :

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)}\cdot(2+2n) = \frac{2(n+1)(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)},$$
$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{(n+1-\omega)(n+1+\omega)}{(n+2-\omega)(n+2+\omega)}\cdot\frac{(1+2n)(2+n)}{2(n+1)},$$
$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2-\omega)(n+2+\omega)}{(n+3-\omega)(n+3+\omega)}\cdot\frac{(2+2n)(3+n)}{3(n+2)},$$
$$\frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3-\omega)(n+3+\omega)}{(n+4-\omega)(n+4+\omega)}\cdot\frac{(3+2n)(4+n)}{4(n+3)}$$

etc.

Therefore we may establish the integral formula :

$$\int z^{n-\omega-1}dz(1-z)^{n-\omega-1} = \Delta,$$

so that there shall become:

$$\mathfrak{A} = \frac{2}{(n-\omega)\Delta},$$

and the remaining coefficients thus may be defined in terms of $\mathfrak{A}$:

$$\mathfrak{B} = 2(n+1)\cdot\frac{nn-\omega\omega}{(n+1)^2-\omega^2}\mathfrak{A},$$
$$\mathfrak{C} = \frac{2(n+2)(2n+1)}{1\cdot 2}\cdot\frac{nn-\omega\omega}{(n+2)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{D} = \frac{2(n+3)(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{nn-\omega\omega}{(n+3)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{E} = \frac{2(n+4)(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{nn-\omega\omega}{(n+4)^2-\omega\omega}\mathfrak{A},$$

etc.

On account of which the equation sought between y and $x = \theta\omega$ will be prepared thus :

$$\frac{n\Delta y}{2(n+\omega)\omega} = \frac{p}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{q}{(n+1)^2-\omega\omega} +$$
$$\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{r}{(n+2)^2-\omega\omega} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{s}{(n+3)^2-\omega\omega} + \text{etc.},$$

from which for any value of $x = \theta\omega$ the value of y itself agreeing may be defined and that by the applied lines p, q, r etc., which are taken agreeing with the abscissas

$n\theta,\ (n+1)\theta,\ (n+2)\theta$ etc. Indeed where it is required to be observed, if ω may be taken equal to some term of the progression $n,\ n+1,\ n+2$ etc., then the denominator of the corresponding given applied line to vanish, thus so that compared with that term, clearly infinite, the rest may vanish. Then truly likewise also the value $\varDelta$ produces an infinitude, and exactly of the same kind, so that then there may become either $y=p,\ y=q,\ y=r$ etc., just as the nature of the problem demands.

COROLLARY 1

31. If the proposed abscissas may denote circular arcs, and indeed the applied lines the sines of the same, so that there shall be

$$p=\sin.n\theta,\ q=\sin.(n+1)\theta,\ r=\sin.(n+2)\theta \text{ etc.},$$

there will be

$$y=\sin.\omega\theta,$$

from which this same general equation results

$$\frac{n\varDelta\sin.\omega\theta}{2(n+\omega)\omega}=\frac{\sin.n\theta}{nn-\omega\omega}-\frac{2n}{1}\cdot\frac{\sin.(n+1)\theta}{(n+1)^2-\omega\omega}+$$
$$\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{\sin.(n+2)\theta}{(n+2)^2-\omega\omega}-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{\sin.(n+3)\theta}{(n+3)^2-\omega\omega}+\text{etc.},$$

where it is especially noteworthy, because the three letters $n,\ \theta$ et ω, may be taken as desired.

COROLLARY 2

32. Therefore if we may suppose

$$\theta=\pi,$$

so that all the sines of the series may be reduced to the same $\sin.n\theta$, there will be

$$\frac{n\varDelta\sin.\omega\pi}{2(n+\omega)\omega\sin.n\pi}=\frac{1}{nn-\omega\omega}+\frac{2n}{1}\cdot\frac{1}{(n+1)^2-\omega^2}+$$
$$\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{1}{(n+2)^2-\omega^2}+\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{1}{(n+3)^2-\omega^2}+\text{etc.},$$

Hence if there shall be

$$n=\tfrac{1}{2} \text{ and } \varDelta=\int z^{-\omega-\frac{1}{2}}dz(1-z)^{-\omega-\frac{1}{2}}$$

or

$$\varDelta=2\int\frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}};$$

there will be had

$$\frac{\Delta\sin.\omega\pi}{8(1+2\omega)\omega}=\frac{1}{1-4\omega^2}+\frac{1}{9-4\omega^2}+\frac{1}{25-4\omega^2}+\frac{1}{49-4\omega^2}+\text{etc.},$$

of which the sum of the series is shown to be

$$=\frac{\pi}{8\omega}\text{tang.}\omega\pi,$$

thus, so that there shall be

$$\frac{\Delta\sin.\omega\pi}{8(1+2\omega)\omega}=\frac{\pi}{8\omega}\text{tang.}\omega\pi$$

and thus,

$$\Delta=\frac{(1+2\omega)\pi}{\cos.\omega\pi}.$$

SCHOLIUM 1

33. But it is not allowed to have excessive confidence with these conclusions on account of the argument advanced above. For with the applied line in place

$$p=\sin.n\theta,\ q=\sin.(n+1)\theta,\ r=\sin.(n+2)\theta\text{ etc.},$$

while the arcs $n\theta,\ (n+1)\theta,\ (n+2)\theta$ etc. may be considered as the abscissas, the equation found presents a curved line of this kind, which passes through all these points ; and truly neither here does it follow this curve itself to be a line of sines, since infinitely many other curved lines may be given passing through that same infinitude of points. Whereby with the letter y serving for the applied line of the abscissa $x=\theta\omega$, with the correspondence indicating the solution for our curve sought, this indeed may supply the equation :

$$\frac{n\Delta y}{2(n+\omega)\omega}=\frac{\sin.n\theta}{n^2-\omega^2}-\frac{2n}{1}\cdot\frac{\sin.(n+1)\theta}{(n+1)^2-\omega^2}+\frac{2n(2n+1)}{1\cdot2}\cdot\frac{\sin.(n+2)\theta}{(n+1)^2-\omega^2}+\frac{2n(2n+1)(2n+2)}{1\cdot2\cdot3}\cdot\frac{\sin.(n+3)\theta}{(n+3)^2-\omega^2}+\text{etc.},$$

thus so that the abscissas

$$x=(n\pm i)\theta$$

may correspond to these applied lines

$$y=\sin.(n\pm i)\theta,$$

but only if i shall be some integer. Truly it may be able to happen, so that for other abscissas, where i is not a whole number and thus generally, if $x=w\theta$, the applied line would not be $y=\sin.\omega\theta$. So that this may be seen more clearly, we will investigate the general equation for all lines plainly passing through the given points, and the value found at this point shall be

$$y=\Theta$$

and a function may be sought vanishing for all the given abscissas, of this kind is

$$\omega(nn-\omega\omega)\frac{((n+1)^2-\omega^2)}{1(2n+1)}\cdot\frac{((n+2)^2-\omega^2)}{2(2n+2)}\cdot\frac{((n+3)^2-\omega^2)}{3(2n+3)} \text{ etc.,}$$

which from the above

$$=\omega(nn-\omega\omega)\mathfrak{A}=\frac{2\omega(n+\omega)}{\varDelta}.$$

This quantity may be called $=\varOmega$ and $f:\varOmega$ shall be a function of this kind of the same $\varOmega$, which may vanish, if $\varOmega=0$, and the general equation for all the satisfying curves will be :

$$y=\varTheta+f:\varOmega=\varTheta+f:\frac{2\omega(n+\omega)}{\varDelta}.$$

And now without any doubt it is certain the equation $y=\sin.\omega\theta$ to be contained in this equation on putting $x=w\theta$, since this equation is satisfied by the required conditions. From which it may be possible therefore to come about, so that the equation $y=\varTheta$ may be different from this same one $y=\sin.\omega\theta$; which can depend especially on the values attributed to the letters θ and *n*, thus so that in some cases the equation found $y=\varTheta$ may agree with this $y=\sin.\omega\theta$, truly in others it may differ from the same.

SCHOLIUM 2

34. We may adapt this to the case, where

$$\theta=\pi \text{ and } n=\tfrac{1}{2}$$

and

$$\varDelta=2\int\frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}};$$

and because the sum of the series found shall be

$$=\frac{\pi}{8\omega}\operatorname{tang}.\omega\pi,$$

this general equation will be had :

$$\frac{\varDelta y}{2(1+2\omega)\omega}=\frac{\pi}{8\omega}\operatorname{tang}.\omega\pi+\frac{\varDelta}{8(1+2\omega)}f:\frac{\omega(1+2\omega)}{2\varDelta}$$

or

$$y=\frac{\pi(1+2\omega)}{\varDelta}\operatorname{tang}.\omega\pi+f:\frac{\omega(1+2\omega)}{2\varDelta},$$

where an added function in general thus has been prepared, so that it may vanish in the cases

$$\omega = 0,\ \ \omega = \pm\tfrac{1}{2},\ \ \omega = \pm\tfrac{3}{2},\ \ \omega = \pm\tfrac{5}{2}\ \text{ etc.},$$

the formulas are of such a kind :

$$\sin.2\omega\pi,\ \ \omega\cos.\omega\pi,\ \text{ likewise } \sin.2i\omega\pi \text{ and } \omega\cos.(2i-1)\omega\pi,$$

with i denoting some whole number ; from which any number of formulas of this kind may be combined as wished. Therefore a certain function of this kind will be given, which shall be called φ, so that there may become

$$y = \sin.\omega\pi$$

and hence

$$\sin.\omega\pi = \tfrac{\pi(1+2\omega)}{\varDelta}\operatorname{tang}.\omega\pi + \varphi,$$

or

$$\varDelta = \tfrac{\pi(1+2\omega)\operatorname{tang}.\omega\pi}{\sin.\omega\pi - \varphi} = 2\int \tfrac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}}.$$

Therefore since in the case $\omega = 0$ the function φ certainly may vanish, and certainly there will be $\varDelta = \pi$, which in the case indicated the function φ contains the factor ω^{λ}, the exponent λ of which shall be greater than unity, because otherwise on taking $\omega = 0$ the quantity φ may not vanish before $\sin.\omega\pi$. And on this account, the conclusions of the second preceding problem are required to be had for the truth.

PROBLEM 4

35. *To find an equation of this kind for the curved line between the abscissa x and the applied line y, so that the given applied lines may agree with the progressions from the abscissas in an interrupted arithmetical progression, evidently :*
If there shall be

$$x = n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta,\ (3-n)\theta\ etc.,$$

so that there may become :

$$y = p,\qquad q,\qquad r,\qquad s,\qquad t,\qquad u\qquad etc.$$

SOLUTION

In general we may put the abscissa

$$x = \theta\omega$$

and for the equation between x and y we may put this equation :

$$\frac{y}{\omega}=\mathfrak{A}\cdot\frac{p}{n}-\mathfrak{B}\cdot\frac{q}{1-n}+\mathfrak{C}\cdot\frac{r}{1+n}-\mathfrak{D}\cdot\frac{s}{2-n}+\mathfrak{E}\cdot\frac{t}{2+n}-\mathfrak{F}\cdot\frac{u}{3-n}+\text{etc.}$$

and from paragraph 25, extended to this general case, there will be had :

$$\mathfrak{A}=\frac{(1-n-\omega)(1-n+\omega)}{1(1-2n)}\cdot\frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)}\cdot\frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)}\cdot\frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)}\cdot\text{etc.},$$

$$\frac{\mathfrak{B}}{\mathfrak{A}}=\frac{(n-\omega)(n+\omega)}{(1-n-\omega)(1-n+\omega)}\cdot\frac{(1-n)}{n},\quad \frac{\mathfrak{C}}{\mathfrak{B}}=\frac{(1-n-\omega)(1-n+\omega)}{(1+n-\omega)(1+n+\omega)}\cdot\frac{1+n}{1-n},$$

$$\frac{\mathfrak{D}}{\mathfrak{C}}=\frac{(1+n-\omega)(1+n+\omega)}{(2-n-\omega)(2-n+\omega)}\cdot\frac{2-n}{1+n},\quad \frac{\mathfrak{E}}{\mathfrak{D}}=\frac{(2-n-\omega)(2-n+\omega)}{(2+n-\omega)(2-n+\omega)}\cdot\frac{2+n}{2-n},$$

etc.

We may establish the value of $\mathfrak{A}$ in the two products :

$$\mathfrak{P}=\frac{(1-n-\omega)(1-n+\omega)}{3(1-2n)}\cdot\frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)}\cdot\frac{(3-n-\omega)(3-n+\omega)}{3(2-3n)}\cdot\text{etc.},$$

$$\mathfrak{Q}=\frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)}\cdot\frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)}\cdot\frac{(3+n-\omega)(3+n+\omega)}{3(2+3n)}\cdot\text{etc.},$$

so that there shall become :

$$\mathfrak{A}=\mathfrak{P}\mathfrak{Q},$$

and we may define each value by the integral formulas, following the precepts in § 17. And indeed initially for the infinite product $\mathfrak{P}$, we may make

$$a=1-n-\omega,\ b=1,\ c=-n+\omega \text{ and } d=1$$

and there will become

$$\mathfrak{P}=\frac{\int dx(1-x)^{-1-n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}}=\frac{1}{\omega-n}\cdot\frac{1}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}},$$

if indeed there shall become:

$$\omega-n>0.$$

For the other infinite product by supposing only n negative, there becomes

$$\mathfrak{Q}=\frac{1}{\omega+n}\cdot\frac{1}{\int x^{n-\omega}dx(1-x)^{n+\omega-1}}.$$

But lest there shall be no need for the condition $\omega-n>0$, we may use from the other distribution and there shall become :

$$\mathfrak{P} = \frac{(1+n+\omega)(1-n-\omega)}{1\cdot 1}\cdot\frac{(2+n+\omega)(2-n-\omega)}{2\cdot 2}\cdot\frac{(3+n+\omega)(3-n-\omega)}{3\cdot 3}\cdot \text{etc.},$$

$$\mathfrak{Q} = \frac{(1+n-\omega)(1-n+\omega)}{(1-2n)(1+2n)}\cdot\frac{(2+n-\omega)(2-n+\omega)}{(2-2n)(2+2n)}\cdot\frac{(3+n-\omega)(3-n+\omega)}{(3-2n)(3+2n)}\cdot \text{etc.},$$

and we may put in place for $\mathfrak{P}$:

$$a = 1-n-\omega,\ \ b = 1,\ \ c = n+\omega,\ \ d = 1,$$

truly for $\mathfrak{Q}$:

$$a = 1+n-\omega,\ b = 1-2n,\ c = n+\omega \ \text{ and }\ d = 1$$

and there will become:

$$\mathfrak{P} = \frac{\int dx(1-x)^{-1+n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}} = \frac{1}{n+\omega}\cdot\frac{1}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}},$$

$$\mathfrak{Q} = \frac{\int x^{-2n}dx(1-x)^{-1+n+\omega}}{\int x^{n-\omega}dx(1-x)^{-1+n+\omega}}.$$

Truly in general there is:

$$\int x^m dx(1-x)^{k-1} = \tfrac{m+k+1}{k}\int x^m dx(1-x)^k,$$

therefore

$$\int x^{-n-\omega}dx(1-x)^{-1+n+\omega} = \tfrac{1}{n+\omega}\int x^{-n-\omega}dx(1-x)^{n+\omega}$$
$$= \tfrac{1}{n+\omega}\int y^{n+\omega}dy(1-y)^{-n-\omega},$$

$$\int x^{-2n}dx(1-x)^{-1+n+\omega} = \tfrac{1-n+\omega}{n+\omega}\int x^{-2n}dx(1-x)^{n+\omega}$$
$$= \tfrac{1-n+\omega}{n+\omega}\int y^{n+\omega}dy(1-y)^{-2n},$$

$$\int x^{n-\omega}dx(1-x)^{-1+n+\omega} = \tfrac{1+2n}{n+\omega}\int x^{n-\omega}dx(1-x)^{n+\omega}$$
$$= \tfrac{1+2n}{n+\omega}\int y^{n+\omega}dx(1-y)^{n-\omega},$$

from which it is concluded

$$\mathfrak{A} = \mathfrak{P}\mathfrak{Q} = \frac{(1-n+\omega)\int y^{n+\omega}dy(1-y)^{-2n}}{\int y^{n+\omega}dy(1-y)^{-n+\omega}\cdot\int y^{n+\omega}dy(1-y)^{n-\omega}}$$

or

$$\mathfrak{A} = \frac{\int y^{n+\omega-1}dy(1-y)^{-2n}}{\int y^{n+\omega}dy(1-y)^{-n-\omega}\cdot\int y^{n+\omega-1}dy(1-y)^{n-\omega}}$$

or

$$\mathfrak{A} = \frac{\int y^{n+\omega-1}dy(1-y)^{-2n}}{(n+\omega)\int y^{n+\omega-1}dy(1-y)^{-n-\omega}\cdot\int y^{n+\omega-1}dy(1-y)^{n-\omega}}$$

Therefore since there shall be

$$\mathfrak{B} = \tfrac{1-n}{n}\cdot\frac{nn-\omega\omega}{(1-n)^2-\omega\omega}\mathfrak{A},\ \mathfrak{C} = \tfrac{1+n}{n}\cdot\frac{nn-\omega\omega}{(1+n)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{C} = \tfrac{2-n}{n}\cdot\frac{nn-\omega\omega}{(2-n)^2-\omega\omega}\mathfrak{A},\ \mathfrak{D} = \tfrac{2+n}{n}\cdot\frac{nn-\omega\omega}{(2+n)^2-\omega\omega}\mathfrak{A}$$

etc.

it will be satisfied concisely enough by the series :

$$\frac{y}{\mathfrak{A}\omega} = \frac{p}{n} - \frac{(nn-\omega\omega)q}{n((1-n)^2-\omega\omega)} + \frac{(nn-\omega\omega)r}{n((1+n)^2-\omega\omega)} - \frac{(nn-\omega\omega)s}{n((2-n)^2-\omega\omega)} + \text{etc.}$$

or

$$\frac{ny}{\mathfrak{A}\omega(nn-\omega\omega)} = \frac{p}{n^2-\omega^2} - \frac{q}{(1-n)^2-\omega^2} + \frac{r}{(1+n)^2-\omega^2} - \text{etc.}$$

But in place of $\mathfrak{A}$ by reconstituting the form of the integral, where indeed in the cause of distinction, I will designate a new variable z , this same series is equal to this expression

$$\frac{ny}{(n-\omega)\omega}\cdot\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}},$$

the integration of which formula being understood extended from the limit $z=0$ to $z=1$.

COROLLARY 1.

36. Therefore if for the sake of brevity we may put this integral form

$$\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}} = \Delta$$

and we may resolve the individual terms of the series into two, we will have

$$\frac{2n\Delta y}{n-\omega} = +\frac{p}{n-\omega} - \frac{q}{1-n-\omega} + \frac{r}{1+n-\omega} - \frac{s}{2-n-\omega} + \frac{t}{2+n-\omega} - \text{etc.}$$
$$-\frac{p}{n+\omega} + \frac{q}{1-n+\omega} - \frac{r}{1+n+\omega} + \frac{s}{2-n+\omega} - \frac{t}{2+n+\omega} + \text{etc.}$$

COROLLARY 2

37. Therefore this equation defines a curved line of this kind, in which the abscissas

$$x = 0,\ n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta \text{ etc.}$$

will correspond to the applied lines

$$y = 0,\quad p,\quad q,\qquad r,\qquad s,\qquad t \quad \text{etc.,}$$

truly with the same negative abscissas taken corresponding the same negative applied lines. But here in general the abscissa $x = \theta\omega$ has been put in place .

COROLLARY 3

38. Because the letter θ has departed from the calculation, in place of this unity may be allowed to be written, so that ω may denote the abscissa itself. Truly if we wish to make an application to the sines of some arcs, it is convenient to retain the letter θ in the calculation.

SCHOLIUM

39. The use of this problem is concerned especially, so that if the above abscissas may be considered as circular arcs and the given abscissas thus may be taken, so that the applied lines *p, q, r, s, t* etc. may become equal to each other, either positive or negative. Where therefore from these cases it may be evident, or the series found may be summed otherwise, they may be called into help, which elsewhere I have commented on concerning similar series, from which indeed the sums of the two following series are deduced :

$$\frac{1}{\alpha}-\frac{1}{\beta-\alpha}+\frac{1}{\beta-\alpha}-\frac{1}{2\beta-\alpha}+\frac{1}{2\beta-\alpha}-\text{etc.}=\frac{\pi}{\beta\text{tang.}\frac{\alpha\pi}{\beta}},$$

$$\frac{1}{\alpha}+\frac{1}{\beta-\alpha}-\frac{1}{\beta-\alpha}+\frac{1}{2\beta-\alpha}-\frac{1}{2\beta-\alpha}+\text{etc.}=\frac{\pi}{\beta\sin.\frac{\alpha\pi}{\beta}}.$$

Hence therefore for our problem we deduce the four following summations :

$$\text{I.}\quad \frac{1}{n-\omega}-\frac{1}{1-n+\omega}+\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}-\text{etc.}=\frac{\pi}{\text{tang.}(n-\omega)\pi},$$

$$\text{II.}\quad \frac{1}{n-\omega}+\frac{1}{1-n+\omega}-\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}+\text{etc.}=\frac{\pi}{\sin.(n-\omega)\pi},$$

$$\text{III.}\quad \frac{1}{n+\omega}-\frac{1}{1-n-\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n-\omega}+\frac{1}{2+n+\omega}-\text{etc.}=\frac{\pi}{\text{tang.}(n+\omega)\pi},$$

$$\text{IV.}\quad \frac{1}{n+\omega}+\frac{1}{1-n-\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n+\omega}+\text{etc.}=\frac{\pi}{\sin.(n+\omega)\pi},$$

From these observations we may set out the cases, which are allowed to be reduced to finite expressions with the aid of these summations.

EXAMPLE I

40. *The applied lines, which correspond to the abscissas*

$$x=0,\ n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta \text{ etc.}$$

shall be

$$y=0,\quad p,\quad q,\quad r,\quad s,\quad t \quad\text{etc.},$$

and by a finite equation the relation between the applied line y and the abscissa $x=\theta\omega$ *may be investigated..*

SOLUTION

The first of the corollaries for this case provides this equation :

$$\frac{2n\Delta y}{f(n-\omega)}=+\frac{1}{n-\omega}-\frac{1}{1-n-\omega}-\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}-\text{etc.}$$
$$-\frac{1}{n+\omega}+\frac{1}{1-n+\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{t}{2+n+\omega}+\text{etc.},$$

which two series are reduced with the aid of the other four above, the summation of which agrees, for II take IV, and thus the equation sought thus itself shall have this finite form

$$\frac{2n\Delta y}{f(n-\omega)}=\frac{\pi}{\sin.(n-\omega)\pi}-\frac{\pi}{\sin.(n+\omega)\pi},$$

which expression is reduced to this :

$$\frac{2\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}=\frac{4\pi\cos.n\pi\sin.\omega\pi}{\cos.2\omega\pi-\cos.2n\pi},$$

thus so that for our curve this equation may be had :

$$\frac{n\Delta y}{f(n-\omega)}=\frac{\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}.$$

We have given the value of Δ before expressed by integral formulas ; but since from the above there shall be

$$\Delta=\frac{1}{\mathfrak{A}(n+\omega)},$$

we will have from the infinite product

$$\Delta = \frac{1}{n+\omega} \cdot \frac{1(1-2n)}{(1-n)^2-\omega^2} \cdot \frac{1(1+2n)}{(1+n)^2-\omega^2} \cdot \frac{2(2-2n)}{(2-n)^2-\omega^2} \cdot \frac{2(2+2n)}{(2+n)^2-\omega^2} \cdot \text{etc.},$$

from which it is agreed more clearly, so that the value Δ may become infinite from the integral formulas, as often as there were

$$\omega = \pm(i \pm n),$$

with i denoting some whole number, truly the same value Δ to vanish in the cases, in which there is

$$n = \pm\tfrac{1}{2}.$$

Then truly also it will help to have noted, if with ω becoming $1+\omega$, the value of Δ may be observed Δ', to become :

$$\Delta' = \frac{(1-n+\omega)\Delta}{n-\omega}.$$

And if in a similar manner Δ'' may come about with $2+\omega$ assumed in place of the value ω, there will become :

$$\Delta'' = \frac{-(2-n+\omega)\Delta'}{-(1-n+\omega)} = \frac{-(2-n+\omega)\Delta}{n-\omega}.$$

COROLLARY 1

41. In as much as the quantity Δ may depend on ω, it may be considered as a function of this, and may be designated in this manner

$$\Delta = f:\omega;$$

therefore then there will become:

$$f:(1+\omega) = \frac{n-1+\omega}{n-\omega} f:\omega$$

and

$$f:(2+\omega) = \frac{n-2+\omega}{n-\omega} f:\omega$$

etc.

Whereby if ω may denote some whole number, this theorem will be had

$$\frac{f:(i+\omega)}{n-i-\omega} = \frac{f:\omega}{n-\omega}.$$

COROLLARY 2

42. Since then with ω assumed negative there shall be :

$$f:(-\omega) = \frac{n+\omega}{n-\omega} f:\omega,$$

there will be

$$\frac{f:-\omega}{n+\omega}=\frac{f:\omega}{n-\omega},$$

hence also in general

$$\frac{f:(i-\omega)}{n-i+\omega}=\frac{f:\omega}{n-\omega}.$$

SCHOLIUM

43. Here the case corresponds to that, which we have set out above in § 25, where the applied lines also were the sines of the abscissas ; and indeed for the present case it will be required to have set

$$\theta=\pi\,,$$

so that there shall be

$$f=\sin.n\pi$$

and all the given points shall be situated on the line of the sines. But hence it does not follow this same curve, which the equation found shows, to be the line of the sines, since innumerable other curves may be able to pass through the same given points. Whereby by no means even now is it certain the value of y of the abscissa $x=\omega\pi$ are in agreement, and defined by this equation

$$\frac{n\Delta y}{(n-\omega)\sin.n\pi}=\frac{\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}$$

to be equal to the sine of the arc $n\omega$, so that there shall become $y=\sin.n\omega$, even if this shall be true in the cases $\omega=\pm(i\pm n)$ and $\omega=0$. Indeed also we have seen in the case above, where ω is the minimum quantity, the equation agreeing with the truth by taking $y=\sin.n\omega$, thus so that there shall become

$$\Delta=\frac{\pi\cos.n\pi}{\sin.n\pi}$$

with there being

$$\Delta=\frac{\int z^{n-1}dz(1-z)^{-n}\cdot\int z^{n-1}dz(1-z)^{n}}{\int z^{n-1}dz(1-z)^{-2n}},$$

just a I have demonstrated there. But so that this matter may be able to be explored more easily in general, for the value Δ being expressed more suitably, I observe to be

$$\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}}=\frac{\int z^{\omega-n}dz(1-z)^{-n-\omega}}{\int dz(1-z)^{-2n}}=(1-2n)\int z^{\omega-n}dz(1-z)^{-n-\omega},$$

since there shall be

$$n<\tfrac{1}{2}\,,$$

from which there shall become :

$$\varDelta = (1-2n)\int z^{\omega-n}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}.$$

Truly if in general there shall be

$$y = \sin.\omega\pi ,$$

there may become :

$$\varDelta = \frac{(n-\omega)\pi\sin.n\pi\cos.n\pi}{n\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}.$$

This question therefore returns, whether or not this equation

$$(1-2n)\int z^{\omega-n}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega} = \frac{(n-\omega)\pi\sin.n\pi\cos.n\pi}{n\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}$$

shall be true also in other cases besides those mentioned above. Towards this end, we will consider the case, where

$$n = \tfrac{1}{4} \text{ and } \omega = \tfrac{1}{2},$$

where indeed the latter part becomes

$$\frac{-\frac{1}{4}\cdot\pi\cdot\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}}{-\frac{1}{4}\cdot\pi\cdot\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}} = \pi ;$$

truly the former part will be

$$= \frac{1}{2}\int\frac{z^{\frac{1}{4}}dz}{(1-z)^{\frac{3}{4}}}\cdot\int\frac{z^{-\frac{1}{4}}dz}{(1-z)^{\frac{1}{4}}},$$

which on putting

$$z = v^4$$

will change into this form

$$8\int\frac{v^4dv}{\sqrt[4]{(1-v^4)^3}}\cdot\int\frac{v^3dv}{\sqrt[4]{(1-v^4)}} = 4\int\frac{dv}{\sqrt[4]{(1-v^4)^3}}\cdot\int\frac{vvdv}{\sqrt[4]{(1-v^4)}},$$

of which the value through that, which I have shown concerning formulas of this kind [E122], actually will become $=\pi$, which now therefore is a significant document for the truth of our equations, just as may be able to be shown perfectly in the following manner.

THEOREM

44. *In whatever manner the two numbers n and ω may be taken, this equation will be agreed to be true*

$$(1-2n)\int\frac{z^{\omega-n}dz}{(1-z)^{n+\omega}}\cdot\int\frac{z^{n+\omega-1}dz}{(1-z)^{\omega-n}}=\frac{(n-\omega)\pi\sin.n\pi\cos.n\pi}{n\sin.(n-\omega)\cdot\pi\sin.(n+\omega)\pi},$$

if indeed the integration of these formulas may be extended from the limit $z=0$ *to the limit* $z=1$.

DEMONSTRATION

So that we may reduce these integral formulas to the form that I have treated [E321], we may put

$$n+\omega=\tfrac{\mu}{\lambda}\ \text{ and }\ \omega-n=\tfrac{\nu}{\lambda},$$

so that there shall be

$$2n=\tfrac{\mu-\nu}{\lambda},$$

and it will be required to demonstrate this equation :

$$\frac{\lambda-\mu+\nu}{\lambda}\int\frac{z^{\frac{\mu}{\lambda}}dz}{\sqrt[\lambda]{(1-z)^{\mu}}}\cdot\int\frac{z^{\frac{\mu-\lambda}{\lambda}}dz}{\sqrt[\lambda]{(1-z)^{\nu}}}=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}\cdot\sin.\frac{\mu\pi}{\lambda}}.$$

Now there may be put $z=v^{\lambda}$, and there will be had :

$$\lambda(\lambda-\mu+\nu)\int\frac{v^{\lambda+\nu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu}}}\cdot\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\nu}}}=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}\cdot\sin.\frac{\mu\pi}{\lambda}};$$

but I undertake these integral formulas being expressed there in the usual manner, the first member will be represented thus

$$\lambda(\lambda-\mu+\nu)(\tfrac{\lambda+\nu}{\lambda-\nu})(\tfrac{\mu}{\lambda-\nu}),$$

which by the first reduction

$$\left(\frac{p}{q}\right)=\frac{p-\lambda}{p+q-\lambda}\left(\frac{p-\lambda}{q}\right)$$

will become

$$\lambda\nu\left(\frac{\nu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right)=\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right).$$

Truly this reduction

$$\frac{\lambda-q}{p}\left(\frac{p+q-\lambda}{q}\right)=\frac{\pi}{\lambda p\sin.\frac{q\pi}{\lambda}},$$

on supposing

$$p = \mu - \nu \text{ and } q = \mu$$

gives

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right) = \frac{\pi}{\lambda(\mu-\nu)\sin.\frac{\mu\pi}{\lambda}}.$$

Truly there is also

$$\left(\frac{\lambda-\nu}{\nu}\right) = \frac{\pi}{\lambda\sin.\frac{\nu\pi}{\lambda}},$$

of which the product is

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\mu}{\mu-\nu}\right) = \frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin.\frac{\mu\pi}{\lambda}\cdot\sin.\frac{\nu\pi}{\lambda}}.$$

Again, since in general there shall be

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right) = \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),$$

on taking

$$p = \lambda - \mu,\ \ q = \mu - \nu,\ \text{ and }\ \ r = \nu$$

there will become

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right) = \left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\mu+\nu}{\mu-\nu}\right)$$

and on account of

$$\left(\frac{\lambda-p}{p}\right) = \frac{\pi}{\lambda\sin.\frac{p\pi}{\lambda}}$$

on taking

$$p = \mu - \nu$$

there will be

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right) = \left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi}$$

and thus

$$\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi} = \frac{\pi\pi}{\lambda\lambda(\lambda-\nu)\sin.\frac{\mu\pi}{\lambda}\sin.\frac{\nu\pi}{\lambda}};$$

from which the first form is reduced to this form

$$\lambda\nu\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right) = \frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}\sin.\frac{\nu\pi}{\lambda}},$$

which is that same equation required to be shown.

COROLLARY 1

45. Therefore in the discussion about integral formulas of this kind

$$\int \frac{v^{p-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-q}}},$$

which I designate by this character

$$\left(\frac{p}{q}\right),$$

to which $\left(\frac{q}{p}\right)$ is equivalent, this reduction is of the greatest importance, from which I have shown to be :

$$\lambda v\left(\frac{\lambda-v}{v}\right)\left(\frac{\lambda-v}{\mu}\right)=\frac{v}{\mu-v}\cdot\frac{\pi\sin.\frac{\mu-v}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}\sin.\frac{v\pi}{\lambda}},$$

thus so that the product of two such integral formulas $\left(\frac{\lambda-v}{v}\right)\left(\frac{\lambda-v}{\mu}\right)$ may be shown by angles alone.

COROLLARY 2

46. If in the value for Δ first found equally there may be put

$$n+\omega=\frac{\mu}{\lambda} \text{ and } n-\omega=\frac{v}{\lambda}$$

then truly

$$z=v^{\lambda},$$

there will be :

$$\Delta=\lambda\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu}}}\cdot\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{v}}}:\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu-v}}}$$

and thus in the manner indicated:

$$\Delta=\frac{\lambda\left(\frac{\mu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-v}\right)}{\left(\frac{\mu}{\lambda-\mu+v}\right)}$$

or

$$\Delta=\frac{\lambda\left(\frac{\lambda-\mu}{\mu}\right)\left(\frac{\lambda-v}{\mu}\right)}{\left(\frac{\lambda-\mu+v}{\mu}\right)}$$

Likewise the true value is also :

$$\Delta=\frac{v\pi}{\mu-v}\cdot\frac{\sin.\frac{\mu-v}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}\sin.\frac{v\pi}{\lambda}}.$$

COROLLARY 3

47. Therefore since for this last formula there shall be at once :

$$\left(\frac{\lambda-\mu}{\mu}\right)=\frac{\pi}{\pi\sin.\frac{\mu\pi}{\lambda}},$$

there will be

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}=\frac{\nu}{\mu-\nu}\cdot\frac{\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}},$$

the truth of which is shown from this general theorem

$$\frac{\left(\frac{q}{p}\right)}{\left(\frac{r}{p}\right)}=\frac{\left(\frac{p+r}{q}\right)}{\left(\frac{p+q}{r}\right)};$$

for there will become:

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}=\frac{\left(\frac{\lambda+\nu}{\lambda-\nu}\right)}{\left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)}=\frac{\nu}{\mu-\nu}\cdot\frac{\left(\frac{\nu}{\lambda-\nu}\right)}{\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)}$$

on account of

$$\left(\frac{\lambda+\nu}{\lambda-\nu}\right)=\frac{\nu}{\lambda}\cdot\left(\frac{\nu}{\lambda-\nu}\right)\text{ and }\left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\mu-\nu}{\lambda}\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right);$$

then truly there is

$$\left(\frac{\nu}{\mu-\nu}\right)=\frac{\pi}{\lambda\sin.\frac{\nu\pi}{\lambda}}\quad\text{et}\quad\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi}.$$

EXAMPLE II

48. *The applied lines will be, to which the abscissas*

$$n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta,\ (3-n)\theta, etc.$$

correspond, will be,

$$p=f,\ q=-f,\ r=+f,\ \ s=-f,\ t=+f,\ u=-f\ \ etc.,$$

and by a finite equation the relation may be investigated in general between the $x=\theta\omega$ *and the applied line* $=y$.

The general equation of paragraph 36 adapted to this case provides

$$\frac{2n\Delta y}{f(n-\omega)}=\frac{1}{n-\omega}+\frac{1}{1-n-\omega}+\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}+\text{etc.}$$
$$-\frac{1}{n+\omega}-\frac{1}{1-n+\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{1}{2+n+\omega}-\text{etc.}$$

where now we recognize to be

$$\Delta = \frac{(n-\omega)\pi \sin.2n\pi}{2n\sin.(n-\omega)\pi \cdot \sin.(n+\omega)\pi}.$$

But that series from § 39 becomes

$$\text{I minus III} = \frac{\pi}{\text{tang}.(n-\omega)\pi} - \frac{\pi}{\text{tang}.(n+\omega)\pi} = \frac{\pi \sin.2\omega\pi}{\sin.(n-\omega)\pi \cdot \sin.(n+\omega)\pi},$$

with which sum substituted there is produced :

$$\frac{y}{f} \cdot \frac{\pi \sin.2n\pi}{\sin.(n-\omega)\pi \cdot \sin.(n+\omega)\pi} = \frac{\pi \sin.2\omega\pi}{\sin.(n-\omega)\pi \cdot \sin.(n+\omega)\pi}$$

or

$$y = \frac{f\sin.2\omega\pi}{\sin.2n\pi} = \frac{f\sin.\frac{2x\pi}{\theta}}{\sin.2n\pi}.$$

Therefore this curve again is the line of the sines, and if there may be taken $\theta = 2\pi$, so that there shall become $f = sin.2n\pi$ *,* the applied line shall be $y = \sin.x$.

COROLLARY 1

49. If there may be taken

$$\theta = \pi \text{ and } f = \text{tang}.n\theta = \text{tang}.n\pi,$$

the given points will be on the line of the tangents; neither yet will the curve found be the line of the tangents ; but its nature will be expressed by this equation:

$$y = \frac{\text{tang}.n\pi \cdot \sin.2x}{\sin.2n\pi} = \frac{\sin.2x}{2\cos.n\pi^2} = \frac{\sin.2x}{1+\cos.2n\pi},$$

and here there will be $y = \text{tang}.x$, as often as there were $x = \pm(i \pm n)\pi$.

COROLLARY 2

50. If in the solution of the earlier example, where there was

$$p = f,\ q = f,\ r = -f,\ s = -f,\ t = f,\ u = f \ \ etc.,$$

we could have put at once the value found in place of Δ , and this equation would have been produced:

$$y = \frac{f\sin.\omega\pi}{\sin.n\pi}$$

From which it would have been seen on taking $\theta = \pi$ and $f = \sin.n\pi$ that curve itself to be the line of the sines.

SCHOLIUM

51. Generally it deserves to be mentioned, since in problem 4, where the abscissas constitute an interrupted arithmetical progression, the value of the quantity Δ will be able to be shown by an absolute angle, since still in problem 3, where the given abscissas constituted a true arithmetic progression, the integral formula Δ in general by no means may be able to be expressed by an angle. Since indeed it shall become there :

$$\Delta = \int z^{n-\omega-1} dz (1-z)^{n-\omega-1},$$

this formula on putting $n-\omega = \frac{\nu}{\lambda}$ and $z = v^{\lambda}$ will change into

$$\Delta = \lambda \int \frac{v^{\nu-1} dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-\nu}}} \text{ or } \Delta = \lambda \left(\frac{\nu}{\nu}\right),$$

which formula being integrated can be implied to be maximally transcending. And if in that problem the given applied lines may be put in place

$$p = f,\ q = -f,\ r = f,\ s = -f,\ t = f,\ u = -f \ etc.$$

and $n = \frac{1}{2}$, the equation for the curve passing through these points will be

$$\frac{\Delta y}{2(1+2\omega)\omega f} = \frac{4}{1-4\omega\omega} + \frac{4}{9-4\omega\omega} + \frac{4}{25-4\omega\omega} + \text{etc.}$$

or

$$\frac{\Delta y}{2 f \omega (1+2\omega)} = \frac{\pi}{2\omega} \operatorname{tang.} \omega\pi,$$

thus so that there shall be

$$y = \frac{\pi f (1+2\omega) \operatorname{tang.} \omega\pi}{\Delta},$$

from which, even if there may be taken:

$$\theta = \pi \text{ and } f = \sin. n\theta = \sin. \tfrac{1}{2}\pi = 1,$$

evidently there does not follow to become $y = sin.\theta\omega = sin.\omega\pi$. Now since in the former example it shall be certain to become :

$$y = \frac{f \sin.\omega\pi}{\sin.n\pi},$$

thus we may set out the same case from the first problem, so that we may investigate the values of the individual coefficients A, B, C, D etc.

PROBLEM 5

52. *The general equation thus set up from Problem I above to determine, so that with these abscissas*

$$x = n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta,\ etc.$$

these applied lines may correspond

$$y = +f,\quad +f,\quad -f,\quad -f,\quad +f,\quad etc.,$$

SOLUTION

There may be put in place as before $x = \theta\omega$ and the equation sought may be considered under this form:

$$\begin{aligned} &y = A\omega + B\omega(\omega\omega - nn) + C\omega(\omega\omega - nn)(\omega\omega - (1-n)^2) \\ &+ D\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2) \\ &+ E\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2)(\omega\omega - (2-n)^2) \\ &+ etc., \end{aligned}$$

from which these equations are deduced :

$$\begin{aligned} &\tfrac{f}{n} = A, \\ &\tfrac{f}{1-n} = A + B\cdot 1(1-2n), \\ &\tfrac{-f}{1+n} = A + B\cdot 1(1+2n) + C\cdot 1(1+2n)\cdot 2\cdot 2n \\ &\tfrac{-f}{1+n} = A + B\cdot 2(2-2n) + C\cdot 2(2-2n)\cdot 1(3-2n) \\ &\quad + d\cdot 2(2-2n)\cdot 1(3-2n)\cdot 3(1-2n) \\ &\qquad \text{etc.} \end{aligned}$$

and hence the following values of the coefficients :

$$A = \tfrac{f}{n},\ B = \tfrac{-f}{n(1-n)},\ C = \tfrac{f}{2n(1-n)(1+n)},\ D = \tfrac{-f}{6n(1-n)(1+n)(2-n)},$$
$$E = \tfrac{f}{24n(1-n)(1+n)(2-n)(2+n)},\ \text{etc.;}$$

which progression since it shall be simple enough, our series for the value of y, which we know now to be

$$= \tfrac{f\sin.\omega\pi}{\sin.n\pi},$$

therefore may merit more attention; and this shall be

$$\frac{\sin.\omega\pi}{\sin.n\pi}=\frac{\omega}{n}-\frac{\omega}{n}\cdot\frac{\omega\omega-nn}{1(1-n)}+\frac{\omega}{n}\cdot\frac{\omega\omega-nn}{1(1-n)}\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}$$
$$-\frac{\omega}{n}\cdot\frac{\omega\omega-nn}{1(1-n)}\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}\cdot\frac{\omega\omega-(1+n)^2}{3(2-n)}+\text{etc.},$$

or if Π may denote continually the preceding term, the whole will become :

$$\frac{\sin.\omega\pi}{\sin.n\pi}=\frac{\omega}{n}-\Pi\cdot\frac{\omega\omega-nn}{1(1-n)}+\Pi\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}$$
$$-\Pi\cdot\frac{\omega\omega-(1+n)^2}{3(2-n)}+\Pi\cdot\frac{\omega\omega-(2-n)^2}{4(2+n)}-\Pi\cdot\frac{\omega\omega-(2+n)^2}{5(3-n)}+\text{etc.}$$

But if all the terms may be desired having the same sign, there will be

$$\frac{\sin.\omega\pi}{\sin.n\pi}=\frac{\omega}{n}+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}$$
$$+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}\cdot\frac{(1+n)^2-\omega\omega}{3(2-n)}+$$
$$\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}\cdot\frac{(1+n)^2-\omega\omega}{3(2-n)}\cdot\frac{(2-n)^2-\omega\omega}{4(2+n)}$$
$$+\text{etc.},$$

This series therefore is seen to be worthy of greater attention, because it recedes the most from the customary ratio of the series, and in that the two arbitrary numbers n and ω occur.

COROLLARY 1

53. If the number ω may vanish, so that there may become $\sin.\ \omega\pi=\omega\pi$, with the division per ω put in place, this equation will be had

$$\frac{\pi}{\sin.n\pi}=\frac{1}{n}+\frac{n}{1(1-n)}+\frac{n(1-n)}{1\cdot2(1+n)}+\frac{n(1-n)(1+n)}{1\cdot2\cdot3(2-n)}$$
$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4(2+n)}+\text{etc.},$$

from which on assuming $n=\frac{1}{2}$ on account of $\sin.\frac{\pi}{2}=1$ there will be :

$$\pi=2+1+\tfrac{1\cdot1\cdot2}{2\cdot4\cdot3}+\tfrac{1\cdot1\cdot3\cdot2}{2\cdot4\cdot6\cdot3}+\tfrac{1\cdot1\cdot3\cdot3\cdot2}{2\cdot4\cdot6\cdot8\cdot5}+\tfrac{1\cdot1\cdot3\cdot3\cdot5\cdot2}{2\cdot4\cdot6\cdot8\cdot10\cdot5}+\text{etc.}$$

or

$$\pi=2+\frac{1}{2\cdot2^1\cdot3}+\frac{1\cdot3}{2\cdot4\cdot2^3\cdot5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot2^5\cdot7}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot2^7\cdot9}+\text{etc.}$$
$$+1+\frac{1}{2\cdot2^2\cdot3}+\frac{1\cdot3}{2\cdot4\cdot2^4\cdot5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot2^6\cdot7}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot2^8\cdot9}+\text{etc.};$$

of which the latter series, since it shall be of half the former, the sum of the latter will be $=\frac{\pi}{3}$, the ratio of which indeed is clear, because there shall be

$$\int\frac{dx}{\sqrt{(1-xx)}}=\text{ang.sin}.x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}+\text{etc.},$$

from which this series becomes $=\frac{\text{ang.sin}.x}{x}$ on assuming $x=\frac{1}{2}$ and thus $=2\frac{\pi}{6}=\frac{\pi}{3}$.

COROLLARY 2

54. But if the other number *n* may vanish, so that there becomes $\sin.n\pi=n\pi$, and the equation may be multiplied by *n*, there may arise :

$$\frac{\sin.\omega\pi}{\pi}=\omega-\frac{\omega^3}{1}+\frac{\omega^3(\omega^2-1)}{1\cdot2\cdot1^2}-\frac{\omega^3(\omega^2-1)(\omega^2-1)}{1\cdot2\cdot3\cdot1^2\cdot2}+\frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)}{1\cdot2\cdot3\cdot4\cdot1^2\cdot2^2}$$
$$-\frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)(\omega^2-4)}{1\cdot2\cdot3\cdot4\cdot5\cdot1^2\cdot2^2\cdot3}+\text{etc.},$$

which series divided by ω is resolved into the two following series

$$\frac{\sin.\omega\pi}{\omega\pi}=1+\frac{\omega^2(\omega^2-1)}{1\cdot2\cdot1^2}+\frac{\omega^2(\omega^2-1)^2(\omega^2-4)}{1\cdot2\cdot3\cdot4\cdot1^2\cdot2^2}+\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot1^2\cdot2^2\cdot3^2}+\text{etc.},$$
$$-\frac{\omega^2}{1}-\frac{\omega^2(\omega^2-1)^2}{1\cdot2\cdot3\cdot1^2\cdot2}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2}{1\cdot2\cdot3\cdot4\cdot5\cdot1^2\cdot2^2\cdot3}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)^2}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot1^2\cdot2^2\cdot3^2\cdot4}-\text{etc.}$$

Here we may take $\omega=\frac{1}{2}$; there will become

$$\frac{2}{\pi}=1-\frac{1\cdot1\cdot3}{1\cdot1\cdot1\cdot2^5}-\frac{1\cdot1\cdot3\cdot1\cdot3\cdot5}{1\cdot1\cdot1\cdot2\cdot2\cdot2^{10}}-\frac{1\cdot1\cdot3\cdot1\cdot3\cdot5\cdot3\cdot5\cdot7}{1\cdot1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot3\cdot3\cdot2^{10}}-\text{etc.},$$
$$-\frac{1\cdot1}{2^2}-\frac{1\cdot1\cdot1\cdot3\cdot1\cdot3}{1\cdot1\cdot1\cdot2\cdot2\cdot3\cdot2^6}-\frac{1\cdot1\cdot1\cdot3\cdot1\cdot3\cdot3\cdot5\cdot3\cdot5}{1\cdot1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot3\cdot4\cdot5\cdot2^{10}}-\text{etc.},$$

thus the latter series can be returned :

$$-\frac{1}{2^2}-\frac{1\cdot1\cdot1\cdot3}{1\cdot1\cdot2\cdot2\cdot2^7}-\frac{1\cdot1\cdot3\cdot3\cdot3\cdot5}{1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot2^{12}}-\frac{1\cdot1\cdot3\cdot3\cdot3\cdot5\cdot5\cdot5\cdot7}{1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot3\cdot3\cdot4\cdot2^{17}}-\text{etc.}$$

COROLLARY 3

55. If there were $n=\frac{1}{2}$, so that there shall be $\sin.n\pi=1$, the factors will be, from which it will be required to have formed the individual terms of the series,

$$\frac{2\omega}{1}\cdot\frac{1-4\omega\omega}{1\cdot2}\cdot\frac{1-4\omega\omega}{3\cdot4}\cdot\frac{9-4\omega\omega}{3\cdot6}\cdot\frac{9-4\omega\omega}{5\cdot8}\cdot\frac{25-4\omega\omega}{5\cdot10}\cdot\frac{25-4\omega\omega}{7\cdot12}\cdot\text{etc.}$$

and the sum of the series will be $\sin.n\pi$, evidently

$$\sin.\omega\pi=2\omega+\frac{2\omega(1-4\omega\omega)}{1\cdot2}+\frac{2\omega(1-4\omega\omega)^2}{1\cdot2\cdot3\cdot4}+\frac{2\omega(1-4\omega\omega)^2(9-4\omega\omega)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}+\text{etc.},$$

from which on taking $\omega=1$ there must become :

$$0=2-3+\frac{3}{2^2}+\frac{5}{2^3\cdot3}+\frac{5}{2^6\cdot3}+\frac{7}{2^7\cdot5}+\frac{7}{2^9\cdot5}+\frac{9}{2^{10}\cdot7}+\frac{5\cdot9}{2^{14}\cdot7}+\text{etc.}$$

of which the true calculation required to be put in place soon will become apparent.

SCHOLIUM

56. For this case also the solution found above deserves to be considered more carefully, which from § 36 on account of

$$\varDelta=\frac{(n-\omega)\pi\sin.2n\pi}{2n\sin.(n-\omega)\pi\cdot\sin(n+\omega)\pi}\text{ and }y=\frac{f\sin.\omega\pi}{\sin.n\pi},$$

since there shall be

$$p=f,\quad q=f,\quad r=-f,\quad s=-f,\quad t=f,\ u=f\quad etc.,$$

may be contained in this equation

$$\frac{\pi\cos.n\pi\cdot\sin.\omega\pi}{\omega\sin.(n-\omega)\pi\cdot\sin(n+\omega)\pi}$$
$$=\frac{1}{nn-\omega\omega}-\frac{1}{(1-n)^2-\omega^2}-\frac{1}{(1+n)^2-\omega^2}+\frac{1}{(2-n)^2-\omega^2}+\frac{1}{(2+n)^2-\omega^2}-\text{etc.},$$

which series may differ maximally from that, as in the manner we have found. But concerning this other series, I observe the following :

I. If ω may vanish, to become

$$\frac{\pi\pi\cos.n\pi}{(\sin.n\pi)^2}=\frac{1}{nn}-\frac{1}{(1-n)^2}-\frac{1}{(1+n)^2}+\frac{1}{(2-n)^2}+\frac{1}{(2+n)^2}-\frac{1}{(3-n)^2}-\text{etc.};$$

but if in addition n may vanish, on account of $\sin.n\pi=n\pi$, the following inconvenience arises :

$$\frac{1}{nn}=\frac{1}{nn}-\frac{2}{1}+\frac{2}{4}-\frac{2}{9}+\frac{2}{16}-\text{etc.}$$

Moreover towards removing this, we may consider only the number n as not vanishing, and since there shall be

$$\cos.n\pi = 1 - \tfrac{1}{2}nn\pi\pi$$

and

$$\sin.n\pi = n\pi - \tfrac{1}{6}n^3\pi^3 = n\pi(1 - \tfrac{1}{6}nn\pi\pi),$$

there will become

$$\frac{\cos.n\pi}{(\sin.n\pi)^2} = \frac{1-\frac{1}{2}nn\pi\pi}{nn\pi\pi(1-\frac{1}{3}nn\pi\pi)} = \frac{1-\frac{1}{6}nn\pi\pi}{nn\pi\pi};$$

from which this true equation is obtained :

$$\frac{1}{nn} - \frac{1}{6}\pi\pi = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \frac{2}{25} + \text{etc.}$$

Indeed there is :

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \text{etc.} = \tfrac{1}{12}\pi\pi.$$

II. Now we may put $n = 0$ and we will have

$$-\frac{\pi}{\omega\sin.\omega\pi} = -\frac{1}{\omega^2} - \frac{1}{1-\omega^2} - \frac{1}{1-\omega^2} + \frac{1}{4-\omega^2} + \frac{1}{4-\omega^2} - \frac{1}{9-\omega^2} - \frac{1}{9-\omega^2} + \text{etc.},$$

or

$$\frac{\pi}{\omega\sin.\omega\pi} = \frac{1}{\omega^2} + \frac{2}{1-\omega^2} - \frac{2}{4-\omega^2} + \frac{2}{9-\omega^2} - \frac{2}{16-\omega^2} + \frac{2}{25-\omega^2} - \text{etc.},$$

from which we may follow with this memorable summation:

$$\frac{1}{1-\omega^2} - \frac{1}{4-\omega^2} + \frac{1}{9-\omega^2} - \frac{1}{16-\omega^2} + \text{etc.} = \frac{\pi}{2\omega\sin.\omega\pi} - \frac{1}{2\omega\omega},$$

from the small amount

$$\sin.\omega\pi = \omega\pi(1 - \tfrac{1}{6}\omega^2\pi^2)$$

the sum of the series

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \text{etc.}$$

is deduced as before :

$$\frac{1}{2\omega\omega(1-\frac{1}{6}\omega^2\pi^2)} - \frac{1}{2\omega\omega} = \frac{1}{12}\pi\pi.$$

III. If there may be taken $n = \frac{1}{2}$, on account of $\cos.\, n\pi = 0$ that series vanishes also, while evidently all the terms actually cancel each other out. But what may eventuate, if n may differ an infinitely small amount from $\frac{1}{2}$, a differentiation may be put in place with n taken to be variable, from which there becomes :

$$\frac{n\pi\sin.n\pi\sin.\omega\pi(1+\cos.(n-\omega)\pi\cdot\cos.(n+\omega)\pi)}{\omega(\sin.(n-\omega)\pi\cdot\sin(n+\omega)\pi)^2}$$
$$=\frac{2n}{(nn-\omega\omega)^2}-\frac{2(1-n)}{((1-n)^2-\omega^2)^2}+\frac{2(1+n)}{((1+n)^2-\omega^2)^2}+\frac{2(2-n)}{((2-n)^2-\omega^2)^2}-\frac{2(2+n)}{((2+n)^2-\omega^2)^2}-\text{etc.},$$

Therefore now we may take $n=\frac{1}{2}$, and there will be

$$-\frac{\pi\pi\sin.\omega\pi}{\omega(\cos.\omega\pi)^2}=-\frac{16}{(1-4\omega^2)^2}-\frac{16}{(1-4\omega^2)^2}+\frac{3\cdot16}{(9-4\omega^2)^2}+\frac{3\cdot16}{(9-4\omega^2)^2}-\text{etc.}$$

or

$$-\frac{\pi\pi\sin.\omega\pi}{32\omega(\cos.\omega\pi)^2}=\frac{1}{(1-4\omega^2)^2}-\frac{3}{(9-4\omega^2)^2}+\frac{5}{(25-4\omega^2)^2}-\frac{7}{(49-4\omega^2)^2}+\text{etc.}$$

from which on taking $\omega=0$ there follows to become :

$$\frac{\pi^3}{12}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.},$$

which indeed agrees from elsewhere.

Truly it is observed the series found in the present problem with much more effort. Thus why not establish the case in corollary 1, even if it is especially particular, the setting out more carefully is worthwhile, as we try to expedite in the following problem.

PROBLEM 6

57. *If n shall be some number, to inquire into the sum of this series*

$$s=\frac{1}{n}+\frac{n}{1(1-n)}+\frac{n(1-n)}{1\cdot2(1+n)}+\frac{n(1-n)(1+n)}{1\cdot2\cdot3(2-n)}+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4(2+n)}+etc.,$$

as indeed we have found before [§ 53] *to be*

$$s=\frac{\pi}{\sin.n\pi}.$$

SOLUTION

Since in this series the law of the progression shall be interrupted, it may be convenient to render this in two parts. Therefore we may put

$$P=\frac{1}{n}+\frac{n(1-n)}{1\cdot2(1+n)}+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4(2+n)}+\frac{n(1-n)(1+n)(2-n)(2+n)(3-n)}{1\cdot2\cdot3\cdot4\cdot5\cdot6(3+n)}+etc.,$$
$$Q=\frac{n}{1(1-n)}+\frac{n(1-n)(1+n)}{1\cdot2\cdot3(2-n)}+\frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot2\cdot3\cdot4\cdot5(3-n)}+etc.,$$

thus so that there shall become

$$s = P + Q.$$

now inquiring into the sum of the series I call into help the following sought from the theory of angles :

$$\frac{\cos.\mu\varphi}{\cos.\varphi} = 1 + \frac{(1-\mu)(1+\mu)}{1\cdot 2}\sin.\varphi^2 + \frac{(1-\mu)(1+\mu)(3-\mu)(3+\mu)}{1\cdot 2\cdot 3\cdot 4}\sin.\varphi^4 + \text{etc.},$$

$$\frac{\sin.\nu\varphi}{\cos.\varphi} = \nu\sin.\varphi + \frac{\nu(2-\nu)(2+\nu)}{1\cdot 2\cdot 3}\sin.\varphi^3 + \frac{\nu(2-\nu)(2+\nu)(4-\nu)(4+\nu)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\sin.\varphi^5 + \text{etc.}$$

and indeed in the first place I will adapt that to the former form P . Therefore since these fractions

$$\frac{(1-\mu)(1+\mu)}{n(1-n)},\quad \frac{(3-\mu)(3+\mu)}{(1+n)(2-n)},\quad \frac{(5-\mu)(5+\mu)}{(2+n)(3-n)}\ \text{etc.}$$

must be equal, I conclude there must be taken $\mu = 1-2n$, from which there becomes

$$\frac{\cos.(1-2n)\varphi}{\cos.\varphi} = 1 + \frac{n(1-n)}{1\cdot 2}\cdot 2^2\sin.\varphi^2 + \frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4}2^4\sin.\varphi^4 + \text{etc.}$$

We may multiply by $d\varphi\sin.\varphi^{2n-1}\cos.\varphi$ and integrate, there becomes

$$\int d\varphi\sin.\varphi^{2n-1}\cos.(1-2n)\varphi = \frac{1}{2n}\sin.\varphi^{2n} + \frac{n(1-n)}{1\cdot 2(n+1)}\cdot 2\sin.\varphi^{2n+2}$$
$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(n+2)}2^3\sin.\varphi^{2n+4} + \text{etc.}$$

Now after the integration there may be put $\sin.\varphi = \frac{1}{2}$ or $\varphi = 30°$, and there will become

$$P = 2^{2n+1}\int d\varphi\sin.\varphi^{2n-1}\cos.(1-2n)\varphi\ ;$$

truly the series Q will be deduced easily from the other known sum by taking $\nu = 2n$, from which there becomes

$$\frac{\sin.2n\varphi}{\cos.\varphi} = n\cdot 2\sin.\varphi + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3}\cdot 2^3\sin.\varphi^3$$
$$+\frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot 2\cdot 3\cdot 4\cdot 5}2^5\sin.\varphi^5 + \text{etc.}$$

This may be multiplied by $d\varphi\sin.\varphi^{-2n}\cos.\varphi$ and integrated; there will become

$$\int d\varphi\sin.\varphi^{-2n}\sin.2n\varphi = \frac{n}{1(1-n)}\sin.\varphi^{2-2n} + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}2^2\sin.\varphi^{4-2n} + \text{etc.}$$

Equally there may be put for the absolute integration $\sin.\varphi = \frac{1}{2}$ or $\varphi = 30°$ and the series will be produced :

$$Q = 2^{2-2n}\int d\varphi \sin.\varphi^{-2n} \sin.2n\varphi.$$

On account of which the sum of the proposed series will be expressed thus, so that there shall be

$$s = 2^{2n+1}\int d\varphi \sin.\varphi^{2n-1} \cos.(1-2n)\varphi \; + 2^{2-2n}\int d\varphi \sin.\varphi^{-2n} \sin.2n\varphi,$$

and because this sum is known from elsewhere, there will be had

$$\frac{\pi}{\sin.n\pi} = 4\int d\varphi \cos.(1-2n)\varphi(2\sin.\varphi)^{2n-1} + 4\int d\varphi \sin.2n\varphi(2\sin.\varphi)^{-2n}.$$

COROLLARY 1

58. If there may be put $2n = \frac{1-\lambda}{2}$, there will be $1-2n = \frac{1+\lambda}{2}$, with which in place our equation becomes more concise, and there will be

$$\frac{\pi}{\sin.\frac{1-\lambda}{4}\pi} = 4\int \frac{d\varphi \cos.\frac{1+\lambda}{2}\varphi}{(2\sin.\varphi)^{\frac{1+\lambda}{2}}} + 4\int \frac{d\varphi \sin.\frac{1-\lambda}{2}}{(2\sin.\varphi)^{\frac{1-\lambda}{2}}} = \frac{\pi\sqrt{2}}{\cos.\frac{\lambda\pi}{4} - \sin.\frac{\lambda\pi}{4}}$$

on putting $\varphi = 30^\circ$ after the integration.

COROLLARY 2

59. In a similar manner on supposing λ negative there will become :

$$\frac{\pi}{\sin.\frac{1+\lambda}{4}\pi} = 4\int \frac{d\varphi \cos.\frac{1-\lambda}{2}\varphi}{(2\sin.\varphi)^{\frac{1-\lambda}{2}}} + 4\int \frac{d\varphi \sin.\frac{1+\lambda}{2}}{(2\sin.\varphi)^{\frac{1+\lambda}{2}}} = \frac{\pi\sqrt{2}}{\cos.\frac{\lambda\pi}{4} + \sin.\frac{\lambda\pi}{4}},$$

where indeed for all the cases it will help in all the cases, which it is possible to establish, the same value of the integral formula is to be found, which we have shown here.

DE EXIMIO USU METHODI INTERPOLATIONUM IN SERIERUM DOCTRINA

Commentatio 555 indicis ENESTROEMIANI
Opuscula analytica 1, 1783, p. 157-210

In methodo interpolationum eiusmodi relatio inter binas variabiles x et y quaeritur, ut, si alteri x successive dati valores *a, b, c, d* etc. tribuantur, altera y inde quoque datos valores *p, q, r, s* etc. sortiatur, seu, quod eodem redit, aequatio pro eiusmodi linea curva quaeritur, quae per quotcunque puncta data transeat. Quo maior ergo fuerit horum punctorum numerus, eo magis linea curva limitatur; interim tamen iam alia occasione observavi, etiamsi punctorum numerus in infinitum augeatur, curvam per ea transeuntem non prorsus determinari, sed semper infinitas adhuc lineas curvas exhiberi posse, quae aeque per cuncta eadem puncta sint transiturae. Quare cum methodus interpolationum pro quovis casu lineam curvam suppeditet determinatam, solutio haec semper pro maxime particulari erit habenda; verum haec ipsa circumstantia singularem quandam indolem solutionis inventae innuit, quae accuratiorem considerationem meretur. Imprimis autem ista solutionis indoles pendet a ratione, qua interpolatio instituitur seu a forma, quae aequationi generali tribuitur, in qua aequationem quaesitam contineri oportet. Quae forma cum infinitis modis constitui possit, investigationes meas ad hanc formam restringam

$$y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \varepsilon x^9 + \text{etc.},$$

quae scilicet tantum potestates impares ipsius x contineat, ita ut, qui ipsius y valores quibuscunque valoribus positivis ipsius x conveniant, iidem negative sumti valoribus iisdem negativis ipsius x respondeant; quo ipso innumerabiles aliae lineae curvae excluduntur, quae per eadem puncta essent transiturae.

PROBLEMA 1

1. *lnvenire aequationem inter binas variabiles x et y huius formae*

$$y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \text{etc.},$$

ut, si ipsi x dati valores

$$a,\ b,\ c,\ d\ \textit{etc.}$$

tribuantur, altera variabilis y itidem datos consequatur valores

$$p,\ q,\ r,\ s\ \textit{etc.}$$

SOLUTIO

Quo aequatio generalis assumta facilius ad hunc casum accommodari possit, ea hac forma exhibeatur

$$\begin{aligned}y = Ax + Bx(xx-aa) &+ Cx(xx-aa)(xx-bb)\\ &+ Dx(xx-aa)(xx-bb)(xx-cc)\\ &+ Ex(xx-aa)(xx-bb)(xx-cc)(xx-dd)\\ &+ \text{etc.},\end{aligned}$$

quae, etsi forte in infinitum progrediatur, si scilicet conditionum numerus sit infinitus, tamen pro singulis conditionibus propositis aequationes suppeditat finitas sequentes:

$$\begin{aligned}&\text{I.}\quad p = Aa,\\ &\text{II.}\quad q = Ab + Bb(bb-aa),\\ &\text{III.}\quad r = Ac + Bc(cc-aa) + Cc(cc-aa)(cc-bb),\\ &\text{IV.}\quad s = Ad + Bd(dd-aa) + Cd(dd-aa)(dd-bb)\\ &\qquad\qquad + Dd(dd-aa)(dd-bb)(dd-cc)\\ &\qquad\qquad \text{etc.},\end{aligned}$$

quae ita repraesententur:

$$\begin{aligned}&\text{I.}\quad \tfrac{p}{a} = A,\\ &\text{II.}\quad \tfrac{q}{b} = A + B(bb-aa),\\ &\text{III.}\quad \tfrac{r}{c} = A + B(cc-aa) + C(cc-aa)(cc-bb),\\ &\text{IV.}\quad \tfrac{s}{d} = A + B(dd-aa) + C(dd-aa)(dd-bb)\\ &\qquad\qquad + D(dd-aa)(dd-bb)(dd-cc)\\ &\qquad\qquad \text{etc.}\end{aligned}$$

Iam prima a singulis sequentibus subtrahatur et differentiae per coefficientes ipsius B dividantur, ut prodeant hae aequationes:

$$\begin{aligned}&\tfrac{aq-bp}{ab(bb-aa)} = q' = B,\\ &\tfrac{ar-cp}{ac(cc-aa)} = r' = B + C(cc-bb),\\ &\tfrac{as-dp}{ad(dd-aa)} = s' = B + C(dd-bb) + D(dd-bb)(dd-cc)\\ &\qquad\qquad \text{etc.}\end{aligned}$$

Nunc simili modo primam a sequentibus subtrahentes et residua per coefficentes ipsius C dividentes perveniemus ad has aequationes:

$$\frac{r'-q'}{cc-bb} = r'' = C,$$

$$\frac{s'-q'}{dd-bb} = s'' = C + D(dd - cc)$$

etc.

porroque ad hanc

$$\frac{s''-r''}{dd-cc} = D.$$

Quamobrem ex quantitatibus datis *a, b, c, d* etc. et *p, q, r, s* etc. coefficientes *A, B, C, D* etc. ita commodissime definientur: Deriventur primo ex quantitatibus datis istae:

$$P = \frac{p}{a},\ Q = \frac{q}{b},\ R = \frac{r}{c},\ S = \frac{s}{d},\ \text{etc.}$$

hincque formentur hae:

$$Q' = \frac{Q-P}{bb-aa},\quad R' = \frac{R-P}{cc-aa},\quad S' = \frac{S-P}{dd-aa},\quad T' = \frac{T-P}{ee-aa},\ \text{etc.,}$$
$$R'' = \frac{R'-Q'}{cc-bb},\quad S'' = \frac{S'-Q'}{dd-bb},\ T'' = \frac{T'-Q}{ee-bb},\ \text{etc.,}$$
$$S''' = \frac{S''-R''}{dd-cc},\ T''' = \frac{T''-R''}{ee-cc},\ \text{etc.,}$$
$$T'''' = \frac{T'''-S'''}{ee-dd},\ \text{etc.}$$

Quibus valoribus inventis habebimus

$$A = P,\ B = Q',\ C = R'',\ D = S''',\ E = T''''\ \text{etc.}$$

COROLLARIUM 1

2. Cum sit $P = \frac{p}{a}$, erit primus coefficiens

$$A = \frac{p}{a},$$

pro sequentibus vero ob

$$Q' = \frac{aq-bp}{ab(bb-aa)},\quad R' = \frac{ar-cp}{ac(cc-aa)},\quad S' = \frac{as-dp}{ad(dd-aa)},\quad T' = \frac{at-ep}{ae(ee-aa)}\quad \text{etc.}$$

erit secundus coefficiens

$$B = \frac{aq-bp}{ab(bb-aa)}$$

seu

$$B = \frac{p}{a(aa-bb)} + \frac{q}{b(bb-aa)}.$$

COROLLARIUM 2

3. Cum porro sit

$$R'' = \frac{ar-cp}{ac(cc-aa)(cc-bb)} - \frac{aq-bp}{ab(bb-aa)(cc-bb)},$$

fiet

$$C = \frac{p}{a(aa-bb)(aa-cc)} + \frac{q}{b(bb-aa)(bb-cc)} + \frac{r}{c(cc-aa)(cc-bb)}.$$

COROLLARIUM 3

4. Simili modo calculum ulterius prosequendo reperietur

$$D = \frac{p}{a(aa-bb)(aa-cc)(aa-dd)} + \frac{q}{b(bb-aa)(bb-cc)(bb-dd)}$$
$$+ \frac{r}{c(cc-aa)(cc-bb)(cc-dd)} + \frac{s}{d(dd-aa)(dd-bb)(dd-cc)},$$

unde formam sequentium quantitatum *E, F, G* etc. iam tuto coniectura assequi licet.

SCHOLION 1

5. Plerumque autem expeditius singulorum coefficientium *A, B, C, D, E* etc. valores ex praecedentibus definiuntur. Ex aequationibus enim fundamentalibus deducuntur formulae sequentes:

$$A = \frac{p}{a},$$
$$B = \frac{q-bA}{b(bb-aa)},$$
$$C = \frac{r-cA}{c(cc-aa)(cc-bb)} - \frac{B}{cc-bb}.$$
$$D = \frac{s-dA}{d(dd-aa)(dd-bb)(dd-cc)} - \frac{B}{(dd-bb)(dd-cc)} - \frac{C}{dd-cc}$$
$$E = \frac{t-eA}{e(ee-aa)(ee-bb)(ee-cc)(ee-dd)} - \frac{B}{(ee-bb)(ee-cc)(ee-dd)}$$
$$- \frac{C}{(ee-cc)(ee-dd)} - \frac{D}{ee-dd}$$

etc.,

ubi plerumque mox eiusmodi ordo observatur, unde sequentes facile derivari possunt, quemadmodum ex sequentibus problematibus, in quibus hanc methodum ad quosdam casus particulares sum accommodaturus, patebit.

SCHOLION 2

6. Antequam autem huiusmodi casus evolvam, in genere observasse iuvabit, quod, si pro quocunque casu oblato aequatio satisfaciens inter binas variabiles *x* et *y* fuerit inventa, quam hoc modo designabo

$$y = X,$$

ita ut sit

$$x = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \text{etc.}$$

tum inde facile aequationem multo latius patentem et pariter satisfacientem formari posse. Statuatur enim

$$Q = x \cdot \tfrac{xx-aa}{aa} \cdot \tfrac{xx-bb}{bb} \cdot \tfrac{xx-cc}{cc} \cdot \tfrac{xx-dd}{dd} \cdot \text{etc.},$$

quae quantitas evanescit pro omnibus valoribus ipsius x propositis

$$x = 0,\ \ x = \pm a,\ \ x = \pm b,\ \ x = \pm c\ \ \text{etc.},$$

idemque praestabunt omnes functiones ipsius Q una cum ipsa Q evanescentes; ex quo manifestum est, si statuatur

$$y = X + Q$$

vel

$$y = X + f : Q,$$

omnibus conditionibus aeque satisfieri. Quoniam igitur haec functio $f : Q$ omnino est arbitraria, dummodo evanescat facto $Q = 0$, haec aequatio

$$y = X + f : Q,$$

solutionem generalissimam exhibere est censenda.

PROBLEMA 2

7. *Sint a, b, c, d etc. quotcunque arcus circulares existente radio* $=1$, *valores autem p, q, r, s etc. sint sinus eorundem arcuum, quandoquidem hoc casu ea proprietas locum habet, ut arcubus negativis iidem sinus negative sumti respondeant, hinc proxime definire rationem inter diametrum et peripheriam.*

SOLUTIO

Cum hic sit

$$p = \sin.a,\ q = \sin.b,\ r = \sin.c\ \ \text{etc.},$$

aequatio inter x et y ita erit comparata, ut sumto pro x arcu circuli quantitas y proxime eius sinum sit expressura fiatque

$$y = \sin.x.$$

Definitis ergo per praecedens problema coefficientibus *A, B, C, D* etc. habebitur haec aequatio

$$\sin.x = Ax + Bx(xx-aa) + Cx(xx-aa)(xx-bb) + \text{etc.},$$

quae adeo veritati est consentanea, quoties fuerit

$$\text{vel } x=0 \text{ vel } x=\pm a \text{ vel } x=\pm b \text{ vel } x=\pm c \text{ etc.}$$

Statuamus nunc arcum x infinite parvum, et quia tum eius sinus, sin. x, ipsi arcui x aequatur, orietur haec aequatio

$$1 = A - Baa + Caabb - Daabbcc + Eaabbccdd - \text{etc.}$$

Substituamus hic pro litteris *A, B, C, D* etc. valores supra inventos ac perveniemus ad hanc aequationem

$$\begin{aligned}
1 &= \tfrac{p}{a}\left(1 - \tfrac{aa}{aa-bb} + \tfrac{aabb}{(aa-bb)(aa-cc)} - \tfrac{aabbcc}{(aa-bb)(aa-cc)(aa-dd)} + \text{etc.}\right) \\
&- \tfrac{q}{b}\left(\tfrac{aa}{bb-aa} - \tfrac{aabb}{(bb-aa)(bb-cc)} + \tfrac{aabbcc}{(bb-aa)(bb-cc)(bb-dd)} - \text{etc.}\right) \\
&+ \tfrac{r}{c}\left(\tfrac{aabb}{(cc-aa)(cc-bb)} - \tfrac{aabbcc}{(cc-aa)(cc-bb)(cc-dd)} + \text{etc.}\right) \\
&- \tfrac{s}{d}\left(\tfrac{aabbcc}{(dd-aa)(dd-bb)(dd-cc)} - \text{etc.}\right) \\
&+ \text{etc.},
\end{aligned}$$

quae reducitur ad hanc, in qua omnes series sibi sunt similes,

$$\begin{aligned}
1 &= \tfrac{p}{a}\left(1 - \tfrac{aa}{bb-aa} + \tfrac{aabb}{(bb-aa)(cc-aa)} + \tfrac{aabbcc}{(bb-aa)(cc-aa)(dd-aa)} + \text{etc.}\right) \\
&- \tfrac{aaq}{b(bb-aa)}\left(1 + \tfrac{bb}{cc-bb} + \tfrac{bbcc}{(cc-bb)(dd-bb)} + \tfrac{bbccdd}{(cc-bb)(dd-bb)(ee-bb)} + \text{etc.}\right) \\
&+ \tfrac{aabbr}{c(cc-aa)(cc-bb)}\left(1 + \tfrac{cc}{dd-cc} + \tfrac{ccdd}{(dd-cc)(ee-cc)} + \text{etc.}\right) \\
&- \tfrac{aabbccs}{d(dd-aa)(dd-bb)(dd-cc)}\left(1 + \tfrac{dd}{ee-dd} + \text{etc.}\right) \\
&+ \text{etc.}
\end{aligned}$$

Quaelibet autem harum serierum sponte est summabilis; primae enim duo termini primi coniuncti dant

$$\frac{bb}{bb-aa};$$

cui si tertius addatur, prodit

$$\frac{bbcc}{(bb-aa)(cc-aa)}$$

hincque porro quartus adiunctus praebet

$$\frac{bbccdd}{(bb-aa)(cc-aa)(dd-aa)}$$

sicque ulterius, ita ut nostrae aequationis prima series evadat

$$\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}$$

Simili vero modo pro secunda reperitur

$$-\frac{q}{b}\cdot\frac{aa}{bb-aa}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}$$

sicque tandem nostra aequatio reducitur ad hanc formam

$$\begin{aligned}1=&\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa}{aa-bb}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa}{aa-cc}\cdot\frac{bb}{bb-cc}\cdot\frac{dd}{dd-cc}\cdot\frac{ee}{ee-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa}{aa-dd}\cdot\frac{bb}{bb-dd}\cdot\frac{cc}{cc-dd}\cdot\frac{ee}{ee-dd}\cdot\text{etc.}\\ &+\frac{t}{e}\cdot\frac{aa}{aa-ee}\cdot\frac{bb}{bb-ee}\cdot\frac{cc}{cc-ee}\cdot\frac{dd}{dd-ee}\cdot\text{etc.}\\ &+\ \text{etc.}\end{aligned}$$

unde, si arcus dati *a, b, c, d* etc. cognitam rationem ad semiperipheriam π teneant, huius quantitatis valor π definietur.

COROLLARIUM 1

8. Si numerus horum arcuum *a, b, c, d* etc. fuerit finitus, tum circuli peripheria eo accuratius definietur, quo maior sit ille numerus et simul quo minores arcus inter eos occurrant. Aucto autem arcuum propositorum numero in infinitum vera ratio peripheriae ad diametrum inde derivabitur.

COROLLARIUM 2

9. Simili modo in genere sinus arcus indefiniti x definiri potest. Subtitutis enim loco coefficientium *A, B, C, D* etc. valoribus inventis aequatio
ad hanc formam redigetur

$$\begin{aligned}\frac{\sin x}{x}=&\frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\text{etc.}\\ &+\ \text{etc.}\end{aligned}$$

quae aequatio sumto arcu x evanescente in illam abit.

COROLLARIUM 3

10. Haec autem reductio multo latius patet, nulla arcuum ratione habita. Si enim quaeratur eiusmodi aequatio inter binas variabiles x et y, ut sumto

$$x = 0,\ a,\ b,\ c,\ d,\ e \text{ etc.}$$

fiat

$$y = 0,\ p,\ q,\ r,\ s,\ t \text{ etc.},$$

aequatio haec in genere ita repraesentari poterit

$$\begin{aligned}\frac{y}{x} = &\frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\frac{ee-xx}{ee-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\frac{ee-xx}{ee-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\frac{ee-xx}{ee-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\frac{ee-xx}{ee-dd}\cdot\text{etc.}\\ &+\text{ etc.},\end{aligned}$$

ex qua forma simul manifestum est, quomodo singulis conditionibus satisfiat.

SCHOLION

11. Casibus hic non immorabor, quibus numerus conditionum praescriptarum *a, b, c, d* etc. assumitur finitus, quoniam inde tantum approximationes pro mensura circuli suppeditantur. Interim tamen observasse haud pigebit, si tantum quatuor arcus accipiantur, qui sint

$$a = \varphi,\ b = 2\varphi,\ c = 3\varphi,\ d = 4\varphi,$$

fore ex solutione problematis

$$\begin{aligned}\varphi = &\frac{\sin.\varphi}{1}\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{4\cdot 4}{3\cdot 5}\\ &-\frac{\sin.2\varphi}{2}\cdot\frac{1\cdot 1}{1\cdot 3}\cdot\frac{3\cdot 3}{1\cdot 5}\cdot\frac{4\cdot 4}{2\cdot 6}\\ &+\frac{\sin.3\varphi}{3}\cdot\frac{1\cdot 1}{2\cdot 4}\cdot\frac{2\cdot 2}{1\cdot 5}\cdot\frac{4\cdot 4}{1\cdot 7}\\ &-\frac{\sin.4\varphi}{4}\cdot\frac{1\cdot 1}{3\cdot 5}\cdot\frac{2\cdot 2}{2\cdot 6}\cdot\frac{3\cdot 3}{1\cdot 7}\\ &=\tfrac{8}{5}\sin.\varphi-\tfrac{2}{5}\sin.2\varphi+\tfrac{8}{105}\sin.3\varphi-\tfrac{1}{140}\sin.4\varphi,\end{aligned}$$

quae expressio eo propius ad veritatem accedit, quo minor capiatur arcus φ; interim tamen, etsi ad quadrantem usque augeatur, ut sit

$$\varphi = \tfrac{\pi}{2},$$

error non fit enormis; prodit enim

$$\frac{\pi}{2} = \frac{8}{5} - \frac{8}{105} = \frac{32}{21}$$

sicque

$$\pi = 3\frac{1}{21}.$$

At si sumamus

$$\varphi = 30^0 = \frac{\pi}{6},$$

fit

$$\frac{\pi}{6} = \frac{8}{5}\cdot\frac{1}{2} - \frac{2}{5}\cdot\frac{\sqrt{3}}{2} + \frac{8}{105} - \frac{1}{140}\cdot\frac{\sqrt{3}}{2}$$

seu

$$\pi = \frac{184}{35} - \frac{171\sqrt{3}}{140},$$

qui valor tantum aliquot particulis centies millesimis unitatis a vero differt. Verum missa hac speculatione casus aliquot, ubi arcuum propositorum *a, b, c, d* etc. certa lege progredientium numerus est infinitus, percurram.

EXEMPLUM I

12. *Progrediantur arcus a, b, c, d etc. secundum seriem numerorum naturalium sitque*

$$a = \varphi,\ \ b = 2\varphi,\ \ c = 3\varphi,\ \ d = 4\varphi \text{ etc. in infinitum;}$$

ex quorum sinibus p, q, r etc. veram longitudinem arcus φ; determinari oporteat.

Solutio ergo problematis pro hoc casu suppeditat hanc aequationem

$$\begin{aligned}\varphi = &\frac{\sin.\varphi}{1}\cdot\frac{2\cdot2}{1\cdot3}\cdot\frac{3\cdot3}{2\cdot4}\cdot\frac{4\cdot4}{3\cdot5}\cdot\frac{5\cdot5}{4\cdot6}\cdot\text{etc.}\\ &-\frac{\sin.2\varphi}{2}\cdot\frac{1\cdot1}{1\cdot3}\cdot\frac{3\cdot3}{1\cdot5}\cdot\frac{4\cdot4}{2\cdot6}\cdot\frac{5\cdot5}{3\cdot7}\cdot\text{etc.}\\ &+\frac{\sin.3\varphi}{3}\cdot\frac{1\cdot1}{2\cdot4}\cdot\frac{2\cdot2}{1\cdot5}\cdot\frac{4\cdot4}{1\cdot7}\cdot\frac{5\cdot5}{2\cdot8}\cdot\text{etc.}\\ &-\frac{\sin.4\varphi}{4}\cdot\frac{1\cdot1}{3\cdot5}\cdot\frac{2\cdot2}{2\cdot6}\cdot\frac{3\cdot3}{1\cdot7}\cdot\frac{5\cdot5}{1\cdot9}\cdot\text{etc.}\\ &+\frac{\sin.5\varphi}{5}\cdot\frac{1\cdot1}{4\cdot6}\cdot\frac{2\cdot2}{3\cdot7}\cdot\frac{3\cdot3}{2\cdot8}\cdot\frac{4\cdot4}{1\cdot9}\cdot\text{etc.}\\ &+\ \text{etc.}\end{aligned}$$

omnia autem haec producta eundem reperiuntur habere valorem $=2$, ita ut sit

$$\frac{1}{2}\varphi = \sin.\varphi - \frac{1}{2}\sin.2\varphi + \frac{1}{3}\sin.3\varphi - \frac{1}{4}\sin.4\varphi + \frac{1}{5}\sin.5\varphi - \text{etc.},$$

cuius seriei veritas casu, quo angulus φ est infinite parvus, per se est manifesta. Evolvamus ergo casus sequentes:

1. Sit

$$\varphi = 90^0 = \tfrac{\pi}{2},$$

ac prodit series LEIBNIZIANA

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.}$$

2. Sit

$$\varphi = 45^0 = \tfrac{\pi}{4}$$

orieturque haec series

$$\tfrac{\pi}{8} = \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3\sqrt{2}} * - \frac{1}{5\sqrt{2}} + \frac{1}{6} - \frac{1}{7\sqrt{2}} * + \frac{1}{\sqrt{2}} - \frac{1}{10} + \frac{1}{11\sqrt{2}} - \text{etc.},$$

quae resolvitur in has duas

$$\begin{aligned} \tfrac{\pi}{8} = {} & \tfrac{1}{\sqrt{2}}\left(1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15} + \text{etc.}\right) \\ & - \tfrac{1}{2}\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}\right), \end{aligned}$$

ita ut sit

$$1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{15} + \text{etc.} = \tfrac{\pi}{2\sqrt{2}}.$$

3. Sit

$$\varphi = 60^0 = \tfrac{\pi}{3}$$

eritque

$$\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} - \tfrac{1}{2} \cdot \tfrac{\sqrt{3}}{2} * + \tfrac{1}{4} \cdot \tfrac{\sqrt{3}}{2} - \tfrac{1}{5} \cdot \tfrac{\sqrt{3}}{2} + \text{etc.}$$

seu

$$\tfrac{\pi}{3\sqrt{3}} = 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{10} - \tfrac{1}{11} + \text{etc.}$$

4. Sit

$$\varphi = 30^0 = \tfrac{\pi}{6}$$

ac fit

$$\tfrac{\pi}{12} = \tfrac{1}{2} - \tfrac{1}{2} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{1}{3} - \tfrac{1}{4} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{1}{5} \cdot \tfrac{1}{2} * - \tfrac{1}{7} \cdot \tfrac{1}{2} + \tfrac{1}{8} \cdot \tfrac{\sqrt{3}}{2} - \text{etc.}$$

seu

$$\begin{aligned} \tfrac{\pi}{12} = {} & \tfrac{1}{2}\left(1 + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19} - \tfrac{1}{23} + \text{etc.}\right) \\ & - \tfrac{\sqrt{3}}{4}\left(1 + \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{1}{10} - \tfrac{1}{11} + \text{etc.}\right) \\ & + \tfrac{1}{3}\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}\right), \end{aligned}$$

quarum serierum ultima fit $= \tfrac{\pi}{12}$; hinc concluditur

$$1+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\text{etc.}=\tfrac{\sqrt{3}}{2}(1+\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{8}-\text{etc.})$$

Utraque autem series aequatur arcui $\frac{\pi}{3}$, quod quidem in priori ex LEIBNIZIANA est manifestum.

COROLLARIUM 1

13. Ex aequatione hic inventa

$$\tfrac{1}{2}\varphi=\sin.\varphi-\tfrac{1}{2}\sin.2\varphi+\tfrac{1}{3}\sin.3\varphi-\tfrac{1}{4}\sin.4\varphi+\tfrac{1}{5}\sin.5\varphi-\text{etc.},$$

plures aliae non minus notatu dignae derivari possunt. Veluti instituta differentiatione prodit

$$\tfrac{1}{2}=\cos.\varphi-\cos.2\varphi+\cos.3\varphi-\cos.4\varphi+\text{etc.},$$

cuius ratio inde est manifesta, quod multiplicando utrinque per $2\cos.\frac{1}{2}\varphi$ prodit aequa identica $\cos.\frac{1}{2}\varphi=\cos.\frac{1}{2}\varphi$.

COROLLARIUM 2

14. At si illam aequationem per $-d\varphi$ multiplicatam integremus, provenit

$$C-\tfrac{1}{4}\varphi\varphi=\cos.\varphi-\tfrac{1}{4}\cos.2\varphi+\tfrac{1}{9}\cos.3\varphi-\tfrac{1}{16}\cos.4\varphi+\text{etc.},$$

ubi ex casu $\varphi=0$ constans per integrationem ingressa determinatur, scilicet

$$C=1-\tfrac{1}{4}+\tfrac{1}{9}-\tfrac{1}{16}+\text{etc.}=\tfrac{\pi\pi}{12},$$

ita ut sit

$$\tfrac{\pi\pi}{12}-\tfrac{1}{4}\varphi\varphi=\cos.\varphi-\tfrac{1}{4}\cos.2\varphi+\tfrac{1}{9}\cos.3\varphi-\tfrac{1}{16}\cos.4\varphi+\text{etc.},$$

quae ergo series sumto $\varphi=\frac{\pi}{\sqrt{3}}$ fit $=0$. Est autem proxime

$$\tfrac{\pi}{\sqrt{3}}=103^\circ\ 55'\ 23''\ \text{et}\ \cos.\tfrac{\pi}{\sqrt{3}}=-\ 0,2406185.$$

COROLLARIUM 3

15. Si hanc aequationem denuo per $d\varphi$ multiplicatam integremus, orietur haec nova summatio

$$\tfrac{1}{12}\pi\pi\varphi-\tfrac{1}{12}\varphi^3=\sin.\varphi-\tfrac{1}{8}\sin.2\varphi+\tfrac{1}{27}\sin.3\varphi-\tfrac{1}{64}\sin.4\varphi+\text{etc.},$$

unde sumto arcu

$$\varphi = 90^0 = \tfrac{\pi}{2}$$

obtinetur

$$\tfrac{1}{32}\pi^3 = 1 - \tfrac{1}{27} + \tfrac{1}{125} - \tfrac{1}{343} + \text{etc.},$$

uti iam aliunde est notum.

SCHOLION

16. Circa seriem inventam

$$\tfrac{1}{2}\varphi = \sin.\varphi - \tfrac{1}{2}\sin.2\varphi + \tfrac{1}{3}\sin.3\varphi - \text{etc.},$$

dubium oriri potest, quod sumto arcu $\varphi = 180° = \pi$ singuli seriei termini evanescant ideoque summa nequeat ipsi $\tfrac{1}{2}\pi$ aequari. Verum ad hoc dubium solvendum statuatur primo $\varphi = \pi - \omega$ et resultabit haec aequatio

$$\tfrac{\pi-\omega}{2} = \sin.\omega + \tfrac{1}{2}\sin.2\omega + \tfrac{1}{3}\sin.3\omega + \tfrac{1}{4}\sin.4\omega + \text{etc.}$$

nunc vero arcus ω infinite parvus sumatur, unde adipiscimur hanc

$$\tfrac{\pi-\omega}{2} = \omega + \omega + \omega + \omega + \omega + \text{ etc.},$$

quae nihil amplius continet absurdi. Quod idem tenendum est, si velimus accipere $\varphi = 2\pi$ vel $\varphi = 3\pi$ etc.

EXEMPLUM II

17. *Si arcus a, b, c, d etc. progressionem arithmeticam quamcunque constituant, ut sit*

$$a = n\varphi,\ b = (n+1)\varphi, c = (n+1)\varphi, d = (n+1)\varphi,\ \ etc.,$$

ex eorum sinibus longitudinem arcus φ definire.

Solutio generalis ante exhibita pro hoc casu dat

$$\begin{aligned}\varphi = &\frac{\sin.n\varphi}{n}\cdot\frac{(n+1)^2}{1(1+2n)}\cdot\frac{(n+2)^2}{2(2+2n)}\cdot\frac{(n+3)^2}{3(3+2n)}\cdot\frac{(n+4)^2}{4(4+2n)}\cdot\frac{(n+5)^2}{5(5+2n)}\cdot\text{etc.}\\
&-\frac{\sin.(n+1)\varphi}{n+1}\cdot\frac{n^2}{1(1+2n)}\cdot\frac{(n+2)^2}{1(3+2n)}\cdot\frac{(n+3)^2}{2(4+2n)}\cdot\frac{(n+4)^2}{3(5+2n)}\cdot\frac{(n+5)^2}{4(6+2n)}\cdot\text{etc.}\\
&+\frac{\sin.(n+2)\varphi}{n+2}\cdot\frac{n^2}{2(2+2n)}\cdot\frac{(n+1)^2}{1(3+2n)}\cdot\frac{(n+3)^2}{1(5+2n)}\cdot\frac{(n+4)^2}{2(6+2n)}\cdot\frac{(n+5)^2}{3(7+2n)}\cdot\text{etc.}\\
&-\frac{\sin.(n+3)\varphi}{n+3}\cdot\frac{n^2}{3(3+2n)}\cdot\frac{(n+1)^2}{2(4+2n)}\cdot\frac{(n+2)^2}{1(5+2n)}\cdot\frac{(n+4)^2}{1(7+2n)}\cdot\frac{(n+5)^2}{2(8+2n)}\cdot\text{etc.}\\
&+\frac{\sin.(n+4)\varphi}{n+4}\cdot\frac{n^2}{4(4+2n)}\cdot\frac{(n+1)^2}{3(5+2n)}\cdot\frac{(n+2)^2}{2(6+2n)}\cdot\frac{(n+3)^2}{1(7+2n)}\cdot\frac{(n+5)^2}{1(9+2n)}\cdot\text{etc.}\\
&+\qquad\qquad\text{etc.}\end{aligned}$$

Ad valores horum productorum in infinitum excurrentium investigandos ponamus brevitatis gratia

$$\varphi = \mathfrak{A}\frac{\sin.n\varphi}{n} - \mathfrak{B}\frac{\sin.(n+1)\varphi}{n+1} + \mathfrak{C}\frac{\sin.(n+2)\varphi}{n+2} - \mathfrak{D}\frac{\sin.(n+3)\varphi}{n+3} + \text{etc.}$$

et hi coefficientes sequenti modo inter se comparentur

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{nn}{(n+1)^2}\cdot\frac{2(2+2n)}{1(3+2n)}\cdot\frac{3(3+2n)}{2(4+2n)}\cdot\frac{4(4+2n)}{3(5+2n)}\cdot\text{etc.}$$

qui valor reducitur ad

$$\frac{nn}{(n+1)^2}\cdot\frac{(i-1)(2+2n)}{1(i+2n)},$$

denotante i numerum infinitum sicque erit

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{2nn}{n+1}.$$

Simili modo colligitur

$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1(1+2n)}{2(2+2n)}\cdot\frac{(n+1)^2}{(n+2)^2}\cdot\frac{(i-3)(4+2n)}{1(i+2n)} = \frac{(n+1)(2n+1)}{2(n+2)},$$

tum vero porro

$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2)(2n+2)}{3(n+3)},\quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3)(2n+3)}{4(n+4)}$$

et sic deinceps; unde sequitur fore

$$\mathfrak{B} = \frac{2nn}{1(n+1)}\mathfrak{A},$$

$$\mathfrak{C} = \frac{2nn(2n+1)}{1\cdot 2(n+2)}\mathfrak{A},$$

$$\mathfrak{D} = \frac{2nn(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)}\mathfrak{A},$$

$$\mathfrak{E} = \frac{2nn(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)}\mathfrak{A}$$

etc.

sicque totum negotium redit ad inventionem primae litterae

$$\mathfrak{A} = \frac{(n+1)^2}{1(2n+1)}\cdot\frac{(n+2)^2}{2(2n+2)}\cdot\frac{(n+3)^2}{3(2n+3)}\cdot\frac{(n+3)^2}{4(2n+4)}\cdot\text{etc.}$$

Iam dudum autem demonstravi, huius producti generalis

$$\frac{a(b+c)}{b(a+c)}\cdot\frac{(a+d)(b+c+d)}{(b+d)(a+c+d)}\cdot\frac{(a+2d)(b+c+2d)}{(b+2d)(a+c+2d)}\cdot\text{etc.}$$

valorem ita exprimi, ut sit

$$= \frac{\int x^{b-1}dx(1-x^d)^{\frac{c-d}{d}}}{\int x^{a-1}dx(1-x^d)^{\frac{c-d}{d}}}$$

utraque scilicet integratione a termino $x=0$ ad $x=1$ extensa. Quare cum pro nostro casu sumi oporteat

$$a = n+1,\ b+c = n+1,\ b = 1,\ c = n \ \text{ et } \ d = 1,$$

habebimus

$$\mathfrak{A} = \frac{\int dx(1-x)^{n-1}}{\int x^n dx(1-x)^{n-1}} = \frac{1}{n\int x^n dx(1-x)^{n-1}}$$

hincque sequentem pro arcu p expressionem

$$\varphi\int x^n dx(1-x)^{n-1} = \frac{1}{nn}\sin.n\varphi - \frac{2n}{1(n+1)^2}\sin.(n+1)\varphi$$
$$+\frac{2n(2n+1)}{1\cdot 2(n+2)^2}\sin.(n+2)\varphi - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}\sin.(n+3)\varphi$$
$$+\frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)^2}\sin.(n+4)\varphi + \text{etc.}$$

Quae series eo maiorem attentionem meretur, quod formulam integralem $\int x^n dx(1-x)^{n-1}$ involvit.

COROLLARIUM 1

18. De hac formula integrali

$$\int x^n dx(1-x)^{n-1}$$

primum observasse iuvabit, si casu $n=\lambda$ eius valor fuerit $=\Delta$, tum eum casu

$$n=\lambda+1$$

fore

$$=\tfrac{\lambda}{2(2\lambda+1)}\Delta.$$

Ita, cum casu $n=1$ sit

$$\int xdx=\tfrac{1}{2},$$

erit

$$\int x^2(1-x)dx=\tfrac{1}{2}\cdot\tfrac{1}{2\cdot 3},\quad \int x^3(1-x)^2dx=\tfrac{1}{2}\cdot\tfrac{1}{2\cdot 3}\cdot\tfrac{2}{2\cdot 5}\ \text{ etc.}$$

COROLLARIUM 2

19. Si ergo in genere ponatur

$$\int x^n dx(1-x)^{n-1}=f:n,$$

quandoquidem eius valor ut functio ipsius *n* spectari potest, erit

$$f:1=\tfrac{1}{2},\ f:2=\tfrac{1}{2}\cdot\tfrac{1}{6},\quad f:3=\tfrac{1}{2}\cdot\tfrac{1}{6}\cdot\tfrac{2}{10},\quad f:4=\tfrac{1}{2}\cdot\tfrac{1}{6}\cdot\tfrac{2}{10}\cdot\tfrac{3}{14}$$

atque in genere

$$f:(n+1)=\tfrac{2}{2(2n+1)}f:n.$$

unde, quoties *n* est numerus integer, valor istius formulae $f:n$ facile assignatur.

COROLLARIUM 3

20. Sit nunc $n=\tfrac{1}{2}$ eritque

$$f:\tfrac{1}{2}=2\int\tfrac{dx\sqrt{x}}{\sqrt{(1-x)}}=2\int\tfrac{yydy}{\sqrt{(1-yy)}}$$

posito $x=yy$*;* at

$$\int\tfrac{yydy}{\sqrt{(1-yy)}}=\tfrac{1}{2}\int\tfrac{dy}{\sqrt{(1-yy)}}=\tfrac{\pi}{4},$$

unde fit

$$f:\tfrac{1}{2}=\tfrac{\pi}{2}$$

hincque porro

$$f:\tfrac{3}{2}=\tfrac{1}{8}\cdot\tfrac{\pi}{2},\ f:\tfrac{5}{2}=\tfrac{1}{8}\cdot\tfrac{3}{16}\cdot\tfrac{\pi}{2},\ f:\tfrac{7}{2}=\tfrac{1}{8}\cdot\tfrac{3}{16}\cdot\tfrac{5}{24}\cdot\tfrac{\pi}{2}\ \text{ etc.}$$

At si in genere sit $n=\frac{\mu}{\nu}$, reperitur

$$f:\tfrac{\mu}{\nu}=\int x^{\frac{\mu}{\nu}}dx(1-x)^{\frac{\mu}{\nu}-1}=\nu\int y^{\mu+\nu-1}dy(1-y^{\nu})^{\frac{\mu}{\nu}-1}$$

posito $x=y^{\nu}$ hincque facta reductione

$$f:\tfrac{\mu}{\nu}=\tfrac{\nu}{2}\int y^{\mu-1}dy(1-y^{\nu})^{\frac{\mu}{\nu}-1},$$

quae forma omnis generis quantitates transcendentes involvit.

COROLLARIUM 4

21. Huius ipsius autem formulae integralis

$$\int x^{n}dx(1-x)^{n-1}$$

valor casu $x=1$ vicissim ex serie inventa satis eleganter determinatur; facta enim differentiatione solum arcum φ ut variabilem spectando prodit

$$\int x^{n}dx(1-x)^{n-1}=\tfrac{1}{n}\cos.n\varphi-\tfrac{2n}{1(n+1)}\cos.(n+1)\varphi$$
$$+\tfrac{2n(2n+1)}{1\cdot2(n+2)}\cos.(n+2)\varphi-\tfrac{2n(2n+1)(2n+2)}{1\cdot2\cdot3(n+3)}\cos.(n+3)\varphi+\text{etc.},$$

quae ergo series aequalis est huic ex ipsa evolutione solita ortae

$$\int x^{n}dx(1-x)^{n-1}=\tfrac{1}{n+1}-\tfrac{n-1}{1(n+2)}+\tfrac{(n-1)(n-2)}{1\cdot2(n+3)}-\tfrac{(n-1)(n-2)(n-3)}{1\cdot2\cdot3(n+4)}+\text{etc.},$$

SCHOLION 1

22. Quoniam casum $n=1$ in praecedente exemplo evolvimus, consideremus hic potissimum casum

$$n=\tfrac{1}{2},$$

quo vidimus esse

$$\int x^{n}dx(1-x)^{n-1}=\tfrac{\pi}{2},$$

eritque propterea

$$\frac{\varphi\pi}{2} = \frac{4}{1}\sin.\frac{1}{2}\varphi - \frac{4}{9}\sin.\frac{3}{2}\varphi + \frac{4}{25}\sin.\frac{5}{2}\varphi - \frac{4}{49}\sin.\frac{7}{2}\varphi + \text{etc.}$$

Ponamus $\varphi = 2\omega$ prodibitque haec series concinnior

$$\frac{\pi\omega}{4} = \frac{1}{2}\sin.\omega - \frac{1}{9}\sin.3\omega + \frac{1}{25}\sin.5\omega - \frac{1}{49}\sin.7\omega + \text{etc.}$$

quae primo, si arcus ω sumatur evanescens, dat

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$$

Sit autem

$$\omega = \frac{\pi}{2}$$

oriturque series etiam cognita

$$\frac{\pi\pi}{8} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.}$$

At sumto arcu

$$\omega = 45^0 = \frac{\pi}{4}$$

provenit

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} + \text{etc.}$$

Sit

$$\omega = 30^0 = \frac{\pi}{6};$$

erit

$$\begin{aligned}\frac{\pi\pi}{24} = {} & \frac{1}{2}(1 + \frac{1}{7^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{25^2} + \text{etc.}) \\ & -1(\frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{etc.}) \\ & +\frac{1}{2}(\frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{17^2} + \frac{1}{23^2} + \text{etc.}),\end{aligned}$$

ubi media est $= \frac{\pi\pi}{72}$; reliquarumque ratio perspicua. Deinde differentiatio nostrae seriei suppeditat hanc formam notatu dignam

$$\frac{\pi}{4} = \frac{1}{1}\cos.\omega - \frac{1}{3}\cos.3\omega + \frac{1}{5}\cos.5\omega - \frac{1}{7}\cos.7\omega + \text{etc.},$$

quoniam onmes plane arcus pro ω assumti eandem praebent summam. Tum vero iterata differentiatio praebet

$$0 = \sin.\omega - \sin.3\omega + \sin.5\omega - \sin.7\omega + \text{etc.}$$

Per integrationem autem elicimus

$$C-\frac{\pi\omega^2}{8}=\frac{1}{1}\cos.\omega-\frac{1}{3^3}\cos.3\omega+\frac{1}{5^3}\cos.5\omega-\frac{1}{7^3}\cos.7\omega+\text{etc.},$$

ubi, cum sumto $\omega=0$ sit

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\text{etc.}=\frac{\pi^3}{32},$$

erit

$$C=\frac{\pi^3}{32},$$

ita ut sit

$$\frac{\pi}{8}\left(\frac{\pi\pi}{4}-\omega^2\right)=\frac{1}{1}\cos.\omega-\frac{1}{3^3}\cos.3\omega+\frac{1}{5^3}\cos.5\omega-\frac{1}{7^3}\cos.7\omega+\text{etc.},$$

SCHOLION 2

23. Ponamus nunc in genere

$$\varphi=\pi,$$

et cum sit

$$\sin.(n+1)\pi=-\sin.n\pi,\ \sin.(n+2)\pi=+\sin.n\pi \text{ etc.},$$

aequatio nostra per $\sin.n\pi$ divisa induet hanc formam

$$\frac{\pi}{\sin.n\pi}\int x^n dx(1-x)^{n-1}=\frac{1}{n^2}+\frac{2n}{1(n+1)^2}+\frac{2n(2n+1)}{1\cdot 2(n+2)^2}+\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}+\text{etc.};$$

sumto autem

$$\varphi=2\pi,$$

erit simili modo

$$\frac{2\pi}{\sin.2n\pi}\int x^n dx(1-x)^{n-1}=\frac{1}{n^2}-\frac{2n}{1(n+1)^2}+\frac{2n(2n+1)}{1\cdot 2(n+2)^2}-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}+\text{etc.};$$

quarum serierum ergo illa per hanc divisa quotum praebet $=\cos.n\pi$, quod incongruum videtur, cum quotus sit unitate maior. At similem difficultatem iam supra resolvimus, quae ex positione $\varphi=2\pi$ est nata; si enim poneremus $\varphi=3\pi$, ipsa series prior emergeret summam habitura

$$\frac{3\pi}{\sin.3n\pi}\int x^n dx(1-x)^{n-1},$$

quae illi non est aequalis, nisi *n* sit ratio evanescens. Quare prima tantum series locum habere est censenda; cuius summam ut ex ipsa eius natura investigemus, statuamus

$$s = \frac{1}{n^2}t^n + \frac{2n}{(n+1)^2}t^{n+1} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2}t^{n+2} + \text{etc.},$$

eritque hinc

$$\frac{d.tds}{dt^2} = 1t^{n-1} + \frac{2n}{1}t^n + \frac{2n(2n+1)}{1\cdot 2}t^{n+2} + \text{etc.},$$

cuius seriei summa manifesto est

$$= t^{n-1}(1-t)^{-2n},$$

ita ut sit

$$\frac{tds}{dt} = \int t^{n-1}dt(1-t)^{-2n}$$

et

$$s = \int \frac{dt}{t}\int \frac{t^{n-1}dt}{(1-t)^{2n}},$$

sicque posito post integrationem $t = 1$ habebitur

$$\frac{\pi}{\sin.n\pi}\int x^n dx(1-x)^{n-1} = \int \frac{dt}{t}\int \frac{t^{n-1}dt}{(1-t)^{2n}}.$$

Quae binarum formularum integralium comparatio eo magis est memorabilis, quod inter plurimas alias, quae adhuc sunt erutae, huius generis non reperiantur.

SCHOLION 3

24. Ponamus in genere

$$\varphi = \frac{\pi}{2}$$

eritque

$$\sin.n\varphi = \sin.\frac{n\pi}{2}, \quad \sin.(n+1)\varphi = \cos.\frac{n\pi}{2},$$
$$\sin.(n+2)\varphi = -\sin.\frac{\pi}{2}, \quad \sin.(n+3)\varphi = -\cos.\frac{n\pi}{2}, \text{ etc.}$$

unde resultat haec aequatio

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1} = \sin.\frac{n\pi}{2}\left(\frac{1}{nn} - \frac{2n(2n+1)}{1\cdot 2(n+2)^2} + \frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)^2} - \text{etc.}\right)$$
$$- \cos.\frac{n\pi}{2}\left(\frac{2n}{1(n+1)^2} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2} + \text{etc.}\right).$$

At ex superiori reductione manifestum est fore

$$1-\frac{2n(2n+1)}{1\cdot 2}t^2+\frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4}t^2-\text{etc.}$$
$$=\frac{(1+t\sqrt{-1})^{-2n}+(1-t\sqrt{-1})^{-2n}}{2},$$
$$\frac{2n}{1}t-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}t^3+\text{etc.}$$
$$=\frac{(1+t\sqrt{-1})^{-2n}-(1-t\sqrt{-1})^{-2n}}{2},$$

hincque colligitur

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1}=$$
$$\frac{1}{2}\sin.\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}}+\frac{1}{2}\sin.\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}}$$
$$-\frac{1}{2\sqrt{-1}}\cos.\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}}+\frac{1}{2\sqrt{-1}}\cos.\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}}$$

ubi quidem post integrationem poni oportet $t=1$. Ut autem hanc expressionem ab imaginariis liberemus, ponamus

$$t=\text{tang.}\omega=\frac{\sin.\omega}{\cos.\omega};$$

erit

$$dt=\frac{d\omega}{\cos.\omega^2},\quad \frac{dt}{t}=\frac{d\omega}{\sin.\omega\cos.\omega},\quad t^{n-1}dt=\frac{d\omega\sin.\omega^{n-1}}{\cos.\omega^{n+1}},$$

tum vero

$$(1+t\sqrt{-1})^{-2n}=\cos.\omega^{2n}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)^{-2n}$$
$$=\cos.\omega^{2n}(\cos.2n\omega-\sqrt{-1}\cdot\sin.2n\omega),$$
$$(1-t\sqrt{-1})^{-2n}=\cos.\omega^{2n}(\cos.\omega-\sqrt{-1}\cdot\sin.\omega)^{-2n}$$
$$=\cos.\omega^{2n}(\cos.2n\omega+\sqrt{-1}\cdot\sin.2n\omega).$$

Quibus valoribus substitutis imaginaria se mutuo tollent prodibitque haec aequatio

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1}=$$
$$\sin.\frac{n\pi}{2}\int\frac{d\omega}{\sin.\omega\cos.\omega}\int d\omega\sin.\omega^{n-1}\cos.\omega^{n-1}\cos.2n\omega$$
$$+\cos.\frac{n\pi}{2}\int\frac{d\omega}{\sin.\omega\cos.\omega}\int d\omega\sin.\omega^{n-1}\cos.\omega^{n-1}\sin.2n\omega,$$

quae in hanc simpliciorem contrahitur

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1}=\int\frac{d\omega}{\sin.\omega\cos.\omega}\int d\omega\sin.\omega^{n-1}\cos.\omega^{n-1}\sin.(\tfrac{n\pi}{2}+2n\omega)$$

vel ob $\sin.\omega\cos.\omega = \frac{1}{2}\sin.2\omega$ in hanc

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1} = \frac{1}{2^n}\int\frac{2d\omega}{\sin.2\omega}\int 2d\omega\sin.2\omega^{n-1}\sin.(\tfrac{n\pi}{2}+2n\omega).$$

Sit nunc angulus $2\omega = \theta$, ut fiat concinnius

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1} = \frac{1}{2^n}\int\frac{d\theta}{\sin.\theta}\int d\theta\sin.\theta^{n-1}\sin.n(\tfrac{\pi}{2}+\theta),$$

ubi post integrationern statui oportet $\theta = 90° = \frac{\pi}{2}$, ut tum fiat $\omega = 45°$ et $t = \text{tang}.\omega = 1$.

EXEMPLUM III

25. *Si arcus a, b, c, d etc. constituant progressionem arithmeticam interruptam, ut sit*

$$a = m\varphi,\ b = n\varphi,\ c = (1+m)\varphi,\ d = (1+n)\varphi,$$
$$e = (2+m)\varphi,\ f = (2+n)\varphi\ etc.,$$

ex eorum sinibus longitudinem arcus p definire.

Solutio generalis supra data (§ 17) suppeditat hanc aequationem

$$\varphi=\frac{\sin.m\varphi}{m}\cdot\frac{nn}{(n-m)(n+m)}\cdot\frac{(1+m)^2}{1(1+2m)}\cdot\frac{(1+n)^2}{(1+n-m)(1+n+m)}$$
$$\cdot\frac{(2+m)^2}{2(2+2m)}\cdot\frac{(n+2)^2}{(2+n-m)(2+n+m)}\cdot\text{etc.}$$
$$-\frac{\sin.n\varphi}{n}\cdot\frac{mm}{(n-m)(n+m)}\cdot\frac{(1+m)^2}{(1+m-n)(1+m+n)}\cdot\frac{(1+n)^2}{1(1+2n)}$$
$$\cdot\frac{(2+m)^2}{(2+m-n)(2+m+n)}\cdot\frac{(2+n)^2}{2(2+2n)}\cdot\text{etc.}$$
$$+\frac{\sin.(1+m)\varphi}{1+m}\cdot\frac{mm}{1(1+2m)}\cdot\frac{nn}{(1+m-n)(1+m+n)}\cdot\frac{(1+n)^2}{(n-m)(2+m+n)}$$
$$\cdot\frac{(2+m)^2}{1(3+2n)}\cdot\frac{(2+n)^2}{(1+n-m)(3+n+m)}\cdot\text{etc.}$$
$$-\frac{\sin.(1+n)\varphi}{1+n}\cdot\frac{mm}{(1+n-m)(1+n+m)}\cdot\frac{nn}{1(1+2n)}\cdot\frac{(1+m)^2}{(n-m)(2+n+m)}$$
$$\cdot\frac{(2+m)^2}{(1+m-n)(3+m+n)}\cdot\frac{(2+n)^2}{1(3+2n)}\cdot\text{etc.}$$
$$+\frac{\sin.(2+m)\varphi}{2+m}\cdot\frac{mm}{2(2+2m)}\cdot\frac{nn}{(2+m-n)(2+m+n)}\cdot\frac{(1+m)^2}{1(3+2m)}$$
$$\cdot\frac{(1+n)^2}{(1+m-n)(3+m+n)}\cdot\frac{(2+n)^2}{(n-m)(4+m+n)}\cdot\text{etc.}$$
$$-\ \text{etc.}$$

Hinc autem in genere nihil attentione dignum concludere licet; unde casum praecipue memorabilem evolvam, quo est

$$n=1-m;$$

pro quo statuo brevitatis gratia

$$\varphi=\frac{\mathfrak{A}\sin.m\varphi}{m}-\frac{\mathfrak{B}\sin.(1-m)\varphi}{1-m}+\frac{\mathfrak{C}\sin.(1+m)\varphi}{1+m}-\frac{\mathfrak{D}\sin.(2-m)\varphi}{2-m}+\text{etc.},$$

ita ut sit

$$\mathfrak{A}=\frac{(1-m)^2}{1(1-2m)}\cdot\frac{(1+m)^2}{1(1+2m)}\cdot\frac{(2-m)^2}{2(2-2m)}\cdot\frac{(2+m)^2}{2(2+2m)}\cdot\frac{(3-m)^2}{3(3-2m)}\cdot\text{etc.},$$
$$\frac{\mathfrak{B}}{\mathfrak{A}}=\frac{mm}{(1-m)^2}\cdot\frac{1(1+2m)}{2\cdot2m}\cdot\frac{2(2-2m)}{1(3-2m)}\cdot\frac{2(2+2m)}{3(1+2m)}\cdot\frac{3(3-2m)}{2(4-2m)}\cdot\text{etc.},$$
$$\frac{\mathfrak{C}}{\mathfrak{B}}=\frac{1(1-2m)}{1(1+2m)}\cdot\frac{(1-m)^2}{(1+m)^2}\cdot\frac{1(3-2m)}{3(1-2m)}\cdot\frac{3(1+2m)}{1(3+2m)}\cdot\frac{2(1-2m)}{4(2-2m)}\cdot\text{etc.},$$
$$\frac{\mathfrak{D}}{\mathfrak{C}}=\frac{1(1+2m)}{2(2-2m)}\cdot\frac{2\cdot2m}{1(3-2m)}\cdot\frac{(1+m)^2}{(2-m)^2}\cdot\frac{1(3+2m)}{4\cdot2m}\cdot\frac{4(2-2m)}{1(5-2m)}\cdot\text{etc.},$$
$$\frac{\mathfrak{E}}{\mathfrak{D}}=\frac{2(2-2m)}{2(2+2m)}\cdot\frac{1(3-2m)}{3(1+2m)}\cdot\frac{3(1-2m)}{1(3+2m)}\cdot\frac{(2-m)^2}{(2+m)^2}\cdot\frac{1(5-2m)}{5(1-2m)}\cdot\text{etc.},$$

etc.

Atque ex superiori reductione reperitur

$$\mathfrak{A} = \frac{\int x^{m-1}dx(1-x)^{-2m}}{m\int x^m dx(1-x)^{m-1}\cdot\int x^{m-1}dx(1-x)^{-m}};$$

tum vero pro reliquis colligitur ex ipsa forma productorum

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{m}{1-m}, \quad \frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1-m}{1+m}, \quad \frac{\mathfrak{D}}{\mathfrak{C}} = \frac{1+m}{2-m}, \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{2-m}{2+m} \text{ etc.,}$$

ita ut sit

$$\mathfrak{B} = \frac{m}{1-m}\mathfrak{A}, \;\; \mathfrak{C} = \frac{m}{1+m}\mathfrak{A}, \;\; \mathfrak{D} = \frac{m}{2-m}\mathfrak{A}, \;\; \mathfrak{E} = \frac{m}{2+m}\mathfrak{A} \text{ etc.}$$

Ponamus ergo brevitatis gratia

$$\int x^m dx(1-x)^{m-1}\cdot\frac{\int x^{m-1}dx(1-x)^{-m}}{\int x^{m-1}dx(1-x)^{-2m}} = M$$

eritque, ut sequitur,

$$M\varphi = \frac{\sin.m\varphi}{m^2} - \frac{\sin.(1-m)\varphi}{(1-m)^2} + \frac{\sin.(1+m)\varphi}{(1+m)^2} - \frac{\sin.(2-m)\varphi}{(2-m)^2} + \frac{\sin.(2+m)\varphi}{(2+m)^2} - \text{etc.,}$$

unde differentiando concludimus fore

$$M = \frac{\cos.m\varphi}{m} - \frac{\cos.(1-m)\varphi}{1-m} + \frac{\cos.(1+m)\varphi}{1+m} - \frac{\cos.(2-m)\varphi}{2-m} + \frac{\cos.(2+m)\varphi}{2+m} - \text{etc.,}$$

quae series ob insignem simplicitatem imprimis est notatu digna, quandoquidem inde ponendo $\varphi = 0$ deducimus

$$M = \frac{1}{m} - \frac{1}{1-m} + \frac{1}{1+m} - \frac{1}{2-m} + \frac{1}{2+m} - \frac{1}{3-m} + \frac{1}{3+m} - \text{etc.,}$$

cuius seriei summam iam olim ostendi esse

$$M = \frac{\pi\cos.m\pi}{\sin.m\pi},$$

unde colligimus hanc elegantem comparationem

$$\int x^m dx(1-x)^{m-1} = \frac{\pi\cos.m\pi}{\sin.m\pi}\cdot\frac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}},$$

quae redigitur porro ad hanc

$$\int x^m dx(1-x)^{m-1} = \frac{(1-m)\pi\cos.m\pi}{\sin.m\pi}\cdot\frac{\int x^m dx(1-x)^{-2m}}{\int x^m dx(1-x)^{-m}},$$

vel ad hanc adhuc concinniorem

$$\int x^m dx(1-x)^{m-1} = \frac{2\pi\cos.m\pi}{\sin.m\pi}\cdot\frac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}}.$$

COROLLARIUM 1

26. En ergo aliquot insignia theoremata, quae huius exempli evolutio nobis suppeditat, quorum primum est:
Si φ denotet angulum quemcunque, erit

$$\frac{\pi\cos.m\pi}{\sin.m\pi} = \frac{\cos.m\varphi}{m} - \frac{\cos.(1-m)\varphi}{1-m} + \frac{\cos.(1+m)\varphi}{1+m} - \frac{\cos.(2-m)\varphi}{2-m} + \text{etc.},$$

quae aequalitas etiam ita exhiberi potest, ut sit

$$\frac{\pi\cos.m\pi}{\sin.m\pi} = \cos.m\varphi\left(\frac{1}{m} - \frac{2m\cos.\varphi}{1-mm} - \frac{2m\cos.2\varphi}{4-mm} - \frac{2m\cos.3\varphi}{9-mm} - \text{etc.}\right)$$
$$-2\sin.m\varphi\left(\frac{\sin.\varphi}{1-mm} + \frac{2\sin.2\varphi}{4-mm} + \frac{3m\sin.3\varphi}{9-mm} + \frac{4m\sin.4\varphi}{16-mm} + \text{etc.}\right),$$

unde, si

$$m\varphi = 90^\circ = \tfrac{\pi}{2} \text{ ideoque } \varphi = \tfrac{\pi}{2m},$$

erit

$$-\frac{\pi\cos.m\pi}{\sin.m\pi} = \frac{\sin.\frac{\pi}{2m}}{1-mm} + \frac{2\sin.\frac{2\pi}{2m}}{4-mm} + \frac{3\sin.\frac{3\pi}{2m}}{9-mm} + \frac{4\sin.\frac{4\pi}{2m}}{16-mm} + \text{etc.}$$

COROLLARIUM 2

27. Secundum theorema ita enunciatur:

Si φ denotet angulum quemcunque, erit

$$\frac{\pi\varphi\cos.m\pi}{\sin.m\pi} = \frac{\sin.m\varphi}{mm} - \frac{\sin.(1-m)\varphi}{(1-m)^2} + \frac{\sin.(1+m)\varphi}{(1+m)^2} - \frac{\sin.(2-m)\varphi}{(2-m)^2} + \text{etc.}$$

Quare sumto $\varphi = \pi$ erit

$$\frac{\pi\pi\cos.m\pi}{\sin.m\pi} = \frac{\sin.m\pi}{mm} - \frac{\sin.m\pi}{(1-m)^2} - \frac{\sin.m\pi}{(1+m)^2} + \frac{\sin.m\pi}{(2-m)^2} + \frac{\sin.m\pi}{(2+m)^2} - \text{etc.}$$

sive

$$\frac{\pi\pi}{\sin.m\pi\text{tang}.m\pi} = \frac{1}{m^2} - \frac{1}{(1-m)^2} - \frac{1}{(1+m)^2} + \frac{1}{(2-m)^2} + \frac{1}{(2+m)^2} - \text{etc.}$$

At posito

$$m\varphi = \pi$$

habebitur

$$\frac{\pi\pi\cos.m\pi}{m\sin.m\pi}=\frac{\sin.\frac{\pi}{m}}{(1-m)^2}-\frac{\sin.\frac{\pi}{m}}{(1+m)^2}+\frac{\sin.\frac{2\pi}{m}}{(2-m)^2}-\frac{\sin.\frac{2\pi}{m}}{(2+m)^2}+\text{etc.}$$

sive hoc modo

$$\frac{\pi\pi\cos.m\pi}{4mm\sin.m\pi}=\frac{1\sin.\frac{\pi}{m}}{(1-mm)^2}+\frac{2\sin.\frac{2\pi}{m}}{(4-mm)^2}+\frac{3\sin.\frac{3\pi}{m}}{(9-m)^2}+\text{etc.}$$

COROLLARIUM 3

28. Tertium theorema spectat ad comparationem formularum integralium et ita enunciatur:
Si sequentium formularum integratio a termino $x=0$ usque ad terminum $x=1$ extendatur, erit semper

$$\int x^{m-1}dx(1-x)^{m-1}\cdot\int x^{m-1}dx(1-x)^{-m}=\frac{2\pi\cos.m\pi}{\sin.m\pi}\cdot\int x^{m-1}dx(1-x)^{-2m},$$

seu si ponatur $m=\frac{\lambda}{n}$ et $x=y^n$, erit

$$\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{n-\lambda}}}\cdot\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{\lambda}}}=\frac{2\pi\cos.\frac{\lambda\pi}{n}}{n\sin.\frac{\lambda\pi}{n}}\cdot\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}$$

SCHOLION

29. Demonstratio huius postremi theorematis non parum ardua videtur; interim tamen per ea, quae olim de huiusmodi formulis integralibus sum commentatus, eius veritas sequenti modo ostendi potest. Indicemus enim, uti ibi feci, hanc formulam integralem

$$\int\frac{y^{p-1}dy}{\sqrt[n]{(1-y^n)^{n-q}}}$$

hoc charactere $\left(\frac{p}{q}\right)$ ac demonstrandum est esse

$$\left(\frac{\lambda}{\lambda}\right)\left(\frac{\lambda}{n-\lambda}\right)=\frac{2\pi\cos.\frac{\lambda\pi}{n}}{n\sin.\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right).$$

Iam primum demonstravi, si fuerit

$$q+r=n,$$

fore

$$\left(\frac{q}{r}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}},$$

unde statim sequitur

$$\left(\frac{\lambda}{n-\lambda}\right)=\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{\lambda}}}=\frac{\pi}{n\sin.\frac{\lambda\pi}{n}},$$

ita ut demonstrandum restet esse

$$\left(\frac{\lambda}{\lambda}\right)=2\cos.\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

Verum ibidem ostendi, si fuerit

$$p+q+r=n,$$

fore

$$\frac{1}{\sin.\frac{r\pi}{n}}\left(\frac{p}{q}\right)=\frac{1}{\sin.\frac{q\pi}{n}}\left(\frac{p}{r}\right)=\frac{1}{\sin.\frac{p\pi}{n}}\left(\frac{q}{r}\right).$$

Sumamus igitur

$$p=\lambda,\quad q=\lambda$$

eritque

$$r=n-2\lambda,$$

quo ob

$$\sin.\frac{(n-2\lambda)\pi}{n}=\sin.\frac{2\lambda\pi}{n}$$

colligimus

$$\frac{1}{\sin.\frac{2\lambda\pi}{n}}\left(\frac{\lambda}{\lambda}\right)=\frac{1}{\sin.\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right),$$

ita ut ob

$$\sin.\frac{2\lambda\pi}{n}=2\sin.\frac{\lambda\pi}{n}\cos.\frac{\lambda\pi}{n}$$

sit revera

$$\left(\frac{\lambda}{\lambda}\right)=2\cos.\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

Multo magis autem abstrusum est theorema supra (§ 23) erutum, quod sub iisdem integrationum terminis sit

$$\frac{\pi}{\sin.n\pi}\int x^n dx(1-x)^{n-1}=\int\frac{dx}{x}\int\frac{x^{n-1}dx}{(1-x)^{2n}}.$$

seu

$$\frac{\pi}{2\sin.n\pi}\int x^{n-1}dx(1-x)^{n-1}=\int\frac{dx}{x}\int\frac{x^{n-1}dx}{(1-x)^{2n}};$$

quae aequatio ut ad illam formam reducatur, loco *n* scribamus $\frac{\lambda}{n}$ sitque $x=y^n$, unde fit

$$\frac{\pi}{2n\sin.\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{n-\lambda}}}=\int\frac{dy}{y}\int\frac{y^{\lambda-1}}{\sqrt[n]{(1-y^n)^{2\lambda}}}.$$

Modo autem vidimus esse

$$\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{n-\lambda}}}=2\cos.\frac{\lambda\pi}{n}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

sicque vi huius theorematis colligimus esse

$$\frac{\pi}{n\operatorname{tang.}\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}=\int\frac{dy}{y}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

hincque porro istud theorema non minus notatu dignum

$$\frac{\pi}{n\operatorname{tang.}\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}=-\int\frac{y^{\lambda-1}dy\cdot ly}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

unde sumto $\lambda=1$ sequentem proportionem elicimus

$$\frac{\pi}{n}:\operatorname{tang.}\frac{\lambda\pi}{n}=\int\frac{dyl\frac{1}{y}}{\sqrt[n]{(1-y^n)^2}}:\int\frac{dy}{\sqrt[n]{(1-y^n)^2}}.$$

PROBLEMA 3

30. *Invenire eiusmodi aequationem pro linea curva inter binas variabiles, abscissam x et applicatam y, ut abscissis in arithmetica progressione sumtis datae conveniant applicatae, scilicet:*
Si sit

$$x=n\theta,\quad(n+1)\theta,\quad(n+2)\theta,\quad(n+3)\theta,\quad(n+4)\theta,\ etc.$$

ut fiat

$$y=p,\qquad q,\qquad r,\qquad s,\qquad t\quad etc.$$

SOLUTIO

Ponamus in genere

$$x=\theta\omega$$

atque ex solutione generali § 10 data consequimur hanc aequationem

$$\begin{aligned}\frac{y}{\omega}=\frac{p}{n}&\cdot\frac{(n+1-\omega)(n+1+\omega)}{1(2n+1)}\cdot\frac{(n+2-\omega)(n+2+\omega)}{2(2n+2)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{3(2n+3)}\cdot\text{etc.}\\ -\frac{q}{n+1}&\cdot\frac{(n-\omega)(n+\omega)}{1(2n+1)}\cdot\frac{(n+2-\omega)(n+2+\omega)}{1(2n+3)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{2(2n+4)}\cdot\text{etc.}\\ +\frac{r}{n+2}&\cdot\frac{(n-\omega)(n+\omega)}{2(2n+2)}\cdot\frac{(n+1-\omega)(n+1+\omega)}{1(2n+3)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{1(2n+5)}\cdot\text{etc.}\\ &+\ \text{etc.},\end{aligned}$$

quam aequationem brevitatis gratia ita repraesentemus

$$\frac{y}{\omega}=\mathfrak{A}\cdot\frac{p}{n}-\mathfrak{B}\cdot\frac{q}{n+1}+\mathfrak{C}\cdot\frac{r}{n+2}-\mathfrak{D}\cdot\frac{s}{n+3}+\text{etc.};$$

ac primo quidem pro valore ipsius $\mathfrak{A}$ eliciendo ex forma generali § 17 allata pro hoc casu habebimus

$$a=n+1-\omega,\ b=1,\ c=n-\omega \text{ et } d=1,$$

unde per formulas integrales a termino $z=0$ ad $z=1$ extendendas colligimus

$$\mathfrak{A}=\frac{\int dz(1-z)^{n-\omega-1}}{\int z^{n-\omega}dz(1-z)^{n-\omega-1}}=\frac{1}{(n-\omega)\int z^{n-\omega}dz(1-z)^{n-\omega-1}}$$

seu

$$\mathfrak{A}=\frac{2}{(n-\omega)\int z^{n-\omega-1}dz(1-z)^{n-\omega-1}},$$

qua integratione concessa reliqua facile expediuntur. Erit enim ut supra § 17

$$\frac{\mathfrak{B}}{\mathfrak{A}}=\frac{(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)}\cdot(2+2n)=\frac{2(n+1)(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)},$$
$$\frac{\mathfrak{C}}{\mathfrak{B}}=\frac{(n+1-\omega)(n+1+\omega)}{(n+2-\omega)(n+2+\omega)}\cdot\frac{(1+2n)(2+n)}{2(n+1)},$$
$$\frac{\mathfrak{D}}{\mathfrak{C}}=\frac{(n+2-\omega)(n+2+\omega)}{(n+3-\omega)(n+3+\omega)}\cdot\frac{(2+2n)(3+n)}{3(n+2)},$$
$$\frac{\mathfrak{E}}{\mathfrak{D}}=\frac{(n+3-\omega)(n+3+\omega)}{(n+4-\omega)(n+4+\omega)}\cdot\frac{(3+2n)(4+n)}{4(n+3)}$$
etc.

Statuamus igitur formulam integralem

$$\int z^{n-\omega-1}dz(1-z)^{n-\omega-1}=\mathit{\Delta},$$

ut sit

$$\mathfrak{A}=\frac{2}{(n-\omega)\mathit{\Delta}},$$

et reliqui coefficientes ita per $\mathfrak{A}$ definientur:

$$\mathfrak{B} = 2(n+1)\cdot\frac{nn-\omega\omega}{(n+1)^2-\omega^2}\mathfrak{A},$$

$$\mathfrak{C} = \frac{2(n+2)(2n+1)}{1\cdot 2}\cdot\frac{nn-\omega\omega}{(n+2)^2-\omega\omega}\mathfrak{A},$$

$$\mathfrak{D} = \frac{2(n+3)(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{nn-\omega\omega}{(n+3)^2-\omega\omega}\mathfrak{A},$$

$$\mathfrak{E} = \frac{2(n+4)(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{nn-\omega\omega}{(n+4)^2-\omega\omega}\mathfrak{A},$$

etc.

Quamobrem aequatio quaesita inter y et $x=\theta\omega$ ita erit comparata:

$$\frac{n\Delta y}{2(n+\omega)\omega} = \frac{p}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{q}{(n+1)^2-\omega\omega} +$$
$$\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{r}{(n+2)^2-\omega\omega} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{s}{(n+3)^2-\omega\omega} + \text{etc.},$$

unde pro quovis valore ipsius $x=\theta\omega$ conveniens ipsius y valor definitur idque per applicatas *p, q, r* etc., quae abscissis $n\theta,\ (n+1)\theta,\ (n+2)\theta$ etc. convenientes assumuntur. Ubi quidem notari oportet, si ω capiatur aequalis cuipiam termino progressionis $n,\ n+1,\ n+2$ etc., tum denominatorem applicatae respondentis datae evanescere, ita ut prae eo termino, quippe infinito, reliqui evanescant. Verum tum simul quoque valor Δ prodit infinitus et praecise eiusmodi, ut tum fiat
vel $y=p$ vel $y=q$ vel $y=r$ etc., quemadmodum rei natura postulat.

COROLLARIUM 1

31. Si abscissae propositae denotent arcus circulares, applicatae vero eorundem sinus, ut sit

$$p=\sin.n\theta,\ q=\sin.(n+1)\theta,\ r=\sin.(n+2)\theta \text{ etc.},$$

erit

$$y=\sin.\omega\theta,$$

unde ista resultat aequatio generalis

$$\frac{n\Delta\sin.\omega\theta}{2(n+\omega)\omega} = \frac{\sin.n\theta}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{\sin.(n+1)\theta}{(n+1)^2-\omega\omega} +$$
$$\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{\sin.(n+2)\theta}{(n+2)^2-\omega\omega} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{\sin.(n+3)\theta}{(n+3)^2-\omega\omega} + \text{etc.},$$

ubi imprimis notatu est dignum, quod tres litterae, $n,\ \theta$ et ω, pro lubitu accipi queant.

COROLLARIUM 2

32. Si ergo sumamus

$$\theta = \pi,$$

ut omnes seriei sinus ad eundem $\sin.n\theta$ reducantur, erit

$$\frac{n\Delta \sin.\omega\pi}{2(n+\omega)\omega \sin.n\pi} = \frac{1}{nn-\omega\omega} + \frac{2n}{1}\cdot\frac{1}{(n+1)^2-\omega^2} +$$
$$\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{1}{(n+2)^2-\omega^2} + \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{1}{(n+3)^2-\omega^2} + \text{etc.},$$

Hinc si sit

$$n = \tfrac{1}{2} \text{ et } \Delta = \int z^{-\omega-\frac{1}{2}} dz (1-z)^{-\omega-\frac{1}{2}}$$

seu

$$\Delta = 2\int \frac{z^{\frac{1}{2}-\omega} dz}{(1-z)^{\frac{1}{2}+\omega}};$$

habebitur

$$\frac{\Delta \sin.\omega\pi}{8(1+2\omega)\omega} = \frac{1}{1-4\omega^2} + \frac{1}{9-4\omega^2} + \frac{1}{25-4\omega^2} + \frac{1}{49-4\omega^2} + \text{etc.},$$

cuius seriei summam ostendi esse

$$= \frac{\pi}{8\omega}\,\text{tang.}\,\omega\pi,$$

ita ut sit

$$\frac{\Delta \sin.\omega\pi}{8(1+2\omega)\omega} = \frac{\pi}{8\omega}\,\text{tang.}\,\omega\pi$$

ideoque

$$\Delta = \frac{(1+2\omega)\pi}{\cos.\omega\pi}.$$

SCHOLION 1

33. His autem conclusionibus nimium confidere non licet ob rationem iam supra allegatam. Positis enim applicatis

$$p = \sin.n\theta,\ q = \sin.(n+1)\theta,\ r = \sin.(n+2)\theta \text{ etc.},$$

dum arcus $n\theta,\ (n+1)\theta,\ (n+2)\theta$ etc. ut abscissae spectantur, aequatio inventa eiusmodi lineam curvam praebet, quae per omnia haec puncta transit; neque vero hinc sequitur hanc curvam ipsam esse lineam sinuum, cum infinitae aliae dentur lineae curvae per eadem illa puncta infinita data transeuntes. Quare servata littera *y* pro applicata abscissae $x = \theta\omega$ respondente significanda solutio nostra pro curva quaesita hanc quidem suppeditat aequationem

$$\frac{n\Delta y}{2(n+\omega)\omega} = \frac{\sin.n\theta}{n^2-\omega^2} - \frac{2n}{1}\cdot\frac{\sin.(n+1)\theta}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1\cdot 2}\cdot\frac{\sin.(n+2)\theta}{(n+1)^2-\omega^2} + \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{\sin.(n+3)\theta}{(n+3)^2-\omega^2} + \text{etc.},$$

ita ut abscissae

$$x = (n \pm i)\theta$$

respondeat haec applicata

$$y = \sin.(n \pm i)\theta,$$

si modo i sit numerus integer quicunque. Fieri vero posset, ut pro aliis abscissis, ubi i non est numerus integer ideoque generatim, si $x = w\theta$, non foret applicata $y =$ $\sin.\omega\theta$. Quo hoc clarius perspiciatur, investigemus aequationem generalem pro omnibus plane lineis per puncta data transeuntibus, sitque valor hactenus repertus

$$y = \Theta$$

et quaeratur functio pro omnibus abscissis datis evanescens, cuiusmodi est

$$\omega(nn - \omega\omega)\frac{(n+1)^2 - \omega^2}{1(2n+1)} \cdot \frac{(n+2)^2 - \omega^2}{2(2n+2)} \cdot \frac{(n+3)^2 - \omega^2}{3(2n+3)} \text{etc.},$$

quae per superiora est

$$= \omega(nn - \omega\omega)\mathfrak{A} = \frac{2\omega(n+\omega)}{\Delta}.$$

Vocetur haec quantitas $= \Omega$ sitque $f : \Omega$ eiusmodi functio ipsius Ω, quae evanescat, si $\Omega = 0$, eritque aequatio generalis pro omnibus curvis satisfacientibus

$$y = \Theta + f : \Omega = \Theta + f : \frac{2\omega(n+\omega)}{\Delta}.$$

Ac iam sine ullo dubio certum est in hac aequatione contineri aequationem $y =$ $\sin.\omega\theta$ posito $x = w\theta$, quandoquidem haec aequatio conditionibus praescriptis satisfacit. Ex quo evenire prorsus posset, ut aequatio $y = \Theta$ ab ista $y =$ $\sin.\omega\theta$ esset diversa; quod imprimis pendere potest a valoribus litteris θ et n tributis, ita ut aliis casibus aequatio inventa $y = \Theta$ cum hac $y =$ $\sin.\omega\theta$ conveniat, aliis vero ab eadem discrepet.

SCHOLION 2

34. Accommodemus haec ad casum, quo

$$\theta = \pi \text{ et } n = \tfrac{1}{2}$$

atque

$$\Delta = 2\int \frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}};$$

et quoniam seriei inventae summa est

$$= \frac{\pi}{8\omega}\tang.\omega\pi,$$

habebitur haec aequatio generalis

$$\frac{\Delta y}{2(1+2\omega)\omega}=\frac{\pi}{8\omega}\text{tang.}\omega\pi+\frac{\Delta}{8(1+2\omega)}f:\frac{\omega(1+2\omega)}{2\Delta}$$

seu

$$y=\frac{\pi(1+2\omega)}{\Delta}\text{tang.}\omega\pi+f:\frac{\omega(1+2\omega)}{2\Delta},$$

ubi functio adiecta in genere ita est comparata, ut casibus

$$\omega=0,\quad \omega=\pm\tfrac{1}{2},\quad \omega=\pm\tfrac{3}{2},\quad \omega=\pm\tfrac{5}{2}\ \text{ etc.}$$

evanescat, cuiusmodi formulae sunt

$$\sin.2\omega\pi,\quad \omega\cos.\omega\pi,\quad \text{item }\sin.2i\omega\pi\text{ et }\omega\cos.(2i-1)\omega\pi,$$

denotante *i* numerum integrum quemcunque; unde quotcunque huiusmodi formulas pro lubitu combinare licet. Huiusmodi ergo dabitur certa quaedam functio, quae sit φ, ut fiat

$$y=\sin.\omega\pi$$

hincque

$$\sin.\omega\pi=\frac{\pi(1+2\omega)}{\Delta}\text{tang.}\omega\pi+\varphi$$

seu

$$\Delta=\frac{\pi(1+2\omega)\text{tang.}\omega\pi}{\sin.\omega\pi-\varphi}=2\int\frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}}.$$

Cum igitur casu $\omega=0$ functio φ certe evanescat, erit utique $\Delta=\pi$, quod indicio est functionem φ factorem continere ω^{λ}, cuius exponens λ sit unitate maior, quia alioquin sumto $\omega=0$ quantitas φ prae $\sin.\omega\pi$ non evanesceret. Atque ob hanc rationem conclusiones praecedentis problematis secundi pro veris sunt habendae.

PROBLEMA 4

35. *lnvenire eiusmodi aequationem pro linea curva inter abscissam x et applicatam y, ut abscissis in arithmetica progressione interrupta progredientibus datae conveniant applicatae, scilicet:*
Si sit

$$x = n\theta,\ \ (1-n)\theta,\ \ (1+n)\theta,\ \ (2-n)\theta,\ \ (2+n)\theta,\ \ (3-n)\theta \ etc.,$$

ut fiat

$$y = p,\quad q,\quad r,\quad s,\quad t,\quad u \quad etc.$$

SOLUTIO

Ponamus in genere abscissam

$$x = \theta\omega$$

et pro aequatione inter x et y statuamus hanc aequationem

$$\frac{y}{\omega} = \mathfrak{A}\cdot\frac{p}{n} - \mathfrak{B}\cdot\frac{q}{1-n} + \mathfrak{C}\cdot\frac{r}{1+n} - \mathfrak{D}\cdot\frac{s}{2-n} + \mathfrak{E}\cdot\frac{t}{2+n} - \mathfrak{F}\cdot\frac{u}{3-n} + \text{etc.}$$

atque ex paragrapho 25 ad hunc casum generalem extenso habebitur

$$\mathfrak{A} = \frac{(1-n-\omega)(1-n+\omega)}{1(1-2n)}\cdot\frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)}\cdot\frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)}\cdot\frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)}\cdot\text{etc.},$$

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{(n-\omega)(n+\omega)}{(1-n-\omega)(1-n+\omega)}\cdot\frac{(1-n)}{n},\quad \frac{\mathfrak{C}}{\mathfrak{B}} = \frac{(1-n-\omega)(1-n+\omega)}{(1+n-\omega)(1+n+\omega)}\cdot\frac{1+n}{1-n},$$

$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(1+n-\omega)(1+n+\omega)}{(2-n-\omega)(2-n+\omega)}\cdot\frac{2-n}{1+n},\quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(2-n-\omega)(2-n+\omega)}{(2+n-\omega)(2-n+\omega)}\cdot\frac{2+n}{2-n},$$

etc.

Evolvamus valorem ipsius $\mathfrak{A}$ in duo producta

$$\mathfrak{P} = \frac{(1-n-\omega)(1-n+\omega)}{3(1-2n)}\cdot\frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)}\cdot\frac{(3-n-\omega)(3-n+\omega)}{3(2-3n)}\cdot\text{etc.},$$

$$\mathfrak{Q} = \frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)}\cdot\frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)}\cdot\frac{(3+n-\omega)(3+n+\omega)}{3(2+3n)}\cdot\text{etc.},$$

ut sit

$$\mathfrak{A} = \mathfrak{P}\mathfrak{Q},$$

et utriusque valorem per formulas integrales secundum praecepta § 17 definiamus. Ac primo quidem pro infinito producto $\mathfrak{P}$ faciamus

$$a = 1-n-\omega,\ b = 1,\ c = -n+\omega \text{ et } d = 1$$

eritque

$$\mathfrak{P} = \frac{\int dx(1-x)^{-1-n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}} = \frac{1}{\omega-n}\cdot\frac{1}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}},$$

siquidem sit

$$\omega - n > 0.$$

Pro altero producto infinito sumendo tantum n negative fiet

$$\mathfrak{Q} = \frac{1}{\omega+n} \cdot \frac{1}{\int x^{n-\omega} dx(1-x)^{n+\omega-1}}.$$

Ne autem conditione $\omega - n > 0$ opus sit, alia distributione utamur sitque

$$\mathfrak{P} = \frac{(1+n+\omega)(1-n-\omega)}{1\cdot1} \cdot \frac{(2+n+\omega)(2-n-\omega)}{2\cdot2} \cdot \frac{(3+n+\omega)(3-n-\omega)}{3\cdot3} \cdot \text{etc.},$$
$$\mathfrak{Q} = \frac{(1+n-\omega)(1-n+\omega)}{(1-2n)(1+2n)} \cdot \frac{(2+n-\omega)(2-n+\omega)}{(2-2n)(2+2n)} \cdot \frac{(3+n-\omega)(3-n+\omega)}{(3-2n)(3+2n)} \cdot \text{etc.},$$

ac pro $\mathfrak{P}$ statuamus

$$a = 1-n-\omega, \quad b = 1, \quad c = n+\omega, \quad d = 1,$$

pro $\mathfrak{Q}$ vero

$$a = 1+n-\omega, \; b = 1-2n, \; c = n+\omega \; \text{ et } \; d = 1$$

eritque

$$\mathfrak{P} = \frac{\int dx(1-x)^{-1+n+\omega}}{\int x^{-n-\omega} dx(1-x)^{-1+n+\omega}} = \frac{1}{\omega+n} \cdot \frac{1}{\int x^{-n-\omega} dx(1-x)^{-1+n+\omega}},$$

$$\mathfrak{Q} = \frac{\int x^{-2n} dx(1-x)^{-1+n+\omega}}{\int x^{n-\omega} dx(1-x)^{-1+n+\omega}}.$$

Est vero in genere

$$\int x^m dx(1-x)^{k-1} = \frac{m+k+1}{k} \int x^m dx(1-x)^k,$$

ergo

$$\int x^{-n-\omega} dx(1-x)^{-1+n+\omega} = \frac{1}{n+\omega} \int x^{-n-\omega} dx(1-x)^{n+\omega}$$
$$= \frac{1}{n+\omega} \int y^{n+\omega} dy(1-y)^{-n-\omega},$$
$$\int x^{-2n} dx(1-x)^{-1+n+\omega} = \frac{1-n+\omega}{n+\omega} \int x^{-2n} dx(1-x)^{n+\omega}$$
$$= \frac{1-n+\omega}{n+\omega} \int y^{n+\omega} dy(1-y)^{-2n,}$$
$$\int x^{n-\omega} dx(1-x)^{-1+n+\omega} = \frac{1+2n}{n+\omega} \int x^{n-\omega} dx(1-x)^{n+\omega}$$
$$= \frac{1+2n}{n+\omega} \int y^{n+\omega} dx(1-y)^{n-\omega}$$

unde concluditur

$$\mathfrak{A} = \mathfrak{P}\mathfrak{Q} = \frac{(1-n+\omega)\int y^{n+\omega} dy(1-y)^{-2n}}{\int y^{n+\omega} dy(1-y)^{-n+\omega} \cdot \int y^{n+\omega} dy(1-y)^{n-\omega}}$$

vel

$$\mathfrak{A} = \frac{\int y^{n+\omega-1} dy(1-y)^{-2n}}{\int y^{n+\omega} dy(1-y)^{-n-\omega} \cdot \int y^{n+\omega-1} dy(1-y)^{n-\omega}}$$

seu

$$\mathfrak{A} = \frac{\int y^{n+\omega-1}dy(1-y)^{-2n}}{(n+\omega)\int y^{n+\omega-1}dy(1-y)^{-n-\omega}\cdot\int y^{n+\omega-1}dy(1-y)^{n-\omega}}$$

Cum igitur sit

$$\mathfrak{B} = \frac{1-n}{n}\cdot\frac{nn-\omega\omega}{(1-n)^2-\omega\omega}\mathfrak{A},\ \mathfrak{C} = \frac{1+n}{n}\cdot\frac{nn-\omega\omega}{(1+n)^2-\omega\omega}\mathfrak{A},$$

$$\mathfrak{C} = \frac{2-n}{n}\cdot\frac{nn-\omega\omega}{(2-n)^2-\omega\omega}\mathfrak{A},\ \mathfrak{D} = \frac{2+n}{n}\cdot\frac{nn-\omega\omega}{(2+n)^2-\omega\omega}\mathfrak{A}$$

etc.

erit per seriem satis concinnam

$$\frac{ny}{\mathfrak{A}\omega(nn-\omega\omega)} = \frac{p}{n^2-\omega^2} - \frac{q}{(1-n)^2-\omega^2} + \frac{r}{(1+n)^2-\omega^2} - \text{etc.}$$

sive

$$\frac{ny}{\mathfrak{A}\omega(nn-\omega\omega)} = \frac{p}{n^2-\omega^2} - \frac{q}{(1-n)^2-\omega^2} + \frac{r}{(1+n)^2-\omega^2} - \text{etc.}$$

Loco $\mathfrak{A}$ autem formam integralem restituendo, ubi quidem novam variabilem distinctionis causa littera z designabo, haec eadem series aequalis est huic expressioni

$$\frac{ny}{(n-\omega)\omega}\cdot\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}},$$

quarum formularum integratio a termino $z=0$ ad $z=1$ extensa est intelligenda.

COROLLARIUM 1.

36. Si ergo brevitatis gratia hanc integralem formam ponamus

$$\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}} = \Delta$$

et singulos seriei terminos in binos resolvamus, habebimus

$$\frac{2n\Delta y}{n-\omega} = +\frac{p}{n-\omega} - \frac{q}{1-n-\omega} + \frac{r}{1+n-\omega} - \frac{s}{2-n-\omega} + \frac{t}{2+n-\omega} - \text{etc.}$$
$$-\frac{p}{n+\omega} + \frac{q}{1-n+\omega} - \frac{r}{1+n+\omega} + \frac{s}{2-n+\omega} - \frac{t}{2+n+\omega} + \text{etc.}$$

COROLLARIUM 2

37. Haec ergo aequatio eiusmodi definit lineam curvam, in qua abscissis

$$x = 0,\ n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta \text{ etc.}$$

respondent applicatae

$$y = 0,\quad p,\quad q,\qquad r,\qquad s,\qquad t \quad \text{etc.,}$$

iisdem vero abscissis negative sumtis respondent eaedem applicatae negative sumtae. In genere autem hic posita est abscissa $x = \theta\omega$.

COROLLARIUM 3

38. Quoniam littera θ ex calculo excessit, eius loco unitatem scribere licuisset, ut littera ω ipsam abscissam denotaret. Verum si applicationem ad arcus eorumque sinus facere velimus, commodum est litteram θ in calculo retinere.

SCHOLION

39. Usus huius problematis imprimis cernitur, si ut supra abscissae tanquam arcus circulares spectentur et abscissae datae ita accipiantur, ut applicatae *p, q, r, s, t* etc. fiant inter se aequales, sive positive sive negative. Quo igitur his casibus appareat, an series inventa aliunde summari possit, in subsidium vocentur, quae olim de similibus seriebus sum commentatus , unde quidem sequentium duarum serierum summae colliguntur

$$\frac{1}{\alpha} - \frac{1}{\beta-\alpha} + \frac{1}{\beta-\alpha} - \frac{1}{2\beta-\alpha} + \frac{1}{2\beta-\alpha} - \text{etc.} = \frac{\pi}{\beta \text{tang.} \frac{\alpha\pi}{\beta}},$$

$$\frac{1}{\alpha} + \frac{1}{\beta-\alpha} - \frac{1}{\beta-\alpha} + \frac{1}{2\beta-\alpha} - \frac{1}{2\beta-\alpha} + \text{etc.} = \frac{\pi}{\beta \sin. \frac{\alpha\pi}{\beta}}.$$

Hinc ergo pro nostro problemate deducimus sequentes summationes quatuor

$$\text{I.}\quad \frac{1}{n-\omega} - \frac{1}{1-n+\omega} + \frac{1}{1+n-\omega} - \frac{1}{2-n+\omega} + \frac{1}{2+n-\omega} - \text{etc.} = \frac{\pi}{\text{tang.}(n-\omega)\pi},$$

$$\text{II.}\quad \frac{1}{n-\omega} + \frac{1}{1-n+\omega} - \frac{1}{1+n-\omega} - \frac{1}{2-n+\omega} + \frac{1}{2+n-\omega} + \text{etc.} = \frac{\pi}{\sin.(n-\omega)\pi},$$

$$\text{III.}\quad \frac{1}{n+\omega} - \frac{1}{1-n-\omega} + \frac{1}{1+n+\omega} - \frac{1}{2-n-\omega} + \frac{1}{2+n+\omega} - \text{etc.} = \frac{\pi}{\text{tang.}(n+\omega)\pi},$$

$$\text{IV.}\quad \frac{1}{n+\omega} + \frac{1}{1-n-\omega} - \frac{1}{1+n+\omega} - \frac{1}{2-n+\omega} + \frac{1}{2+n+\omega} + \text{etc.} = \frac{\pi}{\sin.(n+\omega)\pi},$$

His observatis evolvamus casus, quos harum summationum ope ad expressiones finitas reducere licet.

EXEMPLUM I

40. *Sint applicatae, quae abscissis*

$$x=0,\ n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta \text{ etc.}$$

respondent

$$y=0,\quad p,\quad q,\quad r,\quad s,\quad t \quad \text{etc.,}$$

et per aequationem finitam relatio inter applicatam y et abscissam $x=\theta\omega$ *investigetur.*

SOLUTIO

Corollarium primum pro hoc casu hanc praebet aequationem

$$\frac{2n\Delta y}{f(n-\omega)}=+\frac{1}{n-\omega}-\frac{1}{1-n-\omega}-\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}-\text{etc.}$$
$$-\frac{1}{n+\omega}+\frac{1}{1-n+\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{t}{2+n+\omega}+\text{etc.,}$$

quae binae series reducuntur ope quatuor supra allatarum, quarum summatio constat, ad II minus IV, ideoque aequatio quaesita in forma finita ita se habebit

$$\frac{2n\Delta y}{f(n-\omega)}=\frac{\pi}{\sin.(n-\omega)\pi}-\frac{\pi}{\sin.(n+\omega)\pi},$$

quae expressio redigitur ad hanc

$$\frac{2\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}=\frac{4\pi\cos.n\pi\sin.\omega\pi}{\cos.2\omega\pi-\cos.2n\pi},$$

ita ut pro nostra curva haec habeatur aequatio

$$\frac{n\Delta y}{f(n-\omega)}=\frac{\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}.$$

Valorem ipsius Δ ante per formulas integrales expressum dedimus; cum autem ex superioribus sit

$$\Delta=\frac{1}{\mathfrak{A}(n+\omega)},$$

habebimus per productum infinitum unde clarius constat quam ex formulis integralibus valorem Δ fieri infinitum,

$$\Delta=\frac{1}{n+\omega}\cdot\frac{1(1-2n)}{(1-n)^2-\omega^2}\cdot\frac{1(1+2n)}{(1+n)^2-\omega^2}\cdot\frac{2(2-2n)}{(2-n)^2-\omega^2}\cdot\frac{2(2+2n)}{(2+n)^2-\omega^2}\cdot\text{etc.,}$$

quoties fuerit

$$\omega=\pm(i\pm n)$$

denotante i numerum integrum quemcunque, eundem vero valorem Δ evanescere casibus, quibus est

$$n = \pm \tfrac{1}{2}.$$

Tum vero etiam notasse iuvabit, si abeunte ω in $1+\omega$ valor ipsius Δ notetur Δ', fore

$$\Delta' = \frac{(1-n+\omega)\Delta}{n-\omega}.$$

Ac si simili modo Δ'' conveniat valori $2+\omega$ loco ω assumto, erit

$$\Delta'' = \frac{-(2-n+\omega)\Delta'}{-(1-n+\omega)} = \frac{-(2-n+\omega)\Delta}{n-\omega}.$$

COROLLARIUM 1

41. Quatenus quantitas Δ ab ω pendet, consideretur ut eius functio hocque modo designetur

$$\Delta = f:\omega\,;$$

tum igitur erit

$$f:(1+\omega) = \frac{n-1+\omega}{n-\omega} f:\omega$$

et

$$f:(2+\omega) = \frac{n-2+\omega}{n-\omega} f:\omega$$

etc.

Quare si ω denotet numerum integrum quemcunque, habebitur hoc theorema

$$\frac{f:(i+\omega)}{n-i-\omega} = \frac{f:\omega}{n-\omega}$$

COROLLARIUM 2

42. Cum deinde sumto ω negativo sit

$$f:(-\omega) = \frac{n+\omega}{n-\omega} f:\omega,$$

erit

$$\frac{f:-\omega}{n+\omega} = \frac{f:\omega}{n-\omega},$$

hinc etiam in genere

$$\frac{f:(i-\omega)}{n-i+\omega} = \frac{f:\omega}{n-\omega}.$$

SCHOLION

43. Hic casus respondet illi, quem supra § 25 evolvimus, ubi applicatae datae quoque erant abscissarum sinus; ac pro praesenti quidem casu statui oportet

$$\theta = \pi,$$

ut sit

$$f = \sin.n\pi$$

et puncta omma data in linea sinuum sint sita. Hinc autem non sequitur ipsam curvam, quam aequatio inventa exhibet, esse lineam sinuum, cum innumerabiles aliae curvae per eadem puncta data transire queant. Quare neutiquam etiamnunc certum est valorem ipsius y abscissae $x = \omega\pi$ convenientem et hac aequatione definitum

$$\frac{n\Delta y}{(n-\omega)\sin.n\pi} = \frac{\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}$$

aequari sinui arcus $n\omega$, ut sit $y = \sin.n\omega$, etiamsi hoc verum sit casibus $\omega = \pm(i \pm n)$ et $\omega = 0$. Supra quidem vidimus casu etiam, quo ω est quantitas minima, aequationem fore veritati consentaneam sumendo $y = \sin.n\omega$, ita ut sit

$$\Delta = \frac{\pi\cos.n\pi}{\sin.n\pi}$$

existente

$$\Delta = \frac{\int z^{n-1}dz(1-z)^{-n}\cdot\int z^{n-1}dz(1-z)^{n}}{\int z^{n-1}dz(1-z)^{-2n}},$$

quemadmodum etiam ibi demonstravi. Quo autem haec res facilius in genere explorari possit, pro valore Δ commodius exprimendo observo esse

$$\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}} = \frac{\int z^{\omega-n}dz(1-z)^{-n-\omega}}{\int dz(1-z)^{-2n}} = (1-2n)\int z^{\omega-n}dz(1-z)^{-n-\omega},$$

dum sit

$$n < \tfrac{1}{2},$$

unde erit

$$\Delta = (1-2n)\int z^{\omega-n}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}.$$

Verum si esset in genere

$$y = \sin.\omega\pi,$$

foret

$$\Delta = \frac{(n-\omega)\pi\sin.n\pi\cos.n\pi}{n\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}.$$

Quaestio ergo huc redit, utrum haec aequatio

$$(1-2n)\int z^{\omega-n}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}=\frac{(n-\omega)\pi\sin.n\pi\cos.n\pi}{n\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}$$

etiam aliis casibus praeter supra memoratos sit vera necne. Hunc in finem consideremus casum, quo

$$n=\tfrac{1}{4}\text{ et }\omega=\tfrac{1}{2},$$

ubi posterior quidem pars fit

$$\frac{-\frac{1}{4}\cdot\pi\cdot\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}}{-\frac{1}{4}\cdot\pi\cdot\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}}=\pi$$

prior vero pars erit

$$=\frac{1}{2}\int\frac{z^{\frac{1}{4}}dz}{(1-z)^{\frac{3}{4}}}\cdot\int\frac{z^{-\frac{1}{4}}dz}{(1-z)^{\frac{1}{4}}},$$

quae posito

$$z=v^4$$

abit in hanc formam

$$8\int\frac{v^4dv}{\sqrt[4]{(1-v^4)^3}}\cdot\int\frac{v^3dv}{\sqrt[4]{(1-v^4)}}=4\int\frac{dv}{\sqrt[4]{(1-v^4)^3}}\cdot\int\frac{vvdv}{\sqrt[4]{(1-v^4)}},$$

cuius valor per ea, quae circa huiusmodi formulas demonstravi, revera fit $=\pi$, quod ergo iam est documentum insigne pro veritate nostrae aequationis, quam autem sequenti modo perfecte demonstrare licet.

THEOREMA

44. *Quomodocunque bini numeri n et ω accipiantur, haec aequatio veritati erit consentanea*

$$(1-2n)\int\frac{z^{\omega-n}dz}{(1-z)^{n+\omega}}\cdot\int\frac{z^{n+\omega-1}dz}{(1-z)^{\omega-n}}=\frac{(n-\omega)\pi\sin.n\pi\cos.n\pi}{n\sin.(n-\omega)\cdot\pi\sin.(n+\omega)\pi},$$

si quidem illarum formularum integratio a termino $z=0$ *ad terminum* $z=1$ *extendatur.*

DEMONSTRATIO

Quo has formulas integrales ad formam, quam tractavi, reducamus, ponamus

$$n+\omega=\frac{\mu}{\lambda}\quad\text{et}\quad\omega-n=\frac{\nu}{\lambda},$$

ut sit

$$2n=\frac{\mu-\nu}{\lambda},$$

ac demonstrari oportet hanc aequationem

$$\frac{\lambda-\mu+\nu}{\lambda}\int\frac{z^{\frac{\mu}{\lambda}}dz}{\sqrt[\lambda]{(1-z)^{\mu}}}\cdot\int\frac{z^{\frac{\mu-\lambda}{\lambda}}dz}{\sqrt[\lambda]{(1-z)^{\nu}}}=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}\cdot\sin.\frac{\mu\pi}{\lambda}}.$$

Ponatur nunc $z=v^{\lambda}$ et habebitur

$$\lambda(\lambda-\mu+\nu)\int\frac{v^{\lambda+\nu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu}}}\cdot\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\nu}}}=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}\cdot\sin.\frac{\mu\pi}{\lambda}};$$

more autem has formulas integrales exprimendi ibi recepto membrum prius ita repraesentabitur

$$\lambda(\lambda-\mu+\nu)(\tfrac{\lambda+\nu}{\lambda-\nu})(\tfrac{\mu}{\lambda-\nu}),$$

quod per reductionem primam

$$\left(\frac{p}{q}\right)=\frac{p-\lambda}{p+q-\lambda}\left(\frac{p-\lambda}{q}\right)$$

abit in

$$\lambda\nu\left(\frac{\nu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right)=\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right).$$

Haec vero reductio

$$\frac{\lambda-q}{p}\left(\frac{p+q-\lambda}{q}\right)=\frac{\pi}{\lambda p\sin.\frac{q\pi}{\lambda}}$$

sumto

$$p=\mu-\nu \text{ et } q=\mu$$

dat

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\pi}{\lambda(\mu-\nu)\sin.\frac{\mu\pi}{\lambda}}.$$

Est vero etiam

$$\left(\frac{\lambda-\nu}{\nu}\right)=\frac{\pi}{\lambda\sin.\frac{\nu\pi}{\lambda}},$$

quarum productum est

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\mu}{\mu-\nu}\right)=\frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin.\frac{\mu\pi}{\lambda}\cdot\sin.\frac{\nu\pi}{\lambda}}.$$

Porro, cum in genere sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),$$

sumendo

$$p=\lambda-\mu,\quad q=\mu-\nu,\quad \text{et}\quad r=\nu$$

erit

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right)=\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\mu+\nu}{\mu-\nu}\right)$$

et ob

$$\left(\frac{\lambda-p}{p}\right)=\frac{\pi}{\lambda\sin.\frac{p\pi}{\lambda}}$$

sumto

$$p=\mu-\nu$$

erit

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right)=\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi}$$

ideoque

$$\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi}=\frac{\pi\pi}{\lambda\lambda(\lambda-\nu)\sin.\frac{\mu\pi}{\lambda}\sin.\frac{\nu\pi}{\lambda}};$$

ex quibus prius membrum redigitur ad hanc formam

$$\lambda\nu\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}\sin.\frac{\nu\pi}{\lambda}},$$

quae est ipsa aequatio demonstranda.

COROLLARIUM 1

45. In doctrina ergo de huiusmodi formulis integralibus

$$\int\frac{v^{p-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-q}}},$$

quas hoc charactere designo

$$\left(\frac{p}{q}\right),$$

cui aequivalet $\left(\frac{q}{p}\right)$, haec reductio est gravis momenti, qua demonstravi esse

$$\lambda\nu\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}\sin.\frac{\nu\pi}{\lambda}},$$

ita ut productum binarum talium formularum integralium $\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)$ per solos angulos exhiberi possit.

COROLLARIUM 2

46. Si in valore pro Δ primum invento pariter ponatur

$$n+\omega=\frac{\mu}{\lambda} \quad \text{et} \quad n-\omega=\frac{\nu}{\lambda}$$

tum vero

$$z=v^{\lambda},$$

erit

$$\varDelta=\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu}}}\cdot\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\nu}}}:\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu-\nu}}}$$

ideoque hoc signandi modo

$$\varDelta=\frac{\lambda\left(\frac{\mu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right)}{\left(\frac{\mu}{\lambda-\mu+\nu}\right)}$$

seu

$$\varDelta=\frac{\lambda\left(\frac{\lambda-\mu}{\mu}\right)\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}$$

Idem vero valor est quoque

$$\varDelta=\frac{\nu\pi}{\mu-\nu}\cdot\frac{\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}\sin.\frac{\nu\pi}{\lambda}}.$$

COROLLARIUM 3

47. Cum igitur pro hac postrema formula statim sit

$$\left(\frac{\lambda-\mu}{\mu}\right)=\frac{\pi}{\pi\sin.\frac{\mu\pi}{\lambda}},$$

erit

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}=\frac{\nu}{\mu-\nu}\cdot\frac{\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}},$$

cuius veritas ex hoc theoremate generali ostenditur

$$\frac{\left(\frac{q}{p}\right)}{\left(\frac{r}{p}\right)}=\frac{\left(\frac{p+r}{q}\right)}{\left(\frac{p+q}{r}\right)};$$

erit enim

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}=\frac{\left(\frac{\lambda+\nu}{\lambda-\nu}\right)}{\left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)}=\frac{\nu}{\mu-\nu}\cdot\frac{\left(\frac{\nu}{\lambda-\nu}\right)}{\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)}$$

ob

$$\left(\frac{\lambda+\nu}{\lambda-\nu}\right)=\frac{\nu}{\lambda}\cdot\left(\frac{\nu}{\lambda-\nu}\right) \text{ et } \left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\mu-\nu}{\lambda}\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right);$$

tum vero est

$$\left(\frac{\nu}{\mu-\nu}\right)=\frac{\pi}{\lambda\sin.\frac{\nu\pi}{\lambda}} \quad \text{et} \quad \left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi}.$$

EXEMPLUM II

48. *Sint applicatae, quae abscissis*

$$n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta,\ etc.$$

respondent,

$$p=f,\ q=-f,\ r=+f,\ s=-f,\ t=+f,\ u=-f\ etc.,$$

et per aequationem finitum investigetur relatio in genere inter abscissam $x=\theta\omega$ *et applicatam* $=y$.

Aequatio generalis paragraphi 36 ad hunc casum accommodata praebet

$$\frac{2n\Delta y}{f(n-\omega)}=\frac{1}{n-\omega}+\frac{1}{1-n-\omega}+\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}+\text{etc.}$$
$$-\frac{1}{n+\omega}-\frac{1}{1-n+\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{1}{2+n+\omega}-\text{etc.}$$

ubi nunc quidem novimus esse

$$\Delta=\frac{(n-\omega)\pi\sin.2n\pi}{2n\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}.$$

Illa autem series ex § 39 fit

$$\text{I minus III}=\frac{\pi}{\text{tang.}(n-\omega)\pi}-\frac{\pi}{\text{tang.}(n+\omega)\pi}=\frac{\pi\sin.2\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi},$$

qua summa substituta prodit

$$\frac{y}{f}\cdot\frac{\pi\sin.2n\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}=\frac{\pi\sin.2\omega\pi}{\sin.(n-\omega)\pi\cdot\sin.(n+\omega)\pi}$$

seu

$$y=\frac{f\sin.2\omega\pi}{\sin.2n\pi}=\frac{f\sin.\frac{2x\pi}{\theta}}{\sin.2n\pi}.$$

Haec ergo curva denuo est linea sinuum, ac si sumatur $\theta=2\pi$, ut sit $f=sin.2n\pi$, erit applicata $y=\sin.x$.

COROLLARIUM 1

49. Si sumatur

$$\theta=\pi\ \text{et} f=\text{tang.}n\theta=\text{tang.}n\pi,$$

puncta data erunt in linea tangentium; neque tamen ipsa curva inventa erit linea tangentium; sed eius natura hac exprimetur aequatione

$$y=\frac{\text{tang}.n\pi\cdot\sin.2x}{\sin.2n\pi}=\frac{\sin.2x}{2\cos.n\pi^2}=\frac{\sin.2x}{1+\cos.2n\pi},$$

eritque hic $y=\text{tang}.x$, quoties fuerit $x=\pm(i\pm n)\pi$.

COROLLARIUM 2

50. Si in solutione prioris exempli, quo erat

$$p=f,\ q=f,\ r=-f,\ s=-f,\ t=f,\ u=f\ \ etc.,$$

statim loco Δ valorem inventum posuissemus, prodiisset haec aequatio

$$y=\frac{f\sin.\omega\pi}{\sin.n\pi}$$

Unde perspicuum fuisset sumto $\theta=\pi$ et $f=\sin.n\pi$ curvam illam ipsam fore lineam sinuum.

SCHOLION

51. Omnino notari meretur, quod in problemate 4, ubi abscissae datae progressionem arithmeticam interruptam constituunt, valor quantitatis Δ absolute angulos exhiberi potuerit, cum tamen in problemate 3, ubi abscissae datae veram progressionem arithmeticam constituebant, formula integralis Δ genere neutiquam per angulos exprimi queat. Cum enim ibi esset

$$\varDelta=\int z^{n-\omega-1}dz(1-z)^{n-\omega-1},$$

haec formula posito $n-\omega=\frac{\nu}{\lambda}$ et $z=v^{\lambda}$ abit in

$$\varDelta=\lambda\int\frac{v^{\nu-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-\nu}}}\ \text{ seu } \varDelta=\lambda\left(\frac{\nu}{\nu}\right),$$

quae formula quadraturas maxime transcendentes implicare potest. Ac si in illo problemate statuantur applicatae datae

$$p=f,\ q=-f,\ r=f,\ s=-f,\ t=f,\ u=-f\ etc.$$

et $n=\frac{1}{2}$, aequatio pro curva per ista puncta transeunte erit

$$\frac{\varDelta y}{2(1+2\omega)\omega f}=\frac{4}{1-4\omega\omega}+\frac{4}{9-4\omega\omega}+\frac{4}{25-4\omega\omega}+\text{etc.}$$

seu

$$\frac{\Delta y}{2f\omega(1+2\omega)}=\frac{\pi}{2\omega}\text{tang.}\omega\pi,$$

ita ut sit

$$y=\frac{\pi f(1+2\omega)\text{tang.}\omega\pi}{\Delta},$$

unde, etiamsi sumatur

$$\theta=\pi \text{ et } f=\sin.n\theta=\sin.\tfrac{1}{2}\pi=1,$$

manifesto non sequitur fore $y=sin.\theta\omega=sin.\omega\pi$. Cum priori exemplo iam certum sit esse

$$y=\frac{f\sin.\omega\pi}{\sin.n\pi},$$

eundem casum ex problemate primo ita evolvamus, ut valores singulorum coefficientium A, B, C, D etc. investigemus.

PROBLEMA 5

52. *Aequationem generalem supra Problemate 1 constitutam ita determinare, ut respondeant his abscissis*

$$x=n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta,\ etc.$$

hae applicatae

$$y=+f,\quad +f,\quad -f,\quad -f,\quad +f,\quad etc.,.$$

SOLUTIO

Statuatur ut ante $x=\theta\omega$ et consideretur aequatio quaesita sub hac forma

$$\begin{aligned}&y=A\omega+B\omega(\omega\omega-nn)+C\omega(\omega\omega-nn)(\omega\omega-(1-n)^2)\\&+D\omega(\omega\omega-nn)(\omega\omega-(1-n)^2)(\omega\omega-(1+n)^2)\\&+E\omega(\omega\omega-nn)(\omega\omega-(1-n)^2)(\omega\omega-(1+n)^2)(\omega\omega-(2-n)^2)\\&+etc.,\end{aligned}$$

unde deducuntur hae aequationes

$$\begin{aligned}&\frac{f}{n}=A,\\&\frac{f}{1-n}=A+B\cdot 1(1-2n),\\&\frac{-f}{1+n}=A+B\cdot 1(1+2n)+C\cdot 1(1+2n)\cdot 2\cdot 2n\\&\frac{-f}{1+n}=A+B\cdot 2(2-2n)+C\cdot 2(2-2n)\cdot 1(3-2n)\\&\quad +d\cdot 2(2-2n)\cdot 1(3-2n)\cdot 3(1-2n)\end{aligned}$$

etc.

hincque coefficientium valores sequentes

$$A=\frac{f}{n},\ \ B=\frac{-f}{n(1-n)},\ \ C=\frac{f}{2n(1-n)(1+n)},\ \ D=\frac{-f}{6n(1-n)(1+n)(2-n)},$$
$$E=\frac{f}{24n(1-n)(1+n)(2-n)(2+n)},\ \text{ etc.;}$$

quae progressio cum satis sit simplex, series nostra pro valore ipsius *y*, quem iam novimus esse

$$=\frac{f\sin.\omega\pi}{\sin.n\pi},$$

eo maiorem attentionem meretur; haecque est

$$\frac{\sin.\omega\pi}{\sin.n\pi}=\frac{\omega}{n}-\frac{\omega}{n}\cdot\frac{\omega\omega-nn}{1(1-n)}+\frac{\omega}{n}\cdot\frac{\omega\omega-nn}{1(1-n)}\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}$$
$$-\frac{\omega}{n}\cdot\frac{\omega\omega-nn}{1(1-n)}\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}\cdot\frac{\omega\omega-(1+n)^2}{3(2-n)}+\text{etc.},$$

vel si *Π* iugiter denotet terminum praecedentem, totum erit

$$\frac{\sin.\omega\pi}{\sin.n\pi}=\frac{\omega}{n}-\Pi\cdot\frac{\omega\omega-nn}{1(1-n)}+\Pi\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}$$
$$-\Pi\cdot\frac{\omega\omega-(1+n)^2}{3(2-n)}+\Pi\cdot\frac{\omega\omega-(2-n)^2}{4(2+n)}-\Pi\cdot\frac{\omega\omega-(2+n)^2}{5(3-n)}+\text{etc.}$$

Quodsi omnes termini eodem signo affecti desiderentur, erit

$$\frac{\sin.\omega\pi}{\sin.n\pi}=\frac{\omega}{n}+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}$$
$$+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}\cdot\frac{(1+n)^2-\omega\omega}{3(2-n)}+$$
$$\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}\cdot\frac{(1+n)^2-\omega\omega}{3(2-n)}\cdot\frac{(2-n)^2-\omega\omega}{4(2+n)}$$
$$+\text{etc.},$$

Haec series eo maiori attentione digna videtur, quod a solita serierum ratione plurimum recedit in eaque adeo duo numeri arbitrarii *n* et ω occurrunt.

COROLLARIUM 1

53. Si numerus ω evanescat, ut fiat $sin.\ \omega\pi=\omega\pi$, divisione per ω instituta habebitur haec aequatio

$$\frac{\pi}{\sin.n\pi}=\frac{1}{n}+\frac{n}{1(1-n)}+\frac{n(1-n)}{1\cdot 2(1+n)}+\frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}$$
$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(2+n)}+\text{etc.},$$

unde sumto $n=\frac{1}{2}$ ob $\sin.\frac{\pi}{2}=1$ erit

$$\pi=2+1+\frac{1\cdot 1\cdot 2}{2\cdot 4\cdot 3}+\frac{1\cdot 1\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 5}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 5}+\text{etc.}$$

seu

$$\pi=2+\frac{1}{2\cdot 2^1\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 2^3\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^5\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 2^7\cdot 9}+\text{etc.}$$
$$+1+\frac{1}{2\cdot 2^2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 2^4\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^6\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 2^8\cdot 9}+\text{etc.};$$

quarum serierum posterior cum sit semissis prioris, erit summa posterioris $=\frac{\pi}{3}$, cuius quidem ratio inde est manifesta, quod sit

$$\int\frac{dx}{\sqrt{(1-xx)}}=\text{ang.}\sin.x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\text{etc.},$$

unde illa series fit $=\frac{\text{ang.}\sin.x}{x}$ sumto $x=\frac{1}{2}$ ideoque $=2\frac{\pi}{6}=\frac{\pi}{3}$.

COROLLARIUM 2

54. Si alter numerus n evanescat, ut fiat $\sin.n\pi=n\pi$, et aequatio per n multiplicetur, orietur

$$\frac{\sin.\omega\pi}{\pi}=\omega-\frac{\omega^3}{1}+\frac{\omega^3(\omega^2-1)}{1\cdot 2\cdot 1^2}-\frac{\omega^3(\omega^2-1)(\omega^2-1)}{1\cdot 2\cdot 3\cdot 1^2\cdot 2}+\frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 1^2\cdot 2^2}$$
$$-\frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 1^2\cdot 2^2\cdot 3}+\text{etc.},$$

quae series per ω divisa in binos sequentes resolvitur

$$\frac{\sin.\omega\pi}{\omega\pi}=1+\frac{\omega^2(\omega^2-1)}{1\cdot 2\cdot 1^2}+\frac{\omega^2(\omega^2-1)^2(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 1^2\cdot 2^2}+\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 1^2\cdot 2^2\cdot 3^2}+\text{etc.},$$
$$-\frac{\omega^2}{1}-\frac{\omega^2(\omega^2-1)^2}{1\cdot 2\cdot 3\cdot 1^2\cdot 2}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 1^2\cdot 2^2\cdot 3}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 1^2\cdot 2^2\cdot 3^2\cdot 4}-\text{etc.}$$

Sumamus hic $\omega=\frac{1}{2}$; fiet

$$\frac{2}{\pi}=1-\frac{1\cdot 1\cdot 3}{1\cdot 1\cdot 1\cdot 2^5}-\frac{1\cdot 1\cdot 3\cdot 1\cdot 3\cdot 5}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2^{10}}-\frac{1\cdot 1\cdot 3\cdot 1\cdot 3\cdot 5\cdot 3\cdot 5\cdot 7}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 2^{10}}-\text{etc.},$$
$$-\frac{1\cdot 1}{2^2}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 1\cdot 3}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 2^6}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 1\cdot 3\cdot 3\cdot 5\cdot 3\cdot 5}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 5\cdot 2^{10}}-\text{etc.},$$

postrema series ita referri potest

$$-\frac{1}{2^2}-\frac{1\cdot1\cdot1\cdot3}{1\cdot1\cdot2\cdot2\cdot2^7}-\frac{1\cdot1\cdot3\cdot3\cdot3\cdot5}{1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot2^{12}}-\frac{1\cdot1\cdot3\cdot3\cdot3\cdot5\cdot5\cdot5\cdot7}{1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot3\cdot3\cdot4\cdot2^{17}}-\text{etc.}$$

COROLLARIUM 3

55. Si fuerit $n=\frac{1}{2}$, ut sit $\sin.n\pi=1$, erunt factores, ex quibus singulos seriei terminos formari oportet,

$$\frac{2\omega}{1}\cdot\frac{1-4\omega\omega}{1\cdot2}\cdot\frac{1-4\omega\omega}{3\cdot4}\cdot\frac{9-4\omega\omega}{3\cdot6}\cdot\frac{9-4\omega\omega}{5\cdot8}\cdot\frac{25-4\omega\omega}{5\cdot10}\cdot\frac{25-4\omega\omega}{7\cdot12}\cdot\text{etc.}$$

atque seriei summa erit $\sin.n\pi$, scilicet

$$\sin.\omega\pi=2\omega+\frac{2\omega(1-4\omega\omega)}{1\cdot2}+\frac{2\omega(1-4\omega\omega)^2}{1\cdot2\cdot3\cdot4}+\frac{2\omega(1-4\omega\omega)^2(9-4\omega\omega)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}+\text{etc.},$$

unde sumto $\omega=1$ esse debet

$$0=2-3+\frac{3}{2^2}+\frac{5}{2^3\cdot3}+\frac{5}{2^6\cdot3}+\frac{7}{2^7\cdot5}+\frac{7}{2^9\cdot5}+\frac{9}{2^{10}\cdot7}+\frac{5\cdot9}{2^{14}\cdot7}+\text{etc.}$$

cuius veritas calculum instituenti mox patebit.

SCHOLION

56. Pro hoc casu etiam solutio supra inventa attentius considerari meretur, quae ex § 36 ob

$$\Delta=\frac{(n-\omega)\pi\sin.2n\pi}{2n\sin.(n-\omega)\pi\cdot\sin(n+\omega)\pi}\ \text{et}\ y=\frac{f\sin.\omega\pi}{\sin.n\pi},$$

cum sit

$$p=f,\quad q=f,\quad r=-f,\quad s=-f,\quad t=f,\ u=f\quad etc.,$$

in hac continetur aequatione

$$\frac{\pi\cos.n\pi\cdot\sin.\omega\pi}{\omega\sin.(n-\omega)\pi\cdot\sin(n+\omega)\pi}$$
$$=\frac{1}{nn-\omega\omega}-\frac{1}{(1-n)^2-\omega^2}-\frac{1}{(1+n)^2-\omega^2}+\frac{1}{(2-n)^2-\omega^2}+\frac{1}{(2+n)^2-\omega^2}-\text{etc.},$$

quae series maxime discrepat ab ea, quam modo invenimus. Circa hanc autem seriem sequentia observo:

I. Si ω evanescat, fore

$$\frac{\pi\pi\cos.n\pi}{(\sin.n\pi)^2}=\frac{1}{nn}-\frac{1}{(1-n)^2}-\frac{1}{(1+n)^2}+\frac{1}{(2-n)^2}+\frac{1}{(2+n)^2}-\frac{1}{(3-n)^2}-\text{etc.};$$

at si insuper n evanescat, ob $\sin.n\pi = n\pi$ incommodum nasci sequens

$$\frac{1}{nn} = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \text{etc.}$$

Ad hoc autem tollendum, numerum n ut tantum non evanescentem spectemus. et cum sit

$$\cos.n\pi = 1 - \tfrac{1}{2}nn\pi\pi$$

et

$$\sin.n\pi = n\pi - \tfrac{1}{6}n^3\pi^3 = n\pi(1 - \tfrac{1}{6}nn\pi\pi),$$

erit

$$\frac{\cos.n\pi}{(\sin.n\pi)^2} = \frac{1-\frac{1}{2}nn\pi\pi}{nn\pi\pi(1-\frac{1}{3}nn\pi\pi)} = \frac{1-\frac{1}{6}nn\pi\pi}{nn\pi\pi};$$

unde obtinetur haec vera aequatio

$$\frac{1}{nn} - \frac{1}{6}\pi\pi = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \frac{2}{25} + \text{etc.}$$

Est enim

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{1}{12}\pi\pi.$$

II. Ponamus nunc $n = 0$ et habebimus

$$-\frac{\pi}{\omega\sin.\omega\pi} = -\frac{1}{\omega^2} - \frac{1}{1-\omega^2} - \frac{1}{1-\omega^2} + \frac{1}{4-\omega^2} + \frac{1}{4-\omega^2} - \frac{1}{9-\omega^2} - \frac{1}{9-\omega^2} + \text{etc.},$$

seu

$$\frac{\pi}{\omega\sin.\omega\pi} = \frac{1}{\omega^2} + \frac{2}{1-\omega^2} - \frac{2}{4-\omega^2} + \frac{2}{9-\omega^2} - \frac{2}{16-\omega^2} + \frac{2}{25-\omega^2} - \text{etc.},$$

consequimur hanc memorabilem summationem

$$\frac{1}{1-\omega^2} - \frac{1}{4-\omega^2} + \frac{1}{9-\omega^2} - \frac{1}{16-\omega^2} + \text{etc.} = \frac{\pi}{2\omega\sin.\omega\pi} - \frac{1}{2\omega\omega},$$

parvo ob

$$\sin.\omega\pi = \omega\pi(1 - \tfrac{1}{6}\omega^2\pi^2)$$

seriei

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.}$$

summa ut ante colligitur

$$\frac{1}{2\omega\omega(1-\frac{1}{6}\omega^2\pi^2)} - \frac{1}{2\omega\omega} = \frac{1}{12}\pi\pi.$$

III. Si capiatur $n=\frac{1}{2}$, ob $\cos. n\pi=0$ etiam ipsa series evanescit, dum scilicet omnes termini se mutuo revera destruunt. Quid autem eveniat, si *n* infinite parum a $\frac{1}{2}$ discrepet, differentiatio instituatur sumto *n* variabili, unde fit

$$\frac{n\pi\sin.n\pi\sin.\omega\pi(1+\cos.(n-\omega)\pi\cdot\cos.(n+\omega)\pi)}{\omega(\sin.(n-\omega)\pi\cdot\sin(n+\omega)\pi)^2}$$
$$=\frac{2n}{(nn-\omega\omega)^2}-\frac{2(1-n)}{((1-n)^2-\omega^2)^2}+\frac{2(1+n)}{((1+n)^2-\omega^2)^2}+\frac{2(2-n)}{((2-n)^2-\omega^2)^2}-\frac{2(2+n)}{((2+n)^2-\omega^2)^2}-\text{etc.},$$

Nunc igitur sumatur $n=\frac{1}{2}$ eritque

$$-\frac{\pi\pi\sin.\omega\pi}{\omega(\cos.\omega\pi)^2}=-\frac{16}{(1-4\omega^2)^2}-\frac{16}{(1-4\omega^2)^2}+\frac{3\cdot16}{(9-4\omega^2)^2}+\frac{3\cdot16}{(9-4\omega^2)^2}-\text{etc.}$$

sive

$$-\frac{\pi\pi\sin.\omega\pi}{32\omega(\cos.\omega\pi)^2}=\frac{1}{(1-4\omega^2)^2}-\frac{3}{(9-4\omega^2)^2}+\frac{5}{(25-4\omega^2)^2}-\frac{7}{(49-4\omega^2)^2}+\text{etc.}$$

unde sumto $\omega=0$ sequitur fore

$$\frac{\pi^3}{12}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.},$$

quod quidem aliunde constat.

Verum series in praesente problemate inventa multo magis ardua videtur. Quin adeo casus in corollario 1 evolutus, etsi maxime est particularis, diligentiori evolutione est dignus, quam in problemate sequente expedire conabor.

PROBLEMA 6

57. *Si n sit numerus quicunque, inquirere in summam huius seriei*

$$s=\frac{1}{n}+\frac{n}{1(1-n)}+\frac{n(1-n)}{1\cdot2(1+n)}+\frac{n(1-n)(1+n)}{1\cdot2\cdot3(2-n)}+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4(2+n)}+etc.,$$

quam quidem ante [§ 53] *invenimus esse*

$$s=\frac{\pi}{\sin.n\pi}.$$

SOLUTIO

Cum in hac serie lex progressionis sit interrupta, eam in duas discerpi conveniet. Statuamus ergo

$$P=\frac{1}{n}+\frac{n(1-n)}{1\cdot2(1+n)}+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4(2+n)}+\frac{n(1-n)(1+n)(2-n)(2+n)(3-n)}{1\cdot2\cdot3\cdot4\cdot5\cdot6(3+n)}+etc.,$$
$$Q=\frac{n}{1(1-n)}+\frac{n(1-n)(1+n)}{1\cdot2\cdot3(2-n)}+\frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot2\cdot3\cdot4\cdot5(3-n)}+etc.,$$

ita ut sit

$$s = P + Q.$$

harum iam serierum summas inquisiturus in subsidium voco sequentes ex doctrina angulorum petitas

$$\frac{\cos.\mu\varphi}{\cos.\varphi} = 1 + \frac{(1-\mu)(1+\mu)}{1\cdot 2}\sin.\varphi^2 + \frac{(1-\mu)(1+\mu)(2-\mu)(2+\mu)(3-\mu)(3+\mu)}{1\cdot 2\cdot 3\cdot 4}\sin.\varphi^4 + \text{etc.},$$

$$\frac{\cos.\nu\varphi}{\cos.\varphi} = \nu\sin.\varphi + \frac{\nu(2-\nu)(2+\nu)}{1\cdot 2\cdot 3}\sin.\varphi^3 + \frac{\nu(2-\nu)(2+\nu)(4-\nu)(4+\nu)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\sin.\varphi^5 + \text{etc.}$$

ac primo quidem illam ad formam priorem P accommodabo. Cum igitur hae fractiones

$$\frac{(1-\mu)(1+\mu)}{n(1-n)},\quad \frac{(3-\mu)(3+\mu)}{(1+n)(2-n)},\quad \frac{(5-\mu)(5+\mu)}{(2+n)(3-n)} \text{ etc.}$$

debeant esse aequales, concludo capi debere $\mu = 1 - 2n$, unde erit

$$\frac{\cos.(1-2n)\varphi}{\cos.\varphi} = 1 + \frac{n(1-n)}{1\cdot 2}\cdot 2^2\sin.\varphi^2 + \frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4}2^4\sin.\varphi^4 + \text{etc.}$$

Multiplicemus per $d\varphi\sin.\varphi^{2n-1}\cos.\varphi$ et integremus, fiet

$$\int d\varphi\sin.\varphi^{2n-1}\cos.(1-2n)\varphi = \frac{1}{2n}\sin.\varphi^{2n} + \frac{n(1-n)}{1\cdot 2(n+1)}\cdot 2\sin.\varphi^{2n+2}$$
$$+ \frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(n+2)}2^3\sin.\varphi^{2n+4} + \text{etc.}$$

Nunc post integrationem statuatur $\sin.\varphi = \frac{1}{2}$ seu $\varphi = 30^\circ$ eritque

$$P = 2^{2n+1}\int d\varphi\sin.\varphi^{2n-1}\cos.(1-2n)\varphi\ ;$$

series vero Q facile deducetur ex altera cognita sumendo $\nu = 2n$, unde fit

$$\frac{\sin.2n\varphi}{\cos.\varphi} = n\cdot 2\sin.\varphi + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3}\cdot 2^3\sin.\varphi^3$$
$$+ \frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot 2\cdot 3\cdot 4\cdot 5}2^5\sin.\varphi^5 + \text{etc.}$$

Multiplicetur per $d\varphi\sin.\varphi^{-2n}\cos.\varphi$ et integretur; erit

$$\int d\varphi\sin.\varphi^{-2n}\sin.2n\varphi = \frac{n}{1(1-n)}\sin.\varphi^{2-2n} + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}2^2\sin.\varphi^{4-2n} + \text{etc.}$$

Statuatur pariter integratione absoluta $\sin.\varphi = \frac{1}{2}$ seu $\varphi = 30^\circ$ ac prodibit series

$$Q = 2^{2-2n}\int d\varphi \sin.\varphi^{-2n} \sin.2n\varphi.$$

Quocirca seriei propositae summa ita exprimetur, ut sit

$$s = 2^{2n+1}\int d\varphi \sin.\varphi^{2n-1} \cos.(1-2n)\varphi \ + 2^{2-2n}\int d\varphi \sin.\varphi^{-2n} \sin.2n\varphi,$$

et quia haec summa iam aliunde est cognita, habebitur

$$\frac{\pi}{\sin.n\pi} = 4\int d\varphi \cos.(1-2n)\varphi(2\sin.\varphi)^{2n-1} + 4\int d\varphi \sin.2n\varphi(2\sin.\varphi)^{-2n}.$$

COROLLARIUM 1

58. Si ponatur $2n = \frac{1-\lambda}{2}$, erit $1-2n = \frac{1+\lambda}{2}$, qua positione nostra aequatio fit concinnior, eritque

$$\frac{\pi}{\sin.\frac{1-\lambda}{4}\pi} = 4\int \frac{d\varphi \cos.\frac{1+\lambda}{2}\varphi}{(2\sin.\varphi)^{\frac{1+\lambda}{2}}} + 4\int \frac{d\varphi \sin.\frac{1-\lambda}{2}}{(2\sin.\varphi)^{\frac{1-\lambda}{2}}} = \frac{\pi\sqrt{2}}{\cos.\frac{\lambda\pi}{4} - \sin.\frac{\lambda\pi}{4}}$$

posito post integrationem $\varphi = 30°$.

COROLLARIUM 2

59. Simili modo sumto λ negative erit

$$\frac{\pi}{\sin.\frac{1+\lambda}{4}\pi} = 4\int \frac{d\varphi \cos.\frac{1-\lambda}{2}\varphi}{(2\sin.\varphi)^{\frac{1-\lambda}{2}}} + 4\int \frac{d\varphi \sin.\frac{1+\lambda}{2}}{(2\sin.\varphi)^{\frac{1+\lambda}{2}}} = \frac{\pi\sqrt{2}}{\cos.\frac{\lambda\pi}{4} + \sin.\frac{\lambda\pi}{4}},$$

ubi quidem notasse iuvabit omnibus casibus, quos evolvere licet, eundem formularum integralium valorem actu reperiri, quem hic exhibuimus.